Note

A Correction in the Proof of the Turnpike Theorem for a General Equilibrium Model

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1　Introduction

This short research note gives a correction of the argument used in the standard proof of the turnpike theorem for a general equilibrium model given in Bewley (1980) and Bewley (1982). The argument gives a way to construct a feasible allocation from an initial stock at hand that dynamically converges to a stationary equilibrium allocation from below, with its convergence being dominated by that of a geometric sequence multiplied by the distance between the initial stock at hand and that of the stationary equilibrium, when the distance between the initial stock at hand and that of the stationary equilibrium is small. An error is exposed to exist in guaranteeing the feasibility of the allocation so obtained. A correction proposed in this note relies on the strict convexity of production technologies, and fails to work only with their convexity.

The turnpike theorem asserts that, in a dynamic general equilibrium model, any competitive equilibrium allocation from a given initial stock of produce-able commodities converges to a stationary equilibrium allocation if all agents have a common time-discount rate that is close enough to 1. It implies that, for each economic agent, forming a habit along a stationary equilibrium is as good as an optimizing behavior with a perfect foresight in the long run. Considering an enormous information-processing cost associated with an optimizing behavior with a perfect foresight that is un-modeled, the theorem suggests that a behavior of forming a good habit with a simplified expectation would be far superior to an optimizing behavior with a perfect foresight.

A part of the proof requires a construction of a feasible allocation from the initial stock for a competitive equilibrium allocation that dynamically converges to a stationary equilibrium allocation from below, with its convergence being dominated by that of a geometric sequence multiplied by the distance between the initial stock and that of the stationary equilibrium when the latter is small. The original proof applies the standard argument of D. Gale, which starts from a situation that a stationary production makes all commodities exist positively and constructs a feasible allocation recursively by replacing a tiny part of inputs and consumptions by those of the stationary equilibrium allocation. However, the argument fails to assure a situation that intermediary commodities in the stationary equilibrium allocation exist positively so that the feasibility of the constructed allocation is not guaranteed. An intermediary commodity is the commodity that is produce-able but not consumed, so that its output by a firm is used for inputs by other firms.

A correction proposed in this note relies on the strict convexity of technologies and that zero production is feasible for all firms. Each firm obtains a stationary production plan that is in the interior of its production possibility set by scaling down the stationary equilibrium production plan. Then its output part can be magnified so that the plan stays on the production possibility frontier. An assumption made for the turnpike theorem postulates that all produce-able commodities, including all intermediary commodities, are positively produced in the stationary equilibrium allocation. Hence the procedure gives a situation that all commodities exist positively with such a production.

The note is organized as follows. The section 2 explains the error in the original proof in detail. A correction is proposed and proved to work in the section 3. The necessity of the strict convexity of technologies for the correction is discussed in the section 4.

The notations, the model and the assumptions used in this note follow those in Bewley (1980) and Bewley (1982), unless they are explicitly stated.

2　The Error in the Proof

In Bewley (1980) and Bewley (1982), the hardest part of the proof is to prove that there exist $\epsilon > 0$ and $A > 0$ such that $F_\delta(K) \leq A|K - \bar{K}|^2$ if $|K - \bar{K}| < \epsilon$, where K is the initial stock for a competitive equilibrium allocation, $\bar{K}$ is that for a stationary equilibrium allocation in which each consumer has his marginal utility of money equal to that in the competitive equilibrium, and F_δ is the Lyapunov function whose stability is equivalent to the assertion of the turnpike theorem. This proposition is referred as the second key finding.

The original proof claims that the first step of constructing a feasible allocation from K for the proof of the second key finding is based on the following inequality,

$$\frac{1}{2}\sum_i \bar{x}_i \leq \omega + \sum_j \left(\frac{1}{2}\bar{y}_{j,0} + \frac{1}{4}\bar{y}_{j,0}\right) + \sum_j \bar{y}_{j,1}.$$

It claims that the method of construction pioneered by D. Gale can be applied for this inequality with the replacement coefficient $\theta = \frac{1}{2}$ and the stationary production plan for each j, $y'_j = \left(\frac{1}{2}\bar{y}_{j,0},\ C_j\frac{1}{2}\bar{y}_{j,1}\right)$, where $C_j > 1$ is uniquely determined so that $g_j\left(\frac{1}{2}\bar{y}_{j,0},\ C_j\frac{1}{2}\bar{y}_{j,1}\right) = 0$, and a desired feasible allocation from $\bar{K}$, $((\tilde{x}_i),\ (\tilde{y}_j))$, is obtained. It actually does not give what the vector y'_j is, only states that $y'_{j,0} = \frac{1}{2}\bar{y}_{j,0}$.

Though the inequality is obviously true, the construction from that is not assured to work. The initial stock in period 1 is at most $\Sigma_j C'_j\frac{3}{4}\bar{y}_{j,1}$, where $C'_j > 1$ and $g_j\left(\frac{3}{4}\bar{y}_{j,0},\ C'_j\frac{3}{4}\bar{y}_{j,1}\right) = 0$, so that $C'_j\frac{3}{4} < 1$ for all j. According to the construction, $\tilde{x}_i^1 = \frac{3}{4}\bar{x}_i$ and $\tilde{y}_j^1 = \frac{7}{8}\bar{y}_j$. Then $\Sigma_i \tilde{x}_i^1 = \frac{3}{4}\Sigma_i \bar{x}_i$. Multiplying $\frac{1}{2}$ to the inequality shown above, $\frac{1}{2}$ to the feasibility equation of the stationary allocation, and summing them up, we obtain $\frac{3}{4}\Sigma_i \bar{x}_i \leq \omega + \Sigma_j \frac{7}{8}\bar{y}_{j,0} + \Sigma_j \bar{y}_{j,1}$. This inequality does not tell anything about the feasibility of the construted allocation in period 1. The argument can be refined by using the fact that $\omega + \Sigma_j \frac{1}{2}\bar{y}_{j,0} + \Sigma_j \frac{1}{2}\bar{y}_{j,1} \geq \frac{1}{2}\Sigma_i \bar{x}_t$, making $\bar{y}_{j,1}$ in the inequality be replaced by $\frac{3}{4}\bar{y}_{j,1}$. Then we obtain $\frac{3}{4}\Sigma_i \bar{x}_i \leq \omega + \frac{7}{8}\Sigma_j \bar{y}_{j,0} + \frac{7}{8}\Sigma_j \bar{y}_{j,1}$, which still does not guarantee the feasibility in period 1 unless $C'_j \geq \frac{7}{6}$ for all j.

Suppose that there is an intermediary commodity in the stationary equilibrium allocation and let it k. Then $\Sigma_j(-\bar{y}_{j,0,k}) = \Sigma_j \bar{y}_{j,1,k} > 0$, so that the stock of at least $\frac{7}{8}\Sigma_j \bar{y}_{j,1,k}$ is required for such a commodity k in order to guarantee the feasibility in period 1. It is not guaranteed unless $C'_j \geq \frac{7}{6}$ for all j.

There is no assumption that assures $C'_j \geq \frac{7}{6}$ for all j in any stationary equilibrium allocation with $\delta \geq \underline{\delta}$. It is clearly absurd to assume it directly, since no plausible economic implication can be associated with it.

The error stems from a misunderstanding of the method of construction. It must start from a situation where a total stationary production combined with the total (stationary) initial endowment can make all commodities exist positively in the stationary economy. This situation is expressed by $\omega_k + \Sigma_j y'_{j,0,k} + \Sigma_j y'_{j,1,k} > 0$ for all commodity k's, where y'_j must be feasible for j. Then a tiny part of $y'_{j,0}$, $(1-\theta)$ with $\theta < 1$ very close to 1, can be replaced by $\bar{y}_{j,0}$ for all j with an equally tiny part of consumption $\bar{x}_i$ being still available for i, for all i. By summing up ($\theta\times$the feasibility inequality in period t) and ($(1-\theta)\times$the feasibility inequality for the stationary equilibrium allocation), it is verified that such a replacement is recursively feasible in period ($t+1$). Hence a feasible allocation converging to the stationary allocation is obtained. In the proof of the turnpike theorem, $y'_{j,0}$ and $y'_{j,1}$ must be taken to be co-linear with $\bar{y}_{j,0}$ and $\bar{y}_{j,1}$ respectively. Let $y'_{j,0} = \gamma_j \bar{y}_{j,0}$ where $\gamma_j < 1$. Then it must be the case that $y'_{j,1} \leq C_j(\gamma_j)\gamma_j \bar{y}_{j,1}$, where $g_j(\gamma_j \bar{y}_{j,0},\ C_j(\gamma_j)\gamma_j \bar{y}_{j,1}) = 0$. A natural choice is of course $y'_{j,1} = C_j(\gamma_j)\gamma_j \bar{y}_{j,1}$. Since g_j is strictly convex, $C_j(\gamma_j) > 1$ so that such a choice guarantees that $\omega_k + \Sigma_j y'_{j,0,k} + \Sigma_j y'_{j,1,k} > 0$ even for an intermediary commodity k in the stationary equilibrium allocation. The original proof picks $\gamma_j = \frac{1}{2}$ for all j and postulates that $C_j\left(\frac{1}{2}\right)\frac{1}{2}$ is close enough to 1 for all j to justify that the replacement coefficient θ can be also taken as $\frac{1}{2}$. But such a postulation is not implied directly by the set of natural assumptions for the turnpike theorem.

3 A Correction

A correction can be obtained quite easily, but it relies on the strict convexity of g_j's. Let $\gamma_j = \gamma < 1$ for all j and follow the natural choice of $y'_{j,1}$'s. The strict convexity of g_j's and that $g_j(0) = 0$ for all j's then guarantee that y'_j is feasible for j and all commodities exist positively with such a production. Choose θ sufficiently close to 1 so that the replacement in inputs and consumptions by those in the stationary equilibrium allocation is feasible, $\Sigma_i(1-\theta)\bar{x}_i \leq \omega + \Sigma_j[(1-\theta)\bar{y}_{j,0} + \theta\gamma\bar{y}_{j,0}] + \Sigma_j C_j(\gamma)\gamma\bar{y}_{j,1}$. By repeating the replacement recursively, a feasible allocation from $\Sigma_j C_j(\gamma)\gamma\bar{y}_{j,1} \leq \bar{K}$, $((\underset{\sim}{\tilde{x}_i}), (\underset{\sim}{\tilde{y}_j}))$, is obtained as $\tilde{x}_i^t = (1-\theta^{t+1})\bar{x}_i$ for i and $\tilde{y}_j^t = ((1-\theta^{t+1}(1-\gamma))\bar{y}_{j,0}, (1-\theta^{t+1}(1-C_j(\gamma)\gamma)\bar{y}_{j,1}))$ for j, for all t. Any $\gamma < 1$ serves fine for this particular goal, but it must be chosen so that cutting this allocation out in some period makes its tail to be a feasible allocation from K whose dynamic convergence to the stationary equilibrium allocation is dominated by a geometric sequence of θ multiplied by $|K-\bar{K}|$.

The demand for $k \in L_p$ in period t, the sum of consumptions and inputs of k in period t, is no more than $(1-\theta^{t+1})\bar{K}_k+\theta^{t+1}\gamma\Sigma_j(-\bar{y}_{j,0,k})$. Since what is needed in the proof is a feasible allocation from K, not from $\bar{K}$, this demand must stay below K_k for all $k \in L_p$ in some period if K is taken sufficiently close to $\bar{K}$, by which the allocation can be cut out at a convenient period. Since $K_k \geq \bar{K}_k-|K-\bar{K}|$ for all $k \in L_p$, the upper-bound should stay below $\bar{K}_k-|K-\bar{K}|$ in some period which requires $\bar{K}_k-[(1-\theta^{t+1})\bar{K}_k+\theta^{t+1}\gamma\Sigma_j(-\bar{y}_{j,0,k})]=\theta^{t+1}(\bar{K}_k+\gamma\Sigma_j\bar{y}_{j,0,k})\geq|K-\bar{K}|$. One implication is $\gamma\Sigma_j(-\bar{y}_{j,0,k})<\bar{K}_k$. Since $\bar{K}_k\geq\zeta$ for all $k \in L_p$ by the key assumption for the turnpike theorem, it is sufficient to make $\gamma\Sigma_j(-\bar{y}_{j,0,k})<\zeta$. Then, since θ^t decreases with t, it is sufficient to guarantee that $\theta[\zeta-\gamma\Sigma_j(-\bar{y}_{j,0,k})]\geq|K-\bar{K}|$. Since θ can be chosen arbitrarily close to 1, it is larger than a positive number. Hence it is sufficient to guarantee that $(\zeta-\gamma\Sigma_j(-\bar{y}_{j,0,k}))$ is no less than a positive constant number for all $k \in L_p$, since then ϵ, an upper-bound for $|K-\bar{K}|$, can be chosen adequately to guarantee the condition. Taking γ so that $\gamma\Sigma_j(-\bar{y}_{j,0,k})\leq\frac{\zeta}{2}$ is enough for this purpose, which leads to take γ to satisfy $\gamma|\bar{y}_{j,0}|\leq\frac{\zeta}{2J}$.

Once a period in which all demands for produce-able commodities are covered by K exists, the rest of proof proceeds just as the original one in Bewley (1980). By the choice of γ, the lowest estimate of $\theta^{t+1}(\bar{K}_k+\gamma\Sigma_j\bar{y}_{j,0,k})$ for all $k \in L_p$ is $\theta^{t+1}\frac{\zeta}{2}$. Let τ be the last period t to satisfy $\theta^{t+1}\frac{\zeta}{2} \geq |K - \bar{K}|$. In period τ, $\theta^{\tau+1}(\bar{K}_k + \gamma \Sigma_j \bar{y}_{j,0,k}) \geq |K - \bar{K}|$ is satisfied so that the feasible allocation can be cut out there to make the tail feasible from K. Since $\theta^{\tau+2}\frac{\zeta}{2} < |K - \bar{K}|$, by letting $\theta \geq$ a constant, $\theta^{\tau+1}$ is less than a constant $\times |K - \bar{K}|$. Hence $\theta^{\tau+t+1}$ is less than a constant $\times \theta^t \times |K - \bar{K}|$. This is combined with the condition $|((\bar{x}_i),(\bar{y}_j)) - ((\tilde{x}_i^t),(\tilde{y}_j^t))| \leq \theta^{t+1}|((\bar{x}_i),(\bar{y}_j))|$, which is derived from the construction of $((\tilde{x}_i), (\tilde{y}_j))$, to yield the estimate

$$|((\bar{x}_i),(\bar{y}_j)) - ((\tilde{x}_i^{\tau+t}), (\tilde{y}_j^{\tau+t}))| \leq \theta^{\tau+1}\theta^t B \leq \text{a constant} \times \theta^t \times |K - \bar{K}|,$$

where B is a uniform upper-bound for feasible stationary allocations. Hence the convergence of the tail after τ to the stationary equilibrium allocation is dominated by a geometric sequence of θ multiplied by $|K - \bar{K}|$, which is all required after the first step of the construction of a feasible allocation from K in the proof of the second key finding.

4 A Remark on Strict Convexity

It should be noted that the correction proposed in this article does not work for the case that g_j's are only convex. In such a case, $C_j(\gamma)$ may be 1 for all j. If there is an intermediary commodity in the stationary equilibrium allocation, the choice of y'_j's does not make that commodity exist positively with the production. Replacing a tiny part of inputs by that of stationary equilibrium allocation makes an excess demand on that commodity, so that the replacement of zero consumptions by those of stationary equilibrium allocation is impossible no matter how small the part of replacement is. Hence the correction cannot be applied to works using only concavity of production possibility sets, such as Yano (1984) and Yano (1985). The existence of an intermediary commodity in a stationary equilibrium allocation must not be overlooked since, in reality, the number of such commodities dominates that of primary commodities and consume-able commodities combined.

Relaxing the assumption of strict convexity to mere convexity is actually out of point since the role of the the turnpike theorem is to give a direction in forming a good habit on trades to each economic agent. Without elaborating on details, Bewley (1980) claims that the turnpike theorem would hold with only the concavity of utility functions and the convexity of production transformation functions if stationary equilibria is replaced by the von Neumann facet. The concept of von Neumann facet was formally introduced in McKenzie (1968), and its role as the limit to which optimal growth paths converge was developed without time-discount for an optimal growth model in the same paper. In general equilibrium models, the von Neumann facet is the set of equilibrium allocations under a stationary equilibrium price with the same marginal utilities of income as those in the stationary equilibrium. A simple example in Bewley (1980) shows that the von Neumann facet would contain an allocation that is cyclical in production, if the technologies are of constant returns to scale. The facet actually contains a continuum of allocations in which consumers maximize their utilities and firms maximize their profits under the same stationary price, and that fact is known to each economic agent to whom multiple choices are optimal. Under such a circumstance, for an economic agent to be convinced that a particular consumption or production program is a good habit, he needs to know exactly what all other economic agents adopt as a good habit, which is clearly unrealistic. To make an allocation in the facet to be justified as a good habit, preferences and technologies must be slightly modified so that they are locally strictly convex at the programs in the allocation. Then there is no reason for not choosing the stationary equilibrium allocation itself to be implemented as a good habit, since it proposes the simplest program to each economic agent. So the case just goes back to the standard one with strict convexity.

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