（千葉大学審査学位論文）

Relationships between blocks of finite groups and their centers

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Chiba University
Graduate School of Science
Department of Mathematics and Informatics

Yoshihiro Otokita

Contents

1 Introduction

The present paper deals with some problems on modular representation theory of finite groups. In particular we study the center of a block of a finite group over an algebraically closed field of prime characteristic.

Let G be a finite group, $\mathcal{O}$ a complete discrete valuation ring with quotient field K of characteristic 0 and $F = \mathcal{O}/\mathfrak{p}$ its residue field of characteristic $p > 0$. We assume that K contains all $|G|$-th roots of unity and F is algebraically closed. For a block B of the group algebra FG we denote by $k(B)$ and $l(B)$ the numbers of irreducible ordinary and Brauer characters associated to B, respectively and we let D be a defect group of B of order p^d.

This paper is organized as follows.

In the next chapter we study the Cartan matrix C_B of B. It is well-known that $l(B) \leq k(B)$ with equality if and only if B is a simple algebra. In this case $k(B) = l(B) = 1$ and $C_B = (1)$ (e.g. see Nagao-Tsushima [23, III, Theorem 6.29, 6.37]). So the main purpose of this chapter is to consider blocks with $k(B) - l(B) = 1$. For example, if all the diagonal entries of C_B are two, then B satisfies this condition (see Michler [21]). In general Héthelyi-Kessar-Külshammer-Sambale [9] proved that D is elementary abelian whenever $k(B) - l(B) = 1$ by using the classification of finite simple groups. In this chapter we examine two cases that $p = 2, k(B) - l(B) = 1$ and that $k(B) = 3$. For this purpose we use the fact that blocks of finite groups are symmetric algebras. By this, we review some basic properties of such algebras and describe a result of Héthelyi-Horváth-Külshammer-Murray [8] before the proof of our main theorems.

The third chapter is devoted to improve Brandt's inequality and Okuyama's formula. In [1] Brandt has proved that

$$l(B) + \sum_{S} \dim \operatorname{Ext}^1_B(S, S) \leq k(B) - 1 \tag{1.1}$$

where S ranges over all the isomorphism classes of irreducible right B-modules, if $|D| > 2$. On the other hand Okuyama [26] has characterized the left side of (1.1) by using the center $Z(B)$ and the second socle $\operatorname{soc}^2(B)$ of B as follows:

$$\dim \operatorname{soc}^2(B) \cap Z(B) = l(B) + \sum_{S} \dim \operatorname{Ext}^1_B(S, S). \tag{1.2}$$

Remark that we can obtain (1.1) as a corollary to (1.2) (see Corollary 3.6). The article [26] is written in Japanese, so see Koshitani [13] for the original proof. The studies in this chapter are inspired by these facts. We improve (1.2) and describe relationships between the Loewy structure of B and ideals of $Z(B)$.

In the last chapter we study the structure of B through the Loewy length $LL(Z(B))$ of the center $Z(B)$. A result of Okuyama in [25] states that $LL(Z(B)) \leq |D|$ with equality if and only if B is a nilpotent block and D is cyclic. In this case B is Morita equivalent to the group algebra of a cyclic group of order p^d. In this chapter we improve this inequality. More precisely, we give three upper bounds for $LL(Z(B))$ in terms of $k(B), l(B), D$ and B-subsections. As an application we characterize blocks by using $LL(Z(B))$. Our main theorems in this chapter indicate that we can classify all blocks with $|D| - 3 \leq LL(Z(B)) \leq |D| - 1$ into 8 types.

At the end of this chapter we mention further notation and terminology. Throughout this paper the sets of all the p-elements (resp. p'-elements) in G are denoted by G_p (resp. $G_{p'}$). In addition $\operatorname{Cl}(G)$ (resp. $\operatorname{Cl}(G_{p'})$) denote the sets of all the G-conjugacy classes in G (resp. $G_{p'}$). For two subgroups $H, K \leq G$ we write $H \leq_G K$ if H is G-conjugate to a subgroup of K.

Similarly $h \in_G K$ means that h is G-conjugate to an element in K where $h \in H$. Moreover we use $H \times K$ (resp. $H \rtimes K$) to express a direct product (resp. a non-trivial semi-direct product) of H and K. The *exponent* of G is defined to be the least positive integer $n > 0$ such that $g^n = 1$ for all $g \in G$. For two integers $m, n \geq 1$, C_m denotes a cyclic group of order m and put $C_m^n = C_m \times \cdots \times C_m$ (n-factors). For instance C_p^r is an elementary abelian p-group of p-rank r. Unless otherwise noted we let Λ be a finite-dimensional algebra over F and all Λ-modules are assumed to be finite generated right Λ-modules.

2 Ordinary and Brauer characters

The main purpose of this chapter is to prove the following theorems:

Theorem 2.1 (Otokita [27]). *Let B be a block of FG and C_B its Cartan matrix. Then the following hold.*

(1) If $p = 2$ and $k(B) - l(B) = 1$, then all the diagonal entries of C_B are even.

(2) If $k(B) = 3$, then p is odd.

First of all we note three remarks of this theorem.

- Theorem 2.1 (1) is not true for $p \geq 3$ in general. Let us take the principal block B_0 of $G = PSL(3, 4)$ where $p = 3$ as an example. Then we have $k(B_0) = 6, l(B_0) = 5$ and

$$C_{B_0} = \begin{pmatrix} 5 & 1 & 1 & 1 & 4 \\ 1 & 2 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 4 & 2 & 2 & 2 & 5 \end{pmatrix}$$

- We recall a result of Külshammer [17]. If $l(B) = 1$ and $k(B) = 3$, then $p = 3$ and $D \simeq C_3$. Hence we need only consider the case that $l(B) = 2$ and $k(B) = 3$ in Theorem 2.1 (2).

- Finally, we introduce two results of Héthelyi-Külshammer and Maróti. Héthelyi-Külshammer [10] has proved that $2\sqrt{p-1} \leq |\mathrm{Cl}(G)|$ for all solvable groups, if p divides $|G|$. On the basis of this result they conjectured that $2\sqrt{p-1} \leq k(B)$ for all blocks with non-trivial defect groups. A recent paper Maróti [20] generalizes the first inequality above for all groups. Namely, it is shown that $2\sqrt{p-1} \leq |\mathrm{Cl}(G)|$ for any finite group G and any prime p which divides $|G|$. However the conjecture in [10] for blocks still remains an open problem. If this conjecture is true, then we obtain from Theorem 2.1 (2) that $p = 3$ provided $k(B) = 3$.

In the proof of Theorem 2.1 we use the fact that blocks of finite groups are symmetric algebras. Hence we review some basic properties of such algebras. As mentioned in the first chapter let Λ be a finite-dimensional algebra over an algebraically closed field F of characteristic $p > 0$. We now recall the definitions of Frobenius and symmetric algebras.

Definition 2.2. We say that Λ is a *Frobenius algebra* if there is an F-linear map $\lambda : \Lambda \to F$ such that Ker λ contains no non-zero left or right ideal of Λ. Moreover, a Frobenius algebra Λ is said to be a *symmetric algebra* if $\lambda(ab) = \lambda(ba)$ for all $a, b \in \Lambda$.

Lemma 2.3. *Let Λ be a symmetric algebra with an F-linear map $\lambda : \Lambda \to F$. If e is an idempotent in Λ, then $e\Lambda e$ is also a symmetric algebra through the restriction of λ to $e\Lambda e$.*

Lemma 2.4 (e.g. Külshammer [15] or [16])**.** *The group algebra FG of a finite group G over F is a symmetric algebra through an F-linear map defined by*

$$FG \to F, \quad \sum_{g \in G} a_g g \mapsto a_1.$$

As a consequence, blocks of FG are also symmetric algebras from Lemma 2.3.

We prepare some notation and lemmas in order to describe a result of Héthelyi-Horváth-Külshammer-Murray [8]. In the following we assume that Λ is a symmetric algebra with an F-linear map $\lambda : \Lambda \to F$. We put

$$[\Lambda, \Lambda] = \sum_{a,b \in \Lambda} F(ab - ba),$$

$$T_n(\Lambda) = \{a \in \Lambda \mid a^{p^n} \in [\Lambda, \Lambda]\} \text{ for an integer } n \geq 0, \text{ and}$$

$$T(\Lambda) = \bigcup_{n=0}^{\infty} T_n(\Lambda).$$

Lemma 2.5 (Külshammer [15] or [16]). *Let $a, b \in \Lambda$ and let $n \geq 0$ be an integer. Then the following hold.*

(1) $(a+b)^{p^n} \equiv a^{p^n} + b^{p^n} \mod [\Lambda, \Lambda]$.

(2) If $a \in [\Lambda, \Lambda]$, then $a^{p^n} \in [\Lambda, \Lambda]$.

Therefore

$$[\Lambda, \Lambda] = T_0(\Lambda) \subseteq T_1(\Lambda) \subseteq \cdots \subseteq T_n(\Lambda) \subseteq \cdots \subseteq T(\Lambda)$$

is a chain of F-subspaces of Λ and there exists an integer $m \geq 0$ such that $T_m(\Lambda) = T(\Lambda)$. In particular the next lemma holds for blocks of finite groups.

Lemma 2.6 (Külshammer [16]). *Let B be a block of FG with non-trivial defect group D. If the exponent of D is p^ε, then $T_{\varepsilon-1}(B) \subsetneq T_\varepsilon(B) = T(B)$.*

For a subspace U of Λ we define

$$U^\perp = \{a \in \Lambda \mid \lambda(Ua) = 0\}.$$

Lemma 2.7. *Let U be a subspace of Λ. Then the following hold.*

(1) $U^\perp$ is also a subspace of Λ and $(U^\perp)^\perp = U$.

(2) $\dim U^\perp = \dim \Lambda - \dim U$.

We denote by $Z(\Lambda)$ the center, by $J(\Lambda)$ the Jacobson radical and by $\mathrm{soc}(\Lambda)$ the socle of Λ. We define the *Reynolds ideal* $R(\Lambda)$ by

$$R(\Lambda) = \mathrm{soc}(\Lambda) \cap Z(\Lambda).$$

Moreover let us denote by $l(\Lambda)$ the number of isomorphism classes of irreducible Λ-modules.

Lemma 2.8 (Külshammer [15]). *The following hold.*

(1) $J(\Lambda)^\perp = \mathrm{soc}(\Lambda)$ and $[\Lambda, \Lambda]^\perp = Z(\Lambda)$.

(2) $T(\Lambda) = J(\Lambda) + [\Lambda, \Lambda]$ *and* $T(\Lambda)^\perp = R(\Lambda)$*.*

(3) $l(\Lambda) = \dim \Lambda/T(\Lambda) = \dim R(\Lambda)$*.*

(4) For each $n \geq 0$*,* $T_n(\Lambda)^\perp$ *is an ideal of* $Z(\Lambda)$*.*

We obtain from Lemma 2.8 a chain

$$R(\Lambda) = T(\Lambda)^\perp \subseteq \cdots \subseteq T_n(\Lambda)^\perp \subseteq \cdots \subseteq T_1(\Lambda)^\perp \subseteq T_0(\Lambda)^\perp = Z(\Lambda)$$

of ideals of $Z(\Lambda)$. For each $n \geq 0$, Külshammer [19] defines an F-semilinear map $\zeta_n : Z(\Lambda) \to Z(\Lambda)$ and shows $\operatorname{Im} \zeta_n = T_n(\Lambda)^\perp$. Here we introduce this result.

Lemma 2.9 (Külshammer [19])**.** *Let* Λ *be a symmetric algebra over* F *with an* F*-linear map* $\lambda : \Lambda \to F$*. Then, for any* $n \geq 0$*, there exists an* F*-semilinear map* $\zeta_n : Z(\Lambda) \to Z(\Lambda)$ *which satisfies the following conditions:*

(1) $\lambda(a^{p^n} z) = \{\lambda(a\zeta_n(z))\}^{p^n}$ *for all* $a \in \Lambda, z \in Z(\Lambda)$*.*

(2) $\zeta_n \circ \zeta_m = \zeta_{n+m}$ *for all* $m, n \geq 0$*.*

(3) $\zeta_n(z_1^{p^n} z_2) = z_1 \zeta_n(z_2)$ *for all* $z_1, z_2 \in Z(\Lambda)$*.*

(4) $\operatorname{Im} \zeta_n = T_n(\Lambda)^\perp$*.*

In the following we focus on $T_1(\Lambda)^\perp$.

Lemma 2.10 ([8, Theorem 2.3])**.** $(T_1(\Lambda)^\perp)^2 \subseteq R(\Lambda)$*.*

We express the entries of the Cartan matrix of Λ by using primitive idempotents in Λ. Two idempotents e and f in Λ are said to be Λ-*conjugate* if $e = u^{-1} f u$ for some $u \in \Lambda^\times$. For such idempotents, $e\Lambda$ and $f\Lambda$ are isomorphic as Λ-modules. Thus we can take representatives $\{e_i\}_{1 \leq i \leq l(\Lambda)}$ for the Λ-conjugacy classes of primitive idempotents in Λ and we may assume that $\{e_i \Lambda\}_{1 \leq i \leq l(\Lambda)}$ form a complete set of isomorphism classes of projective indecomposable Λ-modules. Moreover we can define the Cartan matrix $C_\Lambda = (c_{ij})_{1\le i,j\le l(\Lambda)}$ by

$$\begin{aligned} c_{ij} &= \dim \operatorname{Hom}_\Lambda(e_i\Lambda, e_j\Lambda) \\ &= \dim e_i\Lambda e_j. \end{aligned}$$

Here we note a lemma on dual bases of symmetric algebras.

Lemma 2.11. *Let Λ be a symmetric algebra over F with an F-linear map $\lambda : \Lambda \to F$ and let $\{a_i\}_{1\le i\le n}$ be its F-basis, where $n = \dim \Lambda$. Then there exists an F-basis $\{b_i\}_{1\le i\le n}$ such that*

$$\lambda(a_ib_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

We choose an F-basis $\{a_j\}_{l(\Lambda)+1\le j\le n}$ of $T(\Lambda)$, where $n = \dim \Lambda$. Then $\{a_i\}_{1\le i\le n}$ form an F-basis of Λ, where $a_i = e_i$ for all $1 \le i \le l(\Lambda)$. Let $\{b_i\}_{1\le i\le n}$ be a dual basis of $\{a_i\}_{1\le i\le n}$ and put $r_i = b_i$ for $1 \le i \le l(\Lambda)$. Thereby $\{r_i\}_{1\le i\le l(\Lambda)}$ form an F-basis of $R(\Lambda)$ since $r_1, \ldots, r_{l(\Lambda)} \in T(\Lambda)^\perp = R(\Lambda)$ and $l(\Lambda) = \dim R(\Lambda)$. Hence it follows from Lemma 2.10 that we can write $\zeta_1(1)^2$ as an F-linear combination of $r_1, \ldots, r_{l(\Lambda)}$. In particular the next lemma holds.

Lemma 2.12 ([8, Lemma 3.4])**.** *If $p = 2$, then*

$$\zeta_1(1)^2 = \sum_{1\le i\le l(\Lambda)} c_{ii}r_i.$$

Finally, we define the *Higman ideal* and *projective center* of $Z(\Lambda)$. Let $\{a_i\}_{1\le i\le n}$ and $\{b_i\}_{1\le i\le n}$ be a pair of dual bases of Λ. We define an F-linear map

$$\tau : \Lambda \to \Lambda, \quad x \mapsto \sum_{1\le i\le n} b_ixa_i.$$

Then the next lamme holds.

Lemma 2.13 ([8, Lemma 4.1])**.** *The F-linear map $\tau : \Lambda \to \Lambda$ defined above satisfies the following conditions:*

(1) τ *is independent of the choice of dual bases.*

(2) $\mathrm{Im}\ \tau \subseteq R(\Lambda)$ *and* $T(\Lambda) \subseteq \mathrm{Ker}\ \tau$.

The *Higman ideal* of $Z(\Lambda)$ is defined by $H(\Lambda) = \mathrm{Im}\tau$. This definition does not depend on the choice of dual bases from the lemma above.

Now let $e_1, \dots, e_{l(\Lambda)}$ and $r_1, \dots, r_{l(\Lambda)}$ be as in Lemma 2.12.

Lemma 2.14 ([8, Lemma 4.3])**.** *We have*

$$\tau(e_i) = \sum_{1 \le j \le l(\Lambda)} c_{ij} r_j$$

for each $1 \le i \le l(\Lambda)$.

Secondly, we define the *projective center* of $Z(\Lambda)$. We denote by Λ° the opposite algebra and by $\Lambda^\circ \bigotimes_F \Lambda$ the enveloping algebra of Λ. Then Λ is a right $\Lambda^\circ \bigotimes_F \Lambda$-module by the following action:

$$x(a \otimes b) = axb \qquad \text{for } x \in \Lambda \text{ and } a \otimes b \in \Lambda^\circ \bigotimes_F \Lambda.$$

Furthermore

$$\mathrm{End}_{\Lambda^\circ \bigotimes_F \Lambda}(\Lambda) \to Z(\Lambda),\ \rho \mapsto \rho(1)$$

is an algebra isomorphism. The *projective center* $Z^{\mathrm{pr}}(\Lambda)$ is defined by

$$Z^{\mathrm{pr}}(\Lambda) = \{\rho(1) \mid \rho \in \mathrm{End}_{\Lambda^\circ \bigotimes_F \Lambda}(\Lambda) \text{ factors through a projective } \Lambda^\circ \bigotimes_F \Lambda\text{-module}\}.$$

Lemma 2.15 (Broué [3])**.** *Let Λ be a symmetric algebra over F. Then*

$$H(\Lambda) = Z^{\mathrm{pr}}(\Lambda).$$

Now we prove Theorem 2.1.

Proof of Theorem 2.1. Let $S(B)$ be a set of representatives for the G-conjugacy classes of B-subsections. Brauer shows that $k(B) = \sum_{(u,b)\in S(B)} l(b)$ so $S(B)$ consists of two elements, say $(1, B)$ and (u, b) where u is a non-trivial element in D and b is a Brauer correspondent of B in $C_G(u)$. In particular it follows that all the non-trivial elements in D are G-conjugate and thus the exponent of D is 2. Hence we have $T_1(B) = T(B)$ and $R(B) = T_1(B)^{\perp}$ by Lemma 2.6. On the other hand $R(B)$ is contained in the Jacobson radical $J(Z(B))$ of $Z(B)$ since it is a proper ideal. Therefore $(T_1(B)^{\perp})^2 \subseteq R(B) \cdot J(Z(B)) = 0$ and thus the first claim follows from Lemma 2.12. We next prove (2). Seeking a contradiction, we assume $p = 2$. If $l(B) = 1$, then $|D| = 3$ and $p = 3$ by [17], so we may assume $l(B) = 2$ and we can write $C_B = \begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix}$ where $c_1, c_3 \geq 2, c_2 \geq 1$. From (1), c_1 and c_3 are even. Since the determinant of C_B is a power of 2, c_2 is also even and hence $Z^{\mathrm{pr}}(B) = H(B) = 0$ by Lemma 2.14. Consequently, we obtain from Kessar-Linckelmann [11] that $\dim Z^{\mathrm{pr}}(B) = l(B)-1$, a contradiction. We have thus completely proved. □

3 Diagonal entries of Cartan matrices

In this chapter we study some relationships between the Loewy structure of a block B and ideals of its center $Z(B)$. Now let us briefly review the motivation of this chapter.

In [1] Brandt has proved that

$$l(B) + \sum_{S} \dim \operatorname{Ext}^1_B(S,S) \leq k(B) - 1 \tag{3.1}$$

where S ranges over all the isomorphism classes of irreducible B-modules, if $|D| > 2$. On the other hand Okuyama [26] has characterized the left side of (3.1) by using $Z(B)$ and the second socle $\operatorname{soc}^2(B)$ of B as follows:

$$l(B) + \sum_{S} \dim \operatorname{Ext}^1_B(S,S) = \dim \operatorname{soc}^2(B) \cap Z(B). \tag{3.2}$$

In this chapter we improve these results. For an integer $n \geq 1$ we let $\operatorname{soc}^n(B)$ be the n-th socle of B and set $R_n(B) = \operatorname{soc}^n(B) \cap Z(B)$. Then $R_1(B)$ is known as *Reynolds ideal* of $Z(B)$ as mentioned in the previous chapter and its dimension is equal to $l(B)$. Moreover the dimension of $R_2(B)$ is given by (3.2) in relation to the Loewy structure of B.

In the following, for a B-module M and an irreducible B-module S, we denote by $c(M,S)$ the multiplicity of S as composition factors in M. In this chapter we prove the following theorems.

Theorem 3.1 (Otokita [28]). *Let B be a block of FG. Then the following hold.*

(1) For each integer $n \geq 1$,

$$\dim R_n(B) \leq \sum_{S} c(P_S/P_S J^n, S) \tag{3.3}$$

where S ranges over all the isomorphism classes of irreducible B-modules, P_S is the projective cover of S and J is the Jacobson radical of B.

(2) If B has non-trivial defect groups, then there exists an integer $2 \leq m \leq LL(B)$ (the Loewy length of B) such that

$$\dim R_n(B) = \sum_S c(P_S/P_S J^n, S), \tag{3.4}$$

$$\dim R_{n'}(B) < \sum_S c(P_S/P_S J^{n'}, S) \tag{3.5}$$

for all $1 \leq n \leq m < n' \leq LL(B)$.

As a consequence,

$$\dim R_2(B) = l(B) + \sum_S \dim \mathrm{Ext}_B^1(S, S). \tag{3.6}$$

In the proof of these theorems we use some basic facts on symmetric algebras. So we prepare some lemmas. Let Λ be a finite-dimensional symmetric algebra over F with an F-linear map $\lambda : \Lambda \to F$. We put

$$\mathrm{Ann}_\Lambda(U) = \{a \in \Lambda \mid Ua = 0\},$$
$$U^\perp = \{a \in \Lambda \mid \lambda(Ua) = 0\}.$$

Lemma 3.2. *Let U, V be subspaces of Λ. Then the following hold.*

(1) $(U^\perp)^\perp = U$,
$(U + V)^\perp = U^\perp \cap V^\perp$,
$(U \cap V)^\perp = U^\perp + V^\perp$.

(2) If $V \subseteq U$, then $U^\perp \subseteq V^\perp$.

(3) $\dim U^\perp = \dim \Lambda - \dim U$.

(4) If U is an ideal of Λ, then $\mathrm{Ann}_\Lambda(U) = U^\perp$.

We define the *commutator subspace* of subspaces U and V of Λ by

$$[U, V] = \sum_{u \in U, v \in V} F(uv - vu).$$

By the definition above the next lemma is clear.

Lemma 3.3. *Let U, V and W be subspaces of Λ. Then we have*

$$[U+V, W] = [U, W] + [V, W],$$
$$[U, V+W] = [U, V] + [U, W].$$

Now let $\{e_i\}_{1 \le i \le l(B)}$ be representatives for the B-conjugacy classes of primitive idempotents in B. Then $\{S_i = e_iB/e_iJ\}_{1 \le i \le l(B)}$ and $\{P_i = e_iB\}_{1 \le i \le l(B)}$ form complete sets of isomorphism classes of irreducible B-modules and their projective covers, respectively. Furthermore we have

$$\begin{aligned} c_{ij} &= c(P_i, S_j) \\ &= \dim \mathrm{Hom}_B(P_i, P_j) \\ &= \dim e_iBe_j \end{aligned}$$

where $C_B = (c_{ij})_{1 \le i,j \le l(B)}$ is the Cartan matrix of B and the right side of (3.3) is equal to $\sum_i \dim e_iBe_i/e_iJ^ne_i$.

Here we consider the basic algebra eBe of B where $e = e_1 + \cdots + e_{l(B)}$. eBe is also a symmetric algebra and is Morita equivalent to B since $B = BeB$. Hence the next lemma holds.

Lemma 3.4. *For an ideal I of B, eIe is that of eBe and*

$$\dim \mathrm{Ann}_B(I) \cap Z(B) = \dim \mathrm{Ann}_{eBe}(eIe) \cap Z(eBe).$$

Finally we define a subspace

$$B(n) = \sum_{1 \le i \le l(B)} e_iJ^ne_i + \sum_{1 \le i \ne j \le l(B)} e_iBe_j$$

of eBe for each $n \ge 1$. Since $eBe = \sum_{i,j} e_iBe_j$ and $B(n)$ are direct sums we deduce the next lemma from Lemma 3.2 (3).

Lemma 3.5. *The right side of* (3.3) *is equal to* $\dim B(n)^\perp$

Now we prove our main theorems in this chapter.

Proof of Theorem 3.1. We first prove (1). It is clear that $e_iBe_j \subseteq [eBe, eBe]$ whenever $i \neq j$ since we can write $x = xe_j - e_jx$ for all $x \in e_iBe_j$. Therefore we have $B(n) \subseteq eJ^ne + [eBe, eBe]$ and hence $\mathrm{Ann}_{eBe}(eJ^ne) \cap Z(eBe) \subseteq B(n)^\perp$ using Lemma 2.8 and 3.2. So Lemma 3.4 gives us that $\dim R_n(B) = \dim \mathrm{Ann}_B(J^n) \cap Z(B) \leq \dim B(n)^\perp$. Thus our claim follows from Lemma 3.5.

We next prove (2). Remark that $2 \leq LL(B)$ by our assumption. Now we define $m \leq LL(B)$ as the largest non-negative integer which satisfies $[e_iBe_j, e_jBe_i] \subseteq e_iJ^me_i + e_jJ^me_j$ for all $1 \leq i, j \leq l(B)$. We follow three steps.

Step 1: We prove $2 \leq m$.

In the case that $i \neq j$, $e_iBe_j = e_iJe_j$ and thus

$$[e_iBe_j, e_jBe_i] \subseteq e_iJe_jJe_i + e_jJe_iJe_j \subseteq e_iJ^2e_i + e_jJ^2e_j.$$

If $i = j$, then

$$[e_iBe_i, e_iBe_i] = [Fe_i + e_iJe_i, Fe_i + e_iJe_i] \subseteq e_iJ^2e_i$$

since e_iBe_i is local. So we have $2 \leq m$ as claimed.

Step 2: Proof of (3.4).

First of all we obtain

$$\begin{aligned}[eBe, eBe] &= \sum_{1\leq i,j,s,t\leq l(B)} [e_iBe_j, e_sBe_t] \\ &\subseteq \sum_{1\leq i,j\leq l(B)} [e_iBe_j, e_jBe_i] + \sum_{1\leq i\neq j\leq l(B)} e_iBe_j\end{aligned}$$

since $e_te_i = 0$ or $e_je_s = 0$ according to $i \neq t$ or $j \neq s$, respectively. From the proof of (1), equality occurs in (3.3) if and only if $[eBe, eBe] \subseteq B(n)$. Hence our claim follows for n from the definitions of $B(n)$ and m.

Step3: Proof of (3.5).

By the maximality of m, we have that $[e_iBe_j, e_jBe_i] \nsubseteq e_iJ^{n'}e_i + e_jJ^{n'}e_j$ for some $1 \leq i, j \leq l(B)$ and thus $[e_iBe_j, e_jBe_i] \subseteq [eBe, eBe] \nsubseteq B(n')$. Hence equality cannot occur for n' in (3.3) (see Step 2).

Thus the first part is completely proved. The last part is clear by the fact that $\sum_S \dim \mathrm{Ext}_B^1(S, S) = \sum_i \dim e_iJe_i/e_iJ^2e_i$. □

At the end of this chapter we show a corollary to Theorem 3.1 in [26].

Corollary 3.6 (Okuyama [26])**.** (3.1) *is a corollary to* (3.2).

Proof. Suppose $R_2(B) = Z(B)$. Then the unit element of B is contained in $\mathrm{soc}^2(B)$ and hence $B = \mathrm{soc}^2(B)$. This implies $LL(B) \leq 2$, a contradiction. Thus $R_2(B) \neq Z(B)$. Since $J(Z(B))$ is the unique maximal ideal of $Z(B)$, it follows that $R_2(B) \subseteq J(Z(B))$ and (3.1) holds by (3.2).

□

4 Loewy lengths of centers

In this chapter we study the structure of the center $Z(B)$ of B. For this purpose we use its *Loewy length* $LL(Z(B))$. The results in this chapter are based on Otokita [29].

The next proposition is clear by the fact that $Z(B)$ is local in the sense that $J(Z(B))$ has co-dimension 1.

Proposition 4.1. *The following are equivalent.*

(1) D is trivial.

(2) $LL(Z(B)) = 1$.

Moreover we give an upper bound for $LL(Z(B))$ by using $k(B)$ and $l(B)$.

Proposition 4.2.

$$LL(Z(B)) \leq k(B) - l(B) + 1.$$

Proof. Let us denote by $\mathrm{soc}(B)$ and $\mathrm{soc}(Z(B))$ the socles of B and $Z(B)$, respectively. Then

$$k(B) = \dim Z(B),\ l(B) = \dim \mathrm{soc}(B) \cap Z(B) \quad \text{and}$$
$$\mathrm{soc}(B) \cap Z(B) \subseteq \mathrm{soc}(Z(B))$$

are known to hold. Thus we have

$$\begin{aligned} LL(Z(B)) - 1 &\leq \dim Z(B) - \dim \mathrm{soc}(Z(B)) \\ &\leq \dim Z(B) - \dim \mathrm{soc}(B) \cap Z(B) = k(B) - l(B) \end{aligned}$$

as required. □

Let b_D be a root of B, that is, a block of $F[DC_G(D)]$ such that $(b_D)^G = B$. We denote by $N_G(D, b_D)$ the inertial group of b_D in $N_G(D)$, by $I(B) = N_G(D, b_D)/DC_G(D)$ the inertial quotient group and by $e(B) = |I(B)|$ the inertial index of B. In the case D is cyclic the Loewy length $LL(Z(B))$ is given in Koshitani-Külshammer-Sambale [14].

Proposition 4.3 ([14, Corollary 2.8])**.** *If D is cyclic, then*

$$LL(Z(B)) = \frac{|D|-1}{e(B)} + 1.$$

For any algebra Λ over F we denote by $LL(\Lambda)$ its Loewy length. In particular we set $t(P) = LL(FP)$ where FP is the group algebra of a finite p-group P, following Wallace [35].

Lemma 4.4. *If D is normal in G, then $LL(Z(B)) \leq t(D)$. In particular $LL(Z(B)) \leq p^m + p^n - 1$ in the case of $D \simeq C_{p^m} \times C_{p^n}$.*

Proof. By a result of Külshammer [18], B is Morita equivalent to a twisted group algebra $F^\alpha[D \rtimes I(B)]$ for some 2-cocycle α of $D \rtimes I(B)$. Hence $Z(B) \simeq Z(F^\alpha[D \rtimes I(B)])$ as algebras and

$$LL(Z(B)) = LL(Z(F^\alpha[D \rtimes I(B)])) \leq LL(F^\alpha[D \rtimes I(B)]).$$

By Lemmas 1.2, 2.1 and Proposition 1.5 in Passman [30],

$$J(F^\alpha[D \rtimes I(B)]) = J(FD) \cdot F^\alpha[D \rtimes I(B)] = F^\alpha[D \rtimes I(B)] \cdot J(FD)$$

and thus $LL(F^\alpha[D \rtimes I(B)]) = t(D)$. Moreover, by Theorem (3) in Motose [22], we have $t(D) = p^m + p^n - 1$. □

Now we consider the case that $p = 2$.

Proposition 4.5. *If $D \simeq C_{2^m} \times C_{2^n}$ for some $m, n \geq 1$ and $d = m+n$, then one of the following holds:*

(1) B is nilpotent; in this case $LL(Z(B)) = t(D) = 2^m + 2^n - 1$.

(2) $m = n$ and B is Morita equivalent to $F[D \rtimes C_3]$; in this case $LL(Z(B)) \leq t(D) = 2^{m+1} - 1$. In particular $LL(Z(B)) = 2$ provided $m = n = 1$.

(3) $m = n = 1$ and B is Morita equivalent to the principal block of FA_5 where A_5 is the five degree alternating group; in this case $LL(Z(B)) = 2$.

Furthermore, if $2^d - 3 \le LL(Z(B)) \le 2^d - 1$ *then* $D \simeq C_2^2$ *or* $C_4 \times C_2$.

Proof. Without loss of generality, we may assume $m \ge n$. We first calculate the order of automorphism group $\mathrm{Aut}(D)$ of D as follows.

$$|\mathrm{Aut}(D)| = \begin{cases} 3 \cdot 2^{4m-3} & \text{if } m = n \\ 2^{m+3n-2} & \text{if } m > n. \end{cases}$$

We remark that $e(B)$ divides the odd part of $|\mathrm{Aut}(D)|$.

Case 1: $e(B) = 1$.

By Broué-Puig [4] and Puig [31], B is nilpotent and Morita equivalent to FD. Therefore $LL(Z(B)) = t(D) = 2^m + 2^n - 1$.

In the following we may assume $e(B) = 3$ and $m = n$.

Case 2: $m = n = 1$ *and* $e(B) = 3$.

By a result of Erdmann [6], B is Morita equivalent to FA_4 or the principal block of FA_5. In both cases $LL(Z(B)) = 2$ by Proposition 4.2 since $k(B) - l(B) = 1$.

Case 3: $m = n \ge 2$ *and* $e(B) = 3$.

B is Morita equivalent to $F[D \rtimes C_3]$ by Eaton-Kessar-Külshammer-Sambale [5]. Thus (2) follows from Lemma 4.4.

The last part of the proposition is clear by the first part. □

Finally, we study the case that $p = 3$ and $D \simeq C_{3^n} \times C_3$ for some $n \ge 1$.

Proposition 4.6. *If* $D \simeq C_{3^n} \times C_3$ *for some* $n \ge 1$ *and* $d = n + 1$, *then* $LL(Z(B)) \le 3^n + 2$. *In particular* $LL(Z(B)) \le 3^d - 4$.

Proof. We first obtain

$$|\mathrm{Aut}(D)| = \begin{cases} 16 \cdot 3 & \text{if } n = 1 \\ 4 \cdot 3^{n+1} & \text{if } n \ge 2. \end{cases}$$

Case 1: $e(B) \leq 4$.

If $e(B) = 1$, then $LL(Z(B)) = 3^n + 2$ by the same way to Case 1 in the proof of Proposition 4.5. If $2 \leq e(B) \leq 4$, then B is perfectly isometric to its Brauer correspondent $\tilde{B}$ in $N_G(D)$ by Usami [34] and Puig-Usami [32], [33]. Hence $LL(Z(B)) = LL(Z(\tilde{B})) \leq 3^n + 2$ by Lemma 4.4.

Since $e(B)$ divides the $3'$-part of $|\mathrm{Aut}(D)|$, we may assume $n = 1$ in the following.

Case 2: $n = 1$ *and* $5 \leq e(B)$.

$I(B)$ is isomorphic to one of the following groups:

C_8, D_8(dihedral group of order 8)$, Q_8$(quaternion group of order 8),

SD_{16}(semi-dihedral group of order 16).

We first suppose $I(B)$ is isomorphic to D_8 or SD_{16}. By the results of Kiyota [12] and Watanabe [36], $k(B) - l(B)$ is at most 4 and thus $LL(Z(B)) \leq 5$ by Proposition 4.2. Finally, suppose $I(B)$ is isomorphic to C_8 or Q_8. Kiyota [12] has not determined the invariants $k(B)$ and $l(B)$ in general. However, we can compute $k(B) - l(B)$ as follows. Since $I(B)$ acts on $D \backslash \{1\}$ regularly, the conjugacy classes of B-subsections are $(1, B)$ and (u, b_u) for some $u \in D \backslash \{1\}$ where b_u is a Brauer correspondent of B in $C_G(u)$. Moreover $I(b_u) \simeq C_{I(B)}(u)$ is trivial and thus b_u is nilpotent, $k(B) - l(B) = l(b_u) = 1$. Hence $LL(Z(B)) = 2$ as claimed.

The last part of the proposition is clear. □

Now we recall a result of Okuyama [25], the motivation of this chapter.

Theorem 4.7 (Okuyama [25]). *Let D be a defect group of B. Then*

$$LL(Z(B)) \leq |D|. \tag{4.1}$$

Equality occurs in (4.1) *if and only if B is nilpotent and D is cyclic.*

We improve Theorem 4.7 in this chapter. Here we use a set $S(B)$ of representatives for the G-conjugacy classes of B-subsections. Namely, for each $(u, b) \in S(B)$, u is a p-element in G and b is a Brauer correspondent of B in $C_G(u)$. In the following, $|u|$ denotes the order of u and $\bar{b}$ denotes the unique block of $F[C_G(u)/\langle u \rangle]$ dominated by b. First of all we give an upper bound for $LL(Z(B))$ in terms of $S(B)$. The proof below is inspired by Okuyama [25].

Theorem 4.8.

$$LL(Z(B)) \leq \max\{(|u| - 1)LL(Z(\bar{b})) \mid (u, b) \in S(B)\} + 1. \tag{4.2}$$

Proof. We denote by t the first part of the right side of (4.2). Remark that $J(Z(B)) = J(Z(FG)) \cdot 1_B$ where 1_B is the block idempotent of B. We follow three steps.

Step 1: For each $(u, b) \in S(B)$, $(u - 1)J(Z(b))^t = 0$.

Let $\tau : FC_G(u) \to F[C_G(u)/\langle u \rangle]$ be the natural epimorphism. Then

$$\tau(J(Z(b))^{LL(Z(\bar{b}))}) \subseteq J(Z(\bar{b}))^{LL(Z(\bar{b}))} = 0$$

and thus

$$J(Z(b))^{LL(Z(\bar{b}))} \subseteq \mathrm{Ker}\ \tau = (u - 1)FC_G(u).$$

Thereby

$$J(Z(b))^t \subseteq J(Z(b))^{(|u|-1)LL(Z(\bar{b}))} \subseteq \{(u-1)FC_G(u)\}^{|u|-1} = (u-1)^{|u|-1}FC_G(u).$$

Hence the claim follows.

Step 2: Take an element $a = \sum a_g g$ in $J(Z(B))^t$. Then $a_{xy} = a_y$ for all p-elements x in G and p'-elements y in $C_G(x)$.

Let us denote by $\mathrm{Br}_{\langle x \rangle} : Z(FG) \to Z(FC_G(x))$ the Brauer homomorphism. If $\mathrm{Br}_{\langle x \rangle}(1_B) = 0$, then $\mathrm{Br}_{\langle x \rangle}(a) = 0$ and hence $a_{xy} = a_y = 0$

as required. So we may assume $\mathrm{Br}_{\langle x\rangle}(1_B) \neq 0$. Then there exists a B-subsection $(u,b) \in S(B)$ and $t \in G$ such that $x = t^{-1}ut$. Since $a \in Z(FG)$, $a_{xy} = a_{t^{-1}uty} = a_{utyt^{-1}}$ and $a_y = a_{tyt^{-1}}$. Therefore we need only prove the claim above for u and p'-element v in $C_G(u)$. Since $\mathrm{Br}_{\langle u\rangle}$ maps nilpotent elements to nilpotent elements, we have $\mathrm{Br}_{\langle u\rangle}(J(Z(FG))) \subseteq J(Z(FC_G(u)))$ and thus $\mathrm{Br}_{\langle u\rangle}(J(Z(B))^t) \subseteq \sum J(Z(b))^t$ where $\mathrm{Br}_{\langle u\rangle}(1_B)1_b \neq 0$. Hence it follows from Step 1 that $(u-1)\mathrm{Br}_{\langle u\rangle}(a) = 0$. This implies $a_{uv} = a_v$ as asserted.

Step 3: Completion of the proof.

We denote by $Z_{p'}$ the F-subspace of $Z(FG)$ spanned by all p'-section sums. It suffices to prove that $J(Z(B))^t \subseteq Z_{p'}$ since $J(Z(FG)) \cdot Z_{p'} = 0$ (see Brauer [2] or Okuyama [24]). Take an element $a = \sum a_g g \in J(Z(B))^t$. We want to show $a_g = a_h$ for all $g,h \in G$, if the p'-parts of them are G-conjugate. However, it is an immediate consequence of the claim in Step 2 since $a \in Z(FG)$. Thus the theorem is completely proved. □

In addition we give an upper bound for the right side of (4.2) in terms of the defect groups of B.

Corollary 4.9. *Let p^d and p^ε be the order and the exponent of a defect group D of B, respectively. Then*

$$\max\{(|u|-1)LL(Z(\bar{b})) \mid (u,b) \in S(B)\} \leq p^d - p^{d-\varepsilon}. \tag{4.3}$$

If equality occurs in (4.3), *then $D \simeq C_{p^\varepsilon} \times C_{p^{d-\varepsilon}}$.*

As a consequence, we have

$$LL(Z(B)) \leq p^d - p^{d-\varepsilon} + 1. \tag{4.4}$$

Proof. We may assume D is non-trivial. Fix $(u,b) \in S(B)$ associated to the left side of (4.3). We let D' be a defect group of b of order $p^{d'}$ and put $|u| = p^{\varepsilon'}$. Then D' is contained in D up to G-conjugacy since $b^G = B$, $\varepsilon' \leq \varepsilon$ and we may assume that a defect group of $\bar{b}$ is $\bar{D}' = D'/\langle u\rangle$ (see [7, Chapter V, Lemma 4.5]). Hence we obtain from (4.1) that

$$(|u|-1)LL(Z(\bar{b})) \leq (p^{\varepsilon'}-1)p^{d'-\varepsilon'} \leq (p^{\varepsilon'}-1)p^{d-\varepsilon'} = p^d - p^{d-\varepsilon'} \leq p^d - p^{d-\varepsilon}$$

as claimed. We next suppose equality holds in (4.3). Then we have $d = d', \varepsilon = \varepsilon'$ and $\bar{D}$ is cyclic. Since $\langle u \rangle$ is contained in the center of D', D' is abelian. Therefore we deduce $D \simeq D' = \langle u \rangle \times H$ where $H \simeq \bar{D}'$. $\square$

As a corollary to the theorems above we consider a problem of classifying blocks according to $LL(Z(B))$. If $LL(Z(B)) = |D|$, then B is a nilpotent block with cyclic defect group by Theorem 4.7 and thus B is Morita equivalent to the group algebra $F[C_{p^d}]$. Hence we study other three cases that $|D| - 3 \leq LL(Z(B)) \leq |D| - 1$. We remark that the notation given in Corollary 4.9 will be used throughout this chapter.

Theorem 4.10. *Let D be a defect group of B. Then $LL(Z(B)) = |D| - 1$ if and only if one of the following holds:*

(1) $D \simeq C_3$ and $I(B) \simeq C_2$.

(2) B is nilpotent and $D \simeq C_2^2$.

Proof. In the case D is cyclic, (1) follows by Proposition 4.3. So we may assume that $\varepsilon < d$. Then, since

$$LL(Z(B)) = p^d - 1 \leq p^d - p^{d-\varepsilon} + 1 < p^d,$$

we have $D \simeq C_2 \times C_{2^{d-1}}$ by Corollary 4.9. Furthermore we have $d = 2$ and (2) holds by Proposition 4.5. $\square$

The next problem is the case of $LL(Z(B)) = |D| - 2$.

Theorem 4.11. *Let D be a defect group of B. Then $LL(Z(B)) = |D| - 2$ if and only if one of the following holds:*

(1) $D \simeq C_5$ and $I(B) \simeq C_2$.

(2) $D \simeq C_2^2$ and B is Morita equivalent to FA_4.

(3) $D \simeq C_2^2$ and B is Morita equivalent to the principal block of FA_5, where A_4 and A_5 are four and five degree alternating groups, respectively.

Proof. As same reason to the proof of Theorem 4.10, we may assume $\varepsilon < d$ and

$$LL(Z(B)) = p^d - 2 \leq LL(Z(\bar{b}))\ (p^{\varepsilon'} - 1) + 1 \leq p^d - p^{d-\varepsilon} + 1 \leq p^d - 1.$$

Case 1: $LL(Z(B)) = p^d - p^{d-\varepsilon} + 1$.

By Corollary 4.9, $D \simeq C_3 \times C_{3^{d-1}}$. However, this case cannot occur from Proposition 4.6.

Case 2: $LL(Z(\bar{b}))\ (p^{\varepsilon'} - 1) + 1 = p^d - p^{d-\varepsilon} + 1 = p^d - 1$.

We have $D \simeq C_2 \times C_{2^{d-1}}$ and thus (2) or (3) holds by Proposition 4.5.

Case 3: $LL(Z(B)) = LL(Z(\bar{b}))\ (p^{\varepsilon'} - 1) + 1$ *and* $p^d - p^{d-\varepsilon} + 1 = p^d - 1$.

We obtain $p = 2, d - \varepsilon = 1$ and $LL(Z(\bar{b})) = \frac{2^d - 3}{2^{\varepsilon'} - 1}$. Since

$$LL(Z(\bar{b})) \leq |\bar{D}'| = 2^{d' - \varepsilon'} \leq 2^{d - \varepsilon'},$$

$d - \varepsilon' = 1$ (remark that $0 < d - \varepsilon \leq d - \varepsilon'$) and so $LL(Z(\bar{b})) = 1$ or 2. Thus we have $\varepsilon' = 1$ and $d = 2$. In this case (2) or (3) holds by Proposition 4.5. □

Finally, we consider the case of $LL(Z(B)) = |D| - 3$.

Theorem 4.12. *Let D be a defect group of B. Then $LL(Z(B)) = |D| - 3$ if and only if one of the following holds:*

(1) $D \simeq C_5$ and $I(B) \simeq C_4$.

(2) $D \simeq C_7$ and $I(B) \simeq C_2$.

(3) B is nilpotent and $D \simeq C_4 \times C_2$.

Proof. We may assume D is not cyclic, $\varepsilon < d$ and

$$LL(Z(B)) = p^d - 3 \leq LL(Z(\bar{b}))\ (p^{\varepsilon'} - 1) + 1 \leq p^d - p^{d-\varepsilon} + 1 \leq p^d - 1.$$

Case 1: $LL(Z(B)) = p^d - p^{d-\varepsilon} + 1$.

By Corollary 4.9, we have $D \simeq C_4 \times C_{2^{d-2}}$ and hence we obtain $d = 3$ by using Proposition 4.5.

Case 2: $LL(Z(B)) = LL(Z(\bar{b}))\ (p^{\varepsilon'} - 1) + 1, p^d - p^{d-\varepsilon} + 1 = p^d - 2$.

Clearly, $p = 3, d - \varepsilon = 1$ and $LL(Z(\bar{b})) = \frac{3^d - 4}{3^{\varepsilon'} - 1}$. However, this case cannot occur since this is not an integer.

Case 3: $LL(Z(B)) = LL(Z(\bar{b}))\ (p^{\varepsilon'} - 1) + 1, p^d - p^{d-\varepsilon} + 1 = p^d - 1$.

We first obtain $p = 2, d - \varepsilon = 1$ and $LL(Z(\bar{b})) = \frac{2^d - 4}{2^{\varepsilon'} - 1}$. Since $2^{\varepsilon'} - 1$ is odd, we have $\varepsilon' = 1$. Hence

$$2^d - 4 = LL(Z(\bar{b})) \leq |\bar{D}'| \leq 2^{d-1}$$

and thus $d = 3$ (remark $LL(Z(B)) \geq 2$). Moreover, since we have $\bar{D}' \simeq C_4$ and $d = d'$, D' is abelian by the same reason to Corollary 4.9 and thus $D \simeq D' = C_4 \times C_2$.

Case 4: $LL(Z(\bar{b}))\ (p^{\varepsilon'} - 1) + 1 = p^d - p^{d-\varepsilon} + 1 = p^d - 2$.

In this case, $D \simeq C_3 \times C_{3^{d-1}}$. However, $LL(Z(B)) \neq p^d - 3$ by Proposition 4.6.

Case 5: $LL(Z(\bar{b}))\ (p^{\varepsilon'} - 1) + 1 = p^d - 2, p^d - p^{d-\varepsilon} + 1 = p^d - 1$.

We have $p = 2, d - \varepsilon = 1$ and $LL(Z(\bar{b})) = \frac{2^d - 3}{2^{\varepsilon'} - 1}$. Since

$$LL(Z(\bar{b})) \leq |\bar{D}'| = 2^{d' - \varepsilon'} \leq 2^{d - \varepsilon'},$$

we deduce $d - \varepsilon' = 1$ and $LL(Z(\bar{b})) = 1$ or 2. Thus we obtain $d = 2$, but this case cannot occur.

Case 6: $LL(Z(\bar{b}))\ (p^{\varepsilon'} - 1) + 1 = p^d - p^{d-\varepsilon} + 1 = p^d - 1$.

We have $D \simeq C_2 \times C_{2^{d-1}}$ by Corollary 4.9 and hence $d = 3$ in this case using Proposition 4.5. □

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