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Article

A Comprehensive Guide for the Proof of the Turnpike Theorem in General Equilibrium Theory

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1 Introduction

In this article, a nearly complete guide for understanding the proof of the turnpike theorem in general equilibrium theory is presented. The original proof in Bewley (1980) and Bewley (1982) is rather disorganized and partly unintuitive, and has an error. This guide reorganizes it to emphasize that the turnpike theorem is a theorem on dynamics of economic surplus in the spirit of promoting a formation of a good habit over a fully rational optimization for each economic agent. The guide also contains a reproof of the part derived from an error in the original proof, based on the correction proposed in Kaneko (2017). Though the issues related to the existence of an equilibrium are omitted due to the limit on the space, no technical detail is left unexplained for all other issues. The turnpike theorem says that a competitive equilibrium from a strictly positive initial stock of produce-able commodities converges exponentially to a stationary equilibrium allocation as time passes, if the common time-discount rate is close to 1. The guide presents in order that the stationary equilibrium with transfer payments to whose allocation a competitive equilibrium allocation converges must have the same marginal utilities of income as those in the competitive equilibrium for a comparison of social welfare and consumer surplus, the loss in the total market surplus from the stationary equilibrium allocation evaluated at the stationary equilibrium price dominates the the squared distance of competitive equilibrium allocation from the stationary allocation, and the dynamics of the loss in the total market surplus is dominated by a geometric sequence. The reason why the common time-discount rate must be taken close to 1 is clarified as that a Lyapunov stability argument for the case that the time-discount rate is 1 is applied to the dynamics of the loss in the total market surplus. The Lyapunov stability argument requires that the loss in the total market surplus is finite and dominated by the squared distance between initial stock and the stationary equilibrium stock, if the distance is small. The guide calls these properties as key findings and presents that they are corollaries of the fact that a stationary allocation with no consumption in which all commodities are in excess supply can be recursively replaced with the stationary equilibrium allocation by a small ratio to make a feasible allocation that converges exponentially to the stationary equilibrium allocation, where the replacement ratio is independent of the time-discount rate. Hence the guide clarifies that the restriction of the common time-discount rate close to 1 is not essential in proving these key findings.

The turnpike theorem was first found in late 1950s for the one-sector optimal growth theory as the convergence of optimal growth paths from any positive initial capital to a modified golden rule. It was later extended to multi-sector cases and cases with stationary uncertainty in 1970s and early 1980s, all for the optimal growth theory.

Since an optimal growth model was recognized to be just a simplified model of a general equilibrium model with a time-separable utility for the representative agent, an extension to a standard general equilibrium model was conjectured. The first and nearly complete extension of the turnpike theorem to a general equilibrium model without uncertainty appeared in Bewley (1980), which was later published as Bewley (1982). In them, the proof was presented to be a recollection of results found in many literatures on the optimal growth theory, which blurred the economic implication of the theorem in the sense that it was understood to be just an extension of a production theory to the case in which consumers were heterogeneous. Though works such as Bewley (1977) and Bewley (1981) tried to extend the turnpike theorem for general equilibrium theory to an economy with a stationary uncertainty, such literatures did not catch an attention of many economists due to a sudden disinterest in turnpike properties for the optimal growth theory, in mid 1980s. The Ramsey-style models started to dominate in literatures for the optimal growth theory due to a gain in popularity of the dynamic programming method. Lyapunov stability used in literatures on turnpike properties works most effectively for the case that the time-discount rate is equal to 1, as shown in Mckenzie (1976), while analysis of Ramsey-style models requires it to be less than 1, making the former somewhat incompatible with the latter. Since the turnpike theorem for the one-sector growth model with a time-discount rate less than 1 can be proved by a simple application of a dynamic programming method, Ramsey-style models have gained a popularity for their versatility in the optimal growth theory. The reality of economies in 1980s, especially that in U.

S., was also against an existence of a modified golden rule. People recognized that an operation of economy needs a fundamental change in order to sustain their economic welfare. Since the turnpike theorem for the general equilibrium theory was recognized only as an extension of that in the production theory, the disinterest in the optimal growth theory affected negatively on its recognition. For example, a turnpike theorem on the marginal utility of income was proved in Bewley (1977), but not so many economists recognized the result as a turnpike theorem though that was clear once the permanent income hypothesis was properly understood. The negligence was therefore fostered by a false understanding of the economic implication of the turnpike theorem. It is an ergodic theorem in behavioral sciences, and all ergodic theorems in behavioral sciences aim at a reconciliation of a formation of good habits with an optimization in the long run. After 1980s, the economists were rapidly drawn into an obsession with an optimization owning to a gain in popularity of the non-cooperative game theory and the optimal control method. But, in the game theory and the decision theory, there have been many attempts to reconcile a formation of good habits, such as developing and following a convention, with an optimizing behavior. Examples in the game theory are a fictitious play with similarity, the evolutionary stability, a learning with a satisficing criterion, a bargaining for coalition formation in which the intra-coalition allocation of payoff is always the equal division etc. Those in the decision theory are status-quo theories such as the prospect theory, Young's theory of convention formation, the Choque integral representation of a preference with a non-additive measure for the distribution of utilities and the Knightian decision theory.

The true economic implication of the turnpike theorem is to rationalize that making each economic agent to follow a good habit as much as the situation surrounding him/her allows and to share with the society the information that he/she follows a good habit is at least as good as making each economic agent to optimize with a fully rational expectation and to share with the society the information that he/she is a fully rational optimizer, in the long run. Considering the enormous information and surmising cost for being a fully rational optimizer in entire periods, which is usually not modeled in economics, it even suggests that a habit formation theory is superior to the optimization theory in the long run. The point is supported by the use of Lyapunov stability argument in its proof, since the argument picks up a candidate path to which all reasonably behaving paths are supposed to converge dynamically, measures value losses of those paths from the candidate path, checks that the dynamic convergence of the value loss to 0 implies the dynamic convergence to the candidate path, and evaluates a dynamics of the value loss on its convergence to 0. In behavioral sciences, the candidate path represents a hypothetical one in which each agent follows a good habit with the knowledge that all others do the same. By understanding this way, turnpike theorems in economics, not only for general equilibrium theory but also for optimal growth theory, can be seen as a direction for how to operate economies under a stationary environment in the long run. Note that they do not reject an optimizing behavior at all, since the stationary equilibrium allocation is made by it, although its stationary price is hypothetical, not real. They do not claim that the suggested habits are fully implementable either, since the initial stock might be short of the stationary equilibrium level for some produce-able commodity. But an adjustment of production plans combined with a scheme of transfer payments for consumers can be devised so that each economic agent can follow the turnpike after a finite period. If the marginal utilities of income among consumers are intrinsic and known to a social planner, which most macro-economists postulate, a standard way is to restrict income of each consumer in early periods so that his utility measured by the unit of account, which is optimized under the stationary equilibrium price, is at a pre-specified reservation level independent of consumers, allocate residual resources in these periods to firms as inputs to make them produce each of produce-able commodities as much as possible, have a stock of each produce-able commodity no less than that in the stationary equilibrium as early as possible, then make each economic agent to follow the stationary equilibrium and distribute out extra produce-able commodities according to productive sequences along the stationary equilibrium allocation just as that in section 12 of this article. If marginal utilities of income for consumers are endogenously determined in a competitive equilibrium so that they are relative to a competitive equilibrium, a social planner needs to learn them at first by either solving for a competitive equilibrium or making the competitive equilibrium run in early periods and drawing an accurate inference on them based on the observed behavior of consumers. After a social planner obtains an accurate information on them, the same management of allocation as above can be used.

Though this article is a guide for understanding the proof of the turnpike theorem precisely, the main intention of the author in presenting this guide is to correct the recognition of ergodic theorems in behavioral sciences among economists to the right direction. The general equilibrium theory provides a good field to present such a thought, since it is inclusive of all important economic agents and puts an economic welfare derived from final consumptions to be utmost important in an economy so that a production exists only to enhance such an economic welfare. Though the turnpike theorem for general equilibrium theory is obviously related closely to the operation of the production side, it is essentially on the relationship between the economic welfare for consumers and the norm that economic agents are expected to follow. An attention on a habit formation is natural in economics, since it is a common knowledge that social behavior of humans is dominated by forming a good habit and following it. The optimizing behavior is only one type of them, and there are many ways by which an economic agent optimizes. The turnpike theorem tells, even when economic agents are supposed to optimize, one way of optimizing behavior that forms habits to be followed can be rationalized to be at least as good as all other ways of optimizing behavior that is feasible in the same economy, in the long run. The rationalization is given by the convergence of loss in total market surplus of the latter from the former to 0. The explanation for the proof is carefully written to embody this thought.

The precise understanding of the proof of the turnpike theorem for general equilibrium theory eliminates major controversies on the theorem. One is whether the turnpike is insensitive to an initial stock of produce-able commodities in a competitive equilibrium（or an optimal growth path）or not. The answer depends on whether marginal utilities of income for consumers are intrinsic or relative to a competitive equilibrium. Most literatures in the optimal growth theory claim that it is insensitive, and it is rightfully so since marginal utilities of income for consumers are assumed to be intrinsic, so unique, in the optimal growth theory. The theory requires that the social welfare function must be uniquely defined for an economy, which serves as a represen-tative consumer. Hence the stationary equilibrium with transfer payments or the golden rule (modified or not) must be uniquely determined, regardless of an initial stock. In the general equilibrium theory, the competitive equilibrium in the turnpike theorem must be understood as the competitive equilibrium with transfer payments compatible with the unique profile of marginal utilities of income for consumers, where an initial stock varies freely. Then it is far from obvious that such an equilibrium exists for any initial stock of consideration, although a candidate feasible allocation always exists as a solution of the social welfare maximization problem given an initial stock. Such a candidate is determined by a social planner, not as a result of interaction among economic agents, so that the implication of the turnpike theorem as promoting a formation of alternative good habits for economic agents is lost. In contrast, economists for the general equilibrium theory stick to that a competitive equilibrium in the turnpike theorem means as it is. It is rightfully so since such an equilibrium is proved to exist. Then the marginal utilities of income for consumers are endogenously determined in a competitive equilibrium, and there is no guarantee that the same profile can be obtained in a competitive equilibrium from a different initial stock except for the case that the initial stock appears as a stock for the former competitive equilibrium in some period. The proof of the turnpike theorem make it clear that a stationary equilibrium allocation to which a competitive equilibrium allocation converges dynamically must have the same profile of marginal utilities of income as that in the competitive equilibrium. When an initial stock varies, marginal utilities of income for consumers also varies generically so that the turnpike becomes sensitive to an initial stock. The other is whether the turnpike theorem for the undiscounted case and that for the discounted case are essentially different or not. It is well known that the turnpike theorem for the one-sector growth model can be proved by a simple application of dynamic programing. For the undiscounted case in optimal growth theory, Lyapunov stability is combined with either a catching-up criterion or an overtaking criterion on paths of utility sums in order to prove the turnpike theorem. So they look formally different in proofs, but the proof of the turnpike theorem for general equilibrium theory makes it clear that they are essentially the same. Lyapunov stability is useful to analyze the dynamics of optimal control models with the time-discount rate equal to 1. Though an equilibrium in general equilibrium theory is defined for the common time-discount rate less than 1, a loss in total market surplus or a value loss in social welfare net an acquisition cost of initial stock for a feasible allocation takes a finite value without a time-discount. A bound on the incremental decrease of loss in total market surplus by the squared distance of an initial stock to the stationary equilibrium stock when the distance is small is valid for all time-discount rates close to 1, including 1 itself. Hence a standard Lyapunov stability argument for the undiscounted case is applied for all time-discounted rates sufficiently close to 1. With an adequate modification of the definition of competitive equilibrium for the case that the common time-discount rate is 1, it is clear that the turnpike theorem is essentially the same with or without a time-discount if the time-discount rate is close to 1. On the other hand, the proof by dynamic programming is ad-hoc, since it is well known that such a proof cannot be generalized to multi-sector optimal growth models.

This article is organized as follows. This section introduces the purpose of presenting the guide, how the guide is organized, and the implication of a precise understanding brought by the guide. The section 2 gives a guide for the statement of the turnpike theorem. The section 3 explains that feasible allocations from an initial stock and stationary allocations are bounded. The section 4 explains the existence of a solution to the social welfare maximization problem. The section 5 explains several important sequences used later in estimating upper-bounds are bounded and bounded away from 0. The section 6 explains how the stationary equilibrium to whose allocation a competitive equilibrium allocation converges dynamically is chosen. The section 7 gives a guidance for various concepts of surpluses and explain the role of a minimum loss in total market surplus from the stationary equilibrium allocation evaluated at the stationary price. The section 8 gives a guide to prove that incremental forward differences of discounted loss in total market surplus along the competitive equilibrium is bounded away from 0 by a constant × the squared distances of the competitive equilibrium allocation to the stationary equilibrium allocation, locally at the stationary equilibrium allocation. The section 9 gives a guide for the proof that the same incremental forward differences is bounded away from 0 by a constant × the squared distances of the stock in the competitive equilibrium to that in the stationary equilibrium, locally at the stationary equilibrium stock. The section 10 introduces two key findings and gives a guide for how these findings prove that the loss in total market surplus along the competitive equilibrium converges to 0 exponentially. The section 11 gives a guide to prove one of the key finding, that the minimum loss in total market surplus is bounded uniformly for the time-discount rates close to 1. The section 12 gives a guide to prove another key finding, that the minimum loss in total market surplus is bounded by a constant × the squared distance of the initial stock to the stationary equilibrium stock locally at the stationary equilibrium stock. The section 13 gives remarks on extensions of the turnpike theorem in the general equilibrium theory.

The guide for the proof consists of sections 2–12. The guide intentionally refrains from referring to other literatures as much as possible in order to urge readers to complete the proof by themselves in a mathematically rigorous manner. Especially, no reference is given to vast literatures in the optimal growth theory. Obviously the proof itself is in Bewley (1980) and Bewley (1982), but it has an error in the proof of the second key finding. The reproof by correcting it appears in the section 12. The guide also contains improvements on the proof in the argument for the necessity of taking the time-discount rate close to 1 and in the estimates of upper-bounds mostly in the reproof part.

2　The Turnpike Theorem in General Equilibrium Theory

Hereafter I use freely the notations used in Bewley (1980) without defining them. Readers should refer to Bewley (1982) or Bewley

(1980) for the underlying general equilibrium model in an open-end discrete time-series economy without uncertainty. Just note that there are three types of commodities: primary (non-produce-able) commodities (L_o), produce-able commodities (L_p) and consume-able commodities (L_c). Let $L = L_o + L_p$. Any production technology must use some primary commodity(ies) as input(s). Produce-able but non-consume-able commodities exist generally, as intrinsic intermediary commodities. A production cycle takes two periods, making inputs in a period then getting outputs in the next period. Each firm has a pre-specified set of commodities that can be used for inputs, and another pre-specified set of commodities that can be produced. Hence its production possibility set is in $M_{j,0,-} \times M_{j,1,+}$ where $M_{j,0,-}$ is the non-positive orthant of a Euclidean space $M_{j,0}$ for inputs and $M_{j,1,+}$ is the non-negative orthant of a Euclidean space $M_{j,1}$ for outputs. Input vectors are indicated by the subscript 0 and output vectors are indicated by the subscript 1.

A competitive equilibrium is a tuple of a feasible allocation from a given initial stock of produce-able commodities and a price system for which each firm maximizes the infinite sum of profits in periods, each consumer maximizes his infinite discounted sum of utilities in periods under a budget constraint for entire periods, and the complementary slack condition is satisfied so that the price of a commodity in excess supply is 0 in each period. Note that a competitive equilibrium is determined after an initial stock of produce-able commodities is given, a budget constraint may not be dynamically consistent so that a budgetary default in one period is offset by a budgetary surplus in other periods, in the past or the future, and time-discount rates are allowed to vary among consumers. If the total value added in the economy can be freely distributed out as incomes of consumers, a competitive equilibrium in which each consumer has his income equal to the value of his consumption plan becomes viable in the economy. Such an equilibrium is called as a competitive equilibrium with transfer payments.

When all consumers have the same time-discount rate δ, a stationary equilibrium (with transfer payments) is a tuple of stationary allocation of the form $((\bar{x}_i), (\bar{y}_j))_{t=0}^{\infty}$ and a stationary price system of the form $(\delta^t \bar{p})_{t=0}^{\infty}$ that is a competitive equilibrium (with transfer payments) if the initial stock of produce-able commodities is given by the total output in the stationary production. A stationary equilibrium with transfer payments is defined in the same way as a competitive equilibrium with transfer payments.

The turnpike theorem asserts that, assuming that all consumers have the same time-discount rate δ, if a competitive equilibrium allocation starts from positive initial stocks of all produce-able commodities, then there exists a stationary equilibrium with transfer payments such that the squared distance between the competitive equilibrium allocation and the stationary equilibrium allocation converges to 0 exponentially as time passes, uniformly for all common time-discount rates sufficiently close to 1. A collection of sequences of positive numbers is called as converging uniformly to 0 exponentially if all of them are dominated by a geometric sequence multiplied by a constant positive number. The convergence being exponential is considered to be very fast. The stationary equilibrium to whose allocation a competitive equilibrium allocation converges dynamically is the one having the same marginal utilities of income as those in the competitive equilibrium. For a rigorous statement of the theorem, see the theorem 4. 5 in Bewley (1980).

The time-discount rate being equal among consumers is crucial, since otherwise all consumers with their time-discount rates less than the maximum value cease to consume, or die out, within finite periods (the theorem 4. 4 in Bewley (1980)). Consider a consumer with the maximal time-discount rate δ. It will be proved later that competitive allocations are bounded above. Since utility functions are assumed to be continuously differentiable and strictly increasing in each consumeable commodity, the first order condition for his utility maximization problem implies that $(\delta^{-t}p_k^t)_{t=0}^{\infty}$ is uniformly bounded away from 0 for all $k \in L_c$, where $\underset{\sim}{p}$ is the competitive equilibrium price system. Next consider a consumer i with his time-discount rate $\delta_i < \delta$. If the consumer does not die out eventually, he must consume some $k \in L_c$ positively for infinitely many periods. The first order condition for his utility maximization problem becomes an equation in such periods, which implies that $\delta_i^{-t}p_k^t$ must be uniformly bounded from above for such periods. However, since $\delta_i^{-t}p_k^t = \left(\dfrac{\delta}{\delta_i}\right)^t \delta^{-t}p_k^t$ and $\dfrac{\delta}{\delta_i} > 1$, it must explode to ∞ since $(\delta^{-t}p_k^t)_{t=0}^{\infty}$ is uniformly bounded away from 0.

Though it is customary to state the turnpike theorem only for competitive equilibrium without transfer payments, it still holds by replacing all equilibrium with equilibrium with transferable payments, since what is needed in its proof are the characterization of optimization for each economic agent and the complementary slackness condition, which are common in both concepts of equilibrium. This point is not repeated so that readers should understand that the word "competitive equilibrium" in this article actually means "competitive equilibrium with transfer payments" if the existence of the latter is guaranteed.

3　Bounds on Feasible Allocations

Hereafter, $|\cdot|$ applied for a vector means the maximum of the absolute values of its elements, and $\|\cdot\|$ applied for a vector means its Euclidean norm.

The basic fact to be noticed is that the set of feasible allocations given an initial stock vector of produce-able commodities is bounded, where the upper bound depends on the initial stock vector, and that the set of feasible stationary allocation is bounded. These are to be expected since production technologies need input(s) of a primary commodity(ies) and the endowments of primary commodities are bounded. More specifically, there exists an arbitrarily large bound $B>0$ that restricts the expansion of production in the following ways:

1. If $(y_0, y_1)\in Y, |y_0|\geq B$ and $|y_{0k}|\leq|\omega_k|$ for all $k\in L_o$, then $|y_1|<|y_0|$,
2. If $(y_0, y_1)\in Y$, $|y_0|\leq B$ and $|y_{0k}|\leq|\omega_k|$ for all $k\in L_o$, then $|y_1|\leq B$,

where Y denotes the total stationary production set of the economy and ω denotes the total (stationary) initial endowment.

Consider the first statement and suppose, on the contrary, that no such B exists. Then there exists a sequence (y^n) in Y such that $|y^n_{0k}|\leq|\omega_k|$ for all $k\in L_o$ and n, $|y^n_0|$ explodes as n goes to ∞ and $|y^n_1|\geq|y^n_0|$ for all n. Since wasting outputs from a feasible stationary total production is assumed to be feasible, we can make $|y^n_1|=|y^n_0|$ for all n. Since there are only finitely many commodities, by taking a subsequence if necessary, we can assume that there exists a $k_1\in L_p$ for which $|y_1^n| = y_{1,k1}^n$ for all n. Since Y is convex and $0 \in Y$, $\frac{1}{|y_0^n|} y^n \in Y$ for all n. Since the sequence $\left(\frac{1}{|y_0^n|} y^n\right)$ is in the the region of Y bounded by 1 in terms of $|\cdot|$, a subsequence of it must converge to some $y \in Y$ since Y is closed. Since $|y_0^n|$ goes to ∞ as n goes to ∞, $y_{0,k} = 0$ for all $k \in L_o$. Yet $y_{1,k1} = 1$, so that producing a commodity is possible without inputs of primary commodities and that contradicts the assumption of necessity of primary inputs in any production. Similarly, if the second statement is false, there exists a sequence (y^n) in Y such that $|y_1^n|$ explodes as n goes to ∞, $|y_{0,k}^n| \leqq |\omega_k|$ for all $k \in L_o$ and $|y_0^n| \leq |y_1^n| = y_{1,k1}^n$ for some $k_1 \in L_p$. Then the sequence $\left(\frac{1}{|y_1^n|} y^n\right)$ stays in the region of Y in which $|\cdot|$-norm is less than or equal to 1, so that a subsequence of it converges to $y \in Y$. Since $|y_1^n|$ goes to ∞ as n goes to ∞, $y_{0,k} = 0$ for all $k \in L_o$. However, $y_{1,k_1} = 1$ so that the same contradiction as above is obtained.

Since the inputs of primary commodities cannot grow and those of produce-able commodities must shrink after they reach a certain constant amount, demand for inputs and supply of outputs are bounded above through periods in a feasible allocation from a given initial stock, and this upper bound can be taken uniformly over all such allocations. For feasible stationary allocations, making inputs go to infinity is impossible since the outputs cannot cover inputs after the size of inputs become larger than some amount, making the allocation infeasible. Hence the size of inputs is uniformly bounded for all feasible stationary allocations, so that the set of feasible stationary allocations is bounded.

4 Maximization of Social Welfare

Though the issues on the existence of a competitive equilibrium and that of a stationary equilibrium are omitted in this article, a brief guide is provided for the existence of a solution for the social welfare maximization problem. This solution is closely related to a competitive equilibrium with transfer payments through a decentralization of social decision making.

The social welfare maximization problem to be solved is

$$(\mathrm{SP})_K \qquad Max_{((x_i),\,(y_j))} \sum_{t=0}^{\infty} \delta^t \sum_{i \in I} \Lambda_i^{-1} u_i(x_i^t)$$

$$\text{s.t.} \quad \sum_i x_i^t + \sum_j (-y_{j,0}^t) \leq \omega + \sum_j y_{j,1}^{t-1} \quad \text{for all } t \geq 0$$

for each $K \in \mathbb{R}_+^{Lp}$, where $\Lambda_i > 0$ is the marginal utility of income for consumer i, ω is a stationary total endowment of commodities and $(y_{j,1}^{-1}) > 0$ is outputs of firms that are arbitrarily taken so that $\Sigma_j y_{j,1}^{-1} = K$. The second welfare theorem claims that such a solution can be decentralized to a competitive equilibrium with transfer payments if a supporting price for that is found. In the case of finitely many commodities, a supporting price can be found by a straight-forward application of the Kuhn-Tucker theorem. However, in the infinite-horizon time-series economy, approximation by finite-horizon cases may not guarantee that a sequence of supporting prices for finite horizon cases converges to a bounded price so that a proper supporting price for the solution of $(\mathrm{SP})_K$ may not be found.

A solution to the social welfare maximization problem can be constructed as follows. Since the set of feasible allocations from the initial stock K in period 0 is bounded, the values of the social welfare function for such feasible allocations are bounded due to the continuity of u_i's. Hence there exists a sequence of feasible allocations from K, $\{((\underset{\sim}{x_i^n}), (\underset{\sim}{y_j^n}))\}_n$, such that $\Sigma_{t=0}^{\infty} \delta^t \Sigma_{i \in I} \Lambda_i^{-1} u_i(x_i^{n,t})$ converges to sup $\{\Sigma_{t=0}^{\infty} \delta^t \Sigma_{i \in I} \Lambda_i^{-1} u_i(x_i^t) \mid ((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$ is feasible from $K\}$. Since $\{((x_i^{n,0}), (y_j^{n,0}))\}_n$ is bounded, there exists a subsequence of $\{((\underset{\sim}{x_i^n}), (\underset{\sim}{y_j^n}))\}_n$, represented as $\{n_k^0\}_{k=0}^{\infty}$ for a simplicity of description, such that $((x_i^{n_k^0,0}), (y_j^{n_k^0,0}))$ converges as k goes to ∞. Let $((x_i^0), (y_j^0))$ be its limit. Since $\{((x_i^{n_k^0,1}), (y_j^{n_k^0,1}))\}_k$ is bounded, there exists a subsequence of $\{n_k^0\}_{k=0}^{\infty}$, represented as $\{n_k^1\}_{k=0}^{\infty}$, such that $((x_i^{n_k^1,1}), (y_j^{n_k^1,1}))$ converges as k goes to ∞. Let $((x_i^1), (y_j^1))$ be its limit. By continuing recursively, we obtain $\{n_k^t\}_{k=0}^{\infty}$ for all $t \geq 0$ and $\{((x_i^t), (y_j^t))\}_{t=0}^{\infty}$ such that, for each $t \geq 0$, $\{n_k^{t+1}\}_{k=0}^{\infty}$ is a subsequence of $\{n_k^t\}_{k=0}^{\infty}$ and $\{((x_i^{n_k^t,t}), (y_j^{n_k^t,t}))\}_k$ converge to $((x_i^t), (y_j^t))$. Consider the sequence $\{n_s^s\}_{s=0}^{\infty}$. Then $\{n_s^s\}_{s=t}^{\infty}$ is a subsequence of $\{n_k^t\}_{k=0}^{\infty}$ for all t, so that $\{((x_i^{n_s^s,t}), (y_j^{n_s^s,t}))\}_s$ converges to $((x_i^t), (y_j^t))$ for each t. Since consumption sets and production possibility sets are closed and the feasibility conditions are inequalities with equality allowed, $((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$ becomes a feasible allocation from K. Since $\{((\underset{\sim}{x_i^{n_s^s}}), (\underset{\sim}{y_j^{n_s^s}}))\}_{s=0}^{\infty}$ is a subsequence of $\{((\underset{\sim}{x_i^n}), (\underset{\sim}{y_j^n}))\}_n$ and converges to $((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$, the latter realizes the supremum value of the social welfare function over all feasible allocations from K. Hence $((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$ is a solution to $(\mathrm{SP})_K$.

Let $K_t \equiv \Sigma_j \hat{y}_{j,1}^{t-1}$ where $((\underset{\sim}{\hat{x}_i}), (\underset{\sim}{\hat{y}_j}))$ is the solution to $(\mathrm{SP})_K$ just found. Now consider $(\mathrm{SP})_K$ truncated at a finite horizon T, denoted as $(\mathrm{SP})_K^T$, defined by

$$(\mathrm{SP})^T_K \qquad Max_{\{((x_i^t),(y_j^t))\}_{t=0}^T} \sum_{t=0}^{T} \delta^t \sum_{i\in I} \Lambda_i^{-1} u_i(x_i^t)$$

$$\text{s.t.} \begin{cases} \Sigma_i x_i^t + \Sigma_j(-y_{j,0}^t) \leq \omega + \Sigma_j y_{j,1}^{t-1} & \text{for } 0 \leq t \leq T, \\ \Sigma_j y_{j,1}^T & \geq K_{T+1}. \end{cases}$$

The T-period allocation $\{((\hat{x}_i^t),(\hat{y}_j^t))\}_{t=0}^T$ is a solution to this problem. Suppose that there exists a T-period allocation that satisfies all constraints with strict inequalities, though the existence of such an allocation is not guaranteed by the assumptions for the turnpike theorem. Then the Kuhn-Tucker theorem can be applied to this problem so that there exists a T-period non-zero non-negative price system $(p^{T,t})_{t=0}^T$ and a non-zero non-negative q^{T+1} such that $\{((\hat{x}_i^t),(\hat{y}_j^t))\}_{t=0}^T$ solves

$$Max_{\{((x_i^t),(y_j^t))\}_{t=0}^T} \sum_i \sum_{t=0}^{T} [\delta^t \Lambda_i^{-1} u_i(x_i^t) - p^{T,t} x_i^t]$$

$$+ \sum_j \left\{ \sum_{t=0}^{T-1} [p^{T,t} y_{j,0}^t + p^{T,t+1} y_{j,1}^{t+1}] + [p^{T,T} y_{j,0}^T + q^{T+1} y_{j,1}^T] \right\}.$$

This means that the T-period consumption plan $(\hat{x}_i^t)_{t=0}^T$ maximizes $\Sigma_{t=0}^T[\delta^t u_i(x_i^t) - \Lambda_i p^{T,t} x_i^t]$ for each i, and the T-period production plan $(\hat{y}_j^t)_{t=0}^T$ maximizes $\Sigma_{t=0}^{T-1}[p^{T,t} y_{j,0}^t + p^{T,t+1} y_{j,1}^t] + [\mathrm{p}^{T,T} y_{j,0}^T + q^{T+1} y_{j,1}^T]$ for each j. Let $T \to \infty$. By applying an argument similar to that in the section 5 where a concept of productive sequence is used, $(p^{T,t})_{T\geq t}$ is bounded uniformly on t's if $K_{t,k} > 0$ for all produce-able commodity k's and t's. Assuming this, a diagonal sequence argument like the one in the previous paragraph can be applied to have a subsequence $(p^{Ts})_{s=0}^{\infty}$ such that $p^{Ts,t}$ converges as $s \to \infty$ for each $t \geq 0$. Let the limit sequence of spot price be $(\underset{\sim}{p})$. It is uniformly bounded over t's. Hence $\Sigma_{t=0}^{Ts}[\delta^t u_i(x_i^t) - \Lambda_i p^{Ts,t} x_i^t]$ converges to $\Sigma_{t=0}^{\infty}[\delta^t u_i(x_i^t) - \Lambda_i p^t x_i^t]$ for all consumption plans of i as $s \to \infty$, so that it is maximized at $\hat{x}_i$. Similarly, $\Sigma_{t=0}^{Ts-1}[p^{Ts,t}y_{j,0}^{t}+p^{Ts,t+1}y_{j,1}^{t+1}]$ converges to $\Sigma_{t=0}^{\infty}[p^{t}y_{j,0}^{t}+p^{t+1}y_{j,1}^{t+1}]$ for all production plans of j as $s \to \infty$, and it is maximized at $\hat{y}_j$. Since p_s^T satisfies the complementary slackness condition for $\{((\hat{x}_i^t), (\hat{y}_j^t))\}_{t=0}^{Ts}$, p satisfies it for $((\hat{x}_i), (\hat{y}_j))$.

The argument in the previous paragraph relies on the applicability of the Kuhn-Tucker theorem on finite-horizon problems and the uniform boundedness of $(p_s^T)_{s=0}^{\infty}$, which are not guaranteed at all on the assumptions for the turnpike theorem. The additional assumptions are needed, and they are likely to be ad-hoc. If the readers are serious about the equilibrium existence, the competitive equilibrium in the theorem should be read as it is.

5 Positive Boundedness of Marginal Utilities of Income, Marginal Costs of Production Efficiency And Stationary Equilibrium Prices

The property on equilibrium that are relevant to the proof of the turnpike theorem is that marginal utilities of income for consumers in competitive equilibria are uniformly bounded away from 0, and marginal costs of production efficiency for firms and stationary prices in stationary equilibria are bounded above and bounded away from 0 uniformly for all time-discount rates δ sufficiently close to 1.

The first assertion comes from the uniform boundedness of feasible allocations given an initial stock vector of produce-able commodities, which implies that $\dfrac{\partial u_i}{\partial x_k}(\hat{x}_i^t)$ is uniformly bounded above and bounded away from 0 over i, t, competitive equilibrium $((\hat{x}_i), (\hat{y}_j), \hat{p})$ with respect to δ, and δ. Let $a>0$ and $b>a$ be the uniform lower-bound and the uniform upper bound, respectively. Then $a \leq \frac{\partial u_i}{\partial x_k}(\hat{x}_i^t) \leq \Lambda_i \delta^{-t} \hat{p}_k^t$ for all i, t and $k \in L_c$. Throughout the proof it is assumed that the size of $\hat{p}$ is normalized so that $\Sigma_{i \in I} \Lambda_i = 1$, where Λ_i is the marginal utility of income for consumer i in a competitive equilibrium. So there is a consumer i' for whom $\Lambda_{i'} \geq \frac{1}{I}$. It is also assumed that every consumer has a positive endowment of a consume-able primary commodity, so that all consumer have positive incomes in any competitive equilibrium. Hence the consumer i' spends his income on a purchase of some $k \in L_c$ in some period t, and $b \geq \frac{\partial u_{i'}}{\partial x_k}(\hat{x}_{i'}^t) = \Lambda_{i'} \delta^{-t} \hat{p}_k^t \geq \frac{1}{I} \delta^{-t} \hat{p}_k^t$. Hence $\Lambda_i \geq \frac{a}{bI}$ for all i.

To prove the second assertion, let $\underline{\delta}$ be the time-discount rate such that the total output vectors of stationary equilibrium are uniformly bounded away from 0 for all $\delta \geq \underline{\delta}$. The existence of such a $\underline{\delta}$ is a critical assumption for the proof of the turnpike theorem in the presence of intermediary commodities. Let $\zeta > 0$ be such that $\Sigma_{j \in J} \bar{y}_{j,1,k} \geq \zeta$ for all $k \in L_p$ and stationary equilibrium $((\bar{x}_i), (\bar{y}_j), \bar{p})$ for $\delta \geq \underline{\delta}$. A productive sequence is defined as a sequence of the form either ik_0 or $ik_N j_N k_{N-1} \cdots k_1 j_1 k_0$, for which 1) $k_0 \in L_c$ in the case of ik_0, 2) $k_N \in L_c$ and (k_{n-1}, k_n) is a feasible pair of input and output commodities for the firm j_n for all $n = 1, \ldots, N$, and there is no repetition in both $\{k_0, \ldots, k_N\}$ and $\{j_1, \ldots, j_N\}$, in the case of $ik_N j_N k_{N-1} \cdots k_1 j_1 k_0$. If a commodity appears as k_0 in some productive sequence, it is called productive. Note that, since it is assumed for each firm that the input space is the non-positive orthant of an Euclidean space, the output space is the non-negative orthant of an Euclidean space, and the firm's production transformation function is defined on the product of them and strictly increasing, j_n can produce only k_n additionally by using only k_{n-1} additionally as input. Therefore, in any competitive equilibrium, the price of a productive commodity in any period must be positive and non-productive commodities are neither consumed nor used as inputs in production. The latter implies that all primary commodities are productive. Since all produce-able commodities are produced in stationary equilibria for $\delta \geq \underline{\delta}$, by the feasibility of stationary equilibrium allocations, it also implies that they are productive for $\delta \geq \underline{\delta}$. Then the former with the complementary slackness in a competitive equilibrium (not only a stationary equilibrium) proves that, in any competitive equilibrium for $\delta \geq \underline{\delta}$, the equilibrium price system is strictly positive and the equilibrium allocation has no physical slack in resources.

Evaluation of the range of stationary equilibrium prices for $\delta \geq \underline{\delta}$ relies on two properties derived from the first order conditions for utility maximization problems and profit maximization problems, which are:

1. For each $i \in I$, $\bar{p}_k \geq \Lambda_i^{-1} \dfrac{\partial u_i}{\partial x_k}(\bar{x}_i)$ with $=$ if $\bar{x}_{i,k} > 0$, for all $k \in L_c$,
2. For each $j \in J$, $\dfrac{1}{\delta} \dfrac{\bar{p}_k}{\bar{p}_{k'}} \geq MRT^j_{k,k'}(\bar{y}_j)$ with $=$ if $\bar{y}_{j,0,k} < 0$ and $\bar{y}_{j,1,k'} > 0$, for all feasible pairs of input and output commodities (k, k') for j,

where $MRT^j_{k,k'}(\bar{y}_j)$ denotes the marginal rate of transformation of input k to output k' at the stationary production $\bar{y}_j$ for the firm j, and it is defined by $\left(\dfrac{\partial g_j}{\partial y_{0,k}}(\bar{y}_j) \middle/ \dfrac{\partial g_j}{\partial y_{1,k'}}(\bar{y}_j) \right)$. For each commodity k, there exists a productive sequence $ik_Nj_Nk_{N-1} \cdots k_1j_1k_0$ with $k_0 = k$ since it is productive. Associate it a positive number $q\,(ik_Nj_Nk_{N-1} \cdots k_1j_1k_0) \equiv \delta^N \Lambda_i^{-1} \frac{\partial u_i}{\partial x_{k_N}}(\bar{x}_i) \times \Pi_{n=1}^{N} MRT_{k_{n-1},\,k_n}^{j_n}(\bar{y}_j)$. Then $\bar{p}_k \geq q\,(ik_Nj_Nk_{N-1} \cdots k_1j_1k_0)$. If k is a produce-able commodity, it must be produced in the stationary equilibrium allocation for $\delta \geq \delta$ and is either consumed or used as an input for a production because there can be no physical slack in resources for k in that stationary equilibrium. If k is a primary commodity, it must be either consumed or used as input for production by the outset of the model. In either case, there exists a productive sequence $ik_Nj_Nk_{N-1} \cdots k_1j_1k_0$ with $k_0 = k$ such that $\bar{x}_{i,\,k_N} > 0$, and $\bar{y}_{j,\,0,\,k_{n-1}} < 0$ and $\bar{y}_{j,\,1,\,k_n} > 0$ for all $n = 1, \ldots, N$. For such a productive sequence, $\bar{p}_k = q\ (ik_Nj_Nk_{N-1} \cdots k_1j_1k_0)$. ($N$ can be 0.)

Since marginal utilities and marginal rate of transformations are positive-valued and continuous, the uniform boundedness of feasible stationary allocations implies that they are bounded above and bounded away from 0 uniformly over stationary equilibrium allocations for $\delta \geq \underline{\delta}$. Because there is no repetition in a productive sequence, the set of all productive sequences is finite since the set of economic agents and the set of commodities are finite. Hence q-numbers are bounded above and bounded away from 0 uniformly over stationary equilibrium allocations for $\delta \geq \underline{\delta}$. Let $\underline{q}$ be a uniform positive lower-bound and $\bar{q}$ be a uniform upper bound. The property obtained in the previous paragraph then implies that stationary equilibrium prices for $\delta \geq \underline{\delta}$ are bounded above by $\bar{q}$ and bounded away from 0 by $\underline{q}$.

Let ρ_j be the marginal cost of production efficiency for j in a stationary equilibrium $(\bar{x}, \bar{y}, \bar{p})$ for a $\delta \geq \underline{\delta}$. By the first order condition for the stationary profit maximization problem, $\bar{p}_k \geq \rho_j \frac{\partial g_j}{\partial y_{0,k}}(\bar{y}_j)$ with = if $\bar{y}_{j,0,k} < 0$ for all k that can be used as inputs by firm j, and $\delta \bar{p}_k \leq \rho_j \frac{\partial g_j}{\partial y_{1,k}}(\bar{y}_j)$ with = if $\bar{y}_{j,1,k} > 0$ for all k that can be produced by firm j. Since production transformation functions are continuously differentrable and strictly increasing, the uniform boundedness of the feasible stationary allocations implies that $\frac{\partial g_j}{\partial y_k}(\bar{y}_j)$ is bounded above and bounded away from 0 uniformly over j, k, and stationary equilibrium allocations for $\delta \geq \underline{\delta}$. Let $a > 0$ be a uniform positive lower-bound and $b > 0$ be a uniform upper-bound for $\frac{\partial g_j}{\partial y_k}(\bar{y}_j)$'s. Then ρ_j's are uniformly bounded above by $\frac{\bar{q}}{a}$ and bounded away from 0 by $\frac{\delta q}{b}$.

6 Choice of a Stationary Equilibrium

In choosing a stationary equilibrium allocation to which the competitive equilibrium allocation converges, it is crucial that the marginal utility of income for each consumer is the same as that in the competitive equilibrium.

By dividing the utility function by the marginal utility of income, the unit of measurement for utility is unified with the unit of account. This makes a comparison of utilities among consumers possible, and a social welfare function can be defined as the sum of those utilities. The reciprocal of the marginal utility of income for a consumer is the social weight on that consumer in the social welfare function. It also serves as an exchange rate of utility to the unit of account for the consumer, so that a consumer surplus for a spot consumption of a consumer is defined as the benefit minus the cost of that consumption in the unit of account, where the benefit is the utility on the consumption measured by the unit of account and the cost is the expenditure on the consumption. To compare a social welfare of the competitive equilibrium allocation and that of a stationary equilibrium allocation, the social welfare function must be the same for both equilibrium allocations. Similarly, to compare consumer surpluses in the competitive allocation and those in a stationary equilibrium allocation, exchange rates of utility to the unit of account must be the same in both equilibrium allocations.

Let Λ_i be the marginal utility of income for consumer i in the competitive equilibrium. Replacing consumers with one aggregate consumer with the social weights on consumers $\{\Lambda_i^{-1}\}_{i\in I}$, a stationary equilibrium exists if the time-discount rate is sufficiently close to 1. The price system in the stationary equilibrium is normalized so that the marginal utility of income for the aggregate consumer is 1. (For the existence issue, see Bewley (1980).) Then the aggregation rule implies that the first order condition satisfied by the equilibrium aggregate consumption for the aggregate consumer is the same as the collection of the first order conditions that the optimal stationary consumption satisfies for each consumer. These first order conditions are sufficient for utility maximization with transfer payments, due to the concavity of utility functions. Hence the equilibrium aggregate consumption is decentralized to utility maximizing consumption of consumers with transfer payments, where consumer i has the marginal utility of income equal to Λ_i. This decentralized consumptions along with the productions and the price system in the stationary equilibrium with the aggregate consumer form a stationary equilibrium with transfer payments in the original economy.

Hereafter the stationary equilibrium with transfer payments so obtained is called just as the stationary equilibrium.

7 Loss in Market Surplus from the Stationary Equilibrium

The essence of the turnpike theorem is that the loss in total market surplus from the stationary equilibrium allocation satisfies a Lyapunov stability along the path of the competitive equilibrium allocation.

Remind the social welfare maximization problem in section 4, which is

$$(\mathrm{SP})_K \qquad Max_{((\underset{\sim}{x_i}),\,(\underset{\sim}{y_j}))} \sum_{t=0}^{\infty} \delta^t \sum_{i \in I} \Lambda_i^{-1} u_i(x_i^t)$$

$$\text{s.t. } ((\underset{\sim}{x_i}),\ (\underset{\sim}{y_j})) \text{ is feasible from the initial stock of produce-able commodities } K,$$

where $K \in \mathbb{R}_+^{L_p}$. Let $(((\underset{\sim}{\hat{x}_i}),\ (\underset{\sim}{\hat{y}_j})),\ \underset{\sim}{\hat{p}})$ be a competitive equilibrium with transfer payments from K in which the marginal utility of income for consumer i is Λ_i. Then the consumer i maximizes $\Lambda_i^{-1}\delta^t u_i(x_i) - \hat{p}^t x_i$ on his consumption set at $\hat{x}_i^t$ for all t's. The firm j maximizes $\hat{p}^t y_{j,0} + \hat{p}^{t+1} y_{j,1}$ on his production possibility set at $\hat{y}_j^t$ for all t's. By summing up all of these, adding the constant $\Sigma_{t=0}^{\infty} \hat{p}^t \omega$ and rearranging terms under absolute convergence of series involved, $((\underset{\sim}{\hat{x}_i}),\ (\underset{\sim}{\hat{y}_j}))$ maximizes $\Sigma_{t=0}^{\infty} \delta^t \Sigma_i \Lambda_i^{-1} u_i(x_i^t) + \Sigma_{t=0}^{\infty} \hat{p}^t [\Sigma_j y_{j,1}^{t-1} + \Sigma_j y_{j,0}^t + \omega - \Sigma_i x_i^t] - \hat{p}^0 \cdot K$ over all plan $((\underset{\sim}{x_i}),\ (\underset{\sim}{y_j}))$'s bounded by some large number B, where $\Sigma_j y_{j,1}^{-1}$ is understood to be K. Here B is chosen so large that all feasible allocations to be considered are bounded by it. By the complementary slackness at the competitive equilibrium with transfer payments, $\hat{p}^t[\Sigma_j \hat{y}_{j,1}^{t-1} + \Sigma_j \hat{y}_{j,0}^t + \omega - \Sigma_i \hat{x}_i^t] = 0$ in each period t. If $((x_i), (y_j))$ is feasible from K, $\hat{p}^t[\Sigma_j y_{j,1}^{t-1} + \Sigma_j y_{j,0}^t + \omega - \Sigma_i x_i^t] \geq 0$ in each period t. By neglecting the last constant term $(-\hat{p}^0 \cdot K)$, it is observed that the competitive equilibrium allocations $((\hat{x}_i), (\hat{y}_j))$ solves $(\mathrm{SP})_K$. Replacing K by $\overline{K}$, the stationary equilibrium allocation $((\overline{x}_i), (\overline{y}_j))$ solves $(\mathrm{SP})_{\overline{K}}$.

For each consumer i, $\Lambda_i^{-1}\delta^t u_i(x_i^t) - \hat{p}^t x_i^t$ represents the consumer surplus for i evaluated at the competitive equilibrium price in period t. The sum of them for all i's is the consumer surplus evaluated at the competitive equilibrium price in period t. Similarly, for each firm j, $\hat{p}^t y_{j,0}^t + \hat{p}^{t+1} y_{j,1}^t$ is the producer surplus for j evaluated at the competitive equilibrium price in period t. The sum of them for all j's is the producer surplus evaluated at the competitive equilibrium price in period t. The sum of the consumer surplus and producer surplus in period t is then the social surplus evaluated at the competitive equilibrium price in period t. The sums of surpluses over entire periods are total surpluses. The argument in the previous paragraph proves that these surpluses are maximized at the competitive equilibrium allocation for relevant decision variables, and that the sum of the maximized total social surplus and the total value of endowments evaluated at the competitive equilibrium price is equal to the maximized social welfare at the competitive equilibrium allocation in $(\mathrm{SP})_K$ minus the value of initial stock K evaluated at the competitive equilibrium price. These properties are also valid for the stationary equilibrium with its initial stock $\overline{K}$.

The proof of the turnpike theorem picks the stationary equilibrium with transfer payments as a candidate to represent good habits and evaluates the loss in consumer surpluses, producer surpluses and social surplus if economic agents do not follow these good habits but do the best. Since it is hypothesized that the stationary equilibrium with transfer payments represents good habits, the valuation for surpluses is based on the stationary equilibrium price $\bar{p}$. It has been seen that, for any feasible allocation $((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$ from some non-negative initial stock K,

$$\begin{aligned}
&\Sigma_{t=0}^{\infty}\delta^t\Sigma_i[\Lambda_i^{-1}u_i(x_i^t)-\bar{p}x_i^t]+\Sigma_{t=0}^{\infty}\delta^t\Sigma_j[\bar{p}y_{j,0}^t+\delta\bar{p}y_{j,1}^t]\\
&-\Sigma_{t=0}^{\infty}\delta^t\bar{p}[\Sigma_j y_{j,1}^{t-1}+\Sigma_j y_{j,0}^t+\omega-\Sigma_i x_i^t] \qquad (1)\\
&=\Sigma_{t=0}^{\infty}\delta^t\Sigma_i\Lambda_i^{-1}u_i(x_i^t)-\bar{p}K+\frac{1}{1-\delta}\bar{p}\,\omega.
\end{aligned}$$

The left-hand side is called as the total market surplus evaluated at the stationary equilibrium price for $((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$. It is the sum of the total consumer surplus, the total producer surplus, and $(-1)\times$the total value slack in the market. Regarding the market as a virtual economic agent who controls a market price to equalize a market demand and a market supply, the last component represents a surplus for the market evaluated at the stationary equilibrium price. The left-hand side is maximized at the stationary equilibrium allocation $((\bar{x}_i), (\bar{y}_j))$, so as the right-hand side. Hence the loss from this maximized total market surplus evaluated at the stationary price for a feasible allocation from some K is measured non-negatively. The Lyapunov function is defined by the minimum loss in the total market surplus evaluated at the stationary equilibrium price for a initial stock K, where K varies. Namely, it is the function F_δ of initial stock K defined by

$$\begin{aligned}
F_\delta(K)&\equiv\Sigma_{t=0}^{\infty}\delta^t\Sigma_i\Lambda_i^{-1}u_i(\bar{x}_i)-\bar{p}\bar{K}-[\Sigma_{t=0}^{\infty}\delta^t\Sigma_i\Lambda_i^{-1}u_i(x_i^t)-\bar{p}K] \qquad (2)\\
&=\bar{p}(K-\bar{K})-\Sigma_{t=0}^{\infty}\delta^t\Sigma_i\Lambda_i^{-1}[u_i(x_i^t)-u_i(\bar{x}_i)].
\end{aligned}$$

where $((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$ is a solution to $(\mathrm{SP})_K$. In the definition, the right-hand side of (1) is used for simplicity. By using the left-hand side of (1), it becomes

$$\begin{aligned} F_\delta(K) \equiv\ & \Sigma_{t=0}^{\infty}\delta^t\Sigma_{i\in I}[(\Lambda_i^{-1}u_i(\bar{x}_i)-\bar{p}\cdot\bar{x}_i)-(\Lambda_i^{-1}u_i(x_i^t)-\bar{p}\cdot x_i^t)] \\ & +\Sigma_{t=0}^{\infty}\delta^t\Sigma_{j\in J}[(\bar{p}\cdot\bar{y}_{j,0}+\delta\bar{p}\cdot\bar{y}_{j,1})-(\bar{p}\cdot y_{j,0}^t+\delta\bar{p}\cdot y_{j,1}^t)] \\ & +\Sigma_{t=0}^{\infty}\delta^t\bar{p}\cdot[(\Sigma_{j\in J}(y_{j,0}^t+y_{j,1}^{t-1})+\omega-\Sigma_{i\in I}x_i^t)-(\Sigma_{j\in J}(\bar{y}_{j,0}+\bar{y}_{j,1}) \\ & +\omega-\Sigma_{i\in I}\bar{x}_i)]. \end{aligned} \tag{3}$$

where the initial stock K is hypothetically distributed among firms as $(y_{j,1}^{-1})_{j\in J}$.

For each consumer i, given a consumption plan $\underset{\sim}{x_i}$, let $\overline{LCS}_i^t(x_i^t)$ be defined for each t by $(\Lambda_i^{-1}u_i(\bar{x}_i)-\bar{p}\cdot\bar{x}_i)-(\Lambda_i^{-1}u_i(x_i^t)-\bar{p}\cdot x_i^t)$ and $\overline{LCS}_i(\underset{\sim}{x_i})$ by $\Sigma_{t=0}^{\infty}\ \delta^t\overline{LCS}_i^t(x_i^t)$ if the series converge. For a profile of consumption plans $(\underset{\sim}{x_i})$, let $\overline{LCS}^t((x_i^t))$ be defined for each t by $\Sigma_i\overline{LCS}_i^t(x_i^t)$ and $\overline{LCS}((\underset{\sim}{x_i}))$ by $\Sigma_{t=0}^{\infty}\delta^t\overline{LCS}^t((x_i^t))$ if the series converge. Similarly, for each firm j, given a production plan $\underset{\sim}{y_j}$, let $\overline{LPS}_j^t(y_j^t)$ be defined for each t by $(\bar{p}\,\bar{y}_{j,0}+\delta\bar{p}\,\bar{y}_{j,1})-(\bar{p}y_{j,0}^t+\delta\bar{p}y_{j,1}^t)$, and $\overline{LPS}_j(\underset{\sim}{y_j})$ by $\Sigma_{t=0}^{\infty}\delta^t\overline{LPS}_j^t(y_j^t)$ if the series converge. For a profile of production plans $(\underset{\sim}{y_j})$, let $\overline{LPS}^t((y_j^t))$ be defined for each t by $\Sigma_j\overline{LPS}_j^t(y_j^t)$ and $\overline{LPS}((\underset{\sim}{y_j}))$ by $\Sigma_{t=0}^{\infty}\delta^t\overline{LPS}^t((y_j^t))$ if the series converge. For the market, given an allocation $((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$, let $\overline{DVS}^t((x_i^t), (y_{j,1}^{t-1}, y_{j,0}^t))$ be defined for each t by $\bar{p}\cdot[(\Sigma_{j\in J}(y_{j,0}^t+y_{j,1}^{t-1})+\omega-\Sigma_{i\in I}x_i^t)-(\Sigma_{j\in J}(\bar{y}_{j,0}+\bar{y}_{j,1})+\omega-\Sigma_{i\in I}\bar{x}_i)]$, and $\overline{DVS}((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$ by $\Sigma_{t=0}^{\infty}\ \delta^t\overline{DVS}^t((x_i^t), (y_{j,1}^{t-1}, y_{j,0}^t))$ if the series converge. It should be noted that, for each t, $\overline{DVS}^t((x_i^t), (y_{j,1}^{t-1}, y_{j,0}^t))$ is minimized over feasible allocation $((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$'s at $((\bar{x}_i), (\bar{y}_j))$ and that the minimized value is 0. With these definitions, it is clear that $F_\delta(K)=\overline{LCS}((\underset{\sim}{x_i}))+\overline{LPS}((\underset{\sim}{y_j}))+\overline{DVS}((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$ where $((\underset{\sim}{x_i}), (\underset{\sim}{y_j}))$ is a solution to $(\mathrm{SP})_K$.

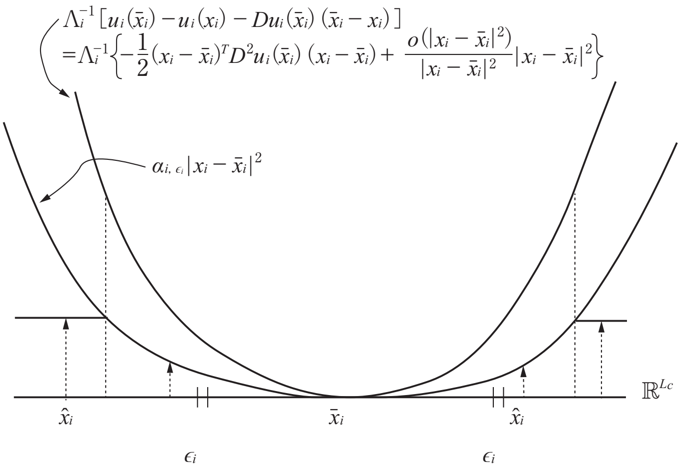
In (3), all three components are non-negative so that $F_\delta(K) \geq 0$ for all initial stock K. It is also clear that $F_\delta(\bar{K})=0$. Letting $\hat{K}^t \equiv \Sigma_{j\in J}\hat{y}^{t-1}_{j,1}$ where $((\hat{x}_i), (\hat{y}_j))$ is the competitive equilibrium allocation from $\hat{K}^0 \equiv \Sigma_{j\in J}y^{-1}_{j,1}$, it is observed that $\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t) = -[\overline{LCS}^t((\hat{x}^t_i)) + \overline{LPS}^t((\hat{y}^t_j)) + \overline{DVS}^t((\hat{x}^t_i), ((\hat{y}^{t-1}_{j,1}, \hat{y}^t_{j,0})))] \leq 0$. Hence the value of the "discounted" Lyapunov function along the competitive equilibrium is dynamically non-increasing.

8 Relationship between Loss in Total Market Surplus and the Competitive Equilibrium Allocation

It is important to derive that the (exponential) dynamic convergence of F_δ to 0 along the competitive equilibrium allocation implies that of the distance between the competitive equilibrium allocation and the stationary equilibrium allocation. A close look at $(\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t))$ proves it. Note that $-(\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t))$ is no less than the sum of the loss in consumer surplus and that in the producer surplus in period t. By the first order conditions for utility maximization at the the stationary equilibrium, for each consumer i, $\Lambda_i\overline{LCS}^t_i(\hat{x}^t_i)$ is approximated from below by $(-1)\times$the second-order term of the Taylor expansion of u_i at $\bar{x}_i$. The differentiable concavity of u_i implies that $(-1)\times$the Hessian of u_i at $\bar{x}_i$ is positive definite, hence its eigenvalues are all positive. This enables to make $\overline{LCS}^t_i(\hat{x}^t_i)$ be bounded from below by a non-negative function of $|\hat{x}^t_i - \hat{x}_i|^2$ locally at $\bar{x}_i$. Similarly, the first order conditions for profit maximization at the stationary equilibrium implies that, for each firm j, $\rho_j^{-1}\overline{LPS}^t_j((\hat{y}^t_j))$ is approximated from below by the second-order term of the Taylor expansion of a function defined on the tangent space of $g_j^{-1}(0)$ at $\bar{y}_j$, $T_{\bar{y}_j}$, that measures the distance of $g_j^{-1}(0)$ from $T_{\bar{y}_j}$. The differentiable convexity of g_j says that the Hessian of this function at $\bar{y}_j$ is positive definite, so that its eigenvalues are all positive. This makes $\overline{LPS}_j^t(\hat{y}_j^t)$ be bounded from below by a non-negative function of $|\hat{y}_j^t - \bar{y}_j|^2$ locally at $\bar{y}_j$. Since Λ_i's are bounded above and ρ_j's are bounded away from 0, the sum of all $\overline{LCS}_i^t(\hat{x}_i^t)$'s and all $\overline{LPS}_j^t(\hat{y}_j^t)$'s is bounded from below by a non-negative function of $|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\bar{x}_i), (\bar{y}_j))|^2$ locally at $((\bar{x}_i), (\bar{y}_j))$, so as $-(\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t))$. Since $F_\delta(\hat{K}^{t+1}) \geq 0$, $F_\delta(\hat{K}^t)) \geq -(\delta F_\delta(\hat{K}^{t+1}) - F_\delta(\hat{K}^t))$, so that $F_\delta(\hat{K}^t))$ is bounded from below by a non-negative function of $|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\bar{x}_i), (\bar{y}_j))|^2$ locally at $((\bar{x}_i), (\bar{y}_j))$.

The argument is made precise as follows. A series of diagrams would help to find lower-bounds. The situation for consumer i can be visualized in $\mathbb{R}^{L_c}$ in the figure 1.

Figure 1: Local Approximation of Loss In the Consumer Surplus from Below



For any $\gamma>0$, there exists $\epsilon_i>0$ such that $\left|\dfrac{o(|z-\bar{x}_i|^2)}{|z-\bar{x}_i|^2}\right|\leq\gamma$ for all z with $|z-\bar{x}_i|\leq\epsilon_i$. Take a γ smaller than $\frac{1}{2}\times$ the smallest eigenvalue for $-\frac{1}{2}D^2u_i(\bar{x})$. Then the positive number α_{i,ϵ_i} is obtained as $\dfrac{\text{the smallest eignevalue for } -\frac{1}{2}D^2u_i(\bar{x})}{2\Lambda_i}$. Since $\bar{p}_k\geq\Lambda_i^{-1}\dfrac{\partial u_i}{\partial x_k}(\bar{x}_i)$ with $=$ if $\bar{x}_{i,k}>0$, $\overline{LCS_i^t}(\hat{x}_i^t)\geq\alpha_{i,\epsilon_i}\min\{\|\hat{x}_i^t-\bar{x}_i|^2,\ \epsilon_i^2\}$.

The situation for firm j can be visualized in $M_{j,0}\times M_{j,1}$ in the figure 2. Regarding $T_{\bar{y}_j}$ as an Euclidean space with the origin at $\bar{y}_j$, a function μ_j can be defined on a neighborhood of 0 by $\mu_j(z)\equiv Dg_j(\bar{y}_j)(\bar{y}_j-y(z))$ where $y(z)$ satisfies $g_j(y(z))=0$ and its orthogonal projection to $T_{\bar{y}_j}$ is $(\bar{y}_j+z)$. The differential strict concavity of g_j asserts that μ_j is twice

Figure 2: Local Approximation of Loss in the Producer Surplus from Below

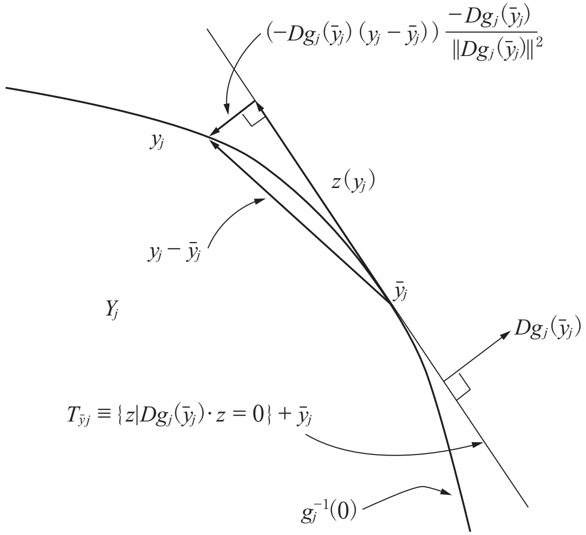
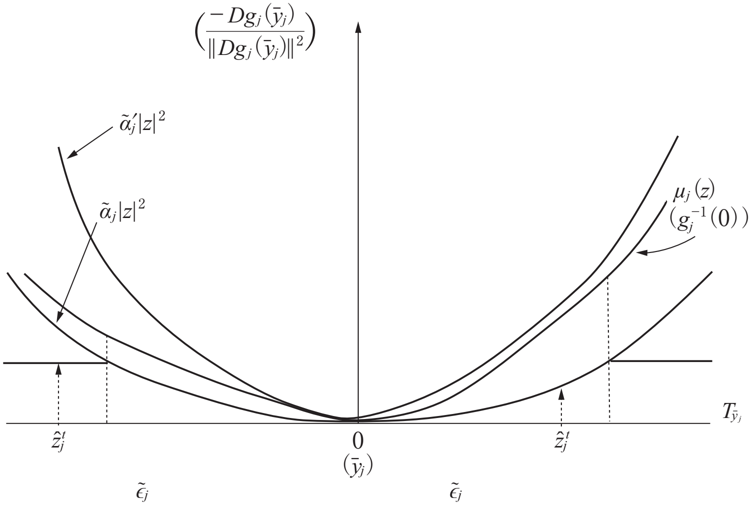


Figure 3: A Look of Production Frontier from the Tangent Space



continuously differentiable and strictly convex locally at $\bar{y}_j$. By looking at the figure 2 from $T_{\bar{y}_j}$ as the horizontal "axis", we have the figure 3. Here $\hat{z}^t_j$ denotes the orthogonal projection of $\hat{y}^t_j$ onto $T_{\bar{y}_j}$. By differentiating both sides of the identity $g_j\left(\bar{y}_j + z + \mu_j(z)\frac{-Dg_j(\bar{y}_j)}{\|Dg_j(\bar{y}_j)\|^2}\right) = 0$ with respect to z and evaluating it at $z=0$, $D\mu_j(0) = Dg_j(\bar{y}_j)$. From the Taylor expansion of μ_j at 0, $\mu_j(z) = \frac{1}{2}z^T D^2\mu_j(0)z + o(|z|^2)$. Since $\frac{1}{2}D^2\mu_j(0)$ is positive definite, its smallest eigenvalue is positive. By choosing small $\tilde{\epsilon}_j$, $\frac{o(|z|^2)}{|z|^2}$ is less than a half of the smallest eigenvalue for all z with $|z| \leq \tilde{\epsilon}_j$. Hence, by letting $\tilde{\alpha}_j \equiv \frac{\text{the smallest eigen value of } \frac{1}{2}D^2\mu_j(0)}{2}$, $\mu_j(z) \geq \tilde{\alpha}_j \min\{|z|^2, \tilde{\epsilon}_j^2\}$ for all z locally at 0.

Recall the property of the marginal cost of production efficiency at the stationary equilibrium for firm j, ρ_j. By the definition of $\mu_j(\cdot)$, $\rho_j\mu_j(\hat{z}_j^t) \leq \overline{LPS}_j^t(\hat{y}_j^t)$. To bound $|(\hat{y}_{j,0}^t, \hat{y}_{j,1}^t) - (\bar{y}_{j,0}, \bar{y}_{j,1})|^2$ by $|\hat{z}^t|^2$ from above, let $\tilde{\alpha}_j'$ be the sum of the dimension of $M_{j0} \times M_{j1}$ multiplied by the largest eigenvalue of $\frac{1}{2}D\mu_j(0)$ and a half of its smallest eigenvalue. Then $\mu_j(z) \leq \tilde{\alpha}_j'|z|^2$ for all z with $|z| \leq \tilde{\epsilon}_j$. With this and the orthogonal decomposition $y(z) - \bar{y}_j = z + \mu_j(z)\frac{-Dg_j(\bar{y})}{\|Dg_j(\bar{y}_j)\|^2}$, it can be derived that

$$|y(z) - \bar{y}_j|^2 \leq \left(\frac{\tilde{\alpha}_j'^2}{\|Dg_j(\bar{y})\|^2} + (\text{dimension of } M_{j,0} \times M_{j,1})\right)|z|^2 \text{ for all } z$$

with $|z| \leq \tilde{\epsilon}_j$, by adjusting $\tilde{\epsilon}_j$ to be smaller than 1. Hence, by letting

$$\alpha_j \equiv \frac{\rho_j\tilde{\alpha}_j}{\frac{\tilde{\alpha}_j'^2}{\|Dg_j(\bar{y}_j)\|^2} + (\text{dimension of } M_{j,0} \times M_{j,1})}, \quad \overline{LPS}_j^t(y(z)) \geq \alpha_j\,|y(z) - \bar{y}_j|^2$$

for all z with $|z| \leq \tilde{\epsilon}_j$. Let a positive number ϵ_j be such that $g_j(y) = 0$ and $|y - \bar{y}_j| \leq \epsilon_j$ imply that $|z(y)| \leq \tilde{\epsilon}_j$, where $z(y)$ is the orthogonal projection of y onto $T_{\bar{y}_j}$. If $|\hat{y}_j^t - \bar{y}_j| > \epsilon_j$, there is a y'' on the line segment between $\hat{y}_j^t$ and $\bar{y}_j$ for which there exists a y' with $g_j(y') = 0$ such that $y'' \leq y'$ and $|y' - \bar{y}_j| = \epsilon_j$. Then $\overline{LPS}_j^t(y'') \geq \overline{LPS}_j^t(y') \geq \alpha_j\epsilon_j^2$. Since $\hat{y}_j^t - \bar{y}_j$ is a strict extension of $y'' - \bar{y}_j$, it follows that $\overline{LPS}_j^t(\hat{y}_j^t) \geq \overline{LPS}_j^t(y'')$. Hence we have $\overline{LPS}_j^t(\hat{y}_j^t) \geq \alpha_j \min\{|\hat{y}_j^t - \bar{y}_j|^2, \epsilon_j^2\}$.

Let the positive number α be the minimum of α_{i,ϵ_i}'s and α_j's, and the positive number ϵ be the minimum of ϵ_i's and ϵ_j's. Then $F_\delta(\hat{K}^t) - \delta F_\delta(\hat{K}^{t+1}) \geq \overline{LCS}^t((\hat{x}_i^t)) + \overline{LPS}^t((\hat{y}_j^t))$ implies that

$$F_\delta(\hat{K}^t) - \delta F_\delta(\hat{K}^{t+1}) \geq \alpha \times \min\{|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\bar{x}_i), (\bar{y}_j))|^2, \epsilon^2\}. \quad (4)$$

The left-hand side is no more than $F_\delta(\hat{K}^t)$, so that a geometric convergence of $F_\delta(\hat{K}^t)$ to 0 as t goes to ∞ implies the turnpike theorem.

9　Relationship between Loss in Total Market Surplus and Stocks in the Competitive Equilibrium

The inequality (4) can be modified into that about the square of difference between the initial stock of produce-able commodities for a competitive equilibrium in period t and that for the stationary equilibrium. Since there is no physical slack in competitive equilibria for $\delta \geq \underline{\delta}$, $|\hat{K}^t - \overline{K}| \leq (I+J)|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\overline{x}_i), (\overline{y}_j))|$. Since $I+J>1$, by letting $\dfrac{\alpha}{(I+J)^2}$ to be renamed as α, $|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\overline{x}_i), (\overline{y}_j))|^2$ can be replaced with $|\hat{K}^t - \overline{K}|^2$ in the right-hand side of (4). Dividing the both sides with δ has no effect on the right-hand side since $\delta \leq 1$, so that

$$\delta^{-1}F_\delta(\hat{K}^t) - F_\delta(\hat{K}^{t+1}) \geq \alpha \times \min\{|\hat{K}^t - \overline{K}|^2, \epsilon^2\} \text{ for all } \delta \geq \underline{\delta}. \tag{5}$$

If δ were 1, this inequality would be found commonly in literatures on Lyapunov stability.

In fact, if δ were 1, the inequality (4) would almost complete the proof of the turnpike theorem if $F_1(\hat{K}^0)$ is finite.[1)] The inequality (4) implies that $F_1(\hat{K}^t)$ is decreasing. If $|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\overline{x}_i), (\overline{y}_j))| \geq \epsilon$ for infinitely often t, the inequality (4) shows that $F_1(\hat{K}^t)$ must become negative eventually, contradicting the non-negativity of F_1. Hence there is a period τ such that $|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\overline{x}_i), (\overline{y}_j))| < \epsilon$ for all $t \geq \tau$. This implies that $F_1(\hat{K}^\tau) \geq F_1(\hat{K}^\tau) - F_1(\hat{K}^{\tau+T}) \geq \alpha \Sigma_{s=0}^{T} |((\hat{x}_i^{\tau+s}), (\hat{y}_j^{\tau+s})) - ((\overline{x}_i), (\overline{y}_j))|^2$ for any large T. If $|((\hat{x}_i^{\tau+s}), (\hat{y}_j^{\tau+s})) - ((\overline{x}_i), (\overline{y}_j))|^2$ does not converge to 0 as t goes to ∞, there exists a positive

1) The argument in the section 11 proves that $F_1(\hat{K}^0)$ is actually finite, since the argument is valid without time-discount.

number γ such that $|((\hat{x}_i^{\tau+s}), (\hat{y}_j^{\tau+s})) - ((\bar{x}_i), (\bar{y}_j))|^2 \geq \gamma$ for infinitely often s. Then, by taking T sufficiently large, the right-hand side becomes larger than $F_1(\hat{K}^\tau)$. This obvious contradiction proves that $|((\hat{x}_i^{\tau+s}), (\hat{y}_j^{\tau+s})) - ((\bar{x}_i), (\bar{y}_j))|^2$ must converge to 0 as s goes to ∞[2)].

However, with $\underline{\delta} \leq \delta < 1$, several difficulties appear aside from the uniform boundedness of $F_\delta(\hat{K}^0)$ over $\delta \geq \underline{\delta}$. First, the inequalities does not guarantee that F_δ is decreasing along a competitive equilibrium stock path, so that it is not even clear whether its values stay in a bounded range or not. This implies a possibility that $|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\bar{x}_i), (\bar{y}_j))|^2$ exceeds ϵ infinitely often may not be excluded. Secondly, the argument for $\delta = 1$ does not require that $F_\delta(\hat{K}^t)$ converges to 0, but the property will be necessary in order to overcome the possibility mentioned above.

10 Completing the Proof with Key Findings

It turns out, in establishing the behavior of $F_\delta(\hat{K}^t)$ as t goes to ∞, the key findings are

1. For any given $\hat{K}^0$, $F_\delta(\hat{K}^0)$ is bounded from above uniformly over $\delta \geq \underline{\delta}$,
2. $F_\delta(K)$ is bounded from above by $|K - \bar{K}|^2$ multiplied by a positive constant for all K sufficiently close to $\bar{K}$, uniformly over $\delta \geq \underline{\delta}$.

In literatures on Lyapunov stability, these properties are derived for $\delta = 1$. The proofs of these key findings in section 11 and section 12

2) There is a conceptual difficulty to define competitive equilibrium for $\delta = 1$ since the objective functions of consumers and firms would be infinite valued for relevant plans. Replacing the series in objective functions with the limit of averaged utility or profit in a finite horizon as the horizon goes to infinity is expected to work.

will reveal that the proofs for $\delta=1$ work since the loss in undiscounted total social welfare and losses in undiscounted total consumer and producer surplus converge. Hence the requirement that $\delta \geq \underline{\delta}$ is not essential for these key findings.

Let's assume temporarily that they have been established. These properties guarantee that the inequality (5) in the case $\delta=1$ is recovered by replacing α and ϵ by smaller positive numbers (if necessary) and retaking $\underline{\delta}$ further closer to 1. This is the only part in the proof of the turnpike theorem where restricting δ to be close to 1 is essential. Let's retake ϵ in the inequality (5) smaller, if necessary, so that $|K-\overline{K}| \leq \epsilon$ implies $F_\delta(K) \leq A|K-\overline{K}|^2$, where A is a positive number. The inequality (5) remains to be valid if the ϵ is replaced by a smaller positive number. Let $C>0$ be such that $F_\delta(\hat{K}^0) \leq C$ for all $\delta \geq \underline{\delta}$. The proof proceeds by mathematical induction. Assume that $F_\delta(\hat{K}^t) \leq C$. Since $F_\delta(\hat{K}^t) - F_\delta(\hat{K}^{t+1}) = (F_\delta(\hat{K}^t) - \delta^{-1}F_\delta(\hat{K}^t)) + (\delta^{-1} F_\delta(\hat{K}^t) - F_\delta(\hat{K}^{t+1}))$ and the first term in the decomposition is non-positive, we need to obtain an upper-bound for $\delta^{-1}F_\delta(\hat{K}^t) - F_\delta(\hat{K}^t)$. If $|\hat{K}^t-\overline{K}| \leq \epsilon$, $\delta^{-1}F_\delta(\hat{K}^t) - F_\delta(\hat{K}^t) \leq \left(\frac{1}{\delta}-1\right) A|\hat{K}^t-\overline{K}|^2$. If $|\hat{K}^t-\overline{K}| > \epsilon$, $\delta^{-1}F_\delta(\hat{K}^t) - F_\delta(\hat{K}^t) \leq \left(\frac{1}{\delta}-1\right)C$. Hence $F_\delta(\hat{K}^t) - F_\delta(\hat{K}^{t+1})$ is bounded from below by $\left[\alpha-\left(\frac{1}{\delta}-1\right)A\right]|\hat{K}^t-\overline{K}|^2$ if $|\hat{K}^t-\overline{K}| \leq \epsilon$, and by $\left[\alpha - \frac{\left(\frac{1}{\delta}-1\right)C}{\epsilon^2}\right]\epsilon^2$ if $|\hat{K}^t-\overline{K}| > \epsilon$. As δ goes to 1, both $\left[\alpha-\left(\frac{1}{\delta}-1\right)A\right]$ and $\left[\alpha - \frac{\left(\frac{1}{\delta}-1\right)C}{\epsilon^2}\right]$ converges to α from below. Hence, for any positive number $\alpha' < \alpha$, there exits a $\delta' \geq \underline{\delta}$ such that these numbers are no less than

α' for all $\delta \geq \delta'$. Rename α' and δ' as new α and $\underline{\delta}$. Then

$$F_\delta(\hat{K}^t) - F_\delta(\hat{K}^{t+1}) \geq \alpha \times \min\{|\hat{K}^t - \overline{K}|^2, \epsilon^2\} \tag{6}$$

for all $\delta \geq \underline{\delta}$. This implies that $F_\delta(\hat{K}^{t+1}) \leq C$.

Replace α, ϵ and $\underline{\delta}$ for (4) and (5) with these new ones. Since new α and ϵ are no more than the original ones and new $\underline{\delta}$ is no less than the original one, those inequalities remains to be valid with this replacement.

The recovery of (6) from (5) for $\delta \geq \underline{\delta}$ enables us to apply a standard asymptotic Lyapunov stability argument for the case that $\delta = 1$. The first key finding and the inequality (6) enable us to make the same argument as the one for the hypothetical case of $\delta = 1$ and conclude that $|\hat{K}^t - \overline{K}|^2$ converges to 0 as t goes to ∞. However, the theorem claims the convergence of $|((\hat{x}_i^t), (\hat{y}_j^t)) - ((\overline{x}_i), (\overline{y}_j))|^2$ to 0, and the convergence of stock difference is not enough for that. The only way to guarantee it is to prove the convergence of $F(\hat{K}^t)$ to 0 as t goes to ∞. To this end, not only the first key ordering but also the second key finding must be used with (6).

The second key finding tells that $A\epsilon^2$ serves as a critical value for $F_\delta(K)$ to know whether $|K - \overline{K}| \leq \epsilon$ or not. So, $F_\delta(\hat{K}^t) > A\epsilon^2$ implies $|\hat{K}^t - \overline{K}| > \epsilon$, which implies, by the inequality (6), that $F_\delta(\hat{K}^{t+1}) \leq F_\delta(\hat{K}^t) - \alpha\epsilon^2$. The second key finding also tells that, if $|\hat{K}^t - \overline{K}| \leq \epsilon$, $|\hat{K}^t - \overline{K}|^2 \geq \frac{1}{A} F_\delta(\hat{K}^t)$, so that the inequality (6) implies $F_\delta(\hat{K}^{t+1}) \leq \left(1 - \frac{\alpha}{A}\right) F_\delta(\hat{K}^t)$. Hence, for all t,

$$F_\delta(\hat{K}^{t+1}) \leq \max\left\{F_\delta(\hat{K}^t) - \alpha\epsilon^2, \left(1 - \frac{\alpha}{A}\right) F_\delta(\hat{K}^t)\right\}. \tag{7}$$

On the right-hand side, which one is larger or not is determined by

$F_\delta(\hat{K}^t) \gtreqless A\epsilon^2$. It is observed that, if $F_\delta(\hat{K}^t) \leq A\epsilon^2$, then $F_\delta(\hat{K}^{t+1}) \leq \left(1-\frac{\alpha}{A}\right)A\epsilon^2 \leq A\epsilon^2$. Hence, once $F_\delta(\hat{K}^t)$ becomes no more than $A\epsilon^2$ at some period t, it stays in that way for all $t' \geq t$ and continue to decrease exponentially by the rate $\left(1-\frac{\alpha}{A}\right)$.

By the first key finding, $F_\delta(\hat{K}^0) \leq C$. Hence, if $F_\delta(\hat{K}^s)$ stays greater than or equal to $A\epsilon^2$ up to t, $F_\delta(\hat{K}^t)$ is bounded from above by $(C - \alpha\epsilon^2 t)$. This upper bound is linearly decreasing with t, so there exits τ that $C - \alpha\epsilon^2\tau < A\epsilon^2$ for the first time. Let $t < \tau$ and suppose $F_\delta(\hat{K}^{t-1}) \leq C - \alpha\epsilon^2(t-1)$. If $F_\delta(\hat{K}^{t-1}) \geq A\epsilon^2$, then the inequality (7) implies $F_\delta(\hat{K}^t) \leq F_\delta(\hat{K}^{t-1}) - \alpha\epsilon^2 \leq C - \alpha\epsilon^2 t$. If $F_\delta(\hat{K}^{t-1}) < A\epsilon^2$, again by (7), $F_\delta(\hat{K}^t) \leq \left(1-\frac{\alpha}{A}\right) F_\delta(\hat{K}^{t-1}) \leq \left(1-\frac{\alpha}{A}\right)(C - \alpha\epsilon^2(t-1))$. Since $t < \tau$, $C - \alpha\epsilon^2(t-1) \geq A\epsilon^2$ so that $\left(1-\frac{\alpha}{A}\right)(C - \alpha\epsilon^2(t-1)) \leq C - \alpha\epsilon^2(t-1) - \alpha\epsilon^2 = C - \alpha\epsilon^2 t$. Hence $F_\delta(\hat{K}^{t-1}) \leq C - \alpha\epsilon^2(t-1)$ implies $F_\delta(\hat{K}^t) \leq \mathrm{C} - \alpha\epsilon^2 t$. Since $F_\delta(\hat{K}^0) \leq C - \alpha\epsilon^2 0$, a mathematical induction proves that $F_\delta(\hat{K}^t) \leq C - \alpha\epsilon^2 t$ for all $t < \tau$.

A similar argument is applied for period τ. If $F_\delta(\hat{K}^{\tau-1}) \geq A\epsilon^2$, the inequality (7) implies $F_\delta(\hat{K}^t) \leq C - \alpha\epsilon^2\tau \leq A\epsilon^2$, where the last inequality is valid by the definition of τ. If $F_\delta(\hat{K}^{\tau-1}) < A\epsilon^2$, the inequality (7) implies $F_\delta(\hat{K}^\tau) \leq \left(1-\frac{\alpha}{A}\right) F_\delta(\hat{K}^{\tau-1}) < A\epsilon^2$, where the last inequality comes from $F_\delta(\hat{K}^{\tau-1}) < A\epsilon^2$ and $\left(1-\frac{\alpha}{A}\right) < 1$. Hence $F_\delta(\hat{K}^\tau) \leq A\epsilon^2$.

The observation after the inequality (7) proves that, for $t > \tau$,

$F_\delta(\hat{K}^t)$ continue to decrease exponentially by the rate $\left(1-\frac{\alpha}{A}\right)$, so the exponential convergence of $F_\delta(\hat{K}^t)$ to 0 as t goes to ∞ is confirmed. The exponential convergence is uniform over $\delta \geq \underline{\delta}$ since the upper bound on $F_\delta(\hat{K}^0)$ is uniform over$\delta \geq \underline{\delta}$. This completes the proof of the turnpike theorem, except for the derivation of two key findings.

11 Derivation of the First Key Finding

To complete the proof of the turnpike theorem, two key findings must be derived. Both rely on the existence of a feasible allocation from the initial stock ($\hat{K}^0$ or K) that converges exponentially to the stationary equilibrium allocation. For the first key finding, it is used to make the series in $F_\delta(K)$ being dominated by a geometric sequence uniformly over $\delta \geq \underline{\delta}$. For the second key finding, it is used to make both $(\overline{LCS^t})$ and $(\overline{LPS^t})$ being dominated by a geometric sequence multiplied by the squared distance between the feasible allocation and the stationary allocation, which is also dominated by a geometric sequence multiplied by $|K-\overline{K}|^2$, only if K is close enough to $\overline{K}$. The estimation of upper-bounds uses the differentiability of u_i's and g_j's, the latter only for the second finding. Only the first order expansion is enough for the first finding while the second order expansion and the differentiable strict concavity or convexity must be used for the second finding. The restriction $\delta \geq \underline{\delta}$ is not essential in proving them since the exponential convergence of the feasible allocation to the stationary equilibrium allocation guarantees that the loss in undiscounted total social welfare and losses in undiscounted total consumer and producer surplus converge.

To derive the first key finding, a feasible allocation from $\hat{K}^0$ that converges to the stationary allocation as time goes is obtained as follows. It is assumed that the economy is capable of producing all produce-able commodities without using up any of primary commodities. Namely, there exists a profile of production plans (y'_j) with $g_j(y_j) \leq 0$ for all $j \in J$ such that $\omega_k + \Sigma_j(y'_{j,0,k} + y'_{j,1,k}) > 0$ for all $k \in L$. In the statement of the turnpike theorem, the attention is restricted to the case that $\hat{K}^0 \gg 0$. Hence there exists a positive number γ less than 1 such that $\Sigma_{j \in J} \gamma y'_{j,1} \leq \hat{K}^0$. Replacing (y'_j) with $(\gamma y'_j)$ retains a positive supply for all commodities after inputs, so let it be renamed as (y'_j). Since there are only finitely many commodities, there exists a positive number θ, less than 1 but sufficiently close to 1, such that $\Sigma_i(1-\theta)\bar{x}_i \leq \omega + \Sigma_{j \in J}[(1-\theta)\bar{y}_{j,0} + \theta y'_{j,0}] + \Sigma_{j \in J} y'_{j,1}$. The inequality tells that a tiny part of inputs profile for (y'_j) can be replaced with that for $(\bar{y}_j)$ in a manner that consumers can consume the same tiny part of the consumption profile $(\bar{x}_i)$ with the initial stock $\Sigma_{j \in J} y'_{j,1}$, and an intuition on that suggests, if this replacement can go on dynamically in a feasible fashion, then a feasible allocation from $\Sigma_{j \in J} y'_{j,1}$ that converges exponentially to the stationary allocation as time passes will be obtained. By the choice of (y'_j), such an allocation must be feasible from $\hat{K}^0$. The intuition is confirmed as follows. Let $\tilde{x}_i^0 \equiv (1-\theta)\ \bar{x}_i$ and $\tilde{y}_j^0 \equiv (1-\theta)\ \bar{y}_j + \theta y'_j$ in period 0. Then $\Sigma_i \tilde{x}_i^0 \leq \omega + \Sigma_j \tilde{y}_{j,0}^0 + \Sigma_{j \in J} y'_{j,1}$. By multiplying θ to both sides of this inequality and $(1-\theta)$ to both sides of the feasibility $\Sigma_i \bar{x}_i \leq \omega + \Sigma_{j \in J}(\bar{y}_{j,0} + \bar{y}_{j,1})$, and summing them up, it is obtained that $\Sigma_{i \in I}[(1-\theta)\bar{x}_i + \theta \tilde{x}_i^0] \leq \omega + \Sigma_{j \in J}[(1-\theta)\bar{y}_{j,0} + \theta \tilde{y}_{j,0}^0] + \Sigma_{j \in J}[(1-\theta)\bar{y}_{j,1} + \theta y'_{j,1}]$. Let $\tilde{x}_i^1 \equiv (1-\theta)\bar{x}_i + \theta \tilde{x}_i^0$ and $\tilde{y}_j^1 \equiv (1-\theta)\bar{y}_j + \theta \tilde{y}_j^0$. Then the inequality is $\Sigma_{i \in I} \tilde{x}_i^1 \leq \omega + \Sigma_{j \in J}(\tilde{y}_{j,0}^1 + \tilde{y}_{j,1}^0)$. By multiplying θ to both sides of this inequality and (1

$-\theta$) to both sides of the inequality that the stationary equilibrium allocation is feasible, and summing them up, it is obtained that $\Sigma_{i\in I}[(1-\theta)\bar{x}_i+\theta\tilde{x}_i^1]\leq\omega+\Sigma_{j\in J}[(1-\theta)\bar{y}_{j,0}+\theta\tilde{y}_{j,0}^1]+\Sigma_{j\in J}[(1-\theta)\ \bar{y}_{j,1}+\theta\tilde{y}_{j,1}^0]$. Let $\tilde{x}_i^2\equiv(1-\theta)\bar{x}_i+\theta\tilde{x}_i^1$ and $\tilde{y}_j^2\equiv(1-\theta)\bar{y}_j+\theta\tilde{y}_j^1$. Then the inequality is $\Sigma_{i\in I}\tilde{x}_i^2\leq\omega+\Sigma_{j\in J}(\tilde{y}_{j,0}^2+\tilde{y}_{j,1}^1)$. Proceeding by an obvious mathematical induction, a feasible allocation from $\Sigma_{j\in J}y'_{j,1}$, $((\underset{\sim}{\tilde{x}_i}),(\underset{\sim}{\tilde{y}_j}))$, is obtained by recursive formulae, $\tilde{x}_i^{t+1}=(1-\theta)\bar{x}_i+\theta\tilde{x}_i^t$ and $\tilde{y}_j^t=(1-\theta)\ \bar{y}_j+\theta\tilde{y}_j^{t-1}$ for all period $t\geq0$, with the initial condition $\tilde{x}_i^0\equiv(1-\theta)\ \bar{x}_i$ and $\tilde{y}_j^{-1}\equiv y'_j$. In closed forms, $\tilde{x}_i^t=(1-\theta^{t+1})\bar{x}_i$ and $\tilde{y}_j^t=(1-\theta^{t+1})\bar{y}_j+\theta^{t+1}y'_j$ for all $t\geq0$. Since $\theta<1$, this allocation converges to $((\bar{x}_i),(\bar{y}_j))$ exponentially as time passes.

Since $F_\delta(K)$ is computed for the solution of $(\mathrm{SP})_K$, $F_\delta(\hat{K}^0)\leq\bar{p}\cdot(\hat{K}^0-\overline{K})-\Sigma_{t=0}^{\infty}\delta^t\Lambda_i^{-1}(u_i(\tilde{x}_i^t)-u_i(\bar{x}_i))$. By the first order Taylor expansion of u_i around $\bar{x}_i$, $u_i(x)-u_i(\bar{x}_i)=Du_i(\bar{x}_i)(x-\bar{x}_i)+\dfrac{o(|x-\bar{x}_i|)}{|x-\bar{x}_i|}|x-\bar{x}_i|$. Hence $|u_i(\tilde{x}_i^t)-u_i(\bar{x}_i)|\leq\left[L_c|Du_i(\bar{x}_i)|+\left|\dfrac{o(\theta^{t+1}|\bar{x}_i|)}{\theta^{t+1}|\bar{x}_i|}\right|\right]\theta^{t+1}|\bar{x}_i|$. Since $lim_{t\to\infty}\theta^{t+1}|\bar{x}_i|=0$, there exists a positive number Γ_i such that $L_c|Du_i(\bar{x}_i)|+\left|\dfrac{o(\theta^{t+1}|\bar{x}_i|)}{\theta^{t+1}|\bar{x}_i|}\right|\leq\Gamma_i$ for all $t\geq0$. Therefore $\Sigma_{t=0}^{\infty}\Lambda_i^{-1}|u_i(\bar{x}_i)-u_i(\tilde{x}_i^t)|\leq\Lambda_i^{-1}\Gamma_i|\bar{x}_i|\dfrac{\theta}{1-\theta}$. Note that the sum of the series in the left-hand side converges without time-discount by δ. Let Γ be the maximum of Γ_i's. Then, by letting λ be a uniform positive lower-bound for Λ_i's and B be a uniform upper-bound for feasible stationary allocations, and reminding that $\bar{q}$ is a uniform upper-bound for stationary equilibrium prices with $\delta\geq\underline{\delta}$, it implies that $\bar{p}\cdot(\hat{K}^0-\overline{K})-\Sigma_{t=0}^{\infty}\delta^t\Sigma_i$

$[\Lambda_i^{-1}(u_i(\tilde{x}_i^t)-u_i(\bar{x}_i))] \leq L_p\bar{q} \mid \hat{K}^0-\bar{K} \mid +\dfrac{\Gamma IB}{\lambda}\dfrac{\theta}{1-\theta}$. This inequality is valid for all $\delta \geq \underline{\delta}$ and the right-hand side of the inequality does not depend on δ. This completes the proof for the first key finding.

12　Derivation of the Second Key Finding

The decomposition

$$F_\delta(K)=\sum_{t=0}^{\infty}\delta^t\overline{LCS}^t((x_i^t))+\sum_{t=0}^{\infty}\delta^t\overline{LPS}^t((y_j^t))+\sum_{t=0}^{\infty}\delta^t\overline{DVS}^t((x_i^t),((y_{j,1}^{t-1},y_{j,0}^t))),$$

the differentiable strict concavity of u_i's and the differentiable strict convexity of g_j's suggest that $F_\delta(K)$ can be bounded from above locally with $|((\underset{\sim}{x_i}),(\underset{\sim}{y_j}))-(\bar{x},\bar{y})|^2$ if the loss in surplus for the market is 0, where $((\underset{\sim}{x_i}),(\underset{\sim}{y_j}))$ is a solution to $(SP)_K$, such as a competitive equilibrium allocation from K. The convergence of $((\underset{\sim}{x_i}),(\underset{\sim}{y_j}))$ to the stationary equilibrium allocation is not yet guaranteed since that itself is basically the claim of the turnpike theorem, so that the behavior of tails of discounted infinite sums involved in the estimation of an upper-bound may not be controlled adequately for δ close to 1. So it should be replaced by a feasible allocation from K that converges to the stationary allocations, just as in the proof of the first key finding. It is desirable that the convergence is independent of δ and exponential, since it would make the undiscounted infinite sums of $\overline{LCS}^t((x_i^t))$'s and $\overline{LPS}^t((y_j^t))$'s to converge so that the upper-bound would become uniform over $\delta \geq \underline{\delta}$. The construction used in the proof of the first key finding reveals that such a feasible allocation may exist if, in the construction, $y'_{j,0}$ and $y'_{j,1}$ can be taken as a small scale multiple of $\bar{y}_0$ and $\bar{y}_1$, respectively. Let this feasible allocation from K be denoted, again, by $((\utilde{x_i}), (\utilde{y_j}))$. To make differentiable concavity or convexity effective for estimating an upper-bound, $\bar{p}\cdot x_i^t=\Lambda_i^{-1}Du_i(\bar{x}_i)x_i^t$ and $(\bar{p}, \delta\bar{p})\cdot(y_{j,0}^t, y_{j,1}^t)=\rho_j Dg_j(\bar{y}_j)y_j^t$ must be satisfied for all i, j and t. The allocation $((\utilde{x_i}), (\utilde{y_j}))$ should not have any physical slack in order to make $\overline{DVS}^t((x_i^t), ((y_{j,1}^{t-1}, y_{j,0}^t)_j))=0$ for all t. To bound $\Sigma_{t=0}^{\infty}\overline{LCS}^t((x_i^t))+\Sigma_{t=0}^{\infty}\overline{LPS}^t((y_j^t))$ locally by $|K-\bar{K}|^2$, $|((\utilde{x_i}), (\utilde{y_j}))-(\bar{x}, \bar{y})|^2$ must be bounded locally by $|K-\bar{K}|^2$. It will be true if a geometric sequence that dominates the convergence of $((\utilde{x_i}), (\utilde{y_j}))$ to the stationary equilibrium allocation has its coefficient as a multiple of $|K-\bar{K}|$.

To make the intuition precise, a proper feasible allocation from K satisfying all properties mentioned above has to be constructed. The construction takes two steps. In the first step, a feasible allocation from K, $((\utilde{x_i}), (\utilde{y_j}))$, is constructed so that $(|((x_i^t), (y_j^t))-((\bar{x}_i), (\bar{y}_j))|)_{t=0}^{\infty}$ is dominated by a geometric sequence multiplied by $|K-\bar{K}|$, and $\bar{p}\cdot x_i^t=\Lambda_i^{-1}Du_i(\bar{x}_i)x_i^t$ and $(\bar{p}, \delta\bar{p})\cdot(y_{j,0}^t, y_{j,1}^t)=\rho_j Dg_j(\bar{y}_j)y_j^t$ are satisfied for all i, j and t. In the second step, it is modified to another feasible allocation from K so as to have no physical slack with all properties guaranteed in the first step retained.

The second finding relies on the following estimates on u_i's and g_j's. For u_i, an implication of the Taylor expansion at $\bar{x}_i$, $u_i(\bar{x}_i)-u_i(x)+$

$$Du_i(\bar{x}_i)(x-\bar{x}_i)=(x-\bar{x}_i)^T\left[-\frac{1}{2}D^2u_i(\bar{x}_i)\right](x-\bar{x}_i)+\frac{o(|x-\bar{x}_i|^2)}{|x-\bar{x}_i|^2}|x-\bar{x}_i|^2,$$

has been already used, where the matrix $-\frac{1}{2}D^2u_i(\bar{x}_i)$ is positive definite. The previous argument used the smallest eigenvalue of the matrix to bound the left-hand side from below by a non-negative quadratic function when $|x-\bar{x}_i|$ is small. This time, the largest eigenvalue of the matrix is used to obtain an upper-bound on the left-hand side. By letting α'_i be its largest eigenvalue and $\tilde{\epsilon}_i$ be a small positive number such that $|x-\bar{x}_i| \leq \tilde{\epsilon}_i$ implies $\frac{o(|x-\bar{x}_i|^2)}{|x-\bar{x}_i|^2} \leq \gamma$ where γ is taken independent of i, the right-hand side is bounded from above by $(L_c\alpha'_i+\gamma)|x-\bar{x}_i|^2$ if $|x-\bar{x}_i| \leq \tilde{\epsilon}_i$. Let $\Gamma_i \equiv L_c\alpha'_i+\gamma$. We have

$$u_i(\bar{x}_i)-u_i(x)+Du_i(\bar{x}_i)(x-\bar{x}_i) \leq \Gamma_i|x-\bar{x}_i|^2 \qquad \text{if } |x-\bar{x}_i| \leq \tilde{\epsilon}_i.$$

Similarly, for g_j, we have already derived that $Dg_j(\bar{y}_j)(\bar{y}_j-y)=\frac{1}{2}z(y)^T D^2\mu_j(0)z(y)+\frac{o(|z(y)|^2)}{|z(y)|^2}|z(y)|^2$ for all y with $g_j(y)=0$ locally at $\bar{y}_j$, where $z(y)$ is the orthogonal projection of y onto $T_{\bar{y}j}$ with the origin at $\bar{y}_j$. The matrix $\frac{1}{2}D^2\mu_j(0)$ is positive definite. The previous argument used the smallest eigenvalue of the matrix in order to obtain a positive lower-bound for the left-hand side for all y close to $\bar{y}$. This time, the largest eigenvalue is used to obtain an upper-bound for the left-hand side. Let α' be the largest eigenvalue of the matrix. Let L_j be the dimension of $M_{j,0}\times M_{j,1}$. For any $\gamma>0$ which is taken independent of j, there exists a small $\tilde{\epsilon}_j>0$ such that $|z| \leq \sqrt{L_j}\,\tilde{\epsilon}_j$ implies $\frac{o(|z|^2)}{|z|^2}<\gamma$. Since $|z(y)|^2 \leq L_j|y-\bar{y}_j|^2$, $Dg_j(\bar{y}_j)(\bar{y}_j-y) \leq (L_j\alpha'+\gamma)\ L_j|y-\bar{y}_j|^2$ for all y with $g_j(\mathrm{y})=0$ and $|y-\bar{y}_j| \leq \tilde{\epsilon}_j$. By letting $\Gamma_j \equiv (L_j\alpha'+\gamma)L_j$,

$$Dg_j(\bar{y}_j)(\bar{y}_j-y) \leq \Gamma_j|y-\bar{y}_j|^2 \qquad \text{if } g_j(y)=0 \text{ and } |y-\bar{y}_j| \leq \tilde{\epsilon}_j.$$

Hereafter $\delta \geq \underline{\delta}$ is assumed without a mention. Remind that it is assumed that $\overline{K}_k \geq \zeta$ for all $k \in L_p$, uniformly over $\delta \geq \underline{\delta}$.

The first step of construction starts with a feasible allocation from $\overline{K}$ just like $((\underset{\sim}{\tilde{x}_i}), (\underset{\sim}{\tilde{y}_j}))$ in the proof of the first key finding. The original proof in Bewley (1980) and Bewley (1982) has an error in the choice of (y_j') satisfying $\Sigma_i(1-\theta)\bar{x}_i \leq \omega + \Sigma_j[(1-\theta)\bar{y}_{j,0} + \theta y_{j,0}'] + \Sigma_j y_{j,1}'$ and $\Sigma_j y_{j,1}' \leq \overline{K}$, which makes $((\underset{\sim}{x_i}),\ (\underset{\sim}{y_j}))$ potentially infeasible. A correction has been proposed in Kaneko (2017). Remind that $\tilde{x}_i^t = (1-\theta^{t+1})\bar{x}_i$ and $\tilde{y}_j^t = (1-\theta^{t+1})\bar{y}_j + \theta^{t+1} y_j'$. To guarantee $(\bar{p},\ \delta\bar{p}) \cdot (\tilde{y}_{j,0}^t,\ \tilde{y}_{j,1}^t) = \rho_j Dg_j(\bar{y}_j)\tilde{y}_j^t$, it must be the case that $(\bar{p},\ \delta\bar{p}) \cdot (y_{j,0}',\ y_{j,1}') = \rho_j Dg_j(\bar{y}_j) y_j'$. This is achieved by taking $y_{j,0}'$ and $y_{j,1}'$ to be co-linear with $\bar{y}_{j,0}$ and $\bar{y}_{j,1}$. The correction takes $(y_{j,0}',\ y_{j,1}') \equiv (\gamma\bar{y}_{j,0},\ C_j(\gamma)\gamma\bar{y}_{j,1})$, where $\gamma > 0$ is taken small enough to satisfy $\gamma|\bar{y}_{j,0}| \leq \frac{\zeta}{2J}$ and $C_j(\gamma) > 1$ is taken to satisfy $g_j(\gamma\bar{y}_{j,0},\ C_j(\gamma)\gamma\bar{y}_{j,1}) = 0$. Such a choice is possible since g_j is strictly convex and $g_j(0) = 0$, for all j. It makes $\tilde{y}_j^t = ((1-\theta^{t+1}(1-\gamma))\bar{y}_{j,0},\ (1-\theta^{t+1}(1-C_j(\gamma)\gamma))\bar{y}_{j,1})$. To keep it on the production possibility frontier, $C_j^t(\gamma) > 1$ is multiplied to the output vector, so that $\tilde{y}_j^t$ becomes $((1-\theta^{t+1}(1-\gamma))\bar{y}_{j,0},\ C_j^t(\gamma)(1-\theta^{t+1}(1-C_j(\gamma)\gamma))\bar{y}_{j,1})$, where $g_j(\tilde{y}_j^t) = 0$. The convergence to the stationary equilibrium allocation is maintained under this modification since it brings $\tilde{y}_j^t$ closer to $\bar{y}_j$ from below. Since θ must be taken sufficiently close to 1, $\theta \geq \frac{1}{2}$ is assumed without loss of generality.

By the choice of γ, demand for k in period t is $\Sigma_i\tilde{x}_{i,k}^t - \Sigma_j\tilde{y}_{j,0,k}^t \leq (1-\theta^{t+1})\overline{K}_k + \theta^{t+1}\frac{\zeta}{2} \leq \overline{K}_k - \theta^{t+1}\frac{\zeta}{2}$ for all produce-able commodity k's. Let τ be the maximum of t's that satisfy $\theta^{t+1}\frac{\zeta}{2} \geq |K - \overline{K}|$. For such a t to exist for all K with $|K - \overline{K}| < \epsilon$, it is sufficient to take $\epsilon < \frac{\zeta}{4}$ since $\theta \geq \frac{1}{2}$. It is clear that the demand for any produce-able commodity k is no more than K_k in such t's. Hence the part of $((\underset{\sim}{\tilde{x}_i}), (\underset{\sim}{\tilde{y}_j}))$ after τ is feasible from K, which is denoted by $((\underset{\sim}{x'_i}), (\underset{\sim}{y'_j}))$. By the choice of τ, $\theta^{\tau+1} < \frac{4}{\zeta}|K - \overline{K}|$ since $\theta \geq \frac{1}{2}$. Hence

$$|((\tilde{x}'^t_i), (\tilde{y}'^t_j)) - ((\bar{x}_i), (\bar{y}_j))| \leq \theta^{\tau+t+1} B < \frac{4B}{\zeta}\theta^t |K - \overline{K}|$$

for each t, where B is a uniform upper-bound for feasible stationary allocations.

The feasible allocation constructed in the first step generally has physical slacks (excess supplies) for many commodities in many periods, and those must be eliminated in order to nullify the effect of the third component in the decomposition of $F_\delta(K)$. Let the sequence of physical slacks in $((\underset{\sim}{x'_i}), (\underset{\sim}{y'_j}))$ be $(\underset{\sim}{\tilde{z}})$. Namely, $\tilde{z}^t \equiv \Sigma_j y'^{t-1}_{j,1} + \omega - (\Sigma_i x'^t_i - \Sigma_j y'^t_{j,0})$ for all $t \geq 0$ with $\Sigma_j y'^{-1}_{j,1} \equiv K$. In each period t, $\tilde{z}^t_k$ must be added to either a consumption of a consumer or an input of a firm. In the latter, it triggers a finite sequence of additional productions which is absorbed by a consumption at the end period. Hence a productive sequence from k starting at t is considered. The consumption in the additional distribution must occur to a consumer i in a consume-able commodity k which is positively consumed by i in the stationary equilibrium allocation $(\bar{x}_{i,k} > 0)$, since $\bar{p}_k = \Lambda_i^{-1} \frac{\partial u_i}{\partial x_k}(\bar{x}_i)$ for such i and k so that the relation $\bar{p} \cdot x'^t_i = \Lambda_i^{-1} Du_i(\bar{x}_i) x'^t_i$ continues to hold after the additional consumption, where t is the period that the additional consumption occurs. Similarly, a production in the additional distribution must occur to a firm j in an input-output pair of commodities (k', k) that is vital to j in the stationary equilibrium allocation $(\bar{y}_{j,0,k'} < 0$ and $\bar{y}_{j,0,k} > 0)$, since $\bar{p}_{k'} = \rho_j \dfrac{\partial g_j{}'}{\partial y_{0,k}}(\bar{y}_j)$ and $\delta \bar{p}_k = \rho_j \dfrac{\partial g_j}{\partial y_{1,k}}(\bar{y}_j)$ for such j and (k', k) so that $(\bar{p}, \delta\bar{p}) \cdot (y'^{t}_{j,0}, y'^{t}_{j,1}) = \rho_j Dg_j(\bar{y}_j) y'^{t}_j$ continues to hold after the additional production, where t is the period that the additional production occurs. Such an additional distribution corresponds to a productive sequence $ik_N j_N k_{N-1} \cdots k_1 j_1 k_0$ with $k_0 = k$ such that $\bar{p}_k = q(ik_N j_N k_{N-1} \cdots k_1 j_1 k_0)$. Let's pick such a productive sequence from k that is the shortest in the length among them, and denote it as $ik_N j_N k_{N-1} \cdots k_1 j_1 k_0$, then modify $\utilde{(x'_i)}, \utilde{(y'_j)})$ as follows. If $N=0$, replace $x'^{t}_{i,k}$ by $x'^{t}_{i,k} + \tilde{z}^t_k$ and rename it as $x'^{t}_{i,k}$ in period t. If $N>1$, modify $(\utilde{(x'_i)}, \utilde{(y'_j)})$ inductively by 1) replace y'^{t}_{j1} by $(y'^{t}_{j1,0} - \tilde{z}^t_{k0} 1_{k0}, y'^{t}_{j1,1} + z^{t+1}_{k1} 1_{k1})$ where $z^{t+1}_{k1} > 0$ is uniquely determined so as to satisfy $g_j(y'^{t}_{j1,0} - \tilde{z}^t_{k0} 1_{k0}, y'^{t}_{j1,1} + z^{t+1}_{k1} 1_{k1}) = 0$, and rename the modified input-output vector as y'^{t}_{j1}, 2) for $n = 2, \ldots, N$, assuming that z^{t+n-1}_{kn-1} is already determined, replace y'^{t+n-1}_{jn} by $(y'^{t+n-1}_{jn,0} - z^{t+n-1}_{kn-1} 1_{k_{n-1}}, y'^{t+n}_{jn,1} + z^{t+n}_{kn} 1_{kn})$ where $z^{t+n}_{kn} > 0$ is uniquely determined so as to satisfy $g_j(y'^{t+n-1}_{jn,0} - z^{t+n-1}_{k_{n-1}} 1_{k_{n-1}}, y'^{t+n}_{jn,1} + z^{t+n}_{kn} 1_{kn}) = 0$, and rename the modified input-output vector as y'^{t+n}_{jn}, 3) assuming that z^{t+N}_{kN} is already determined, replace x'^{t+N}_i by $(x'^{t+N}_i + z^{t+N}_{kN} 1_{kN})$ and rename it as x'^{t+N}_i. As a result, the feasible allocation $(\utilde{(x'_i)}, \utilde{(y'_j)})$ is renewed to have no physical slack for the commodity k in period t.

Starting from $t=0$, $\tilde{z}^0_1$ is distributed out just as explained if it is positive, then $\tilde{z}^0_2$, $\tilde{z}^0_3$ and so forth. After all physical slacks in period 0 is eliminated, move to $t=1$ and distribute out $\tilde{z}^1$ in the same way by the order of commodity. Continuing this way, an allocation in period t is determined once all $(\tilde{z}^s)^t_{s=\max\{t-L,0\}}$ are distributed out, since the length of a productive sequence never exceeds L. Let's denote the feasible allocation determined in this way by $(\utilde{(x_i)}, \utilde{(y_j)})$. By the construction, it has no physical slack.

In distributing out $\tilde{z}_k^t$, a productive sequence $ik_Nj_Nk_{N-1}\cdots k_1j_1k_0$ with $k_0=k$ is chosen so that k_N is positively consumed by i and the input-output pair (k_{n-1}, k_n) is vital to j_n for $n=1, \ldots, N$ in the stationary equilibrium allocation, and a sequence of distribution coefficients $(z_{k_n}^{t+n})_{n=1,\ldots,N}$ is created. Since all g_j's are strictly differentiably convex, $z_{kn}^{t+n} \leq MRT_{k_{n-1}, k_n}^{j_n}(y'^{t+n-1}_{j_n})z_{k_{n-1}}^{t+n-1}$ for all $n=1, \ldots, N$. Hence $z_{k_n}^{t+n} \leq \Pi_{s=1}^n MRT_{k_{s-1}, ks}^{js}(y'^{t+s-1}_{js}) \times \tilde{z}_k^t$. By the construction in the first step, $((\utilde{x}'_i), (\utilde{y}'_j))$, including all renewed ones that appear in the second step, is a part of feasible allocation from $\overline{K}$. Hence $|y'^{t+n-1}_{jn}| \leq B$ for all n, where B is, by retaking larger if necessary, a uniform upper-bound for feasible allocations from $\overline{K}$ and also that for feasible stationary allocations. Since $MRT_{k', k}^j$ is positive valued and continuous on $Y_j \cap \{y \in \mathbb{R}^L \| y| \leq B\}$ for any j and its feasible input-output pair (k', k), they are uniformly bounded from above, say by Q'. A finite set of positive numbers $\{MRT_{k', k}^j(\bar{y}_j) \mid j \in J, \bar{y}_{j,0,k'}<0, \bar{y}_{j,1,k}>0\}$ is clearly uniformly bounded away from 0, say by Q″. Let $Q \equiv \dfrac{Q'}{Q''}$. Since $|\bar{y}_j| \leq B$ for all j, $Q>1$. By the choice of Q, $MRT_{k_{n-1}, k_n}^{j_n}(y'^{t+n-1}_{j_n}) \leq QMRT_{k_{n-1}. k_n}^{j_n}(\bar{y}_{j_n})$ for all n. By the choice of the productive sequence, $MRT_{k_{n-1}, k_n}^{j_n}(\bar{y}_{j_n}) = \dfrac{1}{\delta}\dfrac{\bar{p}_{k_{n-1}}}{\bar{p}_{k_n}} \leq \dfrac{1}{\underline{\delta}}\dfrac{\bar{p}_{k_{n-1}}}{\bar{p}_{k_n}}$. Hence $z_{k_n}^{t+n} \leq Q^n \underline{\delta}^{-n}\left(\dfrac{\bar{p}_{k_0}}{\bar{p}_{k_n}}\right)\tilde{z}_k^t \leq Q^n \underline{\delta}^{-n}\left(\dfrac{\bar{q}}{\underline{q}}\right)\tilde{z}_k^t$. In this estimate, the coefficient on $\tilde{z}_k^t$ does not depend on $((\utilde{x}'_i), (\utilde{y}'_j))$'s that appear in the second step.

Let $((\utilde{x}'_i), (\utilde{y}'_j))$ be that obtained in the first step. In period t, $|(y_j^t)-(y'^t_j)|$ is no more than the sum of all additional distributions for production planned in that period, neglecting the difference of commodities distributed. Similarly, $|(x_i^t)-(x_i'^t)|$ is no more than the sum of all additionally distributed consumptions in period t, neglecting the difference of commodities distributed. Hence a very crude upper-bound on $|((x_i^t),(y_j^t))-((x_i'^t),(y_j'^t))|$ is obtained as

$$|((x_i^t),(y_j^t))-(x_i'^t),(y_j'^t))|$$

$$\le \Sigma_{s=\min\{t-L,0\}}^{t-1} Q^{t-s}\underline{\delta}^{-(t-s)}\left(\frac{\bar{q}}{\underline{q}}\right)(\Sigma_{k=1}^{L}\tilde{z}_k^s)+\Sigma_{k=1}^{L}\tilde{z}_k^t$$

$$\le LQ^L\underline{\delta}^{-L}\left(\frac{\bar{q}}{\underline{q}}\right)\Sigma_{s=\min\{t-L,0\}}^{t}|\tilde{z}^s|.$$

Since the convergence of $(x_i'),(y_j'))$ to the stationary equilibrium allocation is dominated by a geometric sequence multiplied by $|K-\bar{K}|$ and the stationary equilibrium allocation has no physical slack, it is expected that $(|\tilde{z}^t|)_{t=0}^{\infty}$ is also dominated by a geometric sequence multiplied by $|K-\bar{K}|$. Let $k\in L_o$. Then, for all t, $\tilde{z}_k^t=\omega_k-\Sigma_i(1-\theta^{\tau+t+1})$ $\bar{x}_{i,k}+\Sigma_j(1-\theta^{\tau+t+1}(1-\gamma))\bar{y}_{j,0,k}=\theta^{\tau+t+1}(\omega_k+\gamma\Sigma_j\bar{y}_{j,0,k})\le\frac{4B}{\zeta}\theta^t|K-\bar{K}|$

where B is retaken, if necessary, to be also an upper-bound of total endowment of primary commodities. Next, let $k\in L_p$. Then $\tilde{z}_k^0=K_k-\Sigma_i$ $(1-\theta^{\tau+1})\bar{x}_{i,k}+\Sigma_j(1-\theta^{\tau+1}(1-\gamma))\bar{y}_{j,0,k}=K_k-\bar{K}_k+\theta^{\tau+1}[\bar{K}_k+\gamma\Sigma_j\bar{y}_{j,0,k}]\le$ $|K-\bar{K}|+\frac{4}{\zeta}B|K-\bar{K}|=\left(1+\frac{4}{\zeta}B\right)|K-\bar{K}|$, where B is retaken, if necessary, to be also a uniform upper-bound for total outputs of stationary feasible allocations. Similarly, $\tilde{z}_k^t=\Sigma_jC_j^{\tau+t-1}(\gamma)(1-\theta^{\tau+t}(1-C_j(\gamma)\gamma)$ $\bar{y}_{j,1,k}-\Sigma_i(1-\theta^{\tau+t+1})\bar{x}_{i,k}+\Sigma_j(1-\theta^{\tau+t+1}(1-\gamma))\bar{y}_{j,0,k}\le\bar{K}_k-\Sigma_i(1-\theta^{\tau+t+1})$ $\bar{x}_{i,k}+\Sigma_j(1-\theta^{\tau+t+1}(1-\gamma))\bar{y}_{j,0,k}=\theta^{\tau+t+1}[\bar{K}_k+\gamma\Sigma_j\bar{y}_{j,0,k}]\le\frac{4B}{\zeta}\theta^t|K-\bar{K}|$. By combining all these estimates,

$$|\tilde{z}^t| \leq \left(1+\frac{4B}{\zeta}\right)\theta^t|K-\overline{K}|.$$

This estimate is now brought into the the crude estimate of an upper-bound on $|((x_i^t),(y_j^t))-((x_i'^t),(y_j'^t))|$. Noting that $\theta^t+\theta^{t-1}+\cdots+\theta^{\max\{t-L,0\}}=\theta^t(1+\theta^{-1}+\cdots+\theta^{-\min\{L,t\}})\leq\theta^t(1+2+2^2+\cdots+2^L)=2^L\theta^t\dfrac{1-\left(\frac{1}{2}\right)^{L+1}}{1-\frac{1}{2}}\leq(2^{L+1}-1)\theta^t$,

$$|((x_i^t),(y_j^t))-(x_i'^t),(y_j'^t))|\leq LQ^L\underline{\delta}^{-L}\left(\frac{\overline{q}}{\underline{q}}\right)\left(1+\frac{4B}{\zeta}\right)(2^{L+1}-1)\theta^t|K-\overline{K}|.$$

Combined with the previous estimate of an upper-bound for $|((x_i'^t),(y_j'^t))-((\overline{x}_i^t),(\overline{y}_j^t))|$, we have

$$|((x_i^t),(y_j^t))-((\overline{x}_i),(\overline{y}_j))|\leq D\theta^t|K-\overline{K}|$$

where $D\equiv LQ^L\underline{\delta}^{-L}\left(\frac{\overline{q}}{\underline{q}}\right)\left(1+\frac{4B}{\zeta}\right)(2^{L+1}-1)+\frac{4B}{\zeta}$.

By the construction of the feasible allocation $((\underset{\sim}{x_i}),(\underset{\sim}{y_j}))$, $\overline{DVS}^t((x_i^t),((\mathrm{y}_{j,1}^{t-1},y_{j,1}^t)_j))=0$ for all t. Hence $F_\delta(K)\leq\Sigma_{t=0}^{\infty}\ \delta^t(\overline{LCS}^t((x_i^t))+\overline{LPS}^t((y_j^t)))$, where $\overline{LCS}^t((x_i^t))=\Sigma_{j\in I}[(\Lambda_i^{-1}u_i(\overline{x}_i)-\overline{p}\cdot\overline{x}_i)-(\Lambda_i^{-1}u_i(x_i^t)-\overline{p}\cdot x_i^t)]$ and $\overline{LPS}^t((y_j^t))=\Sigma_{j\in J}[(\overline{p}\cdot\overline{y}_{j,0}+\delta\overline{p}\cdot\overline{y}_{j,1})-(\overline{p}\cdot y_{j,0}^t+\delta\overline{p}\cdot y_{j,1}^t)]$. Let $\epsilon\equiv\dfrac{\min\{(\tilde{\epsilon}_i),(\tilde{\epsilon}_j)\}}{D}$. Then $|K-\overline{K}|<\epsilon$ implies $|((x_i^t),(y_j^t))-(\overline{x}_i),(\overline{y}_j))|<\min\{(\tilde{\epsilon}_i),(\tilde{\epsilon}_j)\}$ in all t's. Hence

$$\begin{aligned}
&(\Lambda_i^{-1}u_i(\overline{x}_i)-\overline{p}\cdot\overline{x}_i)-(\Lambda_i^{-1}u_i(x_i^t)-\overline{p}\cdot x_i^t)\\
&=\Lambda_i^{-1}(u_i(\overline{x}_i)-u_i(x_i^t))+\overline{p}\cdot(x_i^t-\overline{x}_i)\\
&=\Lambda_i^{-1}[u_i(\overline{x}_i)-u_i(x_i^t)+Du_i(\overline{x}_i)\cdot(x_i^t-\overline{x}_i)]\\
&\leq\Lambda_i^{-1}\Gamma_i|x_i^t-\overline{x}_i|^2\\
&\leq\Lambda_i^{-1}\Gamma_iD^2\theta^{2t}|K-\overline{K}|^2,
\end{aligned}$$

and

$$\begin{aligned}&(\overline{p}\cdot\overline{y}_{j,0}+\delta\overline{p}\cdot\overline{y}_{j,1})-(\overline{p}\cdot y^t_{j,0}+\delta\overline{p}\cdot y^t_{j,1})\\&=\rho_j Dg_j(\overline{y}_j)(\overline{y}_j-y^t_j)\\&\leq\rho_j\Gamma_j|y^t_j-\overline{y}_j|^2\\&\leq\rho_j\Gamma_j D^2\theta^{2t}|K-\overline{K}|^2,\end{aligned}$$

so that $\overline{LCS}^t((x^t_i))\leq\underline{\Lambda}^{-1}(\Sigma_i\Gamma_i)D^2\theta^{2t}|K-\overline{K}|^2$ and $\overline{LPS}^t((y^t_j))\leq\overline{\rho}(\Sigma_j\Gamma_j)D^2\theta^{2t}|K-\overline{K}|^2$, where $\underline{\Lambda}$ is the uniform positive lower-bound for marginal utilities of income in competitive equilibria and $\overline{\rho}$ is a uniform upper-bound for marginal costs of production efficiency in stationary equilibria. This implies that $\Sigma^{\infty}_{t=0}(\overline{LCS}^t((x^t_i))+\overline{LPS}^t((y^t_j)))$ converges absolutely and is bounded above by $(\underline{\Lambda}^{-1}\Sigma_i\Gamma_i+\overline{\rho}\,\Sigma_j\Gamma_j)D^2\frac{1}{1-\theta^2}|K-\overline{K}|^2$. Note that the series of losses in social surplus converges without time-discount. Since the discounted sums are smaller than undiscounted sums, by letting $A\equiv(\underline{\Lambda}^{-1}\Sigma_i\Gamma_i+\overline{\rho}\,\Sigma_j\Gamma_j)D^2\frac{1}{1-\theta^2}$, $F_\delta(K)\leq A|K-\overline{K}|^2$ for all $\delta\geq\underline{\delta}$ if $|K-\overline{K}|\leq\epsilon$, completing the proof of the second key finding.

13 Concluding Remarks

The proof of the turnpike theorem in the general equilibrium theory without uncertainty reveals that it is the theorem of welfare economics combined with the Lyapunov stability, the latter is used to characterize the dynamics of the loss in total market surplus from the stationary equilibrium allocation that has the same profile of marginal utilities of income for consumers as that in the competitive equilibrium allocation. This loss in total market surplus dominates the square distance between the competitive equilibrium allocation and the stationary equilibrium allocation. A natural definition of competitive equilibrium with transfer payments requires that the time-discount rates among consumers is less than 1, while the Lyapunov stability is developed for the case that it is 1. By making all myopic consumers die out in finite periods, all consumers have the same time-discount rate in a competitive equilibrium, but less than 1. The Lyapunov stability relies on the equation (6) which assumes that $\delta=1$. It holds only for the common time-discount rate close to 1, as the proof of (6) from (5) shows. Both key findings used to complete the proof, one that the loss in total market surplus is bounded, the other that the loss in total market surplus is dominated by the squared distance between initial stock at hand and that in the stationary equilibrium if the former is sufficiently close to the latter, relies on the construction of a feasible allocation as a recursive replacement of a feasible allocation from the initial stock of produce-able commodities in some period in the competitive equilibrium with the stationary equilibrium allocation at the replacement ratio independent of the time-discount rate. So both key findings are, in principle, free of any restriction on the common time-discount rate.

The author believes that there would be a still better organization of the proof of the turnpike theorem than the one presented in this article. For example, the relation between a distance of competitive equilibrium allocation to a stationary equilibrium allocation and that of stock in competitive equilibrium allocation to stationary equilibrium stock may be omitted by dropping the latter, since all surpluses are defined on allocations. The Lyapunov stability argument could rely entirely on the inequality (4) once two key finding in terms of allocations were derived. Such a presentation has an advantage to make the proof to be seen as complete in general equilibrium theory, by cutting any linkage to optimal growth theory. But, for the purpose of eliminating all controversies surrounding the turnpike theorem in general equilibrium theory, the one in this article would be more than sufficient.

Now, where should we go from here ? The obvious direction is to include an uncertainty in the economy. Such efforts have been made in many literatures. The uncertainty must be modeled so that the notion of ergodicity for stochastic processes is viable, and a typical choice is a metrically transitive stationary process on the time space $\{\cdots, -1, 0, 1, \cdots\}$. A possibility of setting such a process to be the underlying uncertainty is shown in Futia (1982). With this underlying uncertainty, an equilibrium that replaces the role of the stationary equilibrium without uncertainty have been searched. A conclusion is given in Bewley (1977), where a notion of stationary equilibrium under an uncertainty is proposed as the replacement. The model used in Bewley (1977) is restricted for monetary saving and consumption of a consumer who does not discount on time, in order to check the validity of the permanent income hypothesis proposed in Friedman (1957). However it is derived in the article that the marginal utility of income for an optimal consumption plan in any period stays above the level at which the expected consumption becomes equal to the expected income (with the price of consumption good normalized as 1 in every period), and converges to the latter as time passes. Namely, the marginal utility of income at which the expected consumption is equal to the expected income serves as the turnpike of the marginal utility of income when a consumer optimizes under an income fluctuation following a metrically transitive stationary process. The result claims essentially that an excessive economization of consumption exists in every period but it disappears in the long run. This turnpike theorem is based on another result, which is called as "main" in the article, that the marginal utility of income for an optimal consumption plan in any period stays above that at which the expected consumption is equal to the expected income for any finite initial money stock in the period, and converges to the latter as the initial money stock in the period goes to ∞. An implication of this result is that the averaged maximized utility over time converges to the expect utility of the consumption whose expected expenditure coincides with the expected income. Based on these findings, a notion of stationary equilibrium with transfer payments is formed and proved to exist without transfer payments and satisfy both welfare theorems in Bewley (1981). This equilibrium requires only that the expected expenditure on consumption must meet the expected income, hence is expected to serve as the turnpike. But I do not know whether a direct extension of the turnpike theorem in general equilibrium theory with such an uncertainty exists or not. As a closely related work on the issue, I know only Marimon (1983), in which a stochastic growth model with its underlying uncertainty as a metrically transitive stationary process is considered. In it, it is found that all optimal interior growth paths converge dynamically to a stationary path. Since an optimal growth model is compatible with a general equilibrium model only under many restrictions on the economy, especially on the production side, the result does not serve as a direct extension. In the spirit of a formation of good habits for economic agents, the turnpike theorem on the marginal utility of income is almost sufficient to claim that following a stationary equilibrium makes good habits. The "almost" part comes from the fact that, on the contrary to the case of a stationary equilibrium without uncertainty, consumers will default in many periods in a stationary equilibrium with uncertainty. The "main" result in Bewley (1977) indicates that keeping money supply sufficient enough to cover all of these defaults would be impossible since an infinite money supply is needed. This point is formally addressed in Bewley (1979). In an economy with uncertainty, the market system must allow consumers to default in many periods in order to be compatible with good habits for economic agents, but such a bold design of a market system has seldom been pursued by economists.

In fact, turnpike results under uncertainty are abundant in optimal growth models, since most of techniques in optimal control dynamics for undiscounted objective can be directly applied to them. However, most of them have few content in terms of economics since an implication on interactions among economic agents is mostly neglected except for production firms, making them on the supply side[3]. The role of allocations in an economy is oversimplified in these literatures. In the proof of the turnpike theorem without uncertainty, a dynamic convergence of stocks in a competitive equilibrium to that in a stationary equilibrium does not imply directly a dynamic convergence of a com-

3) Some works, especially those of W. Brock, make a good contribution to optimal control theory in applied mathematics.

petitive equilibrium allocation to a stationary equilibrium allocation. Even with restrictions on the production side that allow production plans to converge, there is a great degree of freedom left for consumption plans. It is the main point of the proof of the turnpike theorem without uncertainty that a dynamic convergence of a competitive equilibrium allocation to a stationary equilibrium allocation is due to the dynamic convergence of the loss in total market surplus to 0, which can be related to a dynamic convergence of stocks only when it implies a dynamic convergence (to 0) of losses in both total consumer surplus and total producer surplus. An oversimplification on the consumer side makes such a decomposition of economic welfare to be unnecessary, by linking a dynamic convergence of stocks directly to that of a value loss in economic welfare net the acquisition cost of initial stock (cf. the definition of F_δ by (2)). As a result, no complication such as the proof of the second key finding appears in the optimal growth theory though the difficulty in the proof is mostly on the production side. Needless to say, assuming a representative consumer at the outset hides a potential differences in marginal utilities of income among consumers between a competitive equilibrium and a stationary equilibrium, that have led to a confusion over insensitivity of the turnpike with respect to an initial stock.

Another direction is to introduce a non-stationarity into the economy. In general, the ergodicity does not work in such an environment. Hence the nature of non-stationarity must be restricted so that it represents a stochastic jump between stationary economies. The big issue is how an economic system should manage such a jump. It is generally considered that a stochastic nature of a jump is fully expected only by a small group of economic agents. An economic system must be equipped with means to control an incentive for excessive gains by this group of agents and direct them to share the knowledge of a jump through a normal operation of the market system. By assuming, for example, that a group that fully expects a jump can be any coalition of economic agents with equal probability that is independent jump by jump, it seems that such means can be implemented to represent a good habit for such a group in a framework of cooperative game. Money might have some role in the means.

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Summary

A Comprehensive Guide for the Proof of the Turnpike Theorem in General Equilibrium Theory

Fumihiro KANEKO

A comprehensive guide for understanding the proof of the turnpike theorem in general equilibrium theory is presented. The original proof is rather disorganized and partly unintuitive, and has an error. This guide reorganizes it to emphasize that the turnpike theorem is a theorem on dynamics of economic surplus in the spirit of promoting a formation of a good habit over a fully rational optimization for each economic agent. The reason why the common time-discount rate must be taken close to 1 is clarified as that a Lyapunov stability argument for the case that the common time-discount rate is 1 is applied to the dynamics of loss in the total market surplus. The guide also contains a reproof of the part derived from an error in the original proof. No technical detail is left unexplained except for issues related to the existence of an equilibrium, so that the guide is nearly complete.

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