p-saturations of Games

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Contents

Abstract

We introduce the notion of *p-saturations*. We then construct a family of (impartial) games and give explicit formulas for their Sprague-Grundy functions. We also present a connection between games and representations.

The main results are the following:

1. ***p*-saturations of Welter's Game and the Irreducible Representations of Symmetric Groups (Chapter 3)**

   We establish a relation between the Sprague-Grundy function sg of p-saturations of Welter's game and the degrees of the (ordinary) irreducible representations of symmetric groups. In these games, a position can be regarded as a partition λ. Let ρ^λ be the irreducible representation of the symmetric group $\mathrm{Sym}(|\lambda|)$ corresponding to λ. For every prime p, we show the following results:

   a) $\mathrm{sg}(\lambda) \leqslant |\lambda|$ with equality if and only if the degree of ρ^λ is prime to p;

   b) the restriction of ρ^λ to $\mathrm{Sym}(\mathrm{sg}(\lambda))$ has an irreducible component with degree prime to p.

   Further, for every integer p greater than 1, we obtain an explicit formula for $\mathrm{sg}(\lambda)$.

2. **Digit-Separable Sprague-Grundy Functions (Chapter 2)**

   We construct a family of games including Nim and present explicit formulas for their Sprague-Grundy functions. Let Φ be an integer-valued function on the position set $\mathcal{P}$ of Nim. Let p be an integer greater than 1 and let $\Gamma[\Phi]$ be a p-saturation of the subgame of Nim induced in $\{X \in \mathcal{P} : \Phi(X) \geqslant 0\}$. We show that if Φ is *digit-separable* and a *locally Sprague-Grundy function* of $\Gamma[\Phi]$, then Φ is the Sprague-Grundy function of $\Gamma[\Phi]$. The p-saturation indices of some games in this family are also determined.

Acknowledgments

I am deeply grateful to my supervisor Prof. Masaaki Kitazume. He gently encouraged me and gave me a lot of opportunities. The way he happily talks about sporadic simple groups remains engraved in my mind. I was not able to obtain new insights into sporadic simple groups. However, I am delighted with this thesis in which I was able to establish a relation between the representation theory of groups and the game theory. This is because he taught me the former when I was an undergraduate student. He also inspired me to study the latter when I was a doctoral student.

I would like to thank Dr. Kosuke Fukumuro. I have benefited from enjoyable discussions with him. I am indebted to my English teacher Ms. Patricia Becker. She corrected my papers and gave me warm encouragement.

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Finally, if I could tell my gratitude to Mr. Satoki Shirotori, how happy would I be. He passed away, but his words are still heartening me.

1 Games

In this chapter, we recall the basic definitions and results of the (short impartial) game theory. Games can be represented as digraphs. In the game theory, Sprague-Grundy functions play a crucial role. For example, using them, we can describe the winning strategy for games. Sprague-Grundy functions are defined recursively, and explicit formulas for them are not known in most cases. The goal of this thesis is constructing a family of games and presenting explicit formulas for their Sprague-Grundy functions.

1.1 Definitions

Let Γ be a digraph $(\mathcal{P}, \mathcal{A})$, that is, $\mathcal{P}$ is a set and $\mathcal{A}$ is a subset of $\mathcal{P}^2$. We denote $\mathcal{P}$ and $\mathcal{A}$ by $\mathcal{P}(\Gamma)$ and $\mathcal{A}(\Gamma)$, respectively. Let $X_0, \dots, X_n$ be elements of $\mathcal{P}(\Gamma)$. The sequence $(X_0, \dots, X_n)$ is called a *path* of length n from X_0 to X_n if (X_i, X_{i+1}) in $\mathcal{A}(\Gamma)$ for each $i \in [n] = \{0, 1, \dots, n-1\}$. For $X \in \mathcal{P}(\Gamma)$, let $\lg(X)$ denote the maximum length of a path from X. We call $\lg(X)$ the *length* of X.

A digraph Γ is called a *(short impartial) game* if $\lg(X)$ is finite for every $X \in \mathcal{P}(\Gamma)$. Let Γ be a game. The set $\mathcal{P}(\Gamma)$ is called the *position set* of Γ, and an element of $\mathcal{P}(\Gamma)$ is called a *position* in Γ. If X and Y are two positions in Γ and $(X, Y) \in \mathcal{A}(\Gamma)$, then Y is called an *option* of X. If X has no option, then X is called a *terminal* position.

Example 1.1.1. Let us consider the two graphs in Figure 1.1. The lengths of vertices in the left graph are 2, 1, and 0. Hence this graph is a game. In contrast, the lengths of two vertices in the right one are infinite, so this one is not a game.

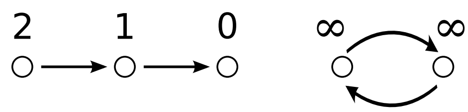


Figure 1.1: The left one is a game, but the right one is not.

Example 1.1.2 (Nim)**.** Let $m \in \mathbb{N}$ and $\mathcal{P} = \mathbb{N}^m$, where $\mathbb{N}$ is the set of non-negative integers. For $X \in \mathcal{P}$ and $i \in [m]$, let X^i denote the i-th component of X, that is, $X = (X^0, \dots, X^{m-1})$. Let

$$\mathcal{A} = \left\{ (X, Y) \in \mathcal{P}^2 : X^i \geqslant Y^i \ \text{ for each } \ i \in [m] \ \text{ and } \ \operatorname{dist}(X, Y) = 1 \right\},$$

where $\operatorname{dist}(X,Y)$ is the Hamming distance between X and Y, that is,

$$\operatorname{dist}(X,Y) = \left|\left\{ i \in [m] : X^i \neq Y^i \right\}\right|.$$

Let $\mathcal{N}^m = (\mathcal{P}, \mathcal{A})$. Then $\mathcal{N}^m$ is a game. This game is called *Nim*.

The game $\mathcal{N}^m$ can be decomposed into m copies of $\mathcal{N}^1$ as follows. Let Γ and Γ' be two games. Let $\mathcal{P} = \mathcal{P}(\Gamma) \times \mathcal{P}(\Gamma')$ and

$$\mathcal{A} = \{ ((X,X'),(Y,X')) \in \mathcal{P}^2 : (X,Y) \in \mathcal{A}(\Gamma) \} \cup \{ ((X,X'),(X,Y')) \in \mathcal{P}^2 : (X',Y') \in \mathcal{A}(\Gamma') \}.$$

Then the game $(\mathcal{P}, \mathcal{A})$ is called the *disjunctive sum* of Γ and Γ', and is denoted by $\Gamma \oplus_2 \Gamma'$. For example,

$$\mathcal{N}^m \quad \text{and} \quad \underbrace{\mathcal{N}^1 \oplus_2 \cdots \oplus_2 \mathcal{N}^1}_{m}$$

are isomorphic as digraphs. As we will see in the next section, the Sprague-Grundy function of $\Gamma \oplus_2 \Gamma'$ is easily deduced from that of Γ and Γ'. In particular, to calculate the Sprague-Grundy function of $\mathcal{N}^m$, we need only compute that of $\mathcal{N}^1$.

In contrast, the following game cannot be decomposed into smaller games.

Example 1.1.3 (Welter's game)**.** Let

$$\mathcal{P} = \left\{ X \in \mathbb{N}^m : X^i \neq X^j \quad \text{for} \quad 0 \leqslant i < j \leqslant m-1 \right\}.$$

Let $\mathcal{W}^m$ be the subgraph of $\mathcal{N}^m$ induced in $\mathcal{P}$, that is, the position set of $\mathcal{W}^m$ is $\mathcal{P}$ and its arrow set $\mathcal{A}(\mathcal{W}^m)$ is

$$\{ (X,Y) \in \mathcal{A}(\mathcal{N}^m) : X,Y \in \mathcal{P} \}.$$

The game $\mathcal{W}^m$ is called *Welter's game*. Since this game can not be decomposed into smaller games, calculating the Sprague-Grundy function of $\mathcal{W}^m$ is more difficult than of $\mathcal{N}^m$.

1.2 How to Play Games

Let us play a game Γ. There are two players, say Player 1 and Player 2. We first choose an initial position and place a coin on it. Two players alternately move the coin to an option of the position where the coin is placed. The winner is the player who has moved the coin to a terminal position.

Example 1.2.1. Let us play Nim $\mathcal{N}^2$. Let $(2,2)$ be the initial position. See Figure 1.2. It is the turn of Player 1. He has the following four options:

$$(1,2), (0,2), (2,1), \text{ and } (2,0).$$

He chooses $(1,2)$. It is the turn of Player 2. He has the following three options

$$(1,1),(1,0), \text{ and } (0,2).$$

He chooses $(1,1)$. Then Player 1 has no choice but to move to $(1,0)$ or $(0,1)$. He moves to $(0,1)$. Finally, Player 2 moves to $(0,0)$ and wins.

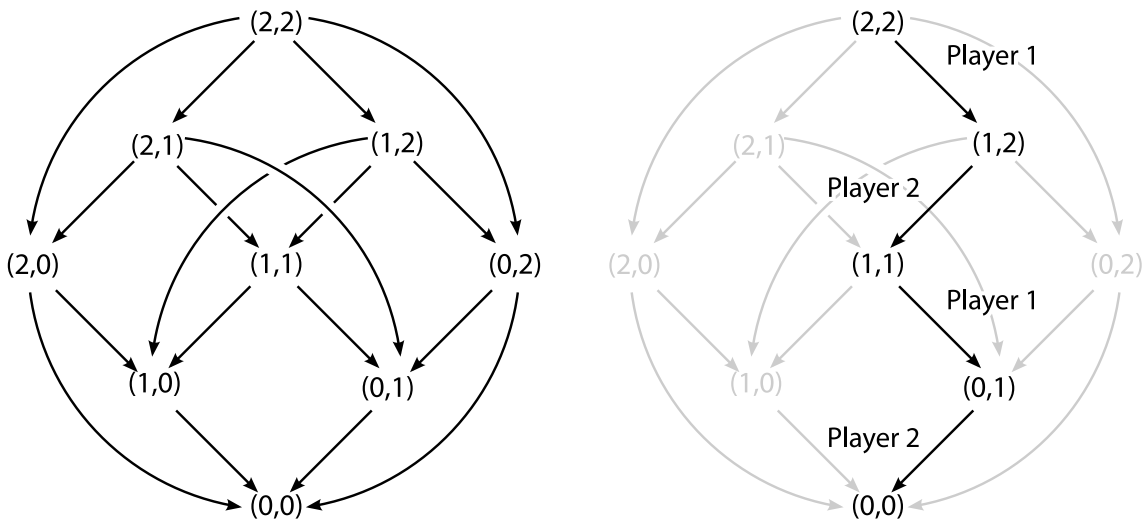


Figure 1.2: The winner is Player 2.

Why could Player 2 win? This is because Player 2 has a winning strategy. We say that a position X in a game is a *winning position* if the previous player has a winning strategy when the initial position is X. For example, (x,x) in $\mathcal{N}^2$ is a winning position.

1.3 Sprague-Grundy Functions

To give the winning strategy, we define Sprague-Grundy numbers. For a proper subset S of $\mathbb{N}$, let $\operatorname{mex} S$ be the smallest non-negative integer not in S. For example, $\operatorname{mex} \varnothing = 0$ and $\operatorname{mex}\{0,1,3\} = 2$. Let X be a position in a game Γ. The *Sprague-Grundy number* of X is defined recursively by

$$\operatorname{sg}(X) = \operatorname{sg}_\Gamma(X) = \operatorname{mex}\{\operatorname{sg}_\Gamma(Y) : Y \text{ is an option of X}\}.$$

The function $\operatorname{sg}_\Gamma : \mathcal{P}(\Gamma) \to \mathbb{N}$ is called the *Sprague-Grundy function* of Γ

Note that if X is a terminal position, then $\operatorname{sg}(X) = \operatorname{mex} \varnothing = 0$. Furthermore, $\operatorname{sg}(X)$ is at most $\lg(X)$.

Example 1.3.1. Let us calculate the Sprague-Grundy numbers of positions in Figure 1.2.

- $(0,0)$ is terminal, so $\mathrm{sg}((0,0)) = 0$.
- $(0,1)$ has one option $(0,0)$, so $\mathrm{sg}((0,1)) = \mathrm{mex}(\{\,\mathrm{sg}((0,0))\,\}) = \mathrm{mex}(\{\,0\,\}) = 1$. Similarly, $\mathrm{sg}((1,0)) = 1$.
- $(1,1)$ has two options $(0,1)$ and $(1,0)$, so $\mathrm{sg}((1,1)) = \mathrm{mex}(\{\,\mathrm{sg}((0,1)), \mathrm{sg}((1,0))\,\}) = \mathrm{mex}(\{\,1\,\}) = 0$.

In this way, we can calculate the Sprague-Grundy numbers of positions recursively. See Figure 1.3.

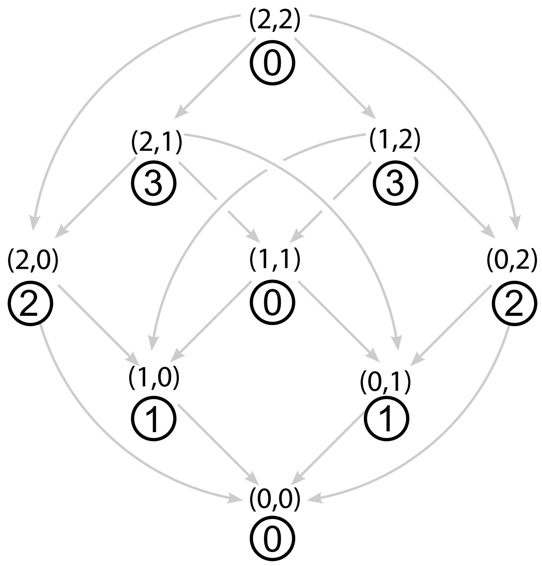


Figure 1.3: Sprague-Grundy numbers.

In fact, the Sprague-Grundy number of X in Nim is

$$X^0 \oplus_2 X^1 \oplus_2 \cdots \oplus_2 X^{m-1},$$

where $\oplus_2$ is binary addition without carry. For example, $3 \oplus_2 5 = (1+2) \oplus_2 (1+4) = 6$. This explicit formula was given by Sprague [12] and Grundy [29] independently. More generally, they proved the following result.

Theorem 1.3.2 (Sprague [12] and Grundy [29])**.** *Let Γ and Γ' be games. Then for $(X,X') \in \mathcal{P}(\Gamma \oplus_2 \Gamma')$,*

$$\mathrm{sg}_{\Gamma \oplus_2 \Gamma'}((X,X')) = \mathrm{sg}_{\Gamma}(X) \oplus_2 \mathrm{sg}_{\Gamma}(X').$$

We now present the winning strategy. Let X be a position in a game Γ. Grundy [12] and Sprague [29] showed that playing X is essentially the same as playing $(\mathrm{sg}_{\Gamma}(X)) \in \mathcal{P}(\mathcal{N}^1)$. In particular, X is a winning position if and only if $\mathrm{sg}_{\Gamma}(X) = 0$. Let us explain this. Let $g = \mathrm{sg}_{\Gamma}(X)$

and $X_g = X$. By definition, X_g has options $X_0, \dots, X_{g-1}$ with $\mathrm{sg}_\Gamma(X_h) = h$ for $h \in [g]$, but has no option Y with $\mathrm{sg}_\Gamma(Y) = g$. The position X_g might have an option X_n with $\mathrm{sg}_\Gamma(X_n) = n > g$. If this is the case, then X_n has an option X'_g with $\mathrm{sg}_\Gamma(X'_g) = g$. Hence the effect of the move X_g to X_n can be immediately reversed by the other player. This implies that if we ignore such reversible moves, then playing X_g is essentially the same as playing (g) in Nim $\mathcal{N}^1$. In particular, X is a winning position if and only if $\mathrm{sg}_\Gamma(X) = 0$.

Example 1.3.3. Let X be the position $(2,2)$ in $\mathcal{N}^2$. Since $\mathrm{sg}(X) = 2 \oplus_2 2 = 0$, every move from $(2,2)$ is reversible, so it is a winning position. Let us verify this. The position X has four options $(0,2), (1,2), (2,0)$, and $(2,1)$. Their Sprague-Grundy numbers are 2, 3, 2, and 3, respectively. Hence each of them has an option Y with $\mathrm{sg}(Y) = 0$. Indeed, for example, $(1,1)$ is an option of $(1,2)$. Similarly, every move from $(1,1)$ is reversible. See Figure 1.4. Hence (2,2) is actually a winning position.

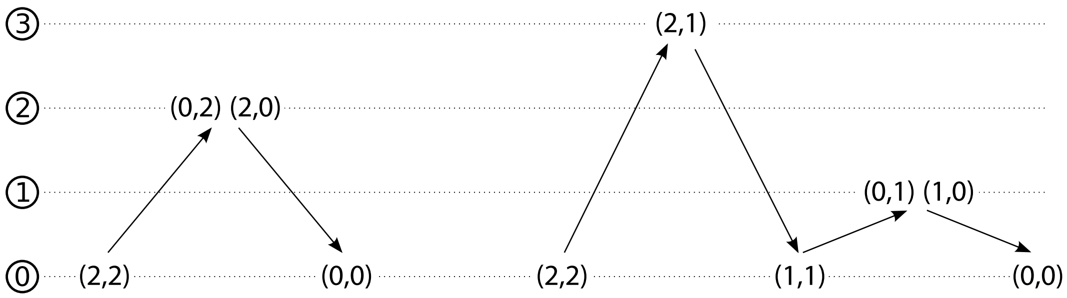


Figure 1.4: Reversible moves.

Sprague-Grundy numbers are defined by recursively. It seems to be almost impossible to present an explicit formula for the Sprague-Grundy function of a given game. For example, let Γ be an induced subgraph of Nim. Excluding trivial cases, such explicit formula was known only when Γ is Nim or Welter's game. The following explicit formula of Welter's game was given by Welter [30] and Sato [25–27] independently.

$$\mathrm{sg}(X) = X^0 \oplus_2 \cdots \oplus_2 X^{m-1} \oplus_2 \bigoplus_{0 \leqslant i < j \leqslant m-1}{}_2 \, \mathfrak{N}_2(X^i - X^j),$$

where $\mathfrak{N}_2(x) = x \oplus_2 (x-1)$.

Appendix 1.A Explicit Formulas for Sprague-Grundy Functions

In this section, we list known explicit formulas for some games.

1.A.1 Subtraction Games

Let $\mathcal{P} = \mathbb{N}$,

$$\mathcal{A} = \{\,(x,y) \in \mathbb{N}^2 : 0 < x - y < 4\,\},$$

and $\Gamma = (\mathcal{P}, \mathcal{A})$. The following table shows the Sprague-Grundy numbers of x with $0 \leqslant x \leqslant 10$ in Γ.

x	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\mathrm{sg}_\Gamma(x)$	0	1	2	3	0	1	2	3	0	1	2	$\cdots$

We see that $\mathrm{sg}(x)$ is equal to the remainder of x divided by 4 because

$$\mathrm{sg}(x) = \mathrm{mex}\,\{\,\mathrm{sg}(x-1), \mathrm{sg}(x-2), \mathrm{sg}(x-3)\,\}\,.$$

We say that Γ has the *nim-sequence*

$$01230123\cdots = \dot{0}12\dot{3}.$$

More generally, let S be a subset of $\mathbb{N}\setminus\{\,0\,\}$. Let $\mathcal{P} = \mathbb{N}$ and

$$\mathcal{A} = \{\,(x,y) \in \mathcal{P}^2 : x - y \in S\,\}\,.$$

The game $(\mathcal{P}, \mathcal{A})$ is called the *subtraction game corresponding to S*. Table 1.1 shows the nim-sequences of for some subtraction games (see Chapter 4 of [1] for more details).

Table 1.1: For example, if $S = \{\,1\,\} \cup S'$ with $S' \subseteq \{\,3,5,7\,\}$, then the subtraction game corresponding to S has the nim-sequence $\dot{0}\dot{1}$.

S	nim-sequence	period
$1(3,5,7,\cdots)$	$\dot{0}\dot{1}$	2
$2(6,10,14,\cdots)$	$\dot{0}01\dot{1}$	4
$1,2(4,5,7,8,10,\cdots)$	$\dot{0}1\dot{2}$	3
$3(9,15,21,\cdots)$	$\dot{0}0011\dot{1}$	6
$2,3(7,8,12,13,\cdots)$	$\dot{0}011\dot{2}$	5
$1,2,3(5,6,7,9,10,11,13,\cdots)$	$\dot{0}012\dot{3}$	4
$4(12,20,28,\cdots)$	$\dot{0}000111\dot{1}$	8
$1,4(6,9,11,14,\cdots)$	$\dot{0}0101\dot{2}$	5
$2,4(3,8,9,10,\cdots)$	$\dot{0}0112\dot{2}$	6
$3,4(10,11,17,\cdots)$	$\dot{0}00111\dot{2}$	7
$1,3,4(6,8,10,11,\cdots)$	$\dot{0}10123\dot{2}$	7
$1,2,3,4(6,7,8,\cdots)$	$\dot{0}123\dot{4}$	5

1.A.2 Take-and-Break Games

Let $d_1, d_2, \ldots \in \mathbb{N}$ and $d_{t,L}$ be the L-th digit in the 2-adic expansion of d_t, that is,

$$d_t = \sum_{L \in \mathbb{N}} d_{t,L} 2^L \quad \text{and} \quad d_{t,L} \in \{0,1\}.$$

We define the game $\cdot d_1 d_2 \cdots$ as follows. The position set of this game is $\bigcup_{m \in \mathbb{N}} \mathbb{N}^m$. A position $X \in \mathbb{N}^m$ has an option Y if and only if

1. $Y = (X^0, \ldots, X^{k-1}, Z^0, \ldots, Z^{L-1}, X^{k+1}, \ldots, X^{m-1})$,
2. $Z^i \geqslant 1$ for each $i \in [L]$, and
3. $d_{t,L} = 1$, where $t = X^k - Z^0 - \cdots - Z^{L-1} > 0$.

In other words, if we take t coins from a heap, then we must break this heap into L non-empty heaps for some L with $d_{t,L} = 1$. Note that if $d_t = 0$, then we cannot take t coins from any heaps.

Example 1.A.1 (Kayles)**.** Let us consider the game $\cdot 77$. This game is called Kayles. In this game, we can take one or two coins. After taking, we can break that heap to two heaps. For example, the options of (4) are

$$(3), (2,1), (1,2), (2), \text{ and } (1,1).$$

Kayles can be viewed as the following games. There is a strip of squares. We can put a block whose length is one or two. Whoever is unable to put a block loses.

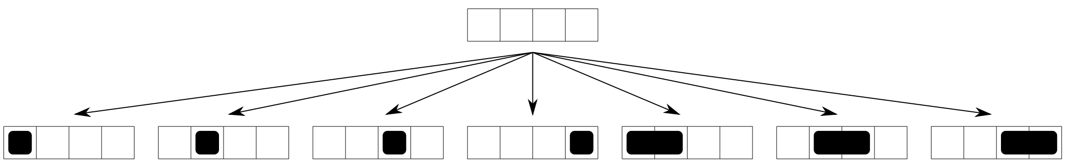


Figure 1.5: Kayles.

The nim-sequence of Kayles has the following periodicity:

Table 1.2: The periodicity of Kayles.

0	**0**	1	2	**3**	1	4	**3**	2	1	**4**	2	**6**
12	4	1	2	**7**	1	4	**3**	2	1	**4**	**6**	7
24	4	1	2	8	**5**	4	7	2	1	8	**6**	7
36	4	1	**3**	8	1	4	7	2	1	8	2	7
48	4	1	2	8	1	4	7	2	1	**4**	2	7
60	4	1	2	8	1	4	7	2	1	8	**6**	7
72	4	1	2	8	1	4	7	2	1	8	2	7
84	4	1	2	8	1	4	7	2	1	8	2	7
96	4	1	2	8	1	4	7	2	1	8	2	7

Example 1.A.2 ($\cdot$007)**.** Let us consider the game $\cdot$007. The options of (6) are

$$(3),(1,2), \text{ and } (2,1).$$

The game $\cdot$007 can be viewed as a variation of Kayles. In $\cdot$007, we can put a block whose length is three.

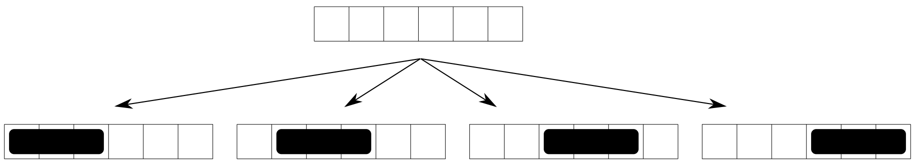


Figure 1.6: $\cdot$007.

In contrast to Kayles, it is an open problem that whether the nim-sequence of $\cdot$007 has a periodicity. See Chapter 4 of [1] for more details.

n	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\mathrm{sg}(n)$	0	0	0	1	1	1	2	2	0	3	3	1	$\cdots$

1.A.3 Rim

Rim was introduced by Flanigan [8]. Let p be an integer greater than 1. The position set of Rim_p is $\mathbb{N}^m$. In Rim_p, we first select two non-negative integers L and t with $1 \leqslant t < 2p^L$. Then we can take t coins from a heap. We also have the option of taking precisely p^L coins from a heap. This option may be exercised up to a total of $p-2$ times in a move. Note that Rim_2 is Nim.

Theorem 1.A.3 ([Flanigan [8])**.** *If X is a position in Rim_p, then*

$$\mathrm{sg}(X) = X^0 \oplus_p \cdots \oplus_p X^{m-1},$$

where $\oplus_p$ is p-ary addition without carry.

1.A.4 Euclid

Euclid, which is based on the Euclidean algorithm for computing the greatest common divisor of two numbers, was introduced by Cole and Davie [4]. The position set of Euclid is $\mathbb{N}^2$ and its arrow set is

$$\begin{aligned} &\left\{\, (X,Y) \in \mathcal{P}^2 : X^0 = Y^0 \ \text{ and } \ X^0 \;\middle|\; (X^1 - Y^1) \,\right\} \\ \cup &\left\{\, (X,Y) \in \mathcal{P}^2 : X^1 = Y^1 \ \text{ and } \ X^1 \;\middle|\; (X^0 - Y^0) \,\right\}. \end{aligned}$$

For example, $(15,65)$ has the following four options:

$$(15,50), (15,35), (15,20), \text{ and } (15,5).$$

Cole and Davie [4] shows that for $X^0 < X^1$, the position (X^0, X^1) is a winning position if and only if

$$X^1 < \frac{1+\sqrt{5}}{2} X^0.$$

Cairns, Ho, and Lengyel [3] found an explicit formula for Euclid. Let $0 < X^0 < X^1$ and let $[c_0, \ldots, c_n]$ be the continued fraction expansion of X^1/X^0, that is,

$$\frac{X^1}{X^0} = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots \, \cfrac{1}{c_{n-1} + \cfrac{1}{c_n}}}}},$$

where $c_n > 1$ if $n > 0$. Let $l(X^0, X^1)$ be the largest non-negative integer i such that

$$c_0 = \cdots = c_{i-1} \leqslant c_i.$$

Theorem 1.A.4 (Cairns, Ho, and Lengyel [3]). *Let X be a position with $0 < X^0 \leqslant X^1$ in Euclid, and let $[c_0, \ldots, c_n]$ be the continued fraction expansion of X^1/X^0. Then*

$$\operatorname{sg}(X) = \left\lfloor \left| \frac{X^1}{X^0} - \frac{X^0}{X^1} \right| \right\rfloor + \begin{cases} (-1)^n & \text{if } c_0 = \cdots = c_n, \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, if $X^0 < X^1$, then

$$\operatorname{sg}(X) = \left\lfloor \frac{X^1}{X^0} \right\rfloor - \begin{cases} 0 & \text{if } l(X^0,X^1) \text{ is even}, \\ 1 & \text{if } l(X^0,X^1) \text{ is odd}. \end{cases}$$

1.A.5 Grossman's Game

Grossman's game was introduced by Grossman [11]. This game is just the misere of Euclid, that is, its position set is $\mathbb{N}^2 \setminus \{(a,0),(0,a) : a \in \mathbb{N}\}$, and its arrow set is

$$\begin{gathered}\{(X,Y) \in \mathcal{P}^2 : X^0 = Y^0 \text{ and } X^0 \mid (X^1 - Y^1)\} \\ \cup \{(X,Y) \in \mathcal{P}^2 : X^1 = Y^1 \text{ and } X^1 \mid (X^0 - Y^0)\}.\end{gathered}$$

For example, in Euclid, $(3,9)$ has the following three options:

$$(3,6),(3,3), \text{ and } (3,0).$$

However, in Grossman's game, $(3,0)$ is not an option of $(3,9)$.

Theorem 1.A.5 (Nivesch [19])**.** *If X is a position in Grossman's game, then*

$$\mathrm{sg}(X) = \left\lfloor \left| \frac{X^1}{X^0} - \frac{X^0}{X^1} \right| \right\rfloor.$$

1.A.6 Nimhoff

Nimhoff was introduced by Fraenkel and Lorberbom [9] to analyze games lying between Nim and Wythoff's game.

Cyclic Nimhoff

Let h be a non-negative integer. The position set of cyclic Nimhoff is $\mathbb{N}^m$ and its arrow set is

$$\mathcal{A}(\mathcal{N}^m) \cup \left\{ (X,Y) \in \mathbb{N}^m : 0 < \sum_{i=0}^{m-1} (X^i - Y^i) < h \right\},$$

where $\mathcal{N}^m$ is Nim. For example, if $h = 4$, then $(2,3)$ has the following eight options:

$$(1,3),(0,3),(2,2),(2,1),(2,0),$$

$$(1,2),(1,1), \text{ and } (0,2).$$

Let a mod b denote the remainder of a divided by b.

Theorem 1.A.6 (Fraenkel and Lorberbom [9])**.** *If X is a position in cyclic Nimhoff, then*

$$\mathrm{sg}(X) = \left(h \bmod \left(\bigoplus_{i=0}^{m-1}{}_2\ \overline{X^i} \right) \right) + \left(\left(\sum_{i=0}^{m-1} X^i \right) \bmod h \right),$$

where $\overline{X^i}$ is the quotient of X^i divided by h.

Balanced Nimhoff with Powers of 2

Let K be a non-negative integer. The position set of 2^K-balanced Nimhoff is $\mathbb{N}^m$ and its arrow set is

$$\mathcal{A}(\mathcal{N}^m) \cup \left\{ (X,Y) \in \mathbb{N}^m : \operatorname{dist}(X,Y) = 2 \text{ and } X^s - Y^s = X^t - Y^t = 2^k \text{ for some } s \neq t \right\}.$$

Let

$$x *_K y = x \oplus_2 y \oplus_2 x_K y_K,$$

where x_K is the K-th digit in the 2-adic expansion of x, that is, $x = \sum_{K \in \mathbb{N}} x_K 2^K$ and $x_K \in \{0,1\}$.

Theorem 1.A.7 (Fraenkel and Lorberbom [9])**.** *If X is a position in 2^K-balanced Nimhoff, then*

$$\operatorname{sg}(X) = X^0 *_K \cdots *_K X^{m-1}.$$

Double Cyclic Nimhoff

Let h be an integer greater than 1. The positions set of double cyclic Nimhoff is $\mathbb{N}^2$ and its arrow set is

$$\begin{aligned} &\mathcal{A}(\mathcal{N}^m) \\ &\cup \left\{ (X,Y) \in \mathcal{P}^2 : X^i \geqslant Y^i \text{ for } i \in [2] \text{ and } \sum_{i=0}^{1} (X^i - Y^i) \in \{1,2,\ldots,h-1,2h\} \right\}. \end{aligned}$$

Let x and y be non-negative integers. If $x \neq y$ and $M = \max\{M \in \mathbb{N} : x_L = y_L \text{ for } 0 \leqslant L < M\}$, then

$$\operatorname{macs}(x,y) = \sum_{L=0}^{M-1} x_L 2^L = \left(\sum_{L=0}^{M-1} y_L 2^L \right).$$

If $x = y$, then $\operatorname{macs}(x,y) = x (= y)$.

Theorem 1.A.8 (Fraenkel and Lorberbom [9])**.** *If X is a position in double cyclic Nimhoff, then*

$$\operatorname{sg}(X) = \left(h \bmod (\overline{X^0} \oplus_2 \overline{X^1}) \right) + \left((X^0 + X^1 - \operatorname{macs}(\overline{X^0}, \overline{X^1})) \bmod h \right),$$

where $\overline{X^i}$ is the quotient of X^i divided by h.

Even Balanced Nimhoff

Let l be a positive integer. The position set of even balanced Nimhoff is $\mathbb{N}^2$ and its arrow set is

$$\mathcal{A}(\mathcal{N}^2) \cup \{ (X,Y) \in \mathcal{P}^2 : \sum_{i=0}^{1}(X^i - Y^i) = 4l \} .$$

For example, if $l = 1$, then (3,4) has the following ten options:

$$(2,4),(1,4),(0,4),(3,3),(3,2),(3,1),(3,0),$$

$$(0,3),(1,2), \text{ and } (2,1).$$

Let x and y be two non-negative integers. Let $\mathrm{macs}_l(x,y)$ denote the number of times it is possible to subtract l from x and y without changing the nim-sum, that is,

$$\mathrm{macs}_l(x,y) = \max\{ d \in \mathbb{N} : x \oplus_2 y = (x-il) \oplus_2 (y-il) \text{ for } 0 \leqslant i \leqslant d \} .$$

Note that

$$\mathrm{macs}_1(x,y) = \mathrm{macs}(x,y).$$

Theorem 1.A.9 (Fraenkel and Lorberbom [9])**.** *If X is a position in even balanced Nimhoff, then*

$$\mathrm{sg}(X) = \begin{cases} X^0 \oplus_2 X^1 & \text{if } \mathrm{macs}_l(\overline{X^0},\overline{X^1}) \text{ is even}, \\ X^0 \oplus_2 X^1 \oplus_2 1 & \text{if } \mathrm{macs}_l(\overline{X^0},\overline{X^1}) \text{ is odd}, \end{cases}$$

where $\overline{X^i}$ is the quotient of X^i divided by 2.

1.A.7 Lim

Lim was introduced and analyzed by Fink, Fraenkel, and Santos [7]. The position set of Lim is $\mathbb{N}^3$ and its arrow set is

$$\{ (X,Y) \in \mathcal{P}^2 : X^i - Y^i = X^j - Y^j = Y^k - X^k > 0 \text{ for some } \{i,j,k\} = [3] \} .$$

For example, the position $(3,4,2)$ has the following seven options:

$$(2,3,3),(1,2,4),(0,1,5),(2,5,1),(1,6,0),(4,3,1), \text{ and } (5,2,0).$$

Theorem 1.A.10 (Fink, Fraenkel, and Santos [7])**.** *If X is a position in Lim, then*

$$\mathrm{sg}(X) = \frac{X^0 + X^1 + X^2 - (X^0 \oplus_2 X^1 \oplus_2 X^2)}{2}.$$

2 Digit-Separable Sprague-Grundy Functions

We construct a family of games including Nim and give explicit formulas for their Sprague-Grundy functions. of all positions can be written explicitly. Let Φ be an integer-valued function on the position set of Nim. Let p be an integer greater than 1 and let $\Gamma[\Phi]$ be a p-saturation of the subgame of Nim induced in $\{X \in \mathcal{P} : \Phi(X) \geqslant 0\}$. We show that if Φ is 'digit-separable' and a 'locally Sprague-Grundy function' of $\Gamma[\Phi]$, then Φ is the Sprague-Grundy function of $\Gamma[\Phi]$. The p-saturation indices of some games in this family are also determined.

2.1 Introduction

In the 1930s, Sprague [29] and Grundy [12] showed that impartial games can be analyzed using Sprague-Grundy functions. Moreover, they gave an explicit formula for the Sprague-Grundy function of Nim. After that, a lot of studies were conducted. Especially, in 1954, Welter [30] presented an explicit formula for the Sprague-Grundy function of Welter's game. As far as the author knows, Nim and Welter's game were the only known nontrivial examples of induced subgames of Nim and their p-saturations whose Sprague-Grundy functions had been written explicitly.

The purpose of this paper is constructing a family of games and presenting explicit formulas for their Sprague-Grundy functions. We first construct finite inverted Nim from a distribution related to Nim and extend this game to inverted Nim. Then, by focusing the fact that the Sprague-Grundy function of inverted Nim is *digit-separable*, we construct a family of games including Nim and inverted Nim, and we present explicit formulas for the Sprague-Grundy functions of games in this family. We also give the p-saturation indices of some of these games.

2.1.1 Inverted Nim

In this subsection, we construct finite inverted Nim using a frequency distribution related to Nim and present an explicit formula for the Sprague-Grundy function of this game. Using this formula, we expand finite inverted Nim to inverted Nim. This process leads us to a family of games. The proofs of the results in this subsection will be given in Section 2.2.

We first construct games by permuting Nim. Let H be a positive integer, and let $\mathcal{N}^{m,H}$ be the subgame of $\mathcal{N}^m$ induced in $[2^H]^m$. Let $W^{m,H}$ be the winning positions of $\mathcal{N}^{m,H}$, that is,

$$W^{m,H} = \{ X \in [2^H]^m : X^0 \oplus_2 \cdots \oplus_2 X^{m-1} = 0 \}.$$

For a permutation $\sigma \in \mathrm{Sym}([2^H])$, let

$$\sigma(W^{m,H}) = \{ (\sigma(X^0), \ldots, \sigma(X^{m-1})) : (X^0, \ldots, X^{m-1}) \in W^{m,H} \},$$

and let $\sigma(\mathcal{N}^{m,H})$ be the maximum induced subgame Δ of $\mathcal{N}^{m,H}$ such that the winning position set of Δ equals $\sigma(W^{m,H})$, that is,

$$\{ X \in \mathcal{P}(\Delta) : \mathrm{sg}_\Delta(X) = 0 \} = \sigma(W^{m,H}).$$

By the definition of Sprague-Grundy functions, we see that

$$\mathcal{P}(\sigma(\mathcal{N}^{m,H})) = \sigma(W^{m,H}) \cup \{ X \in [2^H]^m : (X,Y) \in \mathcal{A}(\mathcal{N}^m) \text{ for some } Y \in \sigma(W^{m,H}) \}.$$

Let $\mathcal{F}^{m,H} = \{ \sigma(\mathcal{N}^{m,H}) : \sigma \in \mathrm{Sym}([2^H]) \}$. We will consider the frequency distribution of $|\Gamma|$ for $\Gamma \in \mathcal{F}^{m,H}$, where $|\Gamma| = |\mathcal{P}(\Gamma)|$.

Example 2.1.1. Let $m = 3$ and $H = 1$. Then $\mathrm{Sym}([2^H]) = \{ (), (0\ 1) \}$ and

$$\mathcal{F}^{3,1} = \{ \mathcal{N}^{3,1}, (0\ 1)(\mathcal{N}^{3,1}) \}.$$

Let us calculate $|\Gamma|$ for $\Gamma \in \mathcal{F}^{3,1}$. We have $\left|\mathcal{N}^{3,1}\right| = \left|[2]^3\right| = 8$. We show that $\left|(0\ 1)(\mathcal{N}^{3,1})\right| = 7$. Let $\sigma = (0\ 1)$. Since

$$\sigma(W^{3,1}) = \sigma(\{ (0,0,0), (0,1,1), (1,0,1), (1,1,0) \}) = \{ (1,1,1), (1,0,0), (0,1,0), (0,0,1) \},$$

we have

$$\{ X \in [2]^3 : (X,Y) \in \mathcal{A}(\mathcal{N}^3) \text{ for some } Y \in \sigma(W^{3,1}) \} = \{ (1,1,0), (1,0,1), (0,1,1) \},$$

so

$$\mathcal{P}((0\ 1)(\mathcal{N}^{3,1})) = [2]^3 \setminus \{ (0,0,0) \}.$$

Hence $\left|(0\ 1)(\mathcal{N}^{3,1})\right| = 7$. Therefore the frequency distribution of $\mathcal{F}^{3,1}$ is as shown in the following table.

7	8
1	1

Proposition 2.1.2. *If m is odd, then the frequency distribution of $\mathcal{F}^{m,H}$ is symmetric.*

For example, the following table shows the frequency distribution of $\mathcal{F}^{3,3}$.

400	406	410	412	$\cdots$	500	502	506	512
1	3	1	5	$\cdots$	5	1	3	1

Let m be odd. Since $\mathcal{N}^{m,H}$ has the largest number of positions in $\mathcal{F}^{m,H}$, it follows from Proposition 2.1.2 that there is a unique game $\mathcal{I}^{m,H}$ that has the smallest number of positions in $\mathcal{F}^{m,H}$. We call this game *m-heap finite inverted Nim with height H*. For example, $\mathcal{I}^{3,1} = (0,1)(\mathcal{N}^{3,1})$. In general, if σ is the bit inversion $x \mapsto x \oplus_2 (2^H - 1)$, then $\mathcal{I}^{m,H} = \sigma(\mathcal{N}^{m,H})$. Figure 2.1 shows $[2^H]^3 \backslash \mathcal{P}(\mathcal{I}^{3,H})$ for $H = 1,2,3,4$.

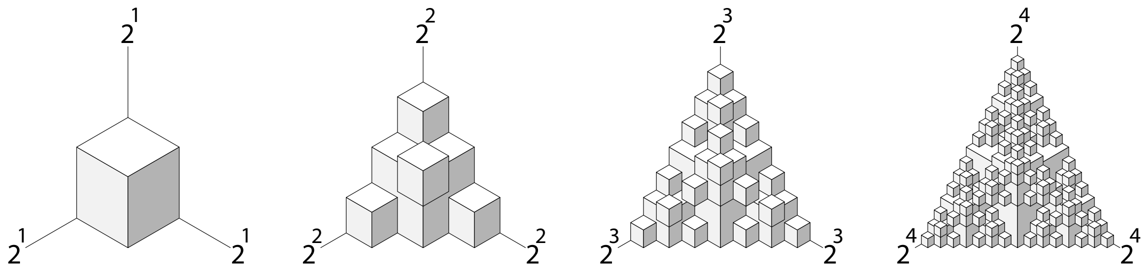


Figure 2.1: For $H = 1,2,3,4$, an excluded position $(X^0, X^1, X^2) \in [2^H]^3 \backslash \mathcal{P}(\mathcal{I}^{3,H})$ is represented by the cube with vertices $(X^0 + \varepsilon^0, X^1 + \varepsilon^1, X^2 + \varepsilon^2)(\varepsilon^i \in \{0,1\})$.

To give an explicit formula for the Sprague-Grundy function of finite inverted Nim, we introduce a notation. Let p be an integer greater than 1. For $x \in \mathbb{N}$, let $x_L^{(p)}$ denote the L-th digit in the p-adic expansion of x, that is,

$$x = \sum_{L \in \mathbb{N}} x_L^{(p)} p^L \quad \text{and} \quad x_L^{(p)} \in [p].$$

For $X \in \mathbb{N}^m$, let

$$X_L^{(p)} = ((X^0)_L^{(p)}, \ldots, (X^{m-1})_L^{(p)}).$$

When no confusion can arise, we will drop $^{(p)}$ and write x_L and X_L instead of $x_L^{(p)}$ and $X_L^{(p)}$.

We define $\Psi^H : \mathbb{N}^m \to \mathbb{Z}$ by

$$\Psi^H(X) = X^0 \oplus_2 \cdots \oplus_2 X^{m-1} \oplus_2 (2^H - 1) - \sum_{L=0}^{H-1} \delta(X_L^{(2)}) 2^{L+1}, \tag{2.1.1}$$

where

$$\delta(X_L) = \begin{cases} 1 & \text{if } X_L = (0, \ldots, 0), \\ 0 & \text{if } X_L \neq (0, \ldots, 0). \end{cases}$$

Theorem 2.1.3. *Let Γ be m-heap finite inverted Nim with height H and X be a position in Γ. If $m \leqslant 3$ or $H \leqslant 3$, then*

$$\mathrm{sg}_\Gamma(X) = \Psi^H(X).$$

Example 2.1.4. Let $H = 3$ and $X = (1, 4, 5)$. We calculate $\Psi^H(X)$. Let us express X^i as 2-adic numbers. Then $X^0 = 1^{[2]}, X^1 = 100^{[2]}, X^2 = 101^{[2]}$. Hence

$$X_0 = (1, 0, 1), \ \ X_1 = (0, 0, 0), \ \ X_2 = (0, 1, 1).$$

This implies that

$$\Psi^H(X) = 1^{[2]} \oplus_2 100^{[2]} \oplus_2 101^{[2]} \oplus_2 111^{[2]} - 100^{[2]} = 011^{[2]}.$$

Unfortunately, if $m > 3$ and $H > 3$, then there exists $X \in \mathcal{P}(\mathcal{I}^{m,H})$ such that $\mathrm{sg}_{\mathcal{I}^{m,H}}(X) \neq \Psi^H(X)$ (see Example 2.2.6 and Remark 2.2.13). However, by considering saturations (see Section 2.3), we can obtain the following similar result. For an induced subgame Γ of $\mathcal{N}^m$, let $\Gamma^{p\text{-sat}}$ be a p-saturation of Γ. Then Ψ^H actually gives the Sprague-Grundy function of $(\mathcal{I}^{m,H})^{2\text{-sat}}$.

Next, we expand finite inverted Nim using Ψ^H. It is clear that if X is a position in finite inverted Nim, then $\Psi^H(X) \geqslant 0$. In fact, the inverse of this is also true.

Proposition 2.1.5. *If $X \in [2^H]^m$, then X is a position in $\mathcal{I}^{m,H}$ if and only if $\Psi^H(X) \geqslant 0$.*

In view of Proposition 2.1.5, we can expand finite inverted Nim as follows. Let Γ be the subgame of $\mathcal{N}^m$ induced in

$$\left\{ X \in \mathbb{N}^m : \Psi^H(X) \geqslant 0 \right\}.$$

The game Γ is called *m-heap inverted Nim with height H*. For example, Nim is inverted Nim with height 0. In fact, Ψ^H gives the Sprague-Grundy function of a 2-saturation of inverted Nim with height H.

We now generalize the above expansion process. Let Γ be a game and Φ be an integer-valued function from $\mathcal{P}(\Gamma)$. Let $\Gamma[\Phi]$ denote the subgame of Γ induced in

$$\{ X \in \mathcal{P}(\Gamma) : \Phi(X) \geqslant 0 \}.$$

For example, if $\Gamma = \mathcal{N}^m$ and $\Phi = \Psi^H$, then $\mathcal{N}^m[\Psi^H]$ is inverted Nim with height H and Ψ^H gives the Sprague-Grundy function of $(\mathcal{N}^m[\Psi^H])^{2\text{-sat}}$. This leads us to the following problem.

Problem 1. Let Φ be an integer-valued function from $\mathbb{N}^m$. When does Φ give the Sprague-Grundy function of $(\mathcal{N}^m[\Phi])^{p\text{-sat}}$?

In the next section, we give a sufficient condition for Φ that satisfies the above condition.

2.1.2 Digit-Separable Functions

We define digit-separable functions and show that the Sprague-Grundy functions of Nim and inverted Nim are digit-separable. We then present the main result, which gives a partial answer to Problem 1. Let p be an integer greater than 1.

We first define digit-separable functions. An integer-valued function Φ from $\mathbb{N}^m$ is said to be *digit-separable in base p* if there exists $\phi_L : \Omega^m \to \mathbb{Z}$ for each $L \in \mathbb{N}$ such that

$$\Phi(X) = \sum_{L\in\mathbb{N}} \phi_L(X_L^{(p)}) \text{ for } X \in \mathbb{N}^m. \tag{2.1.2}$$

Let $[\phi_L]_{L\in\mathbb{N}}$ denote the right-hand side of (2.1.2).

For example, the Sprague-Grundy functions of Nim and inverted Nim are digit-separable in base 2. Indeed, let

$$\psi_L^H(X_L) = \begin{cases} X_L^0 \oplus_2 \cdots \oplus_2 X_L^{m-1} \oplus_2 1 - 2\delta(X_L) & \text{if } L < H \\ X_L^0 \oplus_2 \cdots \oplus_2 X_L^{m-1} & \text{if } L \geqslant H. \end{cases}$$

Then $\Psi^H(X) = [\phi_L^H]_{L\in\mathbb{N}}$.

Remark 2.1.6. The Sprague-Grundy function of Welter's game is not digit-separable. Indeed, let Γ be Welter's game with 3 heaps, and let $X = (1,2,6)$ and $Y = (0,2,7)$. If sg_Γ is digit-separable, then $\mathrm{sg}_\Gamma(X) = \mathrm{sg}_\Gamma(Y)$ since $X_0 = (1,0,0), Y_0 = (0,0,1)$, and $X_L = Y_L$ for $L \geqslant 1$. However, the Sprague-Grundy numbers of X and Y are 2 and 6, respectively.

To state the main result, we introduce some notation. Let α be a non-negative integer and let $\xi_L \subseteq [\alpha_L]^m$ for $L \in \mathbb{N}$. We define an integer-valued function $\phi_L^{\xi,\alpha}$ from $[p]^m$ by

$$\phi_L^{\xi,\alpha}(x) = x^0 \oplus_p \cdots \oplus_p x^{m-1} \ominus_p \alpha_L - p \cdot I^{\xi_L}(x),$$

where I^{ξ_L} is the indicator function of ξ_L, that is,

$$I^{\xi_L}(x) = \begin{cases} 1 & \text{if } x \in \xi_L, \\ 0 & \text{if } x \notin \xi_L. \end{cases}$$

Let $\Phi^{\xi,\alpha}$ denote $[\phi_L^{\xi,\alpha}]_{L\in\mathbb{N}}$. Then

$$\Phi^{\xi,\alpha}(X) = X^0 \oplus_p \cdots \oplus_p X^{m-1} \ominus_p \alpha - p \sum_{L\in\mathbb{N}} I^{\xi_L}(X_L) p^L.$$

Let $\Gamma^{\xi,\alpha} = \mathcal{N}^m[\Phi^{\xi,\alpha}]$ and $\Gamma_L^{\xi,\alpha} = \mathcal{N}^m[\phi_L^{\xi,\alpha}]$ for each $L \in \mathbb{N}$.

Example 2.1.7 (Inverted Nim)**.** Let $p = 2$ and $H \in \mathbb{N}$. Let $\alpha = 2^H - 1$ and

$$\xi_L = \begin{cases} \{(0,\ldots,0)\} & \text{if } L < H \\ \varnothing & \text{if } L \geqslant H. \end{cases}$$

Then $\phi_L^{\xi,\alpha} = \psi_L^H$ for each $L \in \mathbb{N}$. Hence $\Gamma^{\xi,\alpha}$ is inverted Nim with height H and $\Phi^{\xi,\alpha}$ gives the Sprague-Grundy function of $(\Gamma^{\xi,\alpha})^{2\text{-sat}}$. In addition, $\phi_L^{\xi,\alpha}$ gives the Sprague-Grundy function of $(\Gamma_L^{\xi,\alpha})^{2\text{-sat}}$. Indeed, suppose that $H < L$. Then

$$\phi_L^{\xi,\alpha}(x) = \psi_L^H(x) = x^0 \oplus_2 \cdots \oplus_2 x^{m-1} \ominus_2 1 - 2\delta(x).$$

Since $\psi_L^H(x)$ is negative only when $x = (0,\dots,0)$, the game $\Gamma_L^{\xi,\alpha}$ is the subgame of $\mathcal{N}^m$ induced in $[2]^m \backslash (0,\dots,0)$. In other words, $\Gamma_L^{\xi,\alpha}$ is finite inverted Nim with height 1. Hence the Sprague-Grundy function of $(\Gamma_L^{\xi,\alpha})^{2\text{-sat}}$ is given by $\phi_L^{\xi,\alpha}$. Suppose that $L \geqslant H$. Then

$$\phi_L^{\xi,\alpha}(x) = \psi_L^H(x) = x^0 \oplus_2 \cdots \oplus_2 x^{m-1}.$$

Hence $\Gamma_L^{\xi,\alpha}$ is the subgame of $\mathcal{N}^m$ induced in $[2]^m$, so its Sprague-Grundy function is given by $\phi_L^{\xi,\alpha}$.

In fact, $\Phi^{\xi,\alpha}$ always gives the Sprague-Grundy function of $(\Gamma^{\xi,\alpha})^{p\text{-sat}}$ in the above situation.

Theorem 2.1.8. *Let $\alpha \in \mathbb{N}$ and $\xi_L \subseteq [\alpha_L]^m$ for $L \in \mathbb{N}$. If $\phi_L^{\xi,\alpha}$ gives the Sprague-Grundy function of $(\Gamma_L^{\xi,\alpha})^{p\text{-sat}}$ for each $L \in \mathbb{N}$, that is,*

$$\mathrm{sg}_{\Gamma_L^{\xi,a}}(x) = \phi_L^{\xi,a}(x) \ \text{ for } \ x \in \mathcal{P}(\Gamma_L^{\xi,a}), \tag{2.1.3}$$

then $\Phi^{\xi,\alpha}$ gives that of $(\Gamma^{\xi,\alpha})^{p\text{-sat}}$.

We prove this theorem in Section 2.3.

Example 2.1.9. Let α be a non-negative integer. Let

$$\xi_L = \{ (t^0,\dots,t^{m-1}) \in [p]^m : t^0 + \cdots + t^{m-1} < \alpha_L \}.$$

Then $\phi_L^{\xi,\alpha}$ gives the Sprague-Grundy function of $(\Gamma_L^{\xi,\alpha})^{p\text{-sat}}$. It follows from Theorem 2.1.8 that

$$\mathrm{sg}_\Gamma(X) = \Phi^{\xi,\alpha}(X) \qquad \text{for } \ X \in \mathcal{P}(\Gamma),$$

where $\Gamma = (\Gamma^{\xi,\alpha})^{p\text{-sat}}$.

Let $\alpha = p^H - 1$. Then the game $\Gamma^{\xi,\alpha}$ will be called *m-heap p-inverted Nim with height H*. Note that 2-inverted Nim is the ordinary inverted Nim. We can also determine the p-saturation index of p-inverted Nim.

Theorem 2.1.10. *If Γ is m-heap p-inverted Nim with height H, then*

$$\mathrm{sat}_p(\Gamma) = \begin{cases} \min(p+1, m+1) & \text{if } \ p = 2, \\ \min(p, m+1) & \text{if } \ p > 2. \end{cases}$$

The proof of this result is in Section 2.4.

2.1.3 Organization

This chapter is organized as follows. In Section 2.2, we examine finite inverted Nim. Section 2.3 contains the proof of Theorem 2.1.8. In Section 2.4, we present the p-saturation index of p-inverted Nim.

2.2 2-inverted Nim

We investigate 2-inverted Nim. First, we show Proposition 2.1.2, which states that the frequency distribution of $\mathcal{F}^{m,H}$ is symmetric if m is odd. Next, we give a solution formula for $\Psi(X)$. Finally, we show that Theorem 2.1.3. In this section, we write $\oplus$ instead of $\oplus_2$.

2.2.1 Frequency Distributions

We introduce a notation. Let $\mathcal{C} = \mathbb{N}^m \backslash W^{m,H}$ and $C \in \mathcal{C}$. For $i \in [m]$, let

$$\widetilde{C^i} = C^0 \oplus \cdots C^{i-1} \oplus C^{i+1} \oplus \cdots \oplus C^{m-1}. \tag{2.2.1}$$

Then

$$C^0 \oplus \cdots \oplus C^{i-1} \oplus \widetilde{C^i} \oplus C^{i+1} \oplus \cdots C^{m-1} = 0,$$

so

$$(C^0, \ldots, C^{i-1}, \widetilde{C^i}, C^{i+1}, \ldots, C^{m-1}) \in W^{m,H}.$$

For $\sigma \in S[2^H]$, we have

$$\sigma(C) = (\sigma(C^0), \ldots, \sigma(C^{m-1})) \in \sigma(\mathcal{C}) = \mathbb{N}^m \backslash \sigma(W^{m,H})$$

and

$$(\sigma(C^0), \ldots, \sigma(C^{i-1}), \sigma(\widetilde{C^i}), \sigma(C^{i+1}), \ldots, \sigma(C^{m-1})) \in \sigma(W^{m,H}).$$

Let $\alpha_h(\sigma)$ be the number of $C \in \mathcal{C}$ such that the number of i with $\sigma(\widetilde{C^i}) < \sigma(C^i)$ equals h, that is,

$$\alpha_h(\sigma) = \left| \{ C \in \mathcal{C} : |\{ i \in [m] : \sigma(\widetilde{C^i}) < \sigma(C^i) \}| = h \} \right|.$$

Then $\alpha_0(\sigma) = |\mathcal{N}^{m,H} \backslash \sigma(\mathcal{N}^{m,H})|$, so

$$|\sigma(\mathcal{N}^{m,H})| = 2^{mH} - \alpha_0(\sigma).$$

We will show the following equation:

$$\alpha_0(\sigma) + \alpha_m(\sigma) = \frac{2^{M(m-1)}(2^M - 1)}{2^{m-1}} \qquad \text{if } m \text{ is odd.} \tag{2.2.2}$$

From (2.2.2), we can deduce Proposition 2.1.2. Indeed, let $\tau = x \mapsto x \oplus (2^H - 1)(= 2^H - 1 - x)$. Then

$$\sigma(C^i) < \sigma(\widetilde{C}^i) \iff \tau\sigma(C^i) > \tau\sigma(\widetilde{C}^i).$$

It follows that $\alpha_m(\sigma) = \alpha_0(\tau\sigma)$. Hence

$$\alpha_0(\sigma) + \alpha_m(\sigma) = \alpha_0(\sigma) + \alpha_0(\tau\sigma).$$

By (2.2.2), $\alpha_0(\sigma) + \alpha_0(\tau\sigma)$ does not depend on σ. This implies that the frequency distribution of $\mathcal{F}^{m,H}$ is symmetric with respect to $2^{mH} - (\alpha_0(\sigma) + \alpha_0(\tau\sigma))/2$.

To prove (2.2.2), we need a lemma.

Lemma 2.2.1. *If* $0 \leqslant k < m$, *then*

$$\sum_{h=0}^{m} \binom{h}{k} \alpha_h(\sigma) = \frac{2^{M(m-1)}(2^M - 1)}{2^k} \binom{m}{k}.$$

proof. Let

$$S = \left\{ (C,J) \in \mathcal{C} \times \binom{[m]}{k} : \sigma(\widetilde{C}^j) < \sigma(C^j) \ \text{ for each } \ j \in J \right\}.$$

We count $|S|$ in two ways.

By the definition of $\alpha_h(\sigma)$,

$$|S| = \sum_{h=0}^{m} \binom{h}{k} \alpha_h(\sigma).$$

We next show that

$$|S| = \frac{|\mathcal{C} \times \binom{[m]}{k}|}{2^k} = \frac{2^{M(m-1)}(2^M - 1)}{2^k} \binom{m}{k}. \tag{2.2.3}$$

We fix $J \in \binom{[m]}{k}$ and $r \in [m] \backslash J$. Using them, we will give a partition of $\mathcal{C}$ into subsets $\mathcal{C}_1, \ldots, \mathcal{C}_l$ with $|\mathcal{C}_i| = 2^k$ and show that each $\mathcal{C}_i$ contains a unique C with $(C,J) \in S$. Note that this yields (2.2.3).

We construct a partition of $\mathcal{C}$. For $C \in \mathcal{C}$ and $A \subseteq J$, we define $C_{(A)}$ by

$$C^i_{(A)} = \begin{cases} \widetilde{C}^i & \text{if } \ i \in A \text{ or } \ (i = r \ \text{ and } \ |A| \text{ is odd}), \\ C^i & \text{otherwise}. \end{cases} \tag{2.2.4}$$

Let $[C] = \{C_{(A)} : A \subseteq J\}$. We show that $\{[C] : C \in \mathcal{C}\}$ is a partition of $\mathcal{C}$. Since $\bigcup_{C \in \mathcal{C}} [C] = \mathcal{C}$, we need only show that (1) $C_{(A)} \in \mathcal{C}$ and (2) $[C] \cap [C'] = \varnothing$ or $[C]$.

(1) If $|A| = 0$, then $C_{(A)} = C \in \mathcal{C}$. Suppose that $|A| > 0$. By replacing C with $C_{(A')}$, where A' is an arbitrary subset of A with $|A'| = |A| - 1$, we may assume that $|A| = 1$. Let $A = \{a\}$. By (2.2.1),

$$C^0_{(A)} \oplus \cdots \oplus C^{m-1}_{(A)} = C^0 \oplus \cdots \oplus C^m \oplus C^a \oplus C^r \oplus \widetilde{C}^a \oplus \widetilde{C}^r = C^0 \oplus \cdots \oplus C^{m-1} \neq 0. \qquad (2.2.5)$$

Hence $C_{(A)} \in \mathcal{C}$.

(2) If $C'_A = C$ for some $A \subseteq J$, then $[C'] = [C]$. If $C'_A \neq C$ for each $A \subseteq J$, then $[C'] \cap [C] = \varnothing$. Hence $[C] \cap [C'] = \varnothing$ or $[C]$. Therefore $\{[C] : C \in \mathcal{C}\}$ is a partition of $\mathcal{C}$.

We now prove (2.2.3). Let $C \in \mathcal{C}$ and

$$A = \{a \in J : \sigma(\widetilde{C}^a) > \sigma(C^a)\}.$$

By (2.2.4),

$$\sigma(\widetilde{C}^j_{(A)}) < \sigma(C^j_{(A)}) \qquad \text{for each } \ j \in J.$$

This implies that $(C_{(A)}, J) \in S$. We also see that $(C_{(B)}, J) \notin S$ for $B \subseteq J$ with $B \neq A$. Therefore (2.2.3) holds.

□

We now prove (2.2.2). We calculate

$$\sum_{k=0}^{m-1} (-1)^k \sum_{h=0}^{m} \binom{h}{k} \alpha_h(\sigma)$$

in two ways.

If $h < m$, then

$$\sum_{k=0}^{m-1} \binom{h}{k} (-1)^k = (1-1)^h.$$

If $h = m$, then

$$\sum_{k=0}^{m-1} \binom{h}{k} (-1)^k = (1-1)^m - \binom{m}{m} (-1)^m = -(-1)^m.$$

Hence

$$\begin{aligned}\sum_{k=0}^{m-1} (-1)^k \sum_{h=0}^{m} \binom{h}{k} \alpha_h(\sigma) &= \sum_{h=0}^{m} (\sum_{k=0}^{m-1} \binom{h}{k} (-1)^k) \alpha_h(\sigma) \\ &= \alpha_0(\sigma) - (-1)^m \alpha_m(\sigma).\end{aligned}$$

On the other hand, by Lemma (2.2.1),

$$\begin{aligned}\sum_{k=0}^{m-1}(-1)^k\sum_{h=0}^{m}\binom{h}{k}a_h(\sigma) &= 2^{M(m-1)}(2^M-1)\sum_{k=0}^{m-1}\binom{m}{k}\left(\frac{-1}{2}\right)^k\\ &= 2^{M(m-1)}(2^M-1)\left(\left(1-\frac{1}{2}\right)^m-\binom{m}{m}\left(\frac{-1}{2}\right)^m\right)\\ &= \frac{2^{M(m-1)}(2^M-1)}{2^{m-1}}.\end{aligned}$$

Therefore (2.2.2) holds.

2.2.2 Solution Formula

Let X be a position in finite inverted Nim with height H. Let

$$\sigma(X) = X^0 \oplus \cdots \oplus X^{m-1}$$

and

$$\delta(X) = \sum_{L=0}^{H-1}\delta(X_L)2^L.$$

In this and the next section, we devote to prove Theorem 2.1.3.

In this section, we will show that

$$X^i = (\Psi^H(X)+\delta(X^{(i)}))\oplus\delta(X^{(i)})\oplus\sigma(X^{(i)})\oplus(2^H-1), \tag{2.2.6}$$

where $X^{(i)}$ is obtained from X by deleting the i-th component, that is,

$$X^{(i)} = (X^0,\dots,X^{i-1},X^{i+1},\dots,X^{m-1}) \in \mathbb{N}^{m-1}.$$

To prove (2.2.6), we introduce a notation. For a finite subset $\mathcal{S}$ of $\mathbb{N}$, a non-negative integer x is said to be *$\mathcal{S}$-free* if $x_S = 0$ for every $S \in \mathcal{S}$.

Lemma 2.2.2. *If $g \in \mathbb{N}$, then there exist x_S for $S \in \mathcal{S}$ such that $g+\sum_{S\in\mathcal{S}} x_S 2^S$ is $\mathcal{S}$-free. Moreover, if $g+\sum_{S\in\mathcal{S}} x'_S 2^S$ is also $\mathcal{S}$-free, then $\sum_{S\in\mathcal{S}} x_S 2^S = \sum_{S\in\mathcal{S}} x'_S 2^S$.*

proof. We show by induction on $|\mathcal{S}|$. If $g_S = 0$ for each $S \in \mathcal{S}$, then the lemma is trivial. Suppose that $|\mathcal{S}| > 0$ and $g_S = 1$ for some $S \in \mathcal{S}$.

We first show the existence $\sum x_S$. Let $T = \min\mathcal{S}$ and $x_T = 1$. By the induction hypothesis, there exist x_S for $S \in \mathcal{S}\setminus\{T\}$ such that $(g+x_T 2^T)+\sum_{S\in\mathcal{S}\setminus\{T\}} x_S 2^S$ is $\mathcal{S}\setminus\{T\}$-free. Since $T = \min\mathcal{S}$ and $(g+x_T 2^T)_T = 0$, we see that $(g+\sum_{S\in\mathcal{S}} x_S 2^S)_T = 0$. Hence $g+\sum_{S\in\mathcal{S}} x_S 2^S$ is $\mathcal{S}$-free.

We next show the uniqueness of $\sum x_S$. Since $g_T = 1$, we have $x'_T = 1$. Hence

$$(g+2^T) + \sum_{S\in\mathcal{S}\setminus\{T\}} b_S 2^S$$

is also $(\mathcal{S}\setminus\{T\})$-free. By induction hypothesis, $\sum_{S\in\mathcal{S}} x_S 2^S = \sum_{S\in\mathcal{S}} x'_S 2^S$.

□

If $g+\sum_{S\in\mathcal{S}} x_S 2^S$ is $\mathcal{S}$-free, then we denote it $F_{\mathcal{S}}(g)$.

Lemma 2.2.3. *If $e = g+\sum_{S\in\mathcal{S}} 2^S$, then $F_{\mathcal{S}}(g) = e-\sum_{S\in\mathcal{S}} e_S 2^S$.*

proof. It is clear that $e-\sum e_S 2^S$ is $\mathcal{S}$-free. By Lemma 2.2.2,

$$e-\sum_{S\in\mathcal{S}} e_S 2^S = g+\sum_{S\in\mathcal{S}}(1-e_S)2^S = F_{\mathcal{S}}(g).$$

□

Lemma 2.2.4. *If $F_{\mathcal{S}}(g) = g+\sum_{S\in\mathcal{S}} x_S 2^S$, then*

$$F_{\mathcal{S}}(g)+\sum_{S\in\mathcal{S}} x_S 2^S = (g+\sum_{S\in\mathcal{S}} 2^S)\oplus\sum_{S\in\mathcal{S}} 2^S.$$

proof. Let $e = g+\sum_{S\in\mathcal{S}} 2^S$. Since $\sum_{S\in\mathcal{S}} x_S 2^S = F_{\mathcal{S}}(g)-g$, it follows from Lemma 2.2.3 that

$$e+\sum_{S\in\mathcal{S}} x_S 2^S = g+\sum_{S\in\mathcal{S}} 2^S + F_{\mathcal{S}}(g)-g = \sum_{S\in\mathcal{S}} 2^S + F_{\mathcal{S}}(g) = \sum_{S\in\mathcal{S}} 2^S + e - \sum_{S\in\mathcal{S}} e_S 2^S.$$

Hence

$$\sum_{S\in\mathcal{S}} e_S 2^S + \sum_{S\in\mathcal{S}} x_S 2^S = \sum_{S\in\mathcal{S}} 2^S.$$

This implies that there is no $S\in\mathcal{S}$ with $e_S = x_S = 1$. Hence

$$\sum_{S\in\mathcal{S}} e_S 2^S \oplus \sum_{S\in\mathcal{S}} x_S 2^S = \sum_{S\in\mathcal{S}} 2^S.$$

Therefore

$$e\oplus\sum_{S\in\mathcal{S}} 2^S = e\oplus\sum_{S\in\mathcal{S}} e_S 2^S\oplus\sum_{S\in\mathcal{S}} x_S 2^S = F_{\mathcal{S}}(g)\oplus\sum_{S\in\mathcal{S}} x_S 2^S = F_{\mathcal{S}}(g)+\sum_{S\in\mathcal{S}} x_S 2^S.$$

□

Proposition 2.2.5. *If X is a position in finite inverted Nim with height H, then*

$$X^i = (\Psi^H(X)+\delta(X^{(i)}))\oplus\delta(X^{(i)})\oplus\sigma(X^{(i)})\oplus(2^H-1).$$

proof. We may assume that $i = 0$. Let $\overline{X}^0 = X^0 \oplus (2^H - 1)$ and

$$\mathcal{S} = \{L \in [H] : (\delta(X^{(0)}))_L = 1\}.$$

Then $\delta(X^{(0)}) = \sum_{S \in \mathcal{S}} 2^S$. Let $g = \Psi^H(X)$ and $\mathcal{S}^C = \mathbb{N} \backslash \mathcal{S}$. Then

$$\begin{aligned}
g &= \sum_{L \in \mathcal{S}^C} (\overline{X}^0_L \oplus X^0_L \oplus \cdots \oplus X^{m-1}_L) 2^L + \sum_{S \in \mathcal{S}} (\overline{X}^0_S \oplus X^1_S \oplus \cdots \oplus X^{m-1}_S) 2^S - 2 \sum_{S \in \mathcal{S}} \overline{X}^0_S 2^S \\
&= \sum_{L \in \mathcal{S}^C} (\overline{X}^0_L \oplus X^1_L \oplus \cdots \oplus X^{m-1}_L) 2^L + \sum_{S \in \mathcal{S}} \overline{X}^0_S 2^S - 2 \sum_{S \in \mathcal{S}} \overline{X}^0_S 2^S \\
&= \sum_{L \in \mathcal{S}^C} (\overline{X}^0_L \oplus X^1_L \oplus \cdots \oplus X^{m-1}_L) 2^L - \sum_{S \in \mathcal{S}} \overline{X}^0_S 2^S.
\end{aligned}$$

Hence

$$g + \sum_{S \in \mathcal{S}} \overline{X}^0_S 2^S = \sum_{L \in \mathcal{S}^C} (\overline{X}^0_L \oplus X^1_L \oplus \cdots \oplus X^{m-1}_L) 2^L. \tag{2.2.7}$$

Since the right-hand side of (2.2.7) is $\mathcal{S}$-free, it follows from Lemma 2.2.2 that

$$F_{\mathcal{S}}(g) = g + \sum_{S \in \mathcal{S}} \overline{X}^0_S 2^S = \sum_{L \in \mathcal{S}^C} (\overline{X}^0_L \oplus X^1_L \oplus \cdots \oplus X^{m-1}_L) 2^L. \tag{2.2.8}$$

Note that

$$X^1 \oplus \cdots \oplus X^{m-1} = \sum_{L \in \mathcal{S}^C} (X^1_S \oplus \cdots \oplus X^{m-1}_S) 2^S. \tag{2.2.9}$$

Adding (2.2.9) to (2.2.8), we get

$$\sum_{L \in \mathcal{S}^C} \overline{X}^0_L 2^L = F_{\mathcal{S}}(g) \oplus X^1 \oplus \cdots \oplus X^{m-1}. \tag{2.2.10}$$

By adding $\sum_{S \in \mathcal{S}} \overline{X}^0_S 2^S$ to (2.2.10),

$$\overline{X}^0 = F_{\mathcal{S}}(g) \oplus X^1 \oplus \cdots \oplus X^{m-1} \oplus \sum_{S \in \mathcal{S}} \overline{X}^0_S 2^S.$$

It follows from (2.2.8) and Lemma 2.2.4 that

$$\begin{aligned}
\overline{X}^0 &= (F_{\mathcal{S}}(g) \oplus \sum_{S \in \mathcal{S}} \overline{X}^0_S 2^S) \oplus X^1 \oplus \cdots \oplus X^{m-1} \\
&= (g + \sum_{S \in \mathcal{S}} 2^S) \oplus \sum_{S \in \mathcal{S}} 2^S \oplus X^1 \oplus \cdots \oplus X^{m-1} \\
&= (g + \delta(X^{(0)})) \oplus \delta(X^{(0)}) \oplus \sigma(X^{(0)}).
\end{aligned}$$

□

Example 2.2.6. Let $X = (2,5)$. Then

$$\Psi^1(X) = 2 \oplus 5 \oplus (2^1 - 1) = 6.$$

Hence X is a position in inverted Nim with height 1, although it is not a position in finite inverted Nim.

Using Proposition 2.2.5, we can find Y^0 and Y^1 with $\Psi^1((Y^0, X^1)) = \Psi^1((X^0, Y^1)) = h$ for any $h \in \mathbb{N}$. For example, if $h = 3$, then

$$Y^0 = (3 + \pi(X^{(1)})) \oplus \pi(X^{(1)}) \oplus \sigma(X^{(1)}) \oplus (2^1 - 1) = (3+0) \oplus 0 \oplus 5 \oplus 1 = 7 > X^0,$$

$$Y^1 = (3 + \pi(X^{(2)})) \oplus \pi(X^{(2)}) \oplus \sigma(X^{(1)}) \oplus (2^1 - 1) = (3+1) \oplus 1 \oplus 2 \oplus 1 = 6 > X^1.$$

This means that $\Psi^1(X)$ is not equal to the Sprague-Grundy number of X in inverted Nim with height 1. However, it equals the Sprague-Grundy number of X in 2-saturations of inverted Nim with height 1.

2.2.3 Carries

In the proof of Theorem 2.1.3, calculation of carries is important. In this subsection, we introduce a notation on carries and give some easy results.

For $g, a \in \mathbb{N}$, let

$$\gamma(g,a) = (g+a) \oplus g \oplus a \quad \text{and} \quad \text{and} \;\; \gamma_L(g,a) = (\gamma(g,a))_L \;\; \text{for} \;\; L \in \mathbb{N}.$$

Note that $\gamma_L(g,a) = 1$ means that there is a carry in the L-th digit in the calculation of $g + a$. It is clear that $\gamma_0(g,a) = 0$ and

$$\gamma_L(g,a) = 1 \iff g_{L-1} + a_{L-1} + \gamma_{L-1}(g,a) \geqslant 2 \;\; \text{for} \;\; L \geqslant 1.$$

For example, if $g = 1 + 2 + 8$ and $a = 1 + 8$, then

$$\gamma(g,a) = (11+9) \oplus 11 \oplus 9 = 2 \oplus 4 \oplus 16.$$

Let g and h be two distinct non-negative integers. Let

$$R(g,h) = \max\{L \in \mathbb{N} : g_L \neq h_L\}.$$

For example, if $g = 1 + 8 + 16$ and $h = 4 + 16$, then $R(g,h) = 3$.

Remark 2.2.7. Let $h, g \in \mathbb{N}$ with $h < g$. Let $R = R(g,h)$ and $N = R(g+a, h+a)$. If $R < N$, then

$$1 = (g+a)_N = g_N \oplus a_N \oplus \gamma_N(g,a) = h_N \oplus a_N \oplus \gamma_N(g,a) \neq h_N \oplus a_N \oplus \gamma_N(h,a) = 0.$$

This means that $\gamma_L(g,a) = 1$ for $R < L \leqslant N$.

For $g, N \in \mathbb{N}$, let

$$g_{>N} = g^{[2]}_{>N} = \left\lfloor \frac{g}{2^{N+1}} \right\rfloor = \sum_{L>N}^{\infty} g_L 2^{L-N-1}. \tag{2.2.11}$$

Lemma 2.2.8. *If $g - h \leqslant 2^{S+1} - 1$ for some $S < R(g,h)$, then $g_L = 0$ and $h_L = 1$ for $S < L < R(g,h)$.*

proof. Let $S < L < R(g,h)$. Then $g \geqslant g_{>L}2^{L+1} + g_L 2^L$ and $h \leqslant h_{>L}2^{L+1} + h_L 2^L + 2^L - 1$. Since $L < R(g,h)$, we have $g_{>L} - h_{>L} \geqslant 1$. It follows that

$$g - h \geqslant 2^{L+1} + (g_L - h_L)2^L - (2^L - 1) = 2^L + 1 + (g_L - h_L)2^L.$$

Hence

$$2^L + 1 + (g_L - h_L)2^L \leqslant 2^{S+1} - 1.$$

This yields $g_L = 0$ and $h_L = 1$.

□

2.2.4 Explicit Formula

To prove Theorem 2.1.3, we investigate the function Ψ^H. Let

$$\Psi^H_L(X) = (\Psi^H(X))_L, \quad \sigma_L(X) = (\sigma(X))_L, \quad \text{and } \delta_L(X) = (\delta(X))_L.$$

Example 2.2.9. Let $H = 5$ and $X = (11000^{[2]}, 10101^{[2]}, 10000^{[2]})$, where $10101^{[2]} = 2^4 + 2^2 + 2^0$. Then $\Psi^5(X) = 100110^{[2]} > 0$, so X is a position in finite inverted Nim with height 5.

L	0	1	2	3	4	5
X^0_L	0	0	0	0	1	1
X^1_L	1	0	1	0	1	0
X^2_L	0	0	0	0	0	1
$\delta_L(X)$	0	1	0	1	0	0
$\sigma_L(X)$	1	0	1	0	0	0
$\Psi^5_L(X)$	0	1	1	0	0	1

Let $U = 1$. Then $X_U = (0,0,0)$. Since $\Psi^5(X) \geqslant 0$, we see that there exists $V > U$ with $\sigma_V(X) = 0$ and $X_V \neq (0,0,0)$. Indeed, $X_4 = (1,1,0)$, so $V = 4$. Moreover, $\Psi^5_L(X) = \sigma_L(X)$ for $U < L \leqslant V$. We summarize this observation in the following lemma.

Lemma 2.2.10. *Let X be a position in m-heap finite inverted Nim with height H. If $\Psi^H_N(X) = \sigma_N(X)$ or $X_N = (0,\ldots,0)$, then there exist unique $U \leqslant N$ and $V \geqslant N$ satisfying the following three conditions:*

1. $\Psi_U^H(X) = \overline{\sigma_U(X)} = 1$ *and* $X_U = (0,\ldots,0)$, *where* $\overline{\sigma_U(X)} = \sigma_U(X) \oplus 1$.
2. $\Psi_V^H(X) = \sigma_V(X) = 0$ *and* $X_V \in \mathcal{V} = \{x \in [2]^m : \sigma(x) = 0$ *and* $x \neq (0,\ldots,0)\}$.
3. $\Psi_L^H(X) = \sigma_L(X)$ *and* $X_L \notin \mathcal{V}$ *for* $U < L < V$.

proof. We first show the existence of U. If $\Psi_N^H(X) \neq \sigma_N(X)$, then $X_N = (0,\ldots,0)$, so $\Psi_U^H(X) = \overline{\sigma_U(X)} = 1$. Hence $U = N$. Suppose that $\Psi_N^H(X) = \sigma_N(X)$. Let

$$\mathcal{U} = \{U \in \mathbb{N} : U < N \text{ and } X_U = (0,\ldots,0)\}.$$

Since $\Psi_N^H(X) = \sigma_N(X)$, we have $\mathcal{U} \neq \varnothing$. Let

$$\tilde{\mathcal{U}} = \{U \in \mathcal{U} : \Psi_U^H(X) = \overline{\sigma_U(X)} = 1\}.$$

It is easy to see that $\min U \in \tilde{\mathcal{U}}$, so $\tilde{\mathcal{U}} \neq \varnothing$. Let $U = \max \tilde{\mathcal{U}}$. Then U satisfies (1).

We can similarly show the existence of V. Let

$$\mathcal{V} = \{V \in \mathbb{N} : V > N \text{ and } X_V \in \mathcal{V}\}.$$

Since $\Psi_N^H(X) = \sigma_N(X)$, we have $\mathcal{V} \neq \varnothing$. Let $V = \min \mathcal{V}$. Then V satisfies (2). Moreover, U and V satisfies (3).

□

Theorem 2.2.11. *Let X be a position in finite inverted Nim $\mathcal{I}^{m,H}$. If $m \leqslant 3$, then* $\mathrm{sg}_{\mathcal{I}^{m,H}}(X) = \Psi^H(X)$.

proof. It suffices to show when $m = 3$.

Let $g = \Psi^H(X)$ and $0 \leqslant h < g$. For $i \in \{0,1,2\}$, let

$$\sigma^i = \sigma(X^{(i)}), \quad \delta^i = \delta(X^{(i)}), \quad \text{and} \quad \gamma^i = \gamma(g, \delta^i).$$

By Proposition 2.2.5,

$$X^i = (g + \delta^i) \oplus \delta^i \oplus \sigma^i \oplus (2^H - 1). \tag{2.2.12}$$

Let

$$Y^i = (h + \delta^i) \oplus \delta^i \oplus \sigma^i \oplus (2^H - 1). \tag{2.2.13}$$

Then

$$\Psi^H(Y^0, X^1, X^2) = \Psi^H(X^0, Y^1, X^2) = \Psi^H(X^0, X^1, Y^2) = h.$$

We will show that $Y^i < X^i$ for some $i \in \{0,1,2\}$.

For $i \in \{0,1,2\}$, let

$$R^i = R(X^i, Y^i).$$

By (2.2.12) and (2.2.13),

$$R^i = R(g + \delta^i, h + \delta^i).$$

Since $h < g$, we have $h + \delta^i < g + \delta^i$. Hence

$$(h + \delta^i)_{R^i} = 0, \quad (g + \delta^i)_{R^i}.$$

Note that

$$(X^i)_{R^i} = ((g + \delta^i) \oplus \delta^i \oplus \sigma^i \oplus (2^H - 1))_{R^i} = 1 \oplus (\delta^i \oplus \sigma^i)_{R^i} \oplus 1 = (\delta^i \oplus \sigma^i)_{R^i}.$$

We deduce a contradiction assuming $(\delta^i \oplus \sigma^i)_{R^i} = 0$ for all i. Here $X^j_{R^i} = X^k_{R^i} = 1$ and $X^i_{R^i} = 0$. Hence R^i are distinct. By relabeling R^i if necessary, we may assume that $R^0 < R^1 < R^2$. Let $R = R(g,h)$. We split into three cases.

Case 1 ($R < R^1$)**.** We will show $(g + \delta^2)_{\geqslant R^1+1} = (h + \delta^2)_{\geqslant R^1+1}$, which contradicts to $(g + \delta^2)_{R^2} \neq (h + \delta^2)_{R^2}$. Since $R^1 > R$, it follows from Lemma 2.2.7 that $\gamma^1_{R^1} = 1$. Hence

$$1 = (g + \delta^1)_{R^1} = (g \oplus \delta^1 \oplus \gamma^1)_{R^1} = g_{R^1} \oplus 1.$$

This implies $g_{R^1} = 0$ and

$$g_{R^1} + \gamma^2_{R^1} + \delta^2_{R^1} = \gamma^2_{R^1} < 2.$$

Hence $\gamma^2_{R^1+1} = 0$. Therefore $(g + \delta^2)_{\geqslant R^1+1} = (h + \delta^2)_{\geqslant R^1+1}$, which is a contradiction.

Case 2 ($R > R^1$)**.** By Lemma 2.2.8,

$$g_L = 0 \quad \text{for} \quad R^0 < L < R. \tag{2.2.14}$$

We show that $g_U = 1$ for some $R^0 < U < R$. By (2.2.14), we have $g_{R^1} = 0$. Since $\sigma_{R^1}(X) = 0$, Lemma 2.2.10 implies that there exists $U < R^1$ such that $g_U = 1$ and $R^0 < U < R$. Since $X_{R^0} = (0,1,1)$, we have $U > R^0$, contrary to $g_U = 1$ and (2.2.14).

Case 3 ($R = R^1$)**.** By Lemma 2.2.8,

$$g_L = 0 \quad \text{for} \quad R^0 < L < R^1. \tag{2.2.15}$$

We also have $\gamma^2_{R^1+1} = 1$. Hence, by Lemma 2.2.7, there exists $N < R+1$ such that $g_N + \delta_N(X^{(2)}) = 2$ and

$$g_L + \delta_L(X^{(2)}) = 1 \quad \text{for} \quad N < L < R+1. \tag{2.2.16}$$

We show that $g_V + \delta^2_V = 0$ for some $N < V < R+1$. To find V, we show N satisfies the conditions in Lemma 2.2.10. If $X^2_N = 1$, then $X_N = (0,0,1)$, so $\sigma_N = 1 = \Psi^H_N(X)$. If $X^2_N = 0$, then $X_N = (0,0,0)$. Hence, by Lemma 2.2.10, there exists $V > N$ with $g_V + \delta^2_V = 0$. Since $g_V + \delta_V(X^{(2)}) = 0 + 0 = 0$, it suffices to show that $V < R+1$.

Since $\delta^2_{R^1} = \delta^3_{R^0} = 0$, it follows from (2.2.15) that $N < R^0$. By (2.2.16), $g_{R^0} = 1$. Since $X_{R^0} = (0,1,1)$, we see that $V < R^0$ by the definition of V. In particular, $V < R+1$. By (2.2.16), $1 = g_V + \delta_V(X^{(2)}) = 0 + 0 = 0$, which is a contradiction.

□

For $x \in \mathbb{N}$, let $\widehat{x} = x_{>0}$. For $X \in \mathbb{N}^m$, let $\widehat{X} = (\widehat{X^0}, \dots, \widehat{X^{m-1}})$.

Theorem 2.2.12. *Let X be a position in finite inverted Nim $\mathcal{I}^{m,H}$. If $H \leqslant 3$, then* $\mathrm{sg}_{\mathcal{I}^{m,H}}(X) = \Psi^H(X)$.

proof. If $H = 1$, then the assertion is trivial. Suppose that $H > 1$. Assume that

(A) the theorem is true for all positions in $\mathcal{I}^{m,H-1}$.

Note that $\widehat{X}$ is a position in $\mathcal{I}^{m,H-1}$ and $\Psi^{H-1}(\widehat{X}) = \widehat{g} + \delta(X_0)$, where $g = \Phi(X)$. We show that X has an option Y with $\Psi^H(Y) = h$ for any $0 \leqslant h < g$.

Suppose that $h_0 = g_0$. Then $\widehat{h} + \delta(X_0) < \widehat{g} + \delta(X_0) = \Psi^{H-1}(\widehat{X})$, so (A) implies that $\widehat{X}$ has an option Z with $\Psi^{H-1}(Z) = \widehat{h} + \delta(X_0)$. Let

$$Y = X_0 + 2Z = (X_0^0 + 2Z^0, \dots, X_0^{m-1} + 2Z^{m-1}).$$

Then Y is an option of X with $\Psi^H(Y) = h$.

Suppose that $h_0 \neq g_0$. We divide into two cases.

Suppose that the number of $i \in [m]$ with $X_0^i \neq 0$ is not equal to one. Since $\widehat{h} \leqslant \widehat{g} + \delta(X_0)$, it follows from (A) that $\widehat{X}$ has a descendant Z with $\Psi^{H-1}(Z) = \widehat{h}$ and $\mathrm{dist}(\widehat{X}, Z) \leqslant 1$. Suppose that $Z \neq \widehat{X}$. By relabeling X^i, we may assume that $Z^0 < \widehat{X}^0$. Let $Y^0 = 2Z^0$ and $Y^i = X^i$ for $i > 0$. Then $Y_0 \neq (0, \dots, 0)$ and $\Psi^1(Y_0) \neq \Psi^1(X_0) = g_0$. Hence

$$\Psi^H(Y) = 2\Psi^{H-1}(\widehat{Y}) + \Psi^1(Y_0) = 2\widehat{h} + h_0 = h.$$

Suppose that $Z = \widehat{X}$. Then

$$\widehat{\Phi}(\widehat{Y}) = \widehat{h} = \widehat{g} = \widehat{\Phi}(\widehat{X}) = \widehat{g} + \pi(X_0).$$

Hence $\delta(X_0) = 0$. This implies that the number of $i \in [m]$ with $X_0^i \neq 0$ is greater than 1. By relabeling X^i, we may assume that $X_0^0 = 1$. Let $Y^0 = 2Z^0$ and $Y^i = X^i$ for $i > 0$. Then Y is an option of X with the desired properties.

Suppose that the number of $i \in [m]$ with $X_0^i \neq 0$ is equal to one. By relabeling X^i, we may assume that $X_0 = (1, 0, \dots, 0)$. It suffices to show that $\widehat{X}$ has an option Z satisfying one of the following two conditions:

1. $\Psi^{H-1}(Z) = \widehat{h}$ and $Z^j \neq \widehat{X}^j$ for some $j \neq 0$.

2. $\Psi^{H-1}(Z) = \widehat{h} + 1$ and $Z^0 \neq \widehat{X}^0$.

Indeed, suppose that Z satisfies (1). Let $Y^j = 2Z^j$ and $Y^i = X^i$ for $j \neq i$. Then $Y_0 \neq (0,\dots,0)$, so $\Psi^H(Y) = h$. Suppose that Z satisfies (2). Let $Y^0 = 2Z^0$. Then $Y_0 = (0,\dots,0)$, so $\Psi^H(Y) = h$.

Therefore, it suffices to show that there exists Z satisfying (1) or (2) if $H \in \{2,3\}$. Since $\sigma(X) = \sigma(X^{(j)})$ and $\delta(X) = \delta(X^{(j)})$ if $X^j = 0$, we may assume that $X^i > 0$ for $i \in [m]$ by deleting j-th digit with $X^j = 0$. Similarly, since $\sigma(X) = \sigma((X^{(k)})^{(l)})$ and $\delta(X) = \delta((X^{(k)})^{(l)})$ if $X^j = X^k = X^l$ for some distinct j,k,l, we may also assume that

$$\left|\{k \in [m] : X^k = X^j\}\right| \leqslant 2 \ \text{ for } \ j \in [m].$$

Moreover, if $\widehat{X^0} = \widehat{X^i}$ for some $i > 0$, then $\widehat{X}$ has an option Z satisfying (1). Hence we may assume that $\widehat{X^0} \neq \widehat{X^i}$ for $i > 0$.

Let $H = 2$. By the above discussion, the only possibility is $X = (11^{[2]})$. Since $\Psi^2(X) = 0$, there is nothing to prove.

Let $H = 3$. By the above discussion, there are the following nine possibilities:

$$(111,100,100,10,10),(111,100,100,10),(111,100,10,10),$$

$$(101,110,110,10,10),(101,110,110,10),(101,110,10,10),$$

$$(11,110,110,100,100),(11,110,110,100),(11,110,100,100).$$

By direct computation, we see that $\widehat{X}$ has an option with desired properties for each case. For example, let $X = (111,100,100,10)$. Then $\Psi^3(X) = 10$, so we only need to show X has an option Y with $\Psi^3(Y) = 1$. Let $Y = (111,100,100,1)$. Then $\Psi^3(Y) = 1$.

□

Remark 2.2.13. If $H = 4$, then the above argument does not hold. Indeed, consider $X = (1001,1010,100,100)$. Then

$$\Psi^4(X) = 4 \oplus 4 \oplus 9 \oplus 10 \oplus (2^4 - 1) = 12.$$

However, X has no option Y with $\Psi^4(X) = 7$. Indeed,

$$Y^0 = Y^1 = (7 + \pi(X^{(2)})) \oplus \pi(X^{(2)}) \oplus \sigma(X^{(2)}) \oplus 15 = (7+0) \oplus 0 \oplus 7 \oplus 15 = 15,$$

$$Y^2 = (7 + \pi(X^{(3)})) \oplus \pi(X^{(3)}) \oplus \sigma(X^{(3)}) \oplus 15 = (7+1) \oplus 1 \oplus 10 \oplus 15 = 12,$$

$$Y^3 = (7 + \pi(X^{(4)})) \oplus \pi(X^{(4)}) \oplus \sigma(X^{(4)}) \oplus 15 = (7+2) \oplus 2 \oplus 9 \oplus 15 = 13.$$

This implies that the Sprague-Grundy number of X is not 12 in finite inverted Nim with height 4. However, it is equal to 12 in 2-saturations of finite inverted Nim with height 4.

2.3 p-saturations

2.3.1 Games with p-index k and p-saturations

p-saturations was introduced in [13]. Roughly speaking, saturation is a state reached when adding edges by a certain way cannot change the Sprague-Grundy function. For example, Nim is saturated in base 2. If $m \leqslant 3$ or $H \leqslant 3$, then finite inverted Nim $\mathcal{I}^{m,H}$ is also saturated in base 2. However, if $m > 3$ and $H > 3$, then it is not saturated in base 2.

We first explain the way of adding edges. Let $X, Y \in \mathbb{N}^m$ and $D_i = X^i - Y^i$. We consider the following condition:

$(*_p)$ $\operatorname{ord}_p\left(\sum_{i=0}^{m-1} D^i\right) = \min\left\{ \operatorname{ord}_p(D^i) : 0 \leqslant i \leqslant m-1 \right\}$,

where $\operatorname{ord}_p(x)$ is the p-adic order of x, that is,

$$\operatorname{ord}_p(x) = \begin{cases} \max\{ L \in \mathbb{N} : p^L \mid x \} & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

Using the condition $(*_p)$, we define a game $\mathcal{N}^m_{(p,k)}$ as follows. Let $\mathcal{P} = \mathbb{N}^m$ and

$$\mathcal{A}_k = \{ (X,Y) \in \mathcal{P}^2 : X^i \geqslant Y^i \text{ for } i \in [m], \quad 0 < \operatorname{dist}(X,Y) < k, \quad \text{and} \quad (*_p) \text{ is satisfied} \}$$

for $k \in \mathbb{N}$. Let $\mathcal{N}^m_{(p,k)}$ denote the game $(\mathcal{P}, \mathcal{A}_k)$. The greater k, the greater the number of edges. Note that, by definition, $\mathcal{N}^m_{(p,2)} = \mathcal{N}^m$ and $\mathcal{N}^m_{(p,k)} = \mathcal{N}^m_{(p,m+1)}$ for $k \geqslant m+1$.

Example 2.3.1. Let us consider options of $(2,2,2)$ in $\mathcal{N}^3_{(2,k)}$ for $k = 2,3,4$. In $\mathcal{N}^3_{(2,2)}$, the position $(2,2,2)$ has the following six options:

$$(0,2,2), (1,2,2), (2,0,2), (2,1,2), (2,2,0), (2,2,1).$$

In $\mathcal{N}^3_{(2,3)}$, the following six positions are also options of $(2,2,2)$:

$$(0,1,2), (0,2,1), (1,0,2), (1,2,0), (2,0,1), (2,1,0).$$

Indeed, for example, $(2,2,2) - (0,1,2) = (2,1,0)$, and so

$$\operatorname{ord}_2(2+1+0) = 0 = \min\{ \operatorname{ord}_2(2), \operatorname{ord}_2(1), \operatorname{ord}_2(0) \}.$$

In $\mathcal{N}^3_{(2,4)}$, two positions $(0,0,0)$ and $(1,1,1)$ are also options of $(2,2,2)$.

Note that, $\operatorname{sg}_{\mathcal{N}^3}((2,2,2)) = 2 \oplus_2 2 \oplus_2 2 = 2$ and the Nim sums of the above fourteen options of $(2,2,2)$ do not equal 2. This means that $\operatorname{sg}_{\mathcal{N}^3_{(2,k)}}((2,2,2)) = 2$ for $k \in \{2,3,4\}$. In fact,

$$\operatorname{sg}_{\mathcal{N}^m_{(2,k)}}(X) = \operatorname{sg}_{\mathcal{N}^m}(X) \text{ for all } X \in \mathbb{N}^m \text{ and } k \geqslant 2.$$

We now define saturations. Let Γ be an induced subgame of $\mathcal{N}^m$ and $\Gamma_{(p,k)}$ the subgame of $\mathcal{N}_{(p,k)}$ induced in $\mathcal{P}(\Gamma)$ for $k \geqslant 2$. The game $\Gamma_{(p,h)}$ is called a *p-saturation* of Γ (and said to be saturated in base p) if

$$\mathrm{sg}_{\Gamma_{(p,k)}}(X) = \mathrm{sg}_{\Gamma_{(p,h)}}(X) \qquad \text{for all } X \in \mathcal{P}(\Gamma) \text{ and } k \geqslant h. \tag{2.3.1}$$

The smallest k satisfying (2.3.1) is called the *p-saturation index* of Γ and is denoted by $\mathrm{sat}_p(\Gamma)$. Note that $\Gamma_{(p,m+1)}$ is always a p-saturation of Γ.

Example 2.3.2 ([13]). Let Γ be a p-saturation of $\mathcal{N}^m$. Then

$$\mathrm{sg}_\Gamma(X) = X^0 \oplus_p \cdots \oplus_p X^{m-1}.$$

The p-saturation index of $\mathcal{N}^m$ is $\min(p, m+1)$.

Example 2.3.3 ([13]). Let Γ be a p-saturation of Welter's game with m heaps. Then

$$\mathrm{sg}_\Gamma(X) = X^0 \oplus_p \cdots \oplus_p X^{m-1} \ominus_p \left(\bigoplus_{i<j}{}_p \ \mathfrak{N}_p(X^i - X^j) \right),$$

where $\mathfrak{N}_p(x) = x \ominus_p (x-1)$. The 2-saturation index of Welter's game is 2. However, the p-saturation index of this game is not known for $3 \leqslant p \leqslant m$.

2.3.2 Digit-Separable Sprague-Grundy Functions

To prove Theorem 2.1.8, we study $\Phi^{\xi,\alpha}$.

Lemma 2.3.4. *Let ξ and α be as in Theorem 2.1.8. If $\Phi^{\xi,\alpha}$ satisfies (2.1.8), then*

$$\{ x \in [p]^m : x^0 + \cdots + x^{m-1} < \alpha_L \} \subseteq \xi_L.$$

proof. Let $\Delta = \Gamma^{\xi_L, \alpha_L}$. Recall that $\mathcal{P}(\Delta) = \{ x \in [p]^m : I^{\xi_L}(x) = 0 \}$. Let c be the smallest sum of the components of positions in Δ:

$$c = \min \{ x^0 + \cdots + x^{m-1} : x \in \mathcal{P}(\Delta) \}.$$

It suffices to show that $c = \alpha_L$. Choose a position x in Δ such that $x^0 + \cdots + x^{m-1} = c$. Then X is an end position, so $\mathrm{sg}_\Delta(x) = 0$. By (2.1.8),

$$0 = \mathrm{sg}_\Delta(x) = x^0 \oplus_p \cdots \oplus_p x^{m-1} \ominus_p \alpha_L - I^{\xi_L}(x) = c \ominus_p \alpha_L.$$

This yields $c = \alpha_L$.

□

A position $Y \in \mathbb{N}^m$ is called a *proper descendant* of X if $Y \neq X$ and $Y^i \leqslant X^i$ for each $i \in [m]$.

Lemma 2.3.5. *Let $\Gamma = \mathcal{N}^m[\Phi^{\xi,\alpha}]$ and X be a position in $\Gamma_{(p,k)}$. If Y is a proper descendant of X with* $\mathrm{dist}(X,Y) < k$*, then*

$$\mathrm{ord}_p\left(\Phi^{\xi,\alpha}(X) - \Phi^{\xi,\alpha}(Y)\right) \geqslant \min\{\,\mathrm{ord}_p(X^i - Y^i) : i \in [m]\,\}$$

with equality if and only if Y is an option of X.

proof. Since $\mathrm{dist}(X,Y) < k$, the position Y is an option of X if and only if Y satisfies $(*_p)$. Let $N = \min\{\,\mathrm{ord}_p(X^i - Y^i) : i \in [m]\,\}$. Then $X_L = Y_L$ for $0 \leqslant L < N$. Hence

$$\mathrm{ord}_p\left(\Phi^{\xi,\alpha}(X) - \Phi^{\xi,\alpha}(Y)\right) \geqslant N$$

and

$$\left(\Phi^{\xi,\alpha}(X) - \Phi^{\xi,\alpha}(Y)\right)_N = \left(\sum_{i\in[m]} X^i - Y^i\right)_N .$$

This implies that $\mathrm{ord}_p\left(\Phi^{\xi,\alpha}(X) - \Phi^{\xi,\alpha}(Y)\right) = N$ if and only if Y is an option of X.

□

Lemma 2.3.6. *Let $x,y,z \in [p]$. If $x < y$, then $x \oplus_p z > y \oplus_p z$ if and only if $x < p - z$ and $y \geqslant p - z$.*

proof. This follows from the fact that

$$x \oplus_p z = x + y - \begin{cases} 0 & \text{if } x < p - z, \\ p & \text{if } x \geqslant p - z. \end{cases}$$

□

We now show Theorem 2.1.8 by induction on α. If $\alpha = 0$, then this is trivial. Suppose that $\alpha > 0$. Let Δ and $\widehat{\Delta}$ be subgames of $\mathcal{N}^m_{(p,m)}$ induced in $\mathcal{P}(\Gamma^{\xi,\alpha})$ and $\mathcal{P}(\Gamma^{\widehat{\xi},\widehat{\alpha}})$, respectively. Let X be a position in Δ. Then $\widehat{X}$ is a position in $\widehat{\Delta}$. Let $\Phi = \Phi^{\xi,\alpha}$ and $\widehat{\Phi} = \Phi^{\widehat{\xi},\widehat{\alpha}}$. By Lemma 2.3.5, it suffices to show that for $0 \leqslant h < \Phi(X)$, the position X has an option Y with $\Phi(Y) = h$ in Δ.

Step 1. If $h_0 = g_0$, then X has an option with the desired properties.

proof. Since $\widehat{h} + I^{\xi_0}(X_0) < \widehat{g} + I^{\xi_0}(X_0) = \widehat{\Phi}(\widehat{X})$, it follows from the induction hypothesis that $\widehat{X}$ has an option $\widehat{Y}$ with $\widehat{\Phi}(\widehat{Y}) = \widehat{h} + I^{\xi_0}(X_0)$ in $\widehat{\Delta}$. Let $Y = X_0 + p\widehat{Y}$. Then Y is an option of X with the desired properties. Indeed, Y is a proper descendant of X and

$$\begin{aligned} \mathrm{ord}_p\Big(\sum_{i\in[m]} X^i - Y^i\Big) &= \mathrm{ord}_p\Big(\sum_{i\in[m]} \widehat{X}^i - \widehat{Y}^i\Big) + 1 \\ &= \min\left\{\,\mathrm{ord}_p(\widehat{X}^i - \widehat{Y}^i) : i \in [m]\,\right\} + 1 = \min\{\,\mathrm{ord}_p(X^i - Y^i) : i \in [m]\,\}. \end{aligned}$$

Hence Y is an option of X. We also have

$$\Phi(Y) = \phi_0^{\xi,\alpha}(Y) + p\widehat{\Phi}(\widehat{Y}) = h_0 - pI^{\xi_0}(X_0) + p(\widehat{h} + I^{\xi_0}(X_0)) = h.$$

Hence Y satisfies the desired condition.

□

Suppose that $h_0 \neq g_0$. By Lemma 2.3.5, we see that if a proper descendant Y of X satisfies $\Psi^H(Y) = h$, then Y is an option of X. Let

$$\varepsilon = \begin{cases} 1 & \text{if } h_0 \geqslant p - \alpha_0 (\iff h_0 \oplus_p \alpha_0 < \alpha_0), \\ 0 & \text{if } h_0 < p - \alpha_0 (\iff h_0 \oplus_p \alpha_0 \geqslant \alpha_0). \end{cases} \tag{2.3.2}$$

Step 2. If $\widehat{h} + \varepsilon < \widehat{\Phi}(\widehat{X})$, then X has an option with the desired properties.

proof. By the induction hypothesis, $\widehat{X}$ has an option $\widehat{Y}$ with $\widehat{\Phi}(\widehat{Y}) = \widehat{h} + \varepsilon$ in $\widehat{\Delta}$. By relabeling X^i, we may assume that $\widehat{Y}^0 < \widehat{X}^0$. Let

$$Y_0 = (h_0 \oplus_p \alpha_0, 0, \dots, 0).$$

By Lemma 2.3.6, $h_0 \oplus_p a_0 < a_0$ if and only if $h_0 \geqslant p - a_0$. It follows from Lemma 2.3.4 that $I^{\xi_0}(Y_0) = \varepsilon$. Hence $\Phi(Y_0 + p\widehat{Y}) = h$ and $Y_0 + p\widehat{Y}$ is an option of X.

□

Step 3. If

$$\widehat{h} + \varepsilon \geqslant \widehat{\Phi}(\widehat{X})(= \widehat{g} + I^{\xi_0}(X_0)), \tag{2.3.3}$$

then X has an option with the desired properties.

Proof. We divide into two cases.

Case 1 ($h_0 < g_0$ and $X_0 \notin \xi_0$)**.** Since X_0 is a position in Δ^{ξ_0,α_0}, it follows from (2.1.3) that X_0 has an option Y_0 with

$$\phi^{\xi_0,\alpha_0}(Y_0) = h_0.$$

By the induction hypothesis, $\widehat{X}$ has a descendant $\widehat{Y}$ with $\widehat{\Phi}(\widehat{Y}) = \widehat{h}$. Then $Y_0 + p\widehat{Y}$ is an option of X with the desired properties.

Case 2 ($h_0 > g_0$ or $X_0 \in \xi_0$)**.** Suppose that $h_0 > g_0$. Then $\widehat{h} < \widehat{g}$. By (2.3.3), we have $\varepsilon = 1$ and $X_0 \notin \xi_0$. Hence $h_0 \oplus_p \alpha_0 < \alpha_0$ and $\widehat{h} + 1 = \widehat{g}$. Since $X^0 \notin \xi_0$, it follows from Lemma 2.3.4 that

$$X_0^0 + \cdots + X_0^{m-1} \geqslant \alpha_0 > h_0 \oplus_p \alpha_0.$$

This implies that there exists $Y_0^i \leqslant X_0^i$ for each $i \in [m]$ such that

$$Y_0^0 + \cdots + Y_0^{m-1} = h_0 \oplus_p \alpha_0.$$

By Lemma 2.3.4, we have $Y_0 \in \xi_0$. This implies that $Y_0 + p\widehat{X}$ is an option of X with the desired properties.

Suppose that $h_0 < g_0$ and $X_0 \notin P(\Delta^{\xi_0,\alpha_0})$. By (2.3.3), we have $\varepsilon = 1$. Hence $\widehat{h} = \widehat{g} = \widehat{\Phi}(\widehat{X}) - 1$. Since

$$X_0^0 + \cdots + X_0^{m-1} \geqslant g_0 \oplus_p \alpha_0 > h_0 \oplus_p \alpha_0,$$

there exists $Y_0^i \leqslant X_0^i$ for each $i \in [m]$ such that

$$Y_0^0 + \cdots + Y_0^{m-1} = h_0 \oplus_p \alpha_0.$$

Then $Y_0 + p\widehat{X}$ is an option of X with the desired properties. This completes the proof.

2.4 p-saturation Indices

We determine p-saturation indices of games including p-inverted Nim. In this section, we write $\oplus$ instead of $\oplus_p$.

Let $\mathcal{A}$ be a finite subset of $\mathbb{N}$. Let $\alpha = \sum_{L \in \mathcal{A}} p^L$ and

$$\xi_L = \begin{cases} \{ (0, \ldots, 0) \} & \text{if } L \in \mathcal{A}, \\ \varnothing & \text{if } L \notin \mathcal{A}. \end{cases}$$

Then $\phi_L^{\xi,\alpha}$ satisfies (2.1.3). Hence

$$\mathrm{sg}_{\Gamma^{\xi,\alpha}}(X) = \Phi^{\xi,\alpha}(X) = \sigma(X) \ominus a - p\delta(X),$$

where

$$\sigma(X) = X^0 \oplus \cdots \oplus X^{m-1} \qquad \text{and} \qquad \delta(X) = \sum_{L \in \mathbb{N}} I^{\xi_L}(X_L) p^L.$$

Theorem 2.4.1. *Let α and ξ be as above. Then*

$$\mathrm{sat}_p(\Gamma^{\xi,\alpha}) = \begin{cases} \min(3, m) & \textit{if } p = 2 \textit{ and } \alpha \neq 0, \\ \min(p, m) & \textit{otherwise}. \end{cases}$$

We introduce some notation. Let $\Phi(X) = \Phi^{\xi,\alpha}(X)$ and $\Phi_L(X) = (\Phi(X))_L$.

Lemma 2.4.2. *Let $\Gamma = \Gamma^{\xi,\alpha}$. Then*

$$\mathrm{sat}_p(\Gamma) \geqslant \begin{cases} \min(p+1, m+1) & \textit{if } p = 2, \\ \min(p, m+1) & \textit{if } p > 2. \end{cases}$$

proof. If $m=1$, then the assertion is clear. Suppose that $m \geqslant 2$. Since $\alpha \neq 0$, we see that $\alpha_L \neq 0$ for some $L \in \mathbb{N}$. Let N be the maximum of such L:

$$N = \max\{L \in \mathbb{N} : \alpha_L \neq 0\}.$$

We first show the assertion when $p > 2$. Let $k = \min(p, m+1)$. It is sufficient to find a position X in $\Gamma_{(p,m+1)}$ such that if Y is an option of X with $\Phi(Y) = 0$, then $\mathrm{dist}(X,Y) = k-1$. Let

$$X^i = \begin{cases} \alpha + p^{N+1} & \text{if } i = 0, \\ p^{N+1} & \text{if } 1 \leqslant i < k-1, \\ 0 & \text{if } k-1 \leqslant i \leqslant m-1. \end{cases}$$

Since $k \leqslant p$, we have $\Phi(X) = (k-1)p^{N+1}$. Hence X is a position in $\Gamma_{(p,m+1)}$. Let Y be an option of X with $\Phi(Y) = 0$. By Lemma 2.3.5,

$$\min\{\mathrm{ord}_p(X^i - Y^i) : i \in [m]\} = \mathrm{ord}_p((k-1)p^{N+1} - 0) = N+1.$$

Hence $Y_L = X_L$ for $0 \leqslant L < N+1$. This implies that $Y_{N+1} = (0, \ldots, 0)$. Therefore $\mathrm{dist}(X,Y) = k-1$.

Next, suppose that $p = 2$.

Using Example 2.2.6, we construct a position X to show $\mathrm{sat}_2(\Gamma) \geqslant 3$.

Let

$$X^0 = \alpha + 4 \cdot 2^N = (\alpha - 2^N) + 5 \cdot 2^N,$$
$$X^1 = 2 \cdot 2^N,$$
$$X^i = 0 \text{ for } i > 2,$$

and $X = (X^0, \ldots X^{m-1})$. Then

$$\Phi(X) = 6 \cdot 2^N.$$

Let

$$h = 3 \cdot 2^N$$

We define Y^i by

$$\Phi(X^0, \ldots, X^{i-1}, Y^i, X^{i+1}, \ldots, X^{m-1}) = h.$$

Then $Y^i > X^i$. Indeed, if $i > 1$, then this is trivial since $X^i = 0$. We also have

$$Y^0 = (\alpha - 2^N) + 6 \cdot 2^N$$

and

$$Y^1 = 7 \cdot 2^N.$$

Hence $Y^0 > X^0$ and $Y^1 > X^1$. Therefore $\mathrm{sat}_2(\Gamma) \geqslant 3$.

□

We now prove Theorem 2.4.1. Let $k = \max(3, p)$. It is sufficient to show $\mathrm{sat}_p(\Gamma) \leqslant k$. We show by induction on α. If $\alpha = 0$, then this is trivial. Suppose that $\alpha > 0$.

Let $\Phi(X) = \Phi^{\xi,\alpha}(X)$. Let h be an integer with $0 \leqslant h < \Phi(X)$. We show that X has an option Y such that

$$\Phi(Y) = h \text{ and } \mathrm{dist}(X,Y) < k. \tag{2.4.1}$$

Let $g = \Phi(X)$.

Step 1. If $h \equiv \Phi(X) \pmod{p}$, then X has an option Y satisfying (2.4.1).

proof. By the induction hypothesis, $\widehat{X}$ has an option $\widehat{Y}$ such that $\widehat{\Phi}(\widehat{Y}) = \widehat{h} + I^{\xi_0}(X_0)$ and $\mathrm{dist}(\widehat{X},\widehat{Y}) < k$. Then $X_0 + p\widehat{Y}$ is an option of X with the desired properties.

□

Hence we may assume that $h \not\equiv \Phi(X) \pmod{p}$.

By the induction hypothesis, if $\widehat{h} < \widehat{\Phi}(\widehat{X})$, then $\widehat{X}$ has an option $\widehat{Z}$ such that $\widehat{\Phi}(Z') = \widehat{h}$ and $\mathrm{dist}(Z',\widehat{X}) < k$. Let $Z = X_0 + p\widehat{Z}$. When $\widehat{h} = \widehat{\Phi}(\widehat{X})$, let $Z = X$.

Step 2. If one of the following three conditions holds, then X has an option Y satisfying (2.4.1).

1. $Z = X$,
2. $\xi_0 = \varnothing$, or
3. $h_0 \neq p-1$.

proof. Suppose that (1) holds. We first show that

$$h_0 < X^0 + \cdots + X^{m-1} - \alpha_0. \tag{2.4.2}$$

Since $g > h$ and $\widehat{X} = \widehat{Z}$,

$$\widehat{g} \geqslant \widehat{h} = \widehat{\Phi}(\widehat{Z}) = \widehat{\Phi}(\widehat{X}) = \widehat{g} + I^{\xi_0}(X_0).$$

Hence $X_0 \notin \xi_0$ and $\widehat{g} = \widehat{h}$. This implies that $X_0 \neq (0,\ldots,0)$ and $h_0 < g_0$, and hence

$$h_0 < g_0 = X^0 \oplus \cdots \oplus X^{m-1} \ominus \alpha_0 \leqslant X^0 + \cdots + X^{m-1} - \alpha_0.$$

We next construct an option Y of X. By (2.4.2), there exists $Y_0 \in \Omega^m$ such that

$$Y_0^0 \oplus \cdots \oplus Y_0^{m-1} \ominus \alpha_0 = h_0 \quad \text{and} \quad Y_0^i \leqslant X_0^i \text{ for each } i \in [m].$$

Since $g_0 - h_0 < p$, we can take Y_0 so that $\mathrm{dist}(X_0,Y_0) < k$. Let $Y = Y_0 + p\widehat{X}$. Then $\mathrm{dist}(X,Y) < k$. It remains to verify $\Phi(Y) = h$. Since $\Phi(Y) = \phi_0(Y_0) + p\widehat{\Phi}(\widehat{Y}) = h_0 - pI^{\xi_0}(Y_0) + p\widehat{h}$, it is sufficient to show $Y_0 \notin \xi_0$. This is trivial for $\alpha_0 = 0$. Suppose that $\alpha_0 = 1$. Since $h_0 < g_0 \leqslant p-1$, we have $Y_0 \neq (0,\ldots,0)$, and hence $Y_0 \notin \xi_0$.

Suppose that (2) or (3) holds. By (1), we may assume that $\widehat{Z} \neq \widehat{X}$. By relabeling X^i if necessary, we may also assume that $\widehat{Z}^0 < \widehat{X}^0$. Let $Y_0 = (h_0 \oplus \alpha_0 \ominus X_0^1 \ominus \cdots \ominus X_0^{m-1}, X_0^1, \ldots, X_0^{m-1})$ and $Y = Y_0 + p\widehat{Z}$. Then $\operatorname{dist}(X,Y) = \operatorname{dist}(X,Z) < k$ and $Y_i^0 \leqslant X_i^0$ for $i \in [m]$. Since (2) or (3) holds, $Y_0 \notin \xi_0$. Hence $\Phi(Y) = h$.

$\square$

By Step 2, we may assume that

$$\widehat{Z} \neq \widehat{X}, \quad \xi_0 = \{(0,\ldots,0)\}, \quad h_0 = p-1. \tag{2.4.3}$$

Step 3. If X has a descendant W such that $\widehat{\Phi}(\widehat{W}) = \widehat{h}$ and $\operatorname{dist}(X,W) \geqslant 2$, then X has an option Y satisfying (2.4.1).

proof. We may assume that $W^0 < X^0$ and $W^1 < X^1$. Let

$$Y_0 = (1, h_0 \ominus 1 \ominus X_0^2 \ominus \cdots \ominus X_0^{m-1}, X_0^2, \ldots, X_0^{m-1})$$

and $Y = Y_0 + p\widehat{W}$. Then $\operatorname{dist}(X,Y) = \operatorname{dist}(X,W) < k$. Since $Y_0 \notin \xi_0$, we have $\Phi(Y) = h$.

$\square$

By Step 3, we may assume that $\operatorname{dist}(X,Z) = 1$. Let $Z^0 < X^0$.

Step 4. If $Z_0^i \neq 0$ for some $i > 0$, then X has an option Y satisfying (2.4.1).

proof. Let $Y_0 = (\ominus Z_0^1 \ominus \cdots \ominus Z_0^{m-1}, Z_0^1, \ldots, Z_0^{m-1})$. Since $Z_0^i \neq 0$, we have $Y_0 \notin \xi_0$. Therefore $Y_0 + p\widehat{Z}$ is an option of X with the desired properties.

$\square$

By Step 4 and (2.4.3), we may assume that

$$X_0 = Z_0 = (Z_0^0, 0, \ldots, 0) \quad \text{and} \quad X_0^0 = Z_0^0 \neq 0. \tag{2.4.4}$$

Since $\widehat{Z} \neq \widehat{X}$, we have $Z_L^0 \neq X_L^0$ for some $L \in \mathbb{N}$. Let

$$K = \max\{L \in \mathbb{N} : Z_L^0 \neq X_L^0\}.$$

Then $K \geqslant 1$.

Step 5. Suppose that $Z_M^k \neq 0$ for some $1 \leqslant M < K$ and some $k > 0$. If one of the following four conditions holds, then X has an option Y satisfying (2.4.1):

1. $\xi_M = \varnothing$,
2. $Z_M^0 \neq p-1$,
3. $Z_M^k \neq 1$, or

4. $Z^l_M \neq 0$ for some $l \in [m]\backslash\{0,k\}$.

proof. Let

$$W^0 = Z^0 \oplus p^M, \quad W^k = Z^k \ominus p^M = Z^k - p^M, \tag{2.4.5}$$

and $W^i = Y^i$ for $i \in [m]\backslash\{0,k\}$. Put $W = (W^0, \ldots, W^{m-1})$. Then $\mathrm{dist}(X,W) = 2$. It is sufficient to show that W satisfies the condition of Step 3. We first show that W is a descendant of X. By (2.4.5), we have $W^k < Z^k = X^k$ Since $M < K$, we see that $W^0_K = Z^0_K < X^0_K$, and hence $W^0 < X^0$. Thus W is a descendant of X. It remains to show that $\widehat{\Phi}(\widehat{W}) = \widehat{h}$. Since one of the conditions (1) - (4) holds, $I^{\xi_M}(W_M) = I^{\xi_M}(Z_M) = 0$, and hence $\phi_M(W_M) = \phi_M(Z_M)$. Therefore $\widehat{\Phi}(\widehat{W}) = \widehat{\Phi}(\widehat{Z}) = \widehat{h}$.

□

Let

$$N = \min\{\, L \geqslant 1 : Z^0_L \neq p-1 \,\}.$$

Then $K \geqslant N \geqslant 1$ since $Z^0_K < X^0_K \leqslant p-1$.

Step 6. If there exists M such that $1 \leqslant M < N$ and the following two statements do not hold, then X has an option Y satisfying (2.4.1):

(Z1) $Z_M = (p-1, 0, \ldots, 0)$, and

(Z2) $Z_M = (p-1, 0, \ldots, 0, 1, 0, \ldots, 0)$ and $\xi_M = \{(0, \ldots, 0)\}$.

proof. By the definition of N, we have $Z^0_M = p-1$. Since (Z1) does not hold, $Z^i_M \neq 0$ for some $i > 0$. Since (Z2) does not hold, one of the three conditions (1), (3), and (4) in Step 5 holds. Hence X has an option with the desired properties.

□

Therefore we may assume that

$$Z_0 = (Z^1_0, 0, \ldots, 0), \tag{2.4.6}$$

$$Z_L = (p-1, 0, \ldots, 0) \quad \text{or} \quad (p-1, 0, \ldots, 0, 1, 0, \ldots, 0) \text{ and } \alpha_L = 1 \qquad \text{for } 1 \leqslant L < N. \tag{2.4.7}$$

Moreover, if $N < K$, then we may also assume that

$$Z_N = (Z^1_N, 0, \ldots, 0) \quad Z^1_n \neq p-1. \tag{2.4.8}$$

Let

$$Y^0_L = \begin{cases} 0 & \text{if } L = 0, \\ Z^0_L \oplus 1 & \text{if } 0 < L \leqslant N, \\ Z^0_L & \text{if } N \geqslant L. \end{cases}$$

Let $Y^i = X^i$ for $i > 0$ and $Y = (Y^0, \ldots, Y^{m-1})$. The goal is to show that Y is an option of X with $\Phi(Y) = h$.

Step 7. We have $Y^0 < X^0$. In particular, Y is an option of X.

proof. We first show that $Y^0 < X^0$, If $N < K$, then $Y_K^0 = Z_K^0 < X_K^0$, so $Y^0 < X^0$. Suppose that $N = K$. By (2.4.7),

$$Y_0^0 = Y_1^0 = \cdots = Y_{K-1}^0 = 0$$

and $Y_K^0 = Z_K^0 \oplus 1 = Z_K^0 + 1 \leqslant X_K^0$. Hence $Y^0 \leqslant X^0$. By (2.4.4), $X_0^0 \neq 0$, and hence $Y^0 < X^0$.

□

To prove $\Phi(Y) = h$, we define $\beta_L(X)$ by

$$\Phi_L(X) \oplus \beta_L(X) = \sigma_L(X) \ominus \alpha_L(X) \ominus \delta_{L-1}(X). \tag{2.4.9}$$

In other words, $\beta_L(X) = 1$ if the L-th digit is borrowed when subtracting

$$\sigma(X) \ominus \alpha - \sum I^{\xi_L}(X_L) p^{L+1},$$

and $\beta_L(X) = 0$ if otherwise. By definition, $\beta_0(X) = \beta_1(X) = 0$. For example, if $p = 2, \alpha = 7$, and $X = (2,4,4)$, then

$$\Phi(X) = 2 \oplus 4 \oplus 4 \ominus 7 - 2 = 1 \oplus 4 - 2 = 3,$$

and hence $\beta_2(X) = 1$ and $\beta_L(X) = 0$ for $L \neq 2$.

Note that $\beta_L(X) = 1$ if and only if $\sigma_{L-1}(X) \ominus \alpha_{L-1} - \delta_{L-2}(X) - \beta_{L-1}(X) < 0$. Therefore

$$\delta_L(Z) = 0 \;\text{ for }\; 0 \leqslant L \leqslant N-1 \tag{2.4.10}$$

and

$$\beta_L(Z) = 0 \;\text{ for }\; 0 \leqslant L \leqslant N+1, \tag{2.4.11}$$

since $Z_0 \neq (0,\dots,0)$ and $Z_L^0 = p-1$ for $1 \leqslant L < N$.

We give a relation on $\delta(Y)$ and $\beta(Y)$.

Step 8.

$$\delta_{L-1}(Y) + \beta_L(Y) = 1 \qquad \text{for }\; 0 < L \leqslant N. \tag{2.4.12}$$

proof. If $L = 1$, then $\delta_0(Y) = 1$ and $\beta_0(Y) = 0$, so (2.4.12) holds. Suppose that $L > 1$. We divide into two cases.

First, suppose that $\alpha_{L-1} = 0$. Since $\delta_{L-1}(Y) = 0$, it is sufficient to show that $\beta_L(Y) = 1$. By (2.4.7), $Z_{L-1}^0 = (p-1,0,\dots,0)$. Hence $Y_{L-1} = (0,\dots,0)$. This implies that

$$\sigma_{L-1}(Y) \ominus \alpha_{L-1} - \delta_{L-2}(Y) - \beta_{L-1}(Y) = 0 - 1 < 0,$$

and hence $\beta_L(Y) = 1$.

Next, suppose that $\alpha_{L-1} = 1$. There are two possibilities for Z_{L-1}^0.

Suppose that $Z^0_{L-1} = (p-1, 0, \ldots, 0)$. Since $Y_{L-1} = (0, \ldots, 0)$, we have $\delta_{L-1}(Y) = 1$. We also have

$$\sigma_{L-1}(Y) \ominus \alpha_{L-1} - \delta_{L-1}(Y) - \sigma_{L-1}(Y) = 0 \ominus 1 - 1 \geqslant 0.$$

Hence $\beta_L(Y) = 0$.

Suppose that $Z^0_{L-1} = (p-1, 0, \ldots, 0, 1, 0, \ldots, 0)$. Then $Y^0_{L-1} = (0, \ldots, 0, 1, 0, \ldots, 0)$. Hence $\delta_{L-1}(Y) = 0$ and

$$\sigma_{L-1}(Y) \ominus \alpha_{L-1} - \delta_{L-1}(Y) - \sigma_{L-1}(Y) = 1 \ominus 1 - 1 < 0.$$

Therefore $\beta_L(Y) = 1$.

□

Step 9.

$$\delta_N(Z) = \beta_{N+1}(Y) \tag{2.4.13}$$

proof. Suppose that $\delta_N(Z) = 1$. Then $Z_N = (0, \ldots, 0)$ and $\alpha_N = 1$. Hence $\sigma_N(Y) \ominus \alpha_N - \delta_{N-1}(Y) - \beta_N(Y) = 1 \ominus 1 - 1 < 0$, so $\beta_{N+1}(Y) = 1$.

Conversely, suppose that $\beta_{N+1}(Y) = 1$. Then $\sigma_N(Y) \ominus \alpha_N - \delta_{N-1}(Y) - \beta_N(Y) = \sigma_N(Y) \ominus \alpha_N - 1 < 0$. Hence $\sigma_N(Y) = \alpha_N$ and $\sigma_N(Z) = \alpha_N \ominus 1$. Thus

$$h_N = \Phi_N(Z) = \sigma_N(Z) \ominus \alpha_N \ominus \delta_{N-1}(Z) \ominus \beta_N(Z) = p-1. \tag{2.4.14}$$

We divide into two cases.

First, suppose that $N < K$. By (2.4.8), we have $Z_N = (Z^0_N, 0, \ldots, 0)$ and

$$p-1 \neq Z^0_N = \sigma(Z_N) = \alpha_N \ominus 1.$$

It follows that $\alpha_N = 1$ and $Z^0_N = 0$, and hence $\delta_N(Z) = 1$.

Next, suppose that $N = K$. Let

$$X_{\geqslant K+1} = (X^0_{\geqslant K+1}, \ldots, X^{m-1}_{\geqslant K+1})$$

and

$$\Psi = \Phi^{\xi_{\geqslant K+1}, \alpha_{\geqslant K+1}}, \quad \text{where} \quad \xi_{\geqslant K+1} = (\xi_{K+1}, \xi_{K+1}, \ldots).$$

Then

$$\Psi(X_{\geqslant K+1}) = g_{\geqslant K+1} + \delta_K(X) + \beta_{K+1}(X).$$

Since $X_{\geqslant K+1} = Z_{\geqslant K+1}$,

$$\begin{aligned} g_{\geqslant K+1} + \delta_K(X) + \beta_{K+1}(X) &= \Psi(X_{\geqslant K+1}) \\ &= \Psi(Z_{\geqslant K+1}) = h_{\geqslant K+1} + \delta_K(Z) + \beta_{K+1}(Z) = h_{\geqslant K+1} + \delta_K(Z). \end{aligned}$$

Assume that $\delta_K(Z) = 0$. Then $\delta_K(X) = \beta_{K+1}(X) = 0$ and $g_{\geqslant K+1} = h_{\geqslant K+1}$. Since $h_K = h_N = p-1$, we have $g_K = p-1$. Since $X_K^0 \neq Z_K^0$, we have $\sigma_K(X) \ominus \alpha_K \neq \sigma_K(Z) \ominus \alpha_K = p-1$. However,

$$p-1 = g_K = \Phi_K(X) = \sigma_K(X) \ominus \alpha_K \ominus \delta_{K-1}(X) \ominus \beta_K(X).$$

This implies that

$$\sigma_K(X) \ominus \alpha_K - \delta_{K-1}(X) - \beta_K(X) < 0.$$

Therefore $\beta_{K+1}(X) = 1$, which is a contradiction. Hence $\delta_K(Z) = 1$.

□

Step 10.

$$\Phi(Y) = h.$$

proof. It suffices to show $\Phi_L(Y) = h_L$ for $L \in \mathbb{N}$.

If $L = 0$, then $h_0 = p-1 = \Phi_0(Y)$.

Let $1 \leqslant L \leqslant N$. Then $h_L = \Phi_L(Z)$. By (2.4.10) and (2.4.11), $\Phi_L(Z) = \sigma_L(Z) \ominus \alpha_L = \sigma_L(Y) \ominus 1 \ominus \alpha_L$. It follows from (2.4.12) that

$$\begin{aligned}\Phi_L(Y) &= \sigma_L(Y) \ominus \alpha_L \ominus \delta_{L-1}(Y) \ominus \beta_L(Y)\\ &= \sigma_L(Y) \ominus \alpha_L \ominus 1 = \Phi_L(Z) = h_L.\end{aligned}$$

Let $L = N+1$. Then $Y_L = Z_L$. By the definition of N, we have $Z_N^0 \neq p-1$. In particular, $Y_N^0 \neq 0$. This implies that

$$\Phi_{N+1}(Y) = \sigma_{N+1}(Y) \ominus \alpha_{N+1} \ominus \delta_N(Y) \ominus \beta_{N+1}(Y) = \sigma_{N+1}(Y) \ominus \alpha_{N+1} \ominus \beta_{N+1}(Y).$$

By (2.4.11) and (2.4.13),

$$\begin{aligned}\Phi_{N+1}(Z) &= \sigma_{N+1}(Z) \ominus \alpha_{N+1} \ominus \delta_N(Z) \ominus \beta_{N+1}(Z)\\ &= \sigma_{N+1}(Y) \ominus \alpha_{N+1} \ominus \delta_N(Z) = \sigma_{N+1}(Y) \ominus \alpha_{N+1} \ominus \beta_{N+1}(Y).\end{aligned}$$

Hence $\Phi_{N+1}(Y) = \Phi_{N+1}(Z) = h_{N+1}$.

Let $L \geqslant N+2$. It is sufficient to show that $\beta_L(Y) = \beta_L(Z)$. Since $\delta_N(Y) + \beta_{N+1}(Y) = \beta_{N+1}(Y) = \delta_N(Z) = \delta_N(Z) + \beta_{N+1}(Z)$, we have $\beta_{N+2}(Y) = \beta_{N+2}(Z)$. This implies that $\beta_L(Y) = \beta_L(Z)$ for $L \geqslant N+2$. Therefore $\Phi_L(Y) = \Phi_L(Z) = h_L$. This completes the proof.

□

Appendix 2.A Designs and Their Game Distributions

Ryba found a game, called the hexad game, whose winning position set forms a Steiner system $S(5,6,12)$ (see [6, 15]). In this section, we introduce the notion of game distributions and show that the hexad game has the smallest number of positions among games whose winning set position forms an $S(5,6,12)$.

In this section, we use the following set representation of Welter's game. Let

$$P = \binom{\mathbb{N}}{m} = \{\, X \subseteq \mathbb{N} : |X| = m \,\}$$

and

$$A = \left\{ (X,Y) \in P : |X \cap Y| = m-1, \sum_{x \in X} x > \sum_{y \in Y} y \right\}.$$

Then the game (P,A) is called the *set representation of Welter's game*, and is denoted by $\overline{\mathcal{W}^m}$ (see Section 3.2.2 for details).

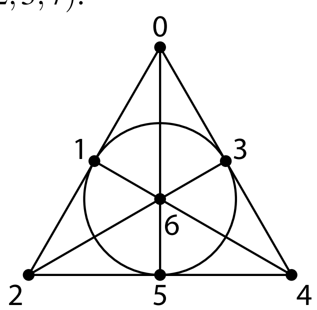
2.A.1 Designs

Definition 2.A.1. Let $t, v, k, \lambda \in \mathbb{N}$. Let $\mathcal{V} = [v] (= \{\, 0, 1, \ldots, v-1 \,\})$ and $\mathcal{B}$ be a subset of $\binom{V}{k} (= \{\, B \subseteq V : |B| = k \,\})$. The pair $(\mathcal{V}, \mathcal{B})$ is called a t-(v,k,λ) *design* if, for each $T \in \binom{V}{t}$, there exist exactly λ elements $B \in \mathcal{B}$ with $T \subseteq B$. The elements of $\mathcal{V}$ is called *points* and those of $\mathcal{B}$ *blocks*. Let D be a t-(v,k,λ) design. If $\lambda = 1$, then D is called a *Steiner system* $S(t,k,v)$.

Example 2.A.2. Some subsets of the wining position set of $\overline{\mathcal{W}^m}$ form Steiner systems. Let $\mathcal{V} = [7] = \{\, 0,1,2,3,4,5,6 \,\}$ and

$$\begin{aligned} \mathcal{B} &= \left\{ B \in \binom{\mathcal{V}}{3} : B \text{ is a winning position in } \overline{\mathcal{W}^3} \right\} \\ &= \{\, \{\,0,1,2\,\}, \{\,0,3,4\,\}, \{\,0,5,6\,\}, \{\,1,3,5\,\}, \{\,1,4,6\,\}, \{\,2,3,6\,\}, \{\,2,4,5\,\} \,\}. \end{aligned}$$

Then $(\mathcal{V}, \mathcal{B})$ is a Steiner system $S(2,3,7)$.



In general, let $\mathcal{V} = [2^N - 1]$ and

$$\begin{aligned}\mathcal{B} &= \left\{ B \in \binom{\mathcal{V}}{3} : B \text{ is a winning position in } \overline{\mathcal{W}^3} \right\} \\ &= \left\{ \{ b_0, b_1, b_2 \} \in \binom{\mathcal{V}}{3} : (b_0 + 1) \oplus_2 (b_1 + 1) \oplus_2 (b_2 + 1) = 0 \right\}.\end{aligned}$$

Then $(\mathcal{V}, \mathcal{B})$ is a Steiner system $S(2, 3, 2^N - 1)$. This Steiner system is called the *projective Steiner triple system*.

Example 2.A.3 (Hexad game)**.** Let $\mathcal{H}$ be the subgame of Welter's game $\overline{\mathcal{W}^6}$ induced in

$$\left\{ X \in \binom{[12]}{6} : \sum_{x \in X} x \geqslant 21 \right\}.$$

This game is called the *hexad game*. As we have mentioned, Ryba found that the winning position set of this game forms a Steiner system $S(5, 6, 12)$.

2.A.2 Game Distributions

We construct games whose winning position set forms a Steiner system and generalize the frequency distribution that was used to construct inverted Nim in Section 2.2.

Let D be a Steiner system $S(k-1, k, v)$ and $\mathcal{B}$ be the block set of D. Let Γ^D denote the maximum induced subgame of $\overline{\mathcal{W}^k}$ with $\mathcal{P}(\Gamma^D) \subseteq \binom{[v]}{k}$ and $W(\Gamma^D) = \mathcal{B}$, where $W(\Gamma^D)$ is the winning position set of Γ^D. Then we can show that Γ^D is the subgame of $\overline{\mathcal{W}^k}$ induced in

$$\begin{aligned}&\mathcal{B} \cup \left\{ X \in \binom{[v]}{k} : (B, X) \in A(\overline{\mathcal{W}^m}) \text{ for some } B \in \mathcal{B} \right\} \\ &= \mathcal{B} \cup \bigcup_{B \in \mathcal{B}} \mathcal{U}(B),\end{aligned}$$

where

$$\mathcal{U}(B) = \left\{ X \in \binom{[v]}{k} : |B \cap X| = k - 1 \text{ and } \sum_{x \in X} x > \sum_{b \in B} b \right\}.$$

Let

$$\mathcal{F}^D = \{ \Gamma^{\sigma(D)} : \sigma \in \mathrm{Sym}([v]) \},$$

where $\sigma(D) = ([v], \sigma(\mathcal{B}))$ (see the example below). We will consider the frequency distribution of $|P(\Gamma)|$ for $\Gamma \in \mathcal{F}^D$. This distribution is called the *game distribution* of D. Let d_n^D denote the number of Γ in $\mathcal{F}^D$ that has exactly n positions:

$$d_n^D = \left| \{ \Gamma \in \mathcal{F}^D : |P(\Gamma)| = n \} \right|.$$

Example 2.A.4 ($S(1,2,2k)$)**.** Let $v = 4, \mathcal{B} = \{\{0,1\},\{2,3\}\}$, and $D = ([4],\mathcal{B})$. Then D is a Steiner system $S(1,2,4)$. We see that

$$\mathcal{P}(\Gamma^D) = \{\{0,1\},\{2,3\}\} \cup \{\{0,2\},\{0,3\},\{1,2\},\{1,3\}\} = \binom{[4]}{2}.$$

Hence the game Γ^D has six positions. There are two different $S(1,2,4)$, that is, $(0\ 2)(D)$ and $(0\ 3)(D)$. We have

$$\mathcal{P}(\Gamma^{(0\ 2)(D)}) = \{\{0,3\},\{1,2\}\} \cup \{\{1,3\},\{2,3\}\}$$

and

$$\mathcal{P}(\Gamma^{(0\ 3)(D)}) = \{\{0,2\},\{1,3\}\} \cup \{\{1,2\},\{0,3\},\{2,3\}\}.$$

Hence they have four and five positions, respectively. Therefore the game distribution of D is as shown in the following table.

4	5	6
1	1	1

Hence $d_4^D = d_5^D = d_6^D = 1$ and $d_n^D = 0$ for $n \neq 4,5,6$.

In general, let D be an $S(1,2,2k)$. Then we can calculate the moment generating function of the game distribution of D as follows.

$$\frac{1}{|\mathcal{F}^D|} \sum_{n \in \mathbb{N}} d_n^D q^n = \frac{q^2}{(2k-1)!!} \prod_{i=1}^{k} \frac{1-q^{2k-1}}{1-q},$$

where

$$(2k-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1).$$

Example 2.A.5 ($S(2,3,7)$)**.** Let D be the projective Steiner triple system $S(2,3,7)$. The game distribution of D is

14	15	16	17	18	19	20	21
1	3	5	6	6	5	3	1

Note that this distribution is symmetric and the game $\Gamma \in \mathcal{F}^D$ with $|P(\Gamma)| = 14$ is related to inverted Nim.

Example 2.A.6 ($S(2,3,9)$)**.** The game distribution of an $S(2,3,9)$ is

68	69	70	71	72	73	74	75	76	77	78	79	80
1	6	16	36	77	94	115	129	131	104	74	39	17

Theorem 2.A.7. *Let D be an $S(5,6,12)$. The game distribution of D is*

905	906	907	908	909	910	911	912	913	914	915	916
1	10	42	150	351	650	1012	1237	939	532	115	1

Let Γ be the game in $\mathcal{F}^D$ with $|P(\Gamma)| = 905$. Then Γ is the hexad game.

3 p-saturations of Welter's Game and the Irreducible Representations of Symmetric Groups

We establish a relation between the Sprague-Grundy function sg of p-saturations of Welter's game and the degrees of the ordinary irreducible representations of symmetric groups. In these games, a position can be regarded as a partition λ. Let ρ^{λ} be the irreducible representation of the symmetric group $\mathrm{Sym}(|\lambda|)$ corresponding to λ. For every prime p, we show the following results: (1) $\mathrm{sg}(\lambda) \leqslant |\lambda|$ with equality if and only if the degree of ρ^{λ} is prime to p; (2) the restriction of ρ^{λ} to $\mathrm{Sym}(\mathrm{sg}(\lambda))$ has an irreducible component with degree prime to p. Further, for every integer p greater than 1, we present an explicit formula for $\mathrm{sg}(\lambda)$.

It should be noted that the notation in this chapter is slightly different from that in the previous chapters. For example, for $X \in \mathbb{N}^m$, the i-th component of X will be denoted by x^i instead of X^i.

3.1 Introduction

Sato [28] conjectured that Welter's game is related to representations of symmetric groups and classical groups. In support of this conjecture, he pointed out that the Sprague-Grundy function of this game can be expressed in a form similar to the hook-length formula. In this chapter, we introduce p-saturations of Welter's game and establish a relation between the Sprague-Grundy function of these games and the degrees of the irreducible representations of symmetric groups. Moreover, we present an explicit formula for this function and a theorem on these degrees.

3.1.1 Welter's game

Welter's game is played with coins or a Young diagram. We review known results on the Sprague-Grundy function of Welter's game. We also give the definitions of games and their Sprague-Grundy functions at the end of this subsection.

Welter's game is played with a finite number of coins. These coins are on a semi-infinite strip of squares numbered $0, 1, 2, \ldots$ with no two coins on the same square. See Figure 3.1. This game has two players. They alternately move a coin to an empty square with a lower number. The first player that is not able to move loses. We now consider the position X when the coins are on the squares numbered $x^1, x^2, \ldots, x^m$. Welter [30] and Sato [25–27] independently showed that its Sprague-Grundy number $\mathrm{sg}(X)$ can be expressed as

$$\mathrm{sg}(X) = x^1 \oplus_2 \cdots \oplus_2 x^m \oplus_2 \left(\bigoplus_{i<j}{}_2 \; \mathfrak{N}_2(x^i - x^j) \right), \tag{3.1.1}$$

where $\oplus_2$ is binary addition without carry and $\mathfrak{N}_2(x) = x \oplus_2 (x-1)$.

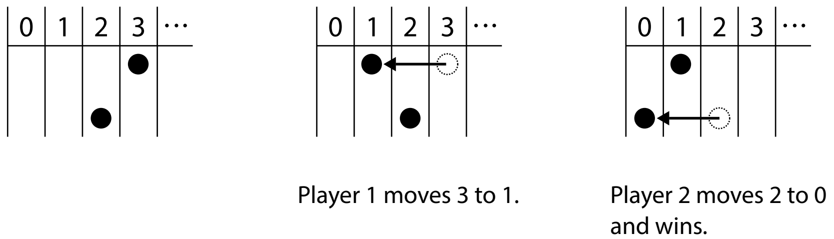


Figure 3.1: An example of Welter's game.

Welter's game can also be played with a Young diagram [25]. Let σ be the permutation of $\{1, 2, \ldots, m\}$ such that $x^{\sigma(1)} > x^{\sigma(2)} > \cdots > x^{\sigma(m)}$. Let $\lambda(X)$ be the partition $(x^{\sigma(1)} - m + 1, x^{\sigma(2)} - m + 2, \ldots, x^{\sigma(m)})$. For example, if $m = 2$ and $(x^1, x^2) = (1, 2)$, then $\lambda(X) = (2-1, 1-0) = (1, 1)$. We identify $\lambda(X)$ with its Young diagram

$$\left\{ (i, j) \in \mathbb{Z}^2 : 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant x^{\sigma(i)} - m + i \right\}.$$

As a result, moving a coin corresponds to removing a hook. From this viewpoint, Sato [25–27] found that (3.1.1) can be written as the following form similar to the hook-length formula:

$$\mathrm{sg}(X) = \sum_{L \in \mathbb{N}} \overline{w}_L(X) 2^L = \bigoplus_{(i,j) \in \lambda(X)}{}_2 \; \mathfrak{N}_2\left(\left| H_{i,j}(X) \right| \right), \tag{3.1.2}$$

where $H_{i,j}(X)$ is the hook

$$H_{i,j}(X) = \left\{ (i', j') \in \lambda(X) : (i' \geqslant i \text{ and } j' = j) \text{ or } (i' = i \text{ and } j' \geqslant j) \right\}$$

and $\overline{w}_L(X)$ is the remainder of 2^L-weight (the number of $H_{i,j}(X)$ whose size is divisible by 2^L) of $\lambda(X)$ divided by 2. In this context, Kawanaka [16] found a family of games that includes Welter's game. This family is deeply related to d-complete posets, which were defined by Proctor [22, 23].

Let us represent Welter's game as a digraph. Note that the position X can be described by the m-tuple $(x^1, \ldots, x^m) \in \mathbb{N}^m$, where $\mathbb{N}$ is the set of all non-negative integers. [1] Let

$$\mathcal{P} = \left\{ (x^1, \ldots, x^m) \in \mathbb{N}^m : x^i \neq x^j \ \text{ for } \ 1 \leqslant i < j \leqslant m \right\}$$

[1]The position X can also be represented by the set $\{x^1, \ldots, x^m\}$. We will use this representation in latter sections.

and

$$\mathcal{A} = \left\{ (X,Y) \in \mathcal{P}^2 : x^i \geqslant y^i \text{ for } 1 \leqslant i \leqslant m \text{ and } \operatorname{dist}(X,Y) = 1 \right\},$$

where $X = (x^1, \ldots, x^m), Y = (y^1, \ldots, y^m)$, and $\operatorname{dist}(X,Y)$ is the Hamming distance between X and Y, that is, $\operatorname{dist}(X,Y) = |\{ i \in \{1, \ldots, m\} : x^i \neq y^i \}|$. The digraph $(\mathcal{P}, \mathcal{A})$ is called *Welter's game with m coins* and is denoted by $\mathcal{W}^m$.

3.1.2 p-saturations

Let p be an integer greater than 1. As we have seen, we can write the Sprague-Grundy function of Welter's game using arithmetic modulo 2. In this subsection, we present a variant of Welter's game whose Sprague-Grundy function can be expressed using arithmetic modulo p.

To state our goal precisely, we introduce some notation. Let X be a position in Welter's game, and let L be a non-negative integer. A hook in $\lambda(X)$ is called a *(p^L)-hook* if its length (size) is divisible by p^L. The number of (p^L)-hooks in $\lambda(X)$ is called the *p^L-weight* of $\lambda(X)$ and is denoted by $w_L(X)$. Let $\overline{w}_L(X)$ denote the remainder of $w_L(X)$ divided by p. We define $w(X) = w^{(p)}(X) = (w_L(X))_{L \in \mathbb{N}}$ and

$$\overline{w}(X) = \overline{w}^{(p)}(X) = \sum_{L \in \mathbb{N}} \overline{w}_L(X) p^L. \tag{3.1.3}$$

We will construct a game whose Sprague-Grundy function is equal to $\overline{w}$. To this end, we need to allow moving multiple coins (removing multiple hooks) with restrictions in one move. Let us give an example.

Example 3.1.1. Let $p = 3$. Let Γ be a game with $\mathcal{P}(\Gamma) = \mathcal{P}(\mathcal{W}^2)$. Assume that for each position X in Γ, the Sprague-Grundy number of X is equal to $\overline{w}^{(3)}(X)$. Let us examine the structure of Γ.

Figure 3.2 shows $\overline{w}^{(3)}(X)$ for some positions in Γ. It is easy to verify that if $|\lambda(X)| \leqslant 3$, then $\overline{w}^{(3)}(X) = |\lambda(X)| = \overline{w}^{(2)}(X)$. Let us consider positions X with $|\lambda(X)| = 4$. First, let $X = (2,3)$. Then, in ordinary Welter's game, X has an option Y with $\overline{w}^{(3)}(Y) = h$ for each $h \in \{1,2,3\}$, but it has no option Y with $\overline{w}^{(3)}(Y) = 0$. Since $\operatorname{sg}_\Gamma(X) = \overline{w}^{(3)}(X) = 4$, it follows from the definition of Sprague-Grundy functions that X must have an option Y with $\overline{w}^{(3)}(Y) = 0$ in Γ. Thus $(X,(0,1)) \in \mathcal{A}(\Gamma)$ or $(X,(1,0)) \in \mathcal{A}(\Gamma)$. In other words, we need to allow moving two coins in one move. Next, let $X = (1,4)$. Since $\operatorname{sg}_\Gamma(X) = \overline{w}^{(3)}(X) = 1$, this position cannot have an option Y with $\overline{w}^{(3)}(Y) = 1$ in Γ. In particular, $(X,(0,2)) \notin \mathcal{A}(\Gamma)$. Hence we must restrict some moves.

Based on the above example, we introduce a game called $\mathcal{W}^m$ *with p-index k*, where k is a positive integer. This game comes from Moore's Nim_k (Nim with index k) [18] and Flanigan's

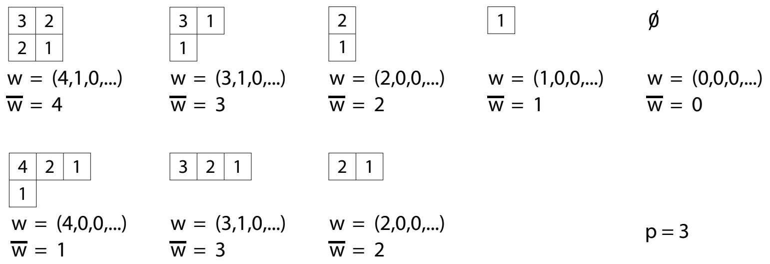


Figure 3.2: $w(X)$ and $\overline{w}(X)$ for some positions. Positions are represented by the corresponding Young diagrams.

Rim_k.[2] Let $\mathcal{D}^m$ be the set of all m-tuples $(d^1,\ldots,d^m) \in \mathbb{N}^m$ such that

$$\operatorname{ord}\left(\sum_{i=1}^{m} d^i\right) = \min\left\{\operatorname{ord}(d^i) : 1 \leqslant i \leqslant m\right\}, \tag{3.1.4}$$

where $\operatorname{ord}(x)$ is the p-adic order of x, that is,

$$\operatorname{ord}(x) = \begin{cases} \max\{L \in \mathbb{N} : p^L \mid x\} & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}$$

Let $\mathcal{P} = \mathcal{P}(\mathcal{W}^m)$ and

$$\mathcal{A}_k = \{(X,Y) \in \mathcal{P}^2 : X - Y \in \mathcal{D}^m,\ 0 < \operatorname{dist}(X,Y) < k\}. \tag{3.1.5}$$

The game $(\mathcal{P},\mathcal{A}_k)$ is called $\mathcal{W}^m$ *with p-index k* and is denoted by $\mathcal{W}^m_{(p,k)}$.

Note that $\mathcal{W}^m_{(p,2)} = \mathcal{W}^m$. In fact, for every $k \geqslant 2$, the game $\mathcal{W}^m_{(2,k)}$ has the same Sprague-Grundy function as the ordinary Welter's game $\mathcal{W}^m_{(2,2)}$ (see Lemma 3.3.5). In contrast, the Sprague-Grundy functions of $\mathcal{W}^2_{(3,3)}$ and $\mathcal{W}^2_{(3,2)}$ are different as we will see in the next example.

Example 3.1.2. Let us consider Example 3.1.1 again. Let $\Gamma = \mathcal{W}^2_{(3,3)}$. We verify that $(1,0)$ and $(0,1)$ are options of $(2,3)$ in Γ. This implies that, $\mathrm{sg}_\Gamma((2,3)) = 4 = \overline{w}^{(3)}((2,3))$. We also show that $(0,2)$ is not an option of $(1,4)$ in Γ.

Note that it is easy to show that $\mathrm{sg}_\Gamma(X) = \overline{w}^{(3)}(X)$ for $X \in \mathcal{P}(\Gamma)$ with $|\lambda(X)| \leqslant 3$. Let $X = (2,3)$. Since $X - (1,0) = (1,3)$ and $X - (0,1) = (2,2)$,

$$\operatorname{ord}(1+3) = 0 = \min\{\operatorname{ord}(1),\operatorname{ord}(3)\} \ \text{ and } \ \operatorname{ord}(2+2) = 0 = \min\{\operatorname{ord}(2),\operatorname{ord}(2)\}.$$

[2] Players can move at most $k-1$ coins in Nim_k and Rim_k. The latter is devised by James A. Flanigan in an unpublished paper entitled "NIM, TRIM and RIM". In Section 3.3, we show that Rim_p and p-saturations of Nim have the same Sprague-Grundy function.

This implies that $(1,0)$ and $(0,1)$ are options of X in Γ. In particular, $\mathrm{sg}_\Gamma(X) = 4 = \overline{w}^{(3)}(X)$. Since $\overline{w}^{(2)}(X) = 0$, the Sprague-Grundy functions of $\mathcal{W}^2_{(3,3)}$ and $\mathcal{W}^2_{(3,2)}$ are different. Next, let $X = (1,4)$. In Γ, the position $(0,2)$ is not an option of X. Indeed, $X - (0,2) = (1,2)$, so

$$\mathrm{ord}(1+2) = 1 \neq 0 = \min\{\mathrm{ord}(1), \mathrm{ord}(2)\}.$$

Hence $\mathrm{sg}_\Gamma(X) = 1 = \overline{w}^{(3)}(X)$.

We now define p-saturations. The game $\mathcal{W}^m_{(p,k)}$ is called a *p-saturation* of $\mathcal{W}^m$ if it has the same Sprague-Grundy function as $\mathcal{W}^m_{(p,m+1)}$. The smallest such k is called the *p-saturation index* of $\mathcal{W}^m$ and is denoted by $\mathrm{sat}_p(\mathcal{W}^m)$. By definition, if $j \geqslant \mathrm{sat}_p(\mathcal{W}^m)$, then $\mathcal{W}^m_{(p,j)}$ also has the same Sprague-Grundy function as $\mathcal{W}^m_{(p,m+1)}$. As we have mentioned above, $\mathrm{sat}_2(\mathcal{W}^m) = 2$ for every positive integer m. In general, $\mathrm{sat}_p(\mathcal{W}^m) \geqslant \min(p, m+1)$, but we do not know its exact value for $3 \leqslant p \leqslant m$ (see Remark 3.3.6).

3.1.3 Main Results

We first present an explicit formula for the Sprague-Grundy function of p-saturations of Welter's game. We begin by introducing some definitions. Let X be a position in Welter's game, and let L be a non-negative integer. A hook in $\lambda(X)$ is called a *(p^L)-hook* if its length (size) is divisible by p^L. The number of (p^L)-hooks in $\lambda(X)$ is called the *p^L-weight* of $\lambda(X)$ and is denoted by $w_L(X)$. Let $\overline{w}_L(X)$ denote the remainder of $w_L(X)$ divided by p. We define $w(X) = (w_L(X))_{L\in\mathbb{N}}$ and

$$\overline{w}(X) = \sum_{L\in\mathbb{N}} \overline{w}_L(X) p^L. \tag{3.1.6}$$

Let $\oplus_p$ and $\ominus_p$ be p-ary addition and subtraction without carry, respectively. For each $x \in \mathbb{Z}$, let $\mathfrak{N}_p(x) = x \ominus_p (x-1) = \sum_{L=0}^{\mathrm{ord}(x)} p^L$.

Theorem 3.1.3. *Let X be a position $(x^1, \ldots, x^m)$ in a p-saturation of $\mathcal{W}^m$. Then the following two assertions hold:*

1. *There exists a position Y such that $\lambda(Y) \subseteq \lambda(X)$ and $|\lambda(Y)| = \overline{w}(Y) = \overline{w}(X)$.*

2. *The Sprague-Grundy number* $\mathrm{sg}(X)$ *of X is expressed as follows:*

$$\begin{aligned} \mathrm{sg}(X) &= \overline{w}(X) \\ &= \bigoplus_{(i,j)\in\lambda(X)}{}_p \; \mathfrak{N}_p\left(\left|H_{i,j}(X)\right|\right) \\ &= x^1 \oplus_p \cdots \oplus_p x^m \ominus_p \left(\bigoplus_{i<j}{}_p \; \mathfrak{N}_p(x^i - x^j) \right). \end{aligned} \tag{3.1.7}$$

As we will see in Section 3.3, the first assertion of Theorem 3.1.3 is the key ingredient of the proof of the second one.

Suppose that p is a prime. We next establish a relation between the Sprague-Grundy function of p-saturations of $\mathcal{W}^m$ and the degrees of the irreducible representations of symmetric groups. Let X be a position in a p-saturation of $\mathcal{W}^m$ and let ρ^X be the ordinary irreducible representation of the symmetric group $\mathrm{Sym}(|\lambda(X)|)$ corresponding to $\lambda(X)$. By Macdonald's result [17], we see that the degree of ρ^X is prime to p if and only if $\overline{w}(X) = |\lambda(X)|$. From Theorem 3.1.3 we obtain the following corollary.

Corollary 3.1.4. *Let X be a position in a p-saturation of Welter's game. If p is a prime, then the following assertions hold:*

1. $\mathrm{sg}(X) \leqslant |\lambda(X)|$ *with equality if and only if the degree of ρ^X is prime to p.*
2. *The restriction of ρ^X to* $\mathrm{Sym}(\mathrm{sg}(X))$ *has an irreducible component with degree prime to p.*

Example 3.1.5. Let $p = 2$ and X be a position $(2,4,6)$ in a 2-saturation of Welter's game. We first calculate the degree of ρ^X and the Sprague-Grundy number of X. Since the multiset of hook-lengths in $\lambda(X)$ is $\{1,1,1,2,3,3,4,5,6\}$ (see Figure 3.3),

$$\deg(\rho^X) = \frac{9!}{6\cdot 5\cdot 4\cdot 3^2\cdot 2} = 168$$

by the hook-length formula [10]. We also have

$$w(X) = (w_0(X), w_1(X), \ldots) = (9,3,1,0,\ldots).$$

Hence $(\overline{w}_0(X), \overline{w}_1(X), \ldots) = (1,1,1,0,\ldots)$. Thus $\mathrm{sg}(X) = \overline{w}(X) = 1+2+4 = 7$.

Corollary 3.1.4 asserts that the restriction $\rho^X|_{\mathrm{Sym}(7)}$ has an irreducible component with odd degree. Indeed, by the branching rule (see, for example, [24]),

$$\rho^X|_{\mathrm{Sym}(7)} = 2\rho^{(2,3,5)} \oplus 2\rho^{(1,4,5)} \oplus 2\rho^{(1,3,6)} \oplus \rho^{(0,4,6)}.$$

We find that the degrees of $\rho^{(2,3,5)}, \rho^{(1,4,5)}$, and $\rho^{(1,3,6)}$ are odd.

By the way, $(2,4,5), (2,3,6)$, and $(1,4,6)$ can be obtained by decreasing one entry in X by 1. Note that the 2^2-weights of $(2,4,5)$ and $(1,4,6)$ are greater than that of X. In Section 3.4, these options will be called 2^*-*options* of X. They will play a key role in the proof of Theorem 3.1.3.

Corollary 3.1.4 suggests the following problem.

Problem 2. Let ρ be an irreducible representation of $\mathrm{Sym}(n)$. What is the greatest integer $\mathrm{msg}(\rho)$ such that the restriction of ρ to $\mathrm{Sym}(\mathrm{msg}(\rho))$ has an irreducible component with degree prime to p?

Note that Corollary 3.1.4 yields $\mathrm{msg}(\rho^X) \geqslant \mathrm{sg}(X)$, where X is a position in a p-saturation of Welter's game. This bound is improved in Remark 3.3.8.

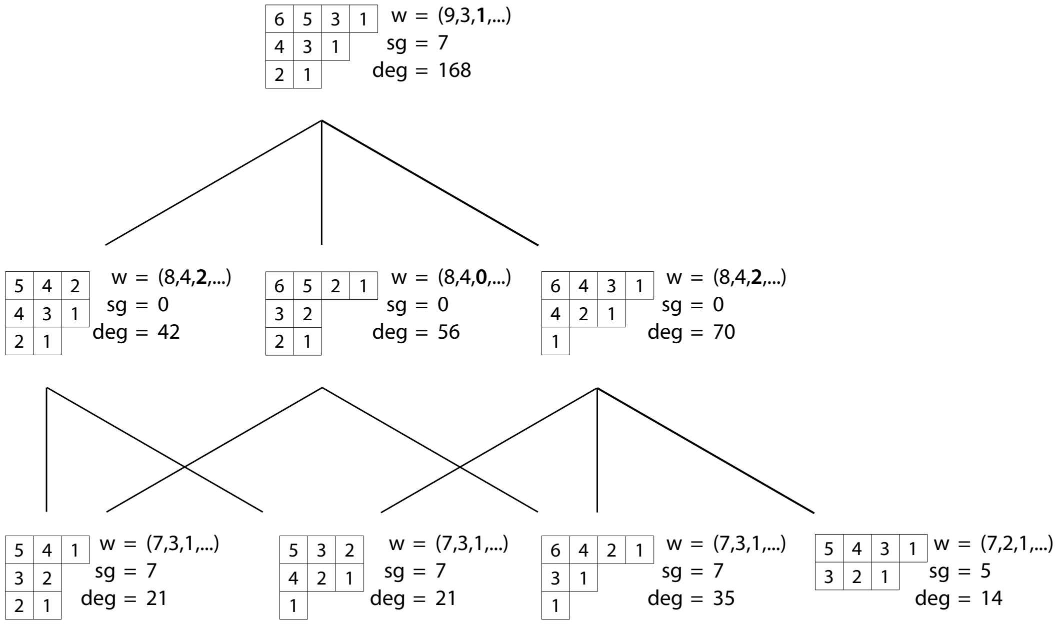


Figure 3.3: The degrees of $\rho^{(2,3,5)}, \rho^{(1,4,5)}$, and $\rho^{(1,3,6)}$ are odd. The 2^2-weights of $(2,4,5)$ and $(1,4,6)$ are 2, which is greater than that of $(2,4,6)$.

3.1.4 Organization

This chapter is organized as follows. In Section 3.2, we introduce a notation and recall the concepts of impartial games and p-core towers. In Section 3.3, we reduce Theorem 3.1.3 to two assertions (A1) and (A2). Section 3.4 contains the definition and basic properties of p^*-options. Using them, we show (A1) in this section. In Section 3.5, we prove (A2).

3.2 Preliminaries

3.2.1 Notation

Throughout this paper, p is an integer greater than 1. We write $\oplus$ instead of $\oplus_p$. We regard $\mathbb{Z}/p^L\mathbb{Z}$ as $\{0,1,\ldots,p^L-1\}$ for $L \in \mathbb{N}$. Let Ω denote $\mathbb{Z}/p\mathbb{Z}$. In this paper, we will frequently use *p-core towers*. For this reason, the following notation is useful.

For a non-negative integer x, let $x_L^{(p)}$ denote the L-th digit in the p-adic expansion of x, that is, $x = \sum_{L\in\mathbb{N}} x_L^{(p)} p^L$ and $x_L^{(p)} \in \Omega$. We write x_L instead of $x_L^{(p)}$ when no confusion can arise.

We identify $x \in \mathbb{N}$ with the infinite sequence $(x_0, x_1, \ldots) \in \Omega^{\mathbb{N}}$. In this notation, for $x, y \in \mathbb{N}$,

$$x \oplus y = (x_0 \oplus y_0, x_1 \oplus y_1, \ldots) = (x_0 + y_0, x_1 + y_1, \ldots).$$

Let $x_{<L}$ denote the residue of x modulo p^L. We also identify $x_{<L} \in \mathbb{Z}/p^L\mathbb{Z}$ with the finite sequence $(x_0, x_1, \ldots, x_{L-1}) \in \Omega^L$. Let $x_{\geqslant L}$ denote the quotient of x divided by p^L, that is, $x_{\geqslant L} = (x_L, x_{L+1}, \ldots)$.

Let $L, N \in \mathbb{N}$. Let $R \in \Omega^L$ and $S \in \Omega^N$. We use (R, S) to denote the concatenation of R and S, that is,

$$(R,S) = (R_0, \ldots, R_{L-1}, S_0, \ldots, S_{N-1}) \in \Omega^{L+N}.$$

For $x \in \mathbb{N}$ and $* \in \{+, -, \oplus, \ominus\}$, we define $R * x$ by

$$R * x = R * x_{<L} \in \Omega^L,$$

where

$$R \oplus x_{<L} = (R_0 \oplus x_0, \ldots, R_{L-1} \oplus x_{L-1}) \quad \text{and} \quad R \ominus x_{<L} = (R_0 \ominus x_0, \ldots, R_{L-1} \ominus x_{L-1}).$$

For example, $(0,0) \ominus 1 = (p-1, 0)$, while $(0,0) - 1 = (p-1, p-1)$ and $(0,0) \ominus p^2 = (0,0)$.

3.2.2 Games

We recall the concept of impartial games. See [1, 5] for details. We also give the definition of p-saturations of Nim and some remarks on Welter's game.

Let Γ be a game. For $X \in \mathcal{P}(\Gamma)$, let $\lg(X)$ denote the maximum length of a path from X. By definition, $\lg(X)$ is finite for each position X in Γ. By an easy inductive argument, $\mathrm{sg}(X) \leqslant \lg(X)$. For example, in Welter's game, $\mathrm{sg}(X) \leqslant \lg(X) = |\lambda(X)|$. Let X and Y be two positions in Γ. If there exists a path from X to Y, then Y is called a *descendant* of X. For example, in Welter's game, Y is a descendant of X if and only if $\lambda(Y) \subseteq \lambda(X)$. A descendant Y of X is said to be *proper* if $Y \neq X$.

We give a characterization of Sprague-Grundy functions. Let σ be a function from $\mathcal{P}(\Gamma)$ to $\mathbb{N}$. If σ satisfies the following two conditions, then σ is the Sprague-Grundy function of Γ.

(SG1) X has no option Y with $\sigma(Y) = \sigma(X)$.

(SG2) X has an option Y with $\sigma(Y) = h$ for $0 \leqslant h < \sigma(X)$.

Example 3.2.1 (Nim and Welter's game)**.** Let $\mathcal{P} = \mathbb{N}^m$ and

$$\mathcal{A} = \left\{ (X,Y) \in \mathcal{P}^2 : x^i \geqslant y^i \ \text{ for } \ 1 \leqslant i \leqslant m \ \text{ and } \ \mathrm{dist}(X,Y) = 1 \right\},$$

where $X = (x^1, \ldots, x^m)$ and $Y = (y^1, \ldots, y^m)$. The game $(\mathcal{P}, \mathcal{A})$ is called *Nim with m-coins* and is denoted by $\mathcal{N}^m$. See Figure 3.4. Nim was first analyzed by Bouton [2]. The following explicit formula for the Sprague-Grundy function of Nim was obtained by Sprague [29] and Grundy [12] independently:

$$\mathrm{sg}(X) = x^1 \oplus_2 \cdots \oplus_2 x^m. \tag{3.2.1}$$

Note that Welter's game $\mathcal{W}^m$ is the subgraph of $\mathcal{N}^m$ induced in $\mathcal{P}(\mathcal{W}^m)$. In other words, Welter's game is Nim with the restriction that the coins are on distinct squares.

Example 3.2.2 (Games with p-index k)**.** In view of Example 3.2.1, we can generalize Welter's game with p-index k. Let $\mathcal{P}$ be a subset of $\mathbb{N}^m$. Let Γ be the subgraph of $\mathcal{N}^m$ induced in $\mathcal{P}$. For each positive integer k, let

$$\mathcal{A}_k = \{\, (X,Y) \in \mathcal{P}^2 : X - Y \in \mathcal{D}^m,\ 0 < \mathrm{dist}(X,Y) < k \},\tag{3.2.2}$$

where $\mathcal{D}^m$ is defined by (3.1.4). The game $(\mathcal{P},\mathcal{A}_k)$ is called Γ *with p-index k* and is denoted by $\Gamma_{(p,k)}$. In other words, $\Gamma_{(p,k)}$ is the subgraph of $\mathcal{N}^m_{(p,k)}$ induced in $\mathcal{P}(\Gamma)$. Figure 3.4 shows a part of $\mathcal{N}^2_{(2,3)}$.

The game $\Gamma_{(p,k)}$ is called a *p-saturation* of Γ if it has the same Sprague-Grundy function as $\Gamma_{(p,m+1)}$. In the next section, we will consider p-saturations not only of Welter's game, but also of Nim.

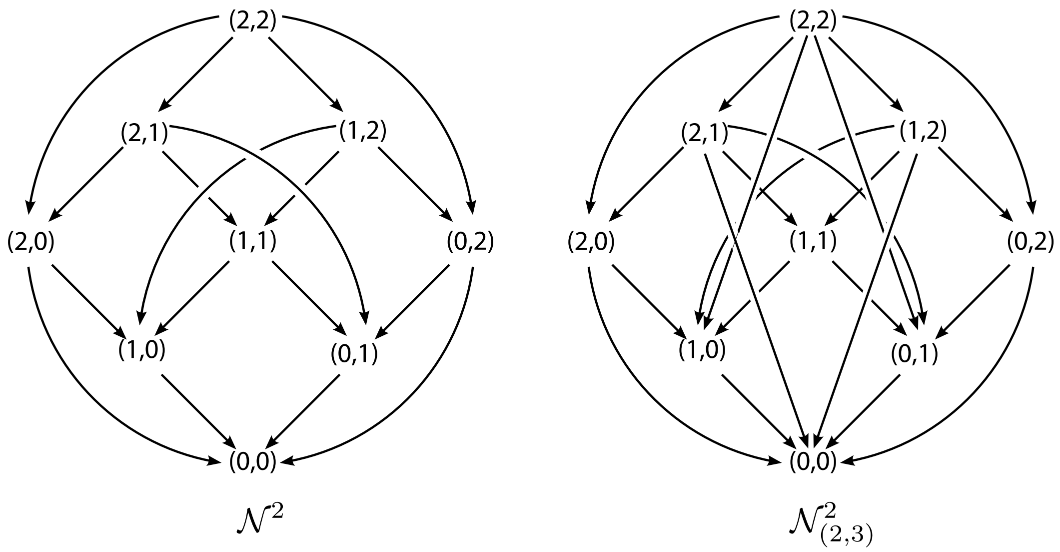


Figure 3.4: Positions in $\mathcal{N}^2$ and $\mathcal{N}^2_{(2,3)}$.

Remark 3.2.3. We give two remarks on positions in Welter's game.

First, we can represent a position in Welter's game as a set. Let us consider two positions (x^1,x^2) and (x^2,x^1) in $\mathcal{W}^2$. If (y^1,y^2) is an option of (x^1,x^2), then (y^2,y^1) is an option of (x^2,x^1). Therefore we can identify these two positions (x^1,x^2) and (x^2,x^1) and represent them as the set $\{x^1,x^2\}$. In general, let

$$\mathcal{P} = \binom{\mathbb{N}}{m} = \{X \subseteq \mathbb{N} : |X| = m\}$$

and $\mathcal{A}_k$ be the set of $(X,Y) \in \mathcal{P}^2$ such that there exists a bijection $\Phi : X \ni x^i \mapsto y^i \in Y$ satisfying

$$((x^1,\ldots,x^m),(y^1,\ldots,y^m)) \in \mathcal{A}(\mathcal{W}^m_{(p,k)}).\tag{3.2.3}$$

The game $(\mathcal{P}, \mathcal{A}_k)$ is called the *set representation of* $\mathcal{W}^m_{(p,k)}$ and is denoted by $\overline{\mathcal{W}^m_{(p,k)}}$. Note that Φ satisfies (3.2.3) if and only if $x \geqslant \Phi(x)$ for $x \in X$ and

$$\operatorname{ord}\left(\sum_{x \in X} x - \sum_{y \in Y} y\right) = \min\{\operatorname{ord}(x - \Phi(x)) : x \in X\}. \tag{3.2.4}$$

We also see that

$$\operatorname{sg}_{\overline{\mathcal{W}^m_{(p,k)}}}(\{x^1, \ldots, x^m\}) = \operatorname{sg}_{\mathcal{W}^m_{(p,k)}}((x^1, \ldots, x^m)).$$

We will use this set representation in the rest of paper. Note that for $X \in \mathcal{P}$, we can define $\lambda(X), w(X)$, and so on.

Second, two positions X and

$$X^{[1]} = \{x + 1 : x \in X\} \cup \{0\}$$

are essentially the same, since Y is an option of X if and only if $Y^{[1]}$ is that of $X^{[1]}$. In general, let

$$X^{[n]} = (X^{[n-1]})^{[1]} \tag{3.2.5}$$

for $n > 1$. Note that $\lambda(X^{[n]}) = \lambda(X)$ for $n \in \mathbb{N}$, where $X^{[0]} = X$.

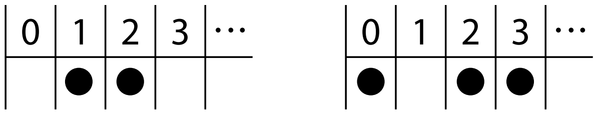


Figure 3.5: $\{1,2\}$ and $\{1,2\}^{[1]}$.

3.2.3 p-core Towers

In this subsection, we define p-core towers and state their properties. Details can be found in, for example, [17, 20, 21]. Using p-core towers, we will show Theorem 3.1.3 in latter sections.

Let X be a position in $\overline{\mathcal{W}^m_{(p,k)}}$, that is, X is a finite subset of $\mathbb{N}$. Let L be a non-negative integer. For each $R \in \Omega^L$, let

$$X_R = \{x_{\geqslant L} : x \in X, x_{<L} = R\}. \tag{3.2.6}$$

For example, if $p = 10$ and $X = \{12345, 67890\}$, then $X_0 = \{6789\}, X_5 = \{1234\}$, and $X_r = \varnothing$ for $r \in \{0, 1, \ldots, 9\} \setminus \{0, 5\}$. The position X is uniquely determined by $(X_R)_{R \in \Omega^L}$. Indeed, let

$$[X_R]_{R \in \Omega^L} = \{(R, \hat{x}) : R \in \Omega^L, \hat{x} \in X_R\}, \tag{3.2.7}$$

where $(R, \hat{x}) = (R_0, R_1, \ldots, R_{L-1}, \hat{x}_0, \hat{x}_1, \ldots)$. Then $X = [X_R]_{R \in \Omega^L}$. With this notation, we define the *p-core* $X_{(p)}$ of X by

$$X_{(p)} = [\{0, 1, \ldots, |X_r| - 1\}]_{r \in \Omega}. \tag{3.2.8}$$

Let $\tau_0(X)$ denote $\left|\lambda(X_{(p)})\right|$. As we will see in the next example, $\lambda(X_{(p)})$ is the partition obtained by removing all (p)-hooks from $\lambda(X)$.

Example 3.2.4. Let $p = 3$ and $X = \{2,4,6,7,10\}$. To calculate $\tau_0(X)$, it is convenient to use the *p-abacus* introduced by James [14]. First, we place five *beads* on 2, 4, 6, 7, and 10 as the left picture in Figure 3.6. Next, we move all beads upwards as high as possible. Then we obtain the 3-core $\{0,1,2,4,7\}$ of X. Therefore $\tau_0(X) = \left|\lambda(X_{(3)})\right| = 4$. Note that moving upward one bead corresponds removing one (3)-hook. Hence $\lambda(X_{(3)})$ contains no (3)-hooks. We also find that $\tau_0(X) = 4 = 19 - 3 \cdot 5 = |\lambda(X)| - pw_1(X) = w_0(X) - pw_1(X)$.

$$\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \\ & X & \end{array} \qquad\qquad \begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \\ & X_{(3)} & \end{array}$$

Figure 3.6: X and $X_{(3)}$ on 3-abacuses.

The sequence $((X_R)_{(p)})_{R\in\Omega^L}$ is called *the L-th row of the p-core tower* of X. We define

$$\tau_L(X) = \sum_{R\in\Omega^L} \tau_0(X_R). \tag{3.2.9}$$

Let $\overline{\tau}_L(X)$ be the remainder of $\tau_L(X)$ divided by p. Let $\tau(X)$ and $\overline{\tau}(X)$ denote the sequences whose L-th terms are $\tau_L(X)$ and $\overline{\tau}_L(X)$, respectively. For example, $\tau_L(\{x\}) = x_L$ for $x \in \mathbb{N}$.

Example 3.2.5. Let us consider Example 3.2.4 again. We calculate $\tau_1(X)$. Since

$$X_0 = \{6_{\geqslant 1}\} = \{2\},\ X_1 = \{4_{\geqslant 1}, 7_{\geqslant 1}, 10_{\geqslant 1}\} = \{1,2,3\}, \quad \text{and} \quad X_2 = \{2_{\geqslant 1}\} = \{0\},$$

we have $(X_0)_{(3)} = \{2\}, (X_1)_{(3)} = \{0,1,2\}$, and $(X_2)_{(3)} = \{0\}$. Hence $\tau_1(X) = 2$. In this way, we obtain

$$\tau(X) = (4,2,1,0,\ldots) \qquad \text{and} \qquad \overline{\tau}(X) = (1,2,1,0,\ldots).$$

We next give the basic properties of $\tau_L(X)$. As we have seen in Example 3.2.4, $\tau_0(X) = w_0(X) - pw_1(X)$. In general,

$$\tau_L(X) = w_L(X) - pw_{L+1}(X). \tag{3.2.10}$$

By (3.2.10), we have $\overline{\tau}_L(X) = \overline{w}_L(X)$ and $\overline{\tau}(X) = \overline{w}(X)$. Moreover,

$$\sum_{L\in\mathbb{N}} \tau_L(X)p^L = (w_0(X) - pw_1(X)) + (w_1(X) - pw_2(X))p + \cdots = w_0(X) = |\lambda(X)|. \tag{3.2.11}$$

Therefore $\overline{\tau}(X) = |\lambda(X)|$ if and only if $\overline{\tau}(X) = \tau(X)$, that is, the total size of the p-cores at each level of the p-core tower of X is at most $p-1$.

Furthermore,

$$\tau_{\geqslant L}(X) = \sum_{R \in \Omega^L} \tau(X_R), \tag{3.2.12}$$

$$w_{\geqslant L}(X) = \sum_{R \in \Omega^L} w(X_R), \tag{3.2.13}$$

where $\tau_{\geqslant L}(X) = (\tau_L(X), \tau_{L+1}(X), \ldots)$ and $w_{\geqslant L}(X) = (w_L(X), w_{L+1}(X), \ldots)$. In particular,

$$\begin{aligned} w_L(X) = \sum_{R \in \Omega^L} w_0(X_R) &= \sum_{R \in \Omega^L} \left(\sum_{x \in X_R} x - (0 + 1 + \cdots + (|X_R| - 1)) \right) \\ &= \sum_{x \in X} x_{\geqslant L} - \sum_{R \in \Omega^L} \binom{|X_R|}{2}. \end{aligned} \tag{3.2.14}$$

Remark 3.2.6. Let X be a position $\left\{ x^1, \ldots, x^m \right\}$ in $\overline{\mathcal{W}^m}$. We close this section by proving

$$\overline{\tau}(X) = x^1 \oplus \cdots \oplus x^m \ominus \left(\bigoplus_{i<j} \mathfrak{N}_p(x^i - x^j) \right) = \bigoplus_{(i,j) \in \lambda(X)} \mathfrak{N}_p\left(\left| H_{i,j}(X) \right| \right).$$

We may assume that $x^1 > \cdots > x^m$. Since $\mathfrak{N}_p(x) = \sum_{L=0}^{\operatorname{ord}(x)} p^L$, it follows that

$$\left(\bigoplus_{(i,j) \in \lambda(X)} \mathfrak{N}_p\left(\left| H_{i,j}(X) \right| \right) \right)_L \equiv w_L(X) \pmod{p}.$$

Hence $\bigoplus_{(i,j) \in \lambda(X)} \mathfrak{N}_p(|H_{i,j}(X)|) = \overline{w}(X) = \overline{\tau}(X)$. We also have

$$\begin{aligned} \bigoplus_{(i,j) \in \lambda(X)} \mathfrak{N}_p\left(\left| H_{i,j}(X) \right| \right) &= \bigoplus_{i=1}^{m} \left(\bigoplus_{0 \leqslant y < x^i,\, y \notin X} \mathfrak{N}_p(x^i - y) \right) \\ &= \bigoplus_{i=1}^{m} \left(\bigoplus_{0 \leqslant y < x^i} \mathfrak{N}_p(x^i - y) \ominus \left(\bigoplus_{i<j} \mathfrak{N}_p(x^i - x^j) \right) \right) \\ &= x^1 \oplus \cdots \oplus x^m \ominus \left(\bigoplus_{i<j} \mathfrak{N}_p(x^i - x^j) \right). \end{aligned}$$

3.3 p-saturations

In this section, we will study p-saturations of Nim and Welter's game. The aim of this section is to reduce Theorem 3.1.3 to two assertions in Subsection 3.3.2.

3.3.1 p-saturations of Nim

In some cases, we can reduce the problem of Welter's game to that of Nim. This is because, by (3.2.12),

$$\overline{\tau}_{\geqslant 1}(X) = \overline{\tau}(X_0) \oplus \cdots \oplus \overline{\tau}(X_{p-1}),$$

where X is a position in $\overline{\mathcal{W}^m}$ and $\overline{\tau}_{\geqslant 1}(X) = (\overline{\tau}_1(X), \overline{\tau}_2(X), \ldots)$. In this subsection, we will prove

$$\operatorname{sg}(X) = x^1 \oplus \cdots \oplus x^m, \tag{3.3.1}$$

where X is a position $(x^1, \ldots, x^m)$ in p-saturations of Nim. Let $\sigma(X)$ be the right-hand side of (3.3.1) for $X \in \mathbb{N}^m$. To prove (3.3.1), it is sufficient to show (SG1) and (SG2).

The next lemma provides a necessary and sufficient condition for a descendant to be an option in $\mathcal{N}^m_{(p,k)}$. In particular, (SG1) holds.

Lemma 3.3.1. *Let X be a position $(x^1, \ldots, x^m)$ in $\mathcal{N}^m_{(p,k)}$. If Y is a proper descendant $(y^1, \ldots, y^m)$ of X such that $\operatorname{dist}(X, Y) < k$, then*

$$\operatorname{ord}(\sigma(X) - \sigma(Y)) \geqslant \min\{\operatorname{ord}(x^i - y^i) : 1 \leqslant i \leqslant m\}$$

with equality if and only if Y is an option of X.

proof. Since $\operatorname{dist}(X, Y) < k$, the position Y is an option of X if and only if $X - Y \in \mathcal{D}^m$. Let $N = \min\{\operatorname{ord}(x^i - y^i) : 1 \leqslant i \leqslant m\}$. Then $x^i_L = y^i_L$ for $0 \leqslant L < N$. Hence

$$\operatorname{ord}(\sigma(X) - \sigma(Y)) \geqslant N$$

and

$$(\sigma(X) - \sigma(Y))_N = \left(\sum_{i=1}^m x^i - y^i\right)_N.$$

Therefore $\operatorname{ord}(\sigma(X) - \sigma(Y)) = N$ if and only if Y is an option of X.

□

It remains to show (SG2).

Lemma 3.3.2. *Let X be a position $(x^1, \ldots, x^m)$ in $\mathcal{N}^m_{p,k}$. Suppose that $k \geqslant \min(p, m+1)$. Then X has an option Y with $\sigma(Y) = h$ for $0 \leqslant h < \sigma(X)$. In particular, $\operatorname{sg}(X) = \sigma(X)$.*

proof. We first construct a descendant Y of X such that $\sigma(Y) = h$. Let $n = \sigma(X)$ and $N = \max\{L \in \mathbb{N} : n_L \neq h_L\}$. Since $h < n$, it follows that $h_N < n_N = x^1_N + \cdots + x^m_N$. Thus there exist $r^1, \ldots, r^m \in \Omega$ such that

$$r^1 + \cdots + r^m = h_N \quad \text{and} \quad r^i \leqslant x^i_N \quad \text{for } 1 \leqslant i \leqslant m.$$

By rearranging x^i if necessary, we may assume that $r^1 < x^1_N$. Since $n_N \leqslant p-1$, we may also assume that

$$\operatorname{dist}\bigl((r^1,\dots,r^m),\,(x^1_N,\dots,x^m_N)\bigr) < k.$$

Let

$$y^1 = \bigl(x^1_0 - n_0 + h_0, \dots, x^1_{N-1} - n_{N-1} + h_{N-1}, r^1, x^1_{N+1}, x^1_{N+2}, \dots\bigr),$$

$$y^i = \bigl(x^i_0, \dots, x^i_{N-1}, r^i, x^i_{N+1}, x^i_{N+2} \dots\bigr) \qquad \text{for } 2 \leqslant i \leqslant m,$$

and $Y = (y^1,\dots,y^m)$. Then Y is a proper descendant of X with $\sigma(Y) = h$.

It remains to verify that Y is an option of X. Since $\operatorname{dist}(X,Y) < k$ and

$$\operatorname{ord}(\sigma(X) - \sigma(Y)) = \operatorname{ord}(x^1 - y^1) = \min\bigl\{\operatorname{ord}(x^i - y^i) : 1 \leqslant i \leqslant m\bigr\},$$

it follows from Lemma 3.3.1 that Y is an option of X.

□

Remark 3.3.3. Using Lemma 3.2.11, we can show that the p-saturation index of $\mathcal{N}^m$ equals $\min(p, m+1)$. Indeed, if $m = 0$, then there is nothing to prove. Suppose that $m \geqslant 1$. Then $\min(p, m+1) \geqslant 2$. Let $k = \min(p, m+1)$ and

$$X = (\underbrace{p,\dots,p}_{k-1}, 0, \dots, 0) \in \mathbb{N}^m.$$

This position X has no option $(y^1,\dots,y^m)$ with $y^1 \oplus \cdots \oplus y^m = 0$ in $\mathcal{N}^m_{p,k-1}$. Thus $\operatorname{sat}_p(\mathcal{N}^m) = k$ by Lemma 3.2.11.

3.3.2 p-saturations of Welter's Game

In the rest of this paper, positions mean positions in $\overline{\mathcal{W}^m_{(p,m+1)}}$, unless otherwise specified.

Theorem 3.1.3 follows immediately from the next result.

Theorem 3.3.4. *Let X be a position.*

1. *The following two assertions hold:*
   (A1) *If $\overline{\tau}(X) = |\lambda(X)| > 0$, then X has a descendant Y with $\overline{\tau}(Y) = \overline{\tau}(X) - 1$.* [3]
   (A2) *If $\overline{\tau}(X) < |\lambda(X)|$, then X has a proper descendant Y with $\overline{\tau}(Y) \geqslant \overline{\tau}(X)$.*
2. $\operatorname{sg}(X) = \overline{\tau}(X)$.

[3] Note that Y satisfies (A1) if and only if $\overline{\tau}(Y) = |\lambda(Y)| = |\lambda(X)| - 1$.

Let X be a position. We will prove the above theorem by induction on $|\lambda(X)|$. If $|\lambda(X)| = 0$, then there is nothing to prove. Suppose that $|\lambda(X)| > 0$. By the induction hypothesis,

$$\operatorname{sg}(Z) = \overline{\tau}(Z) \text{ for } Z \text{ with } |\lambda(Z)| < |\lambda(X)|. \tag{3.3.2}$$

The assertions (A1) and (A2) are proven in Sections 3.4 and 3.5, respectively. In this subsection, we show how (A1) and (A2) imply that $\operatorname{sg}(X) = \overline{\tau}(X)$. It is sufficient to show (SG1) and (SG2).

The next lemma, which is an analog of Lemma 3.3.1, provides a necessary and sufficient condition for a descendant to be an option in $\mathcal{W}^m_{(p,k)}$.

Lemma 3.3.5. *Let X be a position $(x^1, \ldots, x^m)$ in $\mathcal{W}^m_{p,k}$. If Y is a proper descendant $(y^1, \ldots, y^m)$ of X such that* $\operatorname{dist}(X,Y) < k$*, then*

$$\operatorname{ord}(\overline{\tau}(X') - \overline{\tau}(Y')) \geqslant \min\{\operatorname{ord}(x^i - y^i) : 1 \leqslant i \leqslant m\}$$

with equality if and only if Y is an option of X, where $X' = \{x^1, \ldots, x^m\}$ and $Y' = \{y^1, \ldots, y^m\}$.

proof. Since $\operatorname{dist}(X,Y) < k$, the position Y is an option of X if and only if $X - Y \in \mathcal{D}^m$. Let $N = \min\{\operatorname{ord}(x^i - y^i) : 1 \leqslant i \leqslant m\}$.

We first show that $\operatorname{ord}(\overline{\tau}(X') - \overline{\tau}(Y')) \geqslant N$, By the definition of N, we have $x^i_{<N} = y^i_{<N}$ for $1 \leqslant i \leqslant m$. Hence $|X'_R| = |Y'_R|$ for each $R \in \Omega^L$ with $0 \leqslant L \leqslant N$. Thus $\tau_{<N}(X') = \tau_{<N}(Y')$, where $\tau_{<N}(X') = (\tau_0(X'), \tau_1(X'), \ldots, \tau_{N-1}(X'))$. This shows that $\operatorname{ord}(\overline{\tau}(X') - \overline{\tau}(Y')) \geqslant N$. By (3.2.14), we have

$$w_N(X') - w_N(Y') = \sum_{i=1}^{m} x^i_{\geqslant N} - \sum_{i=1}^{m} y^i_{\geqslant N} = \left(\sum_{i=1}^{m} x^i - y^i\right)_{\geqslant N}.$$

It follows that

$$\overline{\tau}_N(X') - \overline{\tau}_N(Y') \equiv w_N(X') - w_N(Y') \equiv \left(\sum_{i=1}^{m} x^i - y^i\right)_N \pmod{p}.$$

Therefore $\operatorname{ord}(\overline{\tau}(X') - \overline{\tau}(Y')) = N$ if and only if Y is an option of X.

□

We now prove (SG1) using Lemma 3.3.5. Let Y be an option of X. Then there exists a bijection $\Phi : X \ni x^i \mapsto y^i \in Y$ such that $(y^1, \ldots, y^m)$ is an option of $(x^1, \ldots, x^m)$ in $\mathcal{W}^m_{(p,m+1)}$. Hence

$$\operatorname{ord}(\overline{\tau}(X) - \overline{\tau}(Y)) = \min\{\operatorname{ord}(x^i - y^i) : 1 \leqslant i \leqslant m\} < \infty$$

by Lemma 3.3.5. This yields $\overline{\tau}(Y) \neq \overline{\tau}(X)$.

Remark 3.3.6. Using Lemma 3.3.5 and Theorem 3.1.3, we can show that

$$\mathrm{sat}_p(\mathcal{W}^m) \geqslant \min(p, m+1).$$

Indeed, if $m = 0$, then the assertion is trivial. Suppose that $m \geqslant 1$. Let $k = \min(p, m+1)$ and

$$X = \{\, p, p+1, \ldots, p+k-2 \,\}^{[m-k+1]}$$

defined in (3.2.5). Then $k \geqslant 2$ and $\overline{\tau}(X) = p(k-1) > 0$, but X has no option Y with $\overline{\tau}(Y) = 0$ in $\mathcal{W}^m_{(p,k-1)}$ by Lemma 3.3.5. Hence $\mathrm{sat}_p(\mathcal{W}^m) \geqslant k$ by Theorem 3.1.3.

We next show (SG2) assuming (A1) and (A2). Recall that we also assume that (3.3.2). Let $n = \overline{\tau}(X)$. If $n = 0$, then (SG2) is trivial. Suppose that $n > 0$. We divide into three cases.

Case 1 ($h = n-1$)**.** It suffices to show that X has a descendant Y with $\overline{\tau}(Y) = n-1$ because if $\overline{\tau}(Y) = n-1$, then

$$\mathrm{ord}(\overline{\tau}(X) - \overline{\tau}(Y)) = \mathrm{ord}(1) = 0,$$

so Y is an option of X by Lemma 3.3.5. If $n = |\lambda(X)|$, then there is nothing to prove by (A1). Suppose that $n < |\lambda(X)|$. By (A2), the position X has a proper descendant Z with $\overline{\tau}(Z) \geqslant \overline{\tau}(X)$. By (3.3.2), we have

$$\mathrm{sg}(Z) = \overline{\tau}(Z) > n-1.$$

Hence Z has an option Y with $\mathrm{sg}(Y) = \overline{\tau}(Y) = n-1$. This position Y is a descendant of X.

Case 2 ($h < n-1$ and $h \not\equiv n \pmod p$)**.** By Case 1, X has an option Y with $\mathrm{sg}(Y) = \overline{\tau}(Y) = n-1$. This position Y has an option Z with $\mathrm{sg}(Z) = \overline{\tau}(Z) = h < n-1$. Since $\mathrm{ord}(n-h) = 0$, the position Z is also an option of X by Lemma 3.3.5.

Case 3 ($h \equiv n \pmod p$)**.** We first construct a descendant Z of X with $\overline{\tau}(Z) = h$ using Lemma 3.2. Let $N = \mathrm{ord}(n-h)$. Since $h \equiv n \pmod p$ and $h < n$, it follows that $N > 0$ and $h_{\geqslant 1} < n_{\geqslant 1}$. Let $a^r = \overline{\tau}(X_r)$ for each $r \in \Omega$. Then

$$a^0 \oplus \cdots \oplus a^{p-1} = n_{\geqslant 1} > h_{\geqslant 1}.$$

By Lemma 3.3.2, there exists $(b^0, \ldots, b^{p-1}) \in \mathbb{N}^p$ such that

1. $\bigoplus_{r \in \Omega} b^r = h_{\geqslant 1}$,
2. $b^r \leqslant a^r$ for each $r \in \Omega$,
3. $N - 1 = \mathrm{ord}(\bigoplus_{r \in \Omega} a^r - \bigoplus_{r \in \Omega} b^r) = \min\{\, \mathrm{ord}(a^r - b^r) : r \in \Omega \,\}$.

Since $|\lambda(X_r)| < |\lambda(X)|$, it follows from (3.3.2) that

$$\mathrm{sg}(X_r) = \overline{\tau}(X_r) = a^r \geqslant b^r.$$

When $b^r < a^r$, let Z_r be an option of X_r with $\mathrm{sg}(Z_r) = \overline{\tau}(Z_r) = b^r$. When $b^r = a^r$, let $Z_r = X_r$. We set $Z = [Z_r]_{r\in\Omega}$. Then $\overline{\tau}(Z) = h$.

We next show that Z is an option of X. By Lemma 3.3.5, it suffices to find a bijection $\Phi : X \to Z$ such that

$$\min\{\,\mathrm{ord}(x - \Phi(x)) : x \in X\,\} = N \tag{3.3.3}$$

and $\Phi(x) \leqslant x$ for $x \in X$.

We first construct a bijection $\Phi : X \to Z$. Since Z_r is an option of X_r when $b^r < a^r$, there exists a bijection $\Phi^r : X_r \to Z_r$ such that

$$\mathrm{ord}(a^r - b^r) = \min\{\,\mathrm{ord}(\hat{x} - \Phi^r(\hat{x})) : \hat{x} \in X_r\,\} \tag{3.3.4}$$

and $\Phi^r(\hat{x}) \leqslant \hat{x}$ for $\hat{x} \in X_r$ by Lemma 3.3.5. Let $\Phi^r : X_r \to Z_r$ be the identity map when $b^r = a^r$. We now define a bijection $\Phi : X \to Z$ by

$$\Phi(x) = (r, \Phi^r(\hat{x})) = (r, (\Phi^r(\hat{x}))_0^{(p)}, (\Phi^r(\hat{x}))_1^{(p)}, \ldots) \in Z,$$

where $r = x_0$ and $\hat{x} = x_{\geqslant 1}$. Since $\Phi^r(\hat{x}) \leqslant \hat{x}$, we have $\Phi(x) \leqslant x$ for $x \in X$.

It remains to show (3.3.3). By (3.3.4) and (3),

$$\min\{\,\mathrm{ord}(\hat{x} - \Phi^r(\hat{x})) : \hat{x} \in X_r,\ r \in \Omega\,\} = \min\{\,\mathrm{ord}(a^r - b^r) : r \in \Omega\,\} = N - 1.$$

Since $\mathrm{ord}(x - \Phi(x)) = \mathrm{ord}(\hat{x} - \Phi^r(\hat{x})) + 1$ if $x \neq \Phi(x)$, we obtain (3.3.3). Therefore Z is an option of X by Lemma 3.3.5. This completes the proof of (SG2).

In the rest of the paper, we will show (A1) and (A2).

Remark 3.3.7. Suppose that p is a prime. Then (A1) follows from the branching rule and the fact that $\overline{\tau}(X) = |\lambda(X)|$ if and only if the degree of ρ^X is prime to p. Indeed, let X be a position such that $\overline{\tau}(X) = |\lambda(X)| > 0$, and let $\deg(\rho^X)$ be the degree of ρ^X. By the branching rule,

$$\deg(\rho^X) = \sum_Y \deg(\rho^Y),$$

where the sum is over all descendants Y of X with $|\lambda(Y)| = |\lambda(X)| - 1$. Since $\overline{\tau}(X) = |\lambda(X)|$ and p is a prime,

$$\deg(\rho^X) \not\equiv 0 \pmod{p}.$$

This implies that $\deg(\rho^Y) \not\equiv 0 \pmod{p}$ for some descendant Y of X with $|\lambda(Y)| = |\lambda(X)| - 1$. Therefore $\overline{\tau}(Y) = \overline{\tau}(X) - 1$.

Remark 3.3.8. For a position X, let

$$\begin{aligned}\operatorname{msg}(X) &= \max\{\,\overline{\tau}(Y) : Y \text{ is a descendant of } X \text{ with } \overline{\tau}(Y) = |\lambda(Y)|\,\}\\ &\quad (= \max\{\,\operatorname{sg}(Y) : Y \text{ is a descendant of } X\,\} \text{ by assuming Theorem 1.1}).\end{aligned}$$

The assertion (A2) yields $\operatorname{msg}(X) \geqslant \overline{\tau}(X)$. As we have mentioned in the introduction, we can improve this bound as follows. If $\overline{\tau}(X) = |\lambda(X)|$, then $\operatorname{msg}(X) = \overline{\tau}(X)$ because when Y is a descendant of X, we see that $\overline{\tau}(Y) = \operatorname{sg}(Y) \leqslant |\lambda(Y)| \leqslant |\lambda(X)|$ by Theorem 3.1.3. Suppose that $\overline{\tau}(X) < |\lambda(X)|$. Then $\overline{\tau}(X) \neq \tau(X)$, so there exists L such that $\tau_L(X) \geqslant p$. Let N be the largest such L. In Section 3.5, we will show that

$$\operatorname{msg}(X) \geqslant (\varepsilon(p-1), \underbrace{p-1, \ldots, p-1}_{N}, \overline{\tau}_{N+1}(X), \overline{\tau}_{N+2}(X), \ldots), \tag{3.3.5}$$

where $\varepsilon = 1$ if $\tau_N(X) \geqslant p+1$ and $\varepsilon = 0$ if $\tau_N(X) = p$. For example, if $p = 3$ and $X = \{3,4,5,9,10,11\}$, then $\tau(X) = (0,0,3,0,\ldots)$ and $\operatorname{msg}(X) = (0,2,2,0,\ldots)$.

3.4 Proof of (A1)

We introduce p^H*-options* for $H \in \mathbb{N}$. p^0-options play a key role in the proof of (A1). In fact, let X be a non-terminal position with $\overline{\tau}(X) = |\lambda(X)|$. We first show that if X has a p^0-option Y, then $\overline{\tau}(Y) = \overline{\tau}(X) - 1$ in Lemma 3.4.3. We then prove that X always has a p^0-option. This implies that (A1) holds.

3.4.1 p^H-options

To define p^H-options, we first introduce a total order. Let $(\alpha_L)_{L\in\mathbb{N}}$ and $(\beta_L)_{L\in\mathbb{N}}$ be two non-negative integer sequences with finitely many nonzero terms. Suppose that $(\alpha_L)_{L\in\mathbb{N}} \neq (\beta_L)_{L\in\mathbb{N}}$. Let $N = \max\{\,L \in \mathbb{N} : \alpha_L \neq \beta_L\,\}$. If $\alpha_N < \beta_N$, then we write

$$(\alpha_L)_{L\in\mathbb{N}} \prec (\beta_L)_{L\in\mathbb{N}}. \tag{3.4.1}$$

For example, for two non-negative integers x and y, we see that $\tau(\{\,x\,\}) \prec \tau(\{\,y\,\})$ if and only if $x < y$. Furthermore, for two positions X and Y,

$$w(X) \prec w(Y) \iff \tau(X) \prec \tau(Y). \tag{3.4.2}$$

Let us show (3.4.2). Note that $w(X) \neq w(Y) \iff \tau(X) \neq \tau(Y)$ since $\tau_L(X) = w_L(X) - pw_{L+1}(X)$. Suppose that $w(X) \neq w(Y)$, and let $N = \max\{\,L \in \mathbb{N} : w_L(X) \neq w_L(Y)\,\}$. Then for $L \geqslant N+1$,

$$\tau_L(X) = w_L(X) - pw_{L+1}(X) = w_L(Y) - pw_{L+1}(Y) = \tau_L(Y).$$

Moreover,

$$\tau_N(X) - \tau_N(Y) = (w_N(X) - pw_{N+1}(X)) - (w_N(Y) - pw_{N+1}(Y)) = w_N(X) - w_N(Y),$$

which gives (3.4.2).

We next define the (p-adic) order of a position. For a non-terminal position X (that is, X has an option), the *order* $\mathrm{ord}(X)$ of X is defined by

$$\mathrm{ord}(X) = \min\{\, L \in \mathbb{N} : \tau_L(X) \neq 0 \,\}. \tag{3.4.3}$$

If X is a terminal position, then we define $\mathrm{ord}(X) = \infty$. For example, $\mathrm{ord}(\{\,x\,\}) = \mathrm{ord}(x)$ for each $x \in \mathbb{N}$.

For a position X, let $(x\ x-d)(X)$ denote the option obtained from X by replacing $x \in X$ with $x - d \in \mathbb{N}\backslash X$, that is,

$$(x\ x-d)(X) = X \cup \{\, x-d \,\} \backslash \{\, x \,\} \quad \text{for } x \in X \text{ and } x-d \in \mathbb{N}\backslash X \text{ with } x-d < x.$$

Definition 3.4.1 (p^H-options)**.** Let $H \in \mathbb{N}$. Let X be a position with order M and Y an option $(x\ x-p^H)(X)$ of X. The position Y is called a *p^H-option* of X if it has the following two properties:

(O1) $\tau_L(Y) \equiv \tau_L(X) - 1 \pmod p$ for $H \leqslant L \leqslant M$.

(O2) $\tau_{\geqslant M+1}(Y) \succeq \tau_{\geqslant M+1}(X)$.

Example 3.4.2. Let $p = 3$. Let $X = \{2,3,5,10\}, Y = (2\ 1)(X), Z = (5\ 4)(X)$, and $W = (10\ 9)(X)$. The position Z is a 3^0-option of X, but Y and W are not. Indeed,

$$\begin{aligned}
\tau(X) &= (2,1,1,0,\ldots),\\
\tau(Y) &= (1,4,0,0,\ldots),\\
\tau(Z) &= (1,1,1,0,\ldots),\\
\tau(W) &= (4,0,1,0,\ldots).
\end{aligned}$$

Hence they satisfy (O1), but only Z satisfies (O2), since $\mathrm{ord}(X) = 0$. Note that $\overline{\tau}(Z) = 13 = 14 - 1 = \overline{\tau}(X) - 1$, so X satisfies (A1). In fact, this is always true by the next lemma.

Lemma 3.4.3. *Let X be a position with $\overline{\tau}(X) = |\lambda(X)|$. If X has a p^0-option Y, then $\overline{\tau}(Y) = \overline{\tau}(X) - 1$.*

proof. By the definition of p^0-options, $|\lambda(Y)| = |\lambda(X)| - 1 = \overline{\tau}(X) - 1$. Hence it is sufficient to show that $\overline{\tau}(Y) = |\lambda(Y)|$. Let $M = \mathrm{ord}(X)$.

We first show that $\tau_{\geqslant M+1}(Y) = |\lambda(Y)|_{\geqslant M+1}$. Since $|\lambda(X)| = \overline{\tau}(X) = \tau(X)$, we have

$$|\lambda(X)|_M = \tau_M(X) \neq 0.$$

This implies that

$$|\lambda(Y)| = |\lambda(X)| - 1 = (\underbrace{p-1,\ldots,p-1}_{M}, \tau_M(X)-1, \tau_{M+1}(X), \tau_{M+2}(X), \ldots).$$

By (O2),

$$\tau_{\geqslant M+1}(Y) \succeq \tau_{\geqslant M+1}(X) = |\lambda(Y)|_{\geqslant M+1}.$$

Hence $\tau_{\geqslant M+1}(Y) = |\lambda(Y)|_{\geqslant M+1}$ for otherwise $\sum_{L \in \mathbb{N}} \tau_L(Y) p^L > |\lambda(Y)|$, contrary to (3.2.11).

It remains to show that $\tau_{<M+1}(Y) = |\lambda(Y)|_{<M+1}$. By (O1),

$$\tau_L(Y) \equiv \tau_L(X) - 1 \equiv |\lambda(Y)|_L \pmod p \text{ for } 0 \leqslant L \leqslant M.$$

Hence $\tau_{<M+1}(Y) = |\lambda(Y)|_{<M+1}$. Therefore $\overline{\tau}(Y) = |\lambda(Y)| = \overline{\tau}(X) - 1$.

□

3.4.2 (A1) for $\mathrm{ord}(X) = 0$

To show (A1) using Lemma 3.4.3, we will show that if X is a non-terminal position with $\overline{\tau}(X) = |\lambda(X)|$, then X has a p^0-option.

Some non-terminal positions have no p^0-options. However, we will prove that every non-terminal position X has a p^M-option, where $M = \mathrm{ord}(X)$. For this reason, we give the next definition.

Definition 3.4.4 (p^*-options)**.** Let X be a position with order M. A p^M-option of X is called a *p^*-option* of X.

The next lemma is an essential property of p^*-options.

Lemma 3.4.5. *Every non-terminal position has a p^*-option.*

By Lemmas 3.4.3 and 3.4.5, (A1) holds when $\mathrm{ord}(X) = 0$. To prove Lemma 3.4.5, we study p^*-options.

We first give a recursive property of p^*-options. From this property, we only need to show Lemma 3.4.5 when $\mathrm{ord}(X) = 0$.

Lemma 3.4.6. *Let X be a position whose order M is positive. Then the following assertions hold:*

(1) $\mathrm{ord}(X_s) = M - 1$ *for some $s \in \Omega$.*

(2) *Let $Y_r = X_r$ for each $r \in \Omega \setminus \{s\}$. If Y_s is a p^*-option of X_s, then $[Y_r]_{r \in \Omega}$ is a p^*-option of X.*

proof. We first show (1). By (3.2.12),

$$\sum_{r\in\Omega}\tau(X_r) = \tau_{\geqslant 1}(X) = (\underbrace{0,\ldots,0}_{M-1},\tau_M(X),\ldots).$$

Hence $\tau_{M-1}(X_s)\neq 0$ for some $s\in\Omega$. In particular, $\mathrm{ord}(X_s) = M-1$.

We next show (2). Let $Y = [Y_r]_{r\in\Omega}$. By (1), the position Y_s is a $p^{(M-1)}$-option of X_s. Hence

$$\tau_M(Y) = \sum_{r\in\Omega}\tau_{M-1}(Y_r) \equiv \sum_{r\in\Omega}\tau_{M-1}(X_r) - 1 \equiv \tau_M(X) - 1 \pmod p$$

and

$$\tau_{\geqslant M+1}(Y) = \sum_{r\in\Omega}\tau_{\geqslant M}(Y_r) \succeq \sum_{r\in\Omega}\tau_{\geqslant M}(X_r) = \tau_{\geqslant M+1}(X).$$

$\square$

To prove Lemma 3.4.5, we compare the difference between p^*-options and non-p^*-options. Let us give an example.

Example 3.4.7. Let us consider Example 3.4.2 again. To clarify the difference between the 3^0-option Z and the other options Y and W, we investigate $\left|X_{(r,R)}\right|$ for $r\in\Omega^1$ and $R\in\Omega^0\cup\Omega^1$. Recall that $Z = (5\ 4)(X), Y = (2\ 1)(X)$, and $W = (10\ 9)(X)$.

R	$\lvert X_{(0,R)}\rvert$	$\lvert X_{(1,R)}\rvert$	$\lvert X_{(2,R)}\rvert$
()	1	1	2
(0)	0	1	1
(1)	1	0	1
(2)	0	0	0

Let $x\in\{2,5,10\}$. Then the following inequality holds only when $x=5$:

$$\left|X_{(x-1)_{<L}}\right| < \left|X_{x)_{<L}}\right| \quad \text{for every } L\geqslant 1. \tag{3.4.4}$$

Indeed,

$$\left|X_{(4_0)}\right| = \left|X_{(1)}\right| = 1 < 2 = \left|X_{(2)}\right| = \left|X_{(5_0)}\right| \text{ and } |X_{4_{<L}}| = 0 < 1 = |X_{5_{<L}}|$$

for every $L\geqslant 2$. We also have

$$\left|X_{(9_0)}\right| = \left|X_{(0)}\right| = 1 = \left|X_{(1)}\right| = \left|X_{(10_0)}\right|$$

and

$$\left|X_{(1_0,1_1)}\right| = \left|X_{(1,0)}\right| = 1 = \left|X_{(2,0)}\right| = \left|X_{(2_0,2_1)}\right|.$$

In fact, we will show that if $x_0\neq 0$ (this assumption is only for simplicity) and (3.4.4) holds, then the option $(x\ x-1)(X)$ is a p^0-option in Lemma 3.4.8. Furthermore, we will show that if

$$\left|X_{r-1}\right| < \left|X_r\right| \text{ for some } r\in\Omega \text{ with } r\neq 0,$$

then there exists x satisfying (3.4.4) in Lemma 3.4.9. Using these results, we will prove Lemma 3.4.5.

As we have mentioned in the example above, we first give a sufficient condition for an option to be a p^*-option. Let $\delta_x = 1$ if $x = 0$, and $\delta_x = 0$ otherwise.

Lemma 3.4.8. *Let X be a position and Y an option $(x\, x - p^H)(X)$ of X. Then the following two assertions hold:*

1. $$w_L(Y) - w_L(X) = \begin{cases} -p^{H-L} & \text{if } L \leqslant H, \\ |X_{x_{<L}}| - |X_{(x-p^H)_{<L}}| - \delta - 1 & \text{if } L \geqslant H+1, \end{cases}$$
   where $\delta = \delta_{x_H}\delta_{x_{H+1}} \cdots \delta_{x_{L-1}}$.
2. *Let $M = \operatorname{ord}(X)$. If $H = M$ and*
   $$\left|X_{(x-p^M)_{<L}}\right| + \delta_{x_M}\delta_{x_{M+1}} \cdots \delta_{x_{L-1}} < \left|X_{x_{<L}}\right| \quad \text{for every } L \geqslant M+1, \tag{3.4.5}$$
   then Y is a p^*-option of X.

proof. We first prove (1). Let $Z = X \setminus \{x\}$. By (3.2.13) and (3.2.14), we have

$$\begin{aligned} w_L(X) - w_L(Z) &= \sum_{R \in \Omega^L} w_0(X_R) - w_0(Z_R) = w_0(X_{x_{<L}}) - w_0(Z_{x_{<L}}) \\ &= x_{\geqslant L} - (|X_{x_{<L}}| - 1). \end{aligned}$$

Since $Z = Y \setminus \{x - p^H\}$, we also have

$$w_L(Y) - w_L(Z) = (x - p^H)_{\geqslant L} - (|Y_{(x-p^H)_{<L}}| - 1).$$

Thus

$$w_L(Y) - w_L(X) = |X_{x_{<L}}| - |Y_{(x-p^H)_{<L}}| + (x - p^H)_{\geqslant L} - x_{\geqslant L}.$$

If $L \leqslant H$, then $x_{<L} = (x - p^H)_{<L}$, and hence $w_L(Y) - w_L(X) = -p^{H-L}$. Suppose that $L \geqslant H+1$. Then $|Y_{(x-p^H)_{<L}}| = |X_{(x-p^H)_{<L}}| + 1$ and $(x - p^H)_{\geqslant L} = x_{\geqslant L} - \delta$, which gives (1).

We next prove (2). By (1) and (3.4.5), $w_L(Y) - w_L(X) \geqslant 0$ for every $L \geqslant M+1$. This shows that $w_{\geqslant M+1}(Y) \succeq w_{\geqslant M+1}(X)$. Hence, by (3.4.2), $\tau_{\geqslant M+1}(Y) \succeq \tau_{\geqslant M+1}(X)$. Since $\tau_M(Y) \equiv \tau_M(X) - 1 \pmod{p}$, the position Y is a p^*-option of X.

□

Finally, we present a sufficient condition for a position to have a p^*-option. In fact, every non-terminal position satisfies this condition as we will show after the next result.

Lemma 3.4.9. *Let X be a position with order M. Let H and N be non-negative integers with $H \leqslant M \leqslant N-1$. Suppose that there exists $S \in \Omega^N$ such that*

$$\left|X_{S-p^H}\right| + \delta < \left|X_S\right|, \tag{3.4.6}$$

where $\delta = \delta_{S_H}\delta_{S_{H+1}} \cdots \delta_{S_{N-1}}$. Then X has an option Y such that $\tau_{\geqslant N}(Y) \succeq \tau_{\geqslant N}(X)$ and $Y = (x\ x - p^H)(X)$ for some $x \in X$ with $x_{<N} = S$. In particular, if $H = M = N-1$, then Y is a p^*-option of X.

proof. We first construct x such that

$$|X_{(x-p^H)_{<L}}| + \delta_{x_H}\delta_{x_{H+1}}\cdots\delta_{x_{L-1}} < |X_{x_{<L}}| \text{ for every } L \geqslant N. \tag{3.4.7}$$

Since $|X_S| = \sum_{r\in\Omega}|X_{(S,r)}|$ and $\sum_{r\in\Omega}|X_{(S,r)-p^H}| = \sum_{r\in\Omega}|X_{(S-p^H,r)}|$, we have

$$\sum_{r\in\Omega}\left|X_{(S,r)-p^H}\right| + \delta = \left|X_{S-p^H}\right| + \delta < \left|X_S\right| = \sum_{r\in\Omega}\left|X_{(S,r)}\right|.$$

This implies that

$$\left|X_{(S,S_N)-p^H}\right| + \delta\delta_{S_N} < \left|X_{(S,S_N)}\right|$$

for some $S_N \in \Omega$. Continuing this process, we obtain $S_N, S_{N+1}, \ldots \in \Omega$. Let $x = (S_0, S_1, \ldots)$. Then $x_{<N} = S$ and (3.4.7) holds.

Let $Y = (x\ x-p^H)(X)$. If Y is an option of X, then $\tau_{\geqslant N}(Y) \succeq \tau_{\geqslant N}(X)$ by Lemma 3.4.8. Hence it suffices to show that Y is an option of X, that is, $x \in X$ and $x - p^H \in \mathbb{N}\backslash X$. Let

$$L = \min\{\, L \in \mathbb{N} : x'_{\geqslant L} = 0 \quad \text{for every} \quad x' \in X \,\}.$$

Then for each $R \in \Omega^L$,

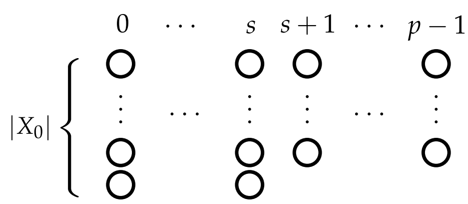
$$X_R = \{\, x'_{\geqslant L} : x' \in X, x'_{<L} = R \,\} \subseteq \{\, 0 \,\},$$

and hence $|X_R| \in \{\, 0, 1 \,\}$. Since $|X_{(x-p^H)_{<L}}| + \delta_{x_H}\delta_{x_{H+1}}\cdots\delta_{x_{L-1}} < |X_{x_{<L}}|$, we have $|X_{(x-p^H)_{<L}}| + \delta_{x_H}\delta_{x_{H+1}}\cdots\delta_{x_{L-1}} = 0$ and $|X_{x_{<L}}| = 1$. This implies that $x - p^H \in \mathbb{N}\backslash X$ and $x \in X$.

□

Lemma 3.4.5. Let X be a non-terminal position. By Lemma 3.4.6, we may assume that the order of X is 0. By Lemma 3.4.9, it suffices to show that $|X_{s-1}| + \delta_s < |X_s|$ for some $s \in \Omega$. Hence we may assume that $|X_0| \geqslant |X_1| \geqslant \cdots \geqslant |X_{p-1}|$. If $|X_0| \in \{\, |X_{p-1}|, |X_{p-1}| + 1 \,\}$, then there exists $s \in \Omega$ such that

$$|X_r| = |X_0| - \begin{cases} 0 & \text{if } r \leqslant s \\ 1 & \text{if } r > s. \end{cases}$$



Hence $\tau_0(X) = 0$. This shows that $|X_0| > |X_{p-1}| + 1$.

□

3.4.3 (A1) for $\operatorname{ord}(X) > 0$

In this subsection, we give a sufficient condition for a position to have a p^0-option and prove that if $|\lambda(X)| = \overline{\tau}(X) > 0$, then X satisfies this condition. This implies that (A1) holds. To this end, it is important to compare the difference between positions that have a p^0-option and those that do not. Let us give an example.

Example 3.4.10. Let $p = 3, X = \{3,4,5,9,10,11\}$, and $\widetilde{X} = \{1,2,3,9,13,14\}$. Then

$$\tau(X) = \tau(\widetilde{X}) = (0,0,3,0,\dots).$$

The position $\widetilde{X}$ has a 3^0-option. Indeed, let $\widetilde{Y} = (1\ 0)(\widetilde{X})$. Then

$$\tau(\widetilde{Y}) = (2,2,2,0,\dots),$$

so $\widetilde{Y}$ is a 3^0-option of $\widetilde{X}$. However, X does not have a 3^0-option. Indeed, if Y is a 3^0-option of X, then Y must be $(3\ 2)(X)$ or $(9\ 8)(X)$. Let Y be one of them. Then

$$\tau(Y) = (5,1,2,0,\dots).$$

Therefore $\widetilde{X}$ does not have a 3^0-option.

To illustrate the difference between X and $\widetilde{X}$, let us calculate $\left|X_{(r,R)}\right|$ and $\left|\widetilde{X}_{(r,R)}\right|$ for $r \in \Omega^1$ and $R \in \Omega^0 \cup \Omega^1 \cup \Omega^2$. The results are in Table 3.1. We see that

$$\left|\widetilde{X}_{(0,0,0)}\right| = 0 \neq 1 = \left|\widetilde{X}_{(1,0,0)}\right|,$$

but

$$\left|X_{(r,R)\ominus 1}\right| = \left|X_{(r,R)}\right| \text{ for } r \in \Omega^1 \text{ and } R \in \Omega^0 \cup \Omega^1 \cup \Omega^2.$$

In view of Example 3.4.10, we introduce an equivalence relation on positions. Let X and X' be two positions, and let N be a non-negative integer. We write

$$X \equiv X' \pmod{p^N} \tag{3.4.8}$$

to mean that $|X_R| = |X'_R|$ for each $R \in \Omega^N$. We see that $X \equiv X' \pmod{p^N}$ if and only if there exists a bijection $\Phi : X \to X'$ with $x \equiv \Phi(x) \pmod{p^N}$ for every $x \in X$.

For example, $\{x\} \equiv \{x'\} \pmod{p^N}$ if and only if $x \equiv x' \pmod{p^N}$ for $x, x' \in \mathbb{N}$. Let X and $\widetilde{X}$ be as in Example 3.4.10. Then

$$X_0 \equiv X_1 \equiv X_2 \pmod{3} \quad \text{and} \quad X_0 \equiv X_1 \equiv X_2 \pmod{3^2},$$

but

$$\widetilde{X}_0 \equiv \widetilde{X}_1 \equiv \widetilde{X}_2 \pmod{3} \quad \text{and} \quad \widetilde{X}_0 \not\equiv \widetilde{X}_1 \equiv \widetilde{X}_2 \pmod{3^2}.$$

By definition, this relation has the following properties.

Table 3.1: $\left|X_{(r,R)}\right|$ and $\left|\widetilde{X}_{(r,R)}\right|$ for $r \in \Omega^1$ and $R \in \Omega^0 \cup \Omega^1 \cup \Omega^2$

R	$\lvert X_{(0,R)}\rvert$	$\lvert X_{(1,R)}\rvert$	$\lvert X_{(2,R)}\rvert$
$()$	2	2	2
(0)	1	1	1
(1)	1	1	1
(2)	0	0	0
$(0,0)$	0	0	0
$(0,1)$	1	1	1
$(0,2)$	0	0	0
$(1,0)$	1	1	1
$(1,1)$	0	0	0
$(1,2)$	0	0	0
$(2,0)$	0	0	0
$(2,1)$	0	0	0
$(2,2)$	0	0	0

R	$\lvert \widetilde{X}_{(0,R)}\rvert$	$\lvert \widetilde{X}_{(1,R)}\rvert$	$\lvert \widetilde{X}_{(2,R)}\rvert$
$()$	2	2	2
(0)	1	1	1
(1)	1	1	1
(2)	0	0	0
$(0,0)$	0	1	1
$(0,1)$	1	0	0
$(0,2)$	0	0	0
$(1,0)$	1	0	0
$(1,1)$	0	1	1
$(1,2)$	0	0	0
$(2,0)$	0	0	0
$(2,1)$	0	0	0
$(2,2)$	0	0	0

Lemma 3.4.11. *Suppose that X and X' are two positions satisfying $X \equiv X' \pmod{p^N}$ for some $N \in \mathbb{N}$. Then $X \equiv X' \pmod{p^L}$ for $0 \leqslant L \leqslant N$, and $\tau_{<N}(X) = \tau_{<N}(X')$.*

The following lemma gives a sufficient condition for a position to have a p^0-option.

Lemma 3.4.12. *Let X be a non-terminal position with order M. If*

(P0) $(X^{[n]})_{s-1} \not\equiv (X^{[n]})_s \pmod{p^M}$ *for some $s \in \Omega$ and some $n \in \mathbb{N}$ with $|X^{[n]}| \equiv 0 \pmod{p^M}$,*

then X has a p^0-option.

proof. If $M = 0$, then the assertion follows from Lemma 3.4.5. Suppose that $M > 0$. By replacing X with $X^{[n]}$, we may assume that $|X| \equiv 0 \pmod{p^M}$

We first show that $|X_R| = |X|/p^L$ when $0 \leqslant L \leqslant M$ and $R \in \Omega^L$ by induction on L. This is trivial for $L = 0$. Suppose that $0 < L \leqslant M$. Note that if Y is a position with $|Y| \equiv 0 \pmod{p}$, then $\tau_0(Y) = 0$ if and only if $|Y_r| = |Y|/p$ for every $r \in \Omega^1$. Let $R' \in \Omega^{L-1}$. Then $|X'_R| = |X|/p^{L-1}$ by the induction hypothesis. In particular, $|X'_R| \equiv 0 \pmod{p}$. We also have $\tau_0(X_{R'}) = 0$, since $\sum_{V \in \Omega^{L-1}} \tau_0(X_V) = \tau_{L-1}(X) = 0$. Hence $|X_{(R',r)}| = |X'_R|/p = |X|/p^L$ for $r \in \Omega^1$.

Assuming the next claim for the moment, we complete the proof.

Claim. There exists $T \in \Omega^{M+1}$ such that $T_0 \neq 0$ and $|X_{T \ominus 1}| < |X_T|$.

Since $T_0 \neq 0$, we have $|X_{T\ominus 1}| = |X_{T-1}| = |X_{T-1}| + \delta_{T_0} \cdots \delta_{T_M}$. By Lemma 3.4.9, X has an option Y such that $\tau_{\geqslant M+1}(Y) \succeq \tau_{\geqslant M+1}(X)$ and $Y = (x\,x-1)(X)$ for some $x \in X$ with $x_{<M+1} = T$.

We show that Y is a p^0-option of X. By Lemma 3.4.8, $\tau_0(Y) - \tau_0(X) \equiv -1 \pmod{p}$ and

$$\begin{aligned}\tau_L(Y) - \tau_L(X) &\equiv |X_{x_{<L}}| - |X_{(x-1)_{<L}}| - \delta_{x_0} \cdots \delta_{x_{L-1}} - 1 \\ &\equiv |X|/p^L - |X|/p^L - 1 \equiv -1 \pmod{p} \quad \text{for } 0 < L \leqslant M.\end{aligned}$$

Thus Y is a p^0-option of X.

It remains to prove the claim. Since X satisfies (P0), we find that $|X_{S\ominus 1}| \neq |X_S|$ for some $S \in \Omega^{M+1}$. Hence there exists $s \in \Omega$ such that $|X_{S\ominus s\ominus 1}| < |X_{S\ominus s}|$, since otherwise $|X_{S\ominus 1}| > |X_S| \geqslant |X_{S\oplus 1}| \geqslant \cdots \geqslant |X_{S\oplus(p-1)}| = |X_{S\ominus 1}|$. Let

$$U = S \ominus s \quad \text{and} \quad \widehat{U} = (U_1, \ldots, U_{M-1}).$$

If $U_0 \neq 0$, then U satisfies the desired condition. Suppose that $U_0 = 0$. We show that there exist $T_0, T_M \in \Omega$ such that $T_0 \neq 0$ and

$$\left|X_{(T_0,\widehat{U},T_M)\ominus 1}\right| < \left|X_{(T_0,\widehat{U},T_M)}\right|, \tag{3.4.9}$$

where $(T_0, \widehat{U}, T_M) = (T_0, U_1, \ldots, U_{M-1}, T_M)$. Since $M > 0$, we have $(U_0, \widehat{U}), (U_0 \ominus 1, \widehat{U}) \in \Omega^M$. Hence

$$\left|X_{(U_0\ominus 1,\widehat{U})}\right| = \left|X\right|/p^M = \left|X_{(U_0,\widehat{U})}\right|.$$

This implies that

$$\sum_{r\in\Omega} \left|X_{(U_0\ominus 1,\widehat{U},r)}\right| = \left|X_{(U_0\ominus 1,\widehat{U})}\right| = \left|X\right|/p^M = \left|X_{(U_0,\widehat{U})}\right| = \sum_{r\in\Omega} \left|X_{(U_0,\widehat{U},r)}\right|.$$

Since $|X_{(U_0\ominus 1,\widehat{U},U_M)}| = |X_{U\ominus 1}| < |X_U| = |X_{(U_0,\widehat{U},U_M)}|$, we find that

$$|X_{(U_0\ominus 1,\widehat{U},T_M)}| > |X_{(U_0,\widehat{U},T_M)}|$$

for some $T_M \in \Omega$. As we have seen above, there exists $T_0 \in \Omega$ satisfying $T_0 \neq U_0 = 0$ and (3.4.9). $\square$

We now prove (A1) for $\operatorname{ord}(X) > 0$. Let M be the order of X. We may assume that $|X| \equiv 0 \pmod{p^M}$. By Lemmas 3.4.3 and 3.4.12, it is sufficient to show that X satisfies (P0). Assume that X does not satisfy (P0). By Lemma 3.4.11, $\tau_{M-1}(X_{r-1}) = \tau_{M-1}(X_r)$ for each $r \in \Omega$. Hence

$$0 \neq \tau_M(X) = \sum_{r\in\Omega} \tau_{M-1}(X_r) = p\tau_{M-1}(X_0) \geqslant p,$$

which contradicts $\tau(X) = \overline{\tau}(X)$. Thus X satisfies (P0). This completes the proof of (A1).

Remark 3.4.13. Let X be a non-terminal position with order M. The condition (P0) is independent of the choice of n, that is, if X satisfies (P0) and $|X^{[h]}| \equiv 0 \pmod{p^M}$, then

$$(X^{[h]})_{t-1} \not\equiv (X^{[h]})_t \pmod{p^M} \text{ for some } t \in \Omega. \tag{3.4.10}$$

Indeed, if $M = 0$, then $X^{[h]}$ always satisfies (3.4.10) since otherwise $\tau_0(X^{[h]}) = 0$. Suppose that $M > 0$. By replacing X with $X^{[h]}$, we may assume that $h = 0$. It suffices to show that

$$\left|(X^{[p^M]})_R\right| - \left|(X^{[p^M]})_{R\ominus 1}\right| = \left|X_{R\ominus p^M}\right| - \left|X_{R\ominus p^M \ominus 1}\right| \text{ for } R \in \Omega^{M+1}.$$

Recall that

$$X^{[p^M]} = \{x + p^M : x \in X\} \cup \{0, 1, \dots p^M - 1\}.$$

Hence

$$(X^{[p^M]})_R = \left\{x + p^M : x \in X\right\}_R \cup \left\{0, 1, \dots p^M - 1\right\}_R \tag{3.4.11}$$

for $R \in \Omega^{M+1}$. Let us calculate the right hand-side of (3.4.11). We have

$$\begin{aligned}\left\{x + p^M : x \in X\right\}_R &= \left\{(x + p^M)_{\geqslant M+1} : x \in X,\ (x + p^M)_{<M+1} = R\right\} \\ &= \left\{(x + p^M)_{\geqslant M+1} : x \in X,\ x_{<M+1} = R \ominus p^M\right\}\end{aligned}$$

and

$$\left\{0, 1, \dots p^M - 1\right\}_R = \begin{cases}\{0\} & \text{if } R < p^M, \\ \varnothing & \text{if } R \geqslant p^M.\end{cases}$$

Hence

$$\left|(X^{[p^M]})_R\right| = \left|X_{R\ominus p^M}\right| + \delta_{R_M}.$$

Note that $R_M = (R \ominus 1)_M$ since $M > 0$. It follows that

$$\begin{aligned}\left|(X^{[p^M]})_R\right| - \left|(X^{[p^M]})_{R\ominus 1}\right| &= \left|X_{R\ominus p^M}\right| + \delta_{R_M} - \left|X_{R\ominus 1\ominus p^M}\right| - \delta_{(R\ominus 1)_M} \\ &= \left|X_{R\ominus p^M}\right| - \left|X_{R\ominus 1\ominus p^M}\right|.\end{aligned}$$

Therefore (P0) is independent of the choice of n.

3.5 Proof of (A2)

In this section, we prove (A2) using *p^*-descendants* and *peak digits*. We first introduce them and present their properties. The key result is Lemma 3.5.7 in Subsection 3.5.2. In this subsection, we prove (A2) assuming this lemma. To prove this lemma, we study the condition (P0) of Lemma 3.4.12 in Subsection 3.5.4. Finally, in Subsection 3.5.5, we prove Lemma 3.5.7.

3.5.1 p^*-descendants and Peak Digits

Definition 3.5.1 (p*-descendants)**.** Let $n \in \mathbb{N}$. Let $(X^0, X^1, \dots, X^n)$ be a position sequence. If X^{i+1} is a p^*-option of X^i for $0 \leqslant i \leqslant n-1$, then this sequence is called a *p^*-path* from X^0 to X^n, and X^n is called a *p^*-descendant* of X^0.

Using p^*-descendants, we might find a descendant satisfying (A2). Let us give an example.

Example 3.5.2. Let $p = 3$. Let $X^0 = \{2,4,5\}$. Let $X^1 = (2\ 1)(X^0)$ and $X^2 = (1\ 0)(X^1)$. Then

$$\begin{aligned}\tau(X^0) &= (5,1,0,\dots),\\ \tau(X^1) &= (4,1,0,\dots),\\ \tau(X^2) &= (0,2,0,\dots),\end{aligned}$$

so (X^0, X^1, X^2) is a 3*-path. We also have

$$\overline{\tau}(X^2) = (0,2,0,\dots) > (2,1,0,\dots) = \overline{\tau}(X^0).$$

Hence X^2 satisfies (A2).

The above example leads us the following definition.

Definition 3.5.3 (peak digits)**.** Let X be a position. The *peak digit* $\mathrm{pk}(X)$ of X is defined by

$$\mathrm{pk}(X) = \max\{\, L \in \mathbb{N} : \tau_{\geqslant L}(Y) \succ \tau_{\geqslant L}(X) \ \text{ for some } \ p^*\text{-descendant } Y \text{ of } X \,\},$$

where $\max \varnothing = -1$.

For example, if X is as in Example 3.5.2, then $\mathrm{pk}(X) = 1$.

In the next subsection, we will deduce (A2) from a lemma on peak-digits. To this end, we give the basic properties of peak-digits. It follows from (3.2.11) that if $\mathrm{pk}(X) > -1$, then $\mathrm{pk}(X) > \mathrm{ord}(X) \geqslant 0$. In particular, $\mathrm{pk}(X) \neq 0$. Peak digits also have the following properties.

Lemma 3.5.4. *If Y is a p^*-option of a position X, then the following assertions hold:*

(1) $\tau_{\geqslant N}(Y) = \tau_{\geqslant N}(X)$*, where* $N = \max\{\mathrm{pk}(X), \mathrm{ord}(X)\} + 1$.

(2) $\mathrm{pk}(Y) \leqslant \mathrm{pk}(X)$.

proof. (1) By definition,

$$\tau_{\geqslant(\mathrm{pk}(X)+1)}(Y) \preceq \tau_{\geqslant(\mathrm{pk}(X)+1)}(X).$$

On the other hand,

$$\tau_{\geqslant(\mathrm{ord}(X)+1)}(Y) \succeq \tau_{\geqslant(\mathrm{ord}(X)+1)}(X),$$

since Y is a p^*-option of X. Thus $\tau_{\geqslant N}(Y) = \tau_{\geqslant N}(X)$.

(2) Let $K = \mathrm{pk}(Y)$. The assertion is trivial if $K = -1$. Suppose that $K > -1$. Then Y has a p^*-descendant Z with $\tau_{\geqslant K}(Z) \succ \tau_{\geqslant K}(Y)$. Since Z is also a p^*-descendant of X, it suffices to show that $\tau_{\geqslant K}(Z) \succ \tau_{\geqslant K}(X)$. We have

$$K = \mathrm{pk}(Y) > \mathrm{ord}(Y) \geqslant \mathrm{ord}(X).$$

Since Y is a p^*-option of X, it follows that

$$\tau_{\geqslant K}(X) \preceq \tau_{\geqslant K}(Y) \prec \tau_{\geqslant K}(Z).$$

□

Corollary 3.5.5. *If $(X^0, \ldots, X^n)$ is a p^*-path with $n > 0$, then $\tau_{\geqslant N}(X^i) = \tau_{\geqslant N}(X^0)$ for $0 \leqslant i \leqslant n$, where $N = \max\{\,\mathrm{pk}(X^0), \mathrm{ord}(X^{n-1})\,\} + 1$.*

proof. For $0 \leqslant i \leqslant n-1$, let $N^i = \max\left\{\,\mathrm{pk}(X^i), \mathrm{ord}(X^i)\,\right\} + 1$. It follows from Lemma 3.5.4 that $\tau_{\geqslant N^i}(X^{i+1}) = \tau_{\geqslant N^i}(X^i)$ and

$$\mathrm{pk}(X^0) \geqslant \mathrm{pk}(X^1) \geqslant \cdots \geqslant \mathrm{pk}(X^{n-1}).$$

Since

$$\mathrm{ord}(X^0) \leqslant \mathrm{ord}(X^1) \leqslant \cdots \leqslant \mathrm{ord}(X^{n-1}),$$

we have $N \geqslant N^i$ for $0 \leqslant i \leqslant n-1$. Hence $\tau_{\geqslant N}(X^i) = \tau_{\geqslant N}(X^0)$ for $0 \leqslant i \leqslant n$.

□

The next result provides a lower bound for $\mathrm{pk}(X)$.

Lemma 3.5.6. *Let X be a position with order M, and let N be an integer with $N \geqslant M+1$. If $|X_{S-p^M}| + \delta_{S_M}\delta_{S_{M+1}}\cdots\delta_{S_{N-1}} + 1 < |X_S|$ for some $S \in \Omega^N$, then $\mathrm{pk}(X) \geqslant N$.*

proof. It suffices to show that X has a p^*-option Y with $\tau_{\geqslant N}(Y) \succ \tau_{\geqslant N}(X)$. By Lemma 3.4.9, X has an option Y such that $\tau_{\geqslant N}(Y) \succeq \tau_{\geqslant N}(X)$ and $Y = (x\ x - p^M)(X)$ for some $x \in X$ with $x_{<N} = S$. We have $w_{\geqslant N}(Y) \succeq w_{\geqslant N}(X)$ and

$$w_N(Y) - w_N(X) = \left|X_{x_{<N}}\right| - \left|X_{(x-p^M)_{<N}}\right| - \delta_{x_M}\delta_{x_{M+1}}\cdots\delta_{x_{N-1}} - 1 > 0$$

by Lemma 3.4.8. It follows that $w_{\geqslant N}(Y) \succ w_{\geqslant N}(X)$, and hence $\tau_{\geqslant N}(Y) \succ \tau_{\geqslant N}(X)$. Therefore Y is a desired p^*-option of X.

□

3.5.2 Proof of (A2)

In this subsection, we will deduce (A2) from the next result.

Lemma 3.5.7. *Let X be a position with peak digit K. If K is positive, then X has a descendant Y with the following four properties:*

1. *(P0) is satisfied.*
2. $\mathrm{ord}(Y) = K$.
3. *Either $\tau_K(X) < \tau_K(Y) < p$ or $\tau_K(Y) = p$.*
4. $\tau_{\geqslant K+1}(Y) = \tau_{\geqslant K+1}(X)$.

We will prove Lemma 3.5.7 in Subsection 3.5.5. To prove (A2) using this lemma, we need a variation of Lemma 3.4.3.

Lemma 3.5.8. *Let X be a position with the following two properties:*

1. *$\tau_M(X) = p$, where $M = \mathrm{ord}(X)$.*
2. $\tau_{\geqslant M+1}(X) = \overline{\tau}_{\geqslant M+1}(X)(= (\overline{\tau}_{M+1}(X), \overline{\tau}_{M+2}(X), \ldots))$.

If X has a p^0-option Y, then $\overline{\tau}(Y) = |\lambda(X)| - 1$.

proof. Since

$$|\lambda(X)| = \sum_{L \in \mathbb{N}} \tau_L(X) p^L = p \cdot p^M + \sum_{L \geqslant M+1} \tau_L(X) p^L$$

and $\tau_{\geqslant M+1}(X) = \overline{\tau}_{\geqslant M+1}(X)$, we have

$$|\lambda(Y)| = |\lambda(X)| - 1 = (\underbrace{p-1, \ldots, p-1}_{M+1}, \tau_{M+1}(X), \tau_{M+2}(X), \ldots).$$

Thus $\tau_{\geqslant M+1}(Y) = |\lambda(Y)|_{\geqslant M+1}$ for otherwise

$$\sum_{L \in \mathbb{N}} \tau_L(Y) p^L > |\lambda(Y)|,$$

which contradicts (3.2.11). Since $\tau_L(Y) \equiv \tau_L(X) - 1 \equiv p - 1 \pmod{p}$ for $0 \leqslant L \leqslant M$, we find that $\overline{\tau}(Y) = |\lambda(Y)| = |\lambda(X)| - 1$.

□

We now prove (A2). Recall that the proof is by induction on $|\lambda(X)|$.

We first show (A2) when $\overline{\tau}(X_s) < |\lambda(X_s)|$ for some $s \in \Omega$. By the induction hypothesis, X_s has a proper descendant Y_s with $\overline{\tau}(Y_s) \geqslant \overline{\tau}(X_s)$. We may assume that $\overline{\tau}(Y_s) = \overline{\tau}(X_s)$. Indeed, suppose that $\overline{\tau}(Y_s) > \overline{\tau}(X_s)$. Since $\overline{\tau}(Y_s) = \mathrm{sg}(Y_s)$ by (3.3.2), it follows that Y_s has an option Y_s' with $\overline{\tau}(Y_s') = \overline{\tau}(X_s)$. Hence we may assume that $\overline{\tau}(Y_s) = \overline{\tau}(X_s)$ by replacing Y_s by Y_s'. Let $Y_r = X_r$ for $r \in \Omega \setminus \{ s \}$ and $Y = [Y_r]_{r \in \Omega}$. Then Y is a proper descendant of X with $\overline{\tau}(Y) = \overline{\tau}(X)$.

We next show (A2) when $\overline{\tau}(X_r) = |\lambda(X_r)|$ for each $r \in \Omega$. Since $\overline{\tau}(X) < |\lambda(X)|$, it follows that $\tau_L(X) \geqslant p$ for some $L \in \mathbb{N}$. Let

$$N = \max \{ L \in \mathbb{N} : \tau_L(X) \geqslant p \}.$$

Then $\tau_{\geqslant N+1}(X) = \overline{\tau}_{\geqslant N+1}(X)$. We divide into two cases.

Case 1 ($N > 0$)**.** Since

$$\overline{\tau}_N(X) < p \leqslant \tau_N(X) = \sum_{r \in \Omega} \tau_{N-1}(X_r),$$

there exist $s^0, \ldots s^{p-1} \in \Omega$ such that $\sum_{r \in \Omega} s^r = \overline{\tau}_N(X)$ and $s^r \leqslant \tau_{N-1}(X_r) = \overline{\tau}_{N-1}(X_r)$ for each $r \in \Omega$. Since $\mathrm{sg}(X_r) = \overline{\tau}(X_r)$, the position X_r has a descendant Y_r such that

$$\tau_L(Y_r) = \begin{cases} s^r & \text{if } L = N-1, \\ \tau_L(X_r) & \text{if } L \neq N-1. \end{cases}$$

Let $Y = [Y_r]_{r \in \Omega}$. Then $\overline{\tau}(Y) = \overline{\tau}(X)$ and $Y \neq X$.

Case 2 ($N = 0$)**.** Since $\tau_0(X) \geqslant p$, we have $\mathrm{ord}(X) = 0$. Let K be the peak digit of X.

Suppose that $K = -1$. By Lemma 3.5.4, if Y is a p^*-option of X, then $\mathrm{pk}(Y) = -1$ and

$$\tau(Y) = (\tau_0(X) - 1, \tau_1(X), \tau_2(X), \ldots).$$

Hence we obtain a descendant with the desired properties by repeatedly applying Lemma 3.4.5.

Suppose that $K > -1$. Then X has a descendant Y with the four properties in Lemma 3.5.7. If $\tau_K(X) < \tau_K(Y) < p$, then $\overline{\tau}(Y) > \overline{\tau}(X)$, so Y satisfies the desired condition. Suppose that $\tau_K(Y) = p$. Since Y satisfies (P0), this position has a p^0-option Z by Lemma 3.4.12. Since $\tau_{\geqslant K+1}(Y) = \tau_{\geqslant K+1}(X) = \overline{\tau}_{\geqslant K+1}(X)$, we have $\tau_{\geqslant K+1}(Y) = \overline{\tau}_{\geqslant K+1}(Y)$. It follows from Lemma 3.5.8 that

$$\begin{aligned} \overline{\tau}(Z) = |\lambda(Y)| - 1 &= (\underbrace{p-1, \ldots, p-1}_{K+1}, \tau_{K+1}(Y), \tau_{K+2}(Y), \ldots) \\ &= (\underbrace{p-1, \ldots, p-1}_{K+1}, \tau_{K+1}(X), \tau_{K+2}(X), \ldots) \geqslant \overline{\tau}(X). \end{aligned}$$

3.5.3 Easy Cases

In this subsection, we will prove Lemma 3.5.7 for some easy cases using the next result, which is a variant of Lemma 3.5.7.

Lemma 3.5.9. *Let X be a position with order M. If $\mathrm{pk}(X) = -1$ and $\tau_M(X) \geqslant p+1$, then X has a descendant Y with the following four properties:*

1. *(P0) is satisfied.*
2. $\mathrm{ord}(Y) = M$.
3. $\tau_M(Y) = p$.
4. $\tau_{\geqslant M+1}(Y) = \tau_{\geqslant M+1}(X)$.

proof. Suppose that $M = 0$. By repeatedly applying Lemma 3.4.5, we obtain a p^*-descendant Y of X such that $\tau_0(Y) = p$ and $\tau_{\geqslant 1}(Y) = \tau_{\geqslant 1}(X)$ since $\mathrm{pk}(X) = -1$. The position Y also satisfies (P0), since otherwise $\tau_0(Y) = 0$.

Suppose that $M > 0$. For each $r \in \Omega$, let $a^r = \tau_{M-1}(X_r)$.

We first show that there exists $(b^0, \dots, b^{p-1}) \in \mathbb{N}^p$ such that

$$\begin{aligned} &\textstyle\sum_{r\in\Omega} b^r = p, \\ &b^r \leqslant a^r \ \text{ for each } \ r \in \Omega, \quad b^s \neq b^t \ \text{ for some } \ s,t \in \Omega. \end{aligned} \tag{3.5.1}$$

It suffices to show the claim when $\sum_{r\in\Omega} a^r = p+1$. By rearranging a^i if necessary, we may assume that $a^0 \geqslant \dots \geqslant a^{p-1}$. Let

$$(b^0, \dots, b^{p-1}) = \begin{cases} (a^0 - 1, a^1, a^2, \dots, a^{p-1}) & \text{if } \ a^1 = 0, \\ (a^0, a^1 - 1, a^2, \dots, a^{p-1}) & \text{if } \ a^1 \neq 0. \end{cases}$$

Then $b^0 \neq b^1$. Hence $(b^0, \dots, b^{p-1})$ satisfies (3.5.1).

To construct a p^*-descendant Y with the desired properties, we next show that X_r has a p^*-descendant Y_r such that

$$\tau_{M-1}(Y_r) = b^r \ \text{ and } \ \tau_{\geqslant M}(Y_r) = \tau_{\geqslant M}(X_r) \tag{3.5.2}$$

for each $r \in \Omega$. If $b^r = a^r$, then X_r itself satisfies (3.5.2). Suppose that $b^r < a^r$. Then $\mathrm{ord}(X_r) = M-1$ since $\tau_{M-1}(X_r) = a^r \neq 0$. Let Z_r be a p^*-option of X_r. It is sufficient to show

$$\tau(Z_r) = (\underbrace{0, \dots, 0}_{M-1}, a^r - 1, \tau_M(X_r), \tau_{M+1}(X_r), \dots). \tag{3.5.3}$$

Let $Z_s = X_s$ for $s \in \Omega \setminus \{ r \}$ and $Z = [Z_s]_{s \in \Omega}$. Then Z is a p^*-option of X by Lemma 3.4.6. Since $\mathrm{pk}(X) = -1$, it follows that

$$\tau(Z) = (\underbrace{0, \ldots, 0}_{M}, \tau_M(X) - 1, \tau_{M+1}(X), \ldots),$$

which gives (3.5.3). Therefore we can obtain a p^*-descendant Y_r of X_r satisfying (3.5.2) by repeatedly applying Lemma 3.4.5.

Let $Y = [Y_r]_{r \in \Omega}$. Then Y is a p^*-descendant of X with

$$\tau_M(Y) = p \quad \text{and} \quad \tau_{\geqslant M+1}(Y) = \tau_{\geqslant M+1}(X).$$

Since $\overline{\tau}_{M-1}(Y_s) = b^s \neq b^t = \overline{\tau}_{M-1}(Y_t)$ for some $s, t \in \Omega$, the position Y also satisfies (P0) by Lemma 3.4.11.

□

Remark 3.5.10. We can now prove Remark 3.3.8 assuming Theorem 3.1.3 and Lemma 3.5.7. Let g be the right-hand side of (3.3.5). It suffices to show that X has a descendant Y with $\mathrm{msg}(Y) \geqslant g$. Let K be the peak digit of X. We split into two cases.

Case 1 ($K \geqslant N$)**.** The position X has a descendant Y satisfying the conditions in Lemma 3.5.7. If $\tau_K(X) < \tau_K(Y) < p$, then $\mathrm{msg}(Y) \geqslant \overline{\tau}(Y) > g$. If $\tau_K(Y) = p$, then $\mathrm{msg}(Y) \geqslant g$ by Lemma 3.5.8.

Case 2 ($K < N$)**.** Let $X^0 = X$. By repeatedly applying Lemma 3.4.5, we obtain a p^*-path $(X^0, \ldots, X^n)$ such that

(R1) $\mathrm{ord}(X^n) \geqslant N$,

(R2) $\mathrm{ord}(X^h) < N$ for $0 \leqslant h < n$.

Corollary 3.5.5 yields $\tau_{\geqslant N}(X^n) = \tau_{\geqslant N}(X)$. Suppose that $\tau_N(X^n) = \tau_N(X) \geqslant p+1$. Since $\mathrm{pk}(X^n) \leqslant \mathrm{pk}(X^0) = K < N = \mathrm{ord}(X^n)$, we have $\mathrm{pk}(X^n) = -1$. Hence X^n has a descendant Y with the four properties in Lemma 3.5.9. Since $\tau_{\geqslant N+1}(Y) = \tau_{\geqslant N+1}(X) = \overline{\tau}_{\geqslant N+1}(X) = \overline{\tau}_{\geqslant N+1}(Y)$, it follows from Lemmas 3.4.12 and 3.5.8 that $\mathrm{msg}(X^n) \geqslant g$. Suppose that $\tau_N(X^n) = \tau_N(X) = p$. If X^n satisfies (P0), then this position has a p^0-option by Lemma 3.4.12, so $\mathrm{msg}(X^n) \geqslant g$. Suppose that X^n does not satisfy (P0). Then $\sum_{r \in \Omega} \tau_{N-1}(X^n_r) = \tau_N(X^n) = p$ and $\tau_{N-1}(X^n_0) = \cdots = \tau_{N-1}(X^n_{p-1})$. Hence

$$\tau(X^n_r) = (\underbrace{0, \ldots, 0}_{N-1}, 1, \tau_N(X^n_r), \cdots) \quad \text{for} \quad r \in \Omega.$$

By Theorem 3.1.3, $\mathrm{sg}(X^n_r) = \overline{\tau}(X^n_r) = \tau(X^n_r)$. Therefore X^n_0 has an option Y_0 such that

$$\overline{\tau}(Y_0) = (\underbrace{p-1, \ldots, p-1}_{N-1}, 0, \tau_N(X^n_0), \cdots)(< \overline{\tau}(X^n_0)).$$

Let $Y_r = X^n_r$ for $r \in \Omega \setminus \{ 0 \}$ and $Y = [Y_r]_{r \in \Omega}$. Then $\mathrm{msg}(Y) \geqslant \overline{\tau}(Y) = g$. □

Let X^0 be a position and $N \in \mathbb{N}$. As we have seen above, there exists a p^*-path $(X^0, \ldots, X^n)$ satisfying (R1) and (R2). We call X^n an $(N-1)$*-rounded* p^**-descendant* of X^0.

We now turn to Lemma 3.5.7. Let X be a position whose peak digit K is positive. We may assume that $|X| \equiv 0 \pmod{p^K}$. Let Z be a p^*-descendant of X such that

$$\tau_K(Z) \geqslant \tau_K(Y) \tag{3.5.4}$$

for every p^*-descendant Y of X. Then $Z \neq X$. We also see that $\tau_{\geqslant K+1}(Z) = \tau_{\geqslant K+1}(X)$ by Corollary 3.5.5. Let Y be a $(K-1)$-rounded p^*-descendant of Z. The choices of Z and Y imply that

$$\operatorname{ord}(Y) = K > \operatorname{pk}(Z) \geqslant \operatorname{pk}(Y).$$

Thus $\operatorname{pk}(Y) = -1$ and $\tau_{\geqslant K}(Y) = \tau_{\geqslant K}(Z)$. If $\tau_K(Y) < p$, then Y satisfies the four properties in Lemma 3.5.7. If $\tau_K(Y) > p$, then Lemma 3.5.9 ensures that Y has a descendant with these four properties. We will show Lemma 3.5.7 for $\tau_K(Y) = p$ in the remaining two subsections.

3.5.4 The Condition (P0)

Let X, Y, Z, and K be as in the previous subsection, that is, X is a position whose peak digit K is positive, Z is a p^*-descendant of X satisfying (3.5.4), and Y is a $(K-1)$-rounded p^*-descendant of Z with $\tau_K(Y) = p$. If Y satisfies (P0), then this position satisfies the four properties in Lemma 3.5.7. Suppose that Y does not satisfy (P0). Then $\tau_0(Y_{(r,R)}) = \tau_0(Y_{(0,R)})$ for $r \in \Omega^1$ and $R \in \Omega^{K-1}$. Therefore

$$\text{there exists } S \in \Omega^{K-1} \text{ such that } \tau_0(Y_{(r,R)}) = \begin{cases} 1 & \text{if } R = S, r \in \Omega^1, \\ 0 & \text{if } R \neq S, r \in \Omega^1. \end{cases} \tag{3.5.5}$$

R	$\tau_0(Y_{(0,R)})$	$\tau_0(Y_{(1,R)})$	$\cdots$	$\tau_0(Y_{(p-1,R)})$
$(0,\ldots,0)$	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$S-1$	0	0	$\cdots$	0
S	1	1	$\cdots$	1
$S+1$	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(p-1,\ldots,p-1)$	0	0	$\cdots$	0

Our goal is to construct another $(K-1)$-rounded p^*-descendant $\widetilde{Y}$ of X such that $\tau_K(\widetilde{Y}) = p$ and $\widetilde{Y}$ does not satisfy (2.4.1). If $\widetilde{Y}$ is such a position, then $\widetilde{Y}$ satisfies (P0) and the other three properties in Lemma 3.5.7.

Let $X^0 = X$ and $X^n = Y$. Let $(X^0, \ldots, X^n)$ be a p^*-path from X^0 to X^n through Z, that is, $X^h = Z$ for some h with $0 < h \leqslant n$. Replacing X^h by X^{h-1} if necessary, we may also assume that $\tau_K(X^{h-1}) < \tau_K(X^h) = p$. Let

$$X^{i+1} = (x^i\, x^i - p^{M^i})(X^i), \quad S^i = x^i_{<K}, \qquad \text{and} \qquad T^i = S^i - p^{M^i} \qquad \text{for } 0 \leqslant i < n.$$

Then

$$\left|X_R^{i+1}\right| = \left|X_R^i\right| + \begin{cases} -1 & \text{if } R = S^i, \\ 1 & \text{if } R = T^i, \\ 0 & \text{if } R \in \Omega^K \setminus \{ S^i, T^i \}. \end{cases} \tag{3.5.6}$$

The next lemma shows that if $\tau_0(X^j_{S^j}) + \tau_0(X^j_{T^j})$ is at least two for some $j \in \{h, \cdots, n-1\}$, then X has a $(K-1)$-rounded p^*-descendant with the desired properties.

Lemma 3.5.11. *Let X be a position and Y a p^*-option $(x\ x - p^M)(X)$ of X. Let N be a non-negative integer with $N \geqslant \max\{M+1, \mathrm{pk}(X)\}$. If $\tau_0(Y_S) + \tau_0(Y_T) \geqslant 2$, where $S = x_{<N}$ and $T = S - p^M$, then X has a p^*-option $\widetilde{Y}$ such that*

1. $\widetilde{Y} \equiv Y \pmod{p^N}$,
2. $\tau_0(\widetilde{Y}_S) \geqslant 2$ *or* $\tau_0(\widetilde{Y}_T) \geqslant 2$.

Before proving Lemma 3.5.11, let us give an easy example.

Let $p = 3$. Let $X = \{1,4,6\}$ and $Y = (1\ 0)(X)$. Then $\tau(X) = (2,2,0,\ldots)$ and $\tau(Y) = (1,2,0,\ldots)$. Hence Y is a 3^*-option of X. Note that $\tau_0(Y_{(0)}) = \tau_0(Y_{(1)}) = 1$. Lemma 3.5.11 asserts that X has another 3^*-option. Indeed, let $\widetilde{Y} = (4\ 3)(X)$. Then $\tau(\widetilde{Y}) = (1,2,0,\ldots)$, so $\widetilde{Y}$ is a 3^*-option of X. Moreover, $\tau_0(\widetilde{Y}_{(0)}) = 2$.

R	$\lvert X_{(0,R)}\rvert$	$\lvert X_{(1,R)}\rvert$	$\lvert X_{(2,R)}\rvert$
(0)	0	1	0
(1)	0	1	0
(2)	1	0	0

R	$\lvert Y_{(0,R)}\rvert$	$\lvert Y_{(1,R)}\rvert$	$\lvert Y_{(2,R)}\rvert$
(0)	$\underline{1}$	$\underline{0}$	0
(1)	0	1	0
(2)	1	0	0

R	$\lvert \widetilde{Y}_{(0,R)}\rvert$	$\lvert \widetilde{Y}_{(1,R)}\rvert$	$\lvert \widetilde{Y}_{(2,R)}\rvert$
(0)	0	1	0
(1)	$\underline{1}$	$\underline{0}$	0
(2)	1	0	0

Lemma 3.5.11. We may assume that $\tau_0(Y_S) = \tau_0(Y_T) = 1$. Then

$$(Y_S)_{(p)} = \{1\}^{[|Y_S|-1]} \quad \text{and} \quad (Y_T)_{(p)} = \{1\}^{[|Y_T|-1]}, \tag{3.5.7}$$

where $(Y_S)_{(p)}$ is the p-core of Y_S. Replacing X by $X^{[p^M]}$ if necessary, we may assume that $x_M \neq 0$. Let $\Delta = \tau_N(Y) - \tau_N(X)$.

We divide the proof into two parts. First, we prove $\Delta \in \{0,1,2\}$ and the following three relations:

$$\left|Y_S\right| = \left|Y_T\right| + \Delta - 1. \tag{3.5.8}$$

$$\left|Y_{(S,r)}\right| \leqslant \left|Y_{(T,r)}\right| + 1 \ \text{ for } \ r \in \Omega. \tag{3.5.9}$$

$$\left|Y_{(S,x_N)}\right| = \left|Y_{(T,x_N)}\right| - 1. \tag{3.5.10}$$

Next, we split into three cases depending on Δ.

Part 1 We first show that $\Delta \in \{0,1,2\}$. Since $N \geqslant \max\{M+1, \mathrm{pk}(X)\}$, it follows from Lemma 3.5.4 that $\tau_{\geqslant N+1}(Y) = \tau_{\geqslant N+1}(X)$. This implies that $\Delta \geqslant 0$ since Y is a p^*-option of X. We also have

$$\Delta = \sum_{R \in \Omega^N} \tau_0(Y_R) - \tau_0(X_R) = \tau_0(Y_S) + \tau_0(Y_T) - \tau_0(X_S) - \tau_0(X_T).$$

Since $\tau_0(Y_S) + \tau_0(Y_T) = 2$ and $\Delta \geqslant 0$, it follows that $\Delta \in \{0,1,2\}$.

We next show (3.5.8). Since $\tau_{\geqslant N+1}(Y) = \tau_{\geqslant N+1}(X)$, it follows that

$$w_{\geqslant N+1}(Y) = w_{\geqslant N+1}(X),$$

so $\Delta = w_N(Y) - w_N(X)$. Hence, by Lemma 3.4.8, $|X_S| = |X_T| + \Delta + 1$. Since $|Y_S| = |X_S| - 1$ and $|Y_T| = |X_T| + 1$, we have (3.5.8).

Finally, we show (3.5.9) and (3.5.10). Since $N+1 > \mathrm{pk}(X)$, we see that

$$|X_{(S,r)}| \leqslant |X_{(S,r)-p^M}| + \delta_{S_M} \cdots \delta_{S_N} + 1 = |X_{(T,r)}| + 1$$

by Lemma 3.5.6. Hence (3.5.9) holds. Since $w_{N+1}(Y) = w_{N+1}(X)$, it follows from Lemm 3.4.8 that $|X_{(S,x_N)}| = |X_{(T,x_N)}| + 1$. Thus we obtain (3.5.10).

Part 2 Let $t = |Y_T|_0$. We split into three cases depending on Δ.

Case 1 ($\Delta = 0$)**.** We have $|Y_S| = |Y_T| - 1$. Suppose that $p = 2$. By (3.5.7),

$$\left|Y_{(S,t)}\right| = |Y_{(T,t)}| - 2 \qquad \text{and} \qquad \left|Y_{(S,t-1)}\right| = |Y_{(T,t-1)}| + 1.$$

See Table 3.2. This contradicts (3.5.10). Suppose that $p > 2$. By (3.5.7),

$$\left|Y_{(S,r)}\right| = \left|Y_{(T,r)}\right| + \begin{cases} -1 & \text{if } r \in \{t, t-2\}, \\ 1 & \text{if } r = t-1, \\ 0 & \text{if } r \in \Omega \setminus \{t, t-1, t-2\}. \end{cases}$$

See Table 3.3. Thus $x_N \in \{t, t-2\}$ and $|X_{(S,t-1)}| = |X_{(S,t-1)-p^M}| + 1$. It follows from Lemma 3.4.9 that X has an option $\widetilde{Y}$ such that $\tau_{\geqslant N+1}(\widetilde{Y}) \succeq \tau_{\geqslant N+1}(X)$ and $\widetilde{Y} = (\tilde{x}\,\tilde{x} - p^M)(X)$ for some $\tilde{x} \in X$ with $\tilde{x}_{<N+1} = (S, t-1)$. Since $\tilde{x}_N = t-1$, we find that

$$(\tau_0(\widetilde{Y}_S), \tau_0(\widetilde{Y}_T)) = \begin{cases} (2,0) & \text{if } x_N = t, \\ (0,2) & \text{if } x_N = t-2. \end{cases}$$

It remains to show that $\widetilde{Y}$ is a p^*-option of X such that $\widetilde{Y} \equiv Y \pmod{p^N}$. Since $\tau_{\geqslant N+1}(\widetilde{Y}) \succeq \tau_{\geqslant N+1}(X)$ and $w_N(\widetilde{Y}) - w_N(X) = |X_S| - |X_T| - 1 = 0$, we have $\tau_{\geqslant N}(\widetilde{Y}) \succeq \tau_{\geqslant N}(X) = \tau_{\geqslant N}(Y)$. Moreover, since $\tilde{x}_{<N} = x_{<N}$, it follows that $\widetilde{Y} \equiv Y \pmod{p^N}$, and hence that $\tau_{<N}(\widetilde{Y}) = \tau_{<N}(Y)$. Therefore $\tau_{\geqslant M+1}(\widetilde{Y}) \succeq \tau_{\geqslant M+1}(Y) \succeq \tau_{\geqslant M+1}(X)$.

Case 2 ($\Delta = 1$)**.** We have $|Y_S| = |Y_T|$. Hence $|Y_{(S,r)}| = |Y_{(T,r)}|$ for each $r \in \Omega$, which is a contradiction.

Case 3 ($\Delta = 2$)**.** We have $|Y_S| = |Y_T| + 1$. If $p = 2$, then by (3.5.7),

$$|Y_{(S,t-1)}| = |Y_{(T,t-1)}| + 2,$$

which contradicts (3.5.9). Suppose that $p > 2$. By (3.5.7),

$$\left|Y_{(S,r)}\right| = \left|Y_{(T,r)}\right| + \begin{cases} -1 & \text{if } r = t, \\ 1 & \text{if } r \in \{t-1, t+1\}, \\ 0 & \text{if } r \in \Omega \setminus \{t, t-1, t+1\}. \end{cases}$$

Thus $x_N = t$. By Lemma 3.4.9, the position X has an option $\widetilde{Y}$ such that $\widetilde{Y} = (\tilde{x}\,\tilde{x} - p^M)(X)$ for some $\tilde{x} \in X$ with $\tilde{x}_{<N+1} = (S, t-1)$. It follows that $\tau_N(\widetilde{Y}) = \tau_N(X) + 2, \widetilde{Y} \equiv Y \pmod{p^N}$, and $\tau_0(\widetilde{Y}_S) = 2$. Therefore $\widetilde{Y}$ is a desired p^*-option of X.

Table 3.2: $p = 2$.

$\lvert Y_T\rvert$	$2a-1$	$2a$	$2a+1$
$\lvert Y_{(T,0)}\rvert$	$a-1$	$a+1$	a
$\lvert Y_{(T,1)}\rvert$	a	$a-1$	$a+1$

Table 3.3: $p > 2$.

$\lvert Y_T\rvert$	$pa-1$	pa	$pa+1$	$\cdots$	$pa+s-1$	$pa+s$
$\lvert Y_{(T,0)}\rvert$	a	$a+1$	a		$a+1$	$a+1$
$\lvert Y_{(T,1)}\rvert$	a	a	$a+1$		$a+1$	$a+1$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\lvert Y_{(T,s-3)}\rvert$	a	a	a		$a+1$	$a+1$
$\lvert Y_{(T,s-2)}\rvert$	a	a	a		a	$a+1$
$\lvert Y_{(T,s-1)}\rvert$	a	a	a		$a+1$	a
$\lvert Y_{(T,s)}\rvert$	a	a	a		a	$a+1$
$\lvert Y_{(T,s+1)}\rvert$	a	a	a		a	a
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$\lvert Y_{(T,p-2)}\rvert$	$a-1$	a	a		a	a
$\lvert Y_{(T,p-1)}\rvert$	a	$a-1$	a		a	a

□

It remains to prove Lemma 3.5.7 when $\tau_0(X^i_{S^i}) + \tau_0(X^i_{T^i}) \leqslant 1$ for $h \leqslant i \leqslant n-1$. We first show that X^{h-1} has another p^*-option $\widetilde{X}^h$ such that $\widetilde{X}^h \equiv X^h \pmod{p^K}$ in Lemma 3.5.12. We next show that there exists a p^*-path $(\widetilde{X}^h, \ldots, \widetilde{X}^n)$ such that $\widetilde{X}^i \equiv X^i$ for every $i \in \{h, \ldots, n\}$ in Lemma 3.5.13. Finally, we prove that $\widetilde{X}^n$ satisfies (P0).

Lemma 3.5.12. *Let X, Y, M, N, S, and T be as in Lemma 3.5.11. Let $\Delta = \tau_N(Y) - \tau_N(X)$. Suppose that $\tau_0(Y_S) + \tau_0(Y_T) = 1$. Then the following assertions hold:*

(1) *If* $\Delta = 1$, *then* X *has a* p^**-option* $\widetilde{Y}$ *such that* $\widetilde{Y} \equiv Y \pmod{p^N}$ *and*

$$\big(\tau_0(\widetilde{Y}_S), \tau_0(\widetilde{Y}_T)\big) = \big(\tau_0(Y_T), \tau_0(Y_S)\big).$$

(2) *If* $\Delta = 0$, *then* $\big(\tau_0(X_S), \tau_0(X_T)\big) = \big(\tau_0(Y_T), \tau_0(Y_S)\big)$.

proof. Suppose that $\Delta = 1$. Then $\tau_0(X_S) = \tau_0(X_T) = 0$. Replacing X by $X^{[p^M]}$ if necessary, we may assume that $x_M \neq 0$. Lemma 3.4.8 yields $|X_S| = |X_T| + 2$. Let $t = |X_T|_0$. Since $(X_S)_{(p)} = \varnothing^{[|X_S|]}$ and $(X_T)_{(p)} = \varnothing^{[|X_T|]}$, we have

$$\big|X_{(S,r)}\big| = \big|X_{(T,r)}\big| + \begin{cases} 1 & \text{if } r \in \{\, t, t+1 \,\}, \\ 0 & \text{if } r \in \Omega \setminus \{\, t, t+1 \,\}. \end{cases}$$

If $x_N = t$, then $\tau_0(Y_S) = 1$. If $x_N = t+1$, then $\tau_0(Y_T) = 1$. It follows from Lemma 3.4.9 that X has a p^*-option with the desired properties.

The proof for $\Delta = 0$ is similar.

□

We next show the existence of another p^*-path.

Lemma 3.5.13. *Let* $(X^0, \dots, X^n)$ *be a* p^**-path and* $\widetilde{X}^0$ *a position such that* $\widetilde{X}^0 \equiv X^0 \pmod{p^N}$ *for some* $N \in \mathbb{N}$. *Suppose that* $\tau_{\geqslant N}(X^i) = \tau_{\geqslant N}(X^0)$ *for* $0 \leqslant i \leqslant n$. *Then there exists a* p^**-path* $(\widetilde{X}^0, \dots, \widetilde{X}^n)$ *such that for* $0 \leqslant i \leqslant n-1$,

1. $\operatorname{ord}(\widetilde{X}^i) = \operatorname{ord}(X^i)$,
2. $\tilde{x}^i \equiv x^i \pmod{p^N}$,

where $X^{i+1} = (x^i\ x^i - p^{M^i})(X^i)$ *and* $\widetilde{X}^{i+1} = (\tilde{x}^i\ \tilde{x}^i - p^{M^i})(\widetilde{X}^i)$. *In particular,* $X^i \equiv \widetilde{X}^i \pmod{p^N}$ *for* $0 \leqslant i \leqslant n$.

proof. The proof is by induction on n. If $n = 0$, then the assertion is trivial. Suppose that $n > 0$. By the induction hypothesis, there exists a p^*-path $(\widetilde{X}^0, \dots, \widetilde{X}^{n-1})$ satisfying (1) and (2). Let $X = X^{n-1}$, $\widetilde{X} = \widetilde{X}^{n-1}$, and $Y = X^n = (x\ x - p^M)(X)$.

We first show that $\operatorname{ord}(\widetilde{X}) = \operatorname{ord}(X) = M$. Since $\widetilde{X} \equiv X \pmod{p^N}$, it follows from Lemma 3.4.11 that $\tau_{<N}(\widetilde{X}) = \tau_{<N}(X)$. Since $\tau_{\geqslant N}(X) = \tau_{\geqslant N}(Y)$, we also have $M < N$. Hence $\operatorname{ord}(\widetilde{X}) = \operatorname{ord}(X) = M$.

We next construct a p^*-option $\widetilde{Y}$ of $\widetilde{X}$ such that $\widetilde{Y} = (\tilde{x}\ \tilde{x} - p^M)(\widetilde{X})$ for some $\tilde{x} \in \widetilde{X}$ with $\tilde{x} \equiv x \pmod{p^N}$. Since $\widetilde{X} \equiv X \pmod{p^N}$ and $w_N(X) = w_N(Y)$, it follows from Lemma 3.4.8 that

$$\begin{aligned} \big|\widetilde{X}_{(x-p^M)_{<N}}\big| + \delta_{x_M} \cdots \delta_{x_{N-1}} + 1 &= \big|X_{(x-p^M)_{<N}}\big| + \delta_{x_M} \cdots \delta_{x_{N-1}} + 1 \\ &= \big|X_{x_{<N}}\big| = \big|\widetilde{X}_{x_{<N}}\big|. \end{aligned}$$

Lemma 3.4.9 implies that $\widetilde{X}$ has an option $\widetilde{Y}$ such that $\tau_{\geqslant N}(\widetilde{Y}) \succeq \tau_{\geqslant N}(\widetilde{X})$ and $\widetilde{Y} = (\tilde{x}\,\tilde{x} - p^M)(X)$ for some $\tilde{x} \in \widetilde{X}$ with $\tilde{x} = x \pmod{p^N}$. Since $\widetilde{X} \equiv X \pmod{p^N}$, we have $\widetilde{Y} \equiv Y \pmod{p^N}$. It remains to verify $\tau_{\geqslant M+1}(\widetilde{Y}) \succeq \tau_{\geqslant M+1}(\widetilde{X})$. Recall that

$$\tau_{\geqslant N}(\widetilde{Y}) \succeq \tau_{\geqslant N}(\widetilde{X}),\ \tau_{\geqslant N}(Y) = \tau_{\geqslant N}(X),$$

$$\tau_{\geqslant M+1}(Y) \succeq \tau_{\geqslant M+1}(X), \text{ and } \tau_{<N}(\widetilde{X}) = \tau_{<N}(X).$$

In addition, since $\widetilde{Y} \equiv Y \pmod{p^N}$, it follows that $\tau_{<N}(\widetilde{Y}) = \tau_{<N}(Y)$. This shows that

$$\tau_{\geqslant M+1}(\widetilde{Y}) \succeq \tau_{\geqslant M+1}(\widetilde{X}).$$

Therefore $\widetilde{Y}$ is a p^*-option of $\widetilde{X}$.

□

3.5.5 Proof of Lemma 3.5.7

proof. Let X, Y, Z, K, and $(X^0, \dots, X^h, \dots, X^n)$ be as in the previous subsection. Let $W = X^{h-1}$ and $Z = (w\ w - p^M)(W)$, and let $S = w_{<K}$ and $T = S - p^M$. Since $\tau_K(W) < \tau_K(Z) = p$ and $\mathrm{ord}(W) < \mathrm{pk}(W) = K$, it follows from Lemma 3.5.11 that we may assume that $\tau_0(Z_S) + \tau_0(Z_T) = 1$. Hence $\tau_K(W) = p - 1$ and

$$\big(\tau_0(Z_S),\, \tau_0(Z_T)\big) \in \{\,(0,1),(1,0)\,\}.$$

Lemma 3.5.12 implies that W has another p^*-option $\widetilde{Z}$ such that

$$\begin{aligned}
&\widetilde{Z} \equiv Z \pmod{p^K},\\
&\big(\tau_0(\widetilde{Z}_S),\, \tau_0(\widetilde{Z}_T)\big) = \big(\tau_0(Z_T),\, \tau_0(Z_S)\big),\\
&\tau_0(\widetilde{Z}_R) = \tau_0(Z_R) \qquad \text{for each } R \in \Omega^K \backslash \{\,S,T\,\}.
\end{aligned} \tag{3.5.11}$$

Note that $\tau_K(\widetilde{Z}) = p$, and that $\tau_{\geqslant K+1}(\widetilde{Z}) = \tau_{\geqslant K+1}(X)$ by Corollary 3.5.5. Let $\widetilde{X}^h = \widetilde{Z}$. Since $\tau_{\geqslant K}(X^i) = \tau_{\geqslant K}(X^h)$ for $h \leqslant i \leqslant n$, it follows from Lemma 3.5.13 that there exists a p^*-path $(\widetilde{X}^h, \dots, \widetilde{X}^n)$ such that for $h \leqslant i \leqslant n-1$,

$$\mathrm{ord}(\widetilde{X}^i) = \mathrm{ord}(X^i) \quad \text{and} \quad \tilde{x}^i \equiv x^i \pmod{p^K},$$

where

$$X^{i+1} = (x^i\ x^i - p^{M^i})(X^i) \quad \text{and} \quad \widetilde{X}^{i+1} = (\tilde{x}^i\ \tilde{x}^i - p^{M^i})(\widetilde{X}^i).$$

Let $\widetilde{Y} = \widetilde{X}^n$. Then $\widetilde{Y} \equiv Y \pmod{p^K}$. We show that $\widetilde{Y}$ satisfies the desired four properties. Since $K \geqslant \max\{\,\mathrm{pk}(\widetilde{X}^h), \mathrm{ord}(\widetilde{X}^{n-1})\,\} + 1$, it follows from Corollary 3.5.5 that $\tau_{\geqslant K}(\widetilde{Y}) = \tau_{\geqslant K}(\widetilde{Z})$. Hence $\tau_K(\widetilde{Y}) = p$, $\mathrm{ord}(\widetilde{Y}) = K$, and $\tau_{\geqslant K+1}(\widetilde{Y}) = \tau_{\geqslant K+1}(X)$.

It remains to show that $\widetilde{Y}$ satisfies (P0). It suffices to show that $\widetilde{Y}$ does not satisfy (2.4.1). Let $S^i = x^i_{<K}$ and $T^i = S^i - p^{M^i}$ for $h \leqslant i \leqslant n-1$. Recall that, for each $R \in \Omega^K \setminus \{ S^i, T^i \}$, we have $X^{i+1}_R = X^i_R$ and $\widetilde{X}^{i+1}_R = \widetilde{X}^i_R$. By Lemma 3.5.11, we may assume that

$$\big(\tau_0(X^i_{S^i}), \tau_0(X^i_{T^i})\big), \big(\tau_0(\widetilde{X}^i_{S^i}), \tau_0(\widetilde{X}^i_{T^i})\big) \in \{ (0,0), (0,1), (1,0) \}$$

for $h \leqslant i \leqslant n-1$. It follows from Lemma 3.5.12 that

$$\begin{aligned} \big(\tau_0(X^{i+1}_{S^i}),\, \tau_0(X^{i+1}_{T^i})\big) &= \big(\tau_0(X^i_{T^i}),\, \tau_0(X^i_{S^i})\big), \\ \big(\tau_0(\widetilde{X}^{i+1}_{S^i}),\, \tau_0(\widetilde{X}^{i+1}_{T^i})\big) &= \big(\tau_0(\widetilde{X}^i_{T^i}),\, \tau_0(\widetilde{X}^i_{S^i})\big). \end{aligned} \tag{3.5.12}$$

By (3.5.11) and (3.5.12),

$$\begin{aligned} &\big(\tau_0(\widetilde{Y}_U),\, \tau_0(\widetilde{Y}_V)\big) = \big(\tau_0(Y_V),\, \tau_0(Y_U)\big) = (0,1) \qquad \text{for some } \ U, V \in \Omega^K, \\ &\tau_0(\widetilde{Y}_R) = \tau_0(Y_R) \qquad \text{for each } \ R \in \Omega^K \setminus \{ U, V \}. \end{aligned}$$

Since Y satisfies (2.4.1), the position $\widetilde{Y}$ does not satisfy this. Therefore $\widetilde{Y}$ satisfies the condition (P0).

□

Bibliography

[1] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning Ways for Your Mathematical Plays*. A.K. Peters, Natick, Mass., 2nd edition, 2001.

[2] Charles L. Bouton. Nim, a game with a complete mathematical theory. *The Annals of Mathematics*, 3(1/4):35–39, 1901.

[3] Grant Cairns, Nhan Bao Ho, and Tamás Lengyel. The Sprague–Grundy function of the real game Euclid. *Discrete Mathematics*, 311(6):457–462, 2011.

[4] Alfred J. Cole and A. J. T. Davie. A game based on the Euclidean algorithm and a winning strategy for it. *The Mathematical Gazette*, 53(386):354–357, 1969.

[5] John H. Conway. *On Numbers and Games*. A.K. Peters, Natick, Mass., 2nd edition, 2001.

[6] John H. Conway and Neil J. A. Sloane. Lexicographic Codes: Error-Correcting Codes from Game Theory. *IEEE Transactions on Information Theory*, 1986.

[7] Alex Fink, Aviezri S. Fraenkel, and Carlos Santos. LIM is not slim. *International Journal of Game Theory*, 43(2):269–281, May 2013.

[8] James A. Flanigan. Nim, Trim and Rim. unpublished document, Mathematics Department, University of California, Los Angeles, 1980.

[9] Aviezri S. Fraenkel and Mordechai Lorberbom. Nimhoff games. *Journal of Combinatorial Theory, Series A*, 58(1):1–25, September 1991.

[10] James S. Frame, Gilbert de B. Robinson, and Robert M. Thrall. The hook graphs of the symmetric group. *Canadian Journal of Mathematics*, 6:316–324, 1954.

[11] Jerrold W. Grossman. A nim-type game, problem# 1537. *Mathematics Magazine*, 70:382, 1997.

[12] Patrick M. Grundy. Mathematics and games. *Eureka*, 2:6–8, 1939.

[13] Yuki Irie. *p*-saturations of Welter's Game and the Irreducible Representations of Symmetric Groups. to appear.

[14] Gordon D. James. Some combinatorial results involving Young diagrams. *Mathematical Proceedings of the Cambridge Philosophical Society*, 83(1):1–10, January 1978.

[15] Joseph Kahane and Alexander J. Ryba. The Hexad Game. *The Electronic Journal of Combinatorics*, 8(2):R11, May 2000.

[16] Noriaki Kawanaka. Games and Algorithms with Hook Structure. *Sugaku*, 63(4):421–441, 2011.

[17] Ian G. Macdonald. On the Degrees of the Irreducible Representations of Symmetric Groups. *Bulletin of the London Mathematical Society*, 3(2):189–192, January 1971.

[18] Eliakim H. Moore. A Generalization of the Game Called Nim. *Annals of Mathematics*, 11(3):93–94, 1910.

[19] Gabriel Nivasch. The Sprague–Grundy function of the game Euclid. *Discrete Mathematics*, 306(21):2798–2800, 11 6, 2006.

[20] Jørn B. Olsson. McKay numbers and heights of characters. *Mathematica Scandinavica*, 38:25–42, June 1976.

[21] Jørn B. Olsson. *Combinatorics and Representations of Finite Groups*. Vorlesungen aus dem FB Mathematik der Univ. Essen, Heft 20, 1993.

[22] Robert A. Proctor. Dynkin Diagram Classification of λ-Minuscule Bruhat Lattices and of d-Complete Posets. *Journal of Algebraic Combinatorics*, 9(1):61–94, January 1999.

[23] Robert A. Proctor. Minuscule Elements of Weyl Groups, the Numbers Game, and d-Complete Posets. *Journal of Algebra*, 213(1):272–303, March 1999.

[24] Bruce E. Sagan. *The Symmetric Group*, volume 203 of *Graduate Texts in Mathematics*. Springer New York, 2001.

[25] Mikio Sato. On a game (notes by Kenji Ueno)(in Japanese). In *Proceedings of the 12th Symposium of the Algebra Section of the Mathematical Society of Japan*, pages 123–136, 1968.

[26] Mikio Sato. Mathematical theory of Maya game (notes by Hikoe Enomoto)(in Japanese). *RIMS Kôkyûroku*, 98:105–135, 1970.

[27] Mikio Sato. On Maya game (notes by Hikoe Enomoto)(in Japansese). *Sugaku no Ayumi*, 15(1):73–84, 1970.

[28] Mikio Sato. *Lecture Notes in Kyoto University (1984-1985)(notes by Toru Umeda)(in Japanese)*. Number 5 in Surikaiseki lecture note. 1989.

[29] Roland P. Sprague. Über mathematische Kampfspiele. *Tohoku Mathematical Journal, First Series*, 41:438–444, 1935.

[30] Cornelius P. Welter. The Theory of a Class of Games on a Sequence of Squares, in Terms of the Advancing Operation in a Special Group. *Indagationes Mathematicae (Proceedings)*, 57:194–200, 1954.