A New Class of Doubly Nonlinear Evolution Equations

M. ASO, T. FUKAO and N. KENMOCHI

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M. Aso, T. Fukao and N. Kenmochi

Abstract. In this paper we consider the solvability for a new class of doubly nonlinear evolution equations. The motivation of this work comes from a transmission problem of two degenerate parabolic equations with convection term, in which the transmission boundary is time-dependent. We give an abstract existence result, and show that the weak variational formulation for the transmission problem can be solved by applying this abstract result. In our existence proof, the abstract theory of pseudo-monotone operators is useful.

1 Introduction

Let $0 < T < \infty$. We consider an evolution equation of the form:

\[ u'(t) + K(t, \theta(t)) + G(t, u(t)) = f(t) \quad \text{in } V^* \text{ for a.e. } t \in [0, T], \quad (1.1) \]

\[ u(0) = u_0, \quad (1.2) \]

where $u' := du/dt$. Here, for each $t \in [0, T]$, $K(t, \cdot)$ is a weakly continuous operator in a reflexive Banach space $V$, $G(t, \cdot)$ is a weakly continuous operator in a Hilbert space $H$, where $V$ is imbedded densely and compactly in $H$, $f$ is a given source function and $u_0$ is an initial datum. Equation (1.1) is considered with the following relation between $\theta$ and $u$:

\[ \theta(t) = \partial\psi^d(u(t)) \quad \text{in } V \text{ for a.e. } t \in [0, T], \quad (1.3) \]

where $\{\psi^d\}$ is a family of proper, lower semicontinuous and convex functions on the dual space $V^*$ of $V$ and $\partial\psi^d$ is the subdifferential of $\psi^d$ from $V^*$ into $V$; especially in our setting the subdifferential $\partial\psi^d$ is assumed to be singlevalued.

Some types of doubly nonlinear evolution equations have been considered so far, for example,

\[ A(u'(t)) + B(u(t)) \ni f(t), \quad (1.4) \]

which was considered by Arai [1], Senba [20], Colli and Visintin [8] and Colli [7], where $A$ and $B$ are maximal monotone, possibly nonlinear and multivalued operators from $V$ into $V^*$; in Colli [7], $A$ is bounded and $B$ is unbounded so that the domain $D(B)$ is contained in a Banach space $W$ imbedded compactly in $V$. Another type of doubly nonlinear evolution equations is of the form:

\[ \frac{d}{dt} A(u(t)) + B(u(t)) \ni f(t), \quad (1.5) \]

which was treated, for instance, by Kenmochi [15], Kenmochi and Pawlow [16], [17], and recently Maitre and Witowski [18]. In our problem, the unknown $u$ can be eliminated by (1.3). In fact, using the conjugate convex function $\psi^{d*}$ of $\psi^d$, we obtain from (1.3) that

\[ u(t) \in \partial\psi^{d*}(\theta(t)) \quad \text{in } V^*, \quad (1.6) \]
where \( \partial \psi^{t*} \) is the subdifferential of \( \psi^{t*} \) from \( V \) into \( V^* \). Hence (1.1) is formally written in the form
\[
\frac{d}{dt} \partial \psi^{t*}(\theta(t)) + K(t, \theta(t)) + G\left(t, \partial \psi^{t*}(\theta(t))\right) \ni f(t) \quad \text{in } V^*.
\] (1.7)

This is a new type of doubly nonlinear evolution equations in respect that the time derivative of \( \partial \psi^{t*}(\theta(t)) \) is included in the equation as well as a highly nonlinear perturbation \( G\left(t, \partial \psi^{t*}(\theta(t))\right) \). We have not noticed any results on this class of evolution equations. In this paper we give an existence result for problem \{(1.1)-(1.3)\}. In our construction of a solution, we approximate (1.1) by a time discretization scheme. After getting some uniform estimates we discuss its convergence to obtain a solution of our problem.

The above type of evolution equations arises from transmission problems of two degenerate parabolic equations. This problem has been studied by Fukao, Kenmochi and Pawlow [11]. In their paper it was treated as a system of transmission-Stefan-type. Two degenerate parabolic equations are combined by the transmission condition. In this case the problem is formulated as the initial value problem for (1.1) in an abstract Banach space, which will be discussed in detail in the last section.

2 Main Result

Throughout this paper, we denote by \( V \) a real reflexive Banach space with norm \( | \cdot |_V \), by \( V^* \) the dual space of \( V \) and by \( \langle \cdot, \cdot \rangle_{V^*,V} \) the duality pairing between \( V^* \) and \( V \). Moreover, let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle_H \) and norm \( | \cdot |_H \) such that the following dense and compact embeddings are satisfied:
\[
V \subset_d H \subset_d V^*.
\]
Without loss of generality we may assume that \( V \) and \( V^* \) are strictly convex spaces.

Now consider the initial value problem (DN): \{(2.1)-(2.3)\}:
\[
u(t) + K(t, \theta(t)) + G(t, \theta(t)) = f(t) \quad \text{in } V^* \text{ for a.e. } t \in [0, T],
\] (2.1)
\[
\theta(t) = \partial \psi^\theta(u(t)) \quad \text{in } V \text{ for a.e. } t \in [0, T],
\] (2.2)
\[
u(0) = u_0,
\] (2.3)
under the following assumptions (A1)-(A3):

(A1) For each \( t \in [0, T] \), \( \psi^\theta \) is a proper, lower semicontinuous and convex function on \( V^* \) with \( D(\psi^\theta) \subset H \), and \( \psi^\theta \) is coercive, that is, there exists a positive constant \( C_0 \), independent of \( t \), such that
\[
\psi^\theta(u) \geq C_0 |u|_H^\alpha \quad \text{for all } u \in D(\psi^\theta).
\] (2.4)
Moreover \( \psi^\theta \) satisfies the following time-dependent condition: For each \( t, s \in [0, T] \) and \( u \in D(\psi^\theta) \), there exists \( \hat{u} \in D(\psi^\theta) \) such that
\[
|\hat{u} - u|_{H^*} \leq C_1 |t - s| (\psi^\theta(u)^{1/2} + 1),
\] (2.5)
\[ \psi^t(u) - \psi^s(u) \leq C_1|t - s|(|\psi^s(u)| + 1), \quad (2.6) \]

where \( C_1 \) is a positive constant independent of \( t, s \) and \( u \). In addition, \( \psi^t \) is strongly monotone in the following sense:

\[ \langle u_1 - u_2, \theta_1 - \theta_2 \rangle_{H^*, H} \geq C_2|\theta_1 - \theta_2|_H^2 \quad \text{for all } \theta_i = \partial \psi^t(u_i) \text{ and } i = 1, 2. \quad (2.7) \]

(A2) The operator \( K(t, \cdot) : D(K(t, \cdot)) = V \to V^* \) satisfies that

\[ \langle K(t, \theta), \theta \rangle_{V^*, V} \geq C_3|\theta|_V^2 - C_4 \quad \text{for all } \theta \in V \text{ and } t \in [0, T], \quad (2.8) \]

(boundedness)

\[ |K(t, \theta)|_{V^*} \leq C_5|\theta|_V + C_6 \quad \text{for all } \theta \in V \text{ and } t \in [0, T], \quad (2.9) \]

where \( C_3, C_4, C_5 \) and \( C_6 \) are positive constants. Moreover \( K(\cdot, \cdot) \) satisfies the following property: If \( \theta_n \to \theta \) in \( L^2(0, T; H) \) and weakly in \( L^2(0, T; V) \) as \( n \to +\infty \), then

\[ K(\cdot, \theta_n(\cdot, \cdot)) \to K(\cdot, \theta(\cdot, \cdot)) \quad \text{weakly in } L^2(0, T; V^*) \text{ as } n \to +\infty. \quad (2.10) \]

In addition, for each \( \theta \in V, K(\cdot, \theta) \) is continuous with respect to \( t \) in the following sense: There exists a continuous function \( \omega : [0, +\infty) \to [0, +\infty) \) with \( \omega(0) = 0 \) such that

\[ |K(t, \theta) - K(s, \theta)|_{V^*} \leq \omega(|t - s|)|\theta|_V + 1 \quad \text{for all } t, s \in [0, T]. \quad (2.11) \]

(A3) The operator \( G(t, \cdot) : D(G(t, \cdot)) = H \to V^* \) satisfies that

\[ |G(t, u)|_{V^*} \leq C_7|u|_H + C_8 \quad \text{for all } u \in H \text{ and } t \in [0, T], \quad (2.12) \]

where \( C_7 \) and \( C_8 \) are positive constants. Moreover \( G(\cdot, \cdot) \) satisfies the following property: If \( t_n \to t \) and \( u_n \to u \) weakly in \( H \) as \( n \to +\infty \), then

\[ G(t_n, u_n) \to G(t, u) \quad \text{weakly in } V^* \text{ as } n \to +\infty. \quad (2.13) \]

Here we give the definition of a solution of (DN).

**Definition 2.1** A pair \( \{u, \theta \} \) of functions \( u, \theta : [0, T] \to H \) is called a solution of (DN), if \( u \in W^{1,2}(0, T; V^*) \cap L^2(0, T; H), \theta \in L^2(0, T; V) \) and \( \{u, \theta \} \) satisfies (2.1), (2.2) and (2.3).

Our main theorem is formulated now:

**Theorem 2.1** Assume that (A1)-(A3) hold. Given \( f \in L^2(0, T; V^*) \) and \( u_0 \in D(\psi^0) \), (DN) has at least one solution \( \{u, \theta \} \).

We shall prove our existence theorem in sections 3 and 4. Our main idea for the construction of a solution is to employ the time discretization method for (DN).
3 Approximation of (DN)

In order to construct a solution of (DN), we use the time discretization method to approximate problem (DN). For an arbitrary \( N \in \mathbb{N} \), we put \( h_N := T/N \), \( t_k^N := kh_N \) for \( k = 0, 1, \ldots, N \) and \( u_k^N := u_{k} \). Then, we consider the following time discretization scheme for (DN): Find a pair \( \{u_k^N, \theta_k^N\} \) of functions satisfying that

\[
\frac{u_k^N - u_{k-1}^N}{h_N} + K(t_k^N, \theta_k^N) = -G(t_{k-1}^N, u_{k-1}^N) + f_{k-1}^N \quad \text{in } V^*, \tag{3.1}
\]

with

\[
\theta_k^N = \partial \psi^N(u_k^N) \quad \text{in } V \quad \text{for } k = 1, 2, \ldots, N, \tag{3.2}
\]

where \( f_k^N \) is the discrete approximation of \( f \) given by

\[
f_k^N := \frac{1}{h_N} \int_{t_k^N}^{t_{k+1}^N} f(s)ds \quad \text{for } k = 0, 1, \ldots, N - 1. \tag{3.3}
\]

Now we discuss the solvability of \((DN)_N := \{ \text{(3.1), (3.2)} \} \). For simplicity, put \( A(t, \cdot) := (\partial \psi^N)^{-1} \). Note that \((DN)_N \) can be written in the form:

\[
A(t_k^N, \theta_k^N) + h_N K(t_k^N, \theta_k^N) \ni u_{k-1}^N - h_N G(t_{k-1}^N, u_{k-1}^N) + h_N f_{k-1}^N \quad \text{in } V^*. \tag{3.4}
\]

In order to construct the time discretization scheme, for each fixed \( t_k^N \), it suffices to check the existence of a solution \( \theta \in V \) of

\[
A(t_k^N, \theta) + h_N K(t_k^N, \theta) \ni g^* \quad \text{in } V^*, \tag{3.5}
\]

where \( g^* \) is given in \( V^* \). Now we recall the following general theory of nonlinear operators.

**Proposition 3.1** Let \( Z \) be a real reflexive Banach space and \( Z^* \) be the dual space of \( Z \). Assume that a multivalued operator \( \mathcal{N} : Z \to Z^* \) satisfies the following conditions \((N1)\) and \((N2)\):

\((N1)\) For each \( z \in Z \), \( \mathcal{N}z \) is a non-empty, bounded, convex and closed set in \( V^* \).

\((N2)\) \( \mathcal{N} \) is weakly sequentially upper semicontinuous and it is coercive, that is,

\[
\lim_{\|z\| \to +\infty} \inf_{z^* \in \mathcal{N}z} \frac{\langle z^*, z \rangle_{Z^*, Z}}{\|z\|_Z} = +\infty.
\]

Then \( \mathcal{N} \) is surjective, that is, \( R(\mathcal{N}) = Z^* \).

This proposition is well-known; for instance, it is an immediate consequence of the abstract results on the surjectiveness of nonlinear multivalued operators of Type M (see Brezis [3], Browder and Hess [6], Kenmochi [14] and so on).

**Lemma 3.1** (i) Let \( t \in [0, T] \) and \( N \in \mathbb{N} \). Then \( D(A(t, \cdot)) = V \) and \( A(t, \theta) + h_N K(t, \theta) \) is a non-empty, bounded, convex and closed set in \( V^* \) for each \( \theta \in V \).
(ii) \( A(t, \cdot) + h_N K(t, \cdot) \) is weakly sequentially upper semicontinuous and coercive as a mapping from \( V \) into \( V^* \).

**Proof.** (i) Since \( A(t, \cdot)^{-1} = \partial \psi^t \), by (2.4) we have \( R(A^{-1}(t, \cdot)) = D(A(t, \cdot)) = V \) and the boundedness of \( A(t, \cdot) \). From the maximal monotonicity of \( A(t, \cdot) \) it follows immediately that \( A(t, \theta) + h_N K(t, \theta) \) is non-empty, bounded, convex and closed for all \( \theta \in V \).

(ii) The coerciveness of \( A(t, \cdot) + h_N K(t, \cdot) \) is seen from (2.8) and the monotonicity of \( A(t, \cdot) \). Lastly we check that \( A(t, \cdot) + h_N K(t, \cdot) \) is weakly sequentially upper semicontinuous in \( V^* \). Now note that \( K(t, \cdot) \) is a singlevalued and weakly continuous operator from \( V \) into \( V^* \), and \( A(t, \cdot) \) is a subdifferential operator with \( R(A(t, \cdot)) = D(A^{-1}(t, \cdot)) \subset D(\psi^t) \subset H \). Therefore, by the compact imbedding \( V \hookrightarrow H \), we see that the graph \( G(A(t, \cdot) + h_N K(t, \cdot)) \) is weakly closed in \( V \times V^* \), which implies the conclusion. \( \square \)

This lemma and Proposition 3.1 imply that for each fixed \( N \in \mathbb{N}, k = 0, 1, \ldots, N \), and any given \( g^* \) in \( V^* \), there exists \( \theta \in V \) which satisfies (3.5). In our setting, the subdifferential operator \( \partial \psi^t \) is singlevalued. Thus, Lemma 3.1 and Proposition 3.1 show that our approximation scheme (DN) has solutions \( u_k^N \in H \) and \( \theta_k^N \in V \) for each \( N \in \mathbb{N} \) and \( k = 0, 1, \ldots, N \).

Moreover we obtain the following estimates for \( \{ u_k^N, \theta_k^N \} \).

**Lemma 3.2** There exists a positive constant \( M_1 \) independent of \( N \) such that

\[
\psi^N_k(u_k^N) \leq M_1 \quad \text{for all} \quad k = 0, 1, \ldots, N, \tag{3.6}
\]

\[
h_N \sum_{k=1}^{N} |\theta_k^N|_V^2 \leq M_1. \tag{3.7}
\]

**Proof.** Multiplying \( (\text{DN})_N \) by \( \theta_k^N \) we obtain

\[
\begin{align*}
\frac{1}{h_N} (u_k^N, \theta_k^N)_{H} - \frac{1}{h_N} (u_{k-1}^N, \theta_k^N)_{H} + \langle K(t_k^N, \theta_k^N), \theta_k^N \rangle_{V^*} & = -\langle G(t_{k-1}^N, u_{k-1}^N), \theta_k^N \rangle_{V^*} + \langle f_{k-1}^N, \theta_k^N \rangle_{V^*}. \tag{3.8}
\end{align*}
\]

By conditions (2.5) and (2.6) in (A1) for \( t \in [t_k^N, \infty) \) and \( u := u_{k-1}^N \in D(\psi^{k-1}) \) we can find an element \( \hat{u}_k^N \in D(\psi^k) \) such that

\[
|\hat{u}_k^N - u_{k-1}^N|_V \leq C_1 h_N \psi^{k-1}(u_{k-1}^N)^{1/2} + 1, \tag{3.9}
\]

\[
\psi^k(\hat{u}_k^N) - \psi^{k-1}(u_{k-1}^N) \leq C_1 h_N \psi^{k-1}(u_{k-1}^N) + 1. \tag{3.10}
\]

By the definition of subdifferential \( \partial \psi^k \) and (3.9) we get

\[
\begin{align*}
\frac{1}{h_N} (u_k^N, \theta_k^N)_{H} - \frac{1}{h_N} (u_{k-1}^N, \theta_k^N)_{H} & = \frac{1}{h_N} (u_k^N - \hat{u}_k^N, \theta_k^N)_{H} + \frac{1}{h_N} (\hat{u}_k^N - u_{k-1}^N, \theta_k^N)_{H} \\
& \geq \frac{1}{h_N} (\psi^k(u_k^N) - \psi^k(\hat{u}_k^N)) - C_1 (\psi^{k-1}(u_{k-1}^N)^{1/2} + 1) |\theta_k^N|_V.
\end{align*}
\]

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Moreover, by Schwarz’s inequality and (3.10),
\[
\frac{1}{h_N} \left( \psi_N^k(u_N^k) - \psi_N^{k-1}(\tilde{u}_N^k) \right) - C_1 \left( \psi_N^{k-1}(u_{k-1}^{N-1})^2 + 1 \right) |\theta_N^k|^V \\
\geq \frac{1}{h_N} \left( \psi_N^k(u_N^k) - \psi_N^{k-1}(u_{k-1}^{N-1}) \right) + \frac{1}{h_N} \left( \psi_N^{k-1}(u_{k-1}^{N-1}) - \psi_N^{k-1}(\tilde{u}_N^k) \right) \\
- C_2^2 C_{\epsilon_1} |\psi_N^{k-1}(u_{k-1}^{N-1}) - \epsilon_1 |\theta_N^k|^V - C_2^2 C_{\epsilon_1} - \epsilon_1 |\theta_N^k|^V \\
\geq \frac{1}{h_N} \left( \psi_N^k(u_N^k) - \psi_N^{k-1}(u_{k-1}^{N-1}) \right) - C_1(\psi_N^{k-1}(u_{k-1}^{N-1}) + 1) \\
- 2\epsilon_1 |\theta_N^k|^V - C_2^2 C_{\epsilon_1} |\psi_N^{k-1}(u_{k-1}^{N-1}) - C_2^2 C_{\epsilon_1} \\
= \frac{1}{h_N} \left( \psi_N^k(u_N^k) - \psi_N^{k-1}(u_{k-1}^{N-1}) \right) - 2\epsilon_1 |\theta_N^k|^V - (C_1 + C_2^2 C_{\epsilon_1}) \psi_N^{k-1}(u_{k-1}^{N-1}) - (C_1 + C_2^2 C_{\epsilon_1});
\]

hereafter \( \epsilon_i \) is an arbitrary positive constant and \( C_{\epsilon_i} := 1/(4\epsilon_i) \) for each \( i \in \mathbb{N} \). Next, by the coerciveness (2.8) of \( K \) we have
\[
\langle K(t_N^k, \theta_N^k), \theta_N^k \rangle_{V^*, V} \geq C_3 |\theta_N^k|^V - C_4.
\]

By the boundedness (2.12) of \( G \) we have
\[
\langle -G(t_{k-1}^{N-1}, u_{k-1}^{N-1}), \theta_N^k \rangle_{V^*, V} + \langle f_{k-1}^{N-1}, \theta_N^k \rangle_{V^*, V} \leq C_7^2 C_{\epsilon_2} |u_{k-1}^{N-1}|^P_H + C_8^2 C_{\epsilon_2} + 3\epsilon_2 |\theta_N^k|^V + C_{\epsilon_2} |f_{k-1}^{N-1}|^P_+.
\]

We obtain from (3.8) with the help of (2.4) that
\[
\frac{1}{h_N} \left( \psi_N^k(u_N^k) - \psi_N^{k-1}(u_{k-1}^{N-1}) \right) + (C_3 - 2\epsilon_1 - 3\epsilon_2) |\theta_N^k|^V \\
\leq \left( C_1 + C_2^2 C_{\epsilon_1} + \frac{C_7^2 C_{\epsilon_2}}{C_0} \right) \psi_N^{k-1}(u_{k-1}^{N-1}) + C_{\epsilon_2} |f_{k-1}^{N-1}|^P_+ + (C_1 + C_2^2 C_{\epsilon_1} + C_4 + C_8^2 C_{\epsilon_2}),
\]

(3.11)

where \( \epsilon_1 \) and \( \epsilon_2 \) are chosen so as to satisfy \( C_3 - 2\epsilon_1 - 3\epsilon_2 > 0 \). This inequality can be written in the form
\[
\psi_N^k(u_N^k) + M_2 \leq \left( 1 + \frac{T M_3}{N} \right) \left( \psi_N^{k-1}(u_{k-1}^{N-1}) + M_2 \right) \quad \text{for all} \ k = 1, \ldots, N,
\]

(3.12)

where \( M_2 \) and \( M_3 \) are positive constants independent of \( N \); \( M_2 \) depends on \( |f|_{L^2(\Omega T; V^*)} \) and \( M_3 := C_1 + C_2^2 C_{\epsilon_1} + (C_7^2 C_{\epsilon_2} / C_0) \). Here, applying the discrete Gronwall’s inequality to (3.12), we obtain
\[
\psi_N^k(u_N^k) \leq \exp(1 + T M_3) (\psi^0(u_0^k) + M_2) \quad \text{for all} \ k = 1, 2, \ldots, N.
\]

(3.13)

Next, we multiply (3.11) and (3.13) by \( h_N \) and sum them up for \( k = 1, 2, \ldots, N \) to obtain
\[
\psi_T(u_N^k) + (C_3 - 2\epsilon_1 - 3\epsilon_2) h_N \sum_{k=1}^N |\theta_N^k|^V \leq M_2 h_N \sum_{k=1}^N \psi_N^{k-1}(u_{k-1}^{N-1}) + M_2 T + \psi^0(u_0^k).
\]

Hence (3.7) is obtained from this inequality together with (3.13). \( \square \)
Now, let \( \{u^N, \theta^N\}_{k=0}^N \) be the discretization scheme of (3.1), (3.2) and (3.3) constructed above and define the piecewise linear \( H \)-valued function \( u^N \) generated by \( \{u^N_k\} \) on \([0, T]\), that is, \( u^N(0) := u_0 \) and

\[
u^N(t) := \frac{(t - t^N_{k-1})u^N_k + (t^N_k - t)u^N_{k-1}}{h_N} \quad \text{if} \ t \in (t^N_{k-1}, t^N_k) \text{ for } k = 1, 2, \ldots, N.
\]

It is clear that \( u^N \in C([0,T];H) \) for each \( N \in \mathbb{N} \). Moreover, we have the following lemma:

**Lemma 3.3** \( \{u^N\}_{N=1}^\infty \) is equicontinuous as a family of functions from \([0, T]\) into \( V^* \) and uniformly bounded on \([0, T]\) as \( H \)-valued functions.

**Proof.** Using Lemma 3.2, we have \( \|u^N_k\|_H \leq (M_1/C_0)^{1/2} \) for \( k = 1, 2, \ldots, N \), so that \( \{u^N\} \) is uniformly bounded on \([0, T]\) as \( H \)-valued functions. Next we take \( s, t \in [0, T] \) with \( s \leq t \) such that \( s \in [t^N_{k-1}, t^N_k] \) and \( t \in [t^N_{j-1}, t^N_j] \), where \( 1 \leq k \leq j \leq N \). We see from the boundedness (2.9) of \( K \) and (2.12) of \( G \) together with (2.4), (3.5) and (3.6) that there exists a positive constant \( M_4 \), which depends on \( C_6 \) and \( C_8 \), such that

\[
\|u^N(t) - u^N(s)\|_{V^*} \\
\leq 2 \sum_{k=k-1}^j \|u^N_k - u^N_{k-1}\|_{V^*} \\
\leq 2 \left( h_N \sum_{k=k-1}^j \|K(t^N_k, \theta^N_k)\|_{V^*} + h_N \sum_{k=k-1}^j \|G(t^N_k, u^N_k)\|_{V^*} + h_N \sum_{k=k-1}^j \|f^N_k\|_{V^*} \right) \\
\leq 2(h_N(j-k+1))^{1/2} \left\{ C_5 \left(h_N \sum_{k=k-1}^N \|\partial^N_k^2\|_V\right)^{1/2} + C_7 \left(h_N \sum_{k=k-1}^N \frac{\|f^N_k(u^N_k)\|}{C_0}\right)^{1/2} + M_4 + |I|_{L^2(0,T;V^*)} \right\} \\
\leq 2(|t-s| + 2h_N)^{1/2} \left\{ C_5 M_4^{1/2} + C_7 \left(TM_1\right)^{1/2} + M_4 + |I|_{L^2(0,T;V^*)} \right\},
\]

which shows the equicontinuity of \( \{u^N\} \) on \([0, T]\). \( \square \)

By virtue of Lemmas 3.2, 3.3 and the Ascoli-Arzela's theorem it is easily seen that there exists a subsequence \( \{N_n\} \subset \{N\} \) and there exists a function \( u \in C([0,T];V^*) \) such that

\( u^{N_n} \to u \) in \( C([0,T];V^*) \) as \( n \to +\infty \). (3.14)

Moreover, define the \( V^* \)-valued step function \( \theta^N \) and the \( V \)-valued step function \( \theta^\infty \) on \([0, T] \) by putting \( \theta^N(0) := u_0, \theta^\infty(0) := \partial v^0(u_0) \) and

\[
\theta^N(t) := \theta^N_k, \quad \theta^\infty(t) := \theta^\infty_k \quad \text{if} \ t \in (t^N_{k-1}, t^N_k) \text{ for } k = 1, 2, \ldots, N.
\]

Then (3.14) implies

\[
\sup_{t \in [0,T]} \|\theta^N(t) - u(t)\|_{V^*} \to 0 \quad \text{as} \ n \to +\infty.
\]

(3.16)
Additionally, we obtain the following convergence and relation between $u$ and $\theta$:

**Lemma 3.4** There exists a subsequence $\{N_m\} \subset \{N_n\}$ with $N_m \to \infty$ as $m \to +\infty$ and there exists a function $\theta \in L^2(0, T; V)$ such that

$$\bar{\theta}^{N_m} \to \theta \quad \text{weakly in } L^2(0, T; V) \text{ as } m \to +\infty.$$  

(3.17)

Moreover the pair $\{u, \theta\}$ satisfies that $\theta(t) = \partial \psi^f(u(t))$ in $V$ for a.e. $t \in [0, T]$.

**Proof.** By (3.7) in Lemma 3.2, we see that $\{\bar{\theta}^{N_m}\}_{m=1}^{\infty} = \{\partial \psi^f(\bar{u}^{N_m})\}_{m=1}^{\infty}$ is bounded in $L^2(0, T; V)$, which implies (3.17) for some subsequence $\{N_m\}$. Next, we show that $\theta(t) = \psi^f(u(t))$ for a.e. $t \in [0, T]$ which is equivalent to

$$\int_0^T \langle \eta(t) - u(t), \theta(t) \rangle_{V^*, V} dt \leq \int_0^T \psi^f(\eta(t)) dt - \int_0^T \psi^f(u(t)) dt \quad \text{for all } \eta \in L^2(0, T; V^*).$$  

(3.18)

Let $\eta$ be any function in $L^2(0, T; V^*)$ such that $\int_0^T \psi^f(\eta(t)) dt < +\infty$. Then there are step functions $\bar{\eta}^{N_m} \in L^2(0, T; V^*)$ for all $m \in \mathbb{N}$ such that

$$\bar{\eta}^{N_m} \to \eta \quad \text{in } L^2(0, T; V^*) \text{ as } m \to +\infty,$$

(3.19)

$$\limsup_{m \to +\infty} \sup_{n \leq m} \sum_{k=1}^{N_m} \int_{t_{k-1}^{N_m}}^{t_k^{N_m}} \psi^f_{k} (\bar{\eta}^{N_m}(t)) dt \leq \int_0^T \psi^f(\eta(t)) dt;$$

(3.20)

see Kenmochi [15]. Now, from the definition of subdifferential it follows that

$$\sum_{k=1}^{N_m} \int_{t_{k-1}^{N_m}}^{t_k^{N_m}} \langle \eta^{N_m}(t) - u^{N_m}(t), \bar{\eta}^{N_m}(t) \rangle_{V^*, V} dt \leq \sum_{k=1}^{N_m} \int_{t_{k-1}^{N_m}}^{t_k^{N_m}} \psi^f_{k} (\bar{\eta}^{N_m}(t)) dt - \sum_{k=1}^{N_m} \int_{t_{k-1}^{N_m}}^{t_k^{N_m}} \psi^f_{k} (\bar{u}^{N_m}(t)) dt.$$  

(3.21)

Let $t$ be any time in $(0, T]$. By virtue of (2.5) and (2.6) in (A1) for $t \in (t_{k-1}^{N_m}, t_k^{N_m}]$, $s := t_k^{N_m}$ and $\bar{u}^{N_m}(t) \in D(\partial \psi^f_{k})$ there exists a function $\bar{z}_m \in D(\psi^f_{k})$ such that

$$|\bar{z}_m - \bar{u}^{N_m}(t_k^{N_m})|_{V^*} \leq C' \left| t - t_k^{N_m} \right| \left( \psi^f_{k} (\bar{u}^{N_m}(t_k^{N_m}))^{\frac{1}{2}} + 1 \right),$$

(3.22)

$$\psi^f(\bar{z}_m) - \psi^f_{k} (\bar{u}^{N_m}(t_k^{N_m})) \leq C' \left| t - t_k^{N_m} \right| \left( \psi^f_{k} (\bar{u}^{N_m}(t_k^{N_m})) + 1 \right).$$

(3.23)

We now observe from (3.16) and (3.22) that

$$\bar{z}_m \to u(t) \quad \text{in } V^* \text{ as } m \to +\infty.$$  

(3.24)

On the other hand, using (3.23), (3.24) and the fact that $\psi^f$ is lower semicontinuous on $V^*$, we see

$$\liminf_{m \to +\infty} \psi^f_{k} (\bar{u}^{N_m}(t_k^{N_m})) \geq \liminf_{m \to +\infty} \left( \frac{1}{1 + C' \left| t - t_k^{N_m} \right|} \psi^f(\bar{z}_m) - \frac{C' \left| t - t_k^{N_m} \right|}{1 + C' \left| t - t_k^{N_m} \right|} \right) \geq \psi^f(u(t)).$$

Thus, taking the limit as $m \to +\infty$ in (3.21) and using (3.16), (3.17), (3.19) and (3.20) with the help of Fatou’s lemma, we get (3.18).
4 Proof of theorem

In this section we prove the main theorem after preparing two lemmas with some lemmas. Let \( \{\bar{n}_m^n, \tilde{n}_m^n\}_{m=1}^\infty \) and \( \{u, \theta\} \) be such as in the last section.

**Lemma 4.1** Let \( \{\theta_j\} \) be any sequence in \( C([0,T]; V) \) such that \( \theta_j \to 0 \) in \( L^2(0,T; V) \) as \( j \to +\infty \). Put

\[
u_j(t) := (\partial \psi^d + F^{-1})(\theta_j(t) + F^{-1}u(t)) \quad \text{in} \quad V^* \quad \text{for all} \quad t \in [0,T],
\]

(4.1) where \( F \) is the duality mapping from \( V \) to \( V^* \). Then \( u_j \in C([0,T]; V^*) \) and

\[ u_j \to \ u \quad \text{weakly in} \quad L^2(0,T; V^*) \quad \text{as} \quad j \to +\infty. \]

(4.2)

**Proof.** First note that \( \partial \psi^d + F^{-1} \) is a singlevalued and surjective operator from \( V^* \) into \( V \) as well as from \( L^2(0,T; V^*) \) into \( L^2(0,T; V) \). Clearly \( u_j \in L^2(0,T; V^*) \). We show \( u_j \in C([0,T]; V^*) \) as follows: For each fixed \( j \in \mathbb{N} \) and \( t \in [0,T] \), let \( \{t_n\} \subset [0,T] \) be any sequence such that \( t_n \to t \) as \( n \to +\infty \). Our claim is to show that

\[ u_j(t_n) \to u_j(t) \quad \text{in} \quad V^* \quad \text{as} \quad n \to +\infty. \]

(4.3)

Take a function \( w \in C([0,T]; V^*) \) with \( w(t) \in D(\psi^d) \) such that

\[ t \to \psi^d(w(t)) \quad \text{is bounded in} \quad [0,T], \]

for example, by the existence result in Kenmochi [15] we can find \( w \) as a solution of the Cauchy problem

\[
\frac{\partial w}{\partial r}(t) + \partial \psi^d(w(t)) = 0 \quad \text{in} \quad H,
\]

\[ w(0) = 0 \quad \text{in} \quad H, \]

where \( \partial \psi^d \) is the subdifferential of \( \psi^d \) in \( H \). Put \( g_j(t) := \theta_j(t) + F^{-1}u(t) \) in \( V \) for all \( t \in [0,T] \). Then, from the definition of subdifferential and assumption (2.4) in (A1) it follows that

\[
\left\langle u_j(t_n) - w(t_n), g_j(t_n) \right\rangle_{V^*; V} = \left\langle u_j(t_n) - w(t_n), \partial \psi^d(w(t_n)) + F^{-1}u_j(t_n) \right\rangle_{V^*; V} + \left| u_j(t_n) \right|_{\bar{V}^*}^2 - \left| w(t_n) \right|_{\bar{V}^*}^2 - \left| u_j(t_n) \right|_{\bar{V}^*}^2.
\]

\[
\geq \left\langle \partial \psi^d(w(t_n)) - \partial \psi^d(w(t_n)), u_j(t_n) \right\rangle_{\bar{V}^*} + \left| u_j(t_n) \right|_{\bar{V}^*}^2 - \left| w(t_n) \right|_{\bar{V}^*}^2 - \left| u_j(t_n) \right|_{\bar{V}^*}^2.
\]

\[
\geq C_0 \left| u_j(t_n) \right|_{\bar{V}^*} - \left| \psi^d(w(t_n)) \right|_{\bar{V}^*} - \left| w(t_n) \right|_{\bar{V}^*} - \left| u_j(t_n) \right|_{\bar{V}^*}.
\]

On the other hand, there exists a positive constant \( M_5 \) such that

\[
\left\langle u_j(t_n) - w(t_n), g_j(t_n) \right\rangle_{V^*; V} \leq M_5 \left| u_j(t_n) \right|_{\bar{V}} + \left| g_j(t_n) \right|_{\bar{V}}.
\]

\[
\leq M_5^2 \varepsilon_0 \left| u_j(t_n) \right|_{\bar{V}}^2 + C_0 \left| g_j(t_n) \right|_{\bar{V}}^2 + \frac{1}{2} \left| w(t_n) \right|_{\bar{V}^*}^2 + \frac{1}{2} \left| g_j(t_n) \right|_{\bar{V}^*}^2.
\]
Since \(|g_j(\tau)|_V \leq \|\theta_j(\tau)|_V + |u(\tau)|_{V^*} =: M_0\) for all \(\tau \in [0,T]\), we have for any \(\varepsilon_3 > 0\) that
\[
(C_0 - M_0^2\varepsilon_3)|u_j(t_n)|^2_H \leq \psi^{\varepsilon_3}(w(t_n)) + |w(t_n)|^2_{V^*} + \left(C_{\varepsilon_3} + \frac{1}{2}\right)|g_j(t_n)|^2_V \\
\leq \max_{\tau \in [0,T]} \left\{\psi^{\varepsilon_3}(w(\tau)) + |w(\tau)|^2_{V^*} + \left(C_{\varepsilon_3} + \frac{1}{2}\right)M_0^2\right\}.
\]
Therefore \(\{u_j(t_n)\}_{n=1}^\infty\) is bounded in \(H\). Hence, the compact imbedding \(H \hookrightarrow V^*\) implies that there exist a subsequence \(\{t_{n_i}\} \subset \{t_n\}\) with \(t_{n_i} \to t\) as \(i \to +\infty\) and an element \(z_j(t) \in V^*\) such that
\[
u_j(t_{n_i}) \to z_j(t) \quad \text{in } V^* \text{ as } i \to +\infty. \tag{4.4}
\]
Now, let \(i \to +\infty\) in the relation
\[
\partial \psi^{\varepsilon_3}(u_j(t_{n_i})) + F^{-1}u_j(t_{n_i}) = g_j(t_{n_i}) \quad \text{in } V \text{ for all } i \in \mathbb{N}. \tag{4.5}
\]
Then, (4.4) implies that
\[
\partial \psi^{\varepsilon_3}(z_j(t)) + F^{-1}z_j(t) = g_j(t) \quad \text{in } V. \tag{4.6}
\]
On the other hand, we have by (4.1) that
\[
g_j(t) := \theta_j(t) + F^{-1}u(t) = \left\{\partial \psi^{\varepsilon_3}(u_j(t)) + F^{-1}u_j(t) - F^{-1}u(t)\right\} + F^{-1}u(t) = \partial \psi^{\varepsilon_3}(u_j(t)) + F^{-1}u_j(t). \tag{4.7}
\]
On account of the strictly monotonicity of \(\partial \psi^{\varepsilon_3} + F^{-1}\), (4.6) and (4.7) imply that \(z_j(t) = u_j(t)\). Therefore we see that \(u_j\) is continuous in \(V^*\) at \(t\). Thus \(u_j \in C([0,T]; V^*)\). Finally letting \(j \to +\infty\) in (4.1), we obtain (4.2). \(\square\)

**Lemma 4.2** There exists a subsequence \(\{N_m\} \subset \{N_n\}\) with \(N_m \to +\infty\) as \(m \to +\infty\) such that
\[
\bar{\theta}^N \to \theta \quad \text{in } L^2(0,T; H) \text{ as } m \to +\infty. \tag{4.8}
\]

**Proof.** Using Lemma 4.1, we consider step functions \(\bar{u}_j^{N_m}\) and \(\bar{\theta}_j^{N_m}\) given by
\[
\bar{u}_j^{N_m}(t) := u_j(t_k^{N_m}) \quad \text{if } t \in (t_{k-1}^{N_m}, t_k^{N_m}] \quad \text{for } k = 1, 2, \ldots, N_m,
\]
\[
\bar{\theta}_j^{N_m}(t) := \theta_j(t_k^{N_m}) = \partial \psi^{N_m}(u_j(t_k^{N_m})) + F^{-1}u_j(t_k^{N_m}) - F^{-1}u(t_k^{N_m}) \quad \text{if } t \in (t_{k-1}^{N_m}, t_k^{N_m}] \quad \text{for } k = 1, 2, \ldots, N_m.
\]
Then, for each \(j \in \mathbb{N}\) we see from the above definitions that
\[
\bar{u}_j^{N_m}(t) \to u_j(t) \quad \text{in } V^* \text{ uniformly } t \in [0,T], \tag{4.9}
\]
\[
\bar{\theta}_j^{N_m}(t) \to \theta_j(t) \quad \text{in } V \text{ uniformly } t \in [0,T], \tag{4.10}
\]
as $m \to +\infty$. Now we note that

$$I := \int_0^T \langle \tilde{u}^N_m(t) - u(t), \tilde{N}_m(t) - \theta(t) \rangle_{V^*, V} dt$$

$$= \int_0^T \langle \tilde{u}^N_m(t) - \tilde{u}^N_j(t), \tilde{N}_m(t) - \theta(t) \rangle_{V^*, V} dt + \int_0^T \langle \tilde{u}^N_j(t) - u(t), \tilde{N}_m(t) - \theta(t) \rangle_{V^*, V} dt$$

$$+ \int_0^T \langle u(t) - u(t), \tilde{N}_m(t) - \theta(t) \rangle_{V^*, V} dt$$

$$= I_1 + I_2 + I_3. \tag{4.11}$$

The first term $I_1$ of the right hand side of (4.11) is estimated as follows: With the help of the strong monotonicity (2.7) in (A1) we have

$$I_1 = \int_0^T \langle \tilde{u}^N_m(t) - \tilde{u}^N_j(t), \tilde{N}_m(t) - \theta(t) \rangle_{V^*, V} dt$$

$$= h_{N_m} \sum_{k=1}^{N_m} \langle \tilde{u}^N_m(t_k^{N_m}) - \tilde{u}^N_j(t_k^{N_m}), \tilde{N}_m(t_k^{N_m}) - \tilde{N}_j(t_k^{N_m}) \rangle_{V^*, V}$$

$$+ \int_0^T \langle \tilde{u}^N_m(t) - \tilde{u}^N_j(t), \tilde{N}_m(t) - \theta(t) \rangle_{V^*, V} dt + \int_0^T \langle \tilde{u}^N_j(t) - \tilde{u}^N_m(t), \theta_j(t) - \theta(t) \rangle_{V^*, V} dt$$

$$\geq C_2 \int_0^T \| \tilde{u}^N_m(t) - \tilde{u}^N_j(t) \|^2_{H} dt$$

$$+ C_2 \int_0^T \langle \tilde{u}^N_m(t) - \tilde{u}^N_j(t), F^{-1}u(t) - F^{-1}\tilde{u}^N_j(t) \rangle_{V^*, V} dt$$

$$+ \int_0^T \langle \tilde{u}^N_m(t) - \tilde{u}^N_j(t), \tilde{N}_m(t) - \theta(t) \rangle_{V^*, V} dt + \int_0^T \langle u(t) - u(t), \tilde{N}_m(t) - \theta(t) \rangle_{V^*, V} dt$$

$$\geq C_2 \int_0^T \| \tilde{u}^N_m(t) - \theta(t) \|^2_{H} dt - C_2 \int_0^T \| \tilde{u}^N_j(t) - \theta(t) \|^2_{H} dt$$

$$+ C_2 \int_0^T \| u(t) - u(t) \|^2_{H} dt$$

$$+ \int_0^T \langle \tilde{u}^N_m(t) - \tilde{u}^N_j(t), \tilde{N}_m(t) - \theta(t) \rangle_{V^*, V} dt + \int_0^T \langle u(t) - u(t), \theta_j(t) - \theta(t) \rangle_{V^*, V} dt.$$

Adding the second and third terms $I_2$ and $I_3$ to the above inequality, we have by (4.11) that

$$\frac{C_2}{3} \int_0^T \| \tilde{u}^N_m(t) - \theta(t) \|^2_{H} dt$$

$$\leq I - I_2 - I_3 + C_2 \int_0^T \| \theta(t) - \theta_j(t) \|^2_{H} dt + C_2 \int_0^T \| \theta_j(t) - \tilde{N}^N_j(t) \|^2_{H} dt$$

$$- C_2 \int_0^T \langle \tilde{u}^N_m(t) - \tilde{u}^N_j(t), F^{-1}u(t) - F^{-1}\tilde{u}^N_j(t) \rangle_{V^*, V} dt$$

$$- \int_0^T \langle \tilde{u}^N_m(t) - \tilde{u}^N_j(t), \tilde{N}_m(t) - \theta(t) \rangle_{V^*, V} dt - \int_0^T \langle u(t) - \tilde{u}^N_j(t), \theta_j(t) - \theta(t) \rangle_{V^*, V} dt.$$

Now, let $m \to +\infty$. Then, by convergences (3.16), (3.17), (4.9), (4.10) and the monotonicity of $F^{-1}$,

$$\lim_{m \to +\infty} \sup_{0 \leq t \leq T} \int_0^T \| \tilde{u}^N_m(t) - \theta(t) \|^2_{H} dt$$

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\[
\begin{align*}
&\leq 3 \int_0^T \| \theta(t) - \theta_j(t) \|_H^2 dt - 3 \int_0^T \langle u(t) - u_j(t), F^{-1}u(t) - F^{-1}u_j(t) \rangle_{V^*,V} dt \\
&\quad - \frac{3}{C_2} \int_0^T \langle u(t) - u_j(t), \theta_j(t) - \theta(t) \rangle_{V^*,V} dt \\
&\leq 3 \int_0^T \| \theta(t) - \theta_j(t) \|_H^2 dt - \frac{3}{C_2} \int_0^T \langle u(t) - u_j(t), \theta_j(t) - \theta(t) \rangle_{V^*,V} dt \quad \text{for all } j \in \mathbb{N}.
\end{align*}
\]

Moreover, let \( j \to +\infty \) in (4.12). Then, by virtue of Lemma 4.1, we obtain that
\[
\limsup_{m \to +\infty} \int_0^T \| \tilde{\theta}^{N_m}(t) - \theta(t) \|_H^2 dt \leq 0.
\]

Thus \( \tilde{\theta}^{N_m} \to \theta \) in \( L^2(0,T;H) \) as \( m \to +\infty \). \( \square \)

**Proof of Theorem 2.1**

As was shown above, there exist a subsequence \( \{N_m\} \subset \{N\} \), \( \theta \in L^2(0,T;V) \) and \( u \in C([0,T];V^*) \cap L^\infty(0,T;H) \) such that
\[
\begin{align*}
\tilde{\theta}^{N_m} &\to \theta \quad \text{in } L^2(0,T;H), \\
\tilde{\varphi}^{N_m} &\to \theta \quad \text{weakly in } L^2(0,T;V), \\
u^{N_m} &\to u \quad \text{in } C([0,T];V^*) \quad \text{as } m \to +\infty,
\end{align*}
\]

and
\[
\theta(t) = \partial u^\beta(u(t)) \quad \text{in } V \quad \text{for a.e. } t \in [0,T].
\]

Now, define \( V^* \)-valued step functions \( \bar{K}^{N_m}, \bar{G}^{N_m} \) and \( \bar{f}^{N_m} \) on \([0,T]\) by
\[
\begin{align*}
\bar{K}^{N_m}(t) &\overset{\text{if } t \in (t_{k-1}^{N_m}, t_k)}{=} K(t_k^{N_m}, \tilde{\theta}^{N_m}_k) \\
\bar{G}^{N_m}(t) &\overset{\text{if } t \in (t_{k-1}^{N_m}, t_k)}{=} G(t_k^{N_m}, u_k^{N_m}) \\
\bar{f}^{N_m}(t) &\overset{\text{if } t \in (t_{k-1}^{N_m}, t_k)}{=} f_k^{N_m}
\end{align*}
\]

Then our approximate problem \( (DN)_N \) can be written in the form:
\[
\frac{d}{dt} u^{N_m}(t) + \bar{K}^{N_m}(t) + \bar{G}^{N_m}(t) = \bar{f}^{N_m}(t) \quad \text{in } V^* \quad \text{for a.e. } t \in [0,T]. \tag{4.13}
\]

Our claim is to show that (4.13) converges to (2.1) as \( m \to +\infty \). To this end, we show that
\[
\bar{K}^{N_m} \to K(\cdot, \theta(\cdot)) \quad \text{weakly in } L^2(0,T;V^*) \quad \text{as } m \to +\infty. \tag{4.14}
\]

By (2.10) in the assumptions (A2) of \( K \) we see that
\[
K(\cdot, \tilde{\varphi}^{N_m}(\cdot)) \to K(\cdot, \theta(\cdot)) \quad \text{weakly in } L^2(0,T;V^*) \quad \text{as } m \to +\infty.
\]

Moreover, by (2.11) in the assumptions (A2) of \( K \),
\[
|\bar{K}^{N_m}(t) - K(t, \tilde{\varphi}^{N_m}(t))|_{V^*} \leq \omega(|t_k^{N_m} - t|)(\| \tilde{\varphi}^{N_m}(t) \|_V + 1)
\]

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for all \( t \in (t_k^{N_m}, t_{k+1}^{N_m}] \) and \( k = 1, 2, \ldots, N_m \).

Hence (4.14) holds. Similarly, by (2.13) in the assumption (A3) of \( G \), we have

\[
\tilde{G}^{N_m} \to G(\cdot, u(\cdot)) \text{ weakly in } L^2(0, T; V^*) \text{ as } m \to +\infty,
\]

(4.15)

and clearly

\[
\tilde{f}^{N_m} \to f \text{ in } L^2(0, T; V^*) \text{ as } m \to +\infty.
\]

These convergences imply that \( \{(d/dt)u^{N_m}\} \) is bounded in \( L^2(0, T; V^*) \), so that

\[
\frac{d}{dt}u^{N_m} \to u' \text{ weakly in } L^2(0, T; V^*) \text{ as } m \to +\infty,
\]

Finally, passing to the limit in \( m \) of (4.13), we have

\[
u'(t) + K(t, \theta(t)) + G(t, u(t)) = f(t) \text{ in } V^* \text{ for a.e. } t \in [0, T].
\]

Thus \( \{u, \theta\} \) is a solution of (DN).

\[\Box\]

5 Application

In this section we give an application of our abstract result.

Let \( 0 < T < +\infty \), and \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary \( \Gamma := \partial \Omega \).

Assume that for each \( t \in [0, T] \), \( \Omega \) is divided into two subdomains \( \Omega_1(t) \) and \( \Omega_2(t) \) by a time-dependent interface \( \Gamma_{12}(t) \), that is,

\[
\Omega = \Omega_1(t) \cup \Gamma_{12}(t) \cup \Omega_2(t) \text{ for all } t \in [0, T].
\]

We denote by \( \Gamma_i(t) \) the set \( \partial \Omega_i(t) \setminus \Gamma_{12}(t) \) for \( i = 1, 2 \) (see Fig. 1).

![Fig.1](image)

Under these setting of the domains, we consider the following system of two degenerate parabolic equations:
\[
\left( \frac{\partial u_1}{\partial t} + v_1 \cdot \nabla u_1 \right) - \text{div}(k_1(\theta_1) \nabla \theta_1) = h \quad \text{in} \quad Q_1 := \bigcup_{t \in (0,T)} \{t\} \times \Omega_1(t), \quad (5.1)
\]
\[
\theta_1 = \theta_1(u_1) \quad \text{in} \quad Q_1, \quad (5.2)
\]
\[
\left( \frac{\partial u_2}{\partial t} + v_2 \cdot \nabla u_2 \right) - \text{div}(k_2(\theta_2) \nabla \theta_2) = h \quad \text{in} \quad Q_2 := \bigcup_{t \in (0,T)} \{t\} \times \Omega_2(t), \quad (5.3)
\]
\[
\theta_2 = \theta_2(u_2) \quad \text{in} \quad Q_2, \quad (5.4)
\]

where \text{div} means the divergence with respect to space variable \(x\), \(h\) is a given function in \(Q := (0,T) \times \Omega\), and \(v_1\) and \(v_2\) are given vector fields on \(Q\). Moreover, for each \(i = 1, 2\), \(\beta_i\) and \(k_i\) are functions such that

- \(\beta_i : \mathbb{R} \to \mathbb{R}\) is Lipschitz continuous and non-decreasing, and satisfies that
  \[
  |\beta_i'(r)| \geq C_{\beta_i} |r| - C_{\beta_i}' \quad \text{for all} \quad r \in \mathbb{R}, \quad (5.5)
  \]
  where \(C_{\beta_i}\) and \(C_{\beta_i}'\) are positive constants;

- \(k_i : \mathbb{R} \to \mathbb{R}\) is positive and non-decreasing, and satisfies that
  \[
  0 < C_{k_i} \leq k_i(r) \leq C_{k_i}' \quad \text{for all} \quad r \in \mathbb{R}, \quad (5.6)
  \]
  where \(C_{k_i}\) and \(C_{k_i}'\) are positive constants.

We combine these two equations (5.1) and (5.3) on the moving boundary \(\Gamma_{12}(t)\) by the transmission condition

\[
\theta_1 = \theta_2, \quad k_1(\theta_1) \frac{\partial \theta_1}{\partial \nu^+} + k_2(\theta_2) \frac{\partial \theta_2}{\partial \nu^-} = 0 \quad \text{on} \quad \Sigma_{12} := \bigcup_{t \in (0,T)} \{t\} \times \Gamma_{12}(t), \quad (5.7)
\]

where \(\nu^+ := \nu^+(t,x)\) is the unit normal vector on \(\Gamma_{12}(t)\) pointing to \(\Omega_2(t)\) and \(\nu^- := -\nu^+\).

We consider the system \{(5.1)-5.7\}, subject to the initial and boundary conditions for \(i = 1, 2\):

\[
u_1(0) = u_{01} \quad \text{in} \quad \Omega_1(0), \quad (5.8)
\]

\[
\nu_i(\theta_i) \frac{\partial \theta_i}{\partial \nu} + n_0 \theta_i = p \quad \text{on} \quad \Sigma_i := \bigcup_{t \in (0,T)} \{t\} \times \Gamma_i(t), \quad (5.9)
\]

where \(u_0\) and \(p\) are given functions, \(n_0\) is a positive constant and \(\nu := \nu(x)\) is the unit outward normal vector on \(\Gamma\). The system \{(5.1)-(5.9)\} is referred as (TDP).

It is well-known that equations (5.1) and (5.2) (resp. (5.3) and (5.4)) describe a Stefan problem in non-cylindrical domain \(Q_1\) (resp. \(Q_2\)), in which there is a convective vector field. We call such problems the transmission-Stefan problem. In Fukao, Kenmochi and Pawlow [10], [11], this problem was discussed under the following assumptions (A4) and (A5):
(A4) The domain $\Omega_1(t)$ and $\Omega_2(t)$ depend smoothly on time $t$ in the following sense: There is a transformation $y = y(t, x) := (y_1(t, x), y_2(t, x), y_3(t, x))$ which is a function of $C^2$-class from $\Omega$ into $\mathbb{R}^3$ satisfying that $y = y(t, x)$ is a $C^2$-diffeomorphism from $\Omega$ onto itself for each $t \in [0, T]$ and

$$y(t, \overline{\Omega(t)}) = \overline{\Omega(0)} \quad \text{for all} \ t \in [0, T] \quad \text{for} \ i = 1, 2, \quad y(0, \cdot) = \mathbf{I} \quad \text{(identity) in} \ \overline{\Omega}.$$ 

(A5) The convective vector fields $\mathbf{v}_1$ and $\mathbf{v}_2$, which may occur by the way of domains, are prescribed so that $\mathbf{v}_i \in C^2(\overline{\Omega})^3$ for $i = 1, 2$ and the following properties are satisfied:

$$\text{div} \mathbf{v}_i(t, \cdot) = 0 \quad \text{in} \ \Omega_i(t) \quad \text{for all} \ t \in (0, T) \quad \text{for} \ i = 1, 2, \quad (5.10)$$

$$\mathbf{v}_1 \cdot \mathbf{v}_+ = -\mathbf{v}_2 \cdot \mathbf{v}_- = v_{\Sigma_{12}} \quad \text{on} \ \Sigma_{12}, \quad (5.11)$$

$$\mathbf{v}_1 \cdot \mathbf{v} = \mathbf{v}_2 \cdot \mathbf{v} = 0 \quad \text{on} \ \Sigma := (0, T) \times \Gamma, \quad (5.12)$$

where $v_{\Sigma_{12}} := v_{\Sigma_{12}}(t, x)$ is the normal speed of $\Gamma_{12}(t)$ at $x \in \Gamma_{12}(t)$.

Using our abstract theory, we show the existence of a weak solution for (TDP). Let $V$ be the Sobolev space $H^1(\Omega)$ and $H$ be the Hilbert space $L^2(\Omega)$. Then we have

$$H^1(\Omega) \overset{d}{\rightarrow} L^2(\Omega) \overset{d}{\rightarrow} (H^1(\Omega))^*,$$

where $(H^1(\Omega))^*$ is the dual space of $H^1(\Omega)$. Moreover $\psi^d$, $K$, $G$ and $f$ are defined as follows:

- $\psi^d$ is a proper, lower semicontinuous and convex function on $V^*$ defined by

$$\psi^d(u) := \begin{cases} \sum_{i=1, 2} \int_{\Omega_i(t)} \beta_i(u) \, dx & \text{if} \ u \in H, \\ +\infty & \text{if} \ u \in V^* \setminus H, \end{cases}$$

where $\beta_i(r) := \int_r^\infty \beta_i(s) \, ds$;

- $K$ is a singlevalued operator from $[0, T] \times V$ into $V^*$ defined by

$$\langle K(t, \theta), z \rangle_{V^*, V} := \int_\Omega k(t, \theta)(\nabla \theta \cdot \nabla z) \, dx + m_0 \int_{\Gamma} \theta z d\Gamma \quad \text{for all} \ z \in V,$$

where $k(t, \theta) := k(t, x, \theta(x))$ and $k(t, x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$k(t, x, r) := \begin{cases} k_1(r) & \text{for all} \ r \in \mathbb{R} \quad \text{if} \ x \in \Omega_1(t), \\ k_2(r) & \text{for all} \ r \in \mathbb{R} \quad \text{if} \ x \in \Omega_2(t); \end{cases}$$

- $G$ is a singlevalued operator from $[0, T] \times H$ into $V^*$ defined by

$$\langle G(t, u), z \rangle_{V^*, V} := -\sum_{i=1, 2} \int_{\Omega_i(t)} u(\mathbf{v}_i(t) \cdot \nabla z) \, dx \quad \text{for all} \ z \in V;$$

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\( f \) is a function in \( L^2(0, T; V^*) \) given by
\[
\langle f(t), z \rangle_{V^*, V} := \int_Q h(t)zdx + \int_\Sigma p(t)z\Gamma \quad \text{for all } z \in V \text{ and a.e. } t \in [0, T],
\]
where \( h \in L^2(Q) \) and \( p \in L^2(\Sigma) \) are prescribed.

It is easily seen that (A1), (A2) and (A3) are satisfied. Now our problem (TDP) is reformulated in the following weak variational form (cf. Fukao, Kenmochi and Pawlow [10]):
\[
- \int_Q u \frac{\partial \eta}{\partial t} dx dt - \int_Q u(v \cdot \nabla \eta) dx dt + \int_Q k(\theta)(\nabla \theta \cdot \nabla \eta) dx dt + r_0 \int_\Sigma \theta \eta d\Gamma dt
\]
\[
= \int_Q h \eta dx dt + \int_\Sigma p \eta d\Gamma dt + \int_\Omega \eta(u_0(0)) dx \quad \text{for all } \eta \in W,
\]
(5.13)
where
\[
k(\theta) := k(t, x, \theta(t, x)),
\]
\[
W := \{ \eta \in H^1(Q); \ \eta(T, \cdot) = 0 \text{ on } \Omega \},
\]
and
\[
u := u_i, \quad \theta := \theta_i, \quad v := v_i \quad \text{on } Q_i \quad \text{and} \quad u_0 := u_0(0) \quad \text{on } \Omega_i(0) \quad \text{for } i = 1, 2.
\]

In fact, the variational identity (5.13) is derived from (5.1), (5.2), (5.3) and (5.4) in the following way. Let \( \eta \) be any test function in \( W \). First, multiply (5.1) and (5.3) by \( \eta \) and integrate their resultants over \( Q_1 \) and \( Q_2 \), respectively. Then, with the help of conditions (5.8), (5.10), (5.11), (5.12) and the Green-Stokes' formula we have
\[
\sum_{i=1,2} \int_Q \left( \frac{\partial u_i}{\partial t} + v_i \cdot \nabla u_i \right) \eta dx dt
\]
\[
= - \int_{Q_1} u_1 \left( \frac{\partial \eta}{\partial t} + v_1 \cdot \nabla \eta \right) dx dt + \int_{\Sigma \cap Q_1} u_1 \eta(v_1 \cdot \nu^+) d\Gamma_1(t) dt - \int_{\Sigma_{Q_1}} u_1(0(0)) dx
\]
\[
- \int_{Q_2} u_2 \left( \frac{\partial \eta}{\partial t} + v_2 \cdot \nabla \eta \right) dx dt + \int_{\Sigma \cap Q_2} u_2 \eta(v_2 \cdot \nu^-) d\Gamma_2(t) dt - \int_{\Sigma_{Q_2}} u_2(0(0)) dx
\]
\[
= - \int_Q u \left( \frac{\partial \eta}{\partial t} + v \cdot \nabla \eta \right) dx dt - \int_\Omega u_0(0) dx.
\]
(5.14)

Next, by (5.7) and (5.9),
\[
- \sum_{i=1,2} \int_{Q_i} \text{div}(k_i(\theta_i) \nabla \theta_i) \eta dx dt
\]
\[
= \int_{Q_1} k_1(\theta_1)(\nabla \theta_1 \cdot \nabla \eta) dx dt - \int_{\Sigma_1} k_1(\theta_1) \frac{\partial \theta_1}{\partial \nu^+} \eta d\Gamma_1(t) dt + \int_{\Sigma \cap Q_1} k_1(\theta_1) \frac{\partial \theta_1}{\partial \nu^-} \eta d\Gamma(t) dt
\]
\[
+ \int_{Q_2} k_2(\theta_2)(\nabla \theta_2 \cdot \nabla \eta) dx dt - \int_{\Sigma_2} k_2(\theta_2) \frac{\partial \theta_2}{\partial \nu^+} \eta d\Gamma_2(t) dt + \int_{\Sigma \cap Q_2} k_2(\theta_2) \frac{\partial \theta_2}{\partial \nu^-} \eta d\Gamma(t) dt
\]
\[
= \int_Q k(\theta)(\nabla \theta \cdot \nabla \eta) dx dt + \int_\Sigma (\rho_0 \theta - p) \eta d\Gamma(t) dt.
\]
(5.15)
Here, add (5.14) and (5.15). Then we get the variational identity (5.13). Moreover by
(5.2), (5.4) and the definition of \( \psi^t \) we see that
\[
\theta(t) = \partial \psi^t(u(t)) \quad \text{in } V \text{ for a.e. } t \in [0,T],
\]
and (5.13) can be transformed to the doubly nonlinear problem (DN) in the space \( V^* \).
Therefore, applying Theorem 2.1, we conclude that there exist \( u \in W^{1,2}(0,T;V^*) \cap L^\infty(0,T;H) \) and \( \theta \in L^2(0,T;V) \) which give a weak solution of (TDP) in the variational
sense (5.13).

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**MASAYASU ASO**
*Graduate School of Science and Technology, Chiba University*
1-33 Yayoi-cho, Inage-ku, Chiba, 263-8522 Japan
E-mail: aso@math.e.chiba-u.ac.jp

**TAKESI FUKAO**
*Graduate School of Science and Technology, Chiba University*
1-33 Yayoi-cho, Inage-ku, Chiba, 263-8522 Japan
E-mail: t-fukao@math.e.chiba-u.ac.jp

**NOBUYUKI KENMOCHE**
*Department of Mathematics, Faculty of Education, Chiba University*
1-33 Yayoi-cho, Inage-ku, Chiba, 263-8522 Japan
E-mail: kenmochi@math.e.chiba-u.ac.jp

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Correspondences concerning these Technical Reports should be addressed to
Koji Nishida
Graduate School of Science and Technology
Chiba University
Yayoi-cho 1-33, Inage-ku, Chiba-shi, 263-8522 JAPAN.
e-mail: nishida@math.s.chiba-u.ac.jp