On Exact Triangles Consisting of Projectively Flat Bundles on Complex Tori

February 2019

Chiba University

Graduate School of Science

Department of Mathematics and Informatics

Kazushi Kobayashi

(千葉大学審査学位論文)

On Exact Triangles Consisting of Projectively Flat Bundles on Complex Tori

February 2019

Chiba University

Graduate School of Science

Department of Mathematics and Informatics

Kazushi Kobayashi

Abstract

The mirror dual objects corresponding to affine Lagrangian multi sections of a trivial special Lagrangian torus fibration $T^{2n} \to T^n$ are holomorphic vector bundles on a mirror dual complex torus of dimension n via the homological mirror symmetry, and in general, it is known that there exists a one-to-one correspondence between these holomorphic vector bundles and a certain kind of projectively flat bundles. In this paper, we study the exact triangles consisting of those projectively flat bundles on higher dimensional complex tori while considering the relation with the exact triangles consisting of stable vector bundles in the derived categories of coherent sheaves on one-dimensional complex tori.

Contents

1 Introduction

In this paper, we construct a mirror pair of tori as an analogue of the SYZ construction [25], and mainly consider the exact triangles consisting of holomorphic vector bundles which appear in the discussions of the homological mirror symmetry [15] for higher dimensional tori. The SYZ construction is conjectured by Strominger, Yau and Zaslow in 1996, and it proposes a way of constructing mirror pairs geometrically. Roughly speaking, this construction is the following. A mirror pair of Calabi-Yau manifolds $(M, \check{M})$ is realized as the special Lagrangian torus fibrations $\pi : M \to B$ and $\check{\pi} : \check{M} \to B$ on the same base space B. In particular, for each $b \in B$, the special Lagrangian torus fibers $\pi^{-1}(b)$ and $\check{\pi}^{-1}(b)$ are related by the T-duality. On the other hand, the homological mirror symmetry is conjectured by Kontsevich in 1994, and it states the following. For each Calabi-Yau manifold M, there exists a Calabi-Yau manifold $\check{M}$ such that the derived category of the Fukaya category [6] on M is equivalent to the derived category of coherent sheaves on $\check{M}$ as triangulated categories. One of the most fundamental examples of mirror pairs is a pair $(T^{2n}, \check{T}^{2n})$ of tori, where T^{2n} is a symplectic torus and $\check{T}^{2n}$ is a complex torus, so there are many studies of the homological mirror symmetry for tori. For example, Polishchuk and Zaslow discuss the homological mirror symmetry in the case $n = 1$, namely, the case of $(T^2, \check{T}^2)$ in [23] (the details of the higher A_∞ product structures are studied in [21]), and Fukaya studied the homological mirror symmetry for abelian varieties via the SYZ construction in [7]. In particular, in [7], he discussed the homological mirror symmetry by focusing on the cases that the objects of the Fukaya category are restricted to affine Lagrangian submanifolds endowed with unitary local systems in the symplectic geometry side. In this setting, the holomorphic vector bundles corresponding to such objects of the Fukaya category are projectively flat. Actually, there are various studies of projectively flat bundles on complex tori. For example, factors of automorphy of projectively flat bundles on complex tori are classified in [8], [19], [14], [26], and in fact, in [11], a one-to-one correspondence between holomorphic vector bundles which appear in the discussions of the homological mirror symmetry for tori and a certain kind of projectively flat bundles is constructed explicitly, by using this classification result. On the other hand, projectively flat bundles are examples of Einstein-Hermitian vector bundles, and Einstein-Hermitian vector bundles relate closely to stable vector bundles via the Kobayashi-Hitchin correspondence [14], [18]. Thus, projectively flat bundles play a fundamental role in the complex (algebraic) geometry, including the homological mirror symmetry for tori.

Now, we explain the statements discussed in the body of this paper briefly. Let $(L, \mathcal{L})$ be an object of the Fukaya category $Fuk(T^{2n})$, where $L \approx T^n$ is an affine Lagrangian (multi) section of the trivial special Lagrangian torus fibration $T^{2n} \to T^n$ and $\mathcal{L} \to L$ is a unitary local system along L. The objects $(L, \mathcal{L})$ correspond to holomorphic vector bundles on $\check{T}^{2n}$ via the homological mirror symmetry, so we denote by $E(L, \mathcal{L})$ the holomorphic vector bundle corresponding to $(L, \mathcal{L})$. As mentioned above, these holomorphic vector bundles $E(L, \mathcal{L})$ are projectively flat. Furthermore, the special Lagrangian torus fibers with unitary local systems along them correspond to skyscraper sheaves on $\check{T}^{2n}$. We can also regard this correspondence as the Fourier-Mukai transform [17], [2]. By using holomorphic vector bundles $E(L, \mathcal{L})$, we can construct a DG-category $DG_{\check{T}^{2n}}$, so we can also obtain the triangulated category $Tr(DG_{\check{T}^{2n}})$ via the Bondal-Kapranov construction [4]. In this triangulated category $Tr(DG_{\check{T}^{2n}})$, we consider the following exact triangle.

$$\cdots \longrightarrow E(L_A, \mathcal{L}_A) \longrightarrow C(\psi) \longrightarrow E(L_B, \mathcal{L}_B) \xrightarrow{\psi} TE(L_A, \mathcal{L}_A) \longrightarrow \cdots . \qquad (1)$$

Here, T is the shift functor and $C(\psi)$ denotes the mapping cone of $\psi : E(L_B, \mathcal{L}_B) \to TE(L_A, \mathcal{L}_A)$. In particular, since the degrees of morphisms between holomorphic vector bundles $E(L, \mathcal{L})$ are equal to or larger than 0 in $DG_{\check{T}^{2n}}$, each exact triangle consisting of projectively flat bundles and their shifts is always expressed as the exact triangle (1). In the above setting, first, we recall the results in the case of 2-dimensional tori, i.e., $(T^2, \check{T}^2)$ which are explained bellow, following [10]. In [10], the exact triangles (1) are studied under the assumptions $gcd(\mathrm{rank}E(L, \mathcal{L}), \mathrm{degree}E(L, \mathcal{L})) = 1$ and $\mathrm{dimExt}^1(E(L_B, \mathcal{L}_B), E(L_A, \mathcal{L}_A)) = 1$, where $gcd(m, n) > 0$ denotes the greatest common divisor of m, $n \in \mathbb{Z}$. For simplicity, we set $r := \mathrm{rank}E(L_A, \mathcal{L}_A)$, $s := \mathrm{rank}E(L_B, \mathcal{L}_B)$, $a := \mathrm{degree}E(L_A, \mathcal{L}_A)$, $b := \mathrm{degree}E(L_B, \mathcal{L}_B)$. Here, note that the slopes of L_A, L_B are $\frac{a}{r}$, $\frac{b}{s}$, respectively. Then, $E(L_A, \mathcal{L}_A)$ and $E(L_B, \mathcal{L}_B)$ are stable by the assumption $gcd(r, a) = gcd(s, b) = 1$. Furthermore, the assumption $\mathrm{dimExt}^1(E(L_B, \mathcal{L}_B), E(L_A, \mathcal{L}_A)) = 1$ implies that $C(\psi)$ also becomes a stable vector bundle whose rank and degree are $r + s$ and $a + b$, respectively. Hence, by Atiyah's classification result of holomorphic vector bundles over an elliptic curve [3], there exists an object $(L, \mathcal{L})$ of the Fukaya category $Fuk(T^2)$ such that $C(\psi) \cong E(L, \mathcal{L})$, where the slope of L is $\frac{a+b}{r+s}$. Actually, in [10], this isomorphism $C(\psi) \cong E(L, \mathcal{L})$ is constructed explicitly, and some geometric interpretations about the isomorphism $C(\psi) \cong E(L, \mathcal{L})$ from the viewpoint of the Fukaya category $Fuk(T^2)$ are also given. Here, we remark that the cases without the assumption $\mathrm{dimExt}^1(E(L_B, \mathcal{L}_B), E(L_A, \mathcal{L}_A)) = 1$ are studied in [5]. Next, we discuss the higher dimensional case, i.e., $(T^{2n}, \check{T}^{2n})$. Here, we focus on the case $\mathrm{rank}E(L_A, \mathcal{L}_A) = 1$, and assume that $C(\psi)$ becomes a simple projectively flat bundle (of course, ψ is a non-trivial morphism), namely, the exact triangle (1) becomes an exact triangle consisting of three projectively flat bundles and their shifts. Then, we prove that the exact triangle (1) is obtained as the pullback of an exact triangle consisting of three projectively flat bundles and their shifts in the derived category of coherent sheaves on a one-dimensional complex torus $\check{T}^2$ by a suitable holomorphic projection $\pi : \check{T}^{2n} \to \check{T}^2$. This result is given in Theorem 5.6.

This paper is organized as follows. In section 2, we explain relations between the objects $(L, \mathcal{L})$ of the Fukaya category $Fuk(T^{2n})$ and holomorphic vector bundles $E(L, \mathcal{L})$. Furthermore, we construct the DG-category $DG_{\check{T}^{2n}}$ consisting of those holomorphic vector bundles $E(L, \mathcal{L})$. In section 3, in order to mention the projective flatness of $E(L, \mathcal{L})$, we recall the results which are given in section 3 of [11]. In sections 4, 5, we consider the exact triangles consisting of three projectively flat bundles and their shifts on complex tori. In section 4, we recall the results in the case of 2-dimensional tori, i.e., $(T^2, \check{T}^2)$ in sections 4, 5, 6 of [10]. In section 5, we study the higher dimensional case, i.e., $(T^{2n}, \check{T}^{2n})$ based on the discussions in section 4. In particular, the main purpose in section 5 is to prove Theorem 5.6 which is explained in the above.

2 Holomorphic vector bundles and Lagrangian submanifolds

In this section, we consider a mirror pair $((T^{2n}, \tilde{\omega}), \check{T}^{2n})$, where $(T^{2n}, \tilde{\omega})$ is a complexified symplectic torus and $\check{T}^{2n}$ is a complex torus, and discuss relations between affine Lagrangian submanifolds in $(T^{2n}, \tilde{\omega})$ and holomorphic vector bundles on $\check{T}^{2n}$. These are based on the SYZ construction [25] (see also [17]). Furthermore, we define the DG-category consisting of these holomorphic vector bundles.

First, we explain the complex geometry side. We define a complex torus $\check{T}^{2n}$ as follows. Let T be a complex matrix of order n such that $\mathrm{Im}T$ is positive definite. We denote by t_{ij} the (i, j)

component of T. Let us consider the lattice L in $\mathbb{C}^n$ generated by

$$\gamma_1 := (2\pi, 0, \cdots, 0)^t, \cdots, \gamma_n := (0, \cdots, 0, 2\pi)^t,$$
$$\gamma_1' := (2\pi t_{11}, \cdots, 2\pi t_{n1})^t, \cdots, \gamma_n' := (2\pi t_{1n}, \cdots, 2\pi t_{nn})^t,$$

and we define $\check{T}^{2n} := \mathbb{C}^n / L = \mathbb{C}^n / 2\pi(\mathbb{Z}^n \oplus T\mathbb{Z}^n)$. Sometimes we regard the n-dimensional complex torus $\check{T}^{2n}$ as a $2n$-dimensional real torus $\mathbb{R}^{2n}/2\pi\mathbb{Z}^{2n}$. In this paper, we further assume that T is a non-singular matrix. Actually, in our setting described below, it turns out that the mirror partner of $\check{T}^{2n}$ does not exist if $\det T = 0$. However, we can avoid this problem and discuss the homological mirror symmetry even if $\det T = 0$ by modifying the definition of the mirror partner of $\check{T}^{2n}$ and a class of holomorphic vector bundles which we treat. This fact will be discussed in [12]. Here, we fix an $\varepsilon > 0$ small enough and let

$$O^{l_1 \cdots l_n}_{m_1 \cdots m_n} := \Bigg\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \check{T}^{2n} \mid \frac{2}{3}\pi(l_j - 1) - \varepsilon < x_j < \frac{2}{3}\pi l_j + \varepsilon,$$
$$\frac{2}{3}\pi(m_k - 1) - \varepsilon < y_k < \frac{2}{3}\pi m_k + \varepsilon, \ j, k = 1, \cdots, n \Bigg\}$$

be a subset of $\check{T}^{2n}$, where $l_j, m_k = 1, 2, 3$,

$$x := (x_1, \cdots, x_n)^t, \ y := (y_1, \cdots, y_n)^t,$$

and we identify $x_i \sim x_i + 2\pi$, $y_i \sim y_i + 2\pi$ for each $i = 1, \cdots, n$. Sometimes we denote $O^{l_1 \cdots (l_j = l) \cdots l_n}_{m_1 \cdots (m_k = m) \cdots m_n}$ instead of $O^{l_1 \cdots l_n}_{m_1 \cdots m_n}$ in order to specify the values $l_j = l$, $m_k = m$. Then, $\{O^{l_1 \cdots l_n}_{m_1 \cdots m_n}\}_{l_j, m_k = 1,2,3}$ is an open cover of $\check{T}^{2n}$, and we define the local coordinates of $O^{l_1 \cdots l_n}_{m_1 \cdots m_n}$ by

$$(x_1, \cdots, x_n, y_1, \cdots, y_n)^t \in \mathbb{R}^{2n}.$$

Furthermore, we locally express the complex coordinates $z := (z_1, \cdots, z_n)^t$ of $\check{T}^{2n}$ by $z = x + Ty$.

Now, we define a class of holomorphic vector bundles $E_{(r,A,\mu,\mathcal{U})}$ on $\check{T}^{2n}$. We first construct it as a complex vector bundle, and then discuss when it becomes a holomorphic vector bundle later in Proposition 2.1. However, since the notations of transition functions of $E_{(r,A,\mu,\mathcal{U})}$ are complicated, before giving the strict definition of $E_{(r,A,\mu,\mathcal{U})}$, we explain the idea of the construction of $E_{(r,A,\mu,\mathcal{U})}$. We assume r, $r' \in \mathbb{N}$, $A = (a_{ij}) \in M(n; \mathbb{Z})$ and $\mu := (\mu_1, \cdots, \mu_n)^t \in \mathbb{C}^n$. This $r' \in \mathbb{N}$ denotes the rank of $E_{(r,A,\mu,\mathcal{U})}$, and it is uniquely defined by using $(r, A) \in \mathbb{N} \times M(n; \mathbb{Z})$. Hereafter, sometimes we denote $\mu = p + T^t q$ with $p := (p_1, \cdots, p_n)^t \in \mathbb{R}^n$, $q := (q_1, \cdots, q_n)^t \in \mathbb{R}^n$. In general, the affine Lagrangian submanifold corresponding to a holomorphic vector bundle $E_{(r,A,\mu,\mathcal{U})}$ is the following (we will explain the details of the symplectic geometry side again later).

$$\left\{ \begin{pmatrix} \check{x} \\ \check{y} \end{pmatrix} \in (T^{2n}, \tilde{\omega}) \mid \check{y} = \frac{1}{r} A \check{x} + \frac{1}{r} \left(\frac{r'}{r} p - \theta \right) \right\}.$$

Here, $\check{x} := (x^1, \cdots, x^n)^t$, $\check{y} := (y^1, \cdots, y^n)^t$ are the coordinates of the complexified symplectic torus $(T^{2n}, \tilde{\omega})$, and we will explain the notation $\theta \in \mathbb{R}^n$ later. In this situation, if $x^j \mapsto x^j + 2\pi$ $(j = 1, \cdots, n)$, then

$$\check{y} \mapsto \check{y} + \frac{2\pi}{r}(a_{1j}, \cdots, a_{nj})^t.$$

We decide the transition functions of $E_{(r,A,\mu,\mathcal{U})}$ by using this $\frac{1}{r}(a_{1j},\cdots,a_{nj})^t\in\mathbb{Q}^n$. This construction is a generalization of the case of $((T^2,\tilde{\omega}),\check{T}^2)$ to the higher dimensional case in the paper [10] (see section 2). In fact, for a given data $(r,A,\mu)\in\mathbb{N}\times M(n;\mathbb{Z})\times(\mathbb{R}^n\oplus T^t\mathbb{R}^n)$, $E_{(r,A,\mu,\mathcal{U})}$ is strictly defined as follows. First, we need to define $r'\in\mathbb{N}$ by using a given pair $(r,A)\in\mathbb{N}\times M(n;\mathbb{Z})$. By the theory of elementary divisors, there exist two matrices $\mathcal{A}$, $\mathcal{B}\in GL(n;\mathbb{Z})$ such that

$$\mathcal{A}A\mathcal{B}=\begin{pmatrix}\tilde{a}_1 & & & & & \\ & \ddots & & & & \\ & & \tilde{a}_s & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0\end{pmatrix}, \tag{2}$$

where $\tilde{a}_i\in\mathbb{N}$ $(i=1,\cdots,s,\ 1\le s\le n)$ and $\tilde{a}_i|\tilde{a}_{i+1}$ $(i=1,\cdots,s-1)$. Then, we define $r'_i\in\mathbb{N}$ and $a'_i\in\mathbb{Z}$ $(i=1,\cdots,s)$ by

$$\frac{\tilde{a}_i}{r}=\frac{a'_i}{r'_i},\ gcd(r'_i,a'_i)=1, \tag{3}$$

where $gcd(m,n)>0$ denotes the greatest common divisor of m, $n\in\mathbb{Z}$. By using these, we set

$$r':=r'_1\cdots r'_s\in\mathbb{N}.$$

This $r'\in\mathbb{N}$ is uniquely defined by a given pair $(r,A)\in\mathbb{N}\times M(n;\mathbb{Z})$, and it is actually the rank of $E_{(r,A,\mu,\mathcal{U})}$ (in this sense, although we should also emphasize $r'\in\mathbb{N}$ when we denote $E_{(r,A,\mu,\mathcal{U})}$, for simplicity, we use the notation $E_{(r,A,\mu,\mathcal{U})}$ in this paper). Now, we define the transition functions of $E_{(r,A,\mu,\mathcal{U})}$. Let

$$\psi^{l_1\cdots l_n}_{m_1\cdots m_n}:O^{l_1\cdots l_n}_{m_1\cdots m_n}\to\mathbb{C}^{r'},\quad l_j,m_k=1,2,3$$

be a smooth section of $E_{(r,A,\mu,\mathcal{U})}$. The transition functions of $E_{(r,A,\mu,\mathcal{U})}$ are non-trivial on

$$\begin{aligned}&O^{(l_1=3)\cdots l_n}_{m_1\cdots m_n}\cap O^{(l_1=1)\cdots l_n}_{m_1\cdots m_n},\ O^{l_1(l_2=3)\cdots l_n}_{m_1\cdots m_n}\cap O^{l_1(l_2=1)\cdots l_n}_{m_1\cdots m_n},\cdots,O^{l_1\cdots(l_n=3)}_{m_1\cdots m_n}\cap O^{l_1\cdots(l_n=1)}_{m_1\cdots m_n},\\ &O^{l_1\cdots l_n}_{(m_1=3)\cdots m_n}\cap O^{l_1\cdots l_n}_{(m_1=1)\cdots m_n},\ O^{l_1\cdots l_n}_{m_1(m_2=3)\cdots m_n}\cap O^{l_1\cdots l_n}_{m_1(m_2=1)\cdots m_n},\cdots,O^{l_1\cdots l_n}_{m_1\cdots(m_n=3)}\cap O^{l_1\cdots l_n}_{m_1\cdots(m_n=1)},\end{aligned}$$

and otherwise are trivial. We define the transition function on $O^{l_1\cdots(l_j=3)\cdots l_n}_{m_1\cdots m_n}\cap O^{l_1\cdots(l_j=1)\cdots l_n}_{m_1\cdots m_n}$ by

$$\psi^{l_1\cdots(l_j=3)\cdots l_n}_{m_1\cdots m_n}\Big|_{O^{l_1\cdots(l_j=3)\cdots l_n}_{m_1\cdots m_n}\cap O^{l_1\cdots(l_j=1)\cdots l_n}_{m_1\cdots m_n}}=e^{\frac{\mathbf{i}}{r}a_jy}V_j\ \psi^{l_1\cdots(l_j=1)\cdots l_n}_{m_1\cdots m_n}\Big|_{O^{l_1\cdots(l_j=3)\cdots l_n}_{m_1\cdots m_n}\cap O^{l_1\cdots(l_j=1)\cdots l_n}_{m_1\cdots m_n}},$$

where $\mathbf{i}=\sqrt{-1}$, $a_j:=(a_{1j},\cdots,a_{nj})$ and $V_j\in U(r')$. Similarly, we define the transition function on $O^{l_1\cdots l_n}_{m_1\cdots(m_k=3)\cdots m_n}\cap O^{l_1\cdots l_n}_{m_1\cdots(m_k=1)\cdots m_n}$ by

$$\psi^{l_1\cdots l_n}_{m_1\cdots(m_k=3)\cdots m_n}\Big|_{O^{l_1\cdots l_n}_{m_1\cdots(m_k=3)\cdots m_n}\cap O^{l_1\cdots l_n}_{m_1\cdots(m_k=1)\cdots m_n}}=U_k\ \psi^{l_1\cdots l_n}_{m_1\cdots(m_k=1)\cdots m_n}\Big|_{O^{l_1\cdots l_n}_{m_1\cdots(m_k=3)\cdots m_n}\cap O^{l_1\cdots l_n}_{m_1\cdots(m_k=1)\cdots m_n}},$$

where $U_k\in U(r')$. In the definitions of these transition functions, actually, we only treat V_j, $U_k\in U(r')$ which satisfy the cocycle condition, so we explain the cocycle condition below. When we define

$$\begin{aligned}
&\psi^{l_1\cdots(l_j=3)\cdots l_n}_{m_1\cdots(m_k=3)\cdots m_n}\Big|_{O^{l_1\cdots(l_j=3)\cdots l_n}_{m_1\cdots(m_k=3)\cdots m_n}\cap O^{l_1\cdots(l_j=1)\cdots l_n}_{m_1\cdots(m_k=1)\cdots m_n}} \\
&= U_k\ \psi^{l_1\cdots(l_j=3)\cdots l_n}_{m_1\cdots(m_k=1)\cdots m_n}\Big|_{O^{l_1\cdots(l_j=3)\cdots l_n}_{m_1\cdots(m_k=3)\cdots m_n}\cap O^{l_1\cdots(l_j=1)\cdots l_n}_{m_1\cdots(m_k=1)\cdots m_n}} \\
&= \Big(U_k\Big)\Big(e^{\frac{\mathbf{i}}{r}a_j y}V_j\Big)\ \psi^{l_1\cdots(l_j=1)\cdots l_n}_{m_1\cdots(m_k=1)\cdots m_n}\Big|_{O^{l_1\cdots(l_j=3)\cdots l_n}_{m_1\cdots(m_k=3)\cdots m_n}\cap O^{l_1\cdots(l_j=1)\cdots l_n}_{m_1\cdots(m_k=1)\cdots m_n}},
\end{aligned}$$

the cocycle condition is expressed as

$$V_jV_k = V_kV_j,\ U_jU_k = U_kU_j,\ \zeta^{-a_{kj}}U_kV_j = V_jU_k,$$

where ζ is the r-th root of 1 and $j,k = 1,\cdots,n$. We define a set $\mathcal{U}$ of unitary matrices by

$$\mathcal{U} := \Big\{V_j, U_k \in U(r') \mid V_jV_k = V_kV_j,\ U_jU_k = U_kU_j,\ \zeta^{-a_{kj}}U_kV_j = V_jU_k,\ j,k = 1,\cdots,n\Big\}. \tag{4}$$

Of course, how to define the set $\mathcal{U}$ relates closely to (in)decomposability of $E_{(r,A,\mu,\mathcal{U})}$. Here, we only treat the set $\mathcal{U}$ such that $E_{(r,A,\mu,\mathcal{U})}$ is simple (we can take such a set $\mathcal{U} \neq \emptyset$ for any $(r,r',A) \in \mathbb{N}^2 \times M(n;\mathbb{Z})$). Furthermore, for each $j = 1,\cdots,n$, we define ξ_j, $\theta_j \in \mathbb{R}$ by

$$e^{\mathbf{i}\xi_j} = \det V_j,\ e^{\mathbf{i}\theta_j} = \det U_j,$$

and set

$$\xi := (\xi_1,\cdots,\xi_n)^t,\ \theta := (\theta_1,\cdots,\theta_n)^t \in \mathbb{R}^n. \tag{5}$$

In particular, for two holomorphic vector bundles $E_{(r,A,\mu,\mathcal{U})}$, $E_{(r,A,\mu,\mathcal{U}')}$ and two pairs (ξ,θ), (ξ',θ') associated to $E_{(r,A,\mu,\mathcal{U})}$, $E_{(r,A,\mu,\mathcal{U}')}$, respectively, $E_{(r,A,\mu,\mathcal{U})} \cong E_{(r,A,\mu,\mathcal{U}')}$ if and only if $\xi \equiv \xi' \pmod{2\pi\mathbb{Z}^n}$, $\theta \equiv \theta' \pmod{2\pi\mathbb{Z}^n}$ hold (see [13]). We use these notations ξ, θ when we define the mirror dual objects corresponding to holomorphic vector bundles $E_{(r,A,\mu,\mathcal{U})}$. Therefore, when we give r, A, μ and $\mathcal{U}$, the complex vector bundle $E_{(r,A,\mu,\mathcal{U})}$ is defined, so next, we consider the condition such that the complex vector bundle $E_{(r,A,\mu,\mathcal{U})}$ becomes a holomorphic vector bundle. We define a connection $\nabla_{(r,A,\mu,\mathcal{U})}$ on $E_{(r,A,\mu,\mathcal{U})}$ locally as

$$\nabla_{(r,A,\mu,\mathcal{U})} = d + \omega_{(r,A,\mu,\mathcal{U})} := d - \frac{\mathbf{i}}{2\pi}\left(\frac{1}{r}x^tA^t + \frac{1}{r}\mu^t\right)dy \cdot I_{r'},$$

where $dy := (dy_1,\cdots,dy_n)^t$ and d denotes the exterior derivative. In fact, $\nabla_{(r,A,\mu,\mathcal{U})}$ is compatible with the transition functions and so defines a global connection. Then, its curvature form $\Omega_{(r,A,\mu,\mathcal{U})}$ is

$$\Omega_{(r,A,\mu,\mathcal{U})} = -\frac{\mathbf{i}}{2\pi r}dx^tA^tdy \cdot I_{r'},$$

where $dx := (dx_1,\cdots,dx_n)^t$. Here, we consider the condition such that $E_{(r,A,\mu,\mathcal{U})}$ is holomorphic. We see that the following proposition holds.

Proposition 2.1. *For a given quadruple $(r,A,\mu,\mathcal{U})$, the complex vector bundle $E_{(r,A,\mu,\mathcal{U})}$ is holomorphic if and only if AT is a symmetric matrix.*

Proof. In general, a complex vector bundle is holomorphic if and only if the (0,2)-part of its curvature form vanishes, so we calculate the (0,2)-part of $\Omega_{(r,A,\mu,\mathcal{U})}$. It turns out to be

$$\Omega^{(0,2)}_{(r,A,\mu,\mathcal{U})} = \frac{\mathbf{i}}{2\pi r} d\bar{z}^t \{T(T-\bar{T})^{-1}\}^t A^t (T-\bar{T})^{-1} d\bar{z} \cdot I_{r'},$$

where $d\bar{z} := (d\bar{z}_1, \cdots, d\bar{z}_n)^t$. Thus, $\Omega^{(0,2)}_{(r,A,\mu,\mathcal{U})} = 0$ is equivalent to that $\{T(T-\bar{T})^{-1}\}^t A^t (T-\bar{T})^{-1}$ is a symmetric matrix, i.e., $AT = (AT)^t$. □

Next, we explain the symplectic geometry side. We consider a $2n$-dimensional real torus $T^{2n} = \mathbb{R}^{2n}/2\pi\mathbb{Z}^{2n}$, and of course, for each point $(x^1, \cdots, x^n, y^1, \cdots y^n)^t \in T^{2n}$, we identify $x^i \sim x^i + 2\pi$, $y^i \sim y^i + 2\pi$, where $i = 1, \cdots, n$. We also denote by $(x^1, \cdots, x^n, y^1, \cdots, y^n)^t$ the local coordinates in the neighborhood of an arbitrary point $(x^1, \cdots, x^n, y^1, \cdots, y^n)^t \in T^{2n}$. Furthermore, we use the same notation $(x^1, \cdots, x^n, y^1, \cdots, y^n)^t$ when we denote the coordinates of the covering space $\mathbb{R}^{2n}$ of T^{2n}. Here, for simplicity, we set

$$\check{x} := (x^1, \cdots, x^n)^t, \ \check{y} := (y^1, \cdots, y^n)^t.$$

We define a complexified symplectic form $\tilde{\omega}$ on T^{2n} by

$$\tilde{\omega} := d\check{x}^t (-T^{-1})^t d\check{y},$$

where $d\check{x} := (dx^1, \cdots, dx^n)^t$ and $d\check{y} := (dy^1, \cdots, dy^n)^t$. We decompose $\tilde{\omega}$ into

$$\tilde{\omega} = d\check{x}^t \mathrm{Re}(-T^{-1})^t d\check{y} + \mathbf{i} d\check{x}^t \mathrm{Im}(-T^{-1})^t d\check{y},$$

and define

$$\omega := \mathrm{Im}(-T^{-1})^t, \ B := \mathrm{Re}(-T^{-1})^t.$$

Sometimes we identify the matrices ω and B with the 2-forms $d\check{x}^t \omega d\check{y}$ and $d\check{x}^t B d\check{y}$, respectively. Then, ω defines a symplectic form on T^{2n}. The closed 2-form B is often called the B-field. We define the objects of the Fukaya category $Fuk(T^{2n}, \tilde{\omega})$ on $(T^{2n}, \tilde{\omega})$ corresponding to holomorphic vector bundles $(E_{(r,A,\mu,\mathcal{U})}, \nabla_{(r,A,\mu,\mathcal{U})})$ on $\check{T}^{2n}$, namely, the pairs of Lagrangian submanifolds and unitary local systems on them. For simplicity, we set

$$p(\theta) := \frac{r'}{r} p - \theta, \ q(\xi) := \frac{r'}{r} q + \xi.$$

Here, since the values $\frac{r'}{r} p - \theta$ and $\frac{r'}{r} q + \xi$ depend on $(r, r') \in \mathbb{N}^2$, strictly speaking, the above notations $p(\theta)$ and $q(\xi)$ are not precise. However, since the notations are complicated, we use the above notations $p(\theta)$ and $q(\xi)$ in this paper. Let us consider the following n-dimensional submanifold $\tilde{L}_{(r,A,p(\theta))}$ in $\mathbb{R}^{2n}$.

$$\tilde{L}_{(r,A,p(\theta))} := \left\{ \begin{pmatrix} \check{x} \\ \check{y} \end{pmatrix} \in \mathbb{R}^{2n} \mid \check{y} = \frac{1}{r} A\check{x} + \frac{1}{r} p(\theta) \right\}.$$

We see that this n-dimensional submanifold $\tilde{L}_{(r,A,p(\theta))}$ becomes a Lagrangian submanifold in $\mathbb{R}^{2n}$ if and only if $\omega A = (\omega A)^t$ holds. Then, for the covering map $\pi : \mathbb{R}^{2n} \to T^{2n}$, $L_{(r,A,p(\theta))} := \pi(\tilde{L}_{(r,A,p(\theta))})$

defines a Lagrangian submanifold in $(T^{2n},\tilde{\omega})$. Furthermore, we consider the trivial complex line bundle $\mathcal{L}_{(r,A,p(\theta),q(\xi))}\to L_{(r,A,p(\theta))}$ with the flat connection

$$\nabla_{\mathcal{L}_{(r,A,p(\theta),q(\xi))}}:=d-\frac{\mathbf{i}}{2\pi}\frac{1}{r}q(\xi)^t d\check{x}.$$

Note that $q(\xi)\in\mathbb{R}^n$ is the holonomy of $\mathcal{L}_{(r,A,p(\theta),q(\xi))}$ along $L_{(r,A,p(\theta))}\approx T^n$. By recalling the definition of the Fukaya category, we have

$$\Omega_{\mathcal{L}_{(r,A,p(\theta),q(\xi))}}=d\check{x}^t Bd\check{y}\big|_{L_{(r,A,p(\theta))}},$$

where $\Omega_{\mathcal{L}_{(r,A,p(\theta),q(\xi))}}$ is the curvature form of the flat connection $\nabla_{\mathcal{L}_{(r,A,p(\theta),q(\xi))}}$, i.e., $\Omega_{\mathcal{L}_{(r,A,p(\theta),q(\xi))}}=0$. Hence, we see

$$d\check{x}^t Bd\check{y}\big|_{L_{(r,A,p(\theta))}}=\frac{1}{r}d\check{x}^t BAd\check{x}=0,$$

so one has $BA=(BA)^t$. Note that $\omega A=(\omega A)^t$ and $BA=(BA)^t$ hold if and only if $AT=(AT)^t$ holds, i.e., $E_{(r,A,\mu,\mathcal{U})}$ becomes a holomorphic vector bundle on $\check{T}^{2n}$ (Proposition 2.1). Here, we give a remark. In the above construction, although two vectors $\xi,\ \theta\in\mathbb{R}^n$ are used in the definition of the object $(L_{(r,A,p(\theta))},\mathcal{L}_{(r,A,p(\theta),q(\xi))})$ of the Fukaya category $Fuk(T^{2n},\tilde{\omega})$, these vectors $\xi,\ \theta\in\mathbb{R}^n$ are determined by the transition functions of the mirror dual holomorphic vector bundle $E_{(r,A,\mu,\mathcal{U})}$. Actually, we need to use these vectors $\xi,\ \theta\in\mathbb{R}^n$ when we define the map $[E_{(r,A,\mu,\mathcal{U})}]\mapsto[(L_{(r,A,p(\theta))},\mathcal{L}_{(r,A,p(\theta),q(\xi))})]$, where $[E_{(r,A,\mu,\mathcal{U})}]$ and $[(L_{(r,A,p(\theta))},\mathcal{L}_{(r,A,p(\theta),q(\xi))})]$ denote the isomorphism class of $E_{(r,A,\mu,\mathcal{U})}$ and the isomorphism class of $(L_{(r,A,p(\theta))},\mathcal{L}_{(r,A,p(\theta),q(\xi))})$, respectively. This fact is found by us, and it will be discussed in [13].

We define the DG-category $DG_{\check{T}^{2n}}$ consisting of holomorphic vector bundles $(E_{(r,A,\mu,\mathcal{U})},\nabla_{(r,A,\mu,\mathcal{U})})$. This definition is an extension of the case of $((T^2,\tilde{\omega}),\check{T}^2)$ to the higher dimensional case in the paper [10] (see section 3). The objects of $DG_{\check{T}^{2n}}$ are holomorphic vector bundles $E_{(r,A,\mu,\mathcal{U})}$ with $U(r')$-connections $\nabla_{(r,A,\mu,\mathcal{U})}$. Of course, we assume $AT=(AT)^t$. Sometimes we simply denote $(E_{(r,A,\mu,\mathcal{U})},\nabla_{(r,A,\mu,\mathcal{U})})$ by $E_{(r,A,\mu,\mathcal{U})}$. For any two objects

$$E_{(r,A,\mu,\mathcal{U})}=(E_{(r,A,\mu,\mathcal{U})},\nabla_{(r,A,\mu,\mathcal{U})}),\ E_{(s,B,\nu,\mathcal{V})}=(E_{(s,B,\nu,\mathcal{V})},\nabla_{(s,B,\nu,\mathcal{V})}),$$

the space of morphisms is defined by

$$\mathrm{Hom}_{DG_{\check{T}^{2n}}}(E_{(r,A,\mu,\mathcal{U})},E_{(s,B,\nu,\mathcal{V})}):=\Gamma(E_{(r,A,\mu,\mathcal{U})},E_{(s,B,\nu,\mathcal{V})})\otimes_{C^\infty(\check{T}^{2n})}\Omega^{0,*}(\check{T}^{2n}),$$

where $\Omega^{0,*}(\check{T}^{2n})$ is the space of anti-holomorphic differential forms, and $\Gamma(E_{(r,A,\mu,\mathcal{U})},E_{(s,B,\nu,\mathcal{V})})$ is the space of homomorphisms from $E_{(r,A,\mu,\mathcal{U})}$ to $E_{(s,B,\nu,\mathcal{V})}$. The space of morphisms $\mathrm{Hom}_{DG_{\check{T}^{2n}}}(E_{(r,A,\mu,\mathcal{U})},E_{(s,B,\nu,\mathcal{V})})$ is a $\mathbb{Z}$-graded vector space, where the grading is defined as the degrees of anti-holomorphic differential forms. The degree r part is denoted $\mathrm{Hom}^r_{DG_{\check{T}^{2n}}}(E_{(r,A,\mu,\mathcal{U})},E_{(s,B,\nu,\mathcal{V})})$. We decompose $\nabla_{(r,A,\mu,\mathcal{U})}$ into its holomorphic part and anti-holomorphic part $\nabla_{(r,A,\mu,\mathcal{U})}=\nabla^{(1,0)}_{(r,A,\mu,\mathcal{U})}+\nabla^{(0,1)}_{(r,A,\mu,\mathcal{U})}$, and define a linear map

$$\mathrm{Hom}^r_{DG_{\check{T}^{2n}}}(E_{(r,A,\mu,\mathcal{U})},E_{(s,B,\nu,\mathcal{V})})\to\mathrm{Hom}^{r+1}_{DG_{\check{T}^{2n}}}(E_{(r,A,\mu,\mathcal{U})},E_{(s,B,\nu,\mathcal{V})})$$

by

$$\psi\mapsto(2\nabla^{(0,1)}_{(s,B,\nu,\mathcal{V})})(\psi)-(-1)^r\psi(2\nabla^{(0,1)}_{(r,A,\mu,\mathcal{U})}).$$

We can check that this linear map is a differential. Furthermore, the product structure is defined by the composition of homomorphisms of vector bundles together with the wedge product for anti-holomorphic differential forms. Then, these differential and product structure satisfy the Leibniz rule. Hence, $DG_{\check{T}^{2n}}$ forms a DG-category.

In general, for a given A_∞ category (DG-category), we can construct a triangulated category via the Bondal-Kapranov-Kontsevich construction [4], [15]. In this paper, we denote by $Tr(\mathscr{C})$ the triangulated category obtained by this construction from a given A_∞ category (DG-category) $\mathscr{C}$. Thus, for the DG-category $DG_{\check{T}^{2n}}$, we can consider the triangulated category $Tr(DG_{\check{T}^{2n}})$.

3 The projective flatness of $E_{(r,A,\mu,\mathcal{U})}$

The main purpose of this section is to recall the results which are given in section 3 of [11]. More precisely, in section 3 of [11], a one-to-one correspondence between holomorphic vector bundles $(E_{(r,A,\mu,\mathcal{U})}, \nabla_{(r,A,\mu,\mathcal{U})})$ and a certain kind of projectively flat bundles is constructed explicitly, by using the result of the classification of factors of automorphy of projectively flat bundles on complex tori. The detail of these discussions is described in subsection 3.2.

3.1 Projectively flat bundles on complex tori

In this subsection, we recall the definition of projectively flat bundles and properties of them.

First, since we need to use in the later discussions, we recall the definition of factors of automorphy for holomorphic vector bundles following [14]. Let M be a complex manifold such that its universal covering space $\tilde{M}$ is a topologically trivial (contractible) Stein manifold ($\mathbb{C}^n$ is an example of a Stein manifold). Let $p : \tilde{M} \to M$ be the covering projection and Γ the covering transformation group acting on $\tilde{M}$ so that $M = \tilde{M}/\Gamma$. Let E be a holomorphic vector bundle of rank r over M. Then its pull-back $\tilde{E} = p^*E$ is a holomorphic vector bundle of the same rank over $\tilde{M}$. Since $\tilde{M}$ is topologically trivial, $\tilde{E}$ is topologically a product bundle. Since $\tilde{M}$ is Stein, by Oka's principle, $\tilde{E}$ is holomorphically a product bundle, i.e., $\tilde{E} = \tilde{M} \times \mathbb{C}^r$. Having fixed this isomorphism, we define a holomorphic map $j : \Gamma \times \tilde{M} \to GL(r;\mathbb{C})$ by the following commutative diagram

$$\begin{array}{ccc} \tilde{E}_{x+\gamma} \cong \mathbb{C}^r & \xleftarrow{\ j(\gamma,x)\ } & \tilde{E}_x \cong \mathbb{C}^r \\ & \searrow \qquad \swarrow & \\ & E_{p(x)} & \end{array},$$

where $x \in \tilde{M}$, $\gamma \in \Gamma$. Then, for $x \in \tilde{M}$, $\gamma, \gamma' \in \Gamma$, the following relation holds.

$$j(\gamma + \gamma', x) = j(\gamma', x + \gamma) \circ j(\gamma, x).$$

The map $j : \Gamma \times \tilde{M} \to GL(r;\mathbb{C})$ is called the factor of automorphy for the holomorphic vector bundle E.

Now, we recall the definition and some properties of projectively flat bundles.

Definition 3.1 (Projectively flat bundles, [8], [19], [14], [26])**.** *Let E be a holomorphic vector bundle of rank r over a compact Kähler manifold M and $P(E)$ its associated principal $GL(r;\mathbb{C})$-bundle. Then $\hat{P}(E) = P(E)/\mathbb{C}^\times I_r$ is a principal $PGL(r;\mathbb{C})$-bundle. We say that E is projectively flat when $\hat{P}(E)$ is provided with a flat structure.*

For a complex vector bundle E of rank r with a connection D over a compact Kähler manifold M, it is known that the following proposition holds.

Proposition 3.2 ([19], [14], [26])**.** *Let R be a curvature of (E, D). Then, E is projectively flat if and only if R takes values in scalar multiples of the identity endmorphism I_E of E, i.e., if and only if there exists a complex 2-form α on M such that $R = \alpha \cdot I_E$.*

There are many studies of projectively flat bundles on complex tori. Let $\mathbb{C}^n/\Gamma$ be a complex torus, where Γ is a nondegenerate lattice of rank $2n$ of $\mathbb{C}^n$. Let us denote the coordinates of $\mathbb{C}^n$ by $z = (z_1, \cdots, z_n)^t$. Hereafter, we focus on projectively flat bundles which admit Hermitian structures over a complex torus $\mathbb{C}^n/\Gamma$ [1]. On the detail of the results which are described below, for example, see [8], [19], [14], [26]. Now, we recall the following theorem (see Theorem 4.7.54 in [14]) which plays an important role in our discussions in subsection 3.2.

Theorem 3.3. *Let E be a holomorphic vector bundle of rank r over a complex torus $\mathbb{C}^n/\Gamma$. If E admits a projectively flat Hermitian structure h, then its factor of automorphy j can be written as follows :*

$$j(\gamma, z) = U(\gamma)\exp\left\{\frac{1}{r}\mathcal{R}(z, \gamma) + \frac{1}{2r}\mathcal{R}(\gamma, \gamma)\right\} \quad (\gamma, z) \in \Gamma \times \mathbb{C}^n,$$

where

(i) *$\mathcal{R}$ is a Hermitian form on $\mathbb{C}^n$ and its imaginary part satisfies*

$$\mathrm{Im}\mathcal{R}(\gamma, \gamma') \in \pi\mathbb{Z} \quad for \quad \gamma, \gamma' \in \Gamma,$$

(ii) *$U : \Gamma \to U(r)$ is a semi-representation in the sense that it satisfies*

$$U(\gamma + \gamma') = U(\gamma)U(\gamma')e^{\frac{\mathbf{i}}{r}\mathrm{Im}\mathcal{R}(\gamma', \gamma)} \quad for \quad \gamma, \gamma' \in \Gamma.$$

Conversely, given a Hermitian form $\mathcal{R}$ on $\mathbb{C}^n$ with property (i) *and a semi-representation $U : \Gamma \to U(r)$, we can define a factor of automorphy $j : \Gamma \times \mathbb{C}^n \to CU(r)$ as above, where*

$$CU(r) := \left\{cU \mid c \in \mathbb{C}^\times \text{ and } U \in U(r)\right\}.$$

The corresponding vector bundle E over $\mathbb{C}^n/\Gamma$ admits a projectively flat Hermitian structure.

In Theorem 3.3, of course, by using a Hermitian matrix R, we can denote

$$\mathcal{R}(z, w) = z^t R\bar{w},$$

where $z = (z_1, \cdots, z_n)^t$ and $w = (w_1, \cdots, w_n)^t$. Then, under the situation of Theorem 3.3, the connection 1-form ω of the Hermitian connection of (E, h) is expressed locally as

$$\omega = -\frac{1}{r}dz^t R\bar{z} \cdot I_r + dz^t b \cdot I_r,$$

where $dz := (dz_1, \cdots, dz_n)^t$ and $b := (b_1, \cdots, b_n)^t \in \mathbb{C}^n$ is a constant vector. Furthermore, the curvature form Ω of the Hermitian connection of (E, h) is expressed locally as

$$\Omega = \frac{1}{r}dz^t Rd\bar{z} \cdot I_r.$$

[1] In fact, since we do not mention about Hermitian structures explicitly in our main discussions, readers do not have to consider about them so much in subsection 3.2.

3.2 The construction of an isomorphism $E_{(r,A,\mu,\mathcal{U})} \cong \mathcal{E}_{(r,A,\mu,\mathcal{U})}$

As mentioned in subsection 3.1, in general, the factors of automorphy of projectively flat bundles on complex tori are classified concretely. Furthermore, by using this classification result, holomorphic vector bundles $(E_{(r,A,\mu,\mathcal{U})}, \nabla_{(r,A,\mu,\mathcal{U})})$ are characterized as a class of projectively flat bundles in section 3 of [11]. The purpose of this subsection is to recall this result.

We consider the case $\check{T}^{2n} = \mathbb{C}^n/L = \mathbb{C}^n/2\pi(\mathbb{Z}^n \oplus T\mathbb{Z}^n)$, and discuss the relations between holomorphic vector bundles $(E_{(r,A,\mu,\mathcal{U})}, \nabla_{(r,A,\mu,\mathcal{U})})$ and projectively flat bundles. Note that the curvature form $\Omega_{(r,A,\mu,\mathcal{U})}$ of a holomorphic vector bundle $E_{(r,A,\mu,\mathcal{U})}$ is expressed locally as

$$\Omega_{(r,A,\mu,\mathcal{U})} = \frac{\mathbf{i}}{2\pi r'}\frac{r'}{r} dz^t \{(T-\bar{T})^{-1}\}^t A d\bar{z} \cdot I_{r'}.$$

Now, we define

$$R := \frac{\mathbf{i}}{2\pi}\frac{r'}{r}\{(T-\bar{T})^{-1}\}^t A,$$

namely,

$$\Omega_{(r,A,\mu,\mathcal{U})} = \frac{1}{r'} dz^t R d\bar{z} \cdot I_{r'}.$$

Then, the matrix R is a real symmetric matrix of order n (see Lemma 3.3 in [11]). By using this real symmetric matrix R of order n, we define a Hermitian bilinear form $\mathcal{R} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ by

$$\mathcal{R}(z,w) := \sum_{i,j=1}^{n} R_{ij} z_i \bar{w}_j,$$

where $z = (z_1, \cdots, z_n)^t$, $w = (w_1, \cdots, w_n)^t$. Here, we give a remark on the matrix $\frac{r'}{r}A$. By the relations (2) and (3), we see

$$\frac{1}{r}\mathcal{A}A\mathcal{B} = \begin{pmatrix} \frac{a'_1}{r'_1} & & & & & \\ & \ddots & & & & \\ & & \frac{a'_s}{r'_s} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

where $\mathcal{A}$, $\mathcal{B} \in GL(n;\mathbb{Z})$. Therefore, one has

$$\frac{r'}{r}\mathcal{A}A\mathcal{B} = \begin{pmatrix} r'_2 \cdots r'_s a'_1 & & & & & \\ & \ddots & & & & \\ & & r'_1 \cdots r'_{s-1} a'_s & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

and since $\mathcal{A}$, $\mathcal{B} \in GL(n;\mathbb{Z})$, we see that

$$\frac{r'}{r}A = \mathcal{A}^{-1} \begin{pmatrix} r'_2 \cdots r'_s a'_1 & & & & & \\ & \ddots & & & & \\ & & r'_1 \cdots r'_{s-1} a'_s & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \mathcal{B}^{-1} \in M(n;\mathbb{Z})$$

holds. Thus, each component of the matrix $\frac{r'}{r}A$ is an integer. In this situation, we can prove that the following propositions hold (see Proposition 3.4 and Proposition 3.5 in [11]).

Proposition 3.4. *For $\gamma_1, \cdots, \gamma_n$ and $\gamma'_1, \cdots, \gamma'_n$, $\mathrm{Im}\mathcal{R}(\gamma_j, \gamma_k) = 0$, $\mathrm{Im}\mathcal{R}(\gamma'_j, \gamma'_k) = 0$, where $j, k = 1, \cdots, n$.*

Proposition 3.5. *For $\gamma_1, \cdots, \gamma_n$ and $\gamma'_1, \cdots, \gamma'_n$, $\mathrm{Im}\mathcal{R}(\gamma_j, \gamma'_k) = \pi(-\frac{r'}{r}a_{kj})$, $\mathrm{Im}\mathcal{R}(\gamma'_k, \gamma_j) = \pi\frac{r'}{r}a_{kj}$, where $j, k = 1, \cdots, n$.*

Here, we consider a projectively flat bundle $\mathcal{E}_{(r,A,\mu,\mathcal{U})}$ of rank r' whose factor of automorphy $j : L \times \mathbb{C}^n \to GL(r';\mathbb{C})$ and connection $\tilde{\nabla}_{(r,A,\mu,\mathcal{U})} = d + \tilde{\omega}_{(r,A,\mu,\mathcal{U})}$ are expressed locally as follows (see also Theorem 3.3).

$$j(\gamma, z) = U(\gamma)\mathrm{exp}\left\{\frac{1}{r'}\mathcal{R}(z,\gamma) + \frac{1}{2r'}\mathcal{R}(\gamma,\gamma)\right\},$$
$$\tilde{\omega}_{(r,A,\mu,\mathcal{U})} = -\frac{1}{r'}dz^t R\bar{z} \cdot I_{r'} + \frac{\mathbf{i}}{2\pi r}\bar{\mu}^t(T-\bar{T})^{-1}dz \cdot I_{r'} - \frac{\mathbf{i}}{2\pi r}\mu^t(T-\bar{T})^{-1}dz \cdot I_{r'}.$$

In particular, $U(\gamma_j), U(\gamma'_k) \in U(r')$ $(j, k = 1, \cdots, n)$ satisfy the following relations (see also Proposition 3.4, Proposition 3.5).

$$U(\gamma_j)U(\gamma_k) = U(\gamma_k)U(\gamma_j),\ U(\gamma'_j)U(\gamma'_k) = U(\gamma'_k)U(\gamma'_j),\ \zeta^{-a_{kj}}U(\gamma'_k)U(\gamma_j) = U(\gamma_j)U(\gamma'_k).$$

Note that these relations are equivalent to the cocycle condition of $E_{(r,A,\mu,\mathcal{U})}$ (see also the definition (4) of $\mathcal{U}$). Therefore, we can denote

$$\mathcal{U} = \Big\{U(\gamma_j), U(\gamma'_k) \in U(r') \mid U(\gamma_j)U(\gamma_k) = U(\gamma_k)U(\gamma_j),\ U(\gamma'_j)U(\gamma'_k) = U(\gamma'_k)U(\gamma'_j),$$
$$\zeta^{-a_{kj}}U(\gamma'_k)U(\gamma_j) = U(\gamma_j)U(\gamma'_k),\ j, k = 1, \cdots, n\Big\}.$$

In fact, the purpose of this subsection is to recall the result that $E_{(r,A,\mu,\mathcal{U})} \cong \mathcal{E}_{(r,A,\mu,\mathcal{U})}$ holds, and this result is given in Theorem 3.6. Similarly as in the case of $E_{(r,A,\mu,\mathcal{U})}$, when we give r, A, μ and $\mathcal{U}$, the projectively flat bundle $\mathcal{E}_{(r,A,\mu,\mathcal{U})}$ is defined. It is clear that the curvature form $\tilde{\Omega}_{(r,A,\mu,\mathcal{U})}$ of $\mathcal{E}_{(r,A,\mu,\mathcal{U})}$ is expressed locally as

$$\tilde{\Omega}_{(r,A,\mu,\mathcal{U})} = \frac{1}{r'}dz^t R d\bar{z} \cdot I_{r'}.$$

Hence, we fix r, A, μ (note that r' is uniquely defined by using r and A) and by comparing the definition of $E_{(r,A,\mu,\mathcal{U})}$ with the definition of $\mathcal{E}_{(r,A,\mu,\mathcal{U})}$, we see that the cardinality of the set

$\{(E_{(r,A,\mu,\mathcal{U})},\nabla_{(r,A,\mu,\mathcal{U})})\}$ is equal to the cardinality of the set $\{(\mathcal{E}_{(r,A,\mu,\mathcal{U})},\tilde{\nabla}_{(r,A,\mu,\mathcal{U})})\}$. Thus, we expect that there exists an isomorphism $\Psi : E_{(r,A,\mu,\mathcal{U})} \stackrel{\sim}{\to} \mathcal{E}_{(r,A,\mu,\mathcal{U})}$ which gives a correspondence between $\{(E_{(r,A,\mu,\mathcal{U})},\nabla_{(r,A,\mu,\mathcal{U})})\}$ and $\{(\mathcal{E}_{(r,A,\mu,\mathcal{U})},\tilde{\nabla}_{(r,A,\mu,\mathcal{U})})\}$. Actually, it is known that the following theorem holds, and this is the main theorem in this subsection (see Theorem 3.6 in [11] [2]).

Theorem 3.6. *One has $E_{(r,A,\mu,\mathcal{U})} \cong \mathcal{E}_{(r,A,\mu,\mathcal{U})}$, where an isomorphism $\Psi : E_{(r,A,\mu,\mathcal{U})} \stackrel{\sim}{\to} \mathcal{E}_{(r,A,\mu,\mathcal{U})}$ is expressed locally as*

$$\Psi(z,\bar{z}) = \exp\left\{\frac{\mathbf{i}}{4\pi r'}z^t\mathcal{A}z + \frac{\mathbf{i}}{4\pi r'}\bar{z}^t\bar{\mathcal{A}}\bar{z} - \frac{\mathbf{i}}{2\pi r'}z^t\mathcal{A}\bar{z} - \frac{\mathbf{i}}{2\pi r}z^t\{(T-\bar{T})^{-1}\}^t\bar{\mu} + \frac{\mathbf{i}}{2\pi r}\bar{z}^t\{(T-\bar{T})^{-1}\}^t\mu\right\}\cdot I_{r'},$$

$$\mathcal{A} := \frac{r'}{r}\{(T-\bar{T})^{-1}\}^t\bar{T}^tA^t(T-\bar{T})^{-1}.$$

4 Exact triangles consisting of stable vector bundles on $\check{T}^2$

In this section, since we treat the case of 2-dimensional tori only, we may assume $gcd(r,a)=1$ in the definition of $E_{(r,a,\mu,\mathcal{U})}$, namely, we may assume that the rank of $E_{(r,a,\mu,\mathcal{U})}$ and the degree of $E_{(r,a,\mu,\mathcal{U})}$ are relatively prime. Furthermore, throughout this section, we fix the set

$$\mathcal{U}_r := \left\{ V_1 := \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \ddots \\ & & & 1 \\ 1 & & & 0 \end{pmatrix},\ U_1 := \begin{pmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{r-1} \end{pmatrix}^{-a} \in U(r) \right\}$$

consisting of unitary matrices when we consider the cocycle conditions (see also the definition (4) of $\mathcal{U}$). In the above definition of $\mathcal{U}_r$, $a\in\mathbb{Z}$ is actually the degree of a holomorphic vector bundle $E_{(r,a,\mu,\mathcal{U}_r)}$ which is defined by using the set $\mathcal{U}_r$. In this section, we take a mirror pair $\left(\left(T^2,\tilde{\omega}=-\frac{1}{T}dx^1\wedge dy^1\right),\check{T}^2=\mathbb{C}/2\pi(\mathbb{Z}\oplus T\mathbb{Z})\right)$ and consider the exact triangle

$$\cdots \longrightarrow E_{(r,a,\mu,\mathcal{U}_r)} \longrightarrow C(\psi) \longrightarrow E_{(s,b,\nu,\mathcal{U}_s)} \stackrel{\psi}{\longrightarrow} TE_{(r,a,\mu,\mathcal{U}_r)} \longrightarrow \cdots \qquad (6)$$

in $Tr(DG_{\check{T}^2})$ under the assumptions

$$\mathrm{dimExt}^1(E_{(s,b,\nu,\mathcal{U}_s)},E_{(r,a,\mu,\mathcal{U}_r)})=1,\ \frac{a}{r}<\frac{b}{s}.$$

Here, $C(\psi)$ and T denote the mapping cone of a non-trivial morphism $\psi\in\mathrm{Ext}^1(E_{(s,b,\nu,\mathcal{U}_s},E_{(r,a,\mu,\mathcal{U}_r)})$ and the shift functor in $Tr(DG_{\check{T}^2})$, respectively. In particular, it is known that the DG-category $DG_{\check{T}^2}$ generates the bounded derived category of coherent sheaves $D^b(Coh(\check{T}^2))$ on $\check{T}^2$, i.e.,

$$Tr(DG_{\check{T}^2})\cong D^b(Coh(\check{T}^2))$$

holds (cf. [16], [1]). Then, we can check that $E_{(r,a,\mu,\mathcal{U}_r)}$ and $E_{(s,b,\nu,\mathcal{U}_s)}$ are stable, and by using the assumption $\mathrm{dimExt}^1(E_{(s,b,\nu,\mathcal{U}_s)},E_{(r,a,\mu,\mathcal{U}_r)})=1$, we see that $C(\psi)$ is also stable. Hence, we can say that

[2] In [11], since the definition of holomorphic vector bundles $E_{(r,A,\mu,\mathcal{U})}$ is restricted to the cases $r=r'$ only, actually, Theorem 3.6 in [11] is the special case of Theorem 3.6 in this paper. However, we are going to revise [11] in the future, and then, we are also going to treat not only the cases $r=r'$ but also the cases $r\neq r'$.

the exact triangle (6) is the most fundamental example of non-trivial exact triangles consisting of stable vector bundles in $Tr(DG_{\check{T}^2}) \cong D^b(Coh(\check{T}^2))$. On the other hand, the isomorphism classes of indecomposable holomorphic vector bundles over an elliptic curve are classified by Atiyah [3]. This result of Atiyah implies that the set of isomorphism classes of indecomposable holomorphic vector bundles are parametrized by $\eta \in \check{T}^2 = \mathbb{C}/2\pi(\mathbb{Z}\oplus T\mathbb{Z})$. Now, we see that $C(\psi)$ is a holomorphic vector bundle whose rank and degree are $r+s$ and $a+b$, respectively, so we expect that there exists a $\eta \in \check{T}^2$ such that $C(\psi) \cong E_{(r+s,a+b,\eta,\mathcal{U}_{r+s})}$ by Atiyah's result. Actually, in section 4 in [10], the value η such that $C(\psi) \cong E_{(r+s,a+b,\eta,\mathcal{U}_{r+s})}$ is determined in the case $\mathrm{dimExt}^1(E_{(s,b,\nu,\mathcal{U}_s)}, E_{(r,a,\mu,\mathcal{U}_r)}) = 1$. The purpose of this section is to recall this result. In particular, the Lagrangian submanifolds corresponding to $C(\psi)$ intersect at one point, and they become a single Lagrangian submanifold corresponding to $E_{(r+s,a+b,\eta,\mathcal{U}_{r+s})}$ by an isomorphism $C(\psi) \cong E_{(r+s,a+b,\eta,\mathcal{U}_{r+s})}$. We expect that this can be regarded as an analogue of the Dehn twist (see [24], [1]). We also remark that the cases without the assumption $\mathrm{dimExt}^1(E_{(s,B,\nu,\mathcal{V})}, E_{(r,A,\mu,\mathcal{U})}) = 1$ are studied in [5].

First, we mention the stability of $E_{(r,a,\mu,\mathcal{U}_r)}$. As discussed in section 2, in $DG_{\check{T}^2}$, the differential

$$\mathrm{Hom}^r_{DG_{\check{T}^2}}(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(s,b,\nu,\mathcal{U}_s)}) \longrightarrow \mathrm{Hom}^{r+1}_{DG_{\check{T}^2}}(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(s,b,\nu,\mathcal{U}_s)})$$

is defined by

$$\psi \longmapsto (2\nabla^{(0,1)}_{(s,b,\nu,\mathcal{U}_s)})(\psi) - (-1)^r \psi(2\nabla^{(0,1)}_{(r,a,\mu,\mathcal{U}_r)}).$$

We denote by $H^r(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(s,b,\nu,\mathcal{U}_s)})$ the r-th cohomology with respect to this differential, and in particular, $H^0(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(s,b,\nu,\mathcal{U}_s)})$ is the space of holomorphic maps. Then, for two holomorphic vector bundles whose ranks and degrees are same, i.e., $(r,a) = (s,b)$, the following proposition holds (see Proposition 3.1 in [10]).

Proposition 4.1. *Let r be a natural number and a an integer. We assume r and a are relatively prime. Then for μ and ν ($\mu, \nu \in \mathbb{R} \oplus T\mathbb{R}$), $\mathrm{dim}H^0(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(r,a,\nu,\mathcal{U}_r)}) = 1$ if and only if $\mu \equiv \nu \pmod{2\pi(\mathbb{Z} \oplus T\mathbb{Z})}$, and, $\mathrm{dim}H^0(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(r,a,\nu,\mathcal{U}_r)}) = 0$ otherwise. Furthermore, in the case $\mathrm{dim}H^0(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(r,a,\nu,\mathcal{U}_r)}) = 1$, a non-trivial element in $H^0(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(r,a,\nu,\mathcal{U}_r)})$ gives an isomorphism $E_{(r,a,\mu,\mathcal{U}_r)} \cong E_{(r,a,\nu,\mathcal{U}_r)}$.*

Moreover, as a corollary of this proposition, we obtain the following (see Corollary 3.2 in [10]).

Corollary 4.2. *For $E_{(r,a,\mu,\mathcal{U}_r)}$, a holomorphic map $\Phi : E_{(r,a,\mu,\mathcal{U}_r)} \to E_{(r,a,\mu,\mathcal{U}_r)}$ is expressed locally as $\Phi = cI_r$, where $c \in \mathbb{C}$.*

Hence $E_{(r,a,\mu,\mathcal{U}_r)}$ is indeed a simple vector bundle, so it is indecomposable. Furthermore, it is known that an indecomposable vector bundle E on an elliptic curve is stable if and only if the rank of E and the degree of E are relatively prime (see [22], p.178). Thus, $E_{(r,a,\mu,\mathcal{U}_r)}$ is also stable. Moreover, for $E_{(r,a,\mu,\mathcal{U}_r)}$ and $E_{(s,b,\nu,\mathcal{U}_s)}$ with $(r,a) \neq (s,b)$, the following proposition is known (see [22], p.179).

Proposition 4.3. *For $E_{(r,a,\mu,\mathcal{U}_r)}$ and $E_{(s,b,\nu,\mathcal{U}_s)}$, if $br - as > 0$ then $\mathrm{dim}H^0(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(s,b,\nu,\mathcal{U}_s)}) = br - as$ and $\mathrm{dim}H^1(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(s,b,\nu,\mathcal{U}_s)}) = 0$, if $br - as < 0$ then $\mathrm{dim}H^0(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(s,b,\nu,\mathcal{U}_s)}) = 0$ and $\mathrm{dim}H^1(E_{(r,a,\mu,\mathcal{U}_r)}, E_{(s,b,\nu,\mathcal{U}_s)}) = as - br$.*

Now, we consider the exact triangle (6), i.e.,

$$\cdots \longrightarrow E_{(r,a,\mu,\mathcal{U}_r)} \longrightarrow C(\psi) \longrightarrow E_{(s,b,\nu,\mathcal{U}_s)} \xrightarrow{\psi} TE_{(r,a,\mu,\mathcal{U}_r)} \longrightarrow \cdots$$

in $Tr(DG_{\check{T}^2}) \cong D^b(Coh(\check{T}^2))$, where

$$\mathrm{dimExt}^1(E_{(s,b,\nu,\mathcal{U}_s)}, E_{(r,a,\mu,\mathcal{U}_r)}) = 1, \ \frac{a}{r} < \frac{b}{s}.$$

In particular,

$$\mathrm{dimExt}^1(E_{(s,b,\nu,\mathcal{U}_s)}, E_{(r,a,\mu,\mathcal{U}_r)}) = \mathrm{dim}H^1(E_{(s,b,\nu,\mathcal{U}_s)}, E_{(r,a,\mu,\mathcal{U}_r)}) = br - as = 1$$

holds by Proposition 4.3. Then, without loss of generality, we may discuss the case $(r,a) = (1,0)$, $(s,b) = (1,1)$ only, because we can consider the $SL(2;\mathbb{Z})$ action on $(T^{2n}, \tilde{\omega})$. This fact is explained in section 6 of [10]. Thus, we may consider the following exact triangle in $Tr(DG_{\check{T}^2})$.

$$\cdots \longrightarrow E_{(1,0,\mu,\mathcal{U}_1)} \longrightarrow C(\psi) \longrightarrow E_{(1,1,\nu,\mathcal{U}_1)} \xrightarrow{\psi} TE_{(r,a,\mu,\mathcal{U}_1)} \longrightarrow \cdots . \qquad (7)$$

Here, we recall Atiyah's result on the classification of the isomorphism classes of indecomposable holomorphic vector bundles over an elliptic curve.

Theorem 4.4 (Atiyah, 1957, [3])**.** *The set of isomorphism classes of indecomposable holomorphic vector bundles over an elliptic curve can be identified with the elliptic curve when the rank and degree of holmorphic vector bundles are relatively prime.*

We explain how to apply this Theorem 4.4 to our discussions. Since $E_{(r,a,\mu,\mathcal{U}_r)} \cong E_{(r,a,\nu,\mathcal{U}_r)}$ holds if and only if $\mu \equiv \nu \pmod{2\pi(\mathbb{Z} \oplus T\mathbb{Z})}$ by Proposition 4.1, the set of isomorphism classes of $E_{(r,a,\mu,\mathcal{U}_r)}$ is parametrized by $\mu \in \mathbb{C}/2\pi(\mathbb{Z} \oplus T\mathbb{Z})$. Now, $C(\psi)$ is a holomorphic vector bundle whose rank and degree are 2 and 1, respectively. So if $C(\psi)$ is indecomposable, we expect that there exists an $\eta \in \mathbb{R} \oplus T\mathbb{R}$ such that $C(\psi) \cong E_{(2,1,\eta,\mathcal{U}_2)}$ by Theorem 4.4. In fact, for $E_{(2,1,\eta,\mathcal{U}_2)}$, it is known that $C(\psi) \cong E_{(2,1,\eta,\mathcal{U}_2)}$ holds if and only if $\eta \equiv \mu + \nu + \pi + \pi T \pmod{2\pi(\mathbb{Z} \oplus T\mathbb{Z})}$ (see Theorem 4.10 in [10]). This is the main theorem which we recall in this section.

We discuss the conditions when there exist non-trivial holomorphic maps $\tilde{\phi} : E_{(2,1,\eta,\mathcal{U}_2)} \to C(\psi)$ and $\phi : C(\psi) \to E_{(2,1,\eta,\mathcal{U}_2)}$. We apply the covariant cohomological functor

$$F := \mathrm{Hom}_{Tr(DG_{\check{T}^2})}(E_{(2,1,\eta,\mathcal{U}_2)}, \ \cdot\ ) = H^0(E_{(2,1,\eta,\mathcal{U}_2)}, \ \cdot\ )$$

and the contravariant cohomological functor

$$G := \mathrm{Hom}_{Tr(DG_{\check{T}^2})}(\ \cdot\ , E_{(2,1,\eta,\mathcal{U}_2)}) = H^0(\ \cdot\ , E_{(2,1,\eta,\mathcal{U}_2)})$$

to the exact triangle (7), and obtain the following long exact sequences.

$$\cdots \longrightarrow F(E_{(1,0,\mu,\mathcal{U}_1)}) \longrightarrow F(C(\psi)) \longrightarrow F(E_{(1,1,\nu,\mathcal{U}_1)}) \xrightarrow{F(\psi)} F(TE_{(1,0,\mu,\mathcal{U}_1)}) \longrightarrow \cdots ,$$

$$\cdots \longrightarrow G(E_{(1,1,\nu,\mathcal{U}_1)}) \longrightarrow G(C(\psi)) \longrightarrow G(E_{(1,0,\mu,\mathcal{U}_1)}) \xrightarrow{G(T^{-1}\psi)} G(T^{-1}E_{(1,1,\nu,\mathcal{U}_1)}) \longrightarrow \cdots .$$

Then, it is known that the following lemmas hold (see Lemma 4.2, Lemma 4.3 in [10]).

Lemma 4.5. *If* $F(\psi) = 0$ *then* $\mathrm{dim}F(C(\psi)) = 1$ *and if* $F(\psi) \neq 0$ *then* $F(C(\psi)) = 0$.

Lemma 4.6. *If* $G(T^{-1}\psi) = 0$ *then* $\mathrm{dim}G(C(\psi)) = 1$ *and if* $G(T^{-1}\psi) \neq 0$ *then* $G(C(\psi)) = 0$.

In particular, although we can consider the Fukaya category $Fuk(T^2,\tilde{\omega})$ in the symplectic geometry side, the conditions $F(\psi)=0$, $G(T^{-1}\psi)=0$ in the above lemmas are equivalent to the condition that an A_∞ product m_2 in $Fuk(T^2,\tilde{\omega})$ vanishes. The detail of these discussions is described in section 5 of [10].

In general, $C(\psi)$ does not depend on the choice of a non-trivial morphism ψ. We explain the local expression of the morphism $\psi=\tilde{\psi}(x_1,y_1)d\bar{z}_1$ briefly. For $\tilde{\psi}(x_1,y_1)$, it can be Fourier-expanded as

$$\tilde{\psi}(x_1,y_1)=\sum_{H\in\mathbb{Z}}\psi_H(x_1)e^{\mathbf{i}Hy_1},$$

and we see that $\psi_H(x_1)$ satisfies

$$\psi_H(x_1+2\pi)=\psi_{H+1}(x_1)$$

by the conditions of transition functions of $E_{(1,0,\mu,\mathcal{U}_1)}$ and $E_{(1,1,\nu,\mathcal{U}_1)}$. So we need to define $\psi_H(x_1)$. Here, as such a function, we take a bump function $\psi_H(x_1)$ which has values at the neighborhoods of x_1 coordinates of the intersection points of Lagrangian submanifolds and their copies in the covering space $\mathbb{R}^2$ of T^2 corresponding to $E_{(1,0,\mu,\mathcal{U}_1)}$, $E_{(1,1,\nu,\mathcal{U}_1)}$ (see also Figure 1 in [10]).

In the above situations, the following theorems hold (see Theorem 4.4, Theorem 4.6 in [10]).

Theorem 4.7. *For* $E_{(2,1,\eta,\mathcal{U}_1)}$, $\mathrm{dim}F(C(\psi))=1$ *holds if and only if* $\eta\equiv\mu+\nu+\pi+\pi T \pmod{2\pi(\mathbb{Z}\oplus T\mathbb{Z})}$.

Theorem 4.8. *For* $E_{(2,1,\eta,\mathcal{U}_1)}$, $\mathrm{dim}G(C(\psi))=1$ *holds if and only if* $\eta\equiv\mu+\nu+\pi+\pi T \pmod{2\pi(\mathbb{Z}\oplus T\mathbb{Z})}$.

Furthermore, the local expressions of the non-trivial morphisms $\tilde{\phi}: E_{(2,1,\eta,\mathcal{U}_2)}\to C(\psi)$, $\phi: C(\psi)\to E_{(2,1,\eta,\mathcal{U}_2)}$ are also given in Corollary 4.5, Corollary 4.7 in section 4 of [10], respectively, and by using these local expressions, we see that the relation

$$\phi\tilde{\phi}=c_T I_2 \tag{8}$$

holds, where $c_T\in\mathbb{C}$ is a constant number. In particular, this c_T is expressed as a derivative of a theta function and is then shown to be nonzero by the Jacobi's differential formula (see Lemma 4.9 in [10]). On the other hand, when we consider the relation (8) from the viewpoint of the symplectic geometry side, the relation (8) is identified with an A_∞ triple product m_3 in the Fukaya category $Fuk(T^2,\tilde{\omega})$ (cf. [9]). Thus, by summarizing the above discussions, we can state the following theorem, and this is the main theorem in this section (see Theorem 4.10 in [10]).

Theorem 4.9. *The mapping cone* $C(\psi)$ *is isomorphic to* $E_{(2,1,\eta,\mathcal{U}_2)}$ *with* $\eta\equiv\mu+\nu+\pi+\pi T \pmod{2\pi(\mathbb{Z}\oplus T\mathbb{Z})}$.

5 Exact triangles consisting of projectively flat bundles on $\check{T}^{2n}$

In this section, we consider exact triangles consisting of three projectively flat bundles and their shifts on an n-dimensional complex torus $\check{T}^{2n}$, as a generalization of the discussions in section

4 to the higher dimensional case. More precisely, in this section, we consider an exact triangle associated to the mapping cone $C(\psi)$ of a non-trivial morphism $\psi \in \mathrm{Ext}^1(E_{(s,B,\nu,\mathcal{V})}, E_{(1,A,\mu,\mathcal{U})})$ [3], and study the case such that $C(\psi)$ becomes a simple projectively flat bundle, namely, consider an exact triangle

$$\cdots \longrightarrow E_{(1,A,\mu,\mathcal{U})} \longrightarrow E_{(t,C,\eta,\mathcal{W})} \longrightarrow E_{(s,B,\nu,\mathcal{V})} \longrightarrow TE_{(1,A,\mu,\mathcal{U})} \longrightarrow \cdots \tag{9}$$

in $Tr(DG_{\check{T}^{2n}})$, where $t' = 1 + s'$, $\frac{t'}{t}C = A + \frac{s'}{s}B$, $A - \frac{1}{s}B \neq O$ and T denotes the shift functor in $Tr(DG_{\check{T}^{2n}})$. Then, we prove that the exact triangle (9) is obtained as the pullback of an exact triangle consisting of three projectively flat bundles and their shifts in $Tr(DG_{\check{T}^2}) \cong D^b(Coh(\check{T}^2))$ by a suitable holomorphic projection $\pi : \check{T}^{2n} \to \check{T}^2$ (Theorem 5.6).

Here, we explain some notations. For a projectively flat bundle $E_{(s,B,\nu,\mathcal{V})} \to \check{T}^{2n}$ with $\nu = u + T^t v$, similarly as in the case of $E_{(1,A,\mu,\mathcal{U})} \to \check{T}^{2n}$ with $\mu = p + T^t q$, we set

$$u(\sigma) := \frac{s'}{s}u - \sigma, \ v(\tau) := \frac{s'}{s}v + \tau,$$

where $s' \in \mathbb{N}$, and $\sigma := (\sigma_1, \cdots, \sigma_n)^t$, $\tau := (\tau_1, \cdots, \tau_n)^t \in \mathbb{R}^n$ are defined by using the determinants of unitary matrices in the set $\mathcal{V}$. Moreover, we set

$$\alpha := A - \frac{1}{s}B \neq O, \ \beta := \frac{1}{s}u(\sigma) - p(\theta),$$

and assume

$$\alpha_{ij} \neq 0.$$

5.1 Preparations in matrix calculations

The purpose of this subsection is to prove Proposition 5.1 and Proposition 5.2 which play an important role in the proof of Theorem 5.6 (this theorem is the main theorem in section 5) in subsection 5.2.

Proposition 5.1. *Assume* $\mathrm{rank}\alpha = 1$. *Then, there exist two matrices* $\mathcal{A}$, $\mathcal{D} \in SL(n;\mathbb{Z})$ *such that*

$$\mathcal{D}^t(s\alpha)\mathcal{A} = -NE_{ij},$$

where $N \in \mathbb{N}$ *and* E_{ij} *denotes the matrix unit.*

Proof. We may assume that there exists a $k \in \mathbb{Z}$ $(0 \leq k \leq n-1)$ such that

$$i_1 \neq i, \cdots, i_k \neq i, \ \alpha_{i_1 j} \neq 0, \cdots, \alpha_{i_k j} \neq 0.$$

For the matrix $s\alpha$, we apply elementary row operations. Set

$$m_1 := gcd((s\alpha)_{i_1 j}, (s\alpha)_{ij}).$$

Then, there exist α'_{ij}, $\alpha'_{i_1 j} \in \mathbb{Z}$ such that

$$(s\alpha)_{ij} = m_1\alpha'_{ij}, \ (s\alpha)_{i_1 j} = m_1\alpha'_{i_1 j}.$$

[3]In fact, we can also obtain the condition such that $\mathrm{Ext}^1(E_{(s,B,\nu,\mathcal{V})}, E_{(1,A,\mu,\mathcal{U})}) \neq 0$ by a direct calculation. See also the relations (33) and (36).

Clearly, α'_{ij} and α'_{i_1j} are relatively prime, so there exist $K_1,\ L_1 \in \mathbb{Z}$ such that

$$-\alpha'_{ij}K_1 - \alpha'_{i_1j}L_1 = 1.$$

By using these four integers α'_{ij}, α'_{i_1j}, K_1 and L_1, we define the following matrix.

$$\mathcal{D}_{i_1} := I_n + (-\alpha'_{ij} - 1)E_{i_1i_1} + (K_1 - 1)E_{ii} + L_1E_{i_1i} + \alpha'_{i_1j}E_{ii_1}.$$

Note

$$\det\mathcal{D}_{i_1} = -\alpha'_{ij}K_1 - \alpha'_{i_1j}L_1 = 1.$$

Then, the (i_1, m) component of $\mathcal{D}^t_{i_1}(s\alpha)$ $(1 \leq m \leq n)$ is calculated as

$$\begin{aligned} -\alpha'_{ij}(s\alpha)_{i_1m} + \alpha'_{i_1j}(s\alpha)_{im} &= -\frac{s^2}{m_1}(\alpha_{ij}\alpha_{i_1m} - \alpha_{i_1j}\alpha_{im}) \\ &= 0, \end{aligned}$$

where the second equality follows from the assumption rank$\alpha = 1$. Similarly, by using the fact $\alpha_{ij} \neq 0$, the (i, m) component of $\mathcal{D}^t_{i_1}(s\alpha)$ $(1 \leq m \leq n)$ is calculated as

$$\begin{aligned} L_1(s\alpha)_{i_1m} + K_1(s\alpha)_{im} &= L_1 \cdot s\frac{\alpha_{i_1j}}{\alpha_{ij}}\alpha_{im} + K_1(s\alpha)_{im} \\ &= \frac{\alpha_{im}}{\alpha_{ij}}(L_1(s\alpha)_{i_1j} + K_1(s\alpha)_{ij}) \\ &= -\frac{\alpha_{im}}{\alpha_{ij}}m_1, \end{aligned}$$

where the first equality follows from the assumption rank$\alpha = 1$, and we put

$$\alpha^{(1)}_{im} := -\frac{\alpha_{im}}{\alpha_{ij}}m_1.$$

As a result, $\mathcal{D}^t_{i_1}(s\alpha)$ is expressed as

$$\mathcal{D}^t_{i_1}(s\alpha) = \begin{pmatrix} \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ \alpha^{(1)}_{i1} & \dots & \alpha^{(1)}_{ij} & \dots & \alpha^{(1)}_{in} \\ \vdots & & \vdots & & \vdots \end{pmatrix},$$

where $(0, \cdots, 0, \cdots, 0)$ is the i_1-th row of $\mathcal{D}^t_{i_1}(s\alpha)$. Next, we set

$$m_2 := gcd((s\alpha)_{i_2j}, \alpha^{(1)}_{ij}),$$

and apply an elementary row operation to $\mathcal{D}^t_{i_1}(s\alpha)$ similarly as above. Hence, by applying elementary row operations to $s\alpha$ for k times, we see that $s\alpha$ is transformed as

$$\begin{aligned}(\mathcal{D}^t_{i_k}\cdots\mathcal{D}^t_{i_1})(s\alpha) &= (\mathcal{D}_{i_1}\cdots\mathcal{D}_{i_k})^t(s\alpha)\\ &= \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \\ \alpha^{(k)}_{i1} & \dots & \alpha^{(k)}_{ij} & \dots & \alpha^{(k)}_{in} \\ 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix},\end{aligned}$$

where

$$\alpha^{(k)}_{im} := -\frac{\alpha_{im}}{\alpha_{ij}} m_k,\ m_k \in \mathbb{N}.$$

Here, we define

$$\mathcal{D} := \mathcal{D}_{i_1}\cdots\mathcal{D}_{i_k},$$

and since $\det\mathcal{D}_{i_1} = \cdots = \det\mathcal{D}_{ik} = 1$, clearly $\det\mathcal{D} = 1$ holds.

We assume that there exists a $l \in \mathbb{Z}$ $(0 \le l \le n-1)$ such that

$$j_1 \neq j, \cdots, j_l \neq j,\ \alpha^{(k)}_{ij_1} \neq 0, \cdots, \alpha^{(k)}_{ij_l} \neq 0.$$

For the matrix $\mathcal{D}^t(s\alpha)$, we apply elementary column operations. Set

$$n_1 := gcd(\alpha^{(k)}_{ij}, \alpha^{(k)}_{ij_1}).$$

Namely, there exist $\alpha'^{(k)}_{ij}$, $\alpha'^{(k)}_{ij_1} \in \mathbb{Z}$ such that

$$\alpha^{(k)}_{ij} = n_1\alpha'^{(k)}_{ij},\ \alpha^{(k)}_{ij_1} = n_1\alpha'^{(k)}_{ij_1}.$$

Clearly, $\alpha'^{(k)}_{ij}$ and $\alpha'^{(k)}_{ij_1}$ are relatively prime, so there exist M_1, $N_1 \in \mathbb{Z}$ such that

$$-\alpha'^{(k)}_{ij}M_1 - \alpha'^{(k)}_{ij_1}N_1 = 1.$$

By using these four integers $\alpha'^{(k)}_{ij}$, $\alpha'^{(k)}_{ij_1}$, M_1 and N_1, we define the following matrix.

$$\mathcal{A}_{j_1} := I_n + (-\alpha'^{(k)}_{ij} - 1)E_{j_1j_1} + (M_1 - 1)E_{jj} + N_1E_{j_1j} + \alpha'^{(k)}_{ij_1}E_{jj_1}.$$

Note

$$\det\mathcal{A}_{j_1} = -\alpha'^{(k)}_{ij}M_1 - \alpha'^{(k)}_{ij_1}N_1 = 1.$$

Then, the (i, j_1) component of $\mathcal{D}^t(s\alpha)\mathcal{A}_{j_1}$ is calculated as

$$\begin{aligned}-\alpha^{(k)}_{ij_1}\alpha'^{(k)}_{ij} + \alpha^{(k)}_{ij}\alpha'^{(k)}_{ij_1} &= -\alpha^{(k)}_{ij_1}\frac{\alpha^{(k)}_{ij}}{n_1} + \alpha^{(k)}_{ij}\frac{\alpha^{(k)}_{ij_1}}{n_1}\\ &= 0,\end{aligned}$$

and the (i,j) component of $\mathcal{D}^t(s\alpha)\mathcal{A}_{j_1}$ is calculated as

$$\alpha^{(k)}_{ij_1}N_1+\alpha^{(k)}_{ij}M_1=-n_1.$$

As a result, $\mathcal{D}^t(s\alpha)\mathcal{A}_{j_1}$ is expressed as

$$\mathcal{D}^t(s\alpha)\mathcal{A}_{j_1}=\begin{pmatrix} \dots & 0 & \dots & 0 & \dots \\ & \vdots & & \vdots & \\ \dots & 0 & \dots & 0 & \dots \\ \dots & 0 & \dots & -n_1 & \dots \\ \dots & 0 & \dots & 0 & \dots \\ & \vdots & & \vdots & \\ \dots & 0 & \dots & 0 & \dots \end{pmatrix},$$

where 0 which is written in the left side of $-n_1$ is the (i,j_1) component. Next, we set

$$n_2:=gcd(\alpha^{(k)}_{ij_2},-n_1),$$

and apply an elementary column operation to $\mathcal{D}^t(s\alpha)\mathcal{A}_{j_1}$ similarly as above. Hence, by applying elementary column operations to $\mathcal{D}^t(s\alpha)$ for l times, we see that $\mathcal{D}^t(s\alpha)$ is transformed as follows.

$$\mathcal{D}^t(s\alpha)\mathcal{A}_{j_1}\cdots\mathcal{A}_{j_l}=-n_lE_{ij}=-NE_{ij},\ N:=n_l\in\mathbb{N}.$$

Here, we define

$$\mathcal{A}:=\mathcal{A}_{j_1}\cdots\mathcal{A}_{j_l},$$

and since $\det\mathcal{A}_{j_1}=\cdots=\det\mathcal{A}_{j_l}=1$, clearly $\det\mathcal{A}=1$ holds. □

Proposition 5.2. *We assume* $\mathrm{rank}\alpha=1$*, and take a pair* $(\mathcal{A},\mathcal{D})$ *of two matrices* $\mathcal{A}$*,* $\mathcal{D}\in SL(n;\mathbb{Z})$ *which satisfy the statement of Proposition 5.1. Then,*

$$(\mathcal{A}^{-1}T\mathcal{D})_{ji'}=0\ \ (1\le i'\ne i\le n)$$

and

$$\mathrm{Im}(\mathcal{A}^{-1}T\mathcal{D})_{ji}\ne 0$$

hold.

Proof. We denote the (i,j) component of $\mathcal{A}$, the (i,j) component of $\mathcal{A}^{-1}$ and the (i,j) component of $\mathcal{D}$ by $\mathcal{A}_{ij}$, $\mathcal{A}^{ij}$ and $\mathcal{D}_{ij}$, respectively.

First, we prove the following equality.

$$\mathcal{A}^{jk}\alpha_{ij}-\mathcal{A}^{jj}\alpha_{ik}=0\ \ (1\le k\le n). \tag{10}$$

It is clear that the equality (10) holds in the case $k=j$, so we consider the case $k\ne j$. In order to prove the equality (10), we should consider the following cases.

(i) $k=j-1$ (ii) $k=j-2$ (iii) $k=j+1$ (iv) $k=j+2$ (v) $1\le k\le j-3$ (vi) $j+3\le k\le n$.

However, since we can prove other cases similarly, we prove the case of (i), i.e.,

$$\mathcal{A}^{jj-1}\alpha_{ij} - \mathcal{A}^{jj}\alpha_{ij-1} = 0 \tag{11}$$

only, here. By a direct calculation, one has

$$\mathcal{A}^{jj-1}\alpha_{ij} = \alpha_{ij}(-1)^{3j-3}\det\begin{pmatrix} \mathcal{A}_{j1} & \dots & \mathcal{A}_{jj-1} & \mathcal{A}_{jj+1} & \dots & \mathcal{A}_{jn} \\ \mathcal{A}_{11} & \dots & \mathcal{A}_{1j-1} & \mathcal{A}_{1j+1} & \dots & \mathcal{A}_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{A}_{j-21} & \dots & \mathcal{A}_{j-2j-1} & \mathcal{A}_{j-2j+1} & \dots & \mathcal{A}_{j-2n} \\ \mathcal{A}_{j+11} & \dots & \mathcal{A}_{j+1j-1} & \mathcal{A}_{j+1j+1} & \dots & \mathcal{A}_{j+1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{A}_{n1} & \dots & \mathcal{A}_{nj-1} & \mathcal{A}_{nj+1} & \dots & \mathcal{A}_{nn} \end{pmatrix}$$

and

$$-\mathcal{A}^{jj}\alpha_{ij-1} = \alpha_{ij-1}(-1)^{3j-3}\det\begin{pmatrix} \mathcal{A}_{j-11} & \dots & \mathcal{A}_{j-1j-1} & \mathcal{A}_{j-1j+1} & \dots & \mathcal{A}_{j-1n} \\ \mathcal{A}_{11} & \dots & \mathcal{A}_{1j-1} & \mathcal{A}_{1j+1} & \dots & \mathcal{A}_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{A}_{j-21} & \dots & \mathcal{A}_{j-2j-1} & \mathcal{A}_{j-2j+1} & \dots & \mathcal{A}_{j-2n} \\ \mathcal{A}_{j+11} & \dots & \mathcal{A}_{j+1j-1} & \mathcal{A}_{j+1j+1} & \dots & \mathcal{A}_{j+1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \mathcal{A}_{n1} & \dots & \mathcal{A}_{nj-1} & \mathcal{A}_{nj+1} & \dots & \mathcal{A}_{nn} \end{pmatrix}.$$

Therefore, by recalling the definition of the determinant of a matrix, the left hand side of the equality (11) turns out to be

$$(-1)^{3j-3}\sum_{\sigma=(l_1,\cdots,l_{j-1},l_{j+1},\cdots,l_n)}\mathrm{sgn}(\sigma)(\alpha_{ij}\mathcal{A}_{jl_1} + \alpha_{ij-1}\mathcal{A}_{j-1l_1})\mathcal{A}_{1l_2}\cdots\mathcal{A}_{j-2l_{j-1}}\mathcal{A}_{j+1l_{j+1}}\cdots\mathcal{A}_{nl_n}, \tag{12}$$

where the sum is computed over all permutations σ of the set $\{1,\cdots,j-1,j+1,\cdots,n\}$, and $\mathrm{sgn}(\sigma)$ denotes the signature of σ. On the other hand, by Proposition 5.1, we see $(\alpha\mathcal{A})_{il_1} = 0$ $(l_1 \neq j)$, i.e.,

$$\sum_{m=1}^{n}\alpha_{im}\mathcal{A}_{ml_1} = 0 \ (l_1 \neq j),$$

and clearly, this equality is equivalent to the following equality.

$$\alpha_{ij}\mathcal{A}_{jl_1} + \alpha_{ij-1}\mathcal{A}_{j-1l_1} = -\sum_{m\neq j-1,\ j}\alpha_{im}\mathcal{A}_{ml_1}. \tag{13}$$

Hence, by using the equality (13), the formula (12) turns out to be

$$-(-1)^{3j-3}\sum_{m\neq j-1,\ j}\sum_{\sigma}\mathrm{sgn}(\sigma)\alpha_{im}\mathcal{A}_{ml_1}\mathcal{A}_{1l_2}\cdots\mathcal{A}_{j-2l_{j-1}}\mathcal{A}_{j+1l_{j+1}}\cdots\mathcal{A}_{nl_n}. \tag{14}$$

For the formula (14), we fix $m \neq j-1,\ j$, and consider the sum of σ only. Then, if $1 \leq m \leq j-2$, we focus on the following two terms.

$$\mathrm{sgn}(l_1,\cdots,l_{m+1},\cdots,l_{j-1},\cdots,l_n)\mathcal{A}_{ml_1}\cdots\mathcal{A}_{ml_{m+1}}\cdots\mathcal{A}_{j-2l_{j-1}}\cdots\mathcal{A}_{nl_n}, \tag{15}$$

$$\mathrm{sgn}(l_{m+1},\cdots,l_1,\cdots,l_{j-1},\cdots,l_n)\mathcal{A}_{ml_{m+1}}\cdots\mathcal{A}_{ml_1}\cdots\mathcal{A}_{j-2l_{j-1}}\cdots\mathcal{A}_{nl_n}. \tag{16}$$

Here, we can check easily that

$$\mathrm{sgn}(l_{m+1},\cdots,l_1,\cdots,l_{j-1},\cdots,l_n)=-\mathrm{sgn}(l_1,\cdots,l_{m+1},\cdots,l_{j-1},\cdots,l_n)$$

holds, so the two terms (15) and (16) are canceled. Similarly, if $j+1\le m\le n$, the following two terms are canceled.

$$\begin{aligned}&\mathrm{sgn}(l_1,\cdots,l_{j+1},\cdots,l_m,\cdots,l_n)\mathcal{A}_{ml_1}\cdots\mathcal{A}_{j+1l_{j+1}}\cdots\mathcal{A}_{ml_m}\cdots\mathcal{A}_{nl_n},\\&\mathrm{sgn}(l_m,\cdots,l_{j+1},\cdots,l_1,\cdots,l_n)\mathcal{A}_{ml_m}\cdots\mathcal{A}_{j+1l_{j+1}}\cdots\mathcal{A}_{ml_1}\cdots\mathcal{A}_{nl_n}.\end{aligned}$$

These facts indicate that the formula (14) vanishes. Thus, the equality (11) holds.

Now we calculate the the (j,i') component $(\mathcal{A}^{-1}T\mathcal{D})_{ji'}$ of $\mathcal{A}^{-1}T\mathcal{D}$, where $1\le i'\ne i\le n$. By a direct calculation, we see

$$\begin{aligned}(\mathcal{A}^{-1}T\mathcal{D})_{ji'}&=\sum_{l=1}^{n}\left(\sum_{k=1}^{n}\mathcal{A}^{jk}t_{kl}\right)\mathcal{D}_{li'}\\&=\sum_{l\ne i}\left(\sum_{k\ne j}\mathcal{A}^{jk}t_{kl}\right)\mathcal{D}_{li'}+\sum_{l\ne i}\mathcal{A}^{jj}t_{jl}\mathcal{D}_{li'}+\sum_{k=1}^{n}\mathcal{A}^{jk}t_{ki}\mathcal{D}_{ii'}.\end{aligned}\tag{17}$$

Since we may assume $AT=(AT)^t$, $BT=(BT)^t$, i.e., $\alpha T=(\alpha T)^t$, where $\alpha_{ij}\ne 0$, the formula (17) turns out to be

$$\begin{aligned}&\sum_{l\ne i}\left(\sum_{k\ne j}\mathcal{A}^{jk}t_{kl}\right)\mathcal{D}_{li'}+\sum_{l\ne i}\mathcal{A}^{jj}\left(-\frac{1}{\alpha_{ij}}\sum_{k\ne j}\alpha_{ik}t_{kl}+\frac{1}{\alpha_{ij}}\sum_{k=1}^{n}\alpha_{lk}t_{ki}\right)\mathcal{D}_{li'}+\sum_{k=1}^{n}\mathcal{A}^{jk}t_{ki}\mathcal{D}_{ii'}\\&=\frac{1}{\alpha_{ij}}\sum_{l\ne i}\sum_{k\ne j}\left(\mathcal{A}^{jk}\alpha_{ij}-\mathcal{A}^{jj}\alpha_{ik}\right)\mathcal{D}_{li'}t_{kl}+\sum_{k=1}^{n}\left(\frac{1}{\alpha_{ij}}\sum_{l\ne i}\mathcal{A}^{jj}\mathcal{D}_{li'}\alpha_{lk}+\mathcal{A}^{jk}\mathcal{D}_{ii'}\right)t_{ki}.\end{aligned}\tag{18}$$

By Proposition 5.1, for $i'\ne i$,

$$(\mathcal{D}^t\alpha)_{i'k}=\sum_{l=1}^{n}\mathcal{D}_{li'}\alpha_{lk}=0$$

holds, so the formula (18) is calculated as

$$\begin{aligned}&\frac{1}{\alpha_{ij}}\sum_{l\ne i}\sum_{k\ne j}\left(\mathcal{A}^{jk}\alpha_{ij}-\mathcal{A}^{jj}\alpha_{ik}\right)\mathcal{D}_{li'}t_{kl}+\sum_{k=1}^{n}\left(-\frac{1}{\alpha_{ij}}\mathcal{A}^{jj}\mathcal{D}_{ii'}\alpha_{ik}+\mathcal{A}^{jk}\mathcal{D}_{ii'}\right)t_{ki}\\&=\frac{1}{\alpha_{ij}}\sum_{l\ne i}\sum_{k\ne j}\left(\mathcal{A}^{jk}\alpha_{ij}-\mathcal{A}^{jj}\alpha_{ik}\right)\mathcal{D}_{li'}t_{kl}+\frac{1}{\alpha_{ij}}\sum_{k=1}^{n}\left(\mathcal{A}^{jk}\alpha_{ij}-\mathcal{A}^{jj}\alpha_{ik}\right)\mathcal{D}_{ii'}t_{ki}.\end{aligned}\tag{19}$$

Therefore, by using the equality (10), one sees that the formula (19) vanishes. Thus,

$$(\mathcal{A}^{-1}T\mathcal{D})_{ji'}=0\ \ (1\le i'\ne i\le n)$$

holds.

Finally, we prove $\mathrm{Im}(\mathcal{A}^{-1}T\mathcal{D})_{ji} \neq 0$. We assume $\mathrm{Im}(\mathcal{A}^{-1}T\mathcal{D})_{ji} = 0$. We proved $(\mathcal{A}^{-1}T\mathcal{D})_{ji'} = 0$ ($1 \leq i' \neq i \leq n$) above, and this fact indicates $\mathrm{Im}(\mathcal{A}^{-1}T\mathcal{D})_{ji'} = 0$. Hence,

$$\det(\mathrm{Im}(\mathcal{A}^{-1}T\mathcal{D})) = \det(\mathrm{Im}T) = 0$$

holds. However, this fact contradicts the positive definiteness of $\mathrm{Im}T$. Thus,

$$\mathrm{Im}(\mathcal{A}^{-1}T\mathcal{D})_{ji} \neq 0.$$

□

5.2 Main result

In this subsection, as stated above, we consider the exact triangle (9), i.e.,

$$\cdots \longrightarrow E_{(1,A,\mu,\mathcal{U})} \longrightarrow E_{(t,C,\eta,\mathcal{W})} \longrightarrow E_{(s,B,\nu,\mathcal{V})} \longrightarrow TE_{(1,A,\mu,\mathcal{U})} \longrightarrow \cdots,$$

where $t' = 1 + s'$, $\frac{t'}{t}C = A + \frac{s'}{s}B$ and T denotes the shift functor in $Tr(DG_{\check{T}^{2n}})$. In order to study the above exact triangle, we use the following theorem (see Theorem 4.1 in [11]) which plays an important role in our discussions. In particular, although we can check easily that $\mathrm{codim}(L_{(1,0,p(\theta))} \cap L_{(1,1,u(\sigma))}) = 1$ holds in Theorem 4.9, this is actually the special case of the following theorem.

Theorem 5.3. *We take two projectively flat bundles $E_{(r,A,\mu,\mathcal{U})}$, $E_{(s,B,\nu,\mathcal{V})}$ ($\mu = p+T^tq,\ \nu = u+T^tv$) on $\check{T}^{2n}$, and consider the following exact triangle associated to the mapping cone of $\psi : E_{(s,B,\nu,\mathcal{V})} \to TE_{(r,A,\mu,\mathcal{U})}$ in $Tr(DG_{\check{T}^{2n}})$,*

$$\cdots \longrightarrow E_{(r,A,\mu,\mathcal{U})} \longrightarrow C(\psi) \longrightarrow E_{(s,B,\nu,\mathcal{V})} \stackrel{\psi}{\longrightarrow} TE_{(r,A,\mu,\mathcal{U})} \longrightarrow \cdots, \qquad (20)$$

where T is the shift functor and $C(\psi)$ denotes the mapping cone of ψ. Furthermore, we assume that there exists a projectively flat bundle $E_{(t,C,\eta,\mathcal{W})}$ such that $C(\psi) \cong E_{(t,C,\eta,\mathcal{W})}$, namely, the exact triangle (20) becomes an exact triangle consisting of three projectively flat bundles and their shifts. Then, $\mathrm{codim}(L_{(r,A,p(\theta))} \cap L_{(s,B,u(\sigma))}) \leq 1$ holds.

Note that the exact triangle (9) satisfies the assumption in Theorem 5.3, and in particular, this Theorem 5.3 states that $\mathrm{codim}(L_{(1,A,p(\theta))} \cap L_{(s,B,u(\sigma))}) = 1$, i.e., $\mathrm{rank}\alpha = \mathrm{rank}(\alpha, \beta) = 1$ holds in the case $\alpha \neq O$ (see the case 2 in the proof of Theorem 5.3). Thus, we may assume

$$\mathrm{rank}\alpha = \mathrm{rank}(\alpha, \beta) = 1.$$

In order to state the main theorem, first, we transform the exact triangle (9) to an exact triangle which is easy to treat, by using Proposition 5.1 and Proposition 5.2, and so on. This process consists of the following two steps.

Step 1.

In Step 1, we construct a biholomorphic map $\varphi : \check{T}_0^{2n} \stackrel{\sim}{\to} \check{T}^{2n}$ for an n-dimensional complex torus $\check{T}_0^{2n}$, and by using this φ, we transform the exact triangle (9) which is defined on $\check{T}^{2n}$ to an exact triangle which is defined on $\check{T}_0^{2n}$. In this transformation, we also consider the homological mirror symmetry setting on a mirror pair $((T^{2n},\tilde{\omega_0}),\check{T}_0^{2n})$, where $(T^{2n},\tilde{\omega_0})$ is a mirror dual complexified symplectic torus of $\check{T}_0^{2n}$.

In general, for two complex tori $\mathbb{C}^n/2\pi(\mathbb{Z}^n\oplus T\mathbb{Z}^n)$, $\mathbb{C}^n/2\pi(\mathbb{Z}^n\oplus T'\mathbb{Z}^n)$, $\mathbb{C}^n/2\pi(\mathbb{Z}^n\oplus T\mathbb{Z}^n)$ and $\mathbb{C}^n/2\pi(\mathbb{Z}^n\oplus T'\mathbb{Z}^n)$ are biholomorphic if and only if $T'=(T\mathscr{C}+\mathscr{A})^{-1}(T\mathscr{D}+\mathscr{B})$, where $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}\in M(n;\mathbb{Z})$ and

$$\begin{pmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{C} & \mathscr{D}\end{pmatrix}\in GL(2n;\mathbb{Z}).$$

Now, we assume $\mathrm{rank}\alpha=1$, so by Proposition 5.1, we can take a pair $(\mathcal{A},\mathcal{D})$ of two matrices $\mathcal{A}$, $\mathcal{D}\in SL(n;\mathbb{Z})$ such that

$$\mathcal{D}^t(s\alpha)\mathcal{A}=-NE_{ij},$$

where $N\in\mathbb{N}$. We fix such a pair $(\mathcal{A},\mathcal{D})$, and set

$$\begin{pmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{C} & \mathscr{D}\end{pmatrix}=\begin{pmatrix}\mathcal{A} & O\\ O & \mathcal{D}\end{pmatrix}\in SL(2n;\mathbb{Z})\subset GL(2n;\mathbb{Z}),$$

namely,

$$T'=(t'_{ij})=\mathcal{A}^{-1}T\mathcal{D}.$$

In particular, by Proposition 5.2, we see that

$$t'_{ji'}=0\ \ (1\leq i'\neq i\leq n)$$

and

$$\mathrm{Im}t_{ji}\neq 0$$

hold. We denote this n-dimensional complex torus $\mathbb{C}^n/2\pi(\mathbb{Z}^n\oplus T'\mathbb{Z}^n)=\mathbb{C}^n/2\pi(\mathbb{Z}^n\oplus\mathcal{A}^{-1}T\mathcal{D}\mathbb{Z}^n)$ by $\check{T}_0^{2n}$. Let us denote the local complex coordinates of $\check{T}_0^{2n}$ by $Z=X+T'Y=X+\mathcal{A}^{-1}T\mathcal{D}Y$, where

$$Z:=(Z_1,\cdots,Z_n)^t,\ X:=(X_1,\cdots,X_n)^t,\ Y:=(Y_1,\cdots,Y_n)^t.$$

Then, a biholomorphic map $\varphi:\check{T}_0^{2n}\stackrel{\sim}{\to}\check{T}^{2n}$ is given by

$$\varphi(Z)=\mathcal{A}Z.$$

When we regard complex manifolds $\check{T}^{2n}$ and $\check{T}_0^{2n}$ as real differentiable manifolds $\mathbb{R}^{2n}/2\pi\mathbb{Z}^{2n}$, the biholomorphic map φ is regarded as the diffeomorphism $\varphi:\mathbb{R}^{2n}/2\pi\mathbb{Z}^{2n}\stackrel{\sim}{\to}\mathbb{R}^{2n}/2\pi\mathbb{Z}^{2n}$ such that

$$\varphi\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}\mathcal{A} & O\\ O & \mathcal{D}\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}.$$

We consider the complexified symplectic torus $(T^{2n},\tilde{\omega_0}:=(-T'^{-1})^t=\mathcal{A}^t\tilde{\omega}(\mathcal{D}^{-1})^t)$ as a mirror partner of the complex torus $\check{T}_0^{2n}$. We denote the local coordinates of $(T^{2n},\tilde{\omega_0})$ by $(X^1,\cdots X^n,Y^1,\cdots,Y^n)^t$, and define

$$\check{X}:=(X^1,\cdots,X^n)^t,\ \check{Y}:=(Y^1,\cdots,Y^n)^t.$$

Here, we define a diffeomorphism $\phi:(T^{2n},\tilde{\omega_0})\stackrel{\sim}{\to}(T^{2n},\tilde{\omega})$ by

$$\phi\begin{pmatrix}\check{X}\\ \check{Y}\end{pmatrix}=\begin{pmatrix}\mathcal{A} & O\\ O & (\mathcal{D}^{-1})^t\end{pmatrix}\begin{pmatrix}\check{X}\\ \check{Y}\end{pmatrix}.$$

By a direct calculation, we see

$$\phi^*(d\check{x}^t\tilde{\omega}d\check{y}) = d\check{X}^t\tilde{\omega_0}d\check{Y},$$

where $d\check{X} := (dX^1, \cdots, dX^n)^t$, $d\check{Y} := (dY^1, \cdots, dY^n)^t$, so this diffeomorphism ϕ is a symplectomorphism.

Now, we explain the homological mirror symmetry setting on $((T^{2n}, \tilde{\omega_0}), \check{T}_0^{2n})$. By using the biholomorphic map $\varphi : \check{T}_0^{2n} \overset{\sim}{\to} \check{T}^{2n}$ and complex vector bundles $E_{(r,A,\mu,\mathcal{U})} \to \check{T}^{2n}$, we can consider the pullback bundles $\varphi^* E_{(r,A,\mu,\mathcal{U})} \to \check{T}_0^{2n}$ of rank r'. Then, the connection $\varphi^*\nabla_{(r,A,\mu,\mathcal{U})}$ on each $\varphi^* E_{(r,A,\mu,\mathcal{U})}$ is expressed locally as

$$\varphi^*\nabla_{(r,A,\mu,\mathcal{U})} = d - \frac{\mathbf{i}}{2\pi}\left(\frac{1}{r}X^t\mathcal{A}^tA^t\mathcal{D} + \frac{1}{r}\mu^t\mathcal{D}\right)dY \cdot I_{r'},$$

where $dY := (dY_1, \cdots, dY_n)^t$. In particular, by Proposition 2.1 and the biholomorphicity of the map $\varphi : \check{T}_0^{2n} \overset{\sim}{\to} \check{T}^{2n}$, we see that a complex vector bundle $\varphi^* E_{(r,A,\mu,\mathcal{U})} \to \check{T}_0^{2n}$ is holomorphic if and only if $AT = (AT)^t$ holds. These holomorphic vector bundles $(\varphi^* E_{(r,A,\mu,\mathcal{U})}, \varphi^*\nabla_{(r,A,\mu,\mathcal{U})})$ again form a DG-category $DG_{\check{T}_0^{2n}}$. Furthermore, $DG_{\check{T}_0^{2n}}$ is equivalent to $DG_{\check{T}^{2n}}$ as DG-categories. Hereafter, since we can also regard $\varphi^* E_{(r,A,\mu,\mathcal{U})} \to \check{T}_0^{2n}$ as $E_{(r,\mathcal{D}^tA\mathcal{A},\mathcal{D}^t\mu,\mathcal{U}')} \to \check{T}_0^{2n}$ by using a suitable set $\mathcal{U}'$ (the definition of $\mathcal{U}'$ depends on the data $(\mathcal{U}, \mathcal{A}, \mathcal{D})$), we denote $\varphi^* E_{(r,A,\mu,\mathcal{U})}$ by $E_{(r,\mathcal{D}^tA\mathcal{A},\mathcal{D}^t\mu,\mathcal{U}')}$. Similarly, by the symplectomorphism $\phi : (T^{2n}, \tilde{\omega_0}) \overset{\sim}{\to} (T^{2n}, \tilde{\omega})$, we see that n-dimensional submanifolds $L_{(r,A,p(\theta))}$ in $(T^{2n}, \tilde{\omega})$ and unitary local systems $\mathcal{L}_{(r,A,p(\theta),q(\xi))} \to L_{(r,A,p(\theta))}$ are mapped to n-dimensional submanifolds

$$\phi^{-1}(L_{(r,A,p(\theta))}) = \left\{ \begin{pmatrix} \check{X} \\ \check{Y} \end{pmatrix} \in (T^{2n}, \tilde{\omega_0}) \mid \check{Y} = \frac{1}{r}\mathcal{D}^tA\mathcal{A}\check{X} + \frac{1}{r}\mathcal{D}^tp(\theta) \right\}$$

and unitary local systems $\phi^*\mathcal{L}_{(r,A,p(\theta),q(\xi))} \to \phi^{-1}(L_{(r,A,p(\theta))})$, respectively. In particular, a pair $(\phi^{-1}(L_{(r,A,p(\theta))}), \phi^*\mathcal{L}_{(r,A,p(\theta),q(\xi))})$ is an object of the Fukaya category $Fuk(T^{2n}, \tilde{\omega_0})$ if and only if $AT = (AT)^t$ (and this is also the condition such that a pair $(L_{(r,A,p(\theta))}, \mathcal{L}_{(r,A,p(\theta),q(\xi))})$ is an object of the Fukaya category $Fuk(T^{2n}, \tilde{\omega})$). Hereafter, since we can also regard $\phi^{-1}(L_{(r,A,p(\theta))}) \subset (T^{2n}, \tilde{\omega_0})$ and $\phi^*\mathcal{L}_{(r,A,p(\theta),q(\xi))} \to \phi^{-1}(L_{(r,A,p(\theta))})$ as $L_{(r,\mathcal{D}^tA\mathcal{A},\mathcal{D}^tp(\theta))} \subset (T^{2n}, \tilde{\omega_0})$ and $\mathcal{L}_{(r,\mathcal{D}^tA\mathcal{A},\mathcal{D}^tp(\theta),\mathcal{A}^tq(\xi))} \to L_{(r,\mathcal{D}^tA\mathcal{A},\mathcal{D}^tp(\theta))}$, respectively, we denote $\phi^{-1}(L_{(r,A,p(\theta))})$ and $\phi^*\mathcal{L}_{(r,A,p(\theta),q(\xi))}$ by $L_{(r,\mathcal{D}^tA\mathcal{A},\mathcal{D}^tp(\theta))}$ and $\mathcal{L}_{(r,\mathcal{D}^tA\mathcal{A},\mathcal{D}^tp(\theta),\mathcal{A}^tq(\xi))}$, respectively. Hence, we obtain the following proposition.

Proposition 5.4. *For a given quadruple $(r, A, \mu, \mathcal{U})$, the complex vector bundle $E_{(r,\mathcal{D}^tA\mathcal{A},\mathcal{D}^t\mu,\mathcal{U}')} \to \check{T}_0^{2n}$ is holomorphic if and only if the pair $(L_{(r,\mathcal{D}^tA\mathcal{A},\mathcal{D}^tp(\theta))}, \mathcal{L}_{(r,\mathcal{D}^tA\mathcal{A},\mathcal{D}^tp(\theta),\mathcal{A}^tq(\xi))})$ is an object of the Fukaya category $Fuk(T^{2n}, \tilde{\omega_0})$.*

Thus, when we set

$$A' := \mathcal{D}^tA\mathcal{A},\ B' := \mathcal{D}^tB\mathcal{A},\ C' := \mathcal{D}^tC\mathcal{A},\ \alpha' := \mathcal{D}^t\alpha\mathcal{A} = -\frac{N}{s}E_{ij},$$
$$\mu' := \mathcal{D}^t\mu,\ \nu' := \mathcal{D}^t\nu,\ \eta' := \mathcal{D}^t\eta,$$
$$p(\theta)' := \mathcal{D}^tp(\theta),\ q(\xi)' := \mathcal{A}^tq(\xi),\ u(\sigma)' := \mathcal{D}^tu(\sigma),\ v(\tau)' := \mathcal{A}^tv(\tau),$$

we may consider the exact triangle

$$\cdots \longrightarrow E_{(1,A',\mu',\mathcal{U}')} \longrightarrow E_{(t,C',\eta',\mathcal{W}')} \longrightarrow E_{(s,B',\nu',\mathcal{V}')} \longrightarrow TE_{(1,A',\mu',\mathcal{U}')} \longrightarrow \cdots \quad (21)$$

in $Tr(DG_{\check{T}^{2n}_0})$, instead of the exact triangle (9) in $Tr(DG_{\check{T}^{2n}})$. In the exact triangle (21), note that the definitions $\mathcal{U}'$, $\mathcal{V}'$, and $\mathcal{W}'$ depend on the data $(\mathcal{U},\mathcal{A},\mathcal{D})$, $(\mathcal{V},\mathcal{A},\mathcal{D})$, and $(\mathcal{W},\mathcal{A},\mathcal{D})$, respectively.

Step 2.

In Step 2, we consider the $SL(2n;\mathbb{Z})$ action on $(T^{2n},\tilde{\omega_0})$ as follows. We take the matrix

$$\begin{pmatrix} I_n & O \\ A' & I_n \end{pmatrix} \in SL(2n;\mathbb{Z}),$$

and define the $SL(2n;\mathbb{Z})$ action on $(T^{2n},\tilde{\omega_0})$ by

$$\begin{pmatrix} \check{X} \\ \check{Y} \end{pmatrix} \in (T^{2n},\tilde{\omega_0}) \longmapsto \begin{pmatrix} I_n & O \\ A' & I_n \end{pmatrix}\begin{pmatrix} \check{X} \\ \check{Y} \end{pmatrix} \in (T^{2n},\tilde{\omega_0}).$$

By using this matrix, we define an automorphism $\tilde{\phi} : (T^{2n},\tilde{\omega_0}) \stackrel{\sim}{\to} (T^{2n},\tilde{\omega_0})$ by

$$\tilde{\phi}\begin{pmatrix} \check{X} \\ \check{Y} \end{pmatrix} = \begin{pmatrix} I_n & O \\ A' & I_n \end{pmatrix}\begin{pmatrix} \check{X} \\ \check{Y} \end{pmatrix}.$$

Since $\tilde{\omega_0}A' = (\tilde{\omega_0}A')^t$, we can check easily that the automorphism $\tilde{\phi}$ preserves the complexified symplectic form $\tilde{\omega_0}$, i.e., $\tilde{\phi}^*\tilde{\omega_0} = \tilde{\omega_0}$. Therefore, the automorphism $\tilde{\phi}$ is a symplectic automorphism.

By using this symplectic automorphism $\tilde{\phi} : (T^{2n},\tilde{\omega_0}) \stackrel{\sim}{\to} (T^{2n},\tilde{\omega_0})$, we can map the objects

$$\mathscr{L}_{(1,A',p(\theta)',q(\xi)')} := (L_{(1,A',p(\theta)')},\mathcal{L}_{(1,A',p(\theta)',q(\xi)')}),\ \mathscr{L}_{(s,B',u(\sigma)',v(\tau)')} := (L_{(s,B',u(\sigma)')},\mathcal{L}_{(s,B',u(\sigma)',v(\tau)')})$$

of the Fukaya category $Fuk(T^{2n},\tilde{\omega_0})$ to the objects

$$\mathscr{L}_{(1,O,p(\theta)',q(\xi)')} := \left(L_{(1,O,p(\theta)')} = \tilde{\phi}^{-1}(L_{(1,A',p(\theta)')}),\mathcal{L}_{(1,O,p(\theta)',q(\xi)')} = \tilde{\phi}^*\mathcal{L}_{(1,A',p(\theta)',q(\xi)')}\right),$$
$$\mathscr{L}_{(s,NE_{ij},u(\sigma)',v(\tau)')} := \left(L_{(s,NE_{ij},u(\sigma)')} = \tilde{\phi}^{-1}(L_{(s,B',u(\sigma)')}),\mathcal{L}_{(s,NE_{ij},u(\sigma)',v(\tau)')} = \tilde{\phi}^*\mathcal{L}_{(s,B',u(\sigma)',v(\tau)')}\right),$$

respectively (note $\alpha' = A' - \frac{1}{s}B' = -\frac{N}{s}E_{ij}$, i.e., $\frac{1}{s}B' = A' + \frac{N}{s}E_{ij}$). Hence, in $Tr(Fuk(T^{2n},\tilde{\omega_0}))$, the exact triangle associated to the mapping cone of a non-trivial morphism $\mathscr{L}_{(s,B',u(\sigma)',v(\tau)')} \to T\mathscr{L}_{(1,A',p(\theta)',q(\xi)')}$ (T denotes the shift functor in $Tr(Fuk(T^{2n},\tilde{\omega_0}))$) is mapped to the exact triangle associated to the mapping cone of a non-trivial morphism $\mathscr{L}_{(s,NE_{ij},u(\sigma)',v(\tau)')} \to T\mathscr{L}_{(1,O,p(\theta)',q(\xi)')}$ by this $SL(2n;\mathbb{Z})$ action. Note that the complex structure of the mirror dual complex torus $\check{T}^{2n}_0$ is also preserved when we consider the $SL(2n;\mathbb{Z})$ action on $(T^{2n},\tilde{\omega_0})$. Thus, by using the homological mirror symmetry $Tr(Fuk(T^{2n},\tilde{\omega_0})) \cong Tr(DG_{\check{T}^{2n}_0})$ (cf. [7], [1] etc.), the exact triangle (21) is mapped to the exact triangle

$$\cdots \longrightarrow E_{(1,O,\mu',\mathcal{U}')} \longrightarrow E_{(t,\frac{s't}{st'}NE_{ij},\eta',\mathcal{W}')} \longrightarrow E_{(s,NE_{ij},\nu',\mathcal{V}')} \longrightarrow TE_{(1,O,\mu',\mathcal{U}')} \longrightarrow \cdots$$

in $Tr(DG_{\check{T}^{2n}_0})$. Here, note $\frac{t'}{t}(\frac{s't}{st'}N) = \frac{s'}{s}N \in \mathbb{Z}$. Furthermore, for two holomorphic vector bundles $E_{(r,A,\mu,\mathcal{U})}$, $E_{(r,A,\mu',\mathcal{U}')} \to \check{T}^{2n}_0$, it is known that

$$E_{(r,A,\mu,\mathcal{U})} \cong E_{(r,A,\mu',\mathcal{U}')}$$

if and only if

$$\mu - \mu' \equiv \frac{r}{r'}(\theta - \theta' + T'^t(\xi' - \xi)) \ \left(\mathrm{mod}\ \frac{r}{r'}2\pi(\mathbb{Z}^n \oplus T'^t\mathbb{Z}^n)\right)$$

holds (see [13]). Here, note that $\xi,\ \theta \in \mathbb{R}^n$ and $\xi',\ \theta' \in \mathbb{R}^n$ are vectors associated to $\mathcal{U}$ and $\mathcal{U}'$, respectively, in the sense of definition (5). By this result, there exist

$$\mu'' = (\mu''_1, \cdots, \mu''_n)^t = (p''_1, \cdots, p''_n)^t + T'^t(q''_1, \cdots, q''_n)^t,$$
$$\nu'' = (\nu''_1, \cdots, \nu''_n)^t,\ \eta'' = (\eta''_1, \cdots, \eta''_n)^t \in \mathbb{R}^n \oplus T'^t\mathbb{R}^n$$

such that

$$E_{(1,O,\mu',\mathcal{U}')} \cong E_{(1,O,\mu'',\mathcal{U}_1)},\ E_{(s,NE_{ij},\nu',\mathcal{V}')} \cong E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})},\ E_{(t,\frac{s't}{st'}NE_{ij},\eta',\mathcal{W}')} \cong E_{(t,\frac{s't}{st'}NE_{ij},\eta'',\mathcal{U}_{t'})},$$

and the definition of $\mathcal{U}_{r'}$ ($r' \in \mathbb{N}$) in these relations is given as follows. When we define a holomorphic vector bundle $E_{(r,aE_{ij},\mu,\mathcal{U})}$ by using a given data $(r, aE_{ij}, \mu) \in \mathbb{N} \times M(n;\mathbb{Z}) \times (\mathbb{R}^n \oplus T'^t\mathbb{R}^n)$ with the condition $\frac{r'}{r}a \in \mathbb{Z}$ (note that we do not assume the condition $\frac{a}{r} = \frac{a'}{r'}$, $gcd(r', a') = 1$ here [4]), we can consider the following set as an example of $\mathcal{U}$.

$$\left\{V_j = V,\ V_l = I_{r'},\ U_i = U^{-\frac{r'}{r}a},\ U_k = I_{r'} \in U(r') \mid l \neq j,\ k \neq i\right\}. \tag{22}$$

Here,

$$V := \begin{pmatrix} 0 & 1 & & \\ & & \ddots & \ddots \\ & & & 1 \\ 1 & & & 0 \end{pmatrix},\ U := \begin{pmatrix} 1 & & & \\ & \zeta' & & \\ & & \ddots & \\ & & & (\zeta')^{r'-1} \end{pmatrix} \in U(r'),\ \zeta' := e^{\frac{2\pi \mathbf{i}}{r'}}.$$

We denote by $\mathcal{U}_{r'}$ the set (22).

As a result, we may consider the exact triangle

$$\cdots \longrightarrow E_{(1,O,\mu'',\mathcal{U}_1)} \longrightarrow E_{(t,\frac{s't}{st'}NE_{ij},\eta'',\mathcal{U}_{t'})} \longrightarrow E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})} \longrightarrow TE_{(1,O,\mu'',\mathcal{U}_1)} \longrightarrow \cdots \tag{23}$$

in $Tr(DG_{\check{T}^{2n}_0})$, instead of the exact triangle (21) in $Tr(DG_{\check{T}^{2n}_0})$.

Here, we give two remarks on the exact triangle (23).

First, we comment on the simplicity of a holomorphic vector bundle $E_{(r,aE_{ij},\mu,\mathcal{U}_{r'})} \to \check{T}^{2n}_0$. In general, calculations on the simplicity of a given holomorphic vector bundle are not easy. However, in the case of $E_{(r,aE_{ij},\mu,\mathcal{U}_{r'})}$, we can calculate morphisms in $\mathrm{End}(E_{(r,aE_{ij},\mu,\mathcal{U}_{r'})})$ directly. We also note that calculations in the proof of the following proposition are analogies of techniques which are used in the proofs of Proposition 4.1 and Corollary 4.2.

Proposition 5.5. *The holomorphic vector bundle $E_{(r,aE_{ij},\mu,\mathcal{U}_{r'})}$ is simple if and only if $gcd\left(r', \frac{r'}{r}a\right) = 1$ holds.*

[4] In this sense, the notation $E_{(r,aE_{ij},\mu,\mathcal{U}_{r'})}$ does not seem precise (see also the relations (2) and (3)). However, we will explain the detail of this problem in p.32, and then, we will also prove the validity of the notation $E_{(r,aE_{ij},\mu,\mathcal{U}_{r'})}$.

Proof. We can compute locally a morphism $\varphi = (\varphi_{KL}) \in \mathrm{End}(E_{(r,aE_{ij},\mu,\mathcal{U}_{r'})})$ as follows. First, for each $k = 1, \cdots, n$, by considering the transition functions of $E_{(r,aE_{ij},\mu,\mathcal{U}_{r'})}$ in the Y_k direction, we see that the morphism φ must satisfy

$$U_k \cdot \varphi(X_1, \cdots, X_n, Y_1, \cdots, Y_n) = \varphi(X_1, \cdots, X_n, Y_1, \cdots, Y_k + 2\pi, \cdots, Y_n) \cdot U_k.$$

Therefore, each (K, L) component φ_{KL} of φ can be Fourier-expanded as

$$\varphi_{KL}(X, Y) = \sum_{I^1_{KL}, \cdots, I^n_{KL} \in \mathbb{Z}} \varphi_{KL, I^1_{KL}, \cdots, I^n_{KL}}(X) e^{\mathbf{i}(I^1_{KL}, \cdots, I^i_{KL} + \frac{L-K}{r}a, \cdots, I^n_{KL})Y}. \tag{24}$$

Next, we consider the Cauchy-Riemann equation

$$\bar{\partial}(\varphi) = 0. \tag{25}$$

By substituting the local expression (24) into the differential equation (25), we obtain the local expression

$$\varphi_{KL, I^1_{KL}, \cdots, I^n_{KL}}(X) = \lambda_{KL, I^1_{KL}, \cdots, I^n_{KL}} e^{\mathbf{i}(I^1_{KL}, \cdots, I^i_{KL} + \frac{L-K}{r}a, \cdots, I^n_{KL})T'^{-1}X},$$

where

$$\lambda_{KL, I^1_{KL}, \cdots, I^n_{KL}} \in \mathbb{C}$$

is an arbitrary constant. Finally, for each $l = 1, \cdots, n$, by considering the transition functions of $E_{(r,aE_{ij},\mu,\mathcal{U}_{r'})}$ in the X_l direction, we see that the morphism φ must satisfy

$$V_l \cdot \varphi(X_1, \cdots, X_n, Y_1, \cdots, Y_n) = \varphi(X_1, \cdots, X_l + 2\pi, \cdots, X_n, Y_1, \cdots, Y_n) \cdot V_l. \tag{26}$$

We consider the relation (26) in the case $l = j$. Then, the relation (26) turns out to be

$$\varphi_{KL}(X_1, \cdots, X_j + 2\pi, \cdots, X_n, Y_1, \cdots, Y_n) = \varphi_{(K+1)(L+1)}(X_1, \cdots, X_n, Y_1, \cdots, Y_n), \tag{27}$$

and this implies

$$\varphi_{KL}(X_1, \cdots, X_j + 2\pi r', \cdots, X_n, Y_1, \cdots, Y_n) = \varphi_{KL}(X_1, \cdots, X_n, Y_1, \cdots, Y_n). \tag{28}$$

On the other hand, the relation (26) in the case $l \neq j$ turns out to be

$$\varphi_{KL}(X_1, \cdots, X_l + 2\pi, \cdots, X_n, Y_1, \cdots, Y_n) = \varphi_{KL}(X_1, \cdots, X_n, Y_1, \cdots, Y_n). \tag{29}$$

Therefore, by the relations (28), (29), we need to consider the condition

$$\left(I^1_{KL}, \cdots, I^i_{KL} + \frac{L-K}{r}a, \cdots, I^n_{KL} \right) T'^{-1} \in \mathbb{Z} \times \cdots \times \frac{\mathbb{Z}}{r'} \times \cdots \times \mathbb{Z} \subset \mathbb{R}^n. \tag{30}$$

However, since $\det \mathrm{Im} T'^{-1} \neq 0$ and the relation (27) need to hold, we may actually consider the condition

$$\left(I^1_{1L}, \cdots, I^i_{1L} + \frac{L-1}{r}a, \cdots, I^n_{1L} \right) = 0 \tag{31}$$

for each $L = 1, \cdots, r'$, instead of the condition (30). Then, it is clear that $I^k_{1L} = 0$ holds for each $k \neq i$, so we focus on the i-th component of the condition (31), i.e.,

$$I^i_{1L} + \frac{L-1}{r} a = 0,$$

and this condition is equivalent to

$$r' I^i_{1L} + (L-1) \frac{r'}{r} a = 0. \tag{32}$$

Clearly, in the case $L = 1$, the condition (32) turns out to be $I^i_{11} = 0$. Hence, we need to set $\lambda_{11, I^1_{11}, \cdots, I^n_{11}} = 0$ for each $(I^1_{11}, \cdots, I^n_{11}) \neq (0, \cdots, 0) \in \mathbb{Z}^n$, and as a result, one sees

$$\varphi_{11} = \lambda_{11,0,\cdots,0} =: \lambda_{11} \in \mathbb{C}.$$

Furthermore, by using the relation (27), we also have the local expressions

$$\varphi_{22} = \varphi_{33} = \cdots = \varphi_{r'r'} = \lambda_{11}.$$

We consider the case $L \neq 1$, i.e., $L = 2, \cdots, r'$. Then, for each $L = 2, \cdots, r'$, if there exists an $I^i_{1L} \in \mathbb{Z}$ such that the condition (32) holds, we need to set $\lambda_{1L, I^1_{1L}, \cdots, I^n_{1L}} = 0$ for each $(I^1_{1L}, \cdots, I^n_{1L}) \neq (0, \cdots, -\frac{L-1}{r} a, \cdots, 0) \in \mathbb{Z}^n$, and as a result, one sees

$$\varphi_{1L} = \lambda_{1L, 0, \cdots, -\frac{L-1}{r} a, \cdots, 0} =: \lambda_{1L} \in \mathbb{C}.$$

Of course, the local expressions

$$\varphi_{2(L+1)} = \varphi_{3(L+2)} = \cdots = \varphi_{r'(L-1)} = \lambda_{1L}$$

also follow from the relation (27). Similarly, for each $L = 2, \cdots, r'$, if there does not exist an $I^i_{1L} \in \mathbb{Z}$ such that the condition (32) holds, we need to set $\lambda_{1L, I^1_{1L}, \cdots, I^n_{1L}} = 0$ for any $(I^1_{1L}, \cdots, I^n_{1L}) \in \mathbb{Z}^n$, namely, one sees

$$\varphi_{1L} = 0.$$

Also in this case, we obtain the local expressions

$$\varphi_{2(L+1)} = \varphi_{3(L+2)} = \cdots = \varphi_{r'(L-1)} = 0$$

by using the relation (27).

Thus, when we set

$$m := \sharp \Big\{ 2 \leq L \leq r' \mid \text{There exists an } I^i_{1L} \in \mathbb{Z} \text{ such that the condition (32) holds.} \Big\},$$

we can conclude

$$\mathrm{End}(E_{(r, aE_{ij}, \mu, \mathcal{U}_{r'})}) \cong \mathbb{C}^{1+m},$$

where $\sharp X$ denotes the cardinality of a given set X.

We assume that $E_{(r, aE_{ij}, \mu, \mathcal{U}_{r'})}$ is simple, namely, $m = 0$. Then, we see

$$\frac{r'}{r} a \notin r'\mathbb{Z},\ 2\frac{r'}{r} a \notin r'\mathbb{Z}, \cdots, (r'-1)\frac{r'}{r} a \notin r'\mathbb{Z},$$

and this fact implies

$$gcd\left(r', \frac{r'}{r}a\right) = 1.$$

Conversely, we assume $gcd\left(r', \frac{r'}{r}a\right) = 1$. Then, by the condition (32), we see

$$L - 1 \in r'\mathbb{Z}.$$

However, this relation contradicts the assumption $2 \leq L \leq r'$. This fact indicates $m = 0$, so we can conclude

$$\mathrm{End}(E_{(r, aE_{ij}, \mu, \mathcal{U}_{r'})}) \cong \mathbb{C}.$$

□

Here, for $E_{(s, NE_{ij}, \nu'', \mathcal{U}_{s'})}$, we consider the relation between the value $s \in \mathbb{N}$ and the value $N \in \mathbb{N}$ by using Proposition 5.5 and so on. When we denote $E_{(s, NE_{ij}, \nu'', \mathcal{U}_{s'})}$, since the rank of $E_{(s, NE_{ij}, \nu'', \mathcal{U}_{s'})}$ is s', under the assumption

$$\frac{N}{s} = \frac{N'}{s''},\ s'', N' \in \mathbb{N},\ gcd(s'', N') = 1,$$

the relation

$$s'' = s'$$

should hold. Actually, we can prove that it is true as follows. First, we see $\frac{s'}{s}N = \frac{s'}{s''}N' \in \mathbb{N}$, so by the assumption $gcd(s'', N') = 1$, there exists a $k \in \mathbb{N}$ such that

$$s' = ks''.$$

Then, $\frac{s'}{s}N = \frac{ks''}{s''}N' = kN'$. Thus, by Proposition 5.5, we have

$$gcd\left(s', \frac{s'}{s}N\right) = gcd(ks'', kN') = k = 1,$$

and this fact indicates $s'' = s'$. Of course, the case of $E_{(t, \frac{s't}{st'}NE_{ij}, \eta'', \mathcal{U}_{t'})}$ is also the same as the case of $E_{(s, NE_{ij}, \nu'', \mathcal{U}_{s'})}$.

Second, we give a remark on the constants μ'', ν'', $\eta'' \in \mathbb{R}^n \oplus T'^t\mathbb{R}^n$. Hereafter, $\mu''^{\vee i}$ denotes the constant vector obtained by eliminating the i-th component from μ''. We also use the notations $\nu''^{\vee i}$, $\eta''^{\vee i}$ in this sense. Furthermore, we denote by $\tilde{T}'_{ji}$ the matrix obtained by eliminating the j-th row and the i-th column from T'. By a direct calculation, we can check that

$$\mathrm{Ext}^1(E_{(s, NE_{ij}, \nu'', \mathcal{U}_{s'})}, E_{(1, O, \mu'', \mathcal{U}_1)}) = H^1(E_{(s, NE_{ij}, \nu'', \mathcal{U}_{s'})}, E_{(1, O, \mu'', \mathcal{U}_1)}) \neq 0$$

if and only if

$$\mu''^{\vee i} \equiv \frac{1}{s}\nu''^{\vee i} \ (\mathrm{mod}\ 2\pi(\mathbb{Z}^{n-1} \oplus \tilde{T}'^t_{ji}\mathbb{Z}^{n-1})) \tag{33}$$

holds. Therefore, since we consider the case $\mathrm{Ext}^1(E_{(s, NE_{ij}, \nu'', \mathcal{U}_{s'})}, E_{(1, O, \mu'', \mathcal{U}_1)}) \neq 0$ only in this paper, hereafter, we may assume that the relation (33) holds. Similarly, we also consider the relation between $\frac{1}{s}\nu''$ and $\frac{1}{t}\eta''$. In order to consider it, we apply the contravariant cohomological functor

$$F := \mathrm{Hom}_{Tr(DG_{\check{T}_0^{2n}})}(\ \cdot\ , E_{(s, NE_{ij}, \nu'', \mathcal{U}_{s'})})$$

to the exact triangle (23).

$$\begin{aligned} \cdots \longrightarrow \quad & F(TE_{(1,O,\mu'',\mathcal{U}_1)}) \quad \longrightarrow F(E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}) \\ \longrightarrow \ & F(E_{(t,\frac{s't}{st'}NE_{ij},\eta'',\mathcal{U}_{t'})}) \longrightarrow \quad F(E_{(1,O,\mu'',\mathcal{U}_1)}) \\ \longrightarrow \ & \qquad \cdots . \end{aligned} \tag{34}$$

In (34), clearly, $F(TE_{(1,O,\mu'',\mathcal{U}_1)}) = H^{-1}(E_{(1,O,\mu'',\mathcal{U}_1)}, E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}) = 0$ holds. Furthermore, by the simplicity of $E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}$, we see $F(E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}) = H^0(E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}, E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}) = \mathrm{End}(E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}) \cong \mathbb{C}$. These facts imply

$$F(E_{(t,\frac{s't}{st'}NE_{ij},\eta'',\mathcal{U}_{t'})}) = H^0(E_{(t,\frac{s't}{st'}NE_{ij},\eta'',\mathcal{U}_{t'})}, E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}) \neq 0, \tag{35}$$

and by a direct calculation, we can check that the condition (35) if and only if

$$\frac{1}{s}\nu''^{\vee i} \equiv \frac{1}{t}\eta''^{\vee i} \ (\mathrm{mod}\ 2\pi(\mathbb{Z}^{n-1} \oplus \tilde{T}'^{t}_{ji}\mathbb{Z}^{n-1})) \tag{36}$$

holds.

Considering the above discussions, we now present the following theorem which is the main theorem in this section. Although the undefined notations $\tilde{\mu}$ and $\mathcal{U}'_{r'}$ are used in the statement of the following theorem, the definitions of them are given in the proof of it.

Theorem 5.6. *There exists a suitable one-dimensional complex torus $\check{T}^2_0$ and a suitable holomorphic projection $\pi : \check{T}^{2n}_0 \to \check{T}^2_0$ such that the exact triangle* (23) *is equivalent to the following exact triangle in $Tr(DG_{\check{T}^{2n}_0})$.*

$$\begin{aligned} \cdots \longrightarrow \ & \pi^* E_{(1,0,\mu''_i,\mathcal{U}'_1)} \otimes E_{(1,O,\tilde{\mu},\mathcal{U}_1)} \ \longrightarrow \pi^* E_{(t,\frac{s't}{st'}N,\eta''_i,\mathcal{U}'_{t'})} \otimes E_{(1,O,\tilde{\mu},\mathcal{U}_1)} \\ \longrightarrow \ & \pi^* E_{(s,N,\nu''_i,\mathcal{U}'_{s'})} \otimes E_{(1,O,\tilde{\mu},\mathcal{U}_1)} \ \longrightarrow \ \pi^* TE_{(1,0,\mu''_i,\mathcal{U}'_1)} \otimes E_{(1,O,\tilde{\mu},\mathcal{U}_1)} \\ \longrightarrow \ & \qquad \cdots . \end{aligned}$$

Proof. Since the relations

$$t'_{ji'} = 0 \ (1 \leq i' \neq i \leq n), \quad \mathrm{Im} t'_{ji} \neq 0$$

hold by Proposition 5.2, we can define the holomorphic projection $\pi : \check{T}^{2n}_0 \to \check{T}^2_0 := \mathbb{C}/2\pi(\mathbb{Z} \oplus t'_{ji}\mathbb{Z})$ by

$$\pi(Z) = Z_j = X_j + t'_{ji} Y_i.$$

In particular, by the non-triviality of $\mathrm{Ext}^1(E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}, E_{(1,O,\mu'',\mathcal{U}_1)}) = H^1(E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}, E_{(1,O,\mu'',\mathcal{U}_1)})$, we can actually check that $\mathrm{Im} t'_{ji} > 0$ holds. Now, $E_{(t,\frac{s't}{st'}NE_{ij},\eta'',\mathcal{U}_{t'})}$ and $E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})}$ are simple by the assumption, and the simplicity of these holomorphic vector bundles indicates

$$gcd\left(t', \frac{t'}{t}\left(\frac{s't}{st'}N\right)\right) = gcd\left(s', \frac{s'}{s}N\right) = 1 \tag{37}$$

by Proposition 5.5. Considering the above discussions, we focus on the exact triangle

$$\cdots \longrightarrow E_{(1,0,\mu''_i,\mathcal{U}'_1)} \longrightarrow E_{(t,\frac{s't}{st'}N,\eta''_i,\mathcal{U}'_{t'})} \longrightarrow E_{(s,N,\nu''_i,\mathcal{U}'_{s'})} \longrightarrow TE_{(1,0,\mu''_i,\mathcal{U}'_1)} \longrightarrow \cdots \tag{38}$$

in $Tr(DG_{\check{T}_0^2}) \cong D^b(Coh(\check{T}_0^2))$. Here, for a holomorphic vector bundle $E_{(r,a,\mu,\mathcal{U})} \to \check{T}_0^2$, we took

$$\mathcal{U}'_{r'} := \left\{ V_j = V,\ U_i = U^{-\frac{r'}{r}a} \in U(r') \right\}$$

as $\mathcal{U}$. In the exact triangle (38), it is clear that $E_{(1,0,\mu''_i,\mathcal{U}'_1)}$ is simple, and since the relation (37) holds, $E_{(t,\frac{s't}{st'}N,\eta''_i,\mathcal{U}'_{t'})}$ and $E_{(s,N,\nu''_i,\mathcal{U}'_{s'})}$ are also simple (cf. Corollary 4.2). In general, for a given vector bundle $E \to \check{T}_0^2$, we can consider the pullback bundle $\pi^* E \to \check{T}_0^{2n}$ by the holomorphic projection $\pi : \check{T}_0^{2n} \to \check{T}_0^2$. Therefore, the following exact triangle in $Tr(DG_{\check{T}_0^{2n}})$ is induced from the exact triangle (38).

$$\cdots \longrightarrow \pi^* E_{(1,0,\mu''_i,\mathcal{U}'_1)} \longrightarrow \pi^* E_{(t,\frac{s't}{st'}N,\eta''_i,\mathcal{U}'_{t'})} \longrightarrow \pi^* E_{(s,N,\nu''_i,\mathcal{U}'_{s'})} \longrightarrow \pi^* T E_{(1,0,\mu''_i,\mathcal{U}'_1)} \longrightarrow \cdots .$$

In particular, $\pi^* E_{(1,0,\mu''_i,\mathcal{U}'_1)}$, $\pi^* E_{(t,\frac{s't}{st'}N,\eta''_i,\mathcal{U}'_{t'})}$ and $\pi^* E_{(s,N,\nu''_i,\mathcal{U}'_{s'})}$ are also simple because $E_{(1,0,\mu''_i,\mathcal{U}'_1)}$, $E_{(t,\frac{s't}{st'}N,\eta''_i,\mathcal{U}'_{t'})}$ and $E_{(s,N,\nu''_i,\mathcal{U}'_{s'})}$ are simple.

On the other hand, it is known that for two simple projectively flat bundles E, E' over a complex torus $\mathbb{C}^n/\Gamma$ ($\Gamma \subset \mathbb{C}^n$ is a lattice) which satisfy $(\mathrm{rank}E, c_1(E)) = (\mathrm{rankE'}, c_1(E'))$, there exists a line bundle $L \in \mathrm{Pic}^0(\mathbb{C}^n/\Gamma)$ such that

$$E' \cong E \otimes L.$$

This fact is described in Theorem 6.1 of [19] (see also Proposition 6.17 (1) in [20]). Now, since the relations (33) and (36) hold, we can actually check that

$$\begin{aligned}
&E_{(1,O,\mu'',\mathcal{U}_1)} \cong \pi^* E_{(1,0,\mu''_i,\mathcal{U}'_1)} \otimes E_{(1,O,\tilde{\mu},\mathcal{U}_1)},\\
&E_{(s,NE_{ij},\nu'',\mathcal{U}_{s'})} \cong \pi^* E_{(s,N,\nu''_i,\mathcal{U}'_{s'})} \otimes E_{(1,O,\tilde{\mu},\mathcal{U}_1)},\\
&E_{(t,\frac{s't}{st'}NE_{ij},\eta'',\mathcal{U}_{t'})} \cong \pi^* E_{(t,\frac{s't}{st'}N,\eta''_i,\mathcal{U}'_{t'})} \otimes E_{(1,O,\tilde{\mu},\mathcal{U}_1)}
\end{aligned}$$

hold by considering the holomorphic line bundle

$$E_{(1,O,\tilde{\mu},\mathcal{U}_1)} \in \mathrm{Pic}^0(\check{T}_0^{2n}),$$

where

$$\tilde{\mu} := (p''_1, \cdots, p''_{i-1}, 0, p''_{i+1}, \cdots, p''_n)^t + T'^t (q''_1, \cdots, q''_{j-1}, 0, q''_{j+1}, \cdots, q''_n)^t.$$

This completes the proof. □

Acknowledgment

I would like to thank my supervisor Professor Hiroshige Kajiura for various advices in writing this paper. I am also grateful to Professor Masahiro Futaki for helpful discussions. This work is supported by Grant-in-Aid for JSPS Research Fellow 18J10909.

References

[1] M. Abouzaid, I. Smith, Homological mirror symmetry for the four-torus, Duke Mathematical Journal, 152.3 (2010), 373-440.

[2] D. Arinkin, A. Polishchuk, Fukaya category and Fourier transform, AMS IP STUDIES IN ADVANCED MATHEMATICS, 2001, 23 : 261-274.

[3] M. F. Atiyah, Vector bundles over an elliptic curve. Proc. London Math. Soc, 1957.

[4] A. Bondal and M. Kapranov, Enhanced triangulated categories, Math. USSR Sbornik 70:93-107, 1991.

[5] K. Chan, Y. H. Suen, SYZ transforms for immersed Lagrangian Multi-sections, arXiv : mathDG/1712.02586.

[6] K. Fukaya, Morse homotopy, A^{∞}-category, and Floer homologies, In: Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993). Lecture Notes in Series, vol. 18, pp. 1-102. Seoul Nat. Univ., Seoul (1993).

[7] K. Fukaya, Mirror symmetry of abelian varieties and multi theta functions, J. Algebr. Geom. 11, 393-512 (2002).

[8] J. Hano, A geometrical characterization of a class of holomorphic vector bundles over a complex torus, Nagoya Math. J. Vol. 61 (1976), 197-202.

[9] H. Kajiura, Homological perturbation theory and homological mirror symmetry. In Higher Structures in Geometry and Physics, volume 287 of Progress in Math., pages 201-226, Birkhäuser/Springer, New York, 2011.

[10] K. Kobayashi, On exact triangles consisting of stable vector bundles on tori, Differential Geometry and its Applications 53 (2017) 268-292, arXiv : mathDG/1610.02821.

[11] K. Kobayashi, Geometric structure of exact triangles consisting of projectively flat bundles on higher dimensional complex tori, arXiv : mathDG/1705.04007.

[12] K. Kobayashi, Remarks on the homological mirror symmetry for tori, In preparation.

[13] K. Kobayashi, The bijectivity of mirror functors on tori, In preparation.

[14] S. Kobayashi, Differential Geometry of Complex Vector Bundles, Princeton University Press, 1987.

[15] M. Kontsevich, Homological algebra of mirror symmetry, In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, 1995, pages 120-139, arXiv : math.AG/9411018.

[16] M. Kontsevich, Y. Soibelman, Homological mirror symmetry and torus fibrations. In Symplectic geometry and mirror symmetry (Seoul, 2000), pages 203-263. World Sci.Publishing, River Edge, NJ, 2001. math.SG/0011041.

[17] N. C. Leung, S.-T. Yau, E. Zaslow, From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform, Adv. Theor. Math. Phys. 4:13191341, 2000.

[18] M. Lübke, A. Teleman, The Kobayashi-Hitchin Correspondence, World Sci. Publ., 1995.

[19] Y. Matsushima, Heisenberg groups and holomorphic vector bundles over a complex torus, Nagoya Math. J. Vol. 61 (1976), 161-195.

[20] S. Mukai, Semi-homogeneous vector bundles on an abelian variety, J. Math. Kyoto Univ. 18 (1978), no. 2, 239-272.

[21] A. Polishchuk, A_∞-structures on an elliptic curve, Comm. Math. Phys. 247, 527 (2004), arXiv : math.AG/0001048.

[22] A. Polishchuk, Abelian Varieties, Theta Functions and the Fourier Transform, Cambridge University Press, 2003.

[23] A. Polishchuk, E. Zaslow, Categorical mirror symmetry : the elliptic curve, Adv. Theor. Math. Phys. 2, 443-470 (1998), arXiv : math.AG/9801119.

[24] P. Seidel, Fukaya categories and Picard-Lefschetz theory, European Math. Soc., 2008.

[25] A. Strominger, S.-T. Yau, and E. Zaslow, Mirror Symmetry is T-duality, Nucl. Phys. B, 479:243-259, 1996.

[26] Jae-Hyun Yang, Holomorphic vector bundles over complex tori, J. Korean Math. Soc. 26(1989), No.1, pp.117-142.