STUDY OF ULRICH IDEALS

January 2020

RYOTARO ISOBE

Graduate School of
Science and Engineering

CHIBA UNIVERSITY

（千葉大学審査学位論文）

STUDY OF ULRICH IDEALS

January 2020

RYOTARO ISOBE

Graduate School of
Science and Engineering

CHIBA UNIVERSITY

Contents

Acknowledgments **5**

Preface **6**

1 Correspondence between trace ideals and birational extensions with application to the analysis of the Gorenstein property of rings **11**

1.1 Introduction . . . 11

1.2 Correspondence between trace ideals and birational extensions of the base ring . . . 15

1.3 The case where R is a Gorenstein ring of dimension one . . . 20

1.4 Modules in which every submodule is a trace module . . . 25

1.5 Surjectivity of the correspondence ρ in dimension one . . . 30

1.6 Anti-stable rings . . . 34

2 The structure of chains of Ulrich ideals in Cohen-Macaulay local ring of dimension one **39**

2.1 Introduction . . . 39

2.2 Preliminaries . . . 42

2.3 Chains of Ulrich ideals . . . 47

2.4 The case where R possesses minimal multiplicity . . . 51

2.5 The case where R is a GGL ring . . . 56

3 Ulrich ideals and 2-AGL rings **62**

3.1 Introduction . . . 62

3.2 Minimal presentations of canonical ideals in 2-AGL rings . . . 65

3.3 2-AGL rings obtained by fiber products . . . 75

3.4 Two-generated Ulrich ideals in 2-AGL rings . . . 78

3.5 G-regularity in 2-AGL rings . . . 92

3.6 Appendix: Ulrich ideals in one-dimensional Gorenstein local rings of finite Cohen-Macaulay representation type . . . 94

4 The structure of Ulrich ideals in hypersurfaces 99

4.1 Introduction . . . 99

4.2 Basic facts . . . 101

4.3 Ulrich ideals in hypersurfaces . . . 102

4.4 Minimal free resolutions . . . 104

4.5 Decomposable Ulrich ideals . . . 110

4.6 The case $R = k[[X, Y]]/(f)$ . . . 112

References 123

Acknowledgments

First of all, I would like to show my greatest appreciation to Professor Shiro Goto who taught me over the years. I have received many support and advice from him since I was an undergraduate. I owe what I am to his enthusiastic guidance.

I also appreciate my supervisor, Professor Koji Nishida for his suggestions. Besides them, I have benefited from comments and help from a lot of professors, co-workers, and friends, including Naoyuki Matsuoka, Naoki Endo, and Shinya Kumashiro.

Finally I would like to thank my family who have supported my study for a long time.

January, 2020 RYOTARO ISOBE

Preface

This thesis deals with the theory of Ulrich ideals in a Cohen-Macaulay local ring and related topics. The notion of Ulrich ideals is a generalization of stable maximal ideals, which dates back to 1971, when the monumental paper [40] of J. Lipman was published. The modern treatment of Ulrich ideals was started by [24, 25] in 2014, and has been explored in connection with the representation theory of rings. In [24], the basic properties of Ulrich ideals are summarized, whereas in [25], Ulrich ideals in two-dimensional rational singularities are closely studied with a concrete classification.

For a moment, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R$ and I an $\mathfrak{m}$-primary ideal of R. Suppose that our ideal I contains a parameter ideal $Q = (a_1, a_2, \cdots, a_d)$ of R as a reduction, that is, the equality $I^{n+1} = QI^n$ holds true for some integer $n \geq 0$. Then the notion of Ulrich ideals is defined as follows.

Definition A ([24])**.** We say that I is an *Ulrich ideal* of R, if the following conditions are satisfied.

(1) $I \neq Q$,

(2) $I^2 = QI$, and

(3) I/I^2 is a free R/I-module.

We notice that Condition (2) together with Condition (1) are equivalent to saying that the associated graded ring $\mathrm{gr}_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ of I is a Cohen-Macaulay ring and $\mathrm{a}(\mathrm{gr}_I(R)) = 1 - d$, where $\mathrm{a}(\mathrm{gr}_I(R))$ denotes the a-invariant of $\mathrm{gr}_I(R)$ ([27, Remark 3.10], [30, Remark (3.1.6)]). Therefore, these two conditions are independent of the choice of reductions Q of I. In addition, assuming Condition (2) is satisfied, Condition (3) is equivalent to saying that I/Q is a free R/I-module ([24, Lemma 2.3]). Therefore, if I is an Ulrich ideal of R, then I satisfies that $I^2 = QI$ and $I = Q :_R I$, that is I is a *good ideal* in the sense of [15]. We also notice that Condition (3) is automatically satisfied if $I = \mathfrak{m}$. Therefore, when the residue class field $R/\mathfrak{m}$ of R is infinite, the maximal ideal $\mathfrak{m}$ is an Ulrich ideal of R if and only if R is not a regular local ring, possessing minimal multiplicity ([43]). From this perspective, Ulrich ideals are a kind of generalization of stable maximal ideals, which Lipman [40] started to analyze in 1971.

Because Ulrich ideals are a very special kind of ideals, it seems natural to expect that, in the behavior of Ulrich ideals, there might be contained ample information on base rings, once they exist. Therefore, we consider the following two questions.

Problem B. Clarify the existence and ubiquity of Ulrich ideals in a given Cohen-Macaulay local ring R.

Problem C. Find the relation between the behavior of Ulrich ideals and the structure of the base ring.

For a commutative ring R, let $\mathrm{Q}(R)$ be the total ring of fractions of R, and let $X : Y = \{a \in \mathrm{Q}(R) \mid aY \subseteq X\}$ for each R-submodules X and Y of $\mathrm{Q}(R)$. We denote by $\mu_R(*)$ the number of elements in a minimal system of generators.

Regarding these two problems, S. Goto, S.-i. Iai, and K.-i. Watanabe [15] found a beautiful correspondence between the set of Ulrich ideals and the set of birational module-finite extensions, when the base ring R is a Gorenstein local ring of dimension one. We denote by $\mathcal{X}_R$ the set of Ulrich ideals of R.

Theorem D (cf. [15, Theorem 4.2], [13, Proposition 3.1])**.** *Let R be a Gorenstein local ring with $\dim R = 1$. We denote by $\mathcal{A}_R^0$ the set of birational module-finite extensions A of R such that A is a Gorenstein ring with $\mu_R(A) = 2$. Then, there exist bijective correspondences*

$$\mathcal{X}_R \to \mathcal{A}_R^0,\ I \mapsto I : I, \quad \textit{and} \quad \mathcal{A}_R^0 \to \mathcal{X}_R,\ A \mapsto R : A.$$

This theorem says that the ubiquity of Ulrich ideals could be grasped through the behavior of Gorenstein R-subalgebra of $\mathrm{Q}(R)$. In [14], they determined all the Ulrich ideals in one-dimensional Gorenstein local rings R of finite CM-representation type, computing all members of $\mathcal{A}_R^0$. However, this theorem says nothing about the case where R is not a Gorenstein ring of dimension one. It seems natural to ask what happens when R is not necessarily Gorenstein or $\dim R \geq 2$.

In Chapter 1 we will extend the correspondence in Theorem D to the correspondence between the set of trace ideals and the set of birational extensions, for arbitrary commutative rings. The aim of this chapter is to explore the structure of (not necessarily Noetherian) commutative rings in connection with their trace ideals. Let R be a commutative ring. For R-modules M and X, let

$$\tau_{M,X} : \mathrm{Hom}_R(M, X) \otimes_R M \to X$$

denote the R-linear map defined by $\tau_{M,X}(f \otimes m) = f(m)$ for all $f \in \mathrm{Hom}_R(M, X)$ and $m \in M$. We set $\tau_X(M) = \mathrm{Im}\,\tau_{M,X}$. Then, $\tau_X(M)$ is an R-submodule of X, and we say that an R-submodule Y of X is a *trace module* in X, if $Y = \tau_X(M)$ for some R-module M. When $X = R$, we call trace modules in R, simply, *trace ideals* in R. There is a recent movement in the theory of trace ideals, raised by H. Lindo and N. Pande [37, 38, 39]. Besides, J. Herzog, T. Hibi, and D. I. Stamate [31] studied the traces of canonical modules, and gave interesting results. The main activity in the present chapter is focused on the study of the structure of the set of regular trace ideals in R. Let I be an ideal of a commutative ring R and suppose that I is *regular*, that is I contains a non-zerodivisor of R. Then, as is essentially shown by [38, Lemma 2.3], I is a trace ideal in R if and only if $R : I = I : I$. We denote by $\mathcal{X}_R^T$ the set of regular trace ideals in R, and explore the structure of $\mathcal{X}_R^T$ in connection with the structure of $\mathcal{Y}_R$, where $\mathcal{Y}_R$ denotes the set of birational extensions A of R such that $aA \subseteq R$ for some non-zerodivisor a of R. We also consider the set $\mathcal{Z}_R$ of regular ideals I of R such that $I^2 = aI$ for some $a \in I$. We then have the following natural correspondences

$$\xi : \mathcal{Z}_R \to \mathcal{Y}_R, \quad \xi(I) = I : I,$$

$$\eta : \mathcal{Y}_R \to \mathcal{X}_R^T, \quad \eta(A) = R : A,$$

$$\rho : \mathcal{X}_R^T \to \mathcal{Y}_R, \quad \rho(I) = I : I$$

among these sets. The basic framework is the following.

Proposition E (cf. Proposition 1.13, Lemma 1.10 (1)). *The correspondence $\xi : \mathcal{Z}_R \to \mathcal{Y}_R$ is surjective, and the following conditions are equivalent.*

(1) $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ *is surjective.*

(2) $\eta : \mathcal{Y}_R \to \mathcal{X}_R^T$ *is injective.*

(3) $A = R : (R : A)$ *for every* $A \in \mathcal{Y}_R$.

Our strategy is to make use of these correspondences in order to analyze the structure of commutative rings R which are not necessarily Noetherian (see, e.g., [11]).

The purpose of Chapter 2 is to investigate the behavior of chains of Ulrich ideals in a one-dimensional Cohen-Macaulay local ring. For a moment, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$. Our main targets of this chapter are chains $I_n \subsetneq$

$I_{n-1} \subsetneq \cdots \subsetneq I_1$ ($n \geq 2$) of Ulrich ideals in R. Let I be an Ulrich ideal of R with a reduction $Q = (a)$. We set $A = I : I$ in the total ring of fractions of R. Hence, A is a birational finite extension of R, and $I = aA$. Firstly, we study the close connection between the structure of the ideal I and the R-algebra A. Secondly, let J be an Ulrich ideal of R and assume that $I \subsetneq J$. Then, we will show that $\mu_R(J) = \mu_R(I)$ and that $J = (b) + I$ for some $a, b \in \mathfrak{m}$ with $I = abA$. Consequently, we have the following, which is one of the main results of this chapter.

Theorem F (= Theorem 2.3)**.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with* $\dim R = 1$*. Then the following assertions hold true.*

(1) *Let I be an Ulrich ideal of R and $A = I : I$. Let $a_1, a_2, \ldots, a_n \in \mathfrak{m}$ ($n \geq 2$) and assume that $I = a_1a_2\cdots a_nA$. For $1 \leq i \leq n$, let $I_i = (a_1a_2\cdots a_i) + I$. Then each I_i is an Ulrich ideal of R and*

$$I = I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1.$$

(2) *Conversely, let $I_1, I_2, \ldots, I_n$ ($n \geq 2$) be Ulrich ideals of R and suppose that*

$$I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1.$$

We set $I = I_n$ and $A = I : I$. Then there exist elements $a_1, a_2, \ldots, a_n \in \mathfrak{m}$ such that $I = a_1a_2\cdots a_nA$ and $I_i = (a_1a_2\cdots a_i) + I$ for all $1 \leq i \leq n-1$.

By using this theorem, we shall study the case where the base rings R are not regular but possess minimal multiplicity ([43]), and R is a GGL ring ([16]). We give a concrete method of computing Ulrich ideals of non-Gorenstein GGL ring, and explore many examples.

In Chapter 3 we discuss three different topics on 2-AGL rings. The notion of 2-almost Gorenstein local ring (2-AGL ring for short) is a generalization of the notion of almost Gorenstein local ring from the point of view of Sally modules of canonical ideals. The first topic is to clarify the structure of minimal presentations of canonical ideals, and the second one is the study of the question of when certain fiber products, so called amalgamated duplications are 2-AGL rings. We also explore Ulrich ideals in 2-AGL rings, mainly two-generated ones. The existence of two-generated Ulrich ideals is basically a substantially strong condition for R, especially in the case where R is a 2-AGL ring.

In Chapter 4 we investigate the structure and ubiquity of Ulrich ideals in a hypersurface ring of arbitrary dimension. Even for the case of hypersurface rings, there seems known only scattered results which give a complete list of Ulrich ideals, except the case of finite CM-representation type and the case of several numerical semigroup rings. Therefore, in this chapter, we focus our attention on a hypersurface ring which is not necessarily finite CM-representation type. In what follows, let $(S, \mathfrak{n})$ be a regular local ring with $\dim S = d+1$ $(d \geq 1)$. We take $0 \neq f \in \mathfrak{n}$ and set $R = S/(f)$. We shall give the following characterization of Ulrich ideals in R, which is one of the main results of this chapter. For each $a \in S$, let $\overline{a}$ denote the image of a in R.

Theorem G (= Theorem 4.5)**.** *Suppose that $(S, \mathfrak{n})$ is a regular local ring with $\dim S = d+1$ $(d \geq 1)$ and $0 \neq f \in \mathfrak{n}$. Set $R = S/(f)$. Then we have*

$$\mathcal{X}_R = \left\{ (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}, \overline{b}) \;\middle|\; \begin{array}{l} a_1, a_2, \ldots, a_d, b \in \mathfrak{n} \text{ be a system of parameters of } S, \\ \text{and there exist } x_1, x_2, \ldots, x_d \in (a_1, a_2, \cdots, a_d, b) \text{ and } \varepsilon \in U(S) \\ \text{such that } b^2 + \displaystyle\sum_{i=1}^{d} a_i x_i = \varepsilon f. \end{array} \right\},$$

where $U(S)$ denotes the set of unit elements of S.

By using this theorem, we compute many examples in the case where $S = k[[X, Y]]$ is the formal power series ring over a field k. On the other hand, the structure of minimal free resolutions of Ulrich ideals was closely explored in [24, 29]. We construct a minimal free resolution of R/I more concretely, for an Ulrich ideal I of a hypersurface ring R. We also give a matrix factorization of the d-th syzygy module of R/I, which is an Ulrich module with respect to I.

Chapter 1 and 2 of this thesis are respectively reproduction of the contents of [13] and [12], which are joint papers with S. Goto and S. Kumashiro already published on journals. Chapter 3 is a reproduction of the contents of [14], which is a joint paper with S. Goto and N. Taniguchi submitted for publication. The results of Chapter 4 are based on the work [34]. Co-authors have already given reproduction permission.

1 Correspondence between trace ideals and birational extensions with application to the analysis of the Gorenstein property of rings

1.1 Introduction

This chapter aims to explore the structure of (not necessarily Noetherian) commutative rings in connection with their trace ideals. Let R be a commutative ring. For R-modules M and X, let

$$\tau_{M,X} : \mathrm{Hom}_R(M, X) \otimes_R M \to X$$

denote the R-linear map defined by $\tau_{M,X}(f \otimes m) = f(m)$ for all $f \in \mathrm{Hom}_R(M, X)$ and $m \in M$. We set $\tau_X(M) = \mathrm{Im}\, \tau_{M,X}$. Then, $\tau_X(M)$ is an R-submodule of X, and we say that an R-submodule Y of X is a *trace module* in X, if $Y = \tau_X(M)$ for some R-module M. When $X = R$, we call trace modules in R, simply, *trace ideals* in R. There is a recent movement in the theory of trace ideals, raised by H. Lindo and N. Pande [37, 38, 39]. Besides, J. Herzog, T. Hibi, and D. I. Stamate [31] studied the traces of canonical modules, and gave interesting results. We explain below our motivation for the present researches and how this chapter is organized, claiming the main results in it.

The present researches are strongly inspired by [37, 38, 39]. In [38] Lindo asked when every ideal of a given ring R is a trace ideal in it, and noted that this is the case when R is a self-injective ring. Subsequently, Lindo and Pande [39] proved that the converse is also true if R is a Noetherian local ring. Our researches have started from the following complete answer to their prediction, which we shall prove in Section 1.4.

Theorem 1.1 (= Theorem 1.20)**.** *Suppose that R is a Noetherian ring and let X be an R-module. Then the following conditions are equivalent.*

(1) *Every R-submodule of X is a trace module in X.*

(2) *Every cyclic R-submodule of X is a trace module in X.*

(3) *There is an embedding*

$$0 \to X \to \bigoplus_{\mathfrak{m} \in \mathrm{Max}\, R} \mathrm{E}_R(R/\mathfrak{m})$$

of R-modules, where for each $\mathfrak{m} \in \operatorname{Max} R$, $\mathrm{E}_R(R/\mathfrak{m})$ *denotes the injective envelope of the cyclic R-module* $R/\mathfrak{m}$.

However, the main activity in the present chapter is focused on the study of the structure of the set of regular trace ideals in R. Let I be an ideal of a commutative ring R and suppose that I is *regular*, that is I contains a non-zerodivisor of R. Then, as is essentially shown by [38, Lemma 2.3], I is a trace ideal in R if and only if $R : I = I : I$, where the colon is considered inside the total ring $\mathrm{Q}(R)$ of fractions of R. We denote by $\mathcal{X}_R^T$ the set of regular trace ideals in R, and explore the structure of $\mathcal{X}_R^T$ in connection with the structure of $\mathcal{Y}_R$, where $\mathcal{Y}_R$ denotes the set of birational extensions A of R such that $aA \subseteq R$ for some non-zerodivisor a of R. We also consider the set $\mathcal{Z}_R$ of regular ideals I of R such that $I^2 = aI$ for some $a \in I$. We then have the following natural correspondences

$$\xi : \mathcal{Z}_R \to \mathcal{Y}_R, \quad \xi(I) = I : I,$$

$$\eta : \mathcal{Y}_R \to \mathcal{X}_R^T, \quad \eta(A) = R : A,$$

$$\rho : \mathcal{X}_R^T \to \mathcal{Y}_R, \quad \rho(I) = I : I$$

among these sets. The basic framework is the following.

Proposition 1.2 (cf. Proposition 1.13, Lemma 1.10 (1))**.** *The correspondence* $\xi : \mathcal{Z}_R \to \mathcal{Y}_R$ *is surjective, and the following conditions are equivalent.*

(1) $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ *is surjective.*

(2) $\eta : \mathcal{Y}_R \to \mathcal{X}_R^T$ *is injective.*

(3) $A = R : (R : A)$ *for every* $A \in \mathcal{Y}_R$.

Our strategy is to make use of these correspondences in order to analyze the structure of commutative rings R which are not necessarily Noetherian (see, e.g., [11]). This approach is partially inspired by and originated in [15], where certain specific ideals (called *good ideals*) in Gorenstein local rings are closely studied. Similarly, as in [15] and as is shown later in Sections 1.2 and 1.3, the above correspondences behave very well, especially in the case where R is a Gorenstein ring of dimension one. We actually have $\eta \circ \rho = 1_{\mathcal{X}_R^T}$ and $\rho \circ \eta = 1_{\mathcal{Y}_R}$ in that case (Lemma 1.10). Nevertheless, being different from [15], our present interest is in the question of when the correspondence $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is bijective.

As is shown in Section 1.2 (Example 1.14), in general there is no hope for the surjectivity of ρ in the case where $\dim R \geq 2$, even if R is a Noetherian integral domain of dimension two. On the other hand, with very specific, so to speak extremal exceptions (Proposition 1.28), the surjectivity of ρ guarantees the Gorenstein property of R, provided R is a Cohen-Macaulay local ring of dimension one. In fact, we will prove in Section 1.5 the following, in which let us refer to [18] for the notion of almost Gorenstein local ring.

Theorem 1.3 (= Theorem 1.29)**.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one. Let $B = \mathfrak{m} : \mathfrak{m}$ and let $\mathrm{J}(B)$ denote the Jacobson radical of B. Then the following assertions are equivalent.*

(1) $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ *is bijective.*

(2) $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ *is surjective.*

(3) *Either R is a Gorenstein ring, or R satisfies the following two conditions.*

   (i) *B is a* DVR *and* $\mathrm{J}(B) = \mathfrak{m}$.

   (ii) *There is no proper intermediate field between $R/\mathfrak{m}$ and $B/\mathrm{J}(B)$.*

When this is the case, R is an almost Gorenstein local ring in the sense of [18]*.*

Therefore, ρ is surjective if and only if R is a Gorenstein ring, provided R is the semigroup ring of a numerical semigroup over a field.

In Section 1.6, we introduce the notion of anti-stable and strongly anti-stable rings. We say that a commutative ring R is *anti-stable* (resp. *strongly anti-stable*), if $\mathrm{Hom}_R(I, R)$ is an invertible module over the ring $\mathrm{End}_R I$ (resp. $\mathrm{Hom}_R(I, R) \cong \mathrm{End}_R I$ as an $\mathrm{End}_R I$-module), for every *regular* ideal I of R. The purpose of Section 1.6 is to provide some basic properties of anti-stable rings and strongly anti-stable rings, mainly in dimension one.

Here, let us remind the reader that R is said to be a *stable* ring, if every ideal I of R is *stable*, that is I is projective over $\mathrm{End}_R I$ ([45]). The notion of stable ideals and rings is originated in the famous articles [1] and [40] of H. Bass and J. Lipman, respectively, and there are known many deep results about them ([45]). Our definition of anti-stable rings is, of course, different from that of stable rings. It requires the projectivity of the dual module $\mathrm{Hom}_R(I, R)$ of I, only for regular ideals I of R, claiming nothing about the projectivity of I itself. Nevertheless, with some additional conditions in dimension one, R is also a stable ring, once it is anti-stable, as we shall show in the following.

Theorem 1.4 (= Theorem 1.43)**.** *Let R be a Cohen-Macaulay ring with* $\dim R_M = 1$ *for every* $M \in \operatorname{Max} R$. *If R is an anti-stable ring, then R is a stable ring.*

The results of Section 1.6 are obtained as applications of the observations developed in Sections 1.2, 1.3, and 1.5. One can also find, in the forthcoming paper [11], further developments of the theory of anti-stable rings of higher dimension.

Similarly as [36], our research is motivated by the works [37, 38, 39] of Lindo and Pande, so that the topics of Section 1.6 are similar to those of [36], but these two researches were done with entire independence of each other. In [39], Lindo and Pande posed a problem what kind of properties a Noetherian ring R enjoys, if *every* ideal of R is isomorphic to a trace ideal in it. In [36], T. Kobayashi and R. Takahashi have given complete answers to the problem. We were also interested in the problem, and thereafter, came to the notion of anti-stable ring. If the ideal I considered is *regular*, the condition (C) that I is isomorphic to a trace ideal is equivalent to saying that $\operatorname{Hom}_R(I, R) \cong \operatorname{End}_R I$ as an $\operatorname{End}_R I$-module (Lemma 1.35). Therefore, if we restrict our attention, say on integral domains R, the condition that every regular ideal satifies condition (C) is equivalent to saying that R is a strongly anti-stable ring. However, in general, these two conditions are apparently different (e.g., consider the case where every non-zerodivisor of the ring is invertible in it, and see [36, Theorem 3.2]). It must be necessary, and might have some significance, to start a basic theory of anti-stable and strongly anti-stable rings in our context, with a different viewpoint from [36], which we have performed in Section 1.6.

In what follows, unless otherwise specified, R denotes a commutative ring. Let $\mathrm{Q}(R)$ be the total ring of fractions of R. For R-submodules X and Y of $\mathrm{Q}(R)$, let

$$X : Y = \{a \in \mathrm{Q}(R) \mid aY \subseteq X\}.$$

If we consider ideals I, J of R, we set $I :_R J = \{a \in R \mid aJ \subseteq I\}$; hence

$$I :_R J = (I : J) \cap R.$$

When $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d, for each finitely generated R-module M, let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators (resp. the length) of M. We denote by

$$\mathrm{e}(M) = \lim_{n\to\infty} d! {\cdot} \frac{\ell_R(M/\mathfrak{m}^{n+1}M)}{n^d}$$

the multiplicity of M. Let $\mathrm{r}(R) = \ell_R(\mathrm{Ext}_R^d(R/\mathfrak{m}, R))$ stand for the Cohen-Macaulay type of R, where we assume the local ring R is Cohen-Macaulay.

1.2 Correspondence between trace ideals and birational extensions of the base ring

Let R be a commutative ring and let M, X be R-modules. We denote by $\tau_{M,X}$: $\mathrm{Hom}_R(M, X) \otimes_R M \to X$ the R-linear map such that $\tau_{M,X}(f \otimes m) = f(m)$ for all $f \in \mathrm{Hom}_R(M, X)$ and $m \in M$. Let $\tau_X(M) = \mathrm{Im}\, \tau_{M,X}$. Then, $\tau_X(M)$ is an R-submodule of X, and we say that an R-submodule Y of X is a trace module in X, if $Y = \tau_X(M)$ for some R-module M. When $X = R$, we simply say that Y is a trace ideal in R. With this notation we have the following.

Proposition 1.5 ([38, Lemma 2.3]). *For an R-submodule Y of X, the following conditions are equivalent.*

(1) *Y is a trace module in X.*

(2) *$Y = \tau_X(Y)$.*

(3) *The embedding $\iota : Y \to X$ induces the isomorphism $\iota_* : \mathrm{Hom}_R(Y, Y) \to \mathrm{Hom}_R(Y, X)$ of R-modules.*

(4) *$f(Y) \subseteq Y$ for all $f \in \mathrm{Hom}_R(Y, X)$.*

We denote by W the set of non-zerodivisors of R. Let $\mathcal{F}_R$ be the set of *regular* ideals of R, that is the ideals I of R with $I \cap W \neq \emptyset$. We then have the following, characterizing trace ideals.

Corollary 1.6. *Let $I \in \mathcal{F}_R$. Then the following conditions are equivalent.*

(1) *I is a trace ideal in R.*

(2) *$I = (R : I)I$.*

(3) *$I : I = R : I$.*

Proof. Since $I \cap W \neq \emptyset$, we have natural identifications $R : I = \mathrm{Hom}_R(I, R)$ and $I : I = \mathrm{Hom}_R(I, I)$, so that the equivalence of conditions (1) and (3) follows from Proposition 1.5. Suppose that $I = (R : I)I$. Then $R : I \subseteq I : I$, whence $R : I = I : I$. Conversely, if $I : I = R : I$, then $(R : I)I = (I : I)I \subseteq I$, while $I \subseteq (R : I)I$, since $1 \in R : I$. Thus $(R : I)I = I$. $\square$

We now consider the following sets:

$$\mathcal{X}_R^T = \{I \in \mathcal{F}_R \mid I \text{ is a trace ideal in } R\},$$
$$\mathcal{Y}_R = \{A \mid R \subseteq A \subseteq \mathrm{Q}(R),\ A \text{ is a subring of } \mathrm{Q}(R) \text{ such that } aA \subseteq R \text{ for some } a \in W\},$$
$$\mathcal{Z}_R = \left\{I \in \mathcal{F}_R \mid I^2 = aI \text{ for some } a \in I\right\}.$$

If R is a Noetherian ring, then $\mathcal{Y}_R$ is the set of birational finite extensions of R. In what follows, we shall clarify the relationship among these sets. We begin with the following.

Proposition 1.7. *The following assertions hold true.*

(1) *Let X be an R-submodule of $\mathrm{Q}(R)$ and set $Y = R : X$. Then $Y = R : (R : Y)$.*

(2) *Let $I \in \mathcal{Z}_R$ and assume that $I^2 = aI$ with $a \in I$. Then $a \in W$ and $I : I = a^{-1}I$.*

Proof. (1) Since $X \subseteq R : Y$, $Y = R : X \supseteq R : (R : Y)$, so that $Y = R : (R : Y)$.

(2) We have $a \in W$, because $I \in \mathcal{F}_R$. Since $a \in I$, $I : I \subseteq a^{-1}I$, while $a^{-1}I \subseteq I : I$, because $a^{-1}I{\cdot}I = a^{-1}I^2 = a^{-1}(aI) = I$. Hence $I : I = a^{-1}I$. $\square$

Lemma 1.8. *The following assertions hold true.*

(1) *Let $I \in \mathcal{X}_R^T$ and $a \in I \cap W$. We set $J = (a) :_R I$. Then, $J \subseteq I$ and $J^2 = aJ$, so that $J \in \mathcal{Z}_R$.*

(2) *Let $I \in \mathcal{Z}_R$ and write $I^2 = aI$ with $a \in I$. We set $J = (a) :_R I$. Then, $I \subseteq J$ and $J \in \mathcal{X}_R^T$.*

Proof. (1) We set $A = I : I$. Then, $A = R : I$ by Corollary 1.6. Hence, $J = (a) :_R I = (a) : I = a(R : I) = aA$, where the second equality follows from the fact that $a \in I \cap W$. Therefore, $J^2 = aJ$ and $J = a(I : I) \subseteq I$.

(2) Notice that $J = (a) : I = a(R : I)$. Let $A = I : I$. Then, $I = aA$, since $A = a^{-1}I$ by Proposition 1.7 (2), so that $R : I = R : aA = a^{-1}(R : A)$. Therefore, $J = R : A$, whence

$$J : J = (R : A) : (R : A) = R : A(R : A) = R : (R : A) = R : J.$$

Thus, $J \in \mathcal{X}_R^T$. $\square$

Let $I \in \mathcal{F}_R$. We say that I is a *good ideal* of R, if $I^2 = aI$ and $I = (a) :_R I$ for some $a \in I$ (cf. [15]). Let $\mathcal{G}_R$ denote the set of good ideals in R. We then have the following, characterizing good ideals.

Proposition 1.9. $\mathcal{X}_R^T \cap \mathcal{Z}_R = \mathcal{G}_R = \{I \in \mathcal{X}_R^T \mid (a) :_R I \in \mathcal{X}_R^T \text{ for some } a \in I \cap W\}$.

Proof. Let $I \in \mathcal{X}_R^T \cap \mathcal{Z}_R$ and set $A = I : I$. We write $I^2 = aI$ with $a \in I$. Then, since $I = aA$ and $A = R : I$ (see Proposition 1.7 and Corollary 1.6), $(a) :_R I = (a) : I = a(R : I) = aA = I$, so that I is a good ideal of R. Conversely, suppose that I is a good ideal of R and assume that $I^2 = aI$ and $I = (a) :_R I$ with $a \in I$. Then $I \in \mathcal{Z}_R$, while $(a) :_R I \in \mathcal{X}_R^T$ by Lemma 1.8 (2). Hence $I \in \mathcal{X}_R^T \cap \mathcal{Z}_R$.

Assume that $I \in \mathcal{X}_R^T$ and that $(a) :_R I \in \mathcal{X}_R^T$ for some $a \in I \cap W$. We set $J = (a) :_R I$. Then, $J^2 = aJ$ and $J \subseteq I$, by Lemma 1.8 (1). For the same reason, we get $(a) :_R J \subseteq J$, because $J \in \mathcal{X}_R^T$ and $a \in J$. Therefore, $I \subseteq (a) :_R J \subseteq J \subseteq I$; hence $I = J$. Thus, $I^2 = aI$ and $I = (a) :_R I$, that is $I \in \mathcal{G}_R$. $\square$

Let us consider three correspondences

$$\xi : \mathcal{Z}_R \to \mathcal{Y}_R, \quad \xi(I) = I : I,$$

$$\eta : \mathcal{Y}_R \to \mathcal{X}_R^T, \quad \eta(A) = R : A,$$

$$\rho : \mathcal{X}_R^T \to \mathcal{Y}_R, \quad \rho(I) = I : I.$$

Here, we briefly confirm the well-definedness of η. Let $A \in \mathcal{Y}_R$ and set $I = R : A$. Since I is an ideal of A, we get $I : I = (R : A) : I = R : AI = R : I$. Therefore, $I \in \mathcal{X}_R^T$, which shows η is well-defined.

With this notation, we have the following, which plays a key role in this chapter.

Lemma 1.10. *The following assertions hold true.*

(1) *The correspondence $\xi : \mathcal{Z}_R \to \mathcal{Y}_R$ is surjective. For each $I, J \in \mathcal{Z}_R$, $\xi(I) = \xi(J)$ if and only if $I \cong J$ as an R-module, so that $\mathcal{Y}_R = \mathcal{Z}_R/\cong$, that is the set of the isomorphism classes in $\mathcal{Z}_R$.*

(2) $\eta(\rho(I)) = R : (R : I)$ *for every* $I \in \mathcal{X}_R^T$.

(3) $\rho(\eta(A)) = R : (R : A)$ *for every* $A \in \mathcal{Y}_R$.

Consequently, $\rho(\mathcal{X}_R^T) = \{A \in \mathcal{Y}_R \mid A = R : (R : A)\}$, $\eta(\mathcal{Y}_R) = \{I \in \mathcal{X}_R^T \mid I = R : (R : I)\}$, *and we have a bijective correspondence* $\eta(\mathcal{Y}_R) \to \rho(\mathcal{X}_R^T)$, $I \mapsto I : I$.

Proof. (1) Let $A \in \mathcal{Y}_R$ and choose $a \in W$ so that $aA \subseteq R$. We set $I = aA$. We then have $I^2 = aI$ and $I : I = aA : aA = A : A = A$, whence $I \in \mathcal{Z}_R$, and ξ is surjective, because $\xi(I) = A$. Let $I, J \in \mathcal{Z}_R$ and choose $a \in I, b \in J$ so that $I^2 = aI$ and $J^2 = bJ$. Then, $I : I = a^{-1}I$ and $J : J = b^{-1}J$. Hence, if $\xi(I) = \xi(J)$, then $a^{-1}I = b^{-1}J$, so that $I \cong J$ as an R-module. Conversely, suppose that $I, J \in \mathcal{Z}_R$ and $I \cong J$. Then $J = \alpha I$ for some invertible element α of $Q(R)$, whence $\xi(J) = J : J = \alpha I : \alpha I = I : I = \xi(I)$.

(2) (3) Notice that $\eta(\rho(I)) = R : (I : I) = R : (R : I)$ for every $I \in \mathcal{X}_R^T$ and

$$\rho(\eta(A)) = (R : A) : (R : A) = R : A(R : A) = R : (R : A)$$

for every $A \in \mathcal{Y}_R$.

The last assertions follow from the fact that $R : (R : Y) = Y$ for every R-submodule Y of $\mathrm{Q}(R)$, once $Y = R : X$ for some R-submodule X of $\mathrm{Q}(R)$ (see Proposition 1.7 (1)). □

Corollary 1.11. *The correspondence* ρ *induces a bijection*

$$\mathcal{G}_R \to \{A \in \mathcal{Y}_R \mid aA = R : A \textit{ for some } a \in W\}, \quad I \mapsto I : I.$$

Proof. Let $I \in \mathcal{G}_R$. We then have, by Proposition 1.9, $I^2 = aI$ and $I = (a) :_R I$ for some $a \in I$. Since $I = (a) : I = R : a^{-1}I$, $I = R : (R : I)$ by Proposition 1.7 (1). Therefore, setting $A = I : I$ $(= a^{-1}I)$, because $A = R : I$ by Corollary 1.6, we get $R : A = R : (R : I) = I = aA$. Hence, $\rho(I) = A$ belongs to the set of the right hand side. By Lemma 1.10, the induced correspondence is automatically injective, since $I = R : (R : I)$ for every $I \in \mathcal{G}_R = \mathcal{X}_R^T \cap \mathcal{Z}_R$.

To see the induced correspondence is surjective, let $A \in \mathcal{Y}_R$ and assume that $aA = R : A$ for some $a \in W$. Let $I = aA$; hence $I = \eta(A) \in \mathcal{X}_R^T$. We then have $I^2 = aI$ and $I : I = aA : aA = A$, so that $I \in \mathcal{X}_R^T \cap \mathcal{Z}_R$ and $\rho(I) = A$. □

If R is a Gorenstein ring of dimension one, $L = R : (R : L)$ for every finitely generated R-submodule L of $\mathrm{Q}(R)$ such that $\mathrm{Q}(R){\cdot}L = \mathrm{Q}(R)$. Therefore, by Lemma 1.10 we readily get the following.

Corollary 1.12. *Suppose that* R *is a Gorenstein ring of dimension one. Then,* $\eta \circ \rho = 1_{\mathcal{X}_R^T}$ *and* $\rho \circ \eta = 1_{\mathcal{Y}_R}$*, so that the correspondence* $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ *is bijective.*

We note the following.

Proposition 1.13. *The following conditions are equivalent.*

(1) $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ *is surjective.*

(2) $\eta : \mathcal{Y}_R \to \mathcal{X}_R^T$ *is injective.*

(3) $A = R : (R : A)$ *for every* $A \in \mathcal{Y}_R$.

Proof. (1) $\Leftrightarrow$ (3) See Lemma 1.10.

(2) $\Rightarrow$ (3) Let $A \in \mathcal{Y}_R$ and set $L = R : (R : A)$. Therefore, $L = \rho(\eta(A)) \in \mathcal{Y}_R$, while $\eta(A) = R : A = R : L = \eta(L)$, where the second equality follows from Proposition 1.7 (1). Hence, $A = L$, because η is injective.

(3) $\Rightarrow$ (2) We have $\rho \circ \eta = 1_{\mathcal{Y}_R}$ by Lemma 1.10, so that ρ is surjective. $\square$

We explore one example, which shows that when $\dim R \geq 2$, in general we cannot expect the bijectivity of the correspondence ρ.

Example 1.14. Let $S = k[X, Y]$ be the polynomial ring over a field k. We set $R = k[X^4, X^3Y, XY^3, Y^4]$ and $T = k[X^4, X^3Y, X^2Y^2, XY^3, Y^4]$ in S. Let $\mathfrak{m} = (X^4, X^3Y, XY^3, Y^4)R$. Then $T = \overline{R}$ and $\mathfrak{m} = R : T$. We have $\dim R = 2$ and $\operatorname{depth} R_\mathfrak{m} = 1$, whence $R_\mathfrak{m}$ is not Cohen-Macaulay. With this setting the following assertions hold true.

(1) $\mathcal{X}_R^T = \{I \mid I$ is an ideal of R with $\operatorname{ht}_R I \geq 2$, and $I \not\subseteq \mathfrak{m}$ or $IT = I\}$. Hence, $\mathfrak{m}^\ell \in \mathcal{X}_R^T$ for all $\ell > 0$.

(2) $\mathcal{Y}_R = \{T, R\}$, and the correspondence $\eta : \mathcal{Y}_R \to \mathcal{X}_R^T$ is injective.

(3) The isomorphism classes in $\mathcal{Z}_R$ are $[(X^4, X^6Y^2)]$ and $[R]$, where for each $I \in \mathcal{Z}_R$, $[I]$ denotes the isomorphism class of I in $\mathcal{Z}_R$.

Proof. $T = \sum_{n \geq 0} S_{4n}$ is the Veronesean subring of S with order 4, whence T is a normal ring with $\dim T = 2$. We get $\mathfrak{m} = T_+ \cap R$, where T_+ is the maximal ideal $(X^4, X^3Y, X^2Y^2, XY^3, Y^4)T$ of T. Because $T = R + kX^2Y^2$ and $\mathfrak{m}{\cdot}X^2Y^2 \subseteq \mathfrak{m}$, $T = \overline{R}$, the normalization of R, and $\mathfrak{m}T = \mathfrak{m}$. Hence, $R : T = \mathfrak{m}$, and $\dim R = \dim R_\mathfrak{m} = 2$. However, because $T/R \cong R/\mathfrak{m}$, $\operatorname{depth} R_\mathfrak{m} = 1$, whence $R_\mathfrak{m}$ is not Cohen-Macaulay. We get $\mathcal{Y}_R = \{T, R\}$, since $\ell_R(T/R) = 1$. Therefore, since $\mathfrak{m} = R : T$, the correspondence $\eta : \mathcal{Y}_R \to \mathcal{X}_R^T$ is injective, and by Lemma 1.10 (1) the isomorphism classes in $\mathcal{Z}_R$ are exactly $[(X^4, X^6Y^2)]$ (notice that $(X^4, X^6Y^2) = X^4T$) and $[R]$.

Let us check Assertion (1). Firstly, let I be an ideal of R with $\mathrm{ht}_R I \geq 2$ such that $I \not\subseteq \mathfrak{m}$ or $IT = I$. We will show that $I \in \mathcal{X}_R^T$. We may assume $I \neq R$. Suppose that $I \not\subseteq \mathfrak{m}$ and let $\mathfrak{p} \in \operatorname{Spec} R$ such that $I \subseteq \mathfrak{p}$. Then, $R_\mathfrak{p} = T_\mathfrak{p}$, since $\mathfrak{p} \neq \mathfrak{m}$, so that $R_\mathfrak{p}$ is a Cohen-Macaulay ring with $\dim R_\mathfrak{p} = 2$. Therefore, $\mathrm{grade}_R I = 2$, and hence $I \in \mathcal{X}_R^T$ by Proposition 1.5.

Suppose that $IT = I$ and let $f \in \mathrm{Hom}_R(I, R)$. Let $\iota : R \to T$ denote the embedding. Then, the composite map $g : I \xrightarrow{f} R \xrightarrow{\iota} T$ is T-linear, because it is the restriction of the homothety of some element of $\mathrm{Q}(R) = \mathrm{Q}(T)$, while $\mathrm{grade}_T I = \mathrm{ht}_T I = 2$, since $\mathrm{ht}_R I = 2$. Consequently, because $T : I = T$, we have $g(I) \subseteq I$, so that $f(I) \subseteq I$. Thus, $I \in \mathcal{X}_R^T$ by Proposition 1.5.

Conversely, let $I \in \mathcal{X}_R^T$. Therefore, I is a non-zero ideal of R with $R : I = I : I$. Hence, $R : I = R$ or $R : I = T$, because $\mathcal{Y}_R = \{R, T\}$. If $R : I = R$, then $\mathrm{grade}_R I \geq 2$. Therefore, $\mathrm{ht}_R I \geq 2$, and $I \not\subseteq \mathfrak{m}$, because $\operatorname{depth} R_\mathfrak{m} = 1$. Suppose that $R : I = T$. Then, I is an ideal of T. We have to show $\mathrm{ht}_R I \geq 2$. Assume the contrary and choose $\mathfrak{p} \in \operatorname{Spec} R$ so that $I \subseteq \mathfrak{p}$ and $\mathrm{ht}_R \mathfrak{p} = 1$. We then have $R_\mathfrak{p} = T_\mathfrak{p}$, and

$$R_\mathfrak{p} : IR_\mathfrak{p} = [R : I]_\mathfrak{p} = [I : I]_\mathfrak{p} = T_\mathfrak{p} = R_\mathfrak{p}.$$

This is impossible, because $IR_\mathfrak{p}$ is a proper ideal in the DVR $R_\mathfrak{p} = T_\mathfrak{p}$. Therefore, $\mathrm{ht}_R I \geq 2$, which completes the proof of Assertion (1). □

1.3 The case where R is a Gorenstein ring of dimension one

We now concentrate our attention on the case where R is a Gorenstein ring of dimension one.

Proposition 1.15. *Assume that R is a Gorenstein ring of dimension one. We then have the following.*

(1) *$I : I$ is a Gorenstein ring for every $I \in \mathcal{G}_R$.*

(2) *Let $A \in \mathcal{Y}_R$ and suppose that A is a Gorenstein ring. Then, $A = I : I$ for some $I \in \mathcal{G}_R$, if R is semi-local.*

Consequently, when R is semi-local, the correspondence ρ induces the bijection

$$\mathcal{G}_R \to \{A \in \mathcal{Y}_R \mid A \text{ is a Gorenstein ring}\}.$$

Proof. (1) Let $A = I : I$. Then, by Corollary 1.11, $R : A = aA$ for some $a \in W$, so that A is a Gorenstein ring (see [32, Satz 5.12]; remember that $\mathrm{Hom}_R(A, R) \cong R : A$.)

(2) We have $R : A = aA$ for some $a \in W$, because $R : A$ is a canonical ideal of A and A is semi-local. Hence, by Corollary 1.11, $A = I : I$ for some $I \in \mathcal{G}_R$. $\square$

When $(R, \mathfrak{m})$ is a Gorenstein local ring of dimension one, we furthermore have the following, which characterizes Gorenstein local rings of dimension one, in which every regular trace ideal is a good ideal. The problem of when A is a Gorenstein ring for every $A \in \mathcal{Y}_R$ is originated in the paper of H. Bass [1], where one can find many deep observations related to the problem. The equivalence of Assertions (1) and (3) in the following theorem is essentially due to [1, (7.7) Theorem].

Theorem 1.16. *Let R be a semi-local Gorenstein ring of dimension one. Then the following conditions are equivalent.*

(1) *Every $A \in \mathcal{Y}_R$ is a Gorenstein ring.*

(2) $\mathcal{X}_R^T = \mathcal{G}_R$.

When $(R, \mathfrak{m})$ is a local ring, one can add the following.

(3) $\mathrm{e}(R) \leq 2$.

Proof. (2) $\Rightarrow$ (1) We have by Lemma 1.10 $A = I : I$ for some $I \in \mathcal{X}_R^T$, so that by Proposition 1.15 (1) A is a Gorenstein ring.

(1) $\Rightarrow$ (2) Every good ideal of R belongs to $\mathcal{X}_R^T$ by Proposition 1.9. Conversely, let $I \in \mathcal{X}_R^T$ and set $A = I : I$. Then, by Proposition 1.15 (2), $A = J : J$ for some $J \in \mathcal{G}_R$, since A is a Gorenstein ring. Therefore, $I = J$, because $I, J \in \mathcal{X}_R^T$ and the correspondence ρ is bijective (Corollary 1.12).

Suppose that $(R, \mathfrak{m})$ is a local ring.

(3) $\Rightarrow$ (1) See [28, Lemma 12.2].

(2) $\Rightarrow$ (3) We may assume that $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$; otherwise, R is a DVR, since $x\mathfrak{m} = R$ for some $x \in R : \mathfrak{m}$. Therefore, $\mathfrak{m} \in \mathcal{X}_R^T = \mathcal{G}_R$, whence $\mathfrak{m}^2 = a\mathfrak{m}$ for some $a \in \mathfrak{m}$. Thus, $\mathrm{e}(R) = 2$, because R is a Gorenstein ring. $\square$

We close this section with a few examples. To state Example 1.17, we need the notion of idealization, which we now briefly explain. Let R be a commutative ring and M an R-module. We set $A = R \oplus M$ as an additive group, and define the multiplication in A by $(a,x) \cdot (b,y) = (ab, ay+bx)$ for $(a,x),(b,y) \in A$. Then, A forms a commutative ring, which is denoted by $A = R \ltimes M$, and called the idealization of M over R.

Example 1.17. Let V be a DVR with t a regular parameter. Let $R = V \ltimes V$ denote the idealization of V over itself. Then, R is a Gorenstein local ring with $\dim R = 1$, $\mathrm{e}(R) = 2$, and $\mathcal{X}_R^T = \{t^n V \times V \mid n \geq 0\}$.

Proof. Because $R \cong V[X]/(X^2)$ where X denotes an indeterminate, R is a Gorenstein local ring with $\dim R = 1$, $\mathrm{e}(R) = 2$. Let $K = \mathrm{Q}(V)$. Then, $\mathrm{Q}(R) = K \ltimes K$, and $\overline{R} = V \ltimes K$. Consequently

$$\mathcal{Y}_R = \{V \ltimes L \mid L \text{ is a finitely generated } V\text{-submodule of } K \text{ such that } V \subseteq L\}.$$

Therefore, $\mathcal{X}_R^T = \{t^n V \times V \mid n \geq 0\}$ by Corollary 1.12, because R is a Gorenstein local ring with $\dim R = 1$ and $R : [V \ltimes L] = \mathrm{Ann}_V(L/V) \times V$ for every finitely generated V-submodule L of K such that $V \subseteq L$. ☐

Example 1.18. Let k be a field.

(1) Let $R = k[[t^4, t^5, t^6]]$. Then R is a Gorenstein ring, possessing

$$\mathcal{X}_R^T = \left\{(t^8, t^9, t^{10}, t^{11}), (t^6, t^8, t^9), (t^5, t^6, t^8), (t^4, t^5, t^6), R\right\} \cup \left\{(t^4 - at^5, t^6) \mid a \in k\right\} \text{ and}$$

$$\mathcal{Y}_R = \left\{k[[t]], k[[t^2, t^3]], k[[t^3, t^4, t^5]], k[[t^4, t^5, t^6, t^7]], R\right\} \cup \left\{k[[t^2 + at^3, t^5]] \mid a \in k\right\},$$

and the correspondence $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is bijective.

(2) Let $R = k[[t^3, t^4, t^5]]$. Then R is not a Gorenstein ring, possessing

$$\mathcal{X}_R^T = \left\{(t^3, t^4, t^5), R\right\} \quad \text{and} \quad \mathcal{Y}_R = \left\{k[[t]], k[[t^2, t^3]], R\right\},$$

and the correspondence $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is not surjective.

Proof. (1) We set $V = k[[t]]$ (the formal power series ring) and $S = k[[t^4, t^5, t^6, t^7]]$. We will show the set $\mathcal{Y}_R$ consists of the rings in the list. Let $\mathfrak{m}$ and $\mathfrak{m}_S$ denote the maximal ideals of R and S, respectively. We begin with the following.

Claim 1. *The following assertions hold true.*

(1) *Set* $B_a = k[[t^2 + at^3, t^5]]$ *for each* $a \in k$. *Then* $S \subsetneq B_a \subseteq k[[t^2, t^3]]$, $B_a = S + S{\cdot}(t^2 + at^3)$, *and* $\ell_S(B_a/S) = 1$.

(2) *Let* $a, b \in k$. *Then* $B_a = B_b$ *if and only if* $a = b$.

Proof. (1) We set $T = k[[(t^2 + at^3)^2, t^5, (t^2 + at^3)^3, (t^2 + at^3){\cdot}t^5]]$. Then, $T \subseteq B_a$, and $T \subseteq S$, since $S = k + t^4V$. Because

$$\mathfrak{m}_T S + \mathfrak{m}_S^2 \supseteq (t^4, t^5, t^6, t^7)S = \mathfrak{m}_S = t^4V,$$

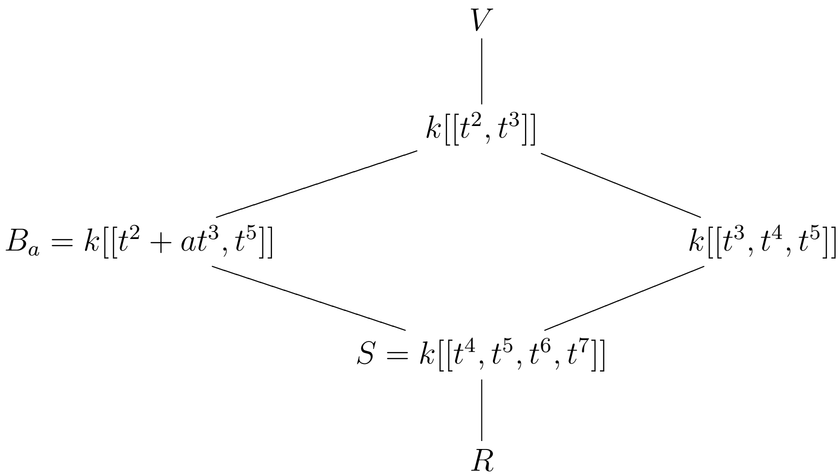
we get $\mathfrak{m}_T S = \mathfrak{m}_S$, whence $T = S$ (remember that $T/\mathfrak{m}_T = S/\mathfrak{m}_S = k$). Consequently, $T = S \subsetneq k[[t^2, t^3]]$, and $B_a = S + S{\cdot}(t^2 + at^3)$, because $t^5 \in \mathfrak{m}_S$. Therefore, $\mu_S(B_a) = 2$, and $\ell_S(B_a/S) = 1$, since $\mathfrak{m}_S B_a = \mathfrak{m}_S \subseteq S$.

(2) Suppose $B_a = B_b$. Then, since the k-space $B_a/\mathfrak{m}_S B_a$ (resp. $B_b/\mathfrak{m}_S B_b$) is spanned by the images of 1 and $t^2 + at^3$ (resp. 1 and $t^2 + bt^3$), we have

$$t^2 + at^3 = \alpha + \beta(t^2 + bt^3) + \gamma$$

for some $\alpha, \beta \in k$ and $\gamma \in t^4V$. Hence, $\alpha = 0$, $\beta = 1$, and $a = b\beta$, so that $a = b$. $\square$

By this claim, we see $R, S, k[[t^3, t^4, t^5]], B_a \ (a \in k), k[[t^2, t^3]], V \in \mathcal{Y}_R$. The relation of embedding among these rings is the following.



We have to show that $\mathcal{Y}_R$ consists of these rings. To see it, let $A \in \mathcal{Y}_R$ and assume that $R \subsetneq A \subsetneq V$. Then, because R is a Gorenstein local ring with $R : \mathfrak{m} = R + kt^7$ and $R \subsetneq A$, we get $S = R + kt^7 \subseteq A$. Let us assume that $S \subsetneq A$ and set $\ell = \ell_S(A/S)$. Then $\ell = 1, 2$, since $\ell_S(V/S) = 3$. We write $\mathfrak{m}_A V = t^n V$ with an integer $n > 0$. We then have $n \leq 4$, since $t^4 \in \mathfrak{m}_A$. Because $A = k + \mathfrak{m}_A \not\subseteq S = k + t^4V$ and $A \neq V$, we furthermore have $n = 2$ or 3.

Suppose that $\ell = 1$. If $n = 3$, then choosing an element $f = t^3 + g$ with $g \in t^4V = \mathfrak{m}_S$, we see $t^3 \in A$, so that $k[[t^3, t^4, t^5]] \subseteq A$. Therefore, $k[[t^3, t^4, t^5]] = A$, because $\ell_S(A/S) = 1$ and $S \subsetneq k[[t^3, t^4, t^5]] \subseteq A$. Let $n = 2$ and choose an element $f = t^2 + at^3 \in A$ with $a \in k$. Then, $B_a \subseteq A$, and $\ell_S(B_a/S) = 1$ by Claim (1), whence $A = B_a$. Suppose now that $\ell = 2$. Then $\ell_A(V/A) = 1$, whence $\mathfrak{m}_A = A : V = t^nV$, so that

$$A = k + t^nV = k[[t^n, t^{n+1}, \dots, t^{2n-1}]]$$

with $n = 2$ or 3. This proves that $\mathcal{Y}_R = \{R, S, k[[t^3, t^4, t^5]], B_a \ (a \in k), k[[t^2, t^3]], V\}$.

Because $\mathcal{X}_R^T = \{R : A \mid A \in \mathcal{Y}_R\}$ by Corollary 1.12 it is direct to show that $\mathcal{X}_R^T$ consists of the following ideals $R : V = (t^8, t^9, t^{10}, t^{11}), R : k[[t^2, t^3]] = (t^6, t^8, t^9), R : k[[t^3, t^4, t^5]] = (t^5, t^6, t^8), R : S = (t^4, t^5, t^6) = \mathfrak{m}, R$, and $R : B_a = (t^4 - at^5, t^6)$ with $a \in k$. Let us note a proof for the fact that $R : B_a = (t^4 - at^5, t^6)$. We set $I = R : B_a$. Firstly, notice that $B_a = R + R{\cdot}(t^2 + at^3)$, since $t^5, (t^2 + at^3)^2 \in \mathfrak{m}$. We then have $t^6 \in I$, since $R : V = t^8V \subseteq I$. Let $\varphi \in I$ and write $\varphi = \alpha t^4 + \beta t^5 + \gamma t^6 + \delta$ with $\alpha, \beta, \gamma \in k$ and $\delta \in R : V$. Then, $\alpha t^4 + \beta t^5 \in I$, and $(\alpha t^4 + \beta t^5)(t^2 + at^3) \in R$ if and only if $(\alpha a + \beta)t^7 \in R$ if and only if $\beta = -\alpha a$, which shows $I = (t^4 - at^5, t^6)$.

(2) The fact $\mathcal{Y}_R = \{k[[t^3, t^4, t^5], k[[t^2, t^3]], V\}$ readily follows from Assertion (1). The assertion on $\mathcal{X}_R^T$ is a special case of the following. $\square$

Proposition 1.19. *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring and let $V = \overline{R}$ denote the integral closure of R in $\mathrm{Q}(R)$. Assume that $R \neq V$ but $\mathfrak{m}V \subseteq R$. Then $\mathcal{X}_R^T = \{\mathfrak{m}, R\}$.*

Proof. Because $R \neq V$, we have $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$, whence $\mathfrak{m} \in \mathcal{X}_R^T$, so that $\{\mathfrak{m}, R\} \subseteq \mathcal{X}_R^T$. Let $I \in \mathcal{X}_R^T$ and set $A = I : I \ (= R : I)$. If $R = A$, then $\mathrm{grade}_R I \geq 2$, and $I = R$. Suppose that $R \subsetneq A$. Then, $I \subseteq \mathfrak{m}$, whence $V \subseteq R : \mathfrak{m} \subseteq A = R : I = I : I \subseteq V$. Therefore, $A = V$. Consequently, I is an ideal of V, whence $I \cong V \cong \mathfrak{m}$ as V-modules (remember that V is a direct product of finitely many principal ideal domains). Therefore, $\tau_R(I) = \tau_R(\mathfrak{m}) = \mathfrak{m}$, because $I \cong \mathfrak{m}$ as an R-module. Hence $\mathcal{X}_R^T = \{\mathfrak{m}, R\}$. $\square$

We will use Proposition 1.19 later in Section 1.5, in order to prove Proposition 1.28.

1.4 Modules in which every submodule is a trace module

In this section, we are interested in the question of, for a given R-module X, when every R-submodule of X is a trace module in it. As is shown in [38], this is the case when $X = R$ and R is a self-injective ring. Our goal is the following, which is known by [39, Theorem 3.5] in the case where R is a Noetherian local ring and $X = R$.

Theorem 1.20. *Suppose that R is a Noetherian ring and let X be an R-module. Then the following conditions are equivalent.*

(1) *Every R-submodule of X is a trace module in X.*

(2) *Every cyclic R-submodule of X is a trace module in X.*

(3) *There is an embedding*

$$0 \to X \to \bigoplus_{\mathfrak{m} \in \operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{m})$$

of R-modules, where for each $\mathfrak{m} \in \operatorname{Max} R$, $\mathrm{E}_R(R/\mathfrak{m})$ denotes the injective envelope of the cyclic R-module $R/\mathfrak{m}$.

To prove Theorem 1.20, we need some preliminaries. The following is a direct consequence of Proposition 1.5.

Proposition 1.21. *The following assertions hold true.*

(1) *Let Y be an R-submodule of X. If every cyclic R-submodule of Y is a trace module in X, then Y is a trace module in X.*

(2) *Let Z and Y be R-submodules of X and assume that $Z \subseteq Y$. If Z is a trace module in X, then Z is a trace module in Y.*

(3) *([38]) If R is a self-injective ring, then every ideal of R is a trace ideal in R.*

We begin with the following.

Lemma 1.22. *Let Y be an R-submodule of X and assume that Y is a finitely presented R-module. Then the following conditions are equivalent.*

(1) *Y is a trace module in X.*

(2) *$Y_{\mathfrak{m}}$ is a trace module in $X_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{Max} R$.*

(3) *$Y_{\mathfrak{p}}$ is a trace module in $X_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. Let $\iota : Y \to X$ denote the embedding and let

$$\iota_* : \operatorname{Hom}_R(Y, Y) \to \operatorname{Hom}_R(Y, X)$$

be the induced homomorphism. We set $C = \operatorname{Coker} \iota_*$. By Proposition 1.5, Y is a trace module in X, if and only if $C = (0)$, that is $C_{\mathfrak{p}} = (0)$ for all $\mathfrak{p} \in \operatorname{Spec} R$. On the other hand, since Y is finitely presented, we have

$$S^{-1} [\operatorname{Hom}_R(Y, Z)] = \operatorname{Hom}_{S^{-1}R}(S^{-1}Y, S^{-1}Z)$$

for every R-module Z and for every multiplicatively closed subset S in R. Hence, the condition that Y is a trace module in X is a local condition. $\square$

For each R-module X, let $\mathrm{E}_R(X)$ stand for the injective envelope of X. We firstly consider the case where R is a local ring.

Theorem 1.23. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and set $E = \mathrm{E}_R(R/\mathfrak{m})$. Let X be an R-module. Then the following conditions are equivalent.*

(1) *Every R-submodule of X is a trace module in X.*

(2) *There is an embedding $0 \to X \to E$ of R-modules.*

Proof. (1) $\Rightarrow$ (2) We may assume that $X \neq (0)$. Let $V = (0) :_X \mathfrak{m}$. We want to show that $\mathrm{E}_R(X) \cong E$, that is, $\ell_R(V) = 1$ and V is an essential R-submodule of X. To do this, it suffices to show that for every non-zero finitely generated R-submodule M of X, $\ell_R(M) < \infty$ and $\ell_R((0) :_M \mathfrak{m}) = 1$. First of all, we show $\operatorname{depth}_R M = 0$. In fact, suppose that $\operatorname{depth}_R M > 0$, and let $a \in \mathfrak{m}$ be a non-zerodivisor on M. We then have by Proposition 1.5 $aM = \tau_X(aM)$ and $M = \tau_X(M)$, since both aM and M are trace modules in X, while $\tau_X(aM) = \tau_X(M)$, because $aM \cong M$. Hence, $aM = M$, which is impossible because $M \neq (0)$. We now fix one socle element $0 \neq x \in (0) :_M \mathfrak{m}$ of M. Let N be an arbitrary non-zero R-submodule of M. Then, since $R/\mathfrak{m}$ is a homomorphic image of $N/\mathfrak{m}N$ and since $R/\mathfrak{m} \cong Rx$, we get a homomorphism $f : N \to M$ such that $f(N) = Rx$, which implies $x \in N$, because N is a trace module in X (see Proposition 1.5). Therefore, if $\dim_R M > 0$, then $x \in \mathfrak{m}^n M$ for all $n > 0$, because $\mathfrak{m}^n M \neq (0)$, so that $x \in \bigcap_{n>0} \mathfrak{m}^n M = (0)$, which is a contradiction. Hence, $\dim_R M = 0$, that is $\ell_R(M) < \infty$.

The above observation also shows that $x \in Ry$ for every $0 \neq y \in (0) :_M \mathfrak{m}$, whence $\ell_R((0) :_M \mathfrak{m}) = 1$, and therefore, X is an R-submodule of E.

(2) ⇒ (1) By Proposition 1.21 (2), we may assume $X = E$. Let Y be an R-submodule of E. It suffices to show that $f(Y) \subseteq Y$ for all $f \in \mathrm{Hom}_R(Y, E)$. We take a homomorphism $g : E \to E$ so that $f = g \circ \iota$, where $\iota : Y \to E$ denotes the embedding. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R, and remember that E is an $\widehat{R}$-module such that

$$\mathrm{Hom}_R(E, E) = \mathrm{Hom}_{\widehat{R}}(E, E) = \widehat{R}.$$

Choose $\alpha \in \widehat{R}$ so that g is the homothety by α. We then have $\alpha Y \subseteq Y$, because every R-submodule of E is actually an $\widehat{R}$-submodule of E. Therefore

$$f(Y) = g(Y) = \alpha Y \subseteq Y,$$

and hence Y is a trace module in E. $\square$

We are now ready to prove Theorem 1.20.

Proof of Theorem 1.20. (1) ⇔ (2) See Proposition 1.21 (1).

(3) ⇒ (1) Let $\mathfrak{m} \in \operatorname{Max} R$. We then have the embedding $0 \to X_\mathfrak{m} \to \mathrm{E}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m})$, since

$$[\bigoplus_{\mathfrak{n} \in \operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{n})]_\mathfrak{m} = \mathrm{E}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}).$$

Therefore, by Theorem 1.23, for every cyclic R-submodule Y of X, $Y_\mathfrak{m}$ is a trace module in $X_\mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Max} R$, so that Lemma 1.22 guarantees that Y is a trace module in X. Hence, by Proposition 1.21 (2), every R-submodule of X is a trace module in X.

(1) ⇒ (3) Let $\mathfrak{m} \in \operatorname{Max} R$. Since every cyclic $R_\mathfrak{m}$-submodule of $X_\mathfrak{m}$ is a localization of a cyclic R-submodule of X, by Lemma 1.22 every $R_\mathfrak{m}$-submodule of $X_\mathfrak{m}$ is a trace module in $X_\mathfrak{m}$. Therefore, by Theorem 1.23, for every $\mathfrak{m} \in \operatorname{Max} R$ we have

$$\mathrm{Ass}_{R_\mathfrak{m}} X_\mathfrak{m} \subseteq \{\mathfrak{m}R_\mathfrak{m}\} \text{ and } \ell_{R_\mathfrak{m}}((0) :_{X_\mathfrak{m}} \mathfrak{m}R_\mathfrak{m}) \leq 1.$$

Consequently, $\mathrm{Ass}_R X \subseteq \operatorname{Max} R$ and $\ell_R((0) :_X \mathfrak{m}) \leq 1$ for all $\mathfrak{m} \in \operatorname{Max} R$, so that

$$\mathrm{E}_R(X) \cong \bigoplus_{\mathfrak{m} \in \operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{m})^{\oplus \mu(\mathfrak{m})}$$

with $\mu(\mathfrak{m}) \in \{0, 1\}$ for each $\mathfrak{m} \in \operatorname{Max} R$. $\square$

The following is a direct consequence of Theorem 1.20.

Corollary 1.24 (cf. [39, Theorem 3.5])**.** *For a Noetherian ring R, the following conditions are equivalent.*

(1) *Every ideal of R is a trace ideal in R.*

(2) *R is a self-injective ring.*

For the implication (1) $\Rightarrow$ (2) in Corollary 1.24, we cannot remove the assumption that R is a Noetherian ring. To explain more precisely about this phenomenon, let R be a commutative ring. We say that R is a *Von Neumann regular* ring, if for each $a \in R$, there exists an element $b \in R$ such that $a = aba$ (cf. [47]). Here, we need only the definition, but interested readers can find in [6] a basic characterization of Von Neumann regular rings.

Lemma 1.25. *Let R be a Von Neumann regular ring. Then $\tau_R(I) = I$ for every ideal I of R.*

Proof. Let $\varphi : I \to R$ be an R-linear map and $a \in I$. Then, $a = aba$ for some $b \in R$, so that $\varphi(a) = a\varphi(ba) \in I$. Thus, $\varphi(I) \subseteq I$. $\square$

We have learned the following example from M. Hashimoto.

Example 1.26. Let K be a commutative ring and assume that there exists an integer $p \geq 2$ such that $a^p = a$ for every $a \in K$. We consider the direct product $S = \prod_{i\in\Lambda} K_i$ of infinitely many copies $\{K_i = K\}_{i\in\Lambda}$ of K, and set $R = \mathbb{Z}{\cdot}1 + \bigoplus_{i\in\Lambda} K_i$ in S. Then, R is a subring of S, and R is Von Neumann regular, since $a^p = a$ for every $a \in S$. We have that S is an essential extension of R, but $R \neq S$, because Λ is infinite. Therefore, R is not a self-injective ring.

Let us note one more example. The following fact is known, when $\mathrm{ch}k = 2$ and $\alpha_i = 1$ for every $i \in \Lambda$. Indeed, with the same notation as Example 1.27, if $\mathrm{ch}k = 2$ and $\alpha_i = 1$ for all $i \in \Lambda$, then $R = k[\{T_i\}_{i\in\Lambda}]/(T_i^2 - 1 \mid i \in \Lambda)$ where $T_i = X_i - 1$ for each $i \in \Lambda$, so that $R = k[G]$, the group algebra of the direct sum $G = \bigoplus_{i\in\Lambda} C_i$ of infinitely many copies of the cyclic group $C_i = \mathbb{Z}/(2)$. Therefore, thanks to [8, Theorem], R is not self-injective. We have learned this result from K. Kurano, and we are grateful to him, since the method of proof given in [8] works also in the setting of Example 1.27, as we will briefly confirm below.

Example 1.27. Let $\Lambda = \{1, 2, 3, \dots\}$ be the set of positive integers. Let $\{X_i\}_{i\in\Lambda}$ be a family of indeterminates and $\{\alpha_i\}_{i\in\Lambda}$ a family of positive integers. We set $S = k[\{X_i\}_{i\in\Lambda}]$ over a field k, $\mathfrak{a} = (X_i^{\alpha_i+1} \mid i \in \Lambda)$, and consider the ring $R = S/\mathfrak{a}$. Then, R is not a self-injective ring, but $\tau_R(I) = I$ for every ideal I of R.

Proof. Let x_i denote, for each $i \in \Lambda$, the image of X_i in R. For each $n \in \Lambda$, we set $R_n = k[x_1, x_2, \dots, x_n]$ in R. Then, $R = \bigcup_{n\in\Lambda} R_n$, and

$$R_n = k[X_1, X_2, \dots, X_n]/(X_1^{\alpha_1+1}, X_2^{\alpha_2+1}, \dots, X_n^{\alpha_n+1}),$$

so that R_n is a self-injective ring for every $n \in \Lambda$. Let $a \in R$ and assume that $a \in R_n$. Then

$$(0) :_R [(0) :_R a] \subseteq \bigcup_{\ell \geq n} \{(0) :_{R_\ell} [(0) :_{R_\ell} a]\},$$

whence $(0) :_R [(0) :_R a] = (a)$, because $(0) :_{R_\ell} [(0) :_{R_\ell} a] = a{\cdot}R_\ell$ for all $\ell \geq n$ (here we use the fact that R_ℓ is a self-injective ring). Therefore, $\tau_R(I) = I$ for every ideal I of R, because $\tau_R((a)) = (0) :_R [(0) :_R a] = (a)$ for each $a \in R$.

To see that R is not self-injective, we set for each $n \in \Lambda$

$$a_n = \begin{cases} 1, & \text{if } n = 1 \\ 1 + x_1 + x_1x_2 + x_1x_2x_3 + \ldots + x_1x_2\cdots x_{n-1}, & \text{if } n > 1 \end{cases}$$

and set $I_n = (x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_n^{\alpha_n})$. Then, $I_n \subseteq I_{n+1}$, and $I = \bigcup_{n\in\Lambda} I_n$, where $I = (x_i^{\alpha_i} \mid i \in \Lambda)$. We then have $a_{n+1}x = a_nx$ for every $x \in I_n$, which one can show by a simple use of induction on n, since $x_i^{\alpha_i+1} = 0$ for all $i \in \Lambda$. Therefore, we may define the R-linear map $\varphi : I \to R$ so that $\varphi(x) = a_nx$ if $x \in I_n$. We now assume that R is a self-injective ring. Then, there must exist an element $a \in R$ such that $ax = \varphi(x)$ for every $x \in I$, namely $ax = a_nx$ for every $x \in I_n$. Choose $n \in \Lambda$ so that $a \in R_n$. Then, because $(a - a_{n+2})x_{n+2}^{\alpha_{n+2}} = 0$, we get $a - a_{n+2} \in (0) :_R x_{n+2}^{\alpha_{n+2}} = (x_{n+2})$. Let $f \in k[X_1, X_2, \dots, X_n]$ such that a is the image of f in R. Then

$$f = 1 + X_1 + X_1X_2 + \ldots + X_1X_2\cdots X_{n+1} + X_{n+2}g + h$$

for some $g \in S$ and $h \in \mathfrak{a}$. Substituting X_i by 0 for all $i \geq n + 2$, we may assume that $g = 0$ and $h \in (X_1^{\alpha_1+1}, X_2^{\alpha_2+1}, \dots, X_{n+1}^{\alpha_{n+1}+1})T$, where $T = k[X_1, X_2, \dots, X_{n+1}]$, that is

$$f = 1 + X_1 + X_1X_2 + \ldots + X_1X_2 \ldots X_{n+1} + \sum_{i=1}^{n+1} X_i^{\alpha_i+1}h_i$$

with $h_i \in T$. This is, however, impossible, because $f \in k[X_1, X_2, \ldots, X_n]$ and the monomial $X_1X_2\cdots X_{n+1}$ is not involved in the polynomial $\sum_{i=1}^{n+1} X_i^{\alpha_i+1}h_i$. Thus, R is not a self-injective ring. $\square$

It seems interesting, but hard, to ask for a complete characterization of (not necessarily Noetherian) commutative rings, in which every ideal is a trace ideal.

1.5 Surjectivity of the correspondence ρ in dimension one

In this section, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one. We are interested in the question of when the correspondence $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is bijective. The second example in Example 1.18 seems to suggest that R is a Gorenstein ring, if $\dim R = 1$ and ρ is bijective. Unfortunately, this is still not the case, as we show in the following. Here, we say that a one-dimensional Cohen-Macaulay local ring $(R, \mathfrak{m})$ has maximal embedding dimension, if $\mathfrak{m}^2 = a\mathfrak{m}$ for some $a \in \mathfrak{m}$ ([43]). We refer to [18, 28] for the notion of almost Gorenstein local ring.

Proposition 1.28 (cf. [35, Example 4.7])**.** *Let K/k be a finite extension of fields. Assume that $k \neq K$ and there is no intermediate field F such that $k \subsetneq F \subsetneq K$. Let $B = K[[t]]$ be the formal power series ring over K and set $R = k[[Kt]]$ in B. Set $n = [K : k]$. We then have the following.*

(1) *R is a Noetherian local ring with $B = \overline{R}$ and $\mathfrak{m} = tB$, where $\mathfrak{m}$ denotes the maximal ideal of R. Hence $B = \mathfrak{m} : \mathfrak{m} = R : \mathfrak{m}$.*

(2) *R is an almost Gorenstein local ring, possessing maximal embedding dimension $n \geq 2$.*

(3) *R is not a Gorenstein ring, if $n \geq 3$.*

(4) *$\mathcal{X}_R^T = \{\mathfrak{m}, R\}$ and $\mathcal{Y}_R = \{B, R\}$, so that $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is a bijection.*

Proof. Let $\omega_1 = 1, \omega_2, \ldots, \omega_n$ be a k-basis of K. Then $R = k[[\omega_1 t, \omega_2 t, \ldots, \omega_n t]]$, whence R is a Noetherian complete local ring. Since $B/\mathfrak{m}B \cong K$, $B = \sum_{i=1}^n R\omega_i$, so that $tB = \mathfrak{m}$. Hence, $\mathfrak{m}$ is also an ideal of B, $\mathfrak{m} = \mathfrak{m}B = tB$, and $\mathfrak{m}^2 = t\mathfrak{m}$. Because B is a module-finite extension of R and $\omega_i = \frac{\omega_i t}{\omega_1 t} \in \mathrm{Q}(R)$ for all $1 \leq i \leq n$, we have $B = \overline{R}$. Therefore, R is an almost Gorenstein ring by [18, Corollary 3.12], possessing maximal embedding dimension $\mathrm{e}(R) = n$. Consequently, R is not a Gorenstein ring, if

$n \geq 3$. We get $\mathcal{X}_R^T = \{\mathfrak{m}, R\}$ by Proposition 1.19, because $R \neq B$ but $\mathfrak{m}B \subseteq R$. The assertion that $\mathcal{Y}_R = \{B, R\}$ is due to [35, Example 4.7]. Let us note a brief proof for the sake of completeness. Let $A \in \mathcal{Y}_R$ and let $\mathfrak{n}$ denote the maximal ideal of A. We then have $\mathfrak{n} = \mathfrak{m}$, because $\mathfrak{n} = \mathfrak{m}_B \cap A = \mathfrak{m} \cap A = \mathfrak{m}$. Consequently, we have an extension $k = R/\mathfrak{m} \subseteq A/\mathfrak{m} \subseteq K = B/\mathfrak{m}$ of fields, so that $R/\mathfrak{m} = A/\mathfrak{m}$, or $A/\mathfrak{m} = B/\mathfrak{m}$ by the choice of the extension K/k. Hence, $R = A$ or $A = B$, and thus $\mathcal{Y}_R = \{R, B\}$. Therefore, because $\mathfrak{m} : \mathfrak{m} = tB : tB = B$ and $R : R = R$, the correspondence $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is a bijection. $\square$

In what follows, we intensively explore the question of when the correspondence $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is bijective. The goal is the following, which essentially shows that except the case of Proposition 1.28, the surjectivity of ρ implies the Gorenstein property of the ring R.

Theorem 1.29. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one. We set $B = \mathfrak{m} : \mathfrak{m}$ and let $\mathrm{J}(B)$ denote the Jacobson radical of B. Then the following assertions are equivalent.*

(1) $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ *is bijective.*

(2) $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ *is surjective.*

(3) *Either R is a Gorenstein ring, or the following two conditions are satisfied.*

 (i) *B is a* DVR *and $\mathrm{J}(B) = \mathfrak{m}$.*

 (ii) *There is no proper intermediate field between $R/\mathfrak{m}$ and $B/\mathrm{J}(B)$.*

When this is the case, R is an almost Gorenstein local ring.

We set $B = \mathfrak{m} : \mathfrak{m}$. Let $\mathrm{J}(B)$ be the Jacobson radical of B. To prove Theorem 1.29, we need some preliminaries. Let us begin with the following.

Lemma 1.30. *Suppose that R is not a* DVR*. Then $R \neq B$ and $\ell_R(B/R) = \mathrm{r}(R)$.*

Proof. We have $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$, since R is not a DVR. The second assertion is clear, since $\ell_R((R : \mathfrak{m})/R) = \mathrm{r}(R)$. $\square$

Proposition 1.31. *Suppose that $R \neq B$ and that $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is surjective. Then there is no proper intermediate ring between R and B.*

Proof. We have $\mathfrak{m}, R \in \mathcal{X}_R^T$ and $B, R \in \mathcal{Y}_R$. Let A be an extension of R such that $R \subsetneq A \subseteq B$. We write $A = \rho(I) = R : I$ with $I \in \mathcal{X}_R^T$. Then $I \subseteq \mathfrak{m}$, since $A \neq R$. Therefore, $A = R : I \supseteq R : \mathfrak{m} = B$, so that $A = B$. □

The following is the heart of our argument.

Theorem 1.32. *Let $(R, \mathfrak{m})$ be a non-Gorenstein Cohen-Macaulay local ring of dimension one. Assume that R is $\mathfrak{m}$-adically complete and there is no proper intermediate ring between R and B. Then the following assertions hold true.*

(1) *$B = \overline{R}$, and B is a* DVR *with* $\mathrm{J}(B) = \mathfrak{m}$.

(2) $[B/\mathfrak{m} : R/\mathfrak{m}] = \mathrm{r}(R) + 1 \geq 3$.

(3) *There is no proper intermediate field between $R/\mathfrak{m}$ and $B/\mathfrak{m}$.*

(4) *$\mathcal{X}_R^T = \{\mathfrak{m}, R\}$ and the correspondence ρ is bijective.*

Proof. We have $\mathfrak{m}B = \mathfrak{m}$, and $R \neq B$, since R is not a DVR (Lemma 1.30). Let $x \in B \setminus R$. Then $B = R[x]$ and $B/\mathfrak{m} = k[\overline{x}]$, where $k = R/\mathfrak{m}$ and $\overline{x}$ denotes the image of x in $B/\mathfrak{m}$. Let n (> 0) be the degree of the minimal polynomial of $\overline{x}$ over k. We then have

$$B = R + Rx + Rx^2 + \cdots + Rx^{n-1}$$

and $n = \mu_R(B)$, so that $n - 1 = \mathrm{r}(R)$ by Lemma 1.30. Therefore, $n \geq 3$ since R is not a Gorenstein ring, so that $x^2 \notin R$ since the elements $1, x, \ldots, x^{n-1}$ form a minimal system of generators of the R-module B. Hence

$$B = R[x^2] = R + Rx^2 + Rx^4 + \cdots + Rx^{2(n-1)}.$$

Let us write $x = \sum_{i=0}^{n-1} c_i x^{2i}$ with $c_i \in R$. We then have $x(1 - ax) = c_0$, where $a = \sum_{i=1}^{n-1} c_i x^{2i-2}$. We will show that $x \notin \mathrm{J}(B)$. If $c_0 \notin \mathfrak{m}$, then x is a unit of B, whence $x \notin \mathrm{J}(B)$. Assume that $c_0 \in \mathfrak{m}$. Then, if $x \in \mathrm{J}(B)$, $1 - ax$ is a unit of B, so that $x = (1 - ax)^{-1} c_0 \in \mathfrak{m}B = \mathfrak{m}$, which is a contradiction. Therefore, $x \notin \mathrm{J}(B)$ for all $x \in B \setminus R$, which shows $\mathrm{J}(B) \subseteq R$, whence $\mathrm{J}(B) = \mathfrak{m}$. Therefore, we have $B = \mathfrak{m} : \mathfrak{m} = \mathrm{J}(B) : \mathrm{J}(B)$. Hence, $B_M = MB_M : MB_M$ for all $M \in \operatorname{Max} B$, which implies the local ring B_M is a DVR (see Lemma 1.30). Therefore, because B is integrally closed in $\mathrm{Q}(B) = \mathrm{Q}(R)$, we get $B = \overline{B} = \overline{R}$.

Since R is $\mathfrak{m}$-adically complete, we have a decomposition

$$B = B_1 \times B_2 \times \cdots \times B_\ell$$

of B into a finite product of DVR's $\{B_j\}_{1 \le j \le \ell}$. We want to show that $\ell = 1$. Let $\mathbf{e}_j = (0, \ldots, 0, 1_{B_j}, 0, \ldots, 0)$ in B and set $\mathbf{e} = \sum_{j=1}^{\ell} \mathbf{e}_j$. Assume now that $\ell \ge 2$. We then have $B = R[\mathbf{e}_1]$, since $\mathbf{e}_1 \notin R$ and since there is no proper intermediate ring between R and B. Hence $B = R\mathbf{e} + R\mathbf{e}_1$, since $\mathbf{e}_1^2 = \mathbf{e}_1$. This is however impossible, because

$$\mu_R(B) = \ell_R(B/\mathfrak{m}B) = 1 + \mathrm{r}(R) > 2.$$

Thus, $\ell = 1$, that is $B = \overline{R}$ is a DVR with the maximal ideal $\mathrm{J}(B) = \mathfrak{m}$. It remains the proof of Assertions (3) and (4). Assume that there is contained a field F such that $R/\mathfrak{m} \subseteq F \subseteq B/\mathfrak{m}$. We consider the natural epimorphism $\varepsilon : B \to B/\mathfrak{m}$ of rings. Then, since $\varepsilon^{-1}(F)$ is an intermediate ring between R and B, either $\varepsilon^{-1}(F) = R$, or $\varepsilon^{-1}(F) = B$, which shows either $F = R/\mathfrak{m}$, or $F = B/\mathfrak{m}$.

Let $I \in \mathcal{X}_R^T$ and assume that $I \ne R$. Then, since $I \subseteq \mathfrak{m}$, we have

$$B = \mathfrak{m} : \mathfrak{m} = R : \mathfrak{m} \subseteq R : I = I : I \subseteq \overline{R} = B,$$

whence $I : I = B$, so that I is an ideal of B. Let us write $I = aB$ with $0 \ne a \in B$. We then have

$$B = R : I = R : aB = a^{-1}(R : B) = a^{-1}\mathfrak{m},$$

since $\mathfrak{m} = R : B$, so that $\mathfrak{m} = aB = I$. Thus, $\mathcal{X}_R^T = \{\mathfrak{m}, R\}$, which shows the correspondence ρ is bijective. This completes the proof of Theorem 1.32. □

We are now ready to prove Theorem 1.29.

Proof of Theorem 1.29. (1) ⇒ (2) This is clear.

(3) ⇒ (1) See Lemma 1.10, Proposition 1.31, and Theorem 1.32 (4).

(2) ⇒ (3) We may assume that R is not a Gorenstein ring. Passing to the $\mathfrak{m}$-adic completion $\widehat{R}$ of R, without loss of generality we may also assume that R is $\mathfrak{m}$-adically complete. Then by Proposition 1.31, there is no proper intermediate ring between R and B, so that the assertion follows from Theorem 1.32.

If ρ is bijective but R is not a Gorenstein ring, we then have $B = \mathfrak{m} : \mathfrak{m}$ is a DVR, so that R is an almost Gorenstein ring by [18, Theorem 5.1]. □

We note the following, which is a direct consequence of Theorem 1.29.

Corollary 1.33. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one. Suppose that one of the following conditions is satisfied.*

(i) *The field $R/\mathfrak{m}$ is algebraically closed.*

(ii) *$\overline{R}$ is a local ring, and $R/\mathfrak{m} \cong \overline{R}/\mathfrak{n}$, where $\mathfrak{n}$ denotes the maximal ideal of $\overline{R}$.*

Then the following assertions are equivalent.

(1) *R is a Gorenstein ring.*

(2) *The correspondence $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is bijective.*

(3) *The correspondence $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is surjective.*

When R is a numerical semigroup ring over a field, Condition (ii) of Corollary 1.33 is always satisfied.

1.6 Anti-stable rings

Let R be a commutative ring and let $\mathcal{F}_R$ denote the set of regular ideals of R. Then, because $(R : I){\cdot}(I : I) \subseteq R : I$, for every $I \in \mathcal{F}_R$ the R-module $R : I$ has also the structure of an $(I : I)$-module. Keeping this fact together with the natural identifications $R : I = \mathrm{Hom}_R(I, R)$ and $I : I = \mathrm{End}_R I$ in our mind, we give the following.

Definition 1.34. We say that R is an *anti-stable* (resp. *strongly anti-stable*) ring, if $R : I$ is an invertible $I : I$-module (resp. $R : I \cong I : I$ as an $(I : I)$-module) for every $I \in \mathcal{F}_R$.

Therefore, every Dedekind domain is anti-stable, and every UFD is a strongly anti-stable ring. Notice that when R is a Noetherian semi-local ring, R is anti-stable if and only if it is strongly anti-stable. Indeed, let $I \in \mathcal{F}_R$, and set $A = I : I$, $M = R : I$. Then, A is also a Noetherian semi-local ring, and therefore, because M has rank one as an A-module, M must be cyclic and free, once it is an invertible module over A.

Let us recall here that R is said to be a *stable* ring, if every ideal I of R is *stable*, that is projective over its endomorphism ring $\mathrm{End}_R I$ ([45]). When R is a Noetherian semi-local ring and $I \in \mathcal{F}_R$, I is a stable ideal of R if and only if $I \in \mathcal{Z}_R$, that is $I^2 = aI$ for some $a \in I$ ([40], [45, Proposition 2.2]). Our definition of anti-stable rings is, of course, different from that of stable rings. However, we shall later show in Corollary 1.43 that the anti-stability of rings implies the stability of rings, under suitable conditions.

First of all, we will show that R is a strongly anti-stable ring if and only if every $I \in \mathcal{F}_R$ is isomorphic to a trace ideal in R.

Lemma 1.35. *Let $I \in \mathcal{F}_R$ and set $A = I : I$. Then the following conditions are equivalent.*

(1) *$I \cong J$ as an R-module for some $J \in \mathcal{X}_R^T$.*

(2) *$I \cong \tau_R(I)$ as an R-module.*

(3) *$R : I \cong A$ as an R-module.*

(4) *$R : I \cong A$ as an A-module.*

(5) *$R : I = aA$ for some unit a of $\mathrm{Q}(R)$.*

Proof. (1) $\Leftrightarrow$ (2) Since $\tau_R(I) \in \mathcal{X}_R^T$, the implication (2) $\Rightarrow$ (1) is clear. Since $J = \tau_R(J)$ for every $J \in \mathcal{X}_R^T$ (Proposition 1.5), we have $\tau_R(I) = J$, if $J \in \mathcal{X}_R^T$ and $I \cong J$ as an R-module, whence the implication (1) $\Rightarrow$ (2) follows.

(4) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (4) Because the given isomorphsim $R : I \to A$ of R-modules is the restriction of the homothety of some unit a of $\mathrm{Q}(R)$, it must be also a homomorphism of A-modules, whence $R : I \cong A$ as an A-module.

(4) $\Leftrightarrow$ (5) This is now clear.

(1) $\Rightarrow$ (3) We have $I = aJ$ for some unit a of $\mathrm{Q}(R)$, whence $R : I = R : aJ = a^{-1}(R : J)$, and $I : I = aJ : aJ = J : J$. Thus, $R : I \cong I : I$ as an R-module, because $R : J = J : J$.

(5) $\Rightarrow$ (2) We have $\tau_R(I) = (R : I)I = aA{\cdot}I = aI$, whence $\tau_R(I) \cong I$ as an R-module. $\square$

For a Noetherian ring R, we set $\mathrm{Ht}_1(R) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{ht}_R\mathfrak{p} = 1\}$. Let us note the following example of strongly anti-stable rings. We include a brief proof.

Example 1.36 ([36, Corollary 3.10])**.** For a Noetherian normal domain R, R is a strongly anti-stable ring if and only if R is a UFD.

Proof. Suppose that R is a strongly anti-stable ring and let $\mathfrak{p} \in \mathrm{Ht}_1(R)$. Then, since R is normal, the R-module $\mathfrak{p}$ is reflexive with $\mathfrak{p} : \mathfrak{p} = R$, while $R : \mathfrak{p} \cong \mathfrak{p} : \mathfrak{p}$ by Lemma 1.35. Hence, $\mathfrak{p} \cong R$, so that R is a UFD. Conversely, suppose that R is a UFD and let $I \in \mathcal{X}_R^T$. Then, $I \cong J$ for some ideal J of R with $\mathrm{grade}_R J \geq 2$, so that $I \cong \tau_R(I)$, since $J \in \mathcal{X}_R^T$ by Corollary 1.6. Thus, R is a strongly anti-stable ring. $\square$

We explore one example of anti-stable rings which are not strongly anti-stable.

Example 1.37. Let k be a field and $S = k[t]$ the polynomial ring. Let $\ell \geq 2$ be an integer and set $R = k[t^2, t^{2\ell+1}]$. We consider the maximal ideal $I = (t^2 - 1, t^{2\ell+1} - 1)$ in R. Then, $\tau_R(I) = R$, and $I \not\cong J$ as an R-module for any $J \in \mathcal{X}_R^T$. Therefore, R is not a strongly anti-stable ring, while R is an anti-stable ring, because $\dim R = 1$ and for every $M \in \mathrm{Max}\, R$, R_M is an anti-stable local ring. See Theorem 1.42 for details.

Proof. Let $\mathfrak{p} \in \mathrm{Spec}\, R$. If $I \not\subseteq \mathfrak{p}$, then $IR_\mathfrak{p} = R_\mathfrak{p}$. If $I \subseteq \mathfrak{p}$, then $t^2 \notin \mathfrak{p} = I$, whence $R_\mathfrak{p} = S_\mathfrak{p}$ is a DVR, because $R : S = (t^2, t^{2\ell+1})R$, so that $IR_\mathfrak{p} \cong R_\mathfrak{p}$. We now notice that $I \subseteq \tau_R(I) \subseteq R$. Hence, either $I = \tau_R(I)$ or $\tau_R(I) = R$. If $I = \tau_R(I)$, then setting $\mathfrak{p} = I$, we get $R_\mathfrak{p}$ is a DVR and $IR_\mathfrak{p} = \tau_{R_\mathfrak{p}}(IR_\mathfrak{p}) \subsetneq R_\mathfrak{p}$, while $\tau_{R_\mathfrak{p}}(IR_\mathfrak{p}) = R_\mathfrak{p}$, because $IR_\mathfrak{p} \cong R_\mathfrak{p}$. This is absurd. Hence $\tau_R(I) = R$. Consequently, $I \not\cong J$ for any $J \in \mathcal{X}_R^T$. In fact, if $I \cong J$ for some $J \in \mathcal{X}_R^T$, then $J = \tau_R(J) = \tau_R(I) = R$, so that $\mu_R(I) = 1$. We write $I = fR$ with some monic polynomial $f \in R$. Let $\overline{k}$ denote the algebraic closure of k and choose $a \in \overline{k}$ so that $f(a) = 0$. Then, since $a^2 = a^{2\ell+1} = 1$, we get $a = 1$, whence $f = (t-1)^n$ with $0 < n \in H$, where $H = \langle 2, 2\ell+1 \rangle$ denotes the numerical semigroup generated by $2, 2\ell+1$. Therefore, $2 - n, (2\ell+1) - n \in H$, because $t^2 - 1, t^{2\ell+1} - 1 \in fR$. Hence, $n = 2$, and $2\ell + 1 \in 2 + H$, which is impossible. Thus, I is not a principal ideal of R, and $I \not\cong J$ for any $J \in \mathcal{X}_R^T$. $\square$

The key in our argument is the following, which plays a key role also in [11].

Lemma 1.38. *Let R be a strongly anti-stable ring. Then the correspondence $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is surjective. More precisely, let $A \in \mathcal{Y}_R$ and set $J = R : A$. Then $J \in \mathcal{G}_R = \mathcal{X}_R^T \cap \mathcal{Z}_R$.*

Proof. Let $A \in \mathcal{Y}_R$ and choose $b \in W$ so that $bA \subseteq R$. Then, since $bA \in \mathcal{F}_R$, by Lemma 1.35 $bA \cong J$ as an R-module for some $J \in \mathcal{X}_R^T$. Let us write $J = aA$ with a a unit of $\mathrm{Q}(R)$ (hence $a \in J \cap W$). We then have $J : J = aA : aA = A : A = A$, whence $A = J : J = R : J = R : aA = a^{-1}(R : A)$, so that $R : A = aA = J \in \mathcal{X}_R^T \cap \mathcal{Z}_R$. Therefore, $\rho(J) = J : J = A$, and the correspondence $\rho : \mathcal{X}_R^T \to \mathcal{Y}_R$ is surjective. $\square$

Let us recall one of the fundamental results on stable rings, which we need to prove Theorem 1.40.

Proposition 1.39 ([45, Lemma 3.2, Theorem 3.4])**.** *Let R be a Cohen-Macaulay semi-local ring and assume that $\dim R_M = 1$ for every $M \in \operatorname{Max} R$. If $\mathrm{e}(R_M) \le 2$ for every $M \in \operatorname{Max} R$, then R is a stable ring.*

We should compare the following theorem with [11, Theorem 3.6].

Theorem 1.40. *Let R be a Cohen-Macaulay local ring of dimension one. Then, R is an anti-stable ring, if and only if $\mathrm{e}(R) \le 2$.*

Proof. Suppose that $\mathrm{e}(R) \le 2$. Let $I \in \mathcal{F}_R$ and set $A = I : I$. Then, by Proposition 1.39 R is a stable ring. Hence, $I^2 = aI$ for some $a \in I$, whence $A = a^{-1}I$. Therefore, $I \cong A$ as an R-module. We now consider $J = (R : I)I$. Then, $J = \tau_R(I) \in \mathcal{X}_R^T$, whence

$$J : J = R : J = R : (R : I)I = [R : (R : I)] : I = I : I,$$

where the last equality follows from the fact that R is a Gorenstein ring. Consequently, $A = J : J \cong J$ (since $J \in \mathcal{F}_R$), so that $I \cong J = \tau_R(J)$. Thus, R is an anti-stable ring.

Conversely, suppose that R is an anti-stable ring. First of all, we will show that R is a Gorenstein ring. Assume the contrary. Then, passing to the $\mathfrak{m}$-adic completion of R, by Proposition 1.31 and Theorem 1.32 we get $\mathcal{X}_R^T = \{\mathfrak{m}, R\}$. Consequently, either $I \cong \mathfrak{m}$ or $I \cong R$, for every ideal $I \in \mathcal{F}_R$. We set $n = \mu_R(\mathfrak{m})$ and write $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ with non-zerodivisors x_i of R. Then, $n > 2$ since R is not a Gorenstein ring, and setting $I = (x_1, x_2, \ldots, x_{n-1})$, we have either $I \cong \mathfrak{m}$ or $I \cong R$, both of which violates the fact that $n = \mu_R(\mathfrak{m}) > 2$. Thus R is a Gorenstein ring. We want to show $\mathrm{e}(R) \le 2$. Assume that $\mathrm{e}(R) \ge 2$ and consider $B = \mathfrak{m} : \mathfrak{m}$. Then, $B \in \mathcal{Y}_R$ and $R \ne B$, because R is not a DVR. Consequently, because $\mathfrak{m} = R : B$, by Lemma 1.38 $\mathfrak{m}^2 = a\mathfrak{m}$ for some $a \in \mathfrak{m}$, which implies $\mathrm{e}(R) = 2$, since R is a Gorenstein ring. $\square$

We say that a Noetherian ring R satisfies the condition (S_1) of Serre, if $\operatorname{depth} R_\mathfrak{p} \ge \inf\{1, \dim R_\mathfrak{p}\}$ for every $\mathfrak{p} \in \operatorname{Spec} R$.

Corollary 1.41. *Let R be a Noetherian ring and suppose that R satisfies (S_1). Then, $\mathrm{e}(R_\mathfrak{p}) \le 2$ for every $\mathfrak{p} \in \operatorname{Ht}_1(R)$, if R is an anti-stable ring.*

Proof. Let $\mathfrak{p} \in \mathrm{Ht}_1(R)$ and set $A = R_\mathfrak{p}$. Hence A is a Cohen-Macaulay local ring of dimension one. Let $I \in \mathcal{F}_A$ and set $J = I \cap R$. We will show that $A : I$ is a cyclic $(I : I)$-module. We may assume that $I \neq A$. Hence, J is a $\mathfrak{p}$-primary ideal of R, since I is a $\mathfrak{p}R_\mathfrak{p}$-primary ideal of $A = R_\mathfrak{p}$. Hence, because $J \in \mathcal{F}_R$ (remember that R satisfies (S_1)), $R : J$ is a projective $(J : J)$-module. Therefore, $A : I = [R : J]_\mathfrak{p}$ is a cyclic module over $I : I = [J : J]_\mathfrak{p}$, since it has rank one over the semi-local ring $I : I$. Thus, $\mathrm{e}(A) \leq 2$ by Theorem 1.40. □

We now come to the main results of this section.

Theorem 1.42. *Let R be a Noetherian ring and suppose that R satisfies (S_1). Let us consider the following four conditions.*

(1) *R is anti-stable.*

(2) *R is strongly anti-stable.*

(3) *Every $I \in \mathcal{F}_R$ is isomorphic to $\tau_R(I)$.*

(4) *$\mathrm{e}(R_\mathfrak{p}) \leq 2$ for every $\mathfrak{p} \in \mathrm{Ht}_1(R)$.*

Then, we have the implications (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (4). If R is semi-local (resp. $\dim R = 1$), then the implication (1) $\Rightarrow$ (2) (resp. (4) $\Rightarrow$ (1)) holds true.

Proof. (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (4) See Lemma 1.35 and Theorem 1.41.

If R is semi-local, then every birational module -finite extension of R is also semi-local, so that the implication (1) $\Rightarrow$ (2) follows.

Suppose that $\dim R = 1$. Let $I \in \mathcal{F}_R$ and set $A = I : I$. Then, by Theorem 1.40 $R_M : IR_M = [R : I]_M$ is a cyclic A_M-module for every $M \in \mathrm{Max}\, R$, so that $R : I$ is an invertible A-module. Hence, the implication (4) $\Rightarrow$ (1) follows. □

Theorem 1.43. *Let R be a Cohen-Macaulay ring with $\dim R_M = 1$ for every $M \in \mathrm{Max}\, R$. If R is an anti-stable ring, then R is a stable ring.*

Proof. For every $M \in \mathrm{Max}\, R$, $\mathrm{e}(R_M) \leq 2$ by Corollary 1.41. Let I be an arbitrary ideal of R and set $A = \mathrm{End}_R I$. Then, because R_M is a stable ring by Proposition 1.39, for every $M \in \mathrm{Max}\, R$ IR_M is a projective A_M-module, so that I is a projective A-module. Thus, R is a stable ring. □

2 The structure of chains of Ulrich ideals in Cohen-Macaulay local ring of dimension one

2.1 Introduction

The purpose of this chapter is to investigate the behavior of chains of Ulrich ideals, in a one-dimensional Cohen-Macaulay local ring, in connection with the structure of birational finite extensions of the base ring.

The notion of Ulrich ideals is a generalization of stable maximal ideals, which dates back to 1971, when the monumental paper [40] of J. Lipman was published. The modern treatment of Ulrich ideals was started by [24, 25] in 2014, and has been explored in connection with the representation theory of rings. In [24], the basic properties of Ulrich ideals are summarized, whereas in [25], Ulrich ideals in two-dimensional rational singularities are closely studied with a concrete classification. However, in contrast to the existing research on Ulrich ideals, the theory pertaining to the one-dimensional case does not seem capable of growth. Some part of the theory, including research on the ubiquity as well as the structure of the chains of Ulrich ideals, seems to have been left unchallenged. In the current chapter, we focus our attention on the one-dimensional case, clarifying the relationship between Ulrich ideals and the birational finite extensions of the base ring. The main objective is to understand the behavior of chains of Ulrich ideals in one-dimensional Cohen-Macaulay local rings.

To explain our objective as well as our main results, let us begin with the definition of Ulrich ideals. Although we shall focus our attention on the one-dimensional case, we would like to state the general definition, in the case of any arbitrary dimension. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R \geq 0$.

Definition 2.1 ([24]). Let I be an $\mathfrak{m}$-primary ideal of R and assume that I contains a parameter ideal $Q = (a_1, a_2, \ldots, a_d)$ of R as a reduction. We say that I is an *Ulrich ideal* of R, if the following conditions are satisfied.

(1) $I \neq Q$,

(2) $I^2 = QI$, and

(3) I/I^2 is a free R/I-module.

We notice that Condition (2) together with Condition (1) are equivalent to saying that the associated graded ring $\mathrm{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ of I is a Cohen-Macaulay ring and $\mathrm{a}(\mathrm{gr}_I(R)) = 1 - d$, where $\mathrm{a}(\mathrm{gr}_I(R))$ denotes the a-invariant of $\mathrm{gr}_I(R)$ ([27, Remark 3.10], [30, Remark (3.1.6)]). Therefore, these two conditions are independent of the choice of reductions Q of I. In addition, assuming Condition (2) is satisfied, Condition (3) is equivalent to saying that I/Q is a free R/I-module ([24, Lemma 2.3]). We also notice that Condition (3) is automatically satisfied if $I = \mathfrak{m}$. Therefore, when the residue class field $R/\mathfrak{m}$ of R is infinite, the maximal ideal $\mathfrak{m}$ is an Ulrich ideal of R if and only if R is not a regular local ring, possessing minimal multiplicity ([43]). From this perspective, Ulrich ideals are a kind of generalization of stable maximal ideals, which Lipman [40] started to analyze in 1971.

Here, let us briefly summarize some basic properties of Ulrich ideals, as seen in [24, 29]. Although we need only a part of them, let us also include some superfluity in order to show what specific properties Ulrich ideals enjoy. Throughout this chapter, let $\mathrm{r}(R)$ denote the Cohen-Macaulay type of R, and let $\mathrm{Syz}_R^i(M)$ denote, for each integer $i \geq 0$ and for each finitely generated R-module M, the i-th syzygy module of M in its minimal free resolution.

Theorem 2.2 ([24, 29]). *Let I be an Ulrich ideal of a Cohen-Macaulay local ring R of dimension $d \geq 0$ and set $t = n - d$ (> 0), where n denotes the number of elements in a minimal system of generators of I. Let*

$$\cdots \to F_i \overset{\partial_i}{\to} F_{i-1} \to \cdots \to F_1 \overset{\partial_1}{\to} F_0 = R \to R/I \to 0$$

be a minimal free resolution of R/I. Then $\mathrm{r}(R) = t{\cdot}\mathrm{r}(R/I)$ and the following assertions hold true.

(1) $\mathbf{I}(\partial_i) = I$ *for* $i \geq 1$.

(2) *For* $i \geq 0$, $\beta_i = \begin{cases} t^{i-d}{\cdot}(t+1)^d & (i \geq d), \\ \binom{d}{i} + t{\cdot}\beta_{i-1} & (1 \leq i \leq d), \\ 1 & (i = 0). \end{cases}$

(3) $\mathrm{Syz}_R^{i+1}(R/I) \cong [\mathrm{Syz}_R^i(R/I)]^{\oplus t}$ *for* $i \geq d$.

(4) *For* $i \in \mathbb{Z}$, $\mathrm{Ext}_R^i(R/I, R) \cong \begin{cases} (0) & (i < d), \\ (R/I)^{\oplus t} & (i = d), \\ (R/I)^{\oplus (t^2-1)\cdot t^{i-(d+1)}} & (i > d). \end{cases}$

Here $\mathbf{I}(\partial_i)$ *denotes the ideal of* R *generated by the entries of the matrix* ∂_i*, and* $\beta_i = \mathrm{rank}_R F_i$.

Because Ulrich ideals are a very special kind of ideals, it seems natural to expect that, in the behavior of Ulrich ideals, there might be contained ample information on base rings, once they exist. As stated above, this is the case of two-dimensional rational singularities, and the present objects of study are rings of dimension one.

In what follows, unless otherwise specified, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$. Our main targets are chains $I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1$ $(n \geq 2)$ of Ulrich ideals in R. Let I be an Ulrich ideal of R with a reduction $Q = (a)$. We set $A = I : I$ in the total ring of fractions of R. Hence, A is a birational finite extension of R, and $I = aA$. Firstly, we study the close connection between the structure of the ideal I and the R-algebra A. Secondly, let J be an Ulrich ideal of R and assume that $I \subsetneq J$. Then, we will show that $\mu_R(J) = \mu_R(I)$, where $\mu_R(*)$ denotes the number of elements in a minimal system of generators, and that $J = (b) + I$ for some $a, b \in \mathfrak{m}$ with $I = abA$. Consequently, we have the following, which is one of the main results of this chapter.

Theorem 2.3. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring with* $\dim R = 1$*. Then the following assertions hold true.*

(1) *Let* I *be an Ulrich ideal of* R *and* $A = I : I$*. Let* $a_1, a_2, \ldots, a_n \in \mathfrak{m}$ $(n \geq 2)$ *and assume that* $I = a_1 a_2 \cdots a_n A$*. For* $1 \leq i \leq n$*, let* $I_i = (a_1 a_2 \cdots a_i) + I$*. Then each* I_i *is an Ulrich ideal of* R *and*

$$I = I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1.$$

(2) *Conversely, let* $I_1, I_2, \ldots, I_n$ $(n \geq 2)$ *be Ulrich ideals of* R *and suppose that*

$$I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1.$$

We set $I = I_n$ *and* $A = I : I$*. Then there exist elements* $a_1, a_2, \ldots, a_n \in \mathfrak{m}$ *such that* $I = a_1 a_2 \cdots a_n A$ *and* $I_i = (a_1 a_2 \cdots a_i) + I$ *for all* $1 \leq i \leq n - 1$.

Let I and J be Ulrich ideals of R and assume that $I \subsetneq J$. We set $B = J : J$. Let us write $J = (b) + I$ for some $b \in \mathfrak{m}$. We then have that $J^2 = bJ$ and that B is a local ring with the maximal ideal $\mathfrak{n} = \mathfrak{m} + \frac{I}{b}$, where $\frac{I}{b} = \left\{ \frac{i}{b} \mid i \in I \right\}$ $(= b^{-1}I)$. We furthermore have the following.

Theorem 2.4. *$\frac{I}{b}$ is an Ulrich ideal of the Cohen-Macaulay local ring B of dimension one and there is a one-to-one correspondence $\mathfrak{a} \mapsto \frac{\mathfrak{a}}{b}$ between the Ulrich ideals $\mathfrak{a}$ of R such that $I \subseteq \mathfrak{a} \subsetneq J$ and the Ulrich ideals $\mathfrak{b}$ of B such that $\frac{I}{b} \subseteq \mathfrak{b}$.*

These two theorems convey to us that the behavior of chains of Ulrich ideals in a given one-dimensional Cohen-Macaulay local ring could be understood via the correspondence, and the relationship between the structure of Cohen-Macaulay local rings R and B could be grasped through the correspondence, which we shall closely discuss in this chapter.

We now explain how this chapter is organized. In Section 2.2, we will summarize some preliminaries, which we shall need later to prove the main results. The proof of Theorems 2.3 and 2.4 will be given in Section 2.3. In Section 2.4, we shall study the case where the base rings R are not regular but possess minimal multiplicity ([43]), and show that the set of Ulrich ideals of R are totally ordered with respect to inclusion. In Section 2.5, we explore the case where R is a GGL ring ([16]).

In what follows, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$. Let $\mathrm{Q}(R)$ (resp. $\mathcal{X}_R$) stand for the total ring of fractions of R (resp. the set of all the Ulrich ideals in R). We denote by $\overline{R}$, the integral closure of R in $\mathrm{Q}(R)$. For a finitely generated R-module M, let $\mu_R(M)$ (resp. $\ell_R(M)$) be the number of elements in a minimal system of generators (resp. the length) of M. For each $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R, let

$$\mathrm{e}^0_{\mathfrak{a}}(R) = \lim_{n \to \infty} \frac{\ell_R(R/\mathfrak{a}^n)}{n}$$

stand for the multiplicity of R with respect to $\mathfrak{a}$. By $\mathrm{v}(R)$ (resp. $\mathrm{e}(R)$) we denote the embedding dimension $\mu_R(\mathfrak{m})$ of R (resp. $\mathrm{e}^0_{\mathfrak{m}}(R)$). Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R.

2.2 Preliminaries

Let us summarize preliminary facts on $\mathfrak{m}$-primary ideals of R, which we need throughout this chapter.

In this section, let I be an $\mathfrak{m}$-primary ideal of R, for which we will assume Condition (C) in Definition 2.6 to be satisfied. This condition is a partial extraction from Definition 2.1 of Ulrich ideals; hence every Ulrich ideal satisfies it (see Remark 2.7).

Firstly, we assume that I contains an element $a \in I$ with $I^2 = aI$. We set $A = I : I$ and

$$\frac{I}{a} = \left\{ \frac{x}{a} \mid x \in I \right\} = a^{-1} I$$

in $\mathrm{Q}(R)$. Therefore, A is a birational finite extension of R such that $R \subseteq A \subseteq \overline{R}$, and $A = \dfrac{I}{a}$, because $I^2 = aI$; hence $I = aA$. We then have the following.

Proposition 2.5. *If $I = (a) :_R I$, then $A = R : I$ and $I = R : A$, whence $R : (R : I) = I$.*

Proof. Notice that $I = (a) :_R I = (a) : I = a[R : I]$ and we have $A = R : I$, because $I = aA$. We get $R : A = I$, since $R : A = R : \dfrac{I}{a} = a[R : I] = aA$. $\square$

Let us now give the following.

Definition 2.6. Let I be an $\mathfrak{m}$-primary ideal of R and set $A = I : I$. We say that I satisfies Condition (C), if

(i) $A/R \cong (R/I)^{\oplus t}$ as an R-module for some $t > 0$, and

(ii) $A = R : I$.

Consequently, $I = R : A$ by Condition (i), when I satisfies Condition (C).

Remark 2.7. Let $I \in \mathcal{X}_R$. Then I satisfies Condition (C). In fact, choose $a \in I$ so that $I^2 = aI$. Then, $I/(a) \cong (R/I)^{\oplus t}$ as an R/I-module, where $t = \mu_R(I) - 1 > 0$ ([24, Lemma 2.3]). Therefore, $I = (a) :_R I$, so that I satisfies the hypothesis in Proposition 2.5, whence $A = R : I$. Notice that $A/R \cong I/(a) \cong (R/I)^{\oplus t}$, because $I = aA$.

We assume, throughout this section, that our $\mathfrak{m}$-primary ideal I satisfies Condition (C). We choose elements $\{f_i\}_{1 \le i \le t}$ of A so that

$$A = R + \sum_{i=1}^{t} R f_i.$$

Therefore, the images $\{\overline{f_i}\}_{1 \le i \le t}$ of $\{f_i\}_{1 \le i \le t}$ in A/R form a free basis of the R/I-module A/R. We then have the following.

Lemma 2.8. *$aA \cap R \subseteq (a) + I$ for all $a \in R$.*

Proof. Let $x \in aA \cap R$ and write $x = ay$ with $y \in A$. We write $y = c_0 + \sum_{i=1}^{t} c_i f_i$ with $c_i \in R$. Then, $ac_i \in I$ for $1 \le i \le t$, since $x = ac_0 + \sum_{i=1}^{t}(ac_i)f_i \in R$. Therefore, $(ac_i)f_i \in IA = I$ for all $1 \le i \le t$, so that $x \in (a) + I$ as claimed. $\square$

Corollary 2.9. *Let J be an $\mathfrak{m}$-primary ideal of R and assume that J contains an element $b \in J$ such that $J^2 = bJ$ and $J = (b) :_R J$. If $I \subseteq J$, then $J = (b) + I$.*

Proof. We set $B = J : J$. Then $B = R : J$ and $J = bB$ by Proposition 2.5, so that $B = R : J \subseteq A = R : I$, since $I \subseteq J$. Consequently, $J = bB \subseteq bA \cap R \subseteq (b) + I$ by Lemma 2.8, whence $J = (b) + I$. $\square$

In what follows, let J be an $\mathfrak{m}$-primary ideal of R and assume that J contains an element $b \in J$ such that $J^2 = bJ$ and $J = (b) :_R J$. We set $B = J : J$. Then $B = R : J = \frac{J}{b}$ by Proposition 2.5. Throughout, suppose that $I \subsetneq J$. Therefore, since $J = (b) + I$ by Corollary 2.9, we get

$$B = \frac{J}{b} = R + \frac{I}{b}.$$

Let $\mathfrak{a} = \frac{I}{b}$. Therefore, $\mathfrak{a}$ is an ideal of A containing I, so that $\mathfrak{a}$ is also an ideal of B with

$$R/(\mathfrak{a} \cap R) \cong B/\mathfrak{a}.$$

With this setting, we have the following.

Lemma 2.10. *The following assertions hold true.*

(1) *$A/B \cong (B/\mathfrak{a})^{\oplus t}$ as a B-module.*

(2) *$\mathfrak{a} \cap R = I :_R J$.*

(3) *$\ell_R([I :_R J]/I) = \ell_R(R/J)$.*

(4) *$I = [b{\cdot}(I :_R J)]A$.*

Proof. (1) Since $A = R + \sum_{i=1}^{t} Rf_i$, we get $A/B = \sum_{i=1}^{t} B\overline{f_i}$ where $\overline{f_i}$ denotes the image of f_i in A/B. Let $\{b_i\}_{1 \le i \le t}$ be elements of $B = \frac{J}{b}$ and assume that $\sum_{i=1}^{t} b_i f_i \in B$. Then, since $\sum_{i=1}^{t}(bb_i)f_i \in R$ and $bb_i \in R$ for all $1 \le i \le t$, we have $bb_i \in I$, so that $b_i \in \frac{I}{b} = \mathfrak{a}$. Hence $A/B \cong (B/\mathfrak{a})^{\oplus t}$ as a B-module.

(2) This is standard, because $J = (b) + I$ and $\mathfrak{a} = \dfrac{I}{b}$.

(3) Since $J/I = [(b) + I]/I \cong R/[I :_R J]$, we get

$$\ell_R([I :_R J]/I) = \ell_R(R/I) - \ell_R(R/[I :_R J]) = \ell_R(R/I) - \ell_R(J/I) = \ell_R(R/J).$$

(4) We have $[b{\cdot}(I :_R J)]A \subseteq I$, since $b{\cdot}(I :_R J) \subseteq I$ and $IA = I$. To see the reverse inclusion, let $x \in I$. Then $x \in J = bB \subseteq bA$. We write $x = b[c_0 + \sum_{i=1}^{t} c_i f_i]$ with $c_i \in R$. Then $bc_i \in I$ for $1 \leq i \leq t$ since $x \in R$, so that $(bc_i)f_i \in I$ for all $1 \leq i \leq t$, because I is an ideal of A. Therefore, $bc_0 \in I$, since $x = bc_0 + \sum_{i=1}^{t}(bc_i)f_i \in I$. Consequently, $c_i \in I :_R b = I :_R J$ for all $0 \leq i \leq t$, so that $x \in [b{\cdot}(I :_R J)]A$ as wanted. □

Corollary 2.11. *$J/(b) \cong ([I :_R J]/I)^{\oplus t}$ as an R-module. Hence $\ell_R(J/(b)) = t{\cdot}\ell_R(R/J)$.*

Proof. We consider the exact sequence

$$0 \to B/R \to A/R \to A/B \to 0$$

of R-modules. By Lemma 2.10 (1), A/B is a free $B/\mathfrak{a}$-module of rank t, possessing the images of $\{f_i\}_{1 \leq i \leq t}$ in A/B as a free basis. Because A/R is a free R/I-module of rank t, also possessing the images of $\{f_i\}_{1 \leq i \leq t}$ in A/R as a free basis, we naturally get an isomorphism between the following two canonical exact sequences;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B/R & \xrightarrow{\ i\ } & A/R & \longrightarrow & A/B & \longrightarrow & 0\\
 & & \downarrow\wr & \circlearrowleft & \downarrow\wr & \circlearrowleft & \downarrow\wr & & \\
0 & \longrightarrow & ([\mathfrak{a} \cap R]/I)^{\oplus t} & \xrightarrow{\ i\ } & (R/I)^{\oplus t} & \longrightarrow & (B/\mathfrak{a})^{\oplus t} & \longrightarrow & 0
\end{array}$$

Since $B/R = \dfrac{J}{b}/R \cong J/(b)$ and $\mathfrak{a} \cap R = I :_R J$ by Lemma 2.10 (2), we get

$$J/(b) \cong ([I :_R J]/I)^{\oplus t}.$$

The second assertion now follows from Lemma 2.10 (3). □

The following is the heart of this section.

Proposition 2.12. *The following conditions are equivalent.*

(1) $J \in \mathcal{X}_R$.

(2) $\mu_R([I :_R J]/I) = 1$.

(3) $[I :_R J]/I \cong R/J$ *as an R-module.*

When this is the case, $\mu_R(J) = t + 1$.

Proof. The implication (3) $\Rightarrow$ (2) is clear, and the reverse implication follows from the equality $\ell_R([I :_R J]/I) = \ell_R(R/J)$ of Lemma 2.10 (3).

(1) $\Rightarrow$ (3) Suppose that $J \in \mathcal{X}_R$. Then $J/(b)$ is R/J-free, so that by Corollary 2.11, $[I :_R J]/I$ is a free R/J-module, whence $[I :_R J]/I \cong R/J$ by Lemma 2.10 (3).

(3) $\Rightarrow$ (1) We have $J/(b) \cong ([I :_R J]/I)^{\oplus t} \cong (R/J)^{\oplus t}$ by Corollary 2.11, so that by Definition 2.1, $J \in \mathcal{X}_R$ with $\mu_R(J) = t + 1$. $\square$

We now come to the main result of this section, which plays a key role in Section 2.5.

Theorem 2.13. *The following assertions hold true.*

(1) *Suppose that $J \in \mathcal{X}_R$. Then there exists an element $c \in \mathfrak{m}$ such that $I = bcA$. Consequently, $I \in \mathcal{X}_R$ and $\mu_R(I) = \mu_R(J) = t + 1$.*

(2) *Suppose that $t \geq 2$. Then $I \in \mathcal{X}_R$ if and only if $J \in \mathcal{X}_R$.*

Proof. (1) Since $J \in \mathcal{X}_R$, by Proposition 2.12 we get an element $c \in \mathfrak{m}$ such that $I :_R J = (c) + I$. Therefore, by Lemma 2.10 (4) we have

$$I = [b{\cdot}(I :_R J)]A = [b{\cdot}((c) + I)]A = bcA + bIA = bcA + bI,$$

whence $I = bcA$ by Nakayama's lemma. Let $a = bc$. Then $I^2 = (aA)^2 = a{\cdot}aA = aI$, so that (a) is a reduction of I; hence $A = \dfrac{I}{a}$. Consequently, $I/(a) \cong A/R \cong (R/I)^{\oplus t}$, so that $I \in \mathcal{X}_R$ with $\mu_R(I) = t + 1$. Therefore, $\mu_R(I) = \mu_R(J)$, because $\mu_R(J) = t + 1$ by Proposition 2.12.

(2) We have only to show the *only if* part. Suppose that $I \in \mathcal{X}_R$ and choose $a \in I$ so that $I^2 = aI$; hence $A = \dfrac{I}{a}$. We then have $\mu_R(I) = t + 1$, since $I/(a) \cong A/R \cong (R/I)^{\oplus t}$. Consequently, since $J = (b) + I$, we get

$$\mu_R(J/(b)) = \mu_R([(b) + I]/(b)) \leq \mu_R(I) = t + 1.$$

On the other hand, we have $\mu_R(J/(b)) = t{\cdot}\mu_R([I :_R J]/I)$, because $J/(b) \cong ([I :_R J]/I)^{\oplus t}$ by Corollary 2.11. Hence

$$t{\cdot}(\mu_R([I :_R J]/I) - 1) \leq 1,$$

so that $\mu_R([I :_R J]/I) = 1$ because $t \geq 2$. Thus by Proposition 2.12, $J \in \mathcal{X}_R$ as claimed. $\square$

2.3 Chains of Ulrich ideals

In this section, we study the structure of chains of Ulrich ideals in R. First of all, remember that all the Ulrich ideals of R satisfy Condition (C) stated in Definition 2.6 (see Remark 2.7), and summarizing the arguments in Section 2.2, we readily get the following.

Theorem 2.14. *Let $I, J \in \mathcal{X}_R$ and suppose that $I \subsetneq J$. Choose $b \in J$ so that $J^2 = bJ$. Then the following assertions hold true.*

(1) $J = (b) + I$.

(2) $\mu_R(J) = \mu_R(I)$.

(3) *There exists an element $c \in \mathfrak{m}$ such that $I = bcA$, so that (bc) is a reduction of I, where $A = I : I$.*

We begin with the following, which shows that Ulrich ideals behave well, if R possesses minimal multiplicity. We shall discuss this phenomenon more closely in Section 2.4.

Corollary 2.15. *Suppose that $\mathrm{v}(R) = \mathrm{e}(R) > 1$ and let $I \in \mathcal{X}_R$. Then $\mu_R(I) = \mathrm{v}(R)$ and R/I is a Gorenstein ring.*

Proof. We have $\mathfrak{m} \in \mathcal{X}_R$ and $\mathrm{r}(R) = \mathrm{v}(R) - 1$, because $\mathrm{v}(R) = \mathrm{e}(R) > 1$. Hence by Theorem 2.14 (2), $\mu_R(I) = \mu_R(\mathfrak{m}) = \mathrm{v}(R)$. The second assertion follows from the equality $\mathrm{r}(R) = [\mu_R(I) - 1]{\cdot}\mathrm{r}(R/I)$ (see [29, Theorem 2.5]). $\square$

For each $I \in \mathcal{X}_R$, Assertion (3) in Theorem 2.14 characterizes those ideals $J \in \mathcal{X}_R$ such that $I \subsetneq J$. Namely, we have the following.

Corollary 2.16. *Let $I \in \mathcal{X}_R$. Then*

$$\{J \in \mathcal{X}_R \mid I \subsetneq J\} = \{(b) + I \mid b \in \mathfrak{m} \text{ such that } (bc) \text{ is a reduction of } I \text{ for some } c \in \mathfrak{m}\}.$$

Proof. Let $b, c \in \mathfrak{m}$ and suppose that (bc) is a reduction of I. We set $J = (b) + I$. We shall show that $J \in \mathcal{X}_R$ and $I \subsetneq J$. Because $bc \notin \mathfrak{m}I$, we have $b, c \notin I$, whence $I \subsetneq J$. If $J = (b)$, we then have $I = bcA \subseteq J = (b)$ where $A = I : I$, so that $cA \subseteq R$. This is impossible, because $c \notin R : A = I$ (see Lemma 2.5). Hence, $(b) \subsetneq J$. Because $I^2 = bcI$, we have $J^2 = bJ + I^2 = bJ + bcI = bJ$. Let us check that $J/(b)$ is a free R/J-module.

Let $\{f_i\}_{1\le i\le t}$ $(t=\mu_R(I)-1>0)$ be elements of A such that $A=R+\sum_{i=1}^{t} Rf_i$, so that their images $\{\overline{f_i}\}_{1\le i\le t}$ in A/R form a free basis of the R/I-module A/R (remember that I satisfies Condition (C) of Definition 2.6). We then have

$$J=(b)+I=(b)+bcA=(b)+\sum_{i=1}^{t} R{\cdot}(bc)f_i.$$

Let $\{c_i\}_{1\le i\le t}$ be elements of R and assume that $\sum_{i=1}^{t} c_i{\cdot}(bcf_i)\in(b)$. Then, since $\sum_{i=1}^{t} c_ic{\cdot}f_i\in R$, we have $c_ic\in I=bcA$, so that $c_i\in bA\cap R$ for all $1\le i\le t$. Therefore, because $bA\cap R\subseteq(b)+I=J$ by Lemma 2.8, we get $c_i\in J$, whence $J/(b)\cong(R/J)^{\oplus t}$. Thus, $J=(b)+I\in\mathcal{X}_R$. $\square$

The equality $\mu_R(I)=\mu_R(J)$ does not hold true in general, if I and J are incomparable, as we show in the following.

Example 2.17. Let $S=k[[X_1,X_2,X_3,X_4]]$ be the formal power series ring over a field k and consider the matrix $\mathbb{M}=\left(\begin{smallmatrix} X_1 & X_2 & X_3\\ X_2 & X_3 & X_1\end{smallmatrix}\right)$. We set $R=S/[\mathfrak{a}+(X_4^2)]$, where $\mathfrak{a}$ denotes the ideal of S generated by the 2×2 minors of $\mathbb{M}$. Let x_i denote the image of X_i in R for each $i=1,2,3,4$. Then, (x_1,x_2,x_3) and (x_1,x_4) are Ulrich ideals of R with different numbers of generators, and they are incomparable with respect to inclusion.

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. (1) This is a direct consequence of Corollary 2.16.

(2) By Theorem 2.14, we may assume that $n>2$ and that our assertion holds true for $n-1$. Therefore, there exist elements $a_1,a_2,\dots,a_{n-1}\in\mathfrak{m}$ such that $(a_1a_2\cdots a_{n-1})$ is a reduction of I_{n-1} and $I_i=(a_1a_2\cdots a_i)+I_{n-1}$ for all $1\le i\le n-2$. Now apply Theorem 2.14 to the chain $I_n\subsetneq I_{n-1}$. We then have $I_{n-1}=(a_1a_2\cdots a_{n-1})+I_n$ together with one more element $a_n\in\mathfrak{m}$ so that $(a_1a_2\cdots a_{n-1}){\cdot}a_nA=I_n$. Hence

$$I_i=(a_1a_2\cdots a_i)+I_{n-1}=(a_1a_2\cdots a_i)+I_n$$

for all $1\le i\le n-1$. $\square$

In order to prove Theorem 2.4, we need more preliminaries. Let us begin with the following.

Theorem 2.18. *Suppose that $I,J\in\mathcal{X}_R$ and $I\subsetneq J$. Let $b\in J$ such that $J^2=bJ$ and $B=J:J$. Then the following assertions hold true.*

(1) $B = R + \frac{I}{b}$ *and* $\frac{I}{b} = I : J$.

(2) *B is a Cohen-Macaulay local ring with* $\dim B = 1$ *and* $\mathfrak{n} = \mathfrak{m} + \frac{I}{b}$ *the maximal ideal. Hence* $R/\mathfrak{m} \cong B/\mathfrak{n}$.

(3) $\frac{I}{b} \in \mathcal{X}_B$ *and* $\mu_B(\frac{I}{b}) = \mu_R(I)$.

(4) $\mathrm{r}(B) = \mathrm{r}(R)$ *and* $\mathrm{e}(B) = \mathrm{e}(R)$. *Therefore,* $\mathrm{v}(B) = \mathrm{e}(B)$ *if and only if* $\mathrm{v}(R) = \mathrm{e}(R)$.

Proof. We set $A = I : I$. Hence $R \subsetneq B \subsetneq A$ by Proposition 2.5. Let $t = \mu_R(I) - 1$.

(1) Because $J = (b) + I$ and $B = \frac{J}{b}$, we get $B = R + \frac{I}{b}$. We have $I : J \subseteq \frac{I}{b}$, since $b \in J$. Therefore, $\frac{I}{b} = I : J$, because

$$J \cdot \frac{I}{b} = I \cdot \frac{J}{b} = IB \subseteq IA = I.$$

(2) It suffices to show that B is a local ring with maximal ideal $\mathfrak{n} = \mathfrak{m} + \frac{I}{b}$. Let $\mathfrak{a} = \frac{I}{b}$. Choose $c \in \mathfrak{m}$ so that $I = bcA$. We then have $\mathfrak{a} = cA \subseteq \mathfrak{m}A \subseteq \mathrm{J}(A)$, where $\mathrm{J}(A)$ denotes the Jacobson radical of A. Therefore, $\mathfrak{n} = \mathfrak{m} + cA$ is an ideal of $B = R + cA$, and $\mathfrak{n} \subseteq \mathrm{J}(B)$, because A is a finite extension of B. On the other hand, because $R/\mathfrak{m} \cong B/\mathfrak{n}$, $\mathfrak{n}$ is a maximal ideal of B, so that $(B, \mathfrak{n})$ is a local ring.

(3) We have $\mathfrak{a}^2 = c\mathfrak{a}$, since $\mathfrak{a} = cA$. Notice that $\mathfrak{a} \neq cB$, since $A \neq B$. Then, because $\mathfrak{a}/cB \cong A/B \cong (B/\mathfrak{a})^{\oplus t}$ by Lemma 2.10 (1), we get $\mathfrak{a} \in \mathcal{X}_B$ and $\mu_B(\mathfrak{a}) = t + 1 = \mu_R(I)$.

(4) We set $L = (c) + I$. Then, since $bcA = I$, $L \in \mathcal{X}_R$ and $\mu_R(L) = \mu_R(I) = t + 1$ by Corollary 2.16 and Theorem 2.14 (2). Therefore, $\mathrm{r}(R) = t{\cdot}\mathrm{r}(R/L)$ by [29, Theorem 2.5], while $\mathrm{r}(B) = t{\cdot}\mathrm{r}(B/\mathfrak{a})$ for the same reason, because $\mathfrak{a} \in \mathcal{X}_B$ by Assertion (3). Remember that the element c is chosen so that $I :_R J = (c) + I$ (see the proof of Theorem 2.13 (1)). We then have $\mathrm{r}(B/\mathfrak{a}) = \mathrm{r}(R/[I :_R J])$, because $B = R + \mathfrak{a}$ and

$$R/L = R/[\mathfrak{a} \cap R] \cong B/\mathfrak{a}$$

where the first equality follows from Lemma 2.10 (2). Thus

$$\mathrm{r}(B) = t{\cdot}\mathrm{r}(B/\mathfrak{a}) = t{\cdot}\mathrm{r}(R/L) = \mathrm{r}(R),$$

as is claimed. To see the equality $\mathrm{e}(B) = \mathrm{e}(R)$, enlarging the residue class field of R, we may assume that $R/\mathfrak{m}$ is infinite. Choose an element $\alpha \in \mathfrak{m}$ so that (α) is a reduction of

$\mathfrak{m}$. Hence αB is a reduction of $\mathfrak{m}B$, while $\mathfrak{m}B$ is a reduction of $\mathfrak{n}$, because

$$\mathfrak{n}A = (\mathfrak{m} + cA)A = \mathfrak{m}A = (\mathfrak{m}B)A.$$

Therefore, αB is a reduction of $\mathfrak{n}$, so that

$$\mathrm{e}(B) = \ell_B(B/\alpha B) = \ell_R(B/\alpha B) = \mathrm{e}^0_{\alpha R}(B) = \mathrm{e}^0_{\alpha R}(R) = \mathrm{e}(R),$$

where the second equality follows from the fact that $R/\mathfrak{m} \cong B/\mathfrak{n}$ and the fourth equality follows from the fact that $\ell_R(B/R) < \infty$. Hence $\mathrm{e}(B) = \mathrm{e}(R)$ and $\mathrm{r}(B) = \mathrm{r}(R)$. Because $\mathrm{v}(R) = \mathrm{e}(R) > 1$ if and only if $\mathrm{r}(R) = \mathrm{e}(R) - 1$, the assertion that $\mathrm{v}(B) = \mathrm{e}(B)$ if and only if $\mathrm{v}(R) = \mathrm{e}(R)$ now follows. $\square$

We need one more lemma.

Lemma 2.19. *Suppose that $I, J \in \mathcal{X}_R$ and $I \subsetneq J$. Let $\alpha \in J$. Then $J = (\alpha) + I$ if and only if $J^2 = \alpha J$.*

Proof. It suffices to show the *only if* part. Suppose $J = (\alpha) + I$. We set $A = I : I$, $B = J : J$, and choose $b \in J$ so that $J^2 = bJ$. Then $J = bB$ and $B \subseteq A$, whence $JA = bA$, while $JA = [(\alpha) + I]A = \alpha A + I$. We now choose $c \in \mathfrak{m}$ so that $I = bcA$ (see Theorem 2.14 (3)). We then have $bA = JA = \alpha A + bcA$, whence $bA = \alpha A$ by Nakayama's lemma. Therefore, $JA = \alpha A$, whence (α) is a reduction of J, so that $J^2 = \alpha J$. $\square$

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. Let $I, J \in \mathcal{X}_R$ such that $I \subsetneq J$. We set $A = I : I$ and $B = J : J$. Let $b \in J$ such that $J = (b) + I$. Then $J^2 = bJ$ by Lemma 2.19 and B is a local ring with $\mathfrak{n} = \mathfrak{m} + \dfrac{I}{b}$ the maximal ideal by Theorem 2.18.

Let $\mathfrak{a} \in \mathcal{X}_R$ such that $I \subseteq \mathfrak{a} \subsetneq J$. First of all, let us check the following.

Claim 2. *$\dfrac{\mathfrak{a}}{b} \in \mathcal{X}_B$ and $\dfrac{\mathfrak{a}}{b} = \mathfrak{a} : J$.*

Proof of Claim 2. Since $b \in J$, $\mathfrak{a} : J \subseteq \dfrac{\mathfrak{a}}{b}$. On the other hand, since

$$B = R : J \subseteq R : \mathfrak{a} = \mathfrak{a} : \mathfrak{a}$$

by Lemma 2.5, we get

$$J \cdot \frac{\mathfrak{a}}{b} = \mathfrak{a} \cdot \frac{J}{b} = \mathfrak{a}B \subseteq \mathfrak{a}{\cdot}(\mathfrak{a} : \mathfrak{a}) = \mathfrak{a},$$

so that $\frac{\mathfrak{a}}{b}$ is an ideal of $B = \frac{J}{b}$ and $\mathfrak{a} : J = \frac{\mathfrak{a}}{b}$. Since $\frac{I}{b} \in \mathcal{X}_B$ by Theorem 2.18 (3), to show that $\frac{\mathfrak{a}}{b} \in \mathcal{X}_B$, we may assume $I \subsetneq \mathfrak{a}$. We then have, by Theorem 2.3 (2), elements $a_1, a_2 \in \mathfrak{m}$ such that $I = ba_1a_2A$ and $\mathfrak{a} = (ba_1) + I$; hence $\frac{\mathfrak{a}}{b} = a_1R + \frac{I}{b}$. We get $\frac{\mathfrak{a}}{b} = a_1B + \frac{I}{b}$, since $\frac{\mathfrak{a}}{b}$ is an ideal of B. Therefore, $\frac{\mathfrak{a}}{b} \in \mathcal{X}_B$ by Corollary 2.16, because a_1a_2B is a reduction of $\frac{I}{b} = a_1a_2A$. $\square$

We now have the correspondence φ defined by $\mathfrak{a} \mapsto \frac{\mathfrak{a}}{b}$, and it is certainly injective. Suppose that $\mathfrak{b} \in \mathcal{X}_B$ and $\frac{I}{b} \subsetneq \mathfrak{b}$. We take $\alpha \in \mathfrak{b}$ so that $\mathfrak{b}^2 = \alpha\mathfrak{b}$. Then, since B is a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m} + \frac{I}{b}$, we have $\mathfrak{b} = \alpha B + \frac{I}{b}$ by Theorem 2.14. Let us write $\alpha = a + x$ with $a \in \mathfrak{m}$ and $x \in \frac{I}{b}$. We then have $\mathfrak{b} = aB + \frac{I}{b}$, so that $\mathfrak{b}^2 = a\mathfrak{b}$ by Lemma 2.19. Set $L = \frac{I}{b}$. Then, since $A = I : I = L : L$, by Theorem 2.14 we have an element $\beta \in \mathfrak{n} = \mathfrak{m} + L$ such that $L = a\beta A$; hence $a\beta \in L$. Let us write $\beta = c + y$ with $c \in \mathfrak{m}$ and $y \in L$. We then have $ac = a\beta - ay \in L$ and $yA \subseteq L$, so that because

$$L = a\beta A \subseteq acA + a{\cdot}yA \subseteq acA + \mathfrak{m}L,$$

we get $L = acA$ by Nakayama's lemma. Therefore, $I = abcA$. On the other hand, since $aB = aR + a{\cdot}\frac{I}{b}$, we get $\mathfrak{b} = aB + \frac{I}{b} = aR + \frac{I}{b}$. Hence, because $b\mathfrak{b} = (ab) + I$ and $I = (ab)cA$, we finally have that $b\mathfrak{b} \in \mathcal{X}_R$ and

$$I = abcA \subsetneq b\mathfrak{b} = (ab) + I \subsetneq J$$

by Theorem 2.3 (1). Thus, the correspondence φ is bijective, which completes the proof of Theorem 2.4. $\square$

2.4 The case where R possesses minimal multiplicity

In this section, we focus our attention on the case where R possesses minimal multiplicity. Throughout, we assume that $\mathrm{v}(R) = \mathrm{e}(R) > 1$. Hence, $\mathfrak{m} \in \mathcal{X}_R$ and $\mu_R(I) = v$ for all $I \in \mathcal{X}_R$ by Corollary 2.15, where $v = \mathrm{v}(R)$. We choose an element $\alpha \in \mathfrak{m}$ so that $\mathfrak{m}^2 = \alpha\mathfrak{m}$.

Let $I, J \in \mathcal{X}_R$ such that $I \subsetneq J$ and assume that there are no Ulrich ideals contained strictly between I and J. Let $b \in J$ with $J^2 = bJ$ and set $B = J : J$. Hence $B = \frac{J}{b}$, and $J = (b) + I$ by Theorem 2.14. Remember that by Theorem 2.18, B is a local ring and

$\mathrm{v}(B) = \mathrm{e}(B) = \mathrm{e}(R) > 1$. We have $\mathfrak{n}^2 = \alpha\mathfrak{n}$ by the proof of Theorem 2.18 (4), where $\mathfrak{n}$ denotes the maximal ideal of B.

We furthermore have the following.

Lemma 2.20. *The following assertions hold true.*

(1) $\ell_R(J/I) = 1$.

(2) $I = b\mathfrak{n} = J\mathfrak{n}$. *Hence, the ideal I is uniquely determined by J, and $I : I = \mathfrak{n} : \mathfrak{n}$.*

(3) $(b\alpha)$ *is a reduction of I. If $I = (b\alpha) + (x_2, x_3, \ldots, x_v)$, then $J = (b, x_2, x_3, \ldots, x_v)$.*

Proof. By Theorem 2.4, we have the one-to-one correspondence

$$\{\mathfrak{a} \in \mathcal{X}_R \mid I \subseteq \mathfrak{a} \subsetneq J\} \xrightarrow{\varphi} \{\mathfrak{b} \in \mathcal{X}_B \mid \frac{I}{b} \subseteq \mathfrak{b}\}, \quad \mathfrak{a} \mapsto \frac{\mathfrak{a}}{b},$$

where the set of the left hand side is a singleton consisting of I, and the set of the right hand side contains $\mathfrak{n}$. Hence $\mathfrak{n} = \frac{I}{b}$, that is $I = b\mathfrak{n} = J\mathfrak{n}$, because $J = bB$. Therefore, $I^2 = b^2\mathfrak{n}^2 = b\alpha{\cdot}b\mathfrak{n} = b\alpha{\cdot}I$, so that $(b\alpha)$ is a reduction of I. Because

$$J/I = bB/b\mathfrak{n} \cong B/\mathfrak{n}$$

and $R/\mathfrak{m} \cong B/\mathfrak{n}$ by Theorem 2.18 (2), we get $\ell_R(J/I) = 1$. Assertion (3) is clear, since $J = (b) + I$. $\square$

Let I, J be ideals of R such that $I \subsetneq J$ and $\ell_R(J/I) < \infty$. Then we say that a chain $I = I_\ell \subsetneq I_{\ell-1} \subsetneq \ldots \subsetneq I_1 = J$ of ideals in R is a composition series which connects I with J, if $\ell_R(I_i/I_{i+1}) = 1$ for all $1 \le i \le \ell - 1$, where $\ell = \ell_R(J/I) + 1$. With this terminology, since $\ell_R(R/I) < \infty$ for all $I \in \mathcal{X}_R$, we have the following.

Corollary 2.21. *Suppose that $I, J \in \mathcal{X}_R$ and $I \subsetneq J$. Then there exists a composition series $I = I_\ell \subsetneq I_{\ell-1} \subsetneq \cdots \subsetneq I_1 = J$ connecting I with J such that $I_i \in \mathcal{X}_R$ for all $1 \le i \le \ell$.*

The following is the heart of this section.

Theorem 2.22. *The set $\mathcal{X}_R$ is totally ordered with respect to inclusion.*

Proof. Suppose that there exist $I, J \in \mathcal{X}_R$ such that $I \nsubseteq J$ and $J \nsubseteq I$. Since $I \subsetneq \mathfrak{m}$ and $J \subsetneq \mathfrak{m}$, thanks to Corollary 2.21, we get composition series

$$I = I_\ell \subsetneq I_{\ell-1} \subsetneq \cdots \subsetneq I_1 = \mathfrak{m} \quad \text{and} \quad J = J_n \subsetneq J_{n-1} \subsetneq \cdots \subsetneq J_1 = \mathfrak{m}$$

connecting I with $\mathfrak{m}$ and J with $\mathfrak{m}$, respectively, such that $I_i, J_j \in \mathcal{X}_R$ for all $1 \le i \le \ell$ and $1 \le j \le n$. We may assume $\ell \le n$. Then Lemma 2.20 (2) shows that $I_i = J_i$ for all $1 \le i \le \ell$, whence $J \subseteq J_\ell = I_\ell \subseteq I$. This is a contradiction. $\square$

Remark 2.23. Theorem 2.22 is no longer true, unless R possesses minimal multiplicity. For example, let k be a field and consider $R = k[[t^3, t^7]]$ in the formal power series ring $k[[t]]$. Then, $\mathcal{X}_R = \{(t^6 - ct^7, t^{10}) \mid 0 \ne c \in k\}$, which is not totally ordered, if $\sharp k > 2$. See Example 2.37 (3) also.

Let us now summarize the results in the case where R possesses minimal multiplicity.

Theorem 2.24. *Let $I \in \mathcal{X}_R$ and take a composition series*

$$(E) \quad I = I_\ell \subsetneq I_{\ell-1} \subsetneq \cdots \subsetneq I_1 = \mathfrak{m}$$

connecting I with $\mathfrak{m}$ such that $I_i \in \mathcal{X}_R$ for every $1 \le i \le \ell = \ell_R(R/I)$. We set $B_0 = R$ and $B_i = I_i : I_i$ for $1 \le i \le \ell$ and let $\mathfrak{n}_i = \mathrm{J}(B_i)$ denote the Jacobson radical of B_i for each $0 \le i \le \ell$. Then we obtain a tower

$$R = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_{\ell-1} \subsetneq B_\ell \subseteq \overline{R}$$

of birational finite extensions of R and furthermore have the following.

(1) *(α^i) is a reduction of I_i for every $1 \le i \le \ell$.*

(2) *$B_i = \mathfrak{n}_{i-1} : \mathfrak{n}_{i-1}$ for every $1 \le i \le \ell$.*

(3) *For $0 \le i \le \ell - 1$, $(B_i, \mathfrak{n}_i)$ is a local ring with $\mathrm{v}(B_i) = \mathrm{e}(B_i) = \mathrm{e}(R) > 1$ and $\mathfrak{n}_i^2 = \alpha \mathfrak{n}_i$.*

(4) *Choose $x_2, x_3, \ldots, x_v \in I$ so that $I = (\alpha^\ell, x_2, \ldots, x_v)$. Then $I_i = (\alpha^i, x_2, x_3, \ldots, x_v)$ for every $1 \le i \le \ell$. In particular, $\mathfrak{m} = (\alpha, x_2, x_3, \ldots, x_v)$, so that the series (E) is a unique composition series of ideals in R which connects I with $\mathfrak{m}$.*

(5) *Let J be an ideal of R and assume that $I \subseteq J \subseteq \mathfrak{m}$. Then $J = I_i$ for some $1 \le i \le \ell$.*

Proof. The uniqueness of composition series in Assertion (4) follows from the fact that the maximal ideal $\mathfrak{m}/I$ of R/I is cyclic, and then, Assertion (5) readily follows from the uniqueness. Assertions (1), (2), (3), and the first part of Assertion (4) follow by standard induction on ℓ. $\square$

Corollary 2.25. *Suppose that there exists a minimal element I in $\mathcal{X}_R$. Then $\sharp\mathcal{X}_R = \ell < \infty$ with $\ell = \ell_R(R/I)$.*

Proof. Since $\mathcal{X}_R$ is totally ordered by Theorem 2.22, I is the smallest element in $\mathcal{X}_R$, so that $I \subseteq J$ for all $J \in \mathcal{X}_R$. Therefore, by Theorem 2.24 (5), J is one of the I_i's in the compoosition series $I = I_\ell \subsetneq I_{\ell-1} \subsetneq \cdots \subsetneq I_1 = \mathfrak{m}$. $\square$

Corollary 2.26. *If $\widehat{R}$ is a reduced ring, then $\mathcal{X}_R$ is a finite set.*

Proof. Since by Theorem 2.24 $\ell_R(R/I) \le \ell_R(\overline{R}/R) < \infty$ for every $I \in \mathcal{X}_R$, the set $\mathcal{X}_R$ contains a minimal element, so that $\mathcal{X}_R$ is a finite set. $\square$

Here let us note the following.

Example 2.27. Let $(S, \mathfrak{n})$ be a two-dimensional regular local ring. Let $\mathfrak{n} = (X, Y)$ and consider the ring $A = S/(Y^2)$. Then $\mathrm{v}(A) = \mathrm{e}(A) = 2$ and

$$\mathcal{X}_A = \{(x^n, y) \mid n \ge 1\}$$

where x, y denote the images of X, Y in A, respectively. Hence $\sharp\mathcal{X}_A = \infty$.

Proof. Let $I_n = (x^n, y)$ for each $n \ge 1$. Then $(x^n) \subsetneq I_n$ and $I_n^2 = x^n I_n$. Let $\mathrm{J}(A) = (x, y)$ be the maximal ideal of A. We then have $J(A)^2 = xJ(A)$, whence $\mathrm{v}(A) = \mathrm{e}(A) = 2$. Because $I_n = (x^n) :_A y$, we get $I_n/(x^n) \cong A/I_n$. Therefore, $I_n \in \mathcal{X}_A$ for all $n \ge 1$. To see that $\mathcal{X}_A$ consists of these ideals I_n's, let $I \in \mathcal{X}_A$ and set $\ell = \ell_A(A/I)$. Then $I \subseteq I_\ell$ or $I \supseteq I_\ell$, since $\mathcal{X}_A$ is totally ordered. In any case, $I = I_\ell$, because $\ell_A(A/I_\ell) = \ell$. Hence $\mathcal{X}_A = \{(x^n, y) \mid n \ge 1\}$. $\square$

We close this section with the following. Here, the hypothesis about the existence of a fractional canonical ideal K is equivalent to saying that R contains an $\mathfrak{m}$-primary ideal I such that $I \cong \mathrm{K}_R$ as an R-module and such that I possesses a reduction $Q = (a)$ generated by a single element a of R ([18, Corollary 2.8]). The latter condition is satisfied, once $\mathrm{Q}(\widehat{R})$ is a Gorenstein ring and the field $R/\mathfrak{m}$ is infinite.

Theorem 2.28. *Suppose that there exists a fractional ideal K of R such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as an R-module. Then the following conditions are equivalent.*

(1) $\sharp \mathcal{X}_R = \infty$.

(2) $\mathrm{e}(R) = 2$ *and* $\widehat{R}$ *is not a reduced ring.*

(3) *The ring* $\widehat{R}$ *has the form* $\widehat{R} \cong S/(Y^2)$ *for some regular local ring* $(S, \mathfrak{n})$ *of dimension two with* $Y \in \mathfrak{n} \setminus \mathfrak{n}^2$.

Proof. (1) $\Rightarrow$ (2) The ring $\widehat{R}$ is not reduced by Corollary 2.26. Suppose R is not a Gorenstein ring; hence $R \subsetneq K$ and $\mathrm{e}(R) > 2$. We set $\mathfrak{a} = R : K$. Let $I \in \mathcal{X}_R$. Then, since $\mu_R(I) = v = \mathrm{e}(R) > 2$ by Corollary 2.15, we have $\mathfrak{a} \subseteq I$ by [29, Corollary 2.12], so that $\ell_R(R/I) \le \ell_R(R/\mathfrak{a}) < \infty$. Therefore, the set $\mathcal{X}_R$ contains a minimal element, which is a contradiction.

(3) $\Rightarrow$ (1) See Example 2.27 and use the fact that there is a one-to-one correspondence $I \mapsto I\widehat{R}$ between Ulrich ideals of R and $\widehat{R}$, respectively.

(2) $\Rightarrow$ (3) Since $\mathrm{v}(R) = \mathrm{e}(R) = 2$, the completion $\widehat{R}$ has the form $\widehat{R} = S/I$, where $(S, \mathfrak{n})$ is a two-dimensional regular local ring and $I = (f)$ a principal ideal of S. Notice that $\mathrm{e}(S/(f)) = 2$ and $\sqrt{(f)} \ne (f)$. We then have $(f) = (Y^2)$ for some $Y \in \mathfrak{n} \setminus \mathfrak{n}^2$, because $f \in \mathfrak{n}^2 \setminus \mathfrak{n}^3$. $\square$

Remark 2.29. In Theorem 2.28, the hypothesis on the existence of fractional canonical ideals K is not superfluous. In fact, let V denote a discrete valuation ring and consider the idealization $R = V \ltimes F$ of the free V-module $F = V^{\oplus n}$ $(n \ge 2)$. Let t be a regular parameter of V. Then for each $n \ge 1$, $I_n = (t^n) \times F$ is an Ulrich ideal of R ([24, Example 2.2]). Hence $\mathcal{X}_R$ is infinite, but $\mathrm{v}(R) = \mathrm{e}(R) = n + 1 \ge 3$.

Higher dimensional cases are much wilder. Even though $(R, \mathfrak{m})$ is a two-dimensional Cohen-Macaulay local ring possessing minimal multiplicity, the set $\mathcal{X}_R$ is not necessarily totally ordered. Before closing this section, let us note examples.

Example 2.30. We consider two examples.

(1) Let $S = k[[X_0, X_1, \ldots, X_n]]$ $(n \ge 3)$ be the formal power series ring over a field k. Let $\ell \ge 1$ be an integer and consider the $2 \times n$ matrix

$$\mathbb{M} = \begin{pmatrix} X_1 & X_2 & \cdots & X_n \\ X_0^{\ell} & X_1 & \cdots & X_{n-1} \end{pmatrix}.$$

We set $R = S/\mathbb{I}_2(\mathbb{M})$, where $\mathbb{I}_2(\mathbb{M})$ denotes the ideal of S generated by the 2×2 minors of the matrix $\mathbb{M}$. Then, R is a Cohen-Macaulay local ring of dimension two, possessing minimal multiplicity. For this ring, we have

$$\mathcal{X}_R = \{(x_0^i, x_1, x_2, \dots, x_n) \mid 1 \leq i \leq \ell\},$$

where x_i denotes the image of X_i in R for each $0 \leq i \leq n$. Therefore, the set $\mathcal{X}_R$ is totally ordered with respect to inclusion.

(2) Let $(S, \mathfrak{n})$ be a regular local ring of dimension three. Let $F, G, H, Z \in \mathfrak{n}$ and assume that $\mathfrak{n} = (F, G, Z) = (G, H, Z) = (H, F, Z)$. (For instance, let $S = k[[X, Y, Z]]$ be the formal power series ring over a field k with $\operatorname{ch} k \neq 2$, and choose $F = X, G = X + Y, H = X - Y$.) We consider the ring $R = S/(Z^2 - FGH)$. Then R is a two-dimensional Cohen-Macaulay local ring of minimal multiplicity two. Let f, g, h, z denote, respectively, the images of F, G, H, Z in R. Then, (f, gh, z), (g, fh, z), (h, fg, z) are Ulrich ideals of R, but any two of them are incomparable.

2.5 The case where R is a GGL ring

In this section, we study the case where R is a GGL ring. The notion of GGL rings is given by [16]. Let us briefly review the definition.

Definition 2.31 ([16]). Suppose that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with $d = \dim R \geq 0$, possessing the canonical module K_R. We say that R is a *generalized Gorenstein local* (GGL for short) ring, if one of the following conditions is satisfied.

(1) R is a Gorenstein ring.

(2) R is not a Gorenstein ring, but there exists an exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules and an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R such that

(i) C is an Ulrich R-module with respect to $\mathfrak{a}$ and

(ii) the induced homomorphism $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \to \mathrm{K}_R/\mathfrak{a}\mathrm{K}_R$ is injective.

When Case (2) occurs, we especially say that R is a GGL ring with respect to $\mathfrak{a}$.

Since our attention is focused on the one-dimensional case, here let us summarize a few results on GGL rings of dimension one. Suppose that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension one, admitting a fractional canonical ideal K. Hence, K is an R-submodule of $\overline{R}$ such that $K \cong \mathrm{K}_R$ as an R-module and $R \subseteq K \subseteq \overline{R}$. One can consult [18, Sections 2, 3] and [32, Vortrag 2] for basic properties of K. We set $S = R[K]$ in $\mathrm{Q}(R)$. Therefore, S is a birational finite extension of R with $S = K^n$ for all $n \gg 0$, and the ring $S = R[K]$ is independent of the choice of K ([4, Theorem 2.5]). We set $\mathfrak{c} = R : S$. First of all, let us note the following.

Lemma 2.32 (cf. [18, Lemma 3.5]). *$\mathfrak{c} = K : S$ and $S = \mathfrak{c} : \mathfrak{c} = R : \mathfrak{c}$.*

Proof. Since $R = K : K$ ([32, Bemerkung 2.5 a)]), we have $\mathfrak{c} = (K : K) : S = K : KS = K : S$, while $R : \mathfrak{c} = (K : K) : \mathfrak{c} = K : K\mathfrak{c} = K : \mathfrak{c}$. Hence $R : \mathfrak{c} = K : \mathfrak{c} = K : (K : S) = S$ ([32, Definition 2.4]). Therefore, $\mathfrak{c} : \mathfrak{c} = (K : S) : \mathfrak{c} = K : S\mathfrak{c} = K : \mathfrak{c} = S$. $\square$

We then have the characterization of GGL rings.

Theorem 2.33 ([16]). *Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) *R is a* GGL *ring with respect to some $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R.*

(2) *K/R is a free $R/\mathfrak{c}$-module.*

(3) *S/R is a free $R/\mathfrak{c}$-module.*

When this is the case, one necessarily has $\mathfrak{a} = \mathfrak{c}$, and the following assertions hold true.

(i) *$R/\mathfrak{c}$ is a Gorenstein ring.*

(ii) *$S/R \cong (R/\mathfrak{c})^{\oplus \mathrm{r}(R)}$ as an R-module.*

The following result is due to [16, 29]. Let us include a brief proof of Assertion (1) for the sake of completeness.

Theorem 2.34 ([16, 29]). *Suppose that R is not a Gorenstein ring. Let $I \in \mathcal{X}_R$. Then the following assertions hold true.*

(1) *If $I \subseteq \mathfrak{c}$, then $I = \mathfrak{c}$.*

(2) *If* $\mu_R(I) \neq 2$, *then* $\mathfrak{c} \subseteq I$.

(3) $\mathfrak{c} \in \mathcal{X}_R$ *if and only if* R *is a* GGL *ring and* S *is a Gorenstein ring.*

Proof. (1) Let $I \in \mathcal{X}_R$ and assume that $I \subseteq \mathfrak{c}$. We choose an element $a \in I$ so that $I^2 = aI$. We then have $I \neq (a)$ and I/I^2 is a free R/I-module. Let $A = I : I$; hence $I = aA$. On the other hand, because $\mathfrak{c} \subseteq I$, by Lemmata 2.5 and 2.32 we have

$$A = R : I \supseteq R : \mathfrak{c} = S \supseteq K.$$

Claim 3. *A is a Gorenstein ring and A/K is the canonical module of R/I.*

Proof of Claim 3. Taking the K-dual of the canonical exact sequence $0 \to I \to R \to R/I \to 0$, we get the exact sequence

$$0 \to K \xrightarrow{\iota} K : I \to \mathrm{Ext}_R^1(R/I, K) \to 0,$$

where $\iota : K \to K : I$ denotes the embedding. On the other hand, $K : I = A$, because

$$I = R : A = (K : K) : A = K : KA = K : A$$

(remember that $K \subseteq A$). Therefore, since $I = K : A$ is a canonical ideal of A ([32, Korollar 5.14]) and $I = aA \cong A$, A is a Gorenstein ring, and $A/K \cong \mathrm{Ext}_R^1(R/I, K)$. $\square$

We consider the exact sequence $0 \to (a)/aI \to I/aI \to I/(a) \to 0$ of R/I-modules. Then, because $I = aA$, we get the canonical isomorphism between the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/I & \xrightarrow{i} & A/I & \longrightarrow & A/R & \longrightarrow & 0 \\ & & \downarrow \wr & \circlearrowleft & \downarrow \wr & \circlearrowleft & \downarrow \wr & & \\ 0 & \longrightarrow & (a)/aI & \xrightarrow{i} & I/aI & \longrightarrow & I/(a) & \longrightarrow & 0 \end{array}$$

of R/I-modules, where A/I is a Gorenstein ring, since A is a Gorenstein ring and $I = aA$. Therefore, since A/I ($\cong I/aI$) is a flat extension of R/I, R/I is a Gorenstein ring, so that $A/K \cong R/I$ by Claim 3. Consequently, the exact sequence

$$0 \to K/R \to A/R \to A/K \to 0$$

of R/I-modules is split, whence K/R is a non-zero free R/I-module, because so is A/R ($\cong I/(a)$). Hence, $\mathfrak{c} = R : S \subseteq R : K = R :_R K = I$, so that $I = \mathfrak{c}$. $\square$

Thanks to Theorem 2.34, we get the following.

Theorem 2.35. *Let R be a* GGL *ring and assume that R is not a Gorenstein ring. Then the following assertions hold true.*

(1) $\{I \in \mathcal{X}_R| \ \mathfrak{c} \subsetneq I\} = \{(a) + \mathfrak{c} \ \mid \ a \in \mathfrak{m}$ *such that* $\mathfrak{c} = abS$ *for some* $b \in \mathfrak{m}\}$.
In particular, $\mathfrak{c} \in \mathcal{X}_R$*, once the set* $\{I \in \mathcal{X}_R| \ \mathfrak{c} \subsetneq I\}$ *is non-empty.*

(2) $\mu_R(I) = \mathrm{r}(R) + 1$ *for all* $I \in \mathcal{X}_R$ *such that* $\mathfrak{c} \subseteq I$.

(3) $\{I \in \mathcal{X}_R| \ \mathfrak{c} \subseteq I\} = \{I \in \mathcal{X}_{\mathcal{R}}| \ \mu_R(I) \neq 2\}$.

Therefore, if R possesses minimal multiplicity, then the set $\mathcal{X}_R$ is totally ordered, and $\mathfrak{c}$ is the smallest element of $\mathcal{X}_R$.

Proof. (1) Let us show the first equality. First of all, assume that $\mathfrak{c} \in \mathcal{X}_R$. Then since $S = \mathfrak{c} : \mathfrak{c}$, for each $\alpha \in \mathfrak{c}$, (α) is a reduction of $\mathfrak{c}$ if and only if $\mathfrak{c} = \alpha S$, so that the required equality follows from Corollary 2.16. Assume that $\mathfrak{c} \notin \mathcal{X}_R$. Hence, by Theorem 2.34 (3), S is not a Gorenstein ring, because R is a GGL ring. Therefore, since $\mathfrak{c} = K : S$ is a canonical module of S (Lemma 2.32 and [32, Korollar 5.14]), we have $\mathfrak{c} \neq \alpha S$ for any $\alpha \in \mathfrak{c}$, whence the set $\{(a) + \mathfrak{c} \mid a \in \mathfrak{m}$ such that $abS = \mathfrak{c}$ for some $b \in \mathfrak{m}\}$ is empty. On the other hand, since $S = \mathfrak{c} : \mathfrak{c} = R : \mathfrak{c}$ and $S/R \cong (R/\mathfrak{c})^{\oplus \mathrm{r}(R)}$ (see Theorem 2.33 (ii)), the $\mathfrak{m}$-primary ideal $\mathfrak{c}$ of R satisfies Condition (C) in Definition 2.6. Therefore, if the set $\{I \in \mathcal{X}_R \mid \mathfrak{c} \subsetneq I\}$ is non-empty, then $\mathfrak{c} \in \mathcal{X}_R$ by Theorem 2.13 (2), because $\mathrm{r}(R) \geq 2$. Thus, $\{I \in \mathcal{X}_R \mid \mathfrak{c} \subsetneq I\} = \emptyset$.

(2) By Assertion (1), we may assume $\mathfrak{c} \in \mathcal{X}_R$. Then, $\mathfrak{c} = \alpha S$ for some $\alpha \in \mathfrak{c}$, and therefore, $\mu_R(\mathfrak{c}) = \mathrm{r}(R) + 1$, since $\mathfrak{c}/(\alpha) \cong S/R \cong (R/\mathfrak{c})^{\oplus \mathrm{r}(R)}$. Thus, by Theorem 2.14, $\mu_R(I) = \mu_R(\mathfrak{c}) = \mathrm{r}(R) + 1$ for every $I \in \mathcal{X}_R$ with $\mathfrak{c} \subseteq I$.

(3) The assertion follows from Assertion (2) and Theorem 2.34 (3).

The last assertion follows from Assertion (3), since $\mu_R(I) = \mathrm{v}(R) > 2$ for every $I \in \mathcal{X}_R$ (see Corollary 2.15). □

Combining Theorems 2.3 and 2.35, we have the following.

Corollary 2.36. *Let R be a* GGL *ring and assume that R is not a Gorenstein ring. Then the following assertions hold true.*

(1) *Let* $a_1, a_2, \dots, a_n, b \in \mathfrak{m}$ $(n \geq 1)$ *and assume that* $\mathfrak{c} = a_1a_2 \cdots a_nbS$. *We set* $I_i = (a_1a_2 \cdots a_i) + \mathfrak{c}$ *for each* $1 \leq i \leq n$. *Then* $\mathfrak{c} \in \mathcal{X}_R$ *and* $I_i \in \mathcal{X}_R$ *for all* $1 \leq i \leq n$, *forming a chain* $\mathfrak{c} \subsetneq I_n \subsetneq I_{n-1} \subsetneq \dots \subsetneq I_1$ *in* $\mathcal{X}_R$.

(2) *Conversely, let* $I_1, I_2, \dots, I_n \in \mathcal{X}_R$ $(n \geq 1)$ *and assume that* $\mathfrak{c} \subsetneq I_n \subsetneq I_{n-1} \subsetneq \dots \subsetneq I_1$. *Then* $\mathfrak{c} \in \mathcal{X}_R$ *and there exist elements* $a_1, a_2, \dots, a_n, b \in \mathfrak{m}$ *such that* $\mathfrak{c} = a_1a_2 \cdots a_nbS$ *and* $I_i = (a_1a_2 \cdots a_i) + \mathfrak{c}$ *for all* $1 \leq i \leq n$.

Concluding this chapter, let us note a few examples of GGL rings.

Example 2.37. Let $k[[t]]$ be the formal power series ring over a field k.

(1) Let $H = \langle 5, 7, 9, 13 \rangle$ denote the numerical semigroup generated by $5, 7, 9, 13$ and $R = k[[t^5, t^7, t^9, t^{13}]]$ the semigroup ring of H over k. Then, R is a GGL ring, possessing $S = k[[t^3, t^5, t^7]]$ and $\mathfrak{c} = (t^7, t^9, t^{10}, t^{13})$. For this ring R, S is not a Gorenstein ring, and $\mathcal{X}_R = \emptyset$.

(2) Let $R = k[[t^4, t^9, t^{15}]]$. Then, R is a GGL ring, possessing $S = k[[t^3, t^4]]$ and $\mathfrak{c} = (t^9, t^{12}, t^{15}) = t^9S$. For this ring R, $\mathcal{X}_R = \{\mathfrak{c}\}$.

(3) Let $R = k[[t^6, t^{13}, t^{28}]]$. Then, R is a GGL ring, possessing $S = k[[t^2, t^{13}]]$ and $\mathfrak{c} = (t^{24}, t^{26}, t^{28}) = t^{24}S$. For this ring R, the set $\{I \in \mathcal{X}_R \mid \mathfrak{c} \subsetneq I\}$ consists of the following families.

    (i) $\{(t^6 + at^{13}) + \mathfrak{c} \mid a \in k\}$,
    (ii) $\{(t^{12} + at^{13} + bt^{19}) + \mathfrak{c} \mid a, b \in k\}$, and
    (iii) $\{(t^{18} + at^{25}) + \mathfrak{c} \mid a \in k\}$.

For each $a \in k$, we have a maximal chain

$$\mathfrak{c} \subsetneq (t^{18} + at^{25}) + \mathfrak{c} \subsetneq (t^{12} + at^{19}) + \mathfrak{c} \subsetneq (t^6 + at^{13}) + \mathfrak{c}$$

in $\mathcal{X}_R$. On the other hand, for $a, b \in k$ such that $a \neq 0$,

$$\mathfrak{c} \subsetneq (t^{12} + at^{13} + bt^{19}) + \mathfrak{c}$$

is also a maximal chain in $\mathcal{X}_R$.

(4) Let $H = \langle 6, 13, 28 \rangle$. Choose integers $0 < \alpha \in H$ and $1 < \beta \in \mathbb{Z}$ so that $\alpha \notin \{6, 13, 28\}$ and $\mathrm{GCD}(\alpha, \beta) = 1$. We consider $R = k[[t^{\alpha}, t^{6\beta}, t^{13\beta}, t^{28\beta}]]$. Then, R is a GGL ring with $\mathrm{v}(R) = 4$ and $\mathrm{r}(R) = 2$. For this ring R, $S = k[[t^{\alpha}, t^{2\beta}, t^{13\beta}]]$, and $\mathfrak{c} = t^{24\beta}S$. For instance, take $\alpha = 12$ and $\beta = 5n$, where $n > 0$ and $\mathrm{GCD}(2, n) = \mathrm{GCD}(3, n) = 1$. Then, $\mathfrak{c} = t^{120n}S = (t^{12})^{10n}S$, so that the set $\{I \in \mathcal{X}_R \mid \mathfrak{c} \subsetneq I\}$ seems rather wild, containing chains of large length.

3 Ulrich ideals and 2-AGL rings

3.1 Introduction

The series [4, 9, 18, 19, 20, 21, 22, 26, 28, 29] of researches are motivated and supported by the strong desire to stratify Cohen-Macaulay rings, finding new and interesting classes which naturally include that of Gorenstein rings. As is already pointed out by these works, the class of *almost Gorenstein local rings* (AGL rings for short) could be a very nice candidate for such classes. The prototype of AGL rings is found in the work [2] of V. Barucci and R. Fröberg in 1997, where they introduced the notion of AGL ring for one-dimensional analytically unramified local rings, developing a beautiful theory on numerical semigroups. In 2013, S. Goto, N. Matsuoka, and T. T. Phuong [18] extended the notion of AGL ring given by [2] to arbitrary one-dimensional Cohen-Macaulay local rings, by means of the first Hilbert coefficients of canonical ideals. They broadly opened up the theory in dimension one, which prepared for the higher dimensional notion of AGL ring provided in 2015 by [28]. Subsequently in 2017, T. D. M. Chau, S. Goto, S. Kumashiro, and N. Matsuoka [4] defined the notion of 2-AGL ring as a possible successor of AGL rings of dimension one. To explain the motivations for the present researches, we need to remind the reader of 2-AGL rings more precisely.

Throughout, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$, possessing the canonical module K_R. We say that an ideal I in R is a *canonical ideal* of R, if $I \neq R$, and $I \cong \mathrm{K}_R$ as an R-module. In what follows, we assume that the ring R possesses a canonical ideal, which contains a parameter ideal $Q = (a)$ of R as a reduction. This assumption is automatically satisfied if R has an infinite residue class field. Let $\mathcal{T} = \mathcal{R}(Q) = R[Qt]$ and $\mathcal{R} = \mathcal{R}(I) = R[It]$ be the Rees algebras of Q and I respectively, where t denotes an indeterminate. We set $\mathcal{S}_Q(I) = I\mathcal{R}/I\mathcal{T}$ and call it the *Sally module* of I with respect to Q ([48]). Let $\mathrm{e}_i(I)$ $(i = 0, 1)$ be the i-th Hilbert coefficients of R with respect to I, that is, the integers satisfy the equality

$$\ell_R(R/I^{n+1}) = \mathrm{e}_0(I)\binom{n+1}{1} - \mathrm{e}_1(I) \quad \text{for all} \quad n \gg 0$$

where $\ell_R(M)$ denotes, for each R-module M, the length of M. We set $\operatorname{rank} \mathcal{S}_Q(I) = \ell_{\mathcal{T}_\mathfrak{p}}([\mathcal{S}_Q(I)]_\mathfrak{p})$ which is called the *rank* of $\mathcal{S}_Q(I)$, where $\mathfrak{p} = \mathfrak{m}\mathcal{T}$. We then have

$$\operatorname{rank} \mathcal{S}_Q(I) = \mathrm{e}_1(I) - [\mathrm{e}_0(I) - \ell_R(R/I)]$$

([23, Proposition 2.2 (3)]). Note that rank $\mathcal{S}_Q(I)$ is an invariant of R, independent of the choice of canonical ideals I and the reductions Q of I (see [4, Theorem 2.5]). With this notation we have the following.

Definition 3.1. ([4, Definition 1.3]) We say that R is a 2-*almost Gorenstein local ring* (2-AGL ring for short), if rank $\mathcal{S}_Q(I) = 2$, that is, $\mathrm{e}_1(I) = \mathrm{e}_0(I) - \ell_R(R/I) + 2$.

Because R is a non-Gorenstein AGL ring if and only if rank $\mathcal{S}_Q(I) = 1$ ([18, Theorem 3.16]), 2-AGL rings could be considered to be one of the successors of AGL rings.

We set $K = a^{-1}I$ in the total ring $\mathrm{Q}(R)$ of fractions of R. Therefore, K is a fractional ideal of R such that $R \subseteq K \subseteq \overline{R}$ (here $\overline{R}$ stands for the integral closure of R in $\mathrm{Q}(R)$) and $K \cong \mathrm{K}_R$, which we call a *canonical fractional ideal* of R. We set $S = R[K]$. Hence, S is a module-finite birational extension of R, and it is independent of the choice of K ([4, Theorem 2.5 (3)]). Let $\mathfrak{c} = R : S$. We are now able to state the characterization of 2-AGL rings given by [4], which we shall often refer to, in the present chapter.

Theorem 3.2 ([4, Theorem 1.4])**.** *The following conditions are equivalent.*

(1) *R is a 2-AGL ring.*

(2) *There is an exact sequence $0 \to \mathcal{B}(-1) \to \mathcal{S}_Q(I) \to \mathcal{B}(-1) \to 0$ of graded $\mathcal{T}$-modules, where $\mathcal{B} = \mathcal{T}/\mathfrak{m}\mathcal{T}$ $(\cong (R/\mathfrak{m})[t])$.*

(3) *$K^2 = K^3$ and $\ell_R(K^2/K) = 2$.*

(4) *$I^3 = QI^2$ and $\ell_R(I^2/QI) = 2$.*

(5) *R is not a Gorenstein ring but $\ell_R(S/[K : \mathfrak{m}]) = 1$.*

(6) *$\ell_R(S/K) = 2$.*

(7) *$\ell_R(R/\mathfrak{c}) = 2$.*

When this is the case, $\mathfrak{m}{\cdot}\mathcal{S}_Q(I) \neq (0)$, whence the exact sequence given by condition (2) is not split, and we have

$$\ell_R(R/I^{n+1}) = \mathrm{e}_0(I)\binom{n+1}{1} - (\mathrm{e}_0(I) - \ell_R(R/I) + 2)$$

for all $n \geq 1$.

As is noted above, the notion of 2-AGL ring could be considered to be one of the successors of the notion of AGL ring. However, if 2-AGL rings claim that they are orthodox successors of AGL rings, it must be proved, showing that they really inherit several distinctive properties which AGL rings usually keep. In the present chapter, to certify the orthodoxy of 2-AGL rings for the further studies, we investigate three topics on 2-AGL rings, which are closely studied already for the case of AGL rings. The first topic concerns minimal presentations of canonical ideals. In Section 3.2, we will give a necessary and sufficient condition for a given one-dimensional Cohen-Macaulay local ring R to be a 2-AGL ring, in terms of minimal presentations of canonical fractional ideals. Our results Theorems 3.5 and 3.12 exactly correspond to those about AGL rings given by [28, Theorem 7.8].

In Section 3.3, we investigate a generalization of so called amalgamated duplications of R ([5]), including certain fiber products, and prove that R is a 2-AGL ring if and only if so is the fiber product $R \times_{R/\mathfrak{c}} R$. By [4, Theorem 4.2] R is a 2-AGL ring if and only if so is the trivial extension $R \ltimes \mathfrak{c}$ of $\mathfrak{c}$ over R, which corresponds to [18, Theotem 6.5] for the case of AGL rings.

In Sections 3.4 and 3.5, we are interested in Ulrich ideals in 2-AGL rings. The existence of two-generated Ulrich ideals is basically a substantially strong condition for R, which we closely discuss in Section 3.4, especially in the case where R is a 2-AGL ring. Here, we should not rush, but should explain about what are Ulrich ideals. The notion of Ulrich ideal/module dates back to the work [24] in 2014, where the authors introduced the notion, generalizing that of MGMCM modules (maximally generated maximal Cohen-Macaulay modules) ([3]), and started the basic theory. The maximal ideal of a Cohen-Macaulay local ring with minimal multiplicity is a typical example of Ulrich ideals, and the higher syzygy modules of Ulrich ideals are Ulrich modules. In [24, 25], all the Ulrich ideals of Gorenstein local rings of finite CM-representation type and of dimension at most 2 are determined, by means of the classification in the representation theory. On the other hand, in [29], the first author, R. Takahashi, and the third author studied the structure of the complex $\mathbf{R}\mathrm{Hom}_R(R/I, R)$ for Ulrich ideals I in a Cohen-Macaulay local ring R of arbitrary dimension, and proved that in a one-dimensional non-Gorenstein AGL ring $(R, \mathfrak{m})$, the only possible Ulrich ideal is the maximal ideal $\mathfrak{m}$ ([29, Theorem 2.14 (1)]). In Section 3.5, we study the natural question of how and what happens about 2-AGL rings. To state our conclusion, let $\mathcal{X}_R$ denote the set of Ulrich ideals in R. We then have the following, which we will prove in Section 3.5. The assertion exactly corresponds to [29, Theorem 2.14 (1)], the result of the case where R is an AGL ring of dimension one.

Theorem 3.3 (= Corollary 3.38)**.** *Suppose that* $(R, \mathfrak{m})$ *is a* 2-AGL *ring with minimal multiplicity, possessing a canonical fractional ideal* K*. Then*

$$\mathcal{X}_R = \begin{cases} \{\mathfrak{c}, \mathfrak{m}\}, & \textit{if } K/R \textit{ is } R/\mathfrak{c}\textit{-free}, \\ \{\mathfrak{m}\}, & \textit{otherwise.} \end{cases}$$

For one-dimensional Gorenstein local rings R of finite CM-representation type, the list of Ulrich ideals is known by [24]. The proof given by [24] is based on the techniques in the representation theory of maximal Cohen-Macaulay modules. It might have some interests to give a straightforward proof, making use of the results in [28, Section 12] from a different point of view. In Section 3.6 we shall perform it as an appendix.

In what follows, unless otherwise specified, let R be a one-dimensional Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$. For each finitely generated R-module M, let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators of M (resp. the length of M). We denote by K_R the canonical module of R.

3.2 Minimal presentations of canonical ideals in 2-AGL rings

In this section, we explore the structure of minimal presentations of canonical ideals of 2-AGL rings. Before going ahead, we summarize some known results on 2-AGL rings, which we shall often refer to throughout this chapter. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$, admitting the canonical module K_R. We assume that R possesses a canonical fractional ideal K, that is an R-submodule of $\mathrm{Q}(R)$ such that $R \subseteq K \subseteq \overline{R}$, where $\overline{R}$ denotes the integral closure of R in $\mathrm{Q}(R)$, and $K \cong \mathrm{K}_R$ as an R-module. Let $S = R[K]$ and set $\mathfrak{c} = R : S$. We denote by $\mathrm{r}(R) = \ell_R(\mathrm{Ext}_R^1(R/\mathfrak{m}, R))$ the Cohen-Macaulay type of R.

Proposition 3.4 ([4, Proposition 3.3])**.** *Suppose that R is a* 2-AGL *ring with* $r = \mathrm{r}(R)$*. Then the following assertions hold true.*

(1) $\mathfrak{c} = K : S = R : K$.

(2) *There is a minimal system* $x_1, x_2, \ldots, x_n$ *of generators of* $\mathfrak{m}$ *such that* $\mathfrak{c} = (x_1^2) + (x_2, x_3, \ldots, x_n)$.

(3) *$S/K \cong R/\mathfrak{c}$ and $S/R \cong K/R \oplus R/\mathfrak{c}$ as $R/\mathfrak{c}$-modules.*

(4) *$K/R \cong (R/\mathfrak{c})^{\oplus \ell} \oplus (R/\mathfrak{m})^{\oplus m}$ as an $R/\mathfrak{c}$-module for some $\ell > 0$, $m \geq 0$ such that $\ell + m = r - 1$.*

(5) *$\mu_R(S) = r + 1$.*

Therefore, if R is a 2-AGL ring, then $\ell_R(K/R) = 2\ell + m$. Hence, K/R is a free $R/\mathfrak{c}$-module if and only if $\ell_R(K/R) = 2(r-1)$.

Let us now fix the setting of this section. In what follows, we assume that $R = T/\mathfrak{a}$, $\mathfrak{m} = \mathfrak{n}/\mathfrak{a}$, for some regular local ring $(T, \mathfrak{n})$ with $\dim T = n \geq 3$ and an ideal $\mathfrak{a}$ of T such that $\mathfrak{a} \subseteq \mathfrak{n}^2$. Suppose that R is not a Gorenstein ring. For each $a \in T$, let $\overline{a}$ denote the image of a in R.

Firstly, suppose that R is a 2-AGL ring, and write $\mathfrak{c} = (x_1^2) + (x_2, x_3, \ldots, x_n)$ with a minimal system $x_1, x_2, \ldots, x_n$ of generators of $\mathfrak{m}$ (see Proposition 3.4 (2)). We choose $X_i \in \mathfrak{n}$ so that $x_i = \overline{X_i}$ in R, whence $\mathfrak{n} = (X_1, X_2, \ldots, X_n)$. Let $J = (X_1^2) + (X_2, X_3, \ldots, X_n)$. We then have $T/J \cong R/\mathfrak{c}$, since $\ell_T(T/J) = \ell_R(R/\mathfrak{c}) = 2$, so that $\mathfrak{a} \subseteq J$ and $\mathfrak{c} = J/\mathfrak{a}$. On the other hand, by Proposition 3.4 (4) we have

$$K/R \cong (R/\mathfrak{c})^{\oplus \ell} \oplus (R/\mathfrak{m})^{\oplus m}$$

with $\ell > 0, m \geq 0$ such that $\ell + m = \mathrm{r}(R) - 1$. Hence, letting $K = R + \sum_{i=1}^{\ell} Rf_i + \sum_{j=1}^{m} Rg_j$ with $f_i, g_j \in K$, we may assume that

$$\sum_{i=1}^{\ell} (R/\mathfrak{c}) \cdot \overline{f_i} \cong (R/\mathfrak{c})^{\oplus \ell} \quad \text{and} \quad \sum_{j=1}^{m} (R/\mathfrak{c}) \cdot \overline{g_j} \cong (R/\mathfrak{m})^{\oplus m},$$

where $\overline{f_i}, \overline{g_j}$ denote the images of f_i, g_j in K/R. With this notation, we have the following, which corresponds to [28, Theorem 7.8] for AGL rings.

Theorem 3.5. *The T-module K has a minimal free presentation of the form*

$$F_1 \xrightarrow{\mathbb{M}} F_0 \xrightarrow{\mathbb{N}} K \to 0,$$

where the matrices $\mathbb{N}$ and $\mathbb{M}$ are given by

$$\mathbb{N} = \left[\begin{smallmatrix} -1 & f_1 f_2 \cdots f_\ell & g_1 g_2 \cdots g_m \end{smallmatrix} \right]$$

and

$$\mathbb{M} = \left[\begin{array}{ccccccc}
a_{11}a_{12}\cdots a_{1n} & \cdots & a_{\ell 1}a_{\ell 2}\cdots a_{\ell n} & b_{11}b_{12}\cdots b_{1n} & \cdots & b_{m1}b_{m2}\cdots b_{mn} & c_1c_2\cdots c_q \\
X_1^2X_2\cdots X_n & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & X_1^2X_2\cdots X_n & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & X_1X_2\cdots X_n & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & X_1X_2\cdots X_n & 0
\end{array}\right]$$

with $a_{ij} \in J$ $(1 \le i \le \ell,\ 1 \le j \le n)$, $b_{ik} \in J$ $(1 \le i \le m,\ 2 \le k \le n)$, *and* $q \ge 0$. *The matrix* $\mathbb{M}$ *involves the information on a system of generators of* $\mathfrak{a}$*, and we have*

$$\mathfrak{a} = \sum_{i=1}^{\ell} \mathrm{I}_2\left(\begin{smallmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ X_1^2 & X_2 & \cdots & X_n \end{smallmatrix}\right) + \sum_{i=1}^{m} \mathrm{I}_2\left(\begin{smallmatrix} b_{i1} & b_{i2} & \cdots & b_{in} \\ X_1 & X_2 & \cdots & X_n \end{smallmatrix}\right) + (c_1, c_2, \ldots, c_q),$$

where $\mathrm{I}_2(\mathbb{L})$ *denotes, for a* $2 \times n$ *matrix* $\mathbb{L}$ *with entries in* T*, the ideal of* T *generated by* 2×2 *minors of* $\mathbb{L}$*.*

Proof. Let

$$F_1 \xrightarrow{\mathbb{A}} F_0 \xrightarrow{[\,-1\ f_1f_2\cdots f_\ell\ g_1g_2\cdots g_m\,]} K \longrightarrow 0$$

be a part of a minimal T-free resolution of K with $F_0 = T \oplus T^{\oplus \ell} \oplus T^{\oplus m}$, which gives rise to a presentation

$$F_1 \xrightarrow{\mathbb{A}'} G_0 \xrightarrow{\mathbb{N}'} K/R \longrightarrow 0$$

of K/R, where $\mathbb{N}' = [\,\bar{f_1}\bar{f_2}\cdots\bar{f_\ell}\ \bar{g_1}\bar{g_2}\cdots\bar{g_m}\,]$, and $\mathbb{A}'$ is the $(\ell+m) \times s$ matrix obtained from $\mathbb{A}$ by deleting the first row. On the other hand, since $K/R \cong (T/J)^{\oplus \ell} \oplus (T/\mathfrak{n})^{\oplus m}$, the T-module K/R has a minimal presentation of the form

$$G_1 = [T^{\oplus \ell} \oplus T^{\oplus m}]^{\oplus n} = T^{\oplus \ell n} \oplus T^{\oplus mn} \xrightarrow{\mathbb{B}} G_0 = T^{\oplus \ell} \oplus T^{\oplus m} \xrightarrow{\mathbb{N}'} K/R \longrightarrow 0,$$

where the matrix $\mathbb{B}$ is given by

$$\mathbb{B} = \left[\begin{array}{cccccc}
X_1^2X_2\cdots X_n & 0 & 0 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & X_1^2X_2\cdots X_n & \vdots & \vdots & \vdots \\
0 & 0 & 0 & X_1X_2\cdots X_n & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & X_1X_2\cdots X_n
\end{array}\right].$$

Therefore, comparing with two presentations of K/R, we get a commutative diagram

$$
\begin{array}{ccccccc}
G_1 & \xrightarrow{\mathbb{B}} & G_0 & \longrightarrow & K/R & \longrightarrow & 0 \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
F_1 & \xrightarrow{\mathbb{A}'} & G_0 & \longrightarrow & K/R & \longrightarrow & 0 \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\
G_1 & \xrightarrow{\mathbb{B}} & G_0 & \longrightarrow & K/R & \longrightarrow & 0
\end{array}
$$

of T-modules, where $\eta \circ \xi$ is an isomorphism. Hence, $\mathbb{A}' \, Q = (\mathbb{B} \mid O)$ for some $s \times s$ invertible matrix Q with entries in T (here O denotes the null matrix). Setting $\mathbb{M} = \mathbb{A}Q$, we get $\mathbb{M} = \left(\begin{array}{c} * \\ \hline \mathbb{A}' \end{array} \right) Q = \left(\begin{array}{c|c} * & * \\ \hline \mathbb{B} & O \end{array} \right)$, whence a required minimal presentation

$$F_1 \xrightarrow{\mathbb{M}} F_0 \xrightarrow{\mathbb{N}} K \longrightarrow 0$$

of K follows.

Let us prove that $a_{ij}, b_{ij} \in J$. We set $Z_1 = X_1^2$, and $Z_i = X_i$ for each $2 \le i \le n$. Then, $a_{ij} \cdot (-1) + Z_j \cdot f_i = 0$ for every $1 \le i \le \ell$ and $1 \le j \le n$, whence $a_{ij} \in J$, because $\mathfrak{c}K \subseteq \mathfrak{c}S = \mathfrak{c}$ and $\mathfrak{c} = J/\mathfrak{a}$. Since $b_{ij} \cdot (-1) + Z_j \cdot g_i = 0$ for $j \ge 2$, we have $b_{ij} \in J$.

The last assertion about the generating system of the defining ideal $\mathfrak{a}$ of R follows from the fact that $Z_1, Z_2, \ldots, Z_n$ forms a regular sequence on T. We refer to [28, Proof of Theorem 7.8] for details. □

As a consequence of Theorem 3.5, we have the following. It exactly corresponds to [28, Corollary 7.10] for AGL rings.

Corollary 3.6. *With the same notation as in Theorem* 3.5*, the following assertions hold true.*

(1) *Suppose that* $n = 3$*. Then,* $\mathrm{r}(R) = 2$*,* $q = 0$*,* $\ell = 1$*, and* $m = 0$*, so that* $\mathbb{M} = \left[\begin{smallmatrix} a_{11} & a_{12} & a_{13} \\ X_1^2 & X_2 & X_3 \end{smallmatrix}\right]$.

(2) *If* R *has minimal multiplicity, then* $q = 0$.

Proof. (1) Consider the minimal T-free resolution

$$0 \longrightarrow F_2 \xrightarrow{{}^t\mathbb{M}} F_1 \longrightarrow F_0 \longrightarrow R \longrightarrow 0,$$

where the matrix $\mathbb{M}$ has the form stated in Theorem 3.5. We then have

$$\mathrm{r}(R) + 1 = \mathrm{rank}_T F_1 = \ell n + mn + q = 3 \cdot [\mathrm{r}(R) - 1] + q,$$

so that $4-2{\cdot}\mathrm{r}(R)=q\geq 0$. Therefore, $\mathrm{r}(R)=2$, and $q=0$, since R is not a Gorenstein ring. Thus, $\ell=1$, $m=0$, because $\ell+m=\mathrm{r}(R)-1$.

(2) Since R has multiplicity n, we have $\mathrm{r}(R)=n-1$, while by [43, Theorem 1 (iii)], $n(n-2)=\ell n+mn+q$. Hence, $q=0$, because $\ell+m+1=n$. $\square$

In this chapter we will refer so often to examples arising from numerical semigroup rings, that let us explain here about a canonical form of generators for their canonical modules. Let $0<a_1,a_2,\ldots,a_\ell\in\mathbb{Z}$ ($\ell>0$) be positive integers such that $\mathrm{GCD}\ (a_1,a_2,\ldots,a_\ell)=1$. We set

$$H=\langle a_1,a_2,\ldots,a_\ell\rangle=\left\{\sum_{i=1}^{\ell}c_ia_i \mid 0\leq c_i\in\mathbb{Z} \text{ for all } 1\leq i\leq\ell\right\}$$

and call it the numerical semigroup generated by the numbers $\{a_i\}_{1\leq i\leq\ell}$. Let $V=k[[t]]$ be the formal power series ring over a field k. We set $R=k[[H]]=k[[t^{a_1},t^{a_2},\ldots,t^{a_\ell}]]$ in V and call it the semigroup ring of H over k. The ring R is a one-dimensional Cohen-Macaulay local domain with $\overline{R}=V$ and $\mathfrak{m}=(t^{a_1},t^{a_2},\ldots,t^{a_\ell})$, the maximal ideal. Let

$$\mathrm{c}(H)=\min\{n\in\mathbb{Z}\mid m\in H \text{ for all } m\in\mathbb{Z} \text{ such that } m\geq n\}$$

be the conductor of H and set $\mathrm{f}(H)=\mathrm{c}(H)-1$. Hence, $\mathrm{f}(H)=\max\ (\mathbb{Z}\setminus H)$, which is called the Frobenius number of H. Let

$$\mathrm{PF}(H)=\{n\in\mathbb{Z}\setminus H\mid n+a_i\in H \text{ for all } 1\leq i\leq\ell\}$$

denote the set of pseudo-Frobenius numbers of H. Therefore, $\mathrm{f}(H)$ coincides with the a-invariant of the graded k-algebra $k[t^{a_1},t^{a_2},\ldots,t^{a_\ell}]$ and $\sharp\mathrm{PF}(H)=\mathrm{r}(R)$ ([30, Example (2.1.9), Definition (3.1.4)]). We set $f=\mathrm{f}(H)$ and

$$K=\sum_{c\in\mathrm{PF}(H)}Rt^{f-c}$$

in V. Then K is a fractional ideal of R such that $R\subseteq K\subseteq\overline{R}$ and

$$K\cong\mathrm{K}_R=\sum_{c\in\mathrm{PF}(H)}Rt^{-c}$$

as an R-module ([30, Example (2.1.9)]). Therefore, K is a canonical fractional ideal of R. Notice that $t^f\notin K$ but $\mathfrak{m}t^f\subseteq R$, whence $K:\mathfrak{m}=K+Rt^f$.

Before stating the concrete example, let us explore the properties of 2-AGL numerical semigroup rings.

Proposition 3.7. *Suppose that R is a 2-AGL ring. Then*

$$K/R = \bigoplus_{c \in \mathrm{PF}(H)\setminus\{f\}} R{\cdot}\overline{t^{f-c}}$$

where $\overline{(*)}$ denotes the image in K/R.

Proof. We set $r = \mathrm{r}(R)$, $f = c_r$ and write $\mathrm{PF}(H) = \{c_1, c_2, \ldots, c_r\}$. Let us consider

$$I = \{i \in \Lambda \mid \mathrm{Ann}_{R/\mathfrak{c}}\, \overline{t^{f-c_i}} = (0)\}, \quad J = \{i \in \Lambda \mid \mathrm{Ann}_{R/\mathfrak{c}}\, \overline{t^{f-c_i}} \neq (0)\}$$

where $\mathfrak{c} = R : R[K]$ and $\Lambda = \{1, 2, \cdots, r-1\}$. Notice that $I \cup J = \Lambda$ and $I \neq \emptyset$. Since R is a 2-AGL ring, there exists $b \in H$ such that $(0) :_{R/\mathfrak{c}} \mathfrak{m} = \mathfrak{m}/\mathfrak{c} = [(t^b) + \mathfrak{c}]/\mathfrak{c}$. Then, for each $i \in I$, we have $t^b{\cdot}\overline{t^{f-c_i}} \neq 0$ and $\mathfrak{m}{\cdot}\overline{t^{f-c_i+b}} = (0)$ in K/R. Hence $f + b - c_i \in \mathrm{PF}(H)$, and the elements $\{t^b{\cdot}\overline{t^{f-c_i}}\}_{i\in I}$ in K/R are linearly independent over k. Therefore

$$K/R = \sum_{i\in I} R{\cdot}\overline{t^{f-c_i}} \bigoplus \sum_{j \in J} R{\cdot}\overline{t^{f-c_j}} = \bigoplus_{i\in\Lambda} R{\cdot}\overline{t^{f-c_i}}$$

as desired. □

For the moment, suppose that R is a 2-AGL ring and we maintain the notation as in the proof of Proposition 3.7. Choose $b = a_j$ for some $1 \leq j \leq \ell$. We then have $f + b - c_i \in \mathrm{PF}(H)$ for each $i \in I$, while $f - c_j \in \mathrm{PF}(H)$ for each $j \in J$ if $J \neq \emptyset$. By writing $I = \{c_1 < c_2 < \cdots < c_p\}$ $(p > 0)$ and $J = \{d_1 < d_2 < \cdots < d_p\}$ $(q \geq 0)$, we have the following.

Theorem 3.8. *The following assertions hold true.*

(1) *$f + b = c_i + c_{p+1-i}$ for every $1 \leq i \leq p$.*

(2) *If $J \neq \emptyset$, then $f = d_j + d_{q+1-j}$ for every $1 \leq j \leq q$.*

Proof. The assertions follow from the fact that the maps

$$\{c_i \mid i \in I\} \to \{c_i \mid i \in I\}, x \mapsto f + b - x, \quad \{c_j \mid j \in J\} \to \{c_j \mid j \in J\}, x \mapsto f - x$$

are well-defined and bijective. □

As a consequence, we get the following, which corresponds to the case where $J = \emptyset$.

Corollary 3.9. *Suppose that R is a 2-AGL ring. Then the following conditions are equivalent.*

(1) $K/R \cong (R/\mathfrak{c})^{\oplus(r-1)}$ *as an R-module.*

(2) *There is an integer* $1 \le j \le \ell$ *such that* $f + a_j = c_i + c_{r-i}$ *for every* $1 \le i \le r-1$.

Let us now go back to state the example of Theorem 3.5. With the notation of Theorem 3.5, we cannot expect $q = 0$ in general, as we show in the following.

Example 3.10 (cf. [4, Example 5.5])**.** Let $V = k[[t]]$ be the formal power series ring over a field k, and set $R = k[[t^5, t^7, t^9, t^{13}]]$. Hence, $R = k[[H]]$, the semigroup ring of the numerical semigroup $H = \langle 5, 7, 9, 13\rangle$. We then have $\mathrm{f}(H) = 11$ and $\mathrm{PF}(H) = \{8, 11\}$, whence $K = R + Rt^3$ and $R[K] = k[[t^3, t^5, t^7]] = R + Rt^3 + Rt^6$. Therefore, $\mathfrak{c} = (t^{10}, t^7, t^9, t^{13})$ and $\ell_R(R/\mathfrak{c}) = 2$, so that by Theorem 3.2, R is a 2-AGL ring with $\mathrm{r}(R) = 2$. We are interested in the defining ideal $\mathfrak{a}$ of R. Let $T = k[[X, Y, Z, W]]$ be the formal power series ring, and let $\varphi : T \to R$ be the k-algebra map defined by $\varphi(X) = t^5, \varphi(Y) = t^7, \varphi(Z) = t^9$, and $\varphi(W) = t^{13}$. Then, R has a minimal T-free resolution of the form

$$\mathbb{F}:\ 0 \to T^2 \xrightarrow{\mathbb{M}} T^6 \xrightarrow{\mathbb{N}} T^5 \xrightarrow{\mathbb{L}} T \to R \to 0,$$

where the matrices $\mathbb{M}, \mathbb{N}$, and $\mathbb{L}$ are given by

$${}^t\mathbb{M} = \begin{bmatrix} W & X^2 & XY & YZ & Y^2-XZ & Z^2-XW \\ X^2 & Y & Z & W & 0 & 0 \end{bmatrix},$$

$$\mathbb{N} = \begin{bmatrix} -Z^2+XW & 0 & X^2Z & -X^3 & 0 & W \\ Y^2-XZ & -X^2Y & X^3 & 0 & -W & 0 \\ 0 & 0 & W & -Z & 0 & Y \\ 0 & -W & 0 & Y & -Z & -X \\ 0 & Z & -Y & 0 & X & 0 \end{bmatrix},$$

$$\mathbb{L} = \begin{bmatrix} Y^2-XZ & Z^2-XW & X^4-YW & X^3Y-ZW & X^2YZ-W^2 \end{bmatrix}.$$

The T-dual of $\mathbb{F}$ gives rise to the presentation

$$T^6 \xrightarrow{{}^t\mathbb{M}} T^2 \to K \to 0$$

of the canonical fractional ideal $K = R + Rt^3$, so that

$$K/R \cong T/(X^2, Y, Z, W) \cong R/\mathfrak{c}.$$

We have $\mathfrak{a} = \operatorname{Ker}\varphi = \mathrm{I}_2\left(\begin{smallmatrix} W & X^2 & XY & YZ \\ X^2 & Y & Z & W \end{smallmatrix}\right) + (Y^2 - XZ, Z^2 - XW)$.

We note one example of 2-AGL rings of minimal multiplicity, whence $q = 0$.

Example 3.11 (cf. [4, Example 5.6]). Let $V = k[[t]]$ be the formal power series ring over a field k, and set $R = k[[H]]$, where $H = \langle 4, 9, 11, 14 \rangle$. Then, $\mathrm{f}(H) = 10$ and $\mathrm{P}F(H) = \{5, 7, 10\}$, whence $K = R + Rt^3 + Rt^5$ and $R[K] = k[[t^3, t^4, t^5]] = R + Rt^3 + Rt^5 + Rt^6$. Therefore, $\mathfrak{c} = (t^8, t^9, t^{11}, t^{14})$ and $\ell_R(R/\mathfrak{c}) = 2$, so that by Theorem 3.2, R is a 2-AGL ring possessing minimal multiplicity 4 and $\mathrm{r}(R) = 3$. We consider the k-algebra map $\varphi : T \to R$ defined by $\varphi(X) = t^4, \varphi(Y) = t^9, \varphi(Z) = t^{11}$, and $\varphi(W) = t^{14}$, where $T = k[[X, Y, Z, W]]$ denotes the formal power series ring. Then, R has a minimal T-free resolution

$$\mathbb{F} :\ 0 \to T^3 \overset{\mathbb{M}}{\to} T^8 \overset{\mathbb{N}}{\to} T^6 \overset{\mathbb{L}}{\to} T \to R \to 0$$

where the matrices $\mathbb{M}, \mathbb{N}$, and $\mathbb{L}$ are given by

$${}^t\mathbb{M} = \begin{bmatrix} -Z & -X^3 & -W & -X^2Y & Y & W & X^4 & X^2Z \\ X^2 & Y & Z & W & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & Y & Z & W \end{bmatrix},$$

$$\mathbb{N} = \begin{bmatrix} -X^2Z & 0 & X^4 & 0 & 0 & 0 & W & -Z \\ 0 & 0 & W & -Z & 0 & W & 0 & -Y \\ 0 & W & 0 & -Y & -X^2Y & X^3 & 0 & 0 \\ -W & 0 & 0 & X^2 & 0 & -Z & Y & 0 \\ 0 & Z & -Y & 0 & -W & 0 & 0 & X \\ Y & -X^2 & 0 & 0 & Z & 0 & -X & 0 \end{bmatrix},$$

$$\mathbb{L} = \begin{bmatrix} Y^2-XW & X^5-YZ & Z^2-X^2W & X^3Z-YW & X^4Y-ZW & X^2YZ-W^2 \end{bmatrix}.$$

Taking T-dual of $\mathbb{F}$, we have the presentation

$$T^8 \overset{{}^t\mathbb{M}}{\to} T^3 \to K \to 0$$

of $K = R + Rt^3 + Rt^5$, so that

$$K/R \cong T/(X^2, Y, Z, W) \oplus T/\mathfrak{n} \cong R/\mathfrak{c} \oplus R/\mathfrak{m}.$$

Hence, K/R is not $R/\mathfrak{c}$-free. We have $\operatorname{Ker} \varphi = \mathrm{I}_2 \left(\begin{smallmatrix} -Z & -X^3 & -W & -X^2Y \\ X^2 & Y & Z & W \end{smallmatrix} \right) + \mathrm{I}_2 \left(\begin{smallmatrix} Y & W & X^4 & X^2Z \\ X & Y & Z & W \end{smallmatrix} \right)$.

We are now asking for a sufficient condition for $R = T/\mathfrak{a}$ to be a 2-AGL ring in terms of the presentation of the canonical ideal. Let us maintain the setting in the preamble of this section, assuming R possesses a canonical fractional ideal K of the form

$$K = R + \sum_{i=1}^{\ell} Rf_i + \sum_{j=1}^{m} Rg_j$$

where $f_i, g_j \in K$, and $\ell > 0$, $m \geq 0$ with $\ell + m + 1 = \mathrm{r}(R)$. We then have the following. We should compare it with [28, Theorem 7.8].

Theorem 3.12. *Let $X_1, X_2, \dots, X_n$ be a regular system of parameters of T and assume that K has a presentation of the form*

$$F_1 \xrightarrow{\mathbb{M}} F_0 \xrightarrow{\mathbb{N}} K \to 0 \qquad (\sharp)$$

where $\mathbb{N}$ and $\mathbb{M}$ are matrices of the form stated in Theorem 3.5, satisfying the condition that $a_{ij}, b_{pk} \in (X_1^2) + (X_2, X_3, \dots, X_n)$ for every $1 \le i \le \ell$, $1 \le j \le n$, $1 \le p \le m$, and $2 \le k \le n$. Then R is a 2-AGL ring.

Proof. The presentation $(\sharp)$ gives rise to a presentation

$$F_1 \xrightarrow{\mathbb{B}} G_0 \xrightarrow{\mathbb{L}} K/R \longrightarrow 0$$

of K/R, where $\mathbb{L} = [\, \bar{f_1} \bar{f_2} \cdots \bar{f_\ell} \ \bar{g_1} \bar{g_2} \cdots \bar{g_m} \,]$ (here $\bar{*}$ denotes the image in K/R), and the matrix $\mathbb{B}$ has the form stated in the proof of Theorem 3.5. Hence

$$K/R \cong (T/J)^{\oplus \ell} \oplus (T/\mathfrak{n})^{\oplus m},$$

so that $\mathfrak{n}{\cdot}(K/R) \neq (0)$, since $\ell > 0$. Therefore, $\mathfrak{c} \subsetneq \mathfrak{m}$. We set $J = (X_1^2) + (X_2, X_3, \dots, X_n)$ and let $I = JR$. Then, since $a_{ik} \in J$, inside of K/R we get

$$X_1^2{\cdot}f_i = \overline{a_{i1}} \quad \text{and} \quad X_k{\cdot}f_i = \overline{a_{ik}}$$

for every $1 \le i \le \ell$ and $2 \le k \le n$. Hence, $X_1^2{\cdot}f_i, X_k{\cdot}f_i \in I$. We similarly have $X_k{\cdot}g_j \in I$ for all $1 \le j \le m$ and $2 \le k \le n$, because $b_{jk} \in J$. Moreover, $X_1^2{\cdot}g_j \in J$ for every $1 \le j \le m$. Thus, $IK \subseteq I$, whence $IS \subseteq I$, because $S = R[K] = K^q$ for $q \gg 0$. Therefore, $I \subseteq \mathfrak{c} \subsetneq \mathfrak{m}$, so that $I = \mathfrak{c}$, since $\ell_R(R/I) \le 2$. Thus, $\ell_R(R/\mathfrak{c}) = 2$, and R is a 2-AGL ring by Theorem 3.2. $\square$

As a consequence of Theorem 3.12, we have the following.

Corollary 3.13. *Let $(T, \mathfrak{n})$ be a regular local ring with $\dim T = n \ge 3$ and $\mathfrak{n} = (X_1, X_2, \dots, X_n)$. Choose positive integers $\ell_1, \ell_2, \dots, \ell_n > 0$ so that $\ell_1 \ge 2$ and set $\mathfrak{a} = \mathrm{I}_2 \begin{pmatrix} X_1^2 & X_2 & \cdots & X_{n-1} & X_n \\ X_2^{\ell_2} & X_3^{\ell_3} & \cdots & X_n^{\ell_n} & X_1^{\ell_1} \end{pmatrix}$. Then $R = T/\mathfrak{a}$ is a 2-AGL ring, for which K/R is a free $R/\mathfrak{c}$-module of rank $n-2$.*

Proof. Since $\sqrt{\mathfrak{a} + (X_1)} = \mathfrak{n}$, $\mathrm{grade}_T \mathfrak{a} = n-1$, so that $\mathfrak{a}$ is a perfect ideal of T, whence $R = T/\mathfrak{a}$ is a Cohen-Macaulay local ring with $\dim R = 1$, and a minimal T-free resolution of R

is given by the Eagon-Northcott complex associated to the matrix $\begin{pmatrix} X_1^2 & X_2 & \cdots & X_{n-1} & X_n \\ X_2^{\ell_2} & X_3^{\ell_3} & \cdots & X_n^{\ell_n} & X_1^{\ell_1} \end{pmatrix}$ ([7]). We take the T-dual of the resolution and get the following presentation

$$T^{\oplus n(n-2)} \xrightarrow{\mathbb{M}'} T^{\oplus(n-1)} \xrightarrow{\varepsilon} K_R \to 0$$

of the canonical module K_R of R, where the matrix $\mathbb{M}'$ is given by

$$\mathbb{M}' = \begin{bmatrix} X_2^{\ell_2}\, -X_3^{\ell_3} \cdots (-1)^n X_n^{\ell_n}\, (-1)^{n+1} X_1^{\ell_1} & & 0 & \\ X_1^2\, -X_2 \cdots (-1)^{n+1} X_n & X_2^{\ell_2}\, -X_3^{\ell_3} \cdots (-1)^n X_n^{\ell_n}\, (-1)^{n+1} X_1^{\ell_1} & & \\ & & \ddots & \\ & & X_1^2\, -X_2 \cdots (-1)^{n+1} X_n & X_2^{\ell_2}\, -X_3^{\ell_3} \cdots (-1)^n X_n^{\ell_n}\, (-1)^{n+1} X_1^{\ell_1} \\ & & 0 & X_1^2\, -X_2 \cdots (-1)^{n+1} X_n \end{bmatrix}.$$

Let x_i denote, for each $1 \le i \le n$, the image of X_i in R. Since $x_1^2 x_1^{\ell_1} = x_2^{\ell_2} x_n$, $x_i x_1^{\ell_1} = x_{i+1}^{\ell_{i+1}} x_n$ for every $2 \le i \le n-1$ and x_1 is a parameter of R, we have that every x_i is a non-zerodivisor in R. We set $y = \frac{x_2^{\ell_2}}{x_1^2}$, and

$$f_i = \begin{cases} x_{i+1}^{\ell_{i+1}} & \text{if } 1 \le i \le n-1 \\ x_1^{\ell_1} & \text{if } i = n \end{cases} \qquad g_i = \begin{cases} x_1^2 & \text{if } i = 1 \\ x_i & \text{if } 2 \le i \le n \end{cases}.$$

Then $f_i = g_i y$ for all $1 \le i \le n$, so that $y^n = \frac{\prod_{i=1}^n f_i}{\prod_{i=1}^n g_i} = x_1^{\ell_1 - 2} x_2^{\ell_2 - 1} \cdots x_n^{\ell_n - 1} \in R$. Hence, $y \in \overline{R}$. Let $K = \sum_{i=0}^{n-2} Ry^i$. Therefore, $R \subseteq K \subseteq \overline{R}$. We will show that K is a canonical fractional ideal of R. Indeed, because $\begin{bmatrix} -1 & y & -y^2 & \cdots (-1)^{n-1} y^{n-2} \end{bmatrix} \cdot \mathbb{M}' = \mathbf{0}$, the T-linear map $\psi : T^{\oplus(n-1)} \to K$ defined by $\psi(\mathbf{e}_i) = (-1)^i y^{i-1}$ for $1 \le i \le n-1$ (here $\{\mathbf{e}_i\}_{1 \le i \le n-1}$ denotes the standard basis of $T^{\oplus n-1}$) factors through K_R. Let $\sigma : K_R \to K$ be the R-linear map such that $\psi = \sigma\varepsilon$. Then, $K = \mathrm{Im}\sigma$, and σ is a monomorphism. Indeed, assume that $X = \operatorname{Ker}\sigma \ne (0)$, and choose $\mathfrak{p} \in \operatorname{Ass}_R X$. Then, $(\mathrm{K}_R)_\mathfrak{p} \cong \mathrm{K}_{R_\mathfrak{p}}$, since $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{K}_R$, while $K_\mathfrak{p} \cong R_\mathfrak{p}$, since K is isomprphic to some $\mathfrak{m}$-primary ideal of R (here $\mathfrak{m}$ denotes the maximal ideal of R). Consequently, we get the exact sequence

$$0 \to X_\mathfrak{p} \to \mathrm{K}_{R_\mathfrak{p}} \to R_\mathfrak{p} \to 0$$

of $R_\mathfrak{p}$-modules, which forces $X_\mathfrak{p} = (0)$, because $\ell_{R_\mathfrak{p}}(\mathrm{K}_{R_\mathfrak{p}}) = \ell_{R_\mathfrak{p}}(R_\mathfrak{p})$. This is a contradiction. Thus, $\mathrm{K}_R \cong K$. We identify $\mathrm{K}_R = K$ and $\varepsilon = \psi$. Then, because $(X_1^{\ell_1}, X_2^{\ell_2}, \ldots, X_n^{\ell_n}) \subseteq (X_1^2, X_2, \ldots, X_n)$, the matrix $\mathbb{M}'$ is transformed with elementary column operations into the following matrix

$$\mathbb{M} = \begin{bmatrix} a_{11} a_{12} \cdots a_{1n} & \cdots & \cdots & a_{n-2,1} a_{n-2,2} \cdots a_{n-2,n} \\ X_1^2 X_2 \cdots X_n & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & X_1^2 X_2 \cdots X_n \end{bmatrix}$$

with $a_{ij} \in (X_1^{\ell_1}, X_2^{\ell_2}, \ldots, X_n^{\ell_n})$, so that Theorem 3.12 shows R is a 2-AGL ring. Since $K/R \cong (T/(X_1^2, X_2, \ldots, X_n))^{\oplus n-2}$, K/R is a free $R/\mathfrak{c}$-module of rank $n-2$. $\square$

3.3 2-AGL rings obtained by fiber products

In this section we study the problem of when certain fiber products, or more generally, quasi-trivial extensions of one-dimensional Cohen-Macaulay local rings are 2-AGL rings.

Let R be a commutative ring and I an ideal of R. For an element $\alpha \in R$, we set $A(\alpha) = R \oplus I$ as an additive group and define the multiplication on $A(\alpha)$ by

$$(a, x) \cdot (b, y) = (ab, ay + bx + \alpha(xy))$$

for $(a, x), (b, y) \in A(\alpha)$. Then, $A(\alpha)$ forms a commutative ring which we denote by $A(\alpha) = R \overset{\alpha}{\ltimes} I$, and call it *the quasi-trivial extension of R by I with respect to α*. We consider $A(\alpha)$ to be an R-algebra via the homomorphism $\xi : R \to A(\alpha)$, $a \mapsto (a, 0)$. Therefore, $A(\alpha)$ is a ring extension of R, and $A(\alpha)$ is a finitely generated R-module, when I is a finitely generated ideal of R. Notice that if $\alpha = 0$, then $A(0) = R \ltimes I$ is the ordinary idealization I over R, introduced by M. Nagata [41, Page 2], and $[(0) \times I]^2 = (0)$ in $A(0)$. If $\alpha = 1$, then $A(1)$ is called in [5] the amalgamated duplication of R along I, and

$$A(1) \cong R \times_{R/I} R,\ (a, i) \mapsto (a, a + i),$$

the fiber product of the two copies of the natural homomorphism $R \to R/I$. Hence, if R is a reduced ring, then so is $A(1)$.

Let us note the following.

Lemma 3.14. *Let $(R, \mathfrak{m})$ be a (not necessarily Noetherian) local ring. Assume that $I \neq R$ or $\alpha \in \mathfrak{m}$. Then $A(\alpha)$ is a local ring with maximal ideal $\mathfrak{m} \times I$.*

Proof. Let $(a, x) \in A(\alpha) \setminus (\mathfrak{m} \times I)$. Then, $a + \alpha x \notin \mathfrak{m}$, since $a \notin \mathfrak{m}$ but $\alpha x \in \mathfrak{m}$. Therefore, setting $b = a^{-1}$ and $y = -(a + \alpha x)^{-1} \cdot xb$, we get $(a, x)(b, y) = 1$ in $A(\alpha)$. Hence, $A(\alpha)$ is a local ring, because $\mathfrak{m} \times I$ is an ideal of $A(\alpha)$. □

Remark 3.15. When $I = R$, $A(-1)$ is not a local ring, even if $(R, \mathfrak{m})$ is a local ring. In fact, assume that $A(-1)$ is a local ring. Then, because $\mathfrak{m} \times R$ is a maximal ideal of $A(-1)$ and $(1, 1) \notin \mathfrak{m} \times R$, we have $(1, 1)(b, y) = (1, 0)$ for some $(b, y) \in A(-1)$, so that $b = 1$ and $y + b + (-1) \cdot 1 \cdot y = 0$. This is absurd.

In what follows, let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring with a canonical fractional ideal K. We set $S = R[K]$ and $\mathfrak{c} = R : S$. Let T be a birational module-finite extension of R (hence $R \subseteq T \subseteq \overline{R}$), and assume that $K \subseteq T$ but $R \neq T$. We set $I = R : T$. Hence, $I = K : T$ by [18, Lemma 3.5 (1)], so that $K : I = T$.

Proposition 3.16. *$T/K \cong \mathrm{K}_{R/I}$. Hence, $\ell_R(T/K) = \ell_R(R/I)$.*

Proof. Take the K-dual of the exact sequence $0 \to I \to R \to R/I \to 0$, and consider the resulting exact sequence $0 \to K \to K : I \to \mathrm{Ext}_R^1(R/I, K) \to 0$. We then have $T/K \cong \mathrm{Ext}_R^1(R/I, K) = \mathrm{K}_{R/I}$, since $K : I = T$. Therefore, $\ell_R(T/K) = \ell_R(\mathrm{K}_{R/I}) = \ell_R(R/I)$. $\square$

Let $\alpha \in R$ and set $A = R \overset{\alpha}{\ltimes} I$. Then, since $I \neq R$, A is a Cohen-Macaulay local ring with $\dim A = 1$ and $\mathfrak{n} = \mathfrak{m} \times I$, the unique maximal ideal (Lemma 3.14). We are now interested in the question of when A is a 2-AGL ring. Notice that we have the extensions

$$A \subseteq T \overset{\alpha}{\ltimes} T \subseteq \mathrm{Q}(R) \overset{\alpha}{\ltimes} \mathrm{Q}(R) = \mathrm{Q}(A)$$

of rings. We set $L = T \times K$ in $T \overset{\alpha}{\ltimes} T$. Hence, L is an A-submodule of $T \overset{\alpha}{\ltimes} T$, and $A \subseteq L \subseteq \overline{A}$.

We begin with the following, which plays a key role in this section.

Proposition 3.17. *$L \cong \mathrm{K}_A$ and $A[L] = T \overset{\alpha}{\ltimes} T$.*

Proof. Since $I = K : T$, $I^\vee \cong T$ where $(-)^\vee = \mathrm{Hom}_R(-, K)$, and we have the natural isomorphism

$$\sigma : A^\vee = \mathrm{Hom}_R(R \oplus I, K) \xrightarrow{\cong} I^\vee \oplus R^\vee \xrightarrow{\cong} T \oplus K = L$$

of R-modules. Let $Z = T \oplus T$. Then, the R-module Z becomes a $T \overset{\alpha}{\ltimes} T$-module by the following action

$$(a, x) \rightharpoonup (b, y) = ((a + \alpha x)b, ay + bx)$$

for each $(a, x) \in T \overset{\alpha}{\ltimes} T$ and $(b, y) \in Z$. It is routine to check that L which is considered inside of Z is an A-submodule of Z, and that the above R-isomorphism $\sigma : A^\vee \to L$ is actually an A-isomorphism. We now consider the homomorphism $\psi : T \overset{\alpha}{\ltimes} T \to Z$ of $T \overset{\alpha}{\ltimes} T$-modules defined by $\psi(1) = (1, 0)$. Then, ψ is an isomorphism, since $\psi(a, x) = (a + \alpha x, x)$ for each $(a, x) \in T \overset{\alpha}{\ltimes} T$. Notice that for each $(a, x) \in T \overset{\alpha}{\ltimes} T$, $(a, x) \in T \times K$ if and only if $\psi(a, x) \in T \times K$. Therefore, L which is considered inside of $T \overset{\alpha}{\ltimes} T$ is isomorphic to K_A, because L which is considered inside of Z is isomorphic to $A^\vee = \mathrm{K}_A$. Since $KT = T$, we have $L^n = T \overset{\alpha}{\ltimes} T$ for every $n \geq 2$. Thus, $A[L] = T \overset{\alpha}{\ltimes} T$, since $A[L] = \bigcup_{\ell \geq 1} L^\ell = L^n$ for $n \gg 0$. $\square$

Let $\mathrm{r}_R(I) = \ell_R(\mathrm{Ext}_R^1(R/\mathfrak{m}, I))$ denote the Cohen-Macaulay type of the R-module I.

Corollary 3.18. $\mathrm{r}(A(\alpha)) = \mu_R(T) + \mathrm{r}(R) = \mathrm{r}_R(I) + \mu_R(K/I)$. *Hence, the Cohen-Macaulay type of $A(\alpha)$ is independent of the choice of $\alpha \in R$.*

Proof. With the same notation as in Proposition 3.17, because $\mathfrak{n}L = (\mathfrak{m} \times I)(T \times K) = \mathfrak{m}T \times \mathfrak{m}K$, we have an R-isomorphism $L/\mathfrak{n}L \cong T/\mathfrak{m}T \oplus K/\mathfrak{m}K$. Therefore, since $R/\mathfrak{m} = A/\mathfrak{n}$, $\mathrm{r}(A) = \mu_R(T) + \mathrm{r}(R)$, which is independent of α. Consequently, because $\sum_{f \in \mathrm{Hom}_R(I,K)} f(I) = (K : I)I = TI = I$ where the second equality follows from the fact that $I = K : T$, by [17, Theorem 3.3] we get $\mathrm{r}(A) = \mathrm{r}_R(I) + \mu_R(K/I)$. □

We now come to the main result of this section.

Theorem 3.19. *With the same notation as above, the following conditions are equivalent.*

(1) *The fiber product $R \times_{R/I} R$ is a* 2-AGL *ring.*

(2) *The idealization $R \ltimes I$ is a* 2-AGL *ring.*

(3) *$A(\alpha) = R \overset{\alpha}{\ltimes} I$ is a* 2-AGL *ring for every $\alpha \in R$.*

(4) *$A(\alpha) = R \overset{\alpha}{\ltimes} I$ is a* 2-AGL *ring for some $\alpha \in R$.*

(5) *$\ell_R(T/K) = 2$.*

(6) *$\ell_R(R/I) = 2$.*

Proof. We maintain the same notation as in Proposition 3.17. Since $A[L] = T \overset{\alpha}{\ltimes} T$, $A[L]/L \cong T/K$ as an R-module, so that $\ell_A(A[L]/L) = \ell_R(T/K)$, because $R/\mathfrak{m} = A/\mathfrak{n}$. Thus, the assertion readily follows from Proposition 3.16, Theorem 3.2, and Proposition 3.4. □

Corollary 3.20. *Suppose that R is a* 2-AGL *ring. If $A(\alpha) = R \overset{\alpha}{\ltimes} I$ is a* 2-AGL *ring for some $\alpha \in R$, then $T = S$ and $I = \mathfrak{c}$.*

Proof. We have $S = R[K] \subseteq T$, since $K \subseteq T$. Therefore, $S = T$, because $\ell_R(T/K) = \ell_R(S/K) = 2$ by Theorems 3.2 and 3.19. □

Choosing $T = S$, we have the following. The equivalence of assertions (2) and (3) covers [4, Theorem 4.2]. We should compare the result with [18, Theorem 6.5] for the assertion on AGL rings.

Corollary 3.21. *Let R be a one-dimensional Cohen-Macaulay local ring with a canonical fractional ideal K and assume that R is not a Gorenstein ring. We set $S = R[K]$ and $\mathfrak{c} = R : S$. Then the following conditions are equivalent.*

(1) *$R \times_{R/\mathfrak{c}} R$ is a* 2-AGL *ring.*

(2) *$R \ltimes \mathfrak{c}$ is a* 2-AGL *ring.*

(3) *R is a* 2-AGL *ring.*

We note one example.

Example 3.22. Let k be a field and set $R = k[[t^4, t^7, t^9]]$. Then $K = R + Rt^5$, so that R is an AGL ring with $\mathrm{r}(R) = 2$, because $\mathfrak{m}K \subseteq R$ ([18, Theorem 3.11]). Hence $\mathfrak{c} = \mathfrak{m}$. Let $T = k[[t^4, t^5, t^6, t^7]]$. Then, $T = R + Rt^5 + Rt^6$, and $I = R : T = (t^7, t^8, t^9)$. Therefore, because $\ell_R(R/I) = 2$, by Theorem 3.19 and Corollary 3.18 $A(\alpha) = R \overset{\alpha}{\ltimes} I$ is a 2-AGL ring with $\mathrm{r}(A(\alpha)) = \mu_R(T) + \mathrm{r}(R) = 3 + 2 = 5$ for every $\alpha \in R$. In particular, $R \times_{R/I} R$ and $R \ltimes I$ are 2-AGL rings.

3.4 Two-generated Ulrich ideals in 2-AGL rings

In this section, we explore Ulrich ideals in 2-AGL rings, mainly two-generated ones. One can find in [12], for arbitrary Cohen-Macaulay local rings of dimension one, a beautiful and complete theory of Ulrich ideals which are not two-generated.

First of all, let us briefly recall the definition of Ulrich ideals. Although we will soon restrict our attention on the one-dimensional case, let us give it for arbitrary dimension. So, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = d \geq 0$, and I an $\mathfrak{m}$-primary ideal of R. We assume that I contains a parameter ideal Q of R as a reduction.

Definition 3.23. ([24, Definition 1.1]) We say that I is an *Ulrich ideal* in R, if the following conditions are satisfied.

(1) $I \neq Q$, but $I^2 = QI$.

(2) I/I^2 is a free R/I-module.

Here let us summarize a few basic result on Ulrich ideals, which we later use in this section. To state them, we need the notion of G-dimension. For the moment, let R be a Noetherian ring. A *totally reflexive* R-module is by definition a finitely generated reflexive

R-module G such that $\mathrm{Ext}_R^p(G,R) = (0)$ and $\mathrm{Ext}_R^p(\mathrm{Hom}_R(G,R),R) = (0)$ for all $p > 0$. Note that every finitely generated free R-module is totally reflexive. The *Gorenstein dimension* (G-dimension for short) of a finitely generated R-module M, denoted by G-$\mathrm{dim}_R M$, is defined as the infimum of integers $n \geq 0$ such that there exists an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$$

of R-modules with each G_i totally reflexive. A Noetherian local ring R is called G-*regular*, if every totally reflexive R-module is free. This is equivalent to saying that the equality G-$\mathrm{dim}_R M = \mathrm{pd}_R M$ holds true for every finitely generated R-modules M ([44]).

Proposition 3.24 ([29, Theorem 2.5, Corollary 2.13, Theorem 2.8])**.** *Let I be an Ulrich ideal in R and set $n = \mu_R(I)$. Then the following assertions hold true.*

(1) $(n-d)\cdot \mathrm{r}(R/I) = \mathrm{r}(R)$.

(2) *Suppose that there exists an exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$ of R-modules where K_R denotes the canonical module of R. If $n \geq d+2$, then $\mathrm{Ann}_R C \subseteq I$.*

(3) $n = d+1$ *if and only if* G-$\mathrm{dim}_R R/I < \infty$.

Let I be an $\mathfrak{m}$-primary ideal of R, containing a parameter ideal Q of R as a reduction. Assume that $I^2 = QI$ and consider the exact sequence

$$0 \to Q/QI \to I/I^2 \to I/Q \to 0$$

of R-modules. We then have that I/I^2 is a free R/I-module if and only if so is I/Q. If $I^2 = QI$ and $\mu_R(I) = d+1$, the latter condition is equivalent to saying that $Q :_R I = I$, that is I is exactly a *good ideal* in the sense of [15]. It is known by [24] that when R is a Gorenstein ring, every Ulrich ideal I in R is $(d+1)$-generated (if it exists), and I is a good ideal of R (see [24, Lemma 2.3, Corollary 2.6]). Similarly as good ideals, Ulrich ideals are characteristic ideals, but behave very well in their nature ([24, 25]). The existence of $(d+1)$-generated Ulrich ideals gives a strong influence to the structure of R, which we shall confirm in this section.

We now be back to the following setting. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$, and let $\mathcal{X}_R$ be the set of Ulrich ideals in R. In general, it is quite difficult to list up the members of $\mathcal{X}_R$ (see, e.g., [24]). Here, to grasp what kind of sets $\mathcal{X}_R$ is, first of all we explore one example. To do this, we need the following.

Lemma 3.25 (cf. [13, Proposition 3.1]). *Let R be a Gorenstein local ring with* $\dim R = 1$. *We denote by $\mathcal{A}_R$ the set of birational module-finite extensions A of R such that A is a Gorenstein ring, and set $\mathcal{A}_R^0 = \{A \in \mathcal{A}_R \mid \mu_R(A) = 2\}$. Then, there exist bijective correspondences*

$$\mathcal{A}_R^0 \to \mathcal{X}_R,\ A \mapsto R : A, \quad \textit{and} \quad \mathcal{X}_R \to \mathcal{A}_R^0,\ I \mapsto I : I.$$

Proof. Let $\mathcal{G}_R$ be the set of ideals I in R such that $I^2 = aI$ and $I = (a) :_R I$ for some non-zerodivisor $a \in I$. We then have by [13, Proposition 3.1] a bijective correspondence $\mathcal{G}_R \to \mathcal{A}_R,\ I \mapsto I : I$. Because $\mathcal{X}_R = \{I \in \mathcal{G}_R \mid \mu_R(I) = 2\}$ and $I : I = a^{-1}I$ for every $I \in \mathcal{G}_R$ and every reduction (a) of I, we get $\mu_R(I : I) = \mu_R(I)$, so that $I : I \in \mathcal{A}_R^0$ for each $I \in \mathcal{X}_R$. Conversely, let $A \in \mathcal{A}_R^0$ and write $A = I : I$ with $I \in \mathcal{G}_R$. Let (a) be a reduction of I. Then, $A = I : I = a^{-1}I$, so that $\mu_R(I) = \mu_R(A) = 2$, while $R : A = I$, because $A = R : I$ by [13, Proposition 2.5] and $I = R : (R : I)$ (remember that R is a Gorenstein ring). Hence, $I \in \mathcal{X}_R$, and the correspondences follow. $\square$

Example 3.26. Let k be a field and set $R = k[[t^n, t^{n+1}, \dots, t^{2n-2}]]$ $(n \geq 3)$, where t is an indeterminate. Then, R is a Gorenstein ring, and

$$\mathcal{X}_R = \begin{cases} \{(t^4, t^6)\} & (n = 3), \\ \{(t^4 - \alpha t^5, t^6) \mid \alpha \in k\} & (n = 4), \\ \quad \emptyset & (n \geq 5). \end{cases}$$

When $n = 4$, we have $(t^4 - \alpha t^5, t^6) = (t^4 - \beta t^5, t^6)$, only if $\alpha = \beta$.

Proof. Our ring R is a Gorenstein ring, since the numerical semigroup $H = \langle n, n+1, \dots, 2n-2 \rangle$ is symmetric ([33]). Therefore, in order to determine the members of $\mathcal{X}_R$, by Lemma 3.25 it suffices to list the members of $\mathcal{A}_R^0$, taking $R : A$ for each $A \in \mathcal{A}_R^0$. We set $V = k[[t]]$.

(1) (*The case where* $n = 3$) Let $A \in \mathcal{A}_R^0$. Then $R \subsetneq A \subsetneq V$, whence $t^5 \in R : \mathfrak{m} \subseteq A$, which follows from the fact that the image of t^5 in $\mathrm{Q}(R)/R$ is a unique socle of $\mathrm{Q}(R)/R$ and $(0) \neq A/R \subseteq \mathrm{Q}(R)/R$. Therefore

$$k[[t^3, t^4, t^5]] \subseteq A \subseteq k[[t^2, t^3]].$$

Hence $A = k[[t^2, t^3]]$, because $k[[t^3, t^4, t^5]]$ is not a Gorenstein ring and $\ell_k(k[[t^2, t^3]]/k[[t^3, t^4, t^5]]) = 1$. It is direct to show $R : A = R : t^2 = (t^4, t^6)$. Hence $\mathcal{X}_R = \{(t^4, t^6)\}$.

(2) (*The case where* $n = 4$) Let $A \in \mathcal{A}_R^0$. Then, $t^7 \in R : \mathfrak{m} \subseteq A$, and $k[[t^4, t^5, t^6, t^7]] \subseteq A \subseteq k[[t^2, t^3]]$. We have $A \not\subseteq k[[t^3, t^4, t^5]]$. Indeed, if $A \subseteq k[[t^3, t^4, t^5]]$, then $A = k[[t^3, t^4, t^5]]$ or $A = k[[t^4, t^5, t^6, t^7]]$, because $\ell_k(k[[t^3, t^4, t^5]]/k[[t^4, t^5, t^6, t^7]]) = 1$. This is, however, impossible, since both $k[[t^3, t^4, t^5]]$ and $k[[t^4, t^5, t^6, t^7]]$ are not a Gorenstein rings. Hence

$$k[[t^4, t^5, t^6, t^7]] \subsetneq A \subseteq k[[t^2, t^3]], \quad A \not\subseteq k[[t^3, t^4, t^5]].$$

We choose $\xi \in A$ so that $\xi \notin k[[t^3, t^4, t^5]]$. Then, since $k[[t^4, t^5, t^6, t^7]] = k + t^4V \subseteq A$, we may assume that $\xi = t^2 + \alpha t^3$ with $\alpha \in k$. Therefore, because

$$k[[t^4, t^5, t^6, t^7]] \subsetneq R[\xi] = k[[t^2 + \alpha t^3, t^4, t^5, t^6]] \subseteq A \subseteq k[[t^2, t^3]]$$

and $\ell_k(k[[t^2, t^3]]/k[[t^4, t^5, t^6, t^7]]) = 2$, we have $\ell_k(k[[t^2, t^3]]/R[\xi]) \leq 1$. Hence, $A = R[\xi]$ or $A = k[[t^2, t^3]]$, where $k[[t^2, t^3]] \notin \mathcal{A}_R^0$ since $\mu_R(k[[t^2, t^3]]) = 3$. Thus, $A = R[\xi]$, and we have $R : A = R : R[\xi] = R : \xi = (t^4 - \alpha t^5, t^6)$. Hence, $\mathcal{X}_R = \{(t^4 - \alpha t^5, t^6) \mid \alpha \in k\}$, because $\mathcal{A}_R^0 = \{R[t^2 + \alpha t^3] \mid \alpha \in k\}$.

(3) (*The case where* $n = 2q+1$ *with* $q \geq 2$) Assume that $\mathcal{X}_R \neq \emptyset$ and choose $I \in \mathcal{X}_R$. We set $A = I : I$. Then $A \in \mathcal{A}_R^0$. We have $t^nV \subseteq k[[t^n, t^{n+1}, \ldots, t^{2n-1}]] \subseteq A$, since the image of t^{2n-1} in $\mathrm{Q}(R)/R$ is a unique socle of $\mathrm{Q}(R)/R$. We set $\mathcal{C} = A : V = t^cV$ $(c \geq 0)$, the conductor of A. Hence, $c \leq n = 2q+1$, because $t^nV \subseteq A$. Let $\ell = \ell_k(V/A)$. We then have $2\ell = c$, since A is a Gorenstein ring ([32, Korollar 3.5]), so that $\ell \leq q$. Let $\mathfrak{m}_A$ denote the maximal ideal of A. Then, $(\mathfrak{m}_A/\mathfrak{m}A)^2 = (0)$, since $\ell_k(A/\mathfrak{m}A) = \ell_R(A/\mathfrak{m}A) = \mu_R(A) = 2$. We look at the chain

$$A/\mathfrak{m}A \supsetneq \mathfrak{m}_A/\mathfrak{m}A \supsetneq (\mathfrak{m}_A/\mathfrak{m}A)^2 = (0)$$

of ideals in the ring $\overline{A} = A/\mathfrak{m}A$, and take $\xi \in \mathfrak{m}_A$, so that $\mathfrak{m}_A/\mathfrak{m}A = (\overline{\xi})$ (here $\overline{\xi}$ denotes the image of ξ in $\overline{A}$). Then, $\overline{\xi} \neq 0$, but $\overline{\xi}^2 = 0$ in $\overline{A}$. Consequently, $\xi^2 \in \mathfrak{m}A \subseteq t^nV$ and $A = R+R\xi$, since $A/\mathfrak{m}A = k+k\overline{\xi}$. Therefore, $2\nu(\xi) \geq n = 2q+1$, so that $\nu(\xi) \geq q+1$ (here $\nu(*)$ denotes the valuation of V). Thus, $A = R + R\xi \subseteq k[[t^{q+1}, t^{q+2}, \ldots, t^{2q+1}]]$, whence $A = k[[t^{q+1}, t^{q+2}, \ldots, t^{2q+1}]]$, because $\ell_k(V/k[[t^{q+1}, t^{q+2}, \ldots, t^{2q+1}]]) = q$ and $\ell_k(V/A) = \ell \leq q$. This is, however, impossible, since A is a Gorenstein ring but $k[[t^{q+1}, t^{q+2}, \ldots, t^{2q+1}]]$ is not. Thus $\mathcal{X}_R = \emptyset$.

(4) (*The case where* $n = 2q$ *with* $q \geq 3$) Assume that $\mathcal{X}_R \neq \emptyset$ and choose $I \in \mathcal{X}_R$. We set $A = I : I$. We then have $t^{2n-1} \in A$, considering the image of t^{2n-1} in $\mathrm{Q}(R)/R$. We

set $\ell = \ell_k(V/A)$ and $\mathcal{C} = A : V$. Then $\mathcal{C} = t^{2\ell}V$, since A is a Gorenstein ring. Therefore, $\ell \leq q$, because $t^n V \subseteq A$ and $n = 2q$. On the other hand, considering the chain

$$A/\mathfrak{m}A \supsetneq \mathfrak{m}_A/\mathfrak{m}A \supsetneq (\mathfrak{m}_A/\mathfrak{m}A)^2 = (0)$$

of ideals in the ring $\overline{A} = A/\mathfrak{m}A$ and taking $\xi \in \mathfrak{m}_A$ so that $\mathfrak{m}_A/\mathfrak{m}A = (\overline{\xi})$, we get $\overline{\xi} \neq 0$ and $\overline{\xi}^2 = 0$ in $\overline{A}$. Therefore, $\xi^2 \in \mathfrak{m}A \subseteq t^n V$ and $A = R + R\xi$, because $A/\mathfrak{m}A = k + k\overline{\xi}$. Consequently, $2\nu(\xi) \geq n = 2q$. Hence, $\nu(\xi) \geq q$, so that

$$A = R + R\xi \subsetneq k[[t^q, t^{q+1}, \ldots, t^{2q-1}]] \subseteq V,$$

where the strictness of the first inclusion follows from the fact that $k[[t^q, t^{q+1}, \ldots, t^{2q-1}]]$ is not a Gorenstein ring. Therefore, because $\ell_k(V/A) = \ell$ and $\ell_k(V/k[[t^q, t^{q+1}, \ldots, t^{2q-1}]]) = q - 1$, we get $q - 1 < \ell$, whence $\ell = q$. We set $T = k[[t^{2q}, t^{q+1}, \ldots, t^{4q-1}]]$. Then $\ell_k(A/T) = q - 1$, since $\ell_k(V/T) = 2q - 1$. Because

$$A/T \subseteq V/T = k\overline{t} + k\overline{t^2} + \cdots + k\overline{t^{2q-1}},$$

where $\overline{*}$ denotes the image in V/T, we obtain elements $\xi_1, \xi_2, \ldots, \xi_{q-1} \in \sum_{i=1}^{2q-1} kt^i$ so that $A = T + \sum_{i=1}^{q-1} k\xi_i$. Therefore

$$A = T[\xi_1, \xi_2, \ldots, \xi_{q-1}] = k[[\xi_1, \xi_2, \ldots, \xi_{q-1}, t^{2q}, \ldots, t^{4q-1}]],$$

whence $\xi_1, \xi_2, \ldots, \xi_{q-1} \in \mathfrak{m}_A$ and $(\overline{\xi_1}, \overline{\xi_2}, \ldots, \overline{\xi}_{q-1}) \subseteq \mathfrak{m}_A/\mathfrak{m}A$. We now notice that if $\sum_{i=1}^{q-1} a_i\xi_i \in \mathfrak{m}A$ with $a_i \in k$, then $\sum_{i=1}^{q-1} a_i\xi_i \in t^{2q}V$, whence $a_i = 0$ for all $1 \leq i \leq q-1$. Therefore, $1 = \ell_k(\mathfrak{m}_A/\mathfrak{m}A) \geq q - 1 \geq 2$. This is a required contradiction.

Let us make sure of the last assertion. Suppose that $n = 4$ and $(t^4 - \alpha t^5, t^6) = (t^4 - \beta t^5, t^6)$ where $\alpha, \beta \in k$. We write $t^4 - \alpha t^5 = f(t^4 - \alpha t^5) + gt^6$ for some $f, g \in R$. By setting $f = c_0 + c_1t^4 + c_2t^5 + c_3t^6 + \eta$, $g = d_0 + d_1t^4 + d_2t^5 + d_3t^6 + \xi$ for some $c_i, d_j \in k$ and $\eta, \xi \in t^8V$, we then have $t^4 - \alpha t^5 = c_0t^4 - \beta c_0t^5 + d_0t^6 +$ (higher terms). Hence, $c_0 = 1$ and $\alpha = \beta c_0 = \beta$, as desired. $\square$

Let us give here simple examples of 2-AGL rings, which contain numerous two-generated Ulrich ideals.

Example 3.27. Let $(R, \mathfrak{m})$ be an AGL ring with $\dim R = 1$ and suppose that R is not a Gorenstein ring, say $R = k[[t^3, t^4, t^5]]$, the semigroup ring of $H = \langle 3, 4, 5\rangle$ over a field k. Let $\alpha \in \mathfrak{m}$ and consider the quasi-trivial extension $A = R \overset{\alpha}{\ltimes} R$ of R with respect to α (see

Section 3) Then, A is a 2-AGL ring by [4, Theorem 3.10], because A is a free R-module with $\ell_R(A/\mathfrak{m}A) = 2$. Let $\mathfrak{q}$ be a parameter ideal of R and assume that $\alpha \in \mathfrak{q}$. We set $I = \mathfrak{q} \times R$. Then, I is an Ulrich ideal of A with $\mu_A(I) = 2$. Therefore, if $\alpha = 0$, then $\mathfrak{q} \times R$ is an Ulrich ideal of A for every parameter ideal $\mathfrak{q}$ of R ([24, Example 2.2]).

Proof. Let $\mathfrak{q} = (a)$ and set $f = (a, 0) \in I$. Then, $I^2 = fI$, since $I^2 = (a^2) \times (aR + \alpha R) = (a^2) \times aR = fI$. Note that $I/fA = [(a) \oplus R]/[(a) \oplus (a)] \cong R/(a)$ and $A/I = [R \oplus R]/[(a) \oplus R] \cong R/(a)$ as R-modules. We then have $\ell_A(A/I) = \ell_A(I/fA) = \ell_R(R/(a))$. Hence, $A/I \cong I/fA$ as an A-module, because I/fA is a cyclic A-module. Thus, $I \in \mathcal{X}_A$ with $\mu_A(I) = 2$. $\square$

Two-generated Ulrich ideals are totally reflexive R-modules (Proposition 3.24 (3)), possessing minimal free resolutions of a very restricted form. Let us note the following, which we need to prove Theorem 3.29. We include a brief proof for the sake of completeness.

Proposition 3.28 (cf. [24, Example 7.3]). *Suppose that I is an Ulrich ideal of R and assume that $\mu_R(I) = 2$. We write $I = (a, b)$ with (a) a reduction of I. Therefore, $b^2 = ac$ for some $c \in I$. With this notation, the minimal free resolution of I is given by*

$$\mathbb{F}: \quad \cdots \to R^2 \xrightarrow{\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} I \to 0,$$

Hence $\operatorname{pd}_R I = \infty$. The ideal I is so called a totally reflexive R-module, because I is reflexive, $\operatorname{Ext}_R^p(I, R) = (0)$, and $\operatorname{Ext}_R^p(\operatorname{Hom}_R(I, R), R) = (0)$ for all $p > 0$.

Proof. Here we don't assume that R is a Gorenstein ring, but the proof given in [24, Example 7.3] still works to get the minimal free resolution $\mathbb{F}$ of I. Since

$$I = (a) :_R I = (a) : I = a(R : I),$$

we have $I \cong R : I \cong I^*$, where $I^* = \operatorname{Hom}_R(I, R)$. Note that the R-dual $\mathbb{F}^*$ of $\mathbb{F}$ remains exact. In fact, assume that $\left(\begin{smallmatrix} -b & a \\ -c & b \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = 0$. Then, since $-bx + ay = 0$, we have $\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right) = \left(\begin{smallmatrix} -b & -c \\ a & b \end{smallmatrix}\right)\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ for some $f, g \in R$. Therefore, $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} -b & a \\ -c & b \end{smallmatrix}\right)\left(\begin{smallmatrix} -g \\ f \end{smallmatrix}\right)$, which shows that $\mathbb{F}^*$ is exact, because $\left(\begin{smallmatrix} -b & a \\ -c & b \end{smallmatrix}\right)^2 = 0$. Consequently, $\operatorname{Ext}_R^p(I, R) = (0)$ for all $p > 0$, whence $\operatorname{Ext}_R^p(I^*, R) = (0)$ for all $p > 0$, because $I \cong I^*$. On the other hand, by the above argument we have an exact sequence

$$0 \to I \to R^{\oplus 2} \to I \to 0$$

whose R-dual $0 \to I^* \to R^{\oplus 2} \to I^* \to 0$ remains exact. Therefore, I is a reflexive R-module. Thus, I is a totally reflexive R-module. $\square$

We now start the analysis of the question of how many two-generated Ulrich ideals are contained in a given 2-AGL ring. Let K be a canonical fractional ideal of R. Let $S = R[K]$ and set $\mathfrak{c} = R : S$. We then have the following, which shows the existence of two-generated Ulrich ideals is a substantially strong restriction.

Theorem 3.29. *Suppose that R is a* 2-AGL *ring and let K be a canonical fractional ideal of R. Let $\mathfrak{c} = (x_1^2) + (x_2, x_3, \ldots, x_n)$ with a minimal system $x_1, x_2, \ldots, x_n$ of generators of $\mathfrak{m}$. Assume that R contains an Ulrich ideal I with $\mu_R(I) = 2$. Then the following assertions hold true.*

(1) *K/R is a free $R/\mathfrak{c}$-module.*

(2) *$I + \mathfrak{c} = \mathfrak{m}$.*

(3) *$\mathfrak{c} = (x_2, x_3, \ldots, x_n)$.*

Consequently, $\mu_R(\mathfrak{c}) = n - 1$, and $x_1^2 \in (x_2, x_3, \ldots, x_n)$.

Proof. (1) We have $K/R \cong (R/\mathfrak{c})^{\oplus \ell} \oplus (R/\mathfrak{m})^{\oplus m}$ with $\ell > 0, m \geq 0$ such that $\ell + m = \mathrm{r}(R) + 1$ (Proposition 3.4 (4)). To show assertion (1), let us assume that $m > 0$. Then, since I is totally reflexive (Proposition 3.28) and $\mathrm{Ext}_R^p(I, K) = (0)$ for every $p > 0$, we get $\mathrm{Ext}_R^p(I, K/R) = (0)$, so that

$$\mathrm{Ext}_R^p(I, R/\mathfrak{m}) = (0) \ \text{ for all } \ p > 0,$$

because $R/\mathfrak{m}$ is a direct summand of K/R. This is impossible, since $\mathrm{pd}_R I = \infty$. Hence, $m = 0$, and K/R is $R/\mathfrak{c}$-free.

(2) Let us use the same notation as in Proposition 3.28. Hence, I has a minimal free resolution of the form

$$\mathbb{F}: \quad \cdots \to R^2 \xrightarrow{\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} -b & -c \\ a & b \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} I \to 0.$$

Remember that $\mathrm{Ext}_R^p(I, R/\mathfrak{c}) = (0)$ for all $p > 0$, because $\mathrm{Ext}_R^p(I, K) = \mathrm{Ext}_R^p(I, R) = (0)$ for all $p > 0$ and $R/\mathfrak{c}$ is a direct summand of K/R. Let $\overline{x}$ denote, for each $x \in R$, the image of x in $R/\mathfrak{c}$. Then, taking the $R/\mathfrak{c}$-dual of the resolution $\mathbb{F}$, we get the exact sequence

$$(E) \quad 0 \to \mathrm{Hom}_R(I, R/\mathfrak{c}) \to (R/\mathfrak{c})^{\oplus 2} \xrightarrow{\begin{pmatrix} -\overline{b} & \overline{a} \\ -\overline{c} & \overline{b} \end{pmatrix}} (R/\mathfrak{c})^{\oplus 2} \xrightarrow{\begin{pmatrix} -\overline{b} & \overline{a} \\ -\overline{c} & \overline{b} \end{pmatrix}} (R/\mathfrak{c})^{\oplus 2} \to \cdots,$$

which shows that $I \nsubseteq \mathfrak{c}$. Therefore, $I + \mathfrak{c} = \mathfrak{m}$, since $\ell_R(R/\mathfrak{c}) = 2$.

(3) To show assertion (3), we notice that $\mathfrak{m}/\mathfrak{c} = (\overline{x_1}) = (\overline{a}, \overline{b})$, and $\ell_R(\mathfrak{m}/\mathfrak{c}) = 1$. Hence $\mathfrak{m}^2 \subseteq \mathfrak{c}$. We set $J = (x_2, x_3, \dots, x_n)$, and consider the following two cases.

Case 1 ($\overline{a} \neq 0$). Let us write $a = \alpha x_1 + \xi$ for some $\alpha \in R$ and $\xi \in J$. Then, since $\overline{a} \neq 0$, $\alpha \notin \mathfrak{m}$ and $\overline{b} \in \mathfrak{m}/\mathfrak{c} = (\overline{a})$. Let $b = \beta a + \gamma$ with $\beta \in R$ and $\gamma \in \mathfrak{c}$. Then, $I = (a, b) = (a, \gamma)$, whence replacing b with γ, we may assume that $\alpha = 1$ and $b \in \mathfrak{c}$. Therefore, $\left(\begin{smallmatrix} -\overline{b} & \overline{a} \\ -\overline{c} & \overline{b} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & \overline{x_1} \\ -\overline{c} & 0 \end{smallmatrix}\right)$, so that $\overline{c} \neq 0$ by the exactness of the sequence (E). Consequently, writing $c = \delta x_1 + \rho$ with $\delta \notin \mathfrak{m}$ and $\rho \in J$, we have $\delta x_1^2 \equiv ac = b^2 \equiv 0 \bmod J$. Hence $x_1^2 \in J$, so that $\mathfrak{c} = J$, as claimed.

Case 2 ($\overline{a} = 0$). Let $a = \alpha x_1^2 + \beta$ with $\alpha \in R$ and $\beta \in J$. Let us write $b = \gamma x_1 + \delta$ with $\gamma \in R$ and $\delta \in J$. Then, since $\mathfrak{m}/\mathfrak{c} = (\overline{b}) \neq (0)$, we get $\gamma \notin \mathfrak{m}$. Let $c = \rho x_1 + \eta$ with $\rho \in R$ and $\eta \in J$. Then, since $b^2 = ac$, we have $\gamma^2 x_1^2 \equiv \alpha\rho x_1^3 \bmod J$. Hence, $x_1^2 \in J$, and $\mathfrak{c} = J$. $\square$

Corollary 3.30. *Suppose that $(R, \mathfrak{m})$ is a* 2-AGL *ring with infinite residue class field. Let I be an Ulrich ideal I in R with $\mu_R(I) = 2$. Then, there exists a minimal system $x_1, x_2, \dots, x_n$ of generators of $\mathfrak{m}$ and $b \in \mathfrak{c}$ such that $\mathfrak{c} = (x_2, x_3, \dots, x_n)$ and $I = (x_1, b)$ with $I^2 = x_1 I$.*

Proof. Choose a minimal system $x_1, x_2, \dots, x_n$ of generators of $\mathfrak{m}$ such that $\mathfrak{c} = (x_1^2) + (x_2, x_3, \dots, x_n)$. Then, $\mathfrak{c} = (x_2, x_3, \dots, x_n)$ by Theorem 3.29. We write $I = (a, b)$ where both the ideals $(a), (b)$ are reductions of I. If $a \notin \mathfrak{c}$, then since $\mathfrak{m}/\mathfrak{c} = (\overline{a})$ where $\overline{*}$ denotes the image in $\mathfrak{m}/\mathfrak{c}$, we get $b = \alpha a + \beta$ with $\alpha \in R$ and $\beta \in \mathfrak{c}$. Hence $I = (a, b) = (a, \beta)$ and $\mathfrak{m} = (a) + \mathfrak{c}$. If $a \in \mathfrak{c}$, then $\mathfrak{m}/\mathfrak{c} = (\overline{b})$, so that $a = \alpha b + \beta$, whence $I = (\beta, b)$. $\square$

The following result is involved in [29, Theorem 2.8], since Cohen-Macaulay local rings of minimal multiplicity are G-regular ([44]). Let us give a brief proof in our context.

Corollary 3.31. *Let R be a* 2-AGL *ring and let K be a canonical fractional ideal of R. Assume that $S = R[K]$ is a Gorenstein ring. If R has minimal multiplicity, then R contains no two-generated Ulrich ideals.*

Proof. Because $R = K : K$ ([32, Bemerkung 2.5]) and $KS = S$,

$$\mathfrak{c} = R : S = K : S \cong \mathrm{Hom}_R(S, K).$$

Therefore, since S is a Gorenstein ring, we have $\mathfrak{c} \cong S$, so that $n - 1 = \mu_R(\mathfrak{c}) = \mu_R(S) = \mathrm{r}(R) + 1$, where the first (resp. third) equality follows from Theorem 3.29 (resp. Proposition 3.4 (4)). Thus, R doesn't have minimal multiplicity, because $\mathrm{r}(R) = n - 1$ otherwise. $\square$

The condition that $\mathfrak{c} \in \mathcal{X}_R$ is a strong restriction on 2-AGL rings R. We need the following, in order to see that 2-AGL rings might contain Ulrich ideals, which are not two-generated.

Proposition 3.32. *Suppose that R is a* 2-AGL *ring, possessing a canonical fractional ideal K. Let $S = R[K]$ and set $\mathfrak{c} = R : S$. Then the following conditions are equivalent.*

(1) $\mathfrak{c} \in \mathcal{X}_R$.

(2) *S is a Gorenstein ring and K/R is a free $R/\mathfrak{c}$-module.*

Proof. (2) $\Rightarrow$ (1) Since $\mathfrak{c} = K : S \cong \mathrm{Hom}_R(S, K)$, we have $\mathfrak{c} = fS$ for some $f \in S$, whence $\mathfrak{c}^2 = f\mathfrak{c}$. Therefore, $\mathfrak{c}/\mathfrak{c}^2$ is a free $R/\mathfrak{c}$-module if and only if so is S/R, because $\mathfrak{c}/fR \cong S/R$. The latter condition is equivalent to saying that K/R is a free $R/\mathfrak{c}$-module, which follows from the exact sequence

$$0 \to R/\mathfrak{c} \to S/\mathfrak{c} \to S/R \to 0$$

and the fact that $S/R \cong K/R \oplus R/\mathfrak{c}$ (Proposition 3.4 (3)).

(1) $\Rightarrow$ (2) By [18, Corollary 3.8], S is a Gorenstein ring, since $\mathfrak{c}^2 = f\mathfrak{c}$ for some $f \in \mathfrak{c}$. Therefore, $\mathfrak{c} = fS$ for some $f \in \mathfrak{c}$, since $\mathfrak{c} = K : S$. Thus, $\mathfrak{c}/\mathfrak{c}^2 \cong S/fS = S/\mathfrak{c}$, whence $S/\mathfrak{c}$ is a free $R/\mathfrak{c}$-module. Consequently, K/R is a free $R/\mathfrak{c}$-module, since $S/R \cong K/R \oplus R/\mathfrak{c}$. $\square$

Let us explore an example, which shows the set $\mathcal{X}_R$ depends on the characteristic of the base fields. For the ring stated in Example 3.33, we have the complete list of Ulrich ideals in it.

Example 3.33. Let $V = k[[t]]$ be the formal power series ring over a field k and set $R = k[[t^6, t^8, t^{10}, t^{11}]]$. Then the following assertions hold true.

(1) R is a 2-AGL ring with $\mathrm{r}(R) = 2$ and $S = k[[t^2, t^{11}]]$ is a Gorenstein ring with $\mathfrak{c} = (t^6, t^8, t^{10}) \in \mathcal{X}_R$. We have $\mathfrak{c} = (x_2, x_3, x_4)$ and $x_1^2 \in \mathfrak{c}$, where $x_1 = t^{11}, x_2 = t^6, x_3 = t^8$, and $x_4 = t^{10}$.

(2) Let $I \in \mathcal{X}_R$ and set $n = \mu_R(I)$. Then, $n = 2, 3$, and $n = 3$ if and only if $I = \mathfrak{c}$.

(3) If $\mathrm{ch}\, k \neq 2$, then the set of two-generated Ulrich ideals is

$$\left\{(t^6 + c_1t^8 + c_2t^{10}, t^{11}) \mid c_1, c_2 \in k\right\} \cup \left\{(t^8 + c_1t^{10} + c_2t^{12}, t^{11}) \mid c_1, c_2 \in k\right\}$$

and we have the following.

(*i*) $(t^6 + c_1t^8 + c_2t^{10}, t^{11}) = (t^6 + d_1t^8 + d_2t^{10}, t^{11})$, only if $c_1 = d_1$ and $c_2 = d_2$.

(*ii*) $(t^8 + c_1t^{10} + c_2t^{12}, t^{11}) = (t^8 + d_1t^{10} + d_2t^{12}, t^{11})$, only if $c_1 = d_1$ and $c_2 = d_2$.

(4) If $\mathrm{ch}\, k = 2$, then the set of two-generated Ulrich ideals is

$$\left\{(t^6 + c_1t^8 + c_2t^{10}, t^{11}) \mid c_1, c_2 \in k\right\} \cup \left\{(t^8 + c_1t^{10} + c_2t^{12}, t^{11} + dt^{12}) \mid c_1, c_2, d \in k\right\}$$
$$\cup \left\{(t^6 + c_1t^8 + c_2t^{11}, t^{10} + dt^{11}) \mid c_1, c_2, d \in k, d \neq 0\right\}$$

and we have the following.

(*i*) $(t^6 + c_1t^8 + c_2t^{10}, t^{11}) = (t^6 + d_1t^8 + d_2t^{10}, t^{11})$, only if $c_1 = d_1$ and $c_2 = d_2$.

(*ii*) $(t^8 + c_1t^{10} + c_2t^{12}, t^{11} + dt^{12}) = (t^8 + d_1t^{10} + d_2t^{12}, t^{11} + et^{12})$, only if $c_1 = d_1$, $c_2 = d_2$, and $d = e$.

(*iii*) $(t^6 + c_1t^8 + c_2t^{11}, t^{10} + dt^{11}) = (t^6 + d_1t^8 + d_2t^{11}, t^{10} + et^{11})$, only if $c_1 = d_1$, $c_2 = d_2$, and $d = e$.

(5) The Ulrich ideals in R generated by monomials in t are $\{(t^6, t^{11}), (t^8, t^{11}), \mathfrak{c} = (t^6, t^8, t^{10})\}$.

Proof. (1) Because $K = R + Rt^2$, we have $\mathrm{r}(R) = 2$ and $S = k[[t^2, t^{11}]]$, so that S is a Gorenstein ring, and $\mathfrak{c} = (t^6, t^8, t^{10})$, since $S = R + Rt^2 + Rt^4$. We have $\ell_R(S/K) = 2$, since $S/K = k{\cdot}\overline{t^4} + k{\cdot}\overline{t^{15}}$, where $\overline{t^4}$ and $\overline{t^{15}}$ denote the images of t^4 and t^{15} in S/K, respectively. Therefore, R is a 2-AGL ring by Theorem 3.2. Because K/R is a cyclic R-module, $K/R \cong R/\mathfrak{c}$, whence $\mathfrak{c} \in \mathcal{X}_R$ by Proposition 3.32.

(2) Since $(n-1){\cdot}\mathrm{r}(R/I) = \mathrm{r}(R) = 2$ by Proposition 3.24 (1), we get $n = 2, 3$. Suppose $n = 3$. Then, $\mathfrak{c} \subseteq I$ by Proposition 3.24 (2), since $K/R \cong R/\mathfrak{c}$. On the other hand, if

$\mathfrak{c} \subsetneq I$, we then have by [12, Theorem 3.1] $\mathfrak{c} = bcS$ for some $b, c \in \mathfrak{m}$. This is, however, impossible, because $\mathfrak{c} = t^6S$ and $b, c \in \mathfrak{m} \subseteq t^6V$. Therefore, $I = \mathfrak{c}$, if $n = 3$.

(3), (4) We denote by $\nu(*)$ the valuation of V. Let $I \in \mathcal{X}_R$ and suppose that $\mu_R(I) = 2$. Let us write $I = (a, b)$ where $a, b \in R$. First we may assume $I^2 = aI$ and $\nu(a) < \nu(b)$. We then have $\nu(a) < 11$. Indeed, if $\nu(a) \geq 12$, then $a, b \in \mathfrak{c} = (t^6, t^8, t^{10})$, so that $I \subseteq \mathfrak{c}$, which is absurd (remember that $I + \mathfrak{c} = \mathfrak{m}$). Besides, we notice that $\nu(a)$ is even, because $I/(a) \cong R/I$ as an R-module. Therefore, $\nu(a) = 6, 8, 10$. In addition, we have the following.

Claim 4. *The following assertions hold true.*

(*i*) *If $\nu(a) = 6$, then $\nu(b) = 10, 11$.*

(*ii*) *If $\nu(a) = 8$, then $\nu(b) = 11$.*

(*iii*) *One has $\nu(a) \neq 10$.*

(*iv*) *If $\mathrm{ch}\, k \neq 2$, then $(\nu(a), \nu(b)) \neq (6, 10)$.*

Proof of Claim 4. (*i*) We first consider the case where $\nu(a) = 6$. Then we get $\nu(b) < 12$. In fact, if $\nu(b) \geq 12$, then the images of $1, t^8, t^{10}, t^{11}$ in R/I are linearly independent over the field k, so that $\ell_R(R/I) \geq 4$. This makes a contradiction, because $I/(a) \cong R/I$. Hence $\nu(b) \leq 11$. We are now assuming that $\nu(b) = 8$. Since $b^2 = ac$ for some $c \in I$, we notice that $\nu(c) = 10$. Let us write $c = a\rho + b\eta$ where $\rho, \xi \in \mathfrak{m}$. We then have $c \in t^{12}V$, which is impossible. Consequently, $\nu(b) = 10, 11$ as claimed.

(*ii*) Suppose that $\nu(a) = 8$. By setting $b^2 = a^2\varphi + ab\psi$ for some $\varphi, \psi \in \mathfrak{m}$, we have $\nu(b) \neq 10$. Let us write $a = t^8 + \alpha t^{10} + \beta t^{11} + \xi$, where $\alpha, \beta \in k$ and $\xi \in R$ with $\nu(\xi) \geq 12$. If $\nu(b) \geq 12$, then $b \in \mathfrak{c}$, so that $a \notin \mathfrak{c}$, because $I + \mathfrak{c} = \mathfrak{m}$. Hence $\beta \neq 0$ (remember that $\mathfrak{c} = (t^6, t^8, t^{10})$). Therefore, if $\nu(b) \geq 14$ (resp. $\nu(b) = 12$), then the images of $1, t^6, t^8, t^{10}, t^{12}$ (resp. $1, t^6, t^8, t^{10}, t^{14}$) in R/I are linearly independent over k, so that $\ell_R(R/I) \geq 5$, which makes a contradiction, because $R/I \cong I/(a)$. Hence $\nu(b) = 11$.

(*iii*) Let us assume that $\nu(a) = 10$. Since $b^2 \in (a^2, ab)$, we have $\nu(b) \neq 11, 12$, whence $\nu(b) \geq 14$. Thus $b \in \mathfrak{c}$ and $a \notin \mathfrak{c}$. Then the images of $1, t^6, t^8, t^{10}, t^{14}, t^{16}$ in R/I are linearly independent over k, which is absurd.

(*iv*) Suppose that $\nu(a) = 6$ and $\nu(a) = 10$. We may assume $a = t^6 + c_1t^8 + c_2t^{11} + c_3t^{19}$ and $b = t^{10} + d_1t^{11} + d_2t^{19}$, where $c_i, d_j \in k$. Look at the isomorphism $R/I \cong k[Y, W]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal of $k[Y, W]$ generated by

$(-c_1Y - c_2W - c_3YW)^3 - Y(-d_1W - d_2YW), Y^2 - (-c_1Y - c_2W - c_3YW)(-d_1W - d_2YW),$
$(-d_1W - d_2YW)^2 - (-c_1Y - c_2W - c_3YW)^2 Y,$ and $W^2 - (-c_1Y - c_2W - c_3YW)^2(-d_1W - d_2YW).$

Hence $(Y,W)^3 + \mathfrak{a} = (Y,W)^3 + (Y^2, d_1YW, W^2)$. If $d_1 = 0$, then $\ell_R(R/I) \geq 4$, which is impossible. Therefore $d_1 \neq 0$. Since $I^2 = aI$, we can write $b^2 = a^2\varphi + ab\psi$ for some $\varphi, \psi \in \mathfrak{m}$. By comparing the coefficients of t^{21}, we have $2d_1 = 0$, so that $\mathrm{ch}\, k = 2$. Consequently, if $\mathrm{ch}\, k \neq 2$, then $(\nu(a), \nu(b)) \neq (6, 10)$, as desired. □

Notice that, for each $0 \neq f \in R$, we have $t^{n+16}V \subseteq (f)$, where $n = \nu(f)$. It follows from the equalities $t^{n+16}V = fV{\cdot}t^{16}V = f{\cdot}(R : V)$ and the fact that $(R : V)$ is an ideal of R.

(3) Suppose that $\mathrm{ch}\, k \neq 2$. First we consider the case where $\nu(a) = 6$ and $\nu(b) = 11$. Then $t^{33}V \subseteq (ab)$, so that $I = (t^6 + c_1t^8 + c_2t^{10}, t^{11})$ for some $c_1, c_2 \in k$. On the other hand, if we set $J = (t^6 + c_1t^8 + c_2t^{10}, t^{11})$ with $c_1, c_2 \in k$, then J is an Ulrich ideal of R. Let $a = t^6 + c_1t^8 + c_2t^{10}$. Notice that $t^n \in aJ$ for each even integer $n \geq 18$, because $t^n = t^{n-12}{\cdot}a^2 - t^{n-12}{\cdot}(c_1^2t^{16} + \cdots + c_2^2t^{20})$. Therefore, $J^2 = aJ + (t^{22}) = aJ$. Moreover, we have the isomorphism $R/J \cong k[Y,Z]/\mathfrak{a}$, where

$$\mathfrak{a} = \big((-c_1Y - c_2Z)^3 - YZ, Y^2 - (-c_1Y - c_2Z)Z, Z^2 - (-c_1Y - c_2Z)^2Y, (-c_1Y - c_2Z)Z\big)$$

which yields $\ell_R(R/J) = 3$, because $\mathfrak{a} + (Y,Z)^3 = (Y,Z)^2$. Hence $R/J \cong J/(a)$, so that $J \in \mathcal{X}_R$.

Let us assume $\nu(a) = 8$ and $\nu(b) = 11$. We may assume $a = t^8 + c_1t^{10} + c_2t^{12}$ and $b = t^{11} + dt^{12}$ where $c_1, c_2, d \in k$. The equality $I^2 = aI$ yields that $2d = 0$ by comparing the coefficients of t^{23}. Hence $d = 0$. Conversely, let $J = (t^8 + c_1t^{10} + c_2t^{12}, t^{11})$ for some $c_1, c_2 \in k$. Then $t^n \in aJ$ for each even integer $n \geq 22$, where $a = t^8 + c_1t^{10} + c_2t^{12}$. We have the isomorphism $R/J \cong k[X,Z]/\mathfrak{a}$, where

$$\mathfrak{a} = \big(X^3 - (-c_1Z - c_2X^2)Z, (-c_1Z - c_2X^2)^2 - XZ, Z^2 - X^2(-c_1Z - c_2X^2), -X^2Z\big)$$

while $\mathfrak{a} = (X^3, Z^2, X^2Z, XZ) = (X^3, XZ, Z^2)$. Therefore, $\ell_R(R/J) = 4$ and $J \in \mathcal{X}_R$. The last assertions follow from the same technique as in the proof of Example 3.26.

(4) Suppose that $\mathrm{ch}\, k = 2$. Thanks to the proof of (3), if $\nu(a) = 6, \nu(b) = 11$ (resp. $\nu(a) = 8, \nu(b) = 11$), then we have $I = (t^6 + c_1t^8 + c_2t^{10}, t^{11})$ (resp. $I = (t^8 + c_1t^{10} + c_2t^{12}, t^{11} + dt^{12})$) where $c_1, c_2 \in k$ (resp. $c_1, c_2, d \in k$).

Let us assume $\nu(a) = 6$ and $\nu(b) = 10$. We then have $I = (t^6 + c_1t^8 + c_2t^{11} + c_3t^{19}, t^{10} + d_1t^{11} + d_2t^{19})$ for some $c_1, c_2, c_3, d_1, d_2 \in k$. Consider the same isomorphism $R/I \cong k[Y,W]/\mathfrak{a}$ as in the proof of Claim 4 (iv). Then $(Y,W)^3 + \mathfrak{a} = (Y,W)^3 + (Y^2, d_1YW, W^2)$. Since $I \in \mathcal{X}_R$, we have $\ell_R(R/I) = 3$, whence $d_1 \neq 0$ and $\mathfrak{a} = (Y,W)^2$. Therefore, $YW \in \mathfrak{a}$ and $t^{19} \in I$. Consequently, $I = (t^6 + c_1t^8 + c_2t^{11}, t^{10} + d_1t^{11})$. For the converse, let $J = (t^6+c_1t^8+c_2t^{11}, t^{10}+d_1t^{11})$ and set $a = t^6+c_1t^8+c_2t^{11}$, where $c_1, c_2, d_1 \in k$ and $d_1 \neq 0$. Since $d_1 \neq 0$, we see that $\ell_R(R/J) = 3$ by the above isomorphism $R/J \cong k[Y,W]/\mathfrak{a}$. The fact that $t^n \in (a^2)$ for each even integer $n \geq 20$ implies $(t^{10} + d_1t^{11})^2 \in (a^2)$. Hence $J^2 = aJ$, so that $J \in \mathcal{X}_R$. Similarly for the proof of Example 3.26, we have the last assertions.

(5) Follows from the assertions (2), (3), and (4). $\square$

Closing this section, since the ring as in Example 3.33 is obtained from the gluing of the numerical semigroup $\langle 3, 4, 5\rangle$, let us explore the 2-AGL rings arising as gluing of numerical semigroup rings.

In what follows, let $0 < a_1, a_2, \dots, a_\ell \in \mathbb{Z}$ $(\ell > 0)$ be positive integers such that $\mathrm{GCD}(a_1, a_2, \dots, a_\ell) = 1$. We set $H_1 = \langle a_1, a_2, \dots, a_\ell\rangle$ and assume that $a_1, a_2, \dots, a_\ell$ forms a minimal system of generators of H_1. Let $0 < \alpha \in H_1$ be an odd integer such that $\alpha \neq a_i$ for every $1 \leq i \leq \ell$. We consider $H = \langle 2a_1, 2a_2, \dots, 2a_\ell, \alpha\rangle$ the gluing of H_1 and the set of non-negative integers $\mathbb{N}$. The reader is referred to [42, Chapter 8] for basic properties of gluing of numerical semigroups. Let $V = k[[t]]$ be the formal power series ring over a field k and set $R_1 = k[[H_1]]$, $R = k[[H]]$ the semigroup rings of H_1 and H, respectively. We denote by $\mathfrak{m}_1$ (resp. $\mathfrak{m}$) the maximal ideal of R_1 (resp. R). Notice that $\mu_R(\mathfrak{m}) = \ell + 1$ and R is a free R_1-module of rank 2. By letting $\mathrm{PF}(H_1) = \{p_1, p_2, \dots, p_r\}$, the canonical fractional ideal K_1 of R_1 has the form $K_1 = \sum_{i=1}^{r} R_1 {\cdot} t^{p_r - p_i}$, while $K = \sum_{i=1}^{r} R {\cdot} t^{2(p_r - p_i)}$ is the canonical fractional ideal of R, where $r = \mathrm{r}(R_1)$ and $p_r = f(H_1)$. We then have $R \otimes_{R_1} K_1 \cong K$ and hence $K/R \cong R \otimes_{R_1} (K_1/R_1)$ as an R-module. We set $\mathfrak{c} = R : R[K]$.

With this notation we have the following.

Proposition 3.34. *Suppose that R_1 is an* AGL *ring, but not a Gorenstein ring. Then the following assertions hold true.*

(1) *R is a* 2-AGL *ring, $\mathfrak{c} = \mathfrak{m}_1R$, and $\mu_R(\mathfrak{c}) = \ell \geq 3$.*

(2) *$\mathfrak{c} \in \mathcal{X}_R$ if and only if R_1 has minimal multiplicity.*

(3) *R doesn't have minimal multiplicity. Therefore, $\mathfrak{m} \notin \mathcal{X}_R$.*

Proof. (1) Since R is a free R_1-module of rank 2 and $\ell_R(R/\mathfrak{m}_1 R) = 2$, we conclude that R is a 2-AGL ring ([4, Theorem 3.10]). Besides, we have $\mathfrak{c} = \operatorname{Ann}_R K/R = (\operatorname{Ann}_{R_1} K_1/R_1)R = \mathfrak{m}_1 R$, whence $\mu_R(\mathfrak{c}) = \ell \geq 3$.

(2) The isomorphisms $\mathfrak{c}/\mathfrak{c}^2 \cong R \otimes_{R_1} (\mathfrak{m}_1/{\mathfrak{m}_1}^2) \cong R \otimes_{R_1} (R_1/\mathfrak{m}_1)^{\oplus \ell} \cong (R/\mathfrak{c})^{\oplus \ell}$ show that $\mathfrak{c}/\mathfrak{c}^2$ is a free $R/\mathfrak{c}$-module. Hence, $\mathfrak{c} \in \mathcal{X}_R$ if and only if $\mathfrak{c}^2 = f\mathfrak{c}$ for some $f \in \mathfrak{c}$. The latter condition is equivalent to saying that $\mathfrak{c}^2 = t^{2a_i}\mathfrak{c}$ for some $1 \leq i \leq \ell$, that is ${\mathfrak{m}_1}^2 = t^{2a_i}\mathfrak{m}_1$, as desired.

(3) We notice that $\mu_R(\mathfrak{m}) = \ell + 1$ and $\mathrm{e}(R) = \min\{2a_1, 2a_2, \ldots, 2a_\ell, \alpha\}$. Suppose that $\mathrm{e}(R) = 2a_i$ for some $1 \leq i \leq \ell$. Since $\ell = \mu_{R_1}(\mathfrak{m}_1) \leq \mathrm{e}(R_1) \leq a_1$, we get

$$\mathrm{e}(R) - \mu_R(\mathfrak{m}) = 2a_i - (\ell + 1) \geq 2\ell - (\ell + 1) = \ell - 1 \geq 2$$

which implies that R doesn't have minimal multiplicity. Thereafter, we consider the case where $\mathrm{e}(R) = \alpha$. Suppose that R has minimal multiplicity, that is $\mathrm{e}(R) = \mu_R(\mathfrak{m})$, in order to seek a contradiction. Since α is an odd integer, we notice that ℓ is even, because $\alpha = \mathrm{e}(R) = \mu_R(\mathfrak{m}) = \ell + 1$. Besides, $\alpha < 2a_i$ for each $1 \leq i \leq \ell$. Let us write $\alpha = \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_\ell a_\ell$ where $\alpha_i \geq 0$. Then one of the $\{\alpha_i\}_{1 \leq i \leq \ell}$ is positive. Therefore, $\alpha = \alpha_i a_i$ for some $1 \leq i \leq \ell$, so that $\alpha_i = 1$ and $\alpha = a_i$. This makes a contradiction. Hence R doesn't have minimal multiplicity. $\square$

Consequently, we have the following.

Theorem 3.35. *Suppose that R_1 is an* AGL *ring, but not a Gorenstein ring. Then the following assertions hold true.*

(1) *Let $I \in \mathcal{X}_R$. Then either $\mu_R(I) = 2$ or $I = \mathfrak{c}$.*

(2) *The set of two-generated Ulrich ideals which are generated by monomials in t is*

$$\left\{(t^{2m}, t^{\alpha}) \mid 0 < m \in H_1,\ \alpha - m \in H_1,\ 2(\alpha - 2m) \in H\right\}.$$

Proof. (1) Thanks to Proposition 3.24 (2), if $\mu_R(I) \geq 3$, then $\mathfrak{c} \subseteq I$. Since R is a 2-AGL ring and $\mathfrak{m} \notin \mathcal{X}_R$, we conclude that $I = \mathfrak{c}$.

(2) Let $I \in \mathcal{X}_R$ such that $\mu_R(I) = 2$ and I is generated by monomials in t. We write $I = (t^p, t^q)$ where $0 < p < q$ and $p, q \in H$. Notice that, for each $0 < h \in H$ with $h \neq \alpha$,

we have that $t^h \in \mathfrak{c}$. Since $I + \mathfrak{c} = \mathfrak{m}$ by Theorem 3.29 (2), we get $I \not\subseteq \mathfrak{c}$, which yields that $p = \alpha$ or $q = \alpha$. The isomorphism $R/I \cong I/(t^p)$ ensures that p is even, so that $\alpha = q$. Therefore $0 < p < \alpha$. Let us write $p = \sum_{i=1}^{\ell} (2a_i) c_i + c\alpha = 2\left(\sum_{i=1}^{\ell} a_i c_i\right) + c\alpha$ where $c_i, c \geq 0$. As $p < \alpha$, we have $c = 0$. Therefore, $p = 2m$ for some $0 < m \in H_1$. Moreover, because $I^2 = t^{2m} I$, we have $2(\alpha - 2m) \in H$, but $\alpha - 2m \notin H$. Since $R/I = R/(t^{2m}, t^\alpha) \cong R_1/(t^m, t^\alpha)$ and $\ell_R(R/I) = m$, we obtain that $t^\alpha \in t^m R_1$. Hence $\alpha - m \in H_1$.

Conversely, let $I = (t^{2m}, t^\alpha)$ where $0 < m_1 \in H_1$, $\alpha - m \in H_1$, and $2(\alpha - 2m) \in H$. We then have $I^2 = t^{2m} I + (t^{2\alpha}) = t^{2m} I$, while $R/I \cong R_1/(t^m, t^\alpha) = R_1/t^m R_1$, so that $\ell_R(R/I) = m$. Therefore $I \in \mathcal{X}_R$, as desired. $\square$

Example 3.36. Let $H_1 = \langle 4, 7, 9 \rangle$ and $\alpha \geq 11$ an odd integer. We set $R_1 = k[[t^4, t^7, t^9]]$ the numerical semigroup ring of H_1 over a field k. By Example 3.22, R_1 is an AGL ring with $\mathrm{r}(R_1) = 2$. Let $H = \langle 8, 14, 18, \alpha \rangle$ and set $R = k[[H]]$. Then $\mu_R(I) = 2$ for each $I \in \mathcal{X}_R$. Moreover, we have the following.

(1) If $\alpha = 11, 13$, then $\mathcal{X}_R = \emptyset$.

(2) If $\alpha \geq 15$, then $(t^8, t^\alpha) \in \mathcal{X}_R$.

(3) If $\alpha = 15$ and $\operatorname{ch} k = 2$, then $(t^8 + ct^{14}, t^\alpha) \in \mathcal{X}_R$ for every $c \in k$, and we have $(t^8 + c_1 t^{14}, t^\alpha) = (t^8 + c_1 t^{14}, t^\alpha)$, only if $c_1 = c_2$.

(4) If $\alpha \geq 17$, then $(t^8 + ct^{14}, t^\alpha) \in \mathcal{X}_R$ for every $c \in k$, and we have $(t^8 + c_1 t^{14}, t^\alpha) = (t^8 + c_1 t^{14}, t^\alpha)$, only if $c_1 = c_2$.

3.5 G-regularity in 2-AGL rings

The condition that K/R is a free $R/\mathfrak{c}$-module gives an agreeable restriction on the behavior of 2-AGL rings, as we have shown in Proposition 3.32 (see also [4, Section 5]). However, even though K/R is not $R/\mathfrak{c}$-free, 2-AGL rings also enjoy nice properties. We will show in the following that every 2-AGL ring R is G-*regular* in the sense of [44], namely, totally reflexive R-modules are all free, provided K/R is not $R/\mathfrak{c}$-free.

Theorem 3.37. *Suppose that R is a* 2-AGL *ring, possessing a canonical fractional ideal K. We set $\mathfrak{c} = R : R[K]$, and assume that K/R is not a free $R/\mathfrak{c}$-module. Let M be a finitely generated R-module. If $\operatorname{Ext}_R^p(M, R) = (0)$ for all $p \gg 0$, then $\operatorname{pd}_R M < \infty$. Hence R is G-regular in the sense of* [44].

Proof. Let $L = \Omega_R^1(M)$ be the first syzygy module of M. For every $p \geq 2$ we have $\mathrm{Ext}_R^{p-1}(L, R) \cong \mathrm{Ext}_R^p(M, R)$, which shows $\mathrm{Ext}_R^p(L, K/R) = (0)$ for all $p \gg 0$, because $\mathrm{Ext}_R^p(L, K) = (0)$. Therefore, since $R/\mathfrak{m}$ is a direct summand of K/R (Proposition 3.4 (4)), $\mathrm{Ext}_R^p(L, R/\mathfrak{m}) = (0)$ for $p \gg 0$, so that $\mathrm{pd}_R L < \infty$. Hence $\mathrm{pd}_R M < \infty$. $\square$

We should compare the following result with [29, Theorem 2.14 (1)], where a corresponding result for one-dimensional AGL rings is given.

Corollary 3.38. *Suppose that* $(R, \mathfrak{m})$ *is a* 2-AGL *ring with minimal multiplicity, possessing a canonical fractional ideal* K *and* $\mathfrak{c} = R : R[K]$. *Then*

$$\mathcal{X}_R = \begin{cases} \{\mathfrak{c}, \mathfrak{m}\}, & \text{if } K/R \text{ is } R/\mathfrak{c}\text{-free}, \\ \{\mathfrak{m}\}, & \text{otherwise.} \end{cases}$$

Proof. Since R has minimal multiplicity, $\mathfrak{m} \in \mathcal{X}_R$, so that $\mathcal{X}_R \neq \emptyset$.

(1) Suppose that K/R is $R/\mathfrak{c}$-free. Then, by [4, Proposition 5.7 (1)], $\mathfrak{m} : \mathfrak{m}$ is a local ring, while $S = R[K]$ is a Gorenstein ring, since R is a 2-AGL ring with minimal multiplicity ([4, Corollary 5.3]). Therefore, thanks to Proposition 3.32, $\mathfrak{c} = R : S \in \mathcal{X}_R$, so that $\{\mathfrak{c}, \mathfrak{m}\} \subseteq \mathcal{X}_R$. Let $I \in \mathcal{X}_R$. Then, because R has minimal multiplicity, $\mu_R(I) \geq 3$ by Corollary 3.31. Therefore, since K/R is $R/\mathfrak{c}$-free, we get $\mathfrak{c} = (0) :_R K/R \subseteq I$ ([29, Corollary 2.13]). Thus, $I = \mathfrak{c}$ or $I = \mathfrak{m}$, because $\ell_R(R/\mathfrak{c}) = 2$.

(2) Suppose that K/R is not $R/\mathfrak{c}$-free and let I be an Ulrich ideal of R. Then, $\mu_R(I) \geq 3$ by Theorem 3.29. Therefore, thanks to the proof of case (1), $I = \mathfrak{c}$ or $I = \mathfrak{m}$. Thus, $I = \mathfrak{m}$, because $\mathfrak{c} \notin \mathcal{X}_R$ by Proposition 3.32. $\square$

We close this chapter with the following, where two kinds of 2-AGL rings of minimal multiplicity are given, one is $R/\mathfrak{c}$-free and the other one is not.

Example 3.39. Let $V = k[[t]]$ denote the formal power series ring over a field k and set $R_1 = k[[t^3, t^7, t^8]]$, $R_2 = k[[t^4, t^9, t^{11}, t^{14}]]$. Let K_i be a canonical fractional ideal of R_i. Then, both R_1 and R_2 are 2-AGL rings. We have K_1/R_1 is a free $R/\mathfrak{c}_1$-module, but K_2/R_2 is not $R/\mathfrak{c}_2$-free, where $\mathfrak{c}_i = R_i : R_i[K_i]$. Therefore, $\mathcal{X}_{R_1} = \{(t^6, t^7, t^8), (t^3, t^7, t^8)\}$, and $\mathcal{X}_{R_2} = \{(t^4, t^9, t^{11}, t^{14})\}$.

Proof. We have $K_1 = R + Rt$ and $K_2 = R + Rt + Rt^5$. Hence, $R_1[K_1] = R[t] = V$, and $R_2[K_2] = R[t^3, t^5] = k[[t^3, t^4, t^5]]$, so that $\ell_{R_1}(R_1[K_1]/K_1) = \ell_{R_2}(R_2[K_2]/K_2) = 2$. Therefore, by Theorem 3.2, both R_1 and R_2 are 2-AGL rings. Because $\ell_{R_1}(K_1/R_1) =$

2 and $\ell_{R_2}(K_2]/R_2) = 3$, K_1/R_1 is a free $R/\mathfrak{c}_1$-module, but K_2/R_2 is not $R/\mathfrak{c}_2$-free (use Proposition 3.4 (4)). Notice that R_1 and R_2 have minimal multiplicity 3 and 4, respectively. Hence, the results readily follow from Corollary 3.38, since $\mathfrak{c}_1 = R_1 : V = t^6V = (t^6, t^7, t^8)$. $\square$

3.6 Appendix: Ulrich ideals in one-dimensional Gorenstein local rings of finite Cohen-Macaulay representation type

In [24], the authors determined all the Ulrich ideals in one-dimensional Gorenstein local rings R of finite CM-representation type, while in [28, Section 12] most birational module-finite extensions of these rings have been searched. Since the proof given by [24] depends on the techniques in the representation theory of maximal Cohen-Macaulay modules, it might have some interests to give a straightforward proof, making use of the results of [28, Section 12] and determining the members of $\mathcal{A}_R^0$ by Lemma 3.25, as well. We note it as an appendix.

In this appendix, let $(R, \mathfrak{m})$ be a Gorenstein complete local ring of dimension one with algebraically closed residue class field k of characteristic 0. Suppose that R has finite CM-representation type. Then, by [49, (8.5), (8.10), and (8.15)] we get

$$R \cong k[[X, Y]]/(f),$$

where $k[[X, Y]]$ is the formal power series ring over k, and f is one of the following polynomials.

(A_n) $X^2 - Y^{n+1}$ $(n \geq 1)$

(D_n) $X^2Y - Y^{n-1}$ $(n \geq 4)$

(E_6) $X^3 - Y^4$

(E_7) $X^3 - XY^3$

(E_8) $X^3 - Y^5$

With this notation we have the following.

Theorem 3.40 ([24, Theorem 1.7]). *The set $\mathcal{X}_R$ is given by the following.*

$$(A_n)\ \mathcal{X}_R = \begin{cases} \{(x, y^q) \mid 0 < q \leq \ell\} & \textit{if } n = 2\ell - 1 \textit{ with } \ell \geq 1, \\ \{(x, y^q) \mid 0 < q \leq \ell\} & \textit{if } n = 2\ell \textit{ with } \ell \geq 1. \end{cases}$$

$$(D_n)\ \mathcal{X}_R = \begin{cases} \{(x^2, y), (x, y^{\ell+1})\} & \textit{if } n = 2\ell + 3 \textit{ with } \ell \geq 1, \\ \{(x^2, y), (x - y^\ell, y(x + y^\ell)), (x + y^\ell, y(x - y^\ell))\} & \textit{if } n = 2(\ell + 1) \textit{ with } \ell \geq 1. \end{cases}$$

(E_6) $\mathcal{X}_R = \{(x, y^2)\}$

(E_7) $\mathcal{X}_R = \{(x, y^3)\}$

(E_8) $\mathcal{X}_R = \emptyset$

where x and y denote the images of X and Y in the corresponding rings, respectively.

Proof. For a ring A, let $J(A)$ denote its Jacobson radical. We denote by $\overline{R}$ the integral closure of R in $\mathrm{Q}(R)$, and by $\mathcal{B}_R$ the set of birational module-finite extensions of R.

(1) (E_6) See Example 3.26.

(2) (E_8) Let $R = k[[t^3, t^5]]$ and $V = k[[t]]$. By [28, Proposition 12.7 (3)], $\mathcal{B}_R = \{R, k[[t^3, t^5, t^7]], k[[t^3, t^4, t^5]], k[[t^2, t^3]], V\}$, among which $k[[t^3, t^5, t^7]], k[[t^3, t^4, t^5]]$ are not Gorenstein rings, and $\mu_R(V) = \mu_R(k[[t^2, t^3]]) = 3$. Hence, $\mathcal{A}_R^0 = \emptyset$, so that $\mathcal{X}_R = \emptyset$ by Lemma 3.25.

(3) (E_7) Let $R = k[[X, Y]]/(X^3 - XY^3)$. We set $S = k[[X, Y]]$, $V = k[[t]]$, and $f = X^3 - XY^3$. Then, since $(f) = (X) \cap (X^2 - Y^3)$, we get the tower

$$R = S/(f) \subseteq S/(X) \oplus S/(X^2 - Y^3) = k[[Y]] \oplus k[[t^2, t^3]] \subseteq k[[Y]] \oplus V = \overline{R}$$

of rings, where we identify $S/(X) = k[[Y]]$ and $S/(X^2 - Y^3) = k[[t^2, t^3]] \subseteq V$.

Claim 5. $\mathcal{A}_R = \{R, k[[Y]] \oplus k[[t^2, t^3]], k[[Y]] \oplus V, k + J(\overline{R})\}$.

Proof of Claim 5. Let $A \in \mathcal{B}_R$ such that $R \neq A$ and let $p_2 : \overline{R} \to V$ denote the projection. We set $B = p_2(A)$. Since $k[[t^2, t^3]] \subseteq B \subseteq V$, $B = k[[t^2, t^3]]$ or $B = V$. Suppose that A is not a local ring. Then, A decomposes into a direct product of local rings, since A is a module-finite extension of the complete local ring R, so that we may choose a non-trivial idempotent $e \in A$. Then, since $\overline{R} = k[[X]] \oplus V$, we get $e = (1, 0)$, or $(0, 1)$, whence $(1, 0), (0, 1) \in A$, so that $A = A(1, 0) + A(0, 1) = k[[Y]] \oplus B$. Suppose that A is a local ring. In this case, the argument in [28, Pages 2708–2710] shows that if $B = V$, then $A \cong k[[Y, Z]]/(Z(Y - Z^2)) = k[[(Y, t^2), (0, t)]] = k + J(\overline{R})$, and that if $B = k[[t^2, t^3]]$, then A is an AGL but not a Gorenstein ring. Thus we have the assertion. □

Since $J(\overline{R}) = R(Y, t^2) + R(0, t) + R(0, t^2)$, we have $k + J(\overline{R}) = R + R(0, t) + R(0, t^2)$, whence $\mu_R(k + J(\overline{R})) = 3$. Therefore, $\mathcal{A}_R^0 = \{k[[Y]] \oplus k[[t^2, t^3]]\}$, so that by Lemma 3.25 $\mathcal{X}_R = \{(x, y^3)\}$, since $R : (k[[Y]] \oplus k[[t^2, t^3]]) = (x, y^3)$.

(4) (D_n) (i) (*The case where* $n = 2\ell+3$ *with* $\ell \geq 1$). Let $R = k[[X, Y]]/(X^2Y - Y^{2\ell+2})$. We set $S = k[[X, Y]]$, $V = k[[t]]$, and $f = Y(X^2 - Y^{2\ell+1})$. We consider the tower

$$R = S/(f) \subseteq S/(Y) \oplus S/(X^2 - Y^{2\ell+1}) = k[[X]] \oplus k[[t^2, t^{2\ell+1}]] \subseteq k[[X]] \oplus V = \overline{R}$$

of rings, where we identify $S/(Y) = k[[X]]$ and $S/(X^2 - Y^{2\ell+1}) = k[[t^2, t^{2\ell+1}]]$. We then have the following.

Claim 6. $\mathcal{A}_R = \left\{R, k + J(\overline{R})\right\} \cup \{k[[X]] \oplus k[[t^2, t^{2q+1}]] \mid 0 \leq q \leq \ell\}$

Proof of Claim 6. Let $A \in \mathcal{B}_R$ such that $R \neq A$ and let $p_2 : \overline{R} \to V$ denote the projection. We set $B = p_2(A)$. Then, by [28, Corollary 12.5 (1)] $B = k[[t^2, t^{2q+1}]]$ for some $0 \leq q \leq \ell$, since $k[[t^2, t^{2\ell+1}]] \subseteq B \subseteq V$. If A is not a local ring, then the same proof as in Claim 5 works, to get $A = k[[X]] \oplus B$. If A is a local ring, then by the argument in [28, Pages 2710–2711] we have $A \cong k[[X, Z]]/[(Z) \cap (X - Z^{2\ell+1})] = k + J(\overline{R})$. $\square$

Consequently, $\mathcal{A}_R^0 = \left\{k[[X]] \oplus k[[t^2, t^{2\ell+1}]], k + J(\overline{R})\right\}$. We have

$$\left(k[[X]] \oplus k[[t^2, t^{2\ell+1}]]\right)/R \cong S/(X^2, Y)$$

and $k + J(\overline{R}) = R + R(0, t)$. Therefore, Lemma 3.25 shows the assertion, because

$$R : \left(k[[X]] \oplus k[[t^2, t^{2\ell+1}]]\right) = (x^2, y) \quad \text{and} \quad R : \left(k + J(\overline{R})\right) = (x, y^{\ell+1}).$$

(4) (D_n) (ii) (*The case where* $n = 2\ell+2$ *with* $\ell \geq 1$). Let $R = k[[X, Y]]/(X^2Y - Y^{2\ell+1})$. We set $S = k[[X, Y]]$, $V = k[[t]]$, and $f = Y(X^2 - Y^{2\ell})$. Consider the tower

$$R = S/(f) \subseteq S/(Y) \oplus T = k[[X]] \oplus \overline{T} = \overline{R}$$

of rings, where $T = S/(X^2 - Y^{2\ell})$. By [28, Page 2711] an intermediate ring $R \subsetneq A \subseteq \overline{R}$ is an AGL ring but not a Gorenstein ring, if A is a local ring. Therefore, every $A \in \mathcal{A}_R$ is not local, if $R \neq A$.

Claim 7. $\mathcal{A}_R = \left\{R, S/(X - Y^\ell) \oplus S/(Y(X + Y^\ell)), S/(X + Y^\ell) \oplus S/(Y(X - Y^\ell))\right\} \cup \left\{k[[X]] \oplus T[\frac{x}{y^q}] \mid 0 \leq q \leq \ell\right\}$

Proof of Claim 7. Let $A \in \mathcal{A}_R$ such that $R \neq A$. Note that $\overline{R} = k[[X]] \oplus S/(X - Y^\ell) \oplus S/(X + Y^\ell)$. Let $\{e_i\}_{i=1,2,3}$ be the orthogonal primitive idempotents of $\overline{R}$. Then, $e_i \in A$ for some $1 \leq i \leq 3$, since A is not a local ring. If $A \neq \overline{R}$, such e_i is unique for A.

(i) (*The case where* $e_1 \in A$). Let $p : \overline{R} \to S/(X - Y^\ell) \oplus S/(X + Y^\ell)$ denote the projection. Then

$$T := S/(X - Y^\ell) \cap (X + Y^\ell) \subseteq p(A) \subseteq \overline{T} = S/(X - Y^\ell) \oplus S/(X + Y^\ell)$$

so that, by [28, Corollary 12.5 (2)] $p(A) = T[\frac{x}{y^q}]$ for some $0 \leq q < \ell$. Hence, $A = k[[X]] \oplus T[\frac{x}{y^q}]$.

(ii) (*The case where* $e_2 \in A$). Let $p : \overline{R} \to k[[X]] \oplus S/(X + Y^\ell)$ denote the projection. Because $A \neq \overline{R}$, we have

$$S/(Y) \cap (X + Y^\ell) \subseteq p(A) \subsetneq k[[X]] \oplus S/(X + Y^\ell),$$

which shows $p(A) = S/(Y) \cap (X + Y^\ell) = S/(Y(X + Y^\ell))$. Thus, $A = S/(X - Y^\ell) \oplus S/(Y(X + Y^\ell))$. Similarly, $A = S/(X + Y^\ell) \oplus S/(Y(X - Y^\ell))$ if $e_3 \in A$, which proves Claim 7. □

Therefore,

$$\mathcal{A}_R^0 = \left\{k[[X]] \oplus T, S/(X - Y^\ell) \oplus S/(Y(X + Y^\ell)), S/(X + Y^\ell) \oplus S/(Y(X - Y^\ell))\right\},$$

so that $\mathcal{X}_R = \left\{(x^2, y), (x - y^\ell, y(x + y^\ell)), (x + y^\ell, y(x - y^\ell))\right\}$.

(5) (A_n) (i) (*The case where* $n = 2\ell$ *with* $\ell \geq 1$). Let $R = k[[t^2, t^{2\ell+1}]]$. Then, $\mathcal{A}_R^0 = \{k[[t^2, t^{2q+1} \mid 0 \leq q \leq \ell - 1]]\}$ by [28, Corollary 12.5 (1)], whence $\mathcal{X}_R = \{(x, y^q) \mid 0 < q \leq \ell\}$.

(5) (A_n) (ii) (*The case where* $n = 2\ell - 1$ *with* $\ell \geq 1$). Let $R = k[[X, Y]]/(X^2 - Y^{2\ell})$. We set $S = k[[X, Y]]$ and $f = X^2 - Y^{2\ell} = (X - Y^\ell)(X + Y^\ell)$. We then have $\ell_R(\overline{R}/R) = \ell$ by the exact sequence

$$0 \to R = S/(f) \longrightarrow \overline{R} = S/(X - Y^\ell) \oplus S/(X + Y^\ell) \longrightarrow S/(X, Y^\ell) \to 0$$

of R-modules. Let $A \in \mathcal{A}_R$ such that $R \neq A$. Then, by [28, Corollary 12.5 (2)] $A = R\left[\frac{x}{y^q}\right]$ for some $0 < q \leq \ell$ in $\mathrm{Q}(R)$. If $n = \ell$, then $A = \overline{R}$ is a Gorenstein ring with $\mu_R(\overline{R}) = 2$, so that $(x, y^\ell) = R : \overline{R} \in \mathcal{X}_R$.

Let us now assume that $0 < q < \ell$. Since $(\frac{x}{y^q})^2 = x^2y^{-2q} = y^{2\ell}y^{-2q} = y^{2(\ell-q)} \in R$, we have $A = R + R{\cdot}\frac{x}{y^q}$. We will show that A is a Gorenstein local ring with $\mu_R(A) = 2$. Indeed, set let $\mathfrak{n} = \mathfrak{m}A + \frac{x}{y^q}A$ of A, and let M be an arbitrary maximal ideal of A. We choose a maximal ideal N of $\overline{R}$ so that $M = N \cap A$. We then have $N \supseteq J(\overline{R}) \supseteq y\overline{R} + \frac{x}{y^q}\overline{R}$, whence $M = N \cap A \supseteq \mathfrak{n}$, so that $M = \mathfrak{n}$ because $\mathfrak{n}$ is a maximal ideal of A. Hence, $(A, \mathfrak{n})$ is a local ring. Consequently, $2 \le \mu_R(A) = \ell_R(A/\mathfrak{m}A) \le \mathrm{e}(A) \le \mathrm{e}(R) = 2$. Thus $A \in \mathcal{X}_R^0$. Note that $R : A = R :_R \frac{x}{y^q}$, because $A = R + R\frac{x}{y^q}$. We now take $a \in R : \frac{x}{y^n}$. Then, setting $b = a{\cdot}\frac{x}{y^q} \in R$, we have $ax = by^q$, so that $AX - BY^q = C(X^2 - Y^{2\ell})$ for some $C \in S$. Here a, b are the images of A, B respectively. Therefore $X(A - CX) = Y^q(B - Y^{2\ell-q})$. Since X, Y^q forms an S-regular sequence, we have $A - CX = Y^qD$ for some $D \in S$. Hence, $a \in (x, y^q)R$, so that $R : A = (x, y^q)$. Therefore $\mathcal{X}_R = \{(x, y^q) \mid 0 < q \le \ell\}$. $\square$

Remark 3.41. The assertion on the ring of type (A_n) also follows from [12, Theorem 4.5]. In fact, the ring R of type (A_n) has minimal multiplicity 2. Hence, by [12, Theorem 4.3] $\mathcal{X}_R$ is totally ordered with respect to inclusion, and $R : \overline{R}$ is the minimal element of $\mathcal{X}_R$.

4 The structure of Ulrich ideals in hypersurfaces

4.1 Introduction

The purpose of this chapter is to investigate the structure and ubiquity of Ulrich ideals in a hypersurface ring.

In a Cohen-Macaulay local ring $(R, \mathfrak{m})$, an $\mathfrak{m}$-primary ideal I is called an Ulrich ideal in R if there exists a parameter ideal Q of R such that $I \supsetneq Q$, $I^2 = QI$, and I/I^2 is R/I-free. The notion of Ulrich ideal/module dates back to the work [24] in 2014, where S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe, and K.-i. Yoshida introduced the notion, generalizing that of maximally generated maximal Cohen-Macaulay modules ([3]), and started the basic theory. The maximal ideal of a Cohen-Macaulay local ring with minimal multiplicity is a typical example of Ulrich ideals, and the higher syzygy modules of Ulrich ideals are Ulrich modules. In [24, 25], all Ulrich ideals of Gorenstein local rings of finite CM-representation type with dimension at most 2 are determined by means of the classification in the representation theory. In [29], S. Goto, R. Takahashi, and N. Taniguchi studied the structure of the complex $\mathbf{R}\mathrm{Hom}_R(R/I, R)$ for Ulrich ideals I in a Cohen-Macaulay local ring of arbitrary dimension, and proved that in a one-dimensional non-Gorenstein almost Gorenstein local ring $(R, \mathfrak{m})$, the only possible Ulrich ideal is the maximal ideal $\mathfrak{m}$ ([29, Theorem 2.14]). On the other hand, in [12], S. Goto, the author, and S. Kumashiro closely explored the structure of chains of Ulrich ideals in a one-dimensional Cohen-Macaulay local ring, and studied the structure of the set $\mathcal{X}_R$ of Ulrich ideals in R. Recently, S. Goto, the author, and N. Taniguchi explored Ulrich ideals in a one-dimensional 2-AGL ring, and proved the result corresponding to [29, Theorem 2.14].

Nevertheless, even for the case of hypersurface rings, there seems known only scattered results which give a complete list of Ulrich ideals, except the case of finite CM-representation type and the case of several numerical semigroup rings. Therefore, in the current chapter, we focus our attention on a hypersurface ring which is not necessarily finite CM-representation type.

In what follows, unless otherwise specified, let $(S, \mathfrak{n})$ be a Cohen-Macaulay local ring with $\dim S = d + 1$ ($d \geq 1$), and $f \in \mathfrak{n}$ a non-zero divisor on S. We set $R = S/(f)$. In

Section 4.2, we will summarize a few results and basic properties of Ulrich ideals, which we shall need later. In Section 4.3, we shall study the structure of Ulrich ideals in R. In Proposition 4.4, we give a sufficient condition for an ideal I of R to be an Ulrich ideal. By using the condition, we can construct many Ulrich ideals in R as images of parameter ideals of S. Furthermore, we have the following, which is one of the main results of this chapter. For each $a \in S$, let $\overline{a}$ denote the image of a in R. We denote by $\mathcal{X}_R$ the set of Ulrich ideals in R. The converse of Proposition 4.4 is also true if S is a regular local ring (i.e. R is a hypersurface ring).

Theorem 4.1. (=Theorem 4.5) *Suppose that* $(S, \mathfrak{n})$ *is a regular local ring with* $\dim S = d+1$ $(d \geq 1)$ *and* $0 \neq f \in \mathfrak{n}$. *Set* $R = S/(f)$. *Then we have*

$$\mathcal{X}_R = \left\{ (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}, \overline{b}) \;\middle|\; \begin{array}{l} a_1, a_2, \ldots, a_d, b \in \mathfrak{n} \text{ be a system of parameters of } S, \\ \text{and there exist } x_1, x_2, \ldots, x_d \in (a_1, a_2, \cdots, a_d, b) \text{ and } \varepsilon \in U(S) \\ \text{such that } b^2 + \sum_{i=1}^{d} a_i x_i = \varepsilon f. \end{array} \right\},$$

where $U(S)$ *denotes the set of unit elements of* S.

On the other hand, the structure of minimal free resolutions of Ulrich ideals was closely explored in [24, 29]. In Section 4.4, we construct a minimal free resolution of R/I more concretely, for an Ulrich ideal I which is obtained in Section 4.3 (Theorem 4.9). We also give a matrix factorization of the d-th syzygy module of R/I, which is an Ulrich module with respect to I (Corollary 4.11). In Section 4.5, we consider the structure of decomposable Ulrich ideals. We shall give a characterization of decomposable 2-generated Ulrich ideals in a one-dimensional Cohen-Macaulay local ring. In the last section, we focus our attention on the case of $S = k[[X, Y]]$ which is the formal power series ring over a field k. The purpose of this section is to make a complete list of Ulrich ideals in R which is not finite CM-representation type. We shall give the list for the case of $f = Y^k$ and $f = X^k Y$ (Proposition 4.17, Theorem 4.19, Corollary 4.21, Theorem 4.23, Theorem 4.26, and Theorem 4.30).

Throughout this chapter, let $\mathrm{r}(R)$ denote the Cohen-Macaulay type of R, and $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators of M (resp. the length of M), for a finitely generated R-module M. We denote by $\mathcal{X}_R$ the set of Ulrich ideals in R.

4.2 Basic facts

In this section, let us recall the definition and basic properties of Ulrich ideals. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = d \geq 0$, and I an $\mathfrak{m}$-primary ideal of R. We assume that I contains a parameter ideal Q of R as a reduction.

Definition 4.2. ([24, Definition 1.1]) We say that I is an *Ulrich ideal* in R, if the following conditions are satisfied.

(1) $I \neq Q$, but $I^2 = QI$.

(2) I/I^2 is a free R/I-module.

In Definition 4.2, Condition (1) is equivalent to saying that the associated graded ring $gr_I(R) = \oplus_{n \geq 0} I^n/I^{n+1}$ is a Cohen-Macaulay ring with $a(gr_I(R)) = 1-d$, where $a(gr_I(R))$ denotes the a-invariant of $gr_I(R)$ ([27, Remark 3.10], [30, Remark 3.1.6]). Therefore, Condition (1) is independent of the choice of reductions Q of I. In addition, Condition (2) is equivalent to saying that I/Q is a free R/I-module, provided Condition (1) is satisfied ([24, Lemma 2.3]). If $I = \mathfrak{m}$, then Condition (2) is automatically satisfied. Hence, when the residue class field $R/\mathfrak{m}$ of R is infinite, the maximal ideal $\mathfrak{m}$ is an Ulrich ideal if and only if R is not a regular local ring, possessing minimal multiplicity ([43]).

For a finitely generated R-module M, we denote by $\text{G-dim}_R\, M$ the Gorenstein dimension (G-dimension for short) of M. With this notation, we then have the following.

Theorem 4.3 ([24, Theorem 7.1, Theorem 7.6], [29, Theorem 2.5, Theorem 2.8]). *Let I be an Ulrich ideal in a Cohen-Macaulay local ring R, and set $n = \mu_R(I)$. Let*

$$\cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 = R \to R/I \to 0$$

be a minimal free resolution of R/I. Then, setting $t = n - d$, the following assertions hold true.

(1) $t \cdot \mathrm{r}(R/I) = \mathrm{r}(R)$.

(2) $\mathbf{I}(\partial_i) = I$ *for all* $i \geq 1$.

(3) *For* $i \geq 0$, $\beta_i = \begin{cases} t^{i-d} \cdot (t+1)^d & (i \geq d), \\ \binom{d}{i} + t \cdot \beta_{i-1} & (1 \leq i \leq d), \\ 1 & (i = 0). \end{cases}$

(4) $n = d+1$ *if and only if* $G\text{-}\dim_R R/I < \infty$.

Here, $\mathbf{I}(\partial_i)$ *denotes the ideal of* R *generated by the entries of the matrix* ∂_i*, and* $\beta_i = \mathrm{rank}_R F_i$.

Therefore, when R is a Gorenstein ring, every Ulrich ideal I is generated by $d+1$ elements, if it exists, and R/I has finite G-dimension but infinite projective dimension. Moreover, because I/Q is a free R/I-module, we have $I = Q :_R I$, that is I is a *good ideal* in the sense of [15]. Similar to good ideals, Ulrich ideals are characteristic ideals, but behave very well in their nature ([24, 25]).

4.3 Ulrich ideals in hypersurfaces

In this section, we give a characterization of Ulrich ideals in a hypersurface ring. Firstly, let $(S, \mathfrak{n})$ be a Cohen-Macaulay local ring with $\dim S = d+1$ $(d \geq 1)$, and $f \in \mathfrak{n}$ a non-zero divisor on S. We set $R = S/(f)$ and $\mathfrak{m} = \mathfrak{n}/(f)$. For each $a \in S$, let $\overline{a}$ denote the image of a in R, and $U(S)$ denote the set of unit elements of S. We then have the following.

Proposition 4.4. *Let* $a_1, a_2, \ldots, a_d, b \in \mathfrak{n}$ *be a system of parameters of* S*. Suppose that there exist* $x_1, x_2, \ldots, x_d \in (a_1, a_2, \cdots, a_d, b)$ *and* $\varepsilon \in U(S)$ *such that* $b^2 + \sum_{i=1}^{d} a_i x_i = \varepsilon f$*. Then* $I = (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}, \overline{b}) \in \mathcal{X}_R$.

Proof. Since $a_1, \ldots, a_d, b$ is a system of parameters of S, I is an $\mathfrak{m}$-primary ideal of R. Let $Q = (\overline{a_1}, \cdots, \overline{a_d})$. Then $\overline{b}^2 \in QI$, since $b^2 + \sum_{i=1}^{d} a_i x_i = \varepsilon f$, therefore $I^2 = QI$. It suffices to show that $I/Q \cong R/I$ (see [24, Lemma 2.3]). Since I/Q is a homomorphic image of R/I, it is enough to show that $\ell_R(R/I) = \ell_R(I/Q)$, which is equivalent to $\ell_R(R/Q) = 2 \cdot \ell_R(R/I)$. In fact, we have

$$\ell_R(R/Q) = \ell_S(S/(a_1, \cdots, a_d, f)) = \ell_S(S/(a_1, \cdots, a_d, b^2)) = 2 \cdot \ell_R(R/I),$$

where the second equality follows from the relation $b^2 + \sum_{i=1}^{d} a_i x_i = \varepsilon f$, and the third equality follows from the fact that $a_1, \ldots, a_d, b$ is a system of parameters of S. □

The converse of Proposition 4.4 is also true if S is a regular local ring. The following is the main result of this section.

Theorem 4.5. *Suppose that $(S, \mathfrak{n})$ is a regular local ring. Then we have*

$$\mathcal{X}_R = \left\{ (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}, \overline{b}) \;\middle|\; \begin{array}{l} a_1, a_2, \ldots, a_d, b \in \mathfrak{n} \textit{ be a system of parameters of } S, \\ \textit{and there exist } x_1, x_2, \ldots, x_d \in (a_1, a_2, \cdots, a_d, b) \textit{ and } \varepsilon \in U(S) \\ \textit{such that } b^2 + \sum_{i=1}^{d} a_i x_i = \varepsilon f. \end{array} \right\}.$$

In order to prove Theorem 4.5, we need the following. We have learned the following lemma from Professor K.-i. Yoshida.

Lemma 4.6. *Suppose that S is a regular local ring. Assume that $a_1, a_2, \ldots, a_d, b \in \mathfrak{n}$ and $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}, \overline{b}) \in \mathcal{X}_R$. Then $f \in (a_1, a_2, \cdots, a_d, b)^2$, and therefore $a_1, a_2, \ldots, a_d, b$ is a system of parameters of S.*

Proof. Set $I = (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}, \overline{b})$. We look at the minimal free resolution

$$F : \cdots \to F_i \overset{\partial_i}{\to} F_{i-1} \to \cdots \to F_1 \overset{\partial_1}{\to} F_0 = R \overset{\varepsilon}{\to} R/I \to 0$$

of R/I, and set $M = \mathrm{Im}\partial_d$. Since $R = S/(f)$ is a hypersurface ring, there exist $A, B \in M_n(S)$ such that $0 \to S^{\oplus n} \overset{A}{\to} S^{\oplus n} \overset{\varepsilon}{\to} M \to 0$ is exact as S-modules and $AB = BA = fE_n$, where $\mathrm{n} = \mu_R(M)$. Whence $\cdots \to R^n \overset{\overline{B}}{\to} R^n \overset{\overline{A}}{\to} R^n \overset{\overline{B}}{\to} R^n \overset{\overline{A}}{\to} R^n \overset{\varepsilon}{\to} M \to 0$ is a minimal free resolution of M. Then $\mathbf{I}(\overline{A}) = \mathbf{I}(\overline{B}) = I$ in R by [24, Theorem 7.6], that is $\mathbf{I}(A) \subseteq (a_1, \cdots, a_d, b) + (f)$ and $\mathbf{I}(B) \subseteq (a_1, \cdots, a_d, b) + (f)$ in S, where $\mathbf{I}(*)$ denotes the ideal of R generated by the entries of the matrix $*$. Since $AB = fE_n$, we get

$$f \in \mathbf{I}(A) \cdot \mathbf{I}(B) \subseteq [(a_1, \cdots, a_d, b) + (f)]^2 = (a_1, \cdots, a_d, b)^2 + f[(a_1, \cdots, a_d, b) + (f)],$$

thus $f \in (a_1, \cdots, a_d, b)^2$ by Nakayama's lemma. $\square$

We are now ready to prove Theorem 4.5.

Proof of Theorem 4.5. Thanks to Proposition 4.4, we have only to show the inclusion $(\subseteq)$. Let $I \in \mathcal{X}_R$. Since $\mu_R(I) = d + 1$ by Theorem 4.3 (1), we can choose $a_1, \ldots, a_d, b \in \mathfrak{n}$ so that $I = (\overline{a_1}, \cdots, \overline{a_d}, \overline{b})$, and $I^2 = (\overline{a_1}, \cdots, \overline{a_d})I$. Then, by using Lemma 4.6, $a_1, \ldots, a_d, b$ is a system of parameters of S and $f \in (a_1, a_2, \cdots, a_d, b)^2$. We write $f = \sum_{i=1}^{d} a_i y_i + \delta b^2$ with $y_1, \ldots, y_d \in (a_1, \cdots, a_d, b)$ and $\delta \in S$. Then we get

$$\begin{aligned} \ell_R(R/Q) &= \ell_S(S/(a_1, \cdots, a_d, f)) = \ell_S(S/(a_1, \cdots, a_d, \delta b^2)) \\ &= \ell_S(S/(a_1, \cdots, a_d, \delta)) + 2 \cdot \ell_R(R/I). \end{aligned}$$

Because $I \in \mathcal{X}_R$ and $\mu_R(I) = d+1$, we have $I/Q \cong R/I$, whence $\ell_R(R/Q) = 2 \cdot \ell_R(R/I)$ (see the proof of Proposition 4.4). Therefore, we have $\ell_S(S/(a_1, \cdots, a_d, \delta)) = 0$, that is $\delta \in U(S)$. Setting $x_i = \delta^{-1} y_i$ ($\in (a_1, \cdots, a_d, b)$) and $\varepsilon = \delta^{-1}$, we get $b^2 + \sum_{i=1}^{d} a_i x_i = \varepsilon f$, which completes the proof of Theorem 4.5. □

The following is a direct consequence of Theorem 4.5, which gives many examples of Ulrich ideals.

Corollary 4.7. *Suppose that $f = b^2$ for some $b \in \mathfrak{n}$. Then, for any system of parameters $a_1, a_2, \ldots, a_d$ of $S/(b)$, we have $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}, \overline{b}) \in \mathcal{X}_R$.*

Proof. We can put $x_i = 0$ and $\varepsilon = 1$. □

We will use Proposition 4.4, Theorem 4.5, and Corollary 4.7 later in Section 4.6.

4.4 Minimal free resolutions

In this section, we construct a minimal free resolution of an Ulrich ideal which is obtained in Section 4.3. Although it is well known that this resolution can be constructed by using Tate's construction ([46, Theorem 4]), let us give another construction by using properties of Ulrich ideals. We begin with the following lemma.

Lemma 4.8. *Suppose that S is a commutative ring and $a_1, \ldots, a_d, x_1, \ldots, x_d \in S$ $(d \geq 1)$. We set $K = \mathrm{K}_\bullet(a_1, \ldots, a_d; S) = (K_\bullet, \partial_\bullet^K)$ and $L = \mathrm{K}_\bullet(x_1, \ldots, x_d; S) = (K_\bullet, \partial_\bullet^L)$ are Koszul complexes of S generated by $a_1, \ldots, a_d$ and $x_1, \ldots, x_d$, and $c = \sum_{i=1}^{d} a_i x_i$. Then*

$$\partial_p^K \cdot {}^t\partial_p^L + {}^t\partial_{p-1}^L \cdot \partial_{p-1}^K = c \cdot id_{K_{p-1}} \text{ for any } p \in \mathbb{Z},$$

where ${}^t*$ denotes the transpose of the matrix $*$.

Proof. We may assume that $1 \leq p \leq d+1$. If $p = 1$,

$$\partial_1^K = \begin{bmatrix} a_1 & a_2 & \cdots & a_d \end{bmatrix}, {}^t\partial_1^L = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}, {}^t\partial_0^L = 0, \text{and } \partial_0^K = 0,$$

hence $\partial_1^K \cdot {}^t\partial_1^L + {}^t\partial_0^L \cdot \partial_0^K = \partial_1^K \cdot {}^t\partial_1^L = c$. If $p = d+1$,

$$\partial_{d+1}^K = 0, {}^t\partial_{d+1}^L = 0, {}^t\partial_d^L = \begin{bmatrix} x_1 & \cdots & (-1)^{i+1}x_i & \cdots & (-1)^{d+1}x_d \end{bmatrix}, \text{and } \partial_d^K = \begin{bmatrix} a_1 \\ \vdots \\ (-1)^{i+1}a_i \\ \vdots \\ (-1)^{d+1}a_d \end{bmatrix},$$

hence $\partial_{d+1}^K \cdot {}^t\partial_{d+1}^L + {}^t\partial_d^L \cdot \partial_d^K = {}^t\partial_d^L \cdot \partial_d^K = c$.

We now assume that $2 \leq p \leq d$. Set $K_1 = \sum_{i=1}^d RT_i$, $\Lambda = \{1, 2, \cdots, d\}$, and $F_i = \{I \subseteq \Lambda \mid \sharp I = i\}$ for $0 \leq i \leq d$. For $I = \{j_1 < j_2 < \cdots < j_p\} \in F_p$, let $T_I = T_{j_1} \wedge T_{j_2} \wedge \cdots \wedge T_{j_p}$. Then $K_p = \oplus_{I \in F_p} RT_I$, and the matrix ∂_p^K(resp. ∂_p^L) has the following form

$$[\partial_p^K]_{I,J} \text{ (resp. } [\partial_p^L]_{I,J}) = \begin{cases} 0 & \text{if } I \not\subseteq J, \\ (-1)^{\alpha+1}a_{j_\alpha} \text{ (resp. } (-1)^{\alpha+1}x_{j_\alpha}) & \text{if } \begin{array}{l} I \subseteq J,\ J = \{j_1 < \cdots < j_p\}, \\ \text{and } I = J \setminus \{j_\alpha\}, \end{array} \end{cases}$$

for $I \in F_{p-1}$ and $J \in F_p$. We need the following Claim.

Claim. *For $I_1, I_2 \in F_{p-1}$, the following assertions hold true.*

(1) $\sharp(I_1 \cup I_2) \geq p+1$ *if and only if* $\sharp(I_1 \cap I_2) \leq p-3$.

(2) $\sharp(I_1 \cup I_2) = p$ *if and only if* $\sharp(I_1 \cap I_2) = p-2$.

(3) $\sharp(I_1 \cup I_2) \leq p-1$ *if and only if* $\sharp(I_1 \cap I_2) \geq p-1$. *When this is the case,* $I_1 = I_2$.

Proof of Claim. Focus on the number $\sharp(I_1 \setminus I_2)$. (1) is the case $\sharp(I_1 \setminus I_2) \geq 2$, (2) is $\sharp(I_1 \setminus I_2) = 1$, otherwise (3). ☐

It suffices to show that

$$[\partial_p^K \cdot {}^t\partial_p^L + {}^t\partial_{p-1}^L \cdot \partial_{p-1}^K]_{I_1,I_2} = \begin{cases} 0 & \text{if } \sharp(I_1 \cup I_2) \geq p+1 \\ 0 & \text{if } \sharp(I_1 \cup I_2) = p \\ c & \text{if } \sharp(I_1 \cup I_2) \leq p-1 \end{cases}$$

for any $I_1, I_2 \in F_{p-1}$ by Claim. We notice that

$$\begin{aligned}
[\partial_p^K \cdot {}^t\partial_p^L + {}^t\partial_{p-1}^L \cdot \partial_{p-1}^K]_{I_1,I_2} &= [\partial_p^K \cdot {}^t\partial_p^L]_{I_1,I_2} + [{}^t\partial_{p-1}^L \cdot \partial_{p-1}^K]_{I_1,I_2} \\
&= \sum_{J \in F_p} [\partial_p^K]_{I_1,J} \cdot [\partial_p^L]_{I_2,J} + \sum_{J' \in F_{p-2}} [\partial_{p-1}^L]_{J',I_1} \cdot [\partial_{p-1}^K]_{J',I_2} \\
&= \sum_{J \in F_p,\ I_1 \cup I_2 \subseteq J} [\partial_p^K]_{I_1,J} \cdot [\partial_p^L]_{I_2,J} + \sum_{J' \in F_{p-2},\ J' \subseteq I_1 \cap I_2} [\partial_{p-1}^L]_{J',I_1} \cdot [\partial_{p-1}^K]_{J',I_2}.
\end{aligned}$$

If $\sharp(I_1 \cup I_2) \geq p+1$, then $\{J \in F_p \mid I_1 \cup I_2 \subseteq J\} = \emptyset$ and $\{J' \in F_{p-2} \mid J' \subseteq I_1 \cap I_2\} = \emptyset$ by Claim. Therefore $[\partial_p^K \cdot {}^t\partial_p^L + {}^t\partial_{p-1}^L \cdot \partial_{p-1}^K]_{I_1,I_2} = 0$.

If $\sharp(I_1 \cup I_2) = p$, we set $I_1 = \{j_1 < j_2 < \cdots < j_{p-1}\}$ and $I_2 = \{\ell_1 < \ell_2 < \cdots < \ell_{p-1}\}$, and take $j_\alpha \in I_1 \setminus I_2$ and $\ell_\beta \in I_2 \setminus I_1$ $(1 \leq \alpha, \beta \leq p-1)$. Then we have

$$\{J \in F_p \mid I_1 \cup I_2 \subseteq J\} = \{I_1 \cup I_2\} = \{I_1 \cup \{\ell_\beta\}\} = \{I_2 \cup \{j_\alpha\}\}, \text{ and}$$

$$\{J' \in F_{p-2} \mid J' \subseteq I_1 \cap I_2\} = \{I_1 \cap I_2\} = \{I_1 \setminus \{j_\alpha\}\} = \{I_2 \setminus \{\ell_\beta\}\},$$

hence we get

$$\begin{aligned}
[\partial_p^K \cdot {}^t\partial_p^L]_{I_1,I_2} &= [\partial_p^K]_{I_1, I_1 \cup \{\ell_\beta\}} \cdot [\partial_p^L]_{I_2, I_2 \cup \{j_\alpha\}} \\
&= \begin{cases} (-1)^{\beta+1} a_{\ell_\beta} \cdot (-1)^{\alpha+2} x_{j_\alpha} & \text{if } j_\alpha > \ell_\beta \\ (-1)^{\beta+2} a_{\ell_\beta} \cdot (-1)^{\alpha+1} x_{j_\alpha} & \text{if } j_\alpha < \ell_\beta \end{cases} \\
&= (-1)^{\alpha+\beta+1} a_{\ell_\beta} x_{j_\alpha}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
[{}^t\partial_{p-1}^L \cdot \partial_{p-1}^K]_{I_1,I_2} &= [\partial_{p-1}^L]_{I_1 \setminus \{j_\alpha\}, I_1} \cdot [\partial_{p-1}^K]_{I_2 \setminus \{\ell_\beta\}, I_2} \\
&= (-1)^{\alpha+1} x_{j_\alpha} \cdot (-1)^{\beta+1} a_{\ell_\beta} \\
&= (-1)^{\alpha+\beta} a_{\ell_\beta} x_{j_\alpha}.
\end{aligned}$$

Therefore $[\partial_p^K \cdot {}^t\partial_p^L + {}^t\partial_{p-1}^L \cdot \partial_{p-1}^K]_{I_1,I_2} = 0$.

If $\sharp(I_1 \cup I_2) \leq p-1$, then $I_1 = I_2$, whence

$$\{J \in F_p \mid I_1 \cup I_2 \subseteq J\} = \{I_1 \cup \{j\} \mid j \in \Lambda \setminus I_1\}, \text{ and}$$

$$\{J' \in F_{p-2} \mid J' \subseteq I_1 \cap I_2\} = \{I_1 \setminus \{j\} \mid j \in I_1\}.$$

Hence we get

$$\begin{aligned}
[\partial_p^K \cdot {}^t\partial_p^L]_{I_1,I_2} &= \sum_{j \in \Lambda \setminus I_1} [\partial_p^K]_{I_1, I_1 \cup \{j\}} \cdot [\partial_p^L]_{I_1, I_1 \cup \{j\}} \\
&= \sum_{j \in \Lambda \setminus I_1} a_j x_j, \text{ and}
\end{aligned}$$

$$\begin{aligned}
[{}^t\partial_{p-1}^L \cdot \partial_{p-1}^K]_{I_1,I_2} &= \sum_{j \in I_1} [\partial_{p-1}^L]_{I_1 \setminus \{j\}, I_1} \cdot [\partial_{p-1}^K]_{I_1 \setminus \{j\}, I_1} \\
&= \sum_{j \in I_1} a_j x_j.
\end{aligned}$$

We then have $[\partial_p^K \cdot {}^t\partial_p^L + {}^t\partial_{p-1}^L \cdot \partial_{p-1}^K]_{I_1,I_2} = \sum_{j \in \Lambda \setminus I_1} a_j x_j + \sum_{j \in I_1} a_j x_j = c$. $\square$

In what follows, throughout this section, we assume that $(S, \mathfrak{n})$ is a Cohen-Macaulay local ring with $\dim S = d+1$ $(d \geq 1)$, and $f \in \mathfrak{n}$ is a non-zero divisor on S. Set $R = S/(f)$. Let $a_1, \ldots, a_d, b \in \mathfrak{n}$ be a system of parameters of S, so that $b^2 + \sum_{i=1}^d a_i x_i = \varepsilon f$ with $x_1, \ldots, x_d \in (a_1, \cdots, a_d, b)$ and $\varepsilon \in U(S)$. Then $I = (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}, \overline{b}) \in \mathcal{X}_R$, with a reduction $Q = (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d})$, by Proposition 4.4. We notice that every Ulrich ideal in R is this form, if S is a regular local ring (Theorem 4.5). We also notice that $I/Q \cong R/I$. By [24, Corollary 7.2], in the exact sequence $0 \to Q \xrightarrow{\iota} I \to R/I \to 0$, the free resolution of I induced from minimal free resolutions of Q and R/I is also minimal. We construct this resolution, by using the relation $b^2 + \sum_{i=1}^d a_i x_i = \varepsilon f$. We set

$$K = \mathrm{K}_\bullet(a_1, \ldots, a_d; S) = (K_\bullet, \partial_\bullet^K) \text{ and } L = \mathrm{K}_\bullet(x_1, \ldots, x_d; S) = (K_\bullet, \partial_\bullet^L)$$

are Koszul complexes of S generated by $a_1, \ldots, a_d$ and $x_1, \ldots, x_d$. We define $G = (G_\bullet, \partial_\bullet)$ by $G_0 = K_0$, $G_i = K_i \oplus G_{i-1} = S^{\oplus \sum_{j=0}^i \binom{d}{j}}$ for $i \geq 1$, and

$$\partial_1 = \left[\begin{array}{c|c} \partial_1^K & b \end{array}\right], \partial_2 = \left[\begin{array}{c|c} \partial_2^K & -bE_d \mid {}^t\partial_1^L \\ \hline O & \partial_1 \end{array}\right], \text{ and}$$

$$\partial_i = \left[\begin{array}{c|c} \partial_i^K & (-1)^{i-1} b E_{\binom{d}{i-1}} \mid {}^t\partial_{i-1}^L \mid O \\ \hline O & \partial_{i-1} \end{array}\right] \text{ for } i \geq 3.$$

We notice that $\partial_i = \partial_{d+1}$ for any $i \geq d+1$. Set $F = (F_\bullet, \overline{\partial_\bullet}) = (G_\bullet \otimes R, \partial_\bullet \otimes R)$. We then have the following, which is the main result of this section.

Theorem 4.9. *$F : \cdots \to F_i \xrightarrow{\overline{\partial_i}} F_{i-1} \to \cdots \to F_1 \xrightarrow{\overline{\partial_1}} F_0 = R \xrightarrow{\varepsilon} R/I \to 0$ is a minimal free resolution of R/I.*

To prove Theorem 4.9, we give the following proposition.

Proposition 4.10. *Set $g = \varepsilon f$ $(= b^2 + \sum_{i=1}^d a_i x_i)$. Then*

$$\partial_i \cdot \partial_{i+1} = \left[\begin{array}{c|c} O & gE_{\sum_{j=0}^{i-1}\binom{d}{j}} \end{array}\right] \text{ for } i \geq 1.$$

In particular, $\partial_{d+1}^2 = gE_{2^d}$.

Proof. We have

$$\begin{aligned}
\partial_1 \cdot \partial_2 &= \left[\begin{array}{c|c} \partial_1^K & b \end{array}\right] \cdot \left[\begin{array}{c|c} \partial_2^K & -bE_d \mid {}^t\partial_1^L \\ \hline O & \partial_1 \end{array}\right] \\
&= \left[\begin{array}{c|c} \partial_1^K & b \end{array}\right] \cdot \left[\begin{array}{c|c|c} \partial_2^K & -bE_d & {}^t\partial_1^L \\ \hline O & \partial_1^K & b \end{array}\right] \\
&= \left[\begin{array}{c|c|c} O & O & \partial_1^K \cdot {}^t\partial_1^L + b^2 \end{array}\right] = \left[\begin{array}{c|c} O & g \end{array}\right],
\end{aligned}$$

$$
\begin{aligned}
\partial_2 \cdot \partial_3 &= \left[\begin{array}{c|c|c} \partial_2^K & -bE_d & {}^t\partial_1^L \\ \hline O & \partial_1^K & b \end{array}\right] \cdot \left[\begin{array}{c|c|c|c} \partial_3^K & bE_{\binom{d}{2}} & {}^t\partial_2^L & O \\ \hline O & \partial_2^K & -bE_d & {}^t\partial_1^L \\ \hline O & O & \partial_1^K & b \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c} O & O & \partial_2^K \cdot {}^t\partial_2^L + {}^t\partial_1^L \cdot \partial_1^K + b^2 E_d & O \\ \hline O & O & O & \partial_1^K \cdot {}^t\partial_1^L + b^2 \end{array}\right] = \left[\begin{array}{c|c} O & gE_d \end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
\partial_3 \cdot \partial_4 &= \left[\begin{array}{c|c|c|c} \partial_3^K & bE_{\binom{d}{2}} & {}^t\partial_2^L & O \\ \hline O & \partial_2^K & -bE_d & {}^t\partial_1^L \\ \hline O & O & \partial_1^K & b \end{array}\right] \cdot \left[\begin{array}{c|c|c|c|c} \partial_4^K & -bE_{\binom{d}{3}} & {}^t\partial_3^L & O & O \\ \hline O & \partial_3^K & bE_{\binom{d}{2}} & {}^t\partial_2^L & O \\ \hline O & O & \partial_2^K & -bE_d & {}^t\partial_1^L \\ \hline O & O & O & \partial_1^K & b \end{array}\right] \\
&= \left[\begin{array}{c|c|c|c|c} O & O & \partial_3^K \cdot {}^t\partial_3^L + {}^t\partial_2^L \cdot \partial_2^K + b^2 E_{\binom{d}{2}} & O & O \\ \hline O & O & O & \partial_2^K \cdot {}^t\partial_2^L + {}^t\partial_1^L \cdot \partial_1^K + b^2 E_d & O \\ \hline O & O & O & O & \partial_1^K \cdot {}^t\partial_1^L + b^2 \end{array}\right] \\
&= \left[\begin{array}{c|c} O & gE_{\sum_{j=0}^{2}\binom{d}{j}} \end{array}\right],
\end{aligned}
$$

by Lemma 4.8. Hence, we may assume that $i \geq 4$ and our assertion holds true for $i-1$. Let $A_j = \left[(-1)^{j-1} bE_{\binom{d}{j-1}} \mid {}^t\partial_{j-1}^L \mid O\right]$ for $j \geq 1$. Then

$$
\begin{aligned}
\partial_i \cdot \partial_{i+1} &= \left[\begin{array}{c|c} \partial_i^K & A_i \\ \hline O & \partial_{i-1} \end{array}\right] \cdot \left[\begin{array}{c|c} \partial_{i+1}^K & A_{i+1} \\ \hline O & \partial_i \end{array}\right] \\
&= \left[\begin{array}{c|c} O & \partial_i^K \cdot A_{i+1} + A_i \cdot \partial_i \\ \hline O & \partial_{i-1} \cdot \partial_i \end{array}\right] = \left[\begin{array}{c|c} O & \partial_i^K \cdot A_{i+1} + A_i \cdot \partial_i \\ \hline O & O \mid gE_{\sum_{j=0}^{i-2}\binom{d}{j}} \end{array}\right], \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
\partial_i^K \cdot A_{i+1} + A_i \cdot \partial_i &= \left[(-1)^i b\partial_i^K \mid \partial_i^K \cdot {}^t\partial_i^L \mid O\right] \\
&\quad + \left[(-1)^{i-1} bE_{\binom{d}{i-1}} \middle| {}^t\partial_{i-1}^L \middle| O\right] \cdot \left[\begin{array}{c|c|c} \partial_i^K & (-1)^{i-1} bE_{\binom{d}{i-1}} & {}^t\partial_{i-1}^L \mid O \\ \hline O & \partial_{i-1}^K & (-1)^{i-2} bE_{\binom{d}{i-2}} \mid O \\ \hline O & O & \partial_{i-2} \end{array}\right] \\
&= \left[(-1)^i b\partial_i^K \mid \partial_i^K \cdot {}^t\partial_i^L \mid O\right] + \left[(-1)^{i-1} b\partial_i^K \mid {}^t\partial_{i-1}^L \cdot \partial_{i-1}^K + b^2 E_{\binom{d}{i-1}} \mid O\right] \\
&= \left[O \middle| \partial_i^K \cdot {}^t\partial_i^L + {}^t\partial_{i-1}^L \cdot \partial_{i-1}^K + b^2 E_{\binom{d}{i-1}} \middle| O\right] = \left[O \middle| gE_{\binom{d}{i-1}} \middle| O\right],
\end{aligned}
$$

by Lemma 4.8. Therefore $\partial_i \cdot \partial_{i+1} = \left[O \middle| gE_{\sum_{j=0}^{i-1}\binom{d}{j}}\right]$. $\square$

We are now ready to prove Theorem 4.9.

Proof of Theorem 4.9. Thanks to Proposition 4.10, $\overline{\partial_i} \cdot \overline{\partial_{i+1}} = 0$ for all $i \geq 1$, hence F is a complex. Let $Q = (\overline{a_1}, \cdots, \overline{a_d})$. Then $\overline{K} = (\overline{K_\bullet}, \overline{\partial_\bullet^K}) = (K_\bullet \otimes R, \partial_\bullet^K \otimes R)$ is a minimal free resolution of R/Q, since Q is a parameter ideal of R, and $\overline{K}$ is a subcomplex of F. On the other hand, $0 \to Q \overset{\iota}{\to} I \to R/I \to 0$ is exact and the following diagrams

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{K_0} & \overset{\iota}{\longrightarrow} & F_1 & \longrightarrow & F_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \overline{\partial_1}} & & \downarrow{\scriptstyle \varepsilon} & & \\
0 & \longrightarrow & Q & \overset{\iota}{\longrightarrow} & I & \longrightarrow & R/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{K_{i+1}} & \overset{\iota}{\longrightarrow} & F_{i+1} & \longrightarrow & F_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{\partial^K_{i+1}}} & & \downarrow{\scriptstyle \overline{\partial_{i+1}}} & & \downarrow{\scriptstyle \overline{\partial_i}} & & \\
0 & \longrightarrow & \overline{K_i} & \overset{\iota}{\longrightarrow} & F_i & \longrightarrow & F_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{\partial^K_i}} & & \downarrow{\scriptstyle \overline{\partial_i}} & & \downarrow{\scriptstyle \overline{\partial_{i-1}}} & & \\
0 & \longrightarrow & \overline{K_{i-1}} & \overset{\iota}{\longrightarrow} & F_{i-1} & \longrightarrow & F_{i-2} & \longrightarrow & 0
\end{array}
$$

are commutative, for all $i \geq 2$. Therefore, F is exact, whence F is a minimal free resolution of R/I, since every entry of $\partial_\bullet$ is not a unit. This completes the proof of Theorem 4.9. $\square$

As a consequence, we get a matrix factorization of d-th syzygy module of R/I, which is an Ulrich module with respect to I (see [24, Definition 1.2]).

Corollary 4.11. *Let $M = \mathrm{Im}\overline{\partial_d}$. Then $0 \to G_{d+1} \overset{\partial_{d+1}}{\to} G_d \overset{\tau}{\to} M \to 0$ is exact as S-modules and $\partial_{d+1}^2 = gE_{2^d}$, where $\tau : G_d \overset{\varepsilon}{\to} F_d \overset{\overline{\partial_d}}{\to} M$. Therefore ∂_{d+1} gives a matrix factorization of M.*

Proof. Set $n = 2^d$. Because $\partial_{d+1}^2 = gE_n$ (Proposition 4.10) and g is a non-zero divisor on S, the map $G_{d+1} \overset{\partial_{d+1}}{\to} G_d$ is injective. $\tau \circ \partial_{d+1} = 0$ is clear. Suppose that

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathrm{Ker}\,\tau. \text{ Then, since } \overline{\partial_d} \cdot \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \\ \vdots \\ \overline{x_n} \end{pmatrix} = 0 \text{ in } R, \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \\ \vdots \\ \overline{x_n} \end{pmatrix} = \overline{\partial_d + 1} \cdot \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \\ \vdots \\ \overline{y_n} \end{pmatrix} \text{ for some } y_i \in S,$$

by Theorem 4.9. Therefore

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \partial_{d+1} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + g \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \partial_{d+1} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \partial_{d+1}^2 \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \partial_{d+1} \cdot \left[\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \partial_{d+1} \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \right],$$

for some $z_i \in S$. Thus $0 \to G_{d+1} \overset{\partial_{d+1}}{\to} G_d \overset{\tau}{\to} M \to 0$ is exact. □

We close this section with Examples.

Example 4.12. (1) If $d = 1$, then

$$\partial_1 = \begin{bmatrix} a_1 & b \end{bmatrix}, \text{ and } \partial_2 = \begin{bmatrix} -b & x_1 \\ a_1 & b \end{bmatrix}.$$

(2) If $d = 2$, then

$$\partial_1 = \begin{bmatrix} a_1 & a_2 & b \end{bmatrix}, \ \partial_2 = \begin{bmatrix} -a_2 & -b & 0 & x_1 \\ a_1 & 0 & -b & x_2 \\ 0 & a_1 & a_2 & b \end{bmatrix}, \text{ and } \partial_3 = \begin{bmatrix} b & -x_2 & x_1 & 0 \\ -a_2 & -b & 0 & x_1 \\ a_1 & 0 & -b & x_2 \\ 0 & a_1 & a_2 & b \end{bmatrix}.$$

(3) If $d = 3$, then

$$\partial_1 = \begin{bmatrix} a_1 & a_2 & a_3 & b \end{bmatrix}, \ \partial_2 = \begin{bmatrix} -a_2 & -a_3 & 0 & -b & 0 & 0 & x_1 \\ a_1 & 0 & -a_3 & 0 & -b & 0 & x_2 \\ 0 & a_1 & a_2 & 0 & 0 & -b & x_3 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 & b \end{bmatrix},$$

$$\partial_3 = \begin{bmatrix} a_3 & b & 0 & 0 & -x_2 & x_1 & 0 & 0 \\ -a_2 & 0 & b & 0 & -x_3 & 0 & x_1 & 0 \\ a_1 & 0 & 0 & b & 0 & x_3 & x_2 & 0 \\ 0 & -a_2 & -a_3 & 0 & -b & 0 & 0 & x_1 \\ 0 & a_1 & 0 & -a_3 & 0 & -b & 0 & x_2 \\ 0 & 0 & a_1 & a_2 & 0 & 0 & -b & x_3 \\ 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 & b \end{bmatrix},$$

$$\text{and } \partial_4 = \begin{bmatrix} -b & x_3 & -x_2 & x_1 & 0 & 0 & 0 & 0 \\ a_3 & b & 0 & 0 & -x_2 & x_1 & 0 & 0 \\ -a_2 & 0 & b & 0 & -x_3 & 0 & x_1 & 0 \\ a_1 & 0 & 0 & b & 0 & x_3 & x_2 & 0 \\ 0 & -a_2 & -a_3 & 0 & -b & 0 & 0 & x_1 \\ 0 & a_1 & 0 & -a_3 & 0 & -b & 0 & x_2 \\ 0 & 0 & a_1 & a_2 & 0 & 0 & -b & x_3 \\ 0 & 0 & 0 & 0 & a_1 & a_2 & a_3 & b \end{bmatrix}.$$

4.5 Decomposable Ulrich ideals

In this section, we explore the structure of decomposable Ulrich ideals in a one-dimensional Cohen-Macaulay local ring R. We begin with the following, which characterizes two-generated decomposable Ulrich ideals.

Proposition 4.13. *Suppose that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with* $\dim R = 1$. *Let I be an $\mathfrak{m}$-primary ideal of R, and assume that $\mu_R(I) = 2$. Then the following conditions are equivalent.*

(1) *$I \in \mathcal{X}_R$ and I is decomposable.*

(2) *There exist $a, b \in \mathfrak{m}$ such that $I = (a, b)$, $(a) = (0) :_R b$, and $(b) = (0) :_R a$.*

Proof. (1) $\Rightarrow$ (2) Choose $a, b \in \mathfrak{m}$ so that $I = (a) \oplus (b) = (a, b)$. Then $ab = 0$, and we have

$$I/I^2 \cong (a)/(a^2) \oplus (b)/(b^2) \cong R/[(a) + (0) :_R a] \oplus R/[(b) + (0) :_R b],$$

while $I/I^2 \cong (R/I)^{\oplus 2}$, since $I \in \mathcal{X}_R$ and $\mu_R(I) = 2$. Therefore, because $I = (a, b) \subseteq (a) + (0) :_R a$ and $I \subseteq (b) + (0) :_R b$, we get $I = (a) + (0) :_R a = (b) + (0) :_R b$. On the other hand, we have

$$I^2 = (a^2, b^2) = (a + b)I,$$

hence $a + b$ is a non-zero divisor on R, since $\sqrt{I} = \mathfrak{m}$.

Claim. $(0) :_R a^2 = (0) :_R a$ *and* $(0) :_R b^2 = (0) :_R b$.

Proof of Claim. $(0) :_R a \subseteq (0) :_R a^2$ is clear. Let $x \in (0) :_R a^2$. Since $(a + b)ax = a^2x + abx = 0$ and $a + b$ is a non-zero divisor on R, we have $ax = 0$, which shows $(0) :_R a^2 = (0) :_R a$. Similarly, $(0) :_R b^2 = (0) :_R b$. $\square$

Let $x \in (0) :_R a$. Because $x \in I = (a, b)$, we write $x = ax_1 + bx_2$ ($x_i \in R$). Then

$$0 = ax = a^2x_1 + abx_2 = a^2x_1,$$

which shows that $x_1 \in (0) :_R a^2 = (0) :_R a$ by Claim. Consequently, we have $x = bx_2 \in (b)$, so that $(0) :_R a = (b)$. We also get $(0) :_R b = (a)$ as well.

(2) $\Rightarrow$ (1) Because $ab = 0$, we have $I^2 = (a + b)I$. Hence $a + b$ is a non-zero divisor on R. Let $x \in (a) \cap (b)$. Then $(a + b)x = 0$, that is $x = 0$. Therefore $I = (a) \oplus (b)$ and we have

$$I/I^2 \cong (a)/(a^2) \oplus (b)/(b^2) \cong R/[(a) + (0) :_R a] \oplus R/[(b) + (0) :_R b] = R/I \oplus R/I,$$

which shows that $I \in \mathcal{X}_R$. $\square$

We now assume that $(S, \mathfrak{n})$ is a regular local ring with $\dim S = 2$, and let $0 \neq f \in \mathfrak{n}$ and $R = S/(f)$. We then have the following, which characterizes decomposable Ulrich ideals in a one-dimensional hypersurface ring.

Theorem 4.14. *Assume that* $f = p_1^{e_1} p_2^{e_2} \cdots p_\ell^{e_\ell}$ $(\ell \geq 1, e_i \geq 1)$ *where* $p_1, p_2, \ldots, p_\ell$ *are different prime elements of* S*. Set* $\Lambda = \{1, 2, \cdots, \ell\}$*. For* $\emptyset \neq J \subsetneq \Lambda$*, we define* $\alpha_J = \prod_{j \in J} p_j^{e_j}$ *and* $\beta_J = \prod_{j \in \Lambda \setminus J} p_j^{e_j}$*. Then*

$$\{I \in \mathcal{X}_R \mid I \text{ is decomposable } \} = \{(\overline{\alpha_J}, \overline{\beta_J}) \mid \emptyset \neq J \subsetneq \Lambda\}.$$

Proof. Suppose that $\emptyset \neq J \subsetneq \Lambda$, and set $a = \alpha_J + \beta_J$, $b = \beta_J$. Then a, b is a system of parameters of S, since α_J, β_J is a system of parameters of S, and we have

$$a^2 \cdot 0 + ab \cdot (-1) + b^2 = -\alpha_J \beta_J = -f.$$

Thus, $(\overline{a}, \overline{b}) = (\overline{\alpha_J}, \overline{\beta_J}) \in \mathcal{X}_R$ by Proposition 4.4, and $(\overline{\alpha_J}, \overline{\beta_J}) = (\overline{\alpha_J}) \oplus (\overline{\beta_J})$.

Conversely, suppose that $I \in \mathcal{X}_R$ and I is decomposable. Then, because R is a Gorenstein ring, $\mu_R(I) = 2$ by Theorem 4.3. We can choose $a, b \in \mathfrak{n}$ so that $I = (\overline{a}, \overline{b})$, $(0) :_R \overline{a} = (\overline{b})$, and $(0) :_R \overline{b} = (\overline{a})$ by Proposition 4.13. Since $\overline{ab} = 0$ in R, we write $ab = \rho f$ with $\rho \in S$. We note that a, b are relatively prime because a, b is a system of parameters of S by Lemma 4.6. Therefore, it suffices to show that $\rho \in U(S)$. Assume that $\rho \in \mathfrak{n}$. Then $\rho = p\rho'$ for some prime element p of S and $\rho' \in S$, hence $ab = p\rho' f \in (p)$, and we may assume that $a \in (p)$. Thus, writing $a = pa'$ with $a' \in S$, we get $a'b = \rho' f$, which means $\overline{a'} \in (0) :_R \overline{b} = (\overline{a})$. This is impossible since $p \notin U(S)$. $\square$

The following is a direct consequence of Theorem 4.14.

Corollary 4.15. *Suppose that* $R = k[[X, Y]]/(X^k Y)$*, where* $k > 0$ *and* $k[[X, Y]]$ *is a formal power series ring over a field* k*. Then*

$$\{I \in \mathcal{X}_R \mid I \text{ is decomposable } \} = \{(x^k, y)\}$$

where x, y *denote the images of* X, Y *in* R*.*

4.6 The case $R = k[[X, Y]]/(f)$

In this section, let $S = k[[X, Y]]$ be a formal power series ring over a field k, and $R = S/(f)$ with $f \in \mathfrak{n} = (X, Y)$. By using Theorem 4.5 and Corollary 4.15, we explore the set $\mathcal{X}_R$, when $f = Y^k$ or $X^{k-1}Y (k \geq 2)$. Let x, y denote the images of X, Y in R.

Firstly, we assume that $f = Y^k$ and $k \geq 2$. Let $I \in \mathcal{X}_R$. Remember that $\mu_R(I) = 2$, since R is a Gorenstein ring.

Proposition 4.16. *$I = (\overline{a}, \overline{b})$ and $I^2 = \overline{a}I$ for some $a = X^n + a_1Y$ and $b = b_1Y$, where $n > 0$, $a_1, b_1 \in S$. Therefore $Y^{k-1} \in (a, b)$.*

Proof. Let us write $I = (\alpha, \beta)$ with $I^2 = \alpha I$ $(\alpha, \beta \in R)$. We set

$$A = I : I = \{\varphi \in Q(R) \mid \varphi I \subseteq I\} \subseteq Q(R),$$

where $Q(R)$ denotes the total ring of fractions of R. Then $A = \frac{I}{\alpha} = R + R\frac{\beta}{\alpha}$, since $I^2 = \alpha I$. On the other hand, let $D = k[[x]] \subseteq R$ and $K = Q(D)$. Then, since A is a module finite birational extension of R and $Q(R) = K[Y]/(Y^k)$, we have

$$R \subseteq A = R + R\frac{\beta}{\alpha} \subseteq \overline{R} = D + \sum_{i=1}^{k-1} Ky^i,$$

where $\overline{R}$ denotes the integral closure of R in $Q(R)$. Because $\frac{\beta}{\alpha} \in D + \sum_{i=1}^{k-1} Ky^i$, we write $\frac{\beta}{\alpha} = d + \rho$ with $d \in D$ and $\rho \in \sum_{i=1}^{k-1} Ky^i$. Therefore, since $\frac{\beta - \alpha d}{\alpha} = \frac{\beta}{\alpha} - d = \rho$ and $A = R + R\rho$, replacing β with $\beta - \alpha d$, from the beginning we may assume that $\frac{\beta}{\alpha} \in \sum_{i=1}^{k-1} Ky^i$. Hence $y^{k-1}\beta = 0$, since $y^{k-1} \cdot \frac{\beta}{\alpha} = 0$ in R. Therefore, we have $y^{k-1} \in I$, because $(\alpha) :_R \beta = I$ (remember that $I/(\alpha) \cong R/I$). Let $a, b \in S$ such that $\overline{a} = \alpha$, $\overline{b} = \beta$ in R. Then a, b is a system of parameters of S by Lemma 4.6. Since $bY^{k-1} \in (Y^k)$ in S, we get $b \in (Y)$, and that $a \notin (Y)$. Consequently, we have that $a = \varepsilon X^n + a_1Y$ and $b = b_1Y$ with $n > 0$, $a_1, b_1 \in S$, and $\varepsilon \in U(S)$, and may assume $\varepsilon = 1$. We also have $Y^{k-1} \in (a, b)$, since $Y^{k-1} \in (a, b) + (Y^k)$. □

Proposition 4.17 ([12, Example 4.8])**.** *Suppose that $R = k[[X, Y]]/(Y^2)$. Then*

$$\mathcal{X}_R = \{(x^\ell, y) \mid \ell > 0\}.$$

Proof. Thanks to Corollary 4.7, $(x^\ell, y) \in \mathcal{X}_R$ for any $\ell > 0$. Conversely, suppose that $I \in \mathcal{X}_R$. Then $I = (\overline{a}, \overline{b})$ for some $a = X^n + a_1Y$ and $b = b_1Y$ with $n > 0$, $a_1, b_1 \in S$, and $Y \in (a, b)$ by Proposition 4.16. Therefore, $(a, b) = (a, b, Y) = (X^n, Y)$. □

If k is odd, we have the following family of Ulrich ideals.

Proposition 4.18. *Suppose that $k = 2m + 1$ $(m \geq 1)$. Let $\ell > 0$ and $\varepsilon \in U(S)$. We consider the ideal $I = (x^{2\ell} + \overline{\varepsilon}y, x^\ell y^m)$ of R. Then the following assertions hold true.*

(1) *$I \in \mathcal{X}_R$.*

(2) *Let $\ell' > 0$, $\varepsilon' \in U(S)$ and suppose that $I = (x^{2\ell'} + \overline{\varepsilon'}y, x^{\ell'}y^m)$. Then $\ell = \ell'$ and $\varepsilon \equiv \varepsilon'$ mod $\mathfrak{n}$.*

Proof. (1) Let $a = X^{2\ell} + \varepsilon Y$ and $b = X^\ell Y^m$. Then a, b is a system of parameter of S, and setting $\varphi = -\varepsilon^{-1}Y^{2m-1}$, $\psi = \varepsilon X^\ell Y^{m-1}$, and $\delta = -1$, we have

$$a^2\varphi + ab\psi + b^2 = \delta Y^{2m+1},$$

so that $I = (\overline{a}, \overline{b}) \in \mathcal{X}_R$ by Proposition 4.4.

(2) Let $\ell, \ell' > 0$ and $\varepsilon, \varepsilon' \in U(S)$, and assume that

$$(x^{2\ell} + \overline{\varepsilon}y, x^\ell y^m) = (x^{2\ell'} + \overline{\varepsilon'}y, x^{\ell'}y^m).$$

Then $(X^{2\ell} + \varepsilon Y, X^\ell Y^m) = (X^{2\ell'} + \varepsilon' Y, X^{\ell'}Y^m)$ by Lemma 4.6, hence we have $\ell = \ell'$, comparing the colength of the ideals. We write $X^{2\ell} + \varepsilon Y = (X^{2\ell} + \varepsilon' Y)\xi + (X^\ell Y^m)\eta$ with $\xi, \eta \in S$. Then $X^{2\ell}(1 - \xi) = Y(-\varepsilon + \varepsilon'\xi + X^\ell Y^{m-1}\eta)$, whence

$$1 - \xi = Y\rho \text{ and } -\varepsilon + \varepsilon'\xi + X^\ell Y^{m-1}\eta = X^{2\ell}\rho$$

for some $\rho \in S$. Therefore, $1 \equiv \xi$ and $-\varepsilon + \varepsilon'\xi \equiv 0$ mod $\mathfrak{n}$, that is $\varepsilon \equiv \varepsilon'$. $\square$

As a consequence, we get the following.

Theorem 4.19. *Suppose that $R = k[[X, Y]]/(Y^3)$. Then*

$$\mathcal{X}_R = \{(x^{2\ell} + \overline{\varepsilon}y, x^\ell y) \mid \ell > 0, \varepsilon \in U(S)\}.$$

Proof. The inclusion ($\supseteq$) follows from Proposition 4.18. Suppose that $I \in \mathcal{X}_R$. By Proposition 4.16, $I = (\overline{a}, \overline{b})$ for some $a = X^n + a_1 Y$ and $b = b_1 Y$ with $n > 0$, $a_1, b_1 \in S$. We notice that $\ell_R(R/(\overline{a})) = 2 \cdot \ell_R(R/I)$, since $I/(\overline{a}) \cong R/I$, and $\ell_R(R/(\overline{a})) = \ell_S(S/(a, Y^3)) = 3n$. If $b_1 \notin \mathfrak{n}$, then $(a, b) = (X^n, Y)$, whence $\ell_R(R/I) = \ell_S(S/(a, b)) = n$. This implies that $3n = 2n$, which is impossible. Hence $b_1 \in \mathfrak{n}$. If $b_1 \in (Y)$, then $y\overline{b} = 0$ in R, thus $y \in (\overline{a}) :_R \overline{b} = I$. This implies that $Y \in (a, b)$ and $(a, b) = (X^n, Y)$, which is also impossible. Therefore, since $b_1 \in \mathfrak{n} \setminus (Y)$, we write $b_1 = \tau X^\ell + b_2 Y$ with $\ell > 0$, $\tau \in U(S)$, and $b_2 \in S$. Because $Y^2 \in (a, b)$ by Proposition 4.16, we have $(a, b) = (a, b, Y^2) = (X^n + a_1 Y, X^\ell Y, Y^2)$, whence $(a, b) = (X^n + a_1 Y, X^\ell Y)$ or $(X^n + a_1 Y, Y^2)$, since $(a, b) \nsubseteq (Y)$. We then have $(a, b) = (X^n + a_1 Y, X^\ell Y)$. Indeed, if $(a, b) = (X^n + a_1 Y, Y^2)$, then $2 \cdot \ell_R(R/I) = 2 \cdot \ell_S(S/(X^n + a_1 Y, Y^2)) = 4n \neq 3n$, which is absurd. Therefore, we may assume that $b_1 = X^\ell$. In addition, we have the following.

Claim. $a_1 \in U(S)$.

Proof of Claim. Because $(\overline{a}, \overline{b}) \in \mathcal{X}_R$,

$$a^2\varphi + ab\psi + b^2 = \varepsilon Y^3$$

for some $\varphi, \psi \in S$ and $\varepsilon \in U(S)$ by Theorem 4.5. Since $a^2\varphi \in (Y)$ and $a \notin (Y)$, $\varphi = Y\varphi_1$ for some $\varphi_1 \in S$. Expanding the equation, we have

$$a_1^2\varphi_1 Y^2 + X^{2\ell}Y + 2a_1\varphi_1 X^{2\ell}Y + a_1\psi X^\ell Y + \varphi_1 X^{4\ell} + \psi X^{3\ell} = \varepsilon Y^2.$$

Therefore, $Y^2(a_1^2\varphi_1 - \varepsilon) \in (X)$, so that $a_1^2\varphi_1 - \varepsilon \in (X)$, whence $a_1 \in U(S)$. □

It suffices to show that $n = 2\ell$. In fact, we have

$$\ell_R(R/I) = \ell_S(S/(X^n + a_1Y, X^\ell Y)) = \ell + n,$$

while $\ell_R(R/(\overline{a})) = 3n$. Consequently, $3n = 2(\ell + n)$, whence $n = 2\ell$. This completes the proof of Theorem 4.19. □

Similarly, if k is even, we have the following.

Proposition 4.20. *Suppose that $k = 2m$ $(m \geq 2)$. Then the following assertions hold true.*

(1) $\{I \in \mathcal{X}_R \mid y^m \in I\} = \{(x^\ell + \alpha y, y^m) \mid \ell > 0, \alpha \in R\}$.

(2) *Let $\ell, \ell' > 0$, $\alpha, \alpha' \in R$ and suppose that $(x^\ell + \alpha y, y^m) = (x^{\ell'} + \alpha' y, y^m)$. Then $\ell = \ell'$ and $\alpha \equiv \alpha' \bmod \mathfrak{m} = \mathfrak{n}/(Y^{2m})$.*

Proof. (1) The inclusion $(\supseteq)$ follows from Corollary 4.7. Suppose that $I \in \mathcal{X}_R$. $I = (\overline{a}, \overline{b})$ for some $a = X^n + a_1Y$ and $b = b_1Y$ with $n > 0$, $a_1, b_1 \in S$. Since $\ell_R(R/(\overline{a})) = 2 \cdot \ell_R(R/I)$ and $\ell_R(R/(\overline{a})) = \ell_S(S/(X^n + a_1Y, Y^{2m})) = 2mn$, we then have $\ell_R(R/I) = mn$. On the other hand, because $y^m \in I$,

$$mn = \ell_R(R/I) = \ell_S(S/(a,b)) = \ell_S(S/(a,b,Y^m)) \leq \ell_S(S/(X^n + a_1Y, Y^m)) = mn,$$

hence $(a,b) = (X^n + a_1Y, Y^m)$. The Assertion (2) follows from the same technique as in the proof of Proposition 4.18 (2). □

Corollary 4.21. *Suppose that $R = k[[X,Y]]/(Y^4)$. Then*

$$\{I \in \mathcal{X}_R \mid y^2 \in I\} = \{(x^\ell + \alpha y, y^2) \mid \ell > 0, \alpha \in R\}.$$

For a moment, suppose that $k = 4$. Let $I \in \mathcal{X}_R$ and assume that $y^2 \notin I$. Then $I = (\overline{a}, \overline{b})$ and $I^2 = \overline{a}I$ for some $a = X^n + a_1 Y$ and $b = b_1 Y$, where $n > 0$, $a_1, b_1 \in S$, by Proposition 4.16. With this notation, we get the following.

Lemma 4.22. $b_1 = X^p + b_2 Y$ *with* $0 < p < n$ *and* $b_2 \in S$.

Proof. Because $y \notin I$, $b_1 \notin U(S)$. We then have $b_1 \in \mathfrak{n} \setminus (Y)$. Indeed, if $b_1 \in (Y)$, then $y^2\overline{b} = 0$ in R, whence $y^2 \in I$. This is impossible. Therefore $b_1 = \tau X^p + b_2 Y$ with $p > 0$, $b_2 \in S$, and $\tau \in U(S)$, and may assume $\tau = 1$. Assume $p \geq n$. Then, because

$$b = X^p Y + b_2 Y^2 \underset{\text{mod} a}{\equiv} X^{p-n} Y(-a_1 Y) + b_2 Y^2 \in (Y^2),$$

we have $y^2 \in (\overline{a}) :_R \overline{b} = I$, which is absurd. Therefore $0 < p < n$. □

Theorem 4.23. *Suppose that $R = k[[X,Y]]/(Y^4)$. Let $I \in \mathcal{X}_R$ and assume that $y^2 \notin I$. We set $I = (\overline{a}, \overline{b})$ with $a, b \in S$. Then the following assertions hold true.*

(1) $(a, b) = (X^n + a_1 Y, Y(X^p + b_2 Y))$ *with* $0 < p < n$, $a_1 \in \mathfrak{n}$, *and* $b_2 \in U(S)$.

(2) *If* $a_1 \in (Y)$, *then* $\operatorname{ch} k = 2$.

(3) *If* $\operatorname{ch} k \neq 2$, *then* $(a, b) = (X^n + \alpha X^r Y, Y(X^p + b_2 Y))$ *with* $0 < r < p < n$, $n - p \leq r$, *and* $\alpha, b_2 \in U(S)$.

Proof. (1) Thanks to Lemma 4.22, $(a, b) = (X^n + a_1 Y, Y(X^p + b_2 Y))$ with $0 < p < n$ and $a_1, b_2 \in S$. We may assume $a = X^n + a_1 Y$ and $b = Y(X^p + b_2 Y)$. Because $\ell_R(R/(\overline{a})) = 2 \cdot \ell_R(R/I)$ and $\ell_R(R/(\overline{a})) = \ell_S(S/(X^n + a_1 Y, Y^4)) = 4n$, we have

$$2n = \ell_R(R/I) = \ell_S(S/(X^n + a_1 Y, Y(X^p + b_2 Y))) = n + \ell_S(S/(X^n + a_1 Y, X^p + b_2 Y)),$$

so that $\ell_S(S/(X^n + a_1 Y, X^p + b_2 Y)) = n$. If $a_1 \in U(S)$, then $(X^n + a_1 Y, X^p + b_2 Y) = (X^n + a_1 Y, X^p(1 - a_1^{-1} b_2 X^{n-p})) = (X^p, Y)$, hence $n = \ell_S(S/(X^n + a_1 Y, X^p + b_2 Y)) = p$, which is impossible. Therefore $a_1 \in \mathfrak{n}$. On the other hand, we have

$$a^2\varphi + ab\psi + b^2 = \varepsilon Y^4$$

for some $\varphi, \psi \in S$ and $\varepsilon \in U(S)$ by Theorem 4.5. Then $\varphi = Y\varphi_1$ for some $\varphi_1 \in S$, since $a^2\varphi \in (Y)$ and $a \notin (Y)$. From the equation, we get

$$\begin{aligned}\varepsilon Y^3 = b_2^2 Y^3 &\\ &+ a_1^2\varphi_1 Y^2 + a_1 b_2 \psi Y^2 + 2b_2 X^p Y^2\\ &+ 2a_1\varphi_1 X^n Y + b_2\psi X^n Y + a_1\psi X^p Y + X^{2p} Y\\ &+ \varphi_1 X^{2n} + \psi X^{n+p}.\end{aligned}$$

Hence $X^{n+p}(\varphi_1 X^{n-p} + \psi) \in (Y)$, so that $\varphi_1 X^{n-p} + \psi \in (Y)$, whence $\psi \in \mathfrak{n}$. Similarly, $Y^2(-\varepsilon Y + b_2^2 Y + a_1^2\varphi_1 + a_1 b_2 \psi) \in (X)$, so that $-\varepsilon Y + b_2^2 Y + a_1^2 \varphi_1 + a_1 b_2 \psi \equiv 0 \bmod (X)$. Because $a_1, \psi \in \mathfrak{n}$, $-\varepsilon Y + b_2^2 Y \equiv 0 \bmod (X, Y^2)$, whence $b_2 \in U(S)$.

(2) Assume $a_1 \in (Y)$. Then, because $0 < p < n$ and $\psi \in \mathfrak{n}$, we have $2b_2 X^p Y^2 \in (X^{p+1}, Y^3)$, therefore $\operatorname{ch} k = 2$, since $b_2 \in U(S)$.

(3) Suppose that $\operatorname{ch} k \neq 2$. Then $a_1 \in \mathfrak{n} \setminus (Y)$ by Assertions (1), (2). We write $a_1 = \alpha X^r + a_2 Y$ with $r > 0$, $\alpha \in U(S)$, and $a_2 \in S$. If $r \geq p$, since $a = X^n + \alpha X^r Y \underset{\mathrm{mod} b}{\equiv} X^n + (-\alpha b_2^{-1} X^{r-p} Y)Y$, replacing αX^r with $-\alpha b_2^{-1} X^{r-p} Y$, we may assume that $a_1 \in (Y)$, which is absurd. Hence $0 < r < p < n$. Because

$$a = X^n + \alpha X^r Y + a_2 Y^2 \underset{\mathrm{mod} b}{\equiv} X^n + \alpha X^r Y - a_2 b_2^{-1} X^p Y = X^n + (\alpha - a_2 b_2^{-1} X^{p-r}) X^r Y,$$

and $\alpha - a_2 b_2^{-1} X^{p-r} \in U(S)$, we may assume that $a_2 = 0$. Since $\ell_S(S/(X^n + a_1 Y, X^p + b_2 Y)) = n$ (see the proof of Assertion (1)), if $n > r + p$,

$$\begin{aligned} n &= \ell_S(S/(X^n + a_1 Y, X^p + b_2 Y)) = \ell_S(S/(X^n + \alpha X^r Y, X^p + b_2 Y))\\ &= \ell_S(S/(X^n - \alpha b_2^{-1} X^{r+p}, X^p + b_2 Y)) = \ell_S(S/(X^{r+p}, X^p + b_2 Y)) = r + p,\end{aligned}$$

which makes a contradiction. Therefore $n \leq r + p$. □

We explore a concrete example.

Example 4.24. Suppose that $R = k[[X, Y]]/(Y^4)$. Let p, n be integers such that $0 < p < n$ and $2n \leq 3p$. We set $a = X^n + 2X^{n-p} Y$, $b = Y(X^p + Y)$. Then the following assertions hold true.

(1) $I = (\overline{a}, \overline{b}) \in \mathcal{X}_R$, for any characteristic of k.

(2) $y^2 \notin I$.

Proof. (1) We set $\varphi = -X^{3p-2n}Y$, $\psi = X^{2p-n}$, and $\varepsilon = 1$. Then a, b is a system of parameters of S, and we have $a^2\varphi + ab\psi + b^2 = \varepsilon Y^4$, therefore $I \in \mathcal{X}_R$ by Proposition 4.4.

(2) If $y^2 \in I$, then $Y^2 \in (a, b)$. We write $Y^2 = (X^n + 2X^{n-p}Y)\xi + Y(X^p + Y)\eta$ with $\xi, \eta \in S$. Hence, since $\xi = Y\xi_1$ for some $\xi_1 \in S$, we have $Y(1 - 2X^{n-p}\xi_1 - \eta) = X^p(X^{n-p}\xi_1 + \eta)$, so that $1 - 2X^{n-p}\xi_1 - \eta = \rho X^p$ and $X^{n-p}\xi_1 + \eta = \rho Y$ for some $\rho \in S$. Therefore, $1 \equiv \eta \bmod \mathfrak{n}$ and $\eta \equiv 0 \bmod \mathfrak{n}$, which is impossible. $\square$

In what follows, we assume that $f = X^kY$ $(k \geq 1)$. Thanks to Corollary 4.15, (x^k, y) is the only decomposable Ulrich ideal in R. Let $I \in \mathcal{X}_R$ and I is indecomposable. We begin with the following.

Proposition 4.25. *$I = (\overline{a}, \overline{b})$ and $I^2 = \overline{a}I$ for some $a = X^n + a_1Y$ and $b = b_1XY$, where $n > 0$, $a_1, b_1 \in S$ such that $a_1 \notin (X)$. In addition, $n < k$, if $k \geq 2$.*

Proof. We identify $R \subseteq S/(X^k) \times S/(Y)$ and let x_1, y_1 (resp. x_2) denote the images of X, Y (resp. X) in $S/(X^k)$ (resp. $S/(Y)$). Hence $S/(Y) = k[[x_2]]$ and $Q(R) = (K_1 + \sum_{i=1}^{k-1} K_1x_1^i) \times K_2$, where $K_1 = Q(k[[y_1]])$ and $K_2 = Q(k[[x_2]])$. We set $A = I : I$. Then

$$R \subseteq A \subseteq \overline{R} = (k[[y_1]] + \sum_{i=1}^{k-1} K_1x_1^i) \times k[[x_2]],$$

since A is a module finite birational extension of R. Let us write $I = (\alpha, \beta)$ with $I^2 = \alpha I$. Then $A = R + R\frac{\beta}{\alpha}$. Remember now that A is a local ring, since $A \cong I$ is indecomposable. Let $J, \mathfrak{m}$, and $J(\overline{R})$ denote the maximal ideals of A, R, and the Jacobson radical of $\overline{R}$. Then, since

$$k = R/\mathfrak{m} \subseteq A/J \subseteq \overline{R}/J(\overline{R}) = k \times k,$$

we have $R/\mathfrak{m} = A/J$. Take $r \in R$ so that $\frac{\beta}{\alpha} \equiv r \bmod J$. Then, replacing β with $\beta - r\alpha$, we can assume that $\frac{\beta}{\alpha} \in J$. Since $J \subseteq J(\overline{R}) = (y_1k[[y_1]] + \sum_{i=1}^{k-1} K_1x_1^i) \times x_2k[[x_2]]$, we get $\frac{\beta}{\alpha} = r' + \rho$ for some $r' \in R$ and $\rho \in (\sum_{i=1}^{k-1} K_1x_1^i) \times (0)$. Therefore, replacing β with $\beta - \alpha r'$, from the beginning we may assume that $\frac{\beta}{\alpha} \in (\sum_{i=1}^{k-1} K_1x_1^i) \times (0)$. Let us now write $\alpha = \overline{a}$ and $\beta = \overline{b}$ with $a, b \in S$. Then, since $\beta^k = 0$ in R, we have $b^k \in (X^kY)$, so that $b \in (XY)$. We write $b = b_1XY$ with $b_1 \in S$. Notice that a, b is a system of parameters of S by Lemma 4.6. Consequently, $a \notin (X) \cup (Y)$, so that we may assume that $a = X^n + a_1Y$ with $n > 0$ and $a_1 \in S$ such that $a_1 \notin (X)$. If $k \geq 2$, we have $X^{k-1} \in (a, b)$, since $x^{k-1} \in (\alpha) :_R \beta = I$. Thus, because $X^{k-1} \in (a, b, Y) = (X^n, Y)$, we get $n < k$. $\square$

Theorem 4.26. *Suppose that* $R = k[[X,Y]]/(X^kY)$ *with* $1 \le k \le 2$. *Then*

$$\mathcal{X}_R = \{(x^k, y)\}.$$

Proof. Suppose that $I \in \mathcal{X}_R$ and I is indecomposable. Assume that $k = 1$. Then, since $\overline{R} = S/(X) \times S/(Y)$ and $\ell_R(\overline{R}/R) = 1$, $A = \overline{R}$ where $A = I : I$, which is impossible because A is a local ring (see the proof of Proposition 4.25). Assume that $k = 2$. By Proposition 4.26, $I = (\overline{a}, \overline{b})$ for some $a = X + a_1Y$ and $b = b_1XY$ with $a_1, b_1 \in S$ such that $a_1 \notin (X)$. Since $X \in (a, b)$ (see the proof of Proposition 4.25), we can write $X = (X + a_1Y)\varphi + b_1XY\psi$ with $\varphi, \psi \in S$. Then $a_1Y\varphi \in (X)$ and $a_1 \notin (X)$, whence $\varphi \in (X)$. Therefore, writing $\varphi = X\varphi_1$ with $\varphi_1 \in S$, we get $1 = (X + a_1Y)\varphi_1 + b_1Y\psi \in \mathfrak{n}$, which is impossible. Consequently, if $k \le 2$, R has no indecomposable Ulrich ideal. Thanks to Corollary 4.15, this completes the proof of this Theorem. $\square$

In what follows, suppose that $k \ge 3$. Let $I \in \mathcal{X}_R$ and assume that I is indecomposable. Then $I = (\overline{a}, \overline{b})$ and $I^2 = \overline{a}I$ for some $a = X^n + a_1Y$ and $b = b_1XY$ with $n > 0$ and $a_1, b_1 \in S$ such that $a_1 \notin (X)$ by Proposition 4.25. With this notation, we have the following.

Proposition 4.27. *The following assertions hold true.*

(1) $n \le k - 2$.

(2) *If* $k \ge 4$ *and* $n = k - 2$, *then* $xy \in I$.

Proof. Because $(\overline{a}, \overline{b}) \in \mathcal{X}_R$,

$$a^2\varphi + ab\psi + b^2 = \varepsilon X^kY$$

for some $\varphi, \psi \in S$ and $\varepsilon \in U(S)$ by Theorem 4.5. Since $a^2\varphi \in (XY)$ and $a \notin (X) \cup (Y)$, $\varphi = XY\varphi_1$ for some $\varphi_1 \in S$. We then have

$$\begin{aligned}\varepsilon X^{k-1} = {} & a_1^2\varphi_1Y^2 \\ & + 2a_1\varphi_1X^nY + a_1b_1\psi Y + b_1^2XY \quad \cdots \text{(A)} \\ & + \varphi_1X^{2n} + b_1\psi X^n.\end{aligned}$$

(1) Assume that $n > k - 2$. Then $n = k - 1$ by Proposition 4.25. Hence $X^{k-1}(\varepsilon - b_1\psi - \varphi_1X^{k-1}) \in (Y)$, so that $\varepsilon - b_1\psi \in \mathfrak{n}$, whence $b_1 \in U(S)$. Therefore, we may assume that $b_1 = 1$. Since $a_1 \notin (X)$, we write $a_1 = \tau Y^\ell + a_2X$ with $\ell \ge 0$, $a_2 \in S$, and

$\tau \in U(S)$. We then have $(a, b) = (X^{k-1} + \tau Y^{\ell+1} + a_2XY, XY) = (X^{k-1} + \tau Y^{\ell+1}, XY)$. Thus, from the beginning we may assume $a_1 = \tau Y^{\ell}$. From the above equation (A), we get $\tau\psi Y^{\ell+1} + XY \equiv 0 \bmod (X^2, Y^2)$, hence $\ell = 0$. On the other hand, because $\ell_R(R/(\overline{a})) = 2 \cdot \ell_R(R/I)$, we have

$$\ell_R(R/(\overline{a})) = \ell_S(S/(X^{k-1} + \tau Y, X^kY)) = k + k - 1 = 2k - 1, \text{ and}$$
$$\ell_R(R/I) = \ell_S(S/(X^{k-1} + \tau Y, XY)) = 1 + k - 1 = k.$$

Hence $2k - 1 = 2k$, which is impossible. Therefore $n \leq k - 2$.

(2) Suppose that $k \geq 4$ and $n = k - 2$. From the equation (A), we have $X^{k-2}(\varepsilon X - \varphi_1 X^{k-2} - b_1\psi) \in (Y)$, whence $b_1\psi \equiv \delta X \bmod (Y)$, where $\delta = \varepsilon - \varphi_1 X^{k-3} \in U(S)$. Assume that $b_1 \in \mathfrak{n}$. Then $\psi \in U(S)$ and $b_1 = \rho X + b_2 Y$ for some $\rho \in U(S)$ and $b_2 \in S$. We may assume that $\rho = 1$. We also get $a_1Y(a_1\varphi_1 Y + b_1\psi) \in (X)$ from the equation (A). Since $a_1 \notin (X)$, we have $a_1\varphi_1 Y + b_1\psi \in (X)$, so that $a_1\varphi_1 Y + b_2\psi Y = Y(a_1\varphi_1 + b_2\psi) \in (X)$. Whence $b_2 \in (a_1, X)$(notice that $\psi \in U(S)$). Writing $b_2 = a_1\xi + X\eta$ with $\xi, \eta \in S$, we get

$$b = XY(X + a_1\xi Y + \eta XY) \underset{\text{mod}a}{\equiv} X^2Y(1 - \xi X^{k-3} + \eta XY),$$

hence we may assume that $b = X^2Y$ ($b_2 = 0$). Let $\ell = \ell_S(S/(a_1, X))$. Then

$$\ell_R(R/(\overline{a})) = \ell_S(S/(X^{k-2} + a_1Y, X^kY)) = k(\ell + 1) + k - 2 = k \cdot \ell + 2k - 2, \text{ and}$$
$$\ell_R(R/I) = \ell_S(S/(X^{k-2} + a_1Y, X^2Y)) = 2(\ell + 1) + k - 2 = 2\ell + k.$$

Since $\ell_R(R/(\overline{a})) = 2 \cdot \ell_R(R/I)$, we have $k \cdot \ell + 2k - 2 = 2(2\ell + k)$, so that $(k - 4)\ell = 2$. Thus, $k = 6$, $\ell = 1$ or $k = 5$, $\ell = 2$.

If $k = 6$ and $\ell = 1$, we can write $a_1 = \tau Y + a_2 X$ with $\tau \in U(S)$ and $a_2 \in S$ (notice that $\ell = \ell_S(S/(a_1, X))$). From the equation (A), we get $\tau\psi XY^2 \equiv 0 \bmod (X^2, Y^3)$, which makes a contradiction.

If $k = 5$ and $\ell = 2$, we can write $a_1 = \tau Y^2 + a_2 X$ with $\tau \in U(S)$ and $a_2 \in S$. Similarly, we get $\tau\psi XY^3 \equiv 0 \bmod (X^2, Y^4)$, which is impossible. Consequently, we have $b_1 \in U(S)$, therefore $xy \in I$. □

We get the following family of Ulrich ideals.

Proposition 4.28. *Suppose that $k \geq 3$. Then the following assertions hold true.*

(1) $\{I \in \mathcal{X}_R \mid xy \in I\} = \{(x^{k-2} + \overline{\varepsilon} y, xy) \mid \varepsilon \in U(S)\}$.

(2) *Let $\varepsilon, \varepsilon' \in U(S)$ and suppose that $(x^{k-2} + \overline{\varepsilon} y, xy) = (x^{k-2} + \overline{\varepsilon'} y, xy)$. Then $\varepsilon \equiv \varepsilon'$ mod $\mathfrak{n}$.*

Proof. (1) Let $a = X^{k-2} + \varepsilon Y$ with $\varepsilon \in U(S)$ and $b = XY$. Then a, b is a system of parameters of S. Setting $\varphi = 0$, $\psi = -\varepsilon^{-1} X$, and $\delta = -\varepsilon^{-1}$, we have $a^2\varphi + ab\psi + b^2 = \delta X^k Y$, thus $(\overline{a}, \overline{b}) \in \mathcal{X}_R$ by Proposition 4.4. Conversely, suppose that $I \in \mathcal{X}_R$ and $xy \in I$. Then $I = (\overline{a}, \overline{b})$ and $I^2 = \overline{a} I$ for some $a = X^n + a_1 Y$ and $b = b_1 XY$ with $n > 0$ and $a_1, b_1 \in S$ by Proposition 4.25, and $XY \in (a, b)$, hence $(a, b) = (a, XY)$. Let $\ell = \ell_S(S/(a_1, X))$. Because $\ell_R(R/(\overline{a})) = 2 \cdot \ell_R(R/I)$,

$$\ell_R(R/(\overline{a})) = \ell_S(S/(X^n + a_1 Y, X^k Y)) = k \cdot (\ell + 1) + n, \text{ and}$$
$$\ell_R(R/I) = \ell_S(S/(X^n + a_1 Y, XY)) = \ell + 1 + n,$$

we have $k \cdot (\ell + 1) + n = 2(\ell + 1 + n)$, so that $(k-2)\ell = n - (k-2)$. Since $k \geq 3$ and $k - 2 \geq n$ (Proposition 4.27), we get $n = k - 2$ and $\ell = 0$, therefore $(a, b) = (X^{k-2} + a_1 Y, XY)$ with $a_1 \in U(S)$ as desired. The Assertion (2) follows from the same technique as in the proof of Proposition 4.18 (2). ☐

Let $I \in \mathcal{X}_R$ and assume that I is indecomposable. We choose $a = X^n + a_1 Y$ and $b = b_1 XY$ as in Proposition 4.25. We then have the following.

Proposition 4.29. *The following assertions hold true.*

(1) *If $n = 1$, then k is odd, and $(a, b) = (X + \varepsilon Y^\ell, XY^p)$ where $\varepsilon \in U(S)$ and $\ell, p > 0$ such that $(k-2)\ell = 2p - 1$.*

(2) *Suppose that k is odd. Let $\ell, p > 0$ such that $(k-2)\ell = 2p - 1$ and $\varepsilon \in U(S)$. Then $(x + \overline{\varepsilon} y^\ell, xy^p) \in \mathcal{X}_R$.*

(3) *Let $\ell, p > 0$ (resp. $\ell', p' > 0$) such that $(k-2)\ell = 2p - 1$ (resp. $(k-2)\ell' = 2p' - 1$) and $\varepsilon, \varepsilon' \in U(S)$. If $(x + \overline{\varepsilon} y^\ell, xy^p) = (x + \overline{\varepsilon'} y^{\ell'}, xy^{p'})$, then $\ell = \ell'$, $p = p'$, and $\varepsilon \equiv \varepsilon'$ mod $\mathfrak{n}$.*

Proof. (1) Suppose that $n = 1$. Since $(a, Y) = \mathfrak{n}$ and $S/(a)$ is a DVR, $b_1 = \rho Y^{p-1} + ab_2$ for some $p > 0$, $\rho \in U(S)$, and $b_2 \in S$ (notice that $b_1 \notin (a)$, since $b \notin (a)$). Then $(a, b) = (a, XY^p)$. On the other hand, because $a_1 \notin (X)$, we can write $a_1 = \tau Y^{\ell - 1} + a_2 X$ for some $\ell > 0$ and $a_2 \in S$. We then have $a = X + a_2 XY + \tau Y^\ell = (1 + a_2 Y)X + \tau Y^\ell$, hence we may assume $a = X + \varepsilon Y^\ell$ with $\ell > 0$ and $\varepsilon \in U(S)$. Now notice that $\ell_S(S/(a, X^k Y)) =$

$\ell_S(S/(X+\varepsilon Y^\ell, X^kY)) = k\ell+1$ and $\ell_S(S/(a,b)) = \ell_S(S/(X+\varepsilon Y^\ell, XY^p)) = \ell+p$, so that $k\ell+1 = 2(\ell+p)$, whence $(k-2)\ell = 2p-1$ and k is odd.

(2) Let $a = X+\varepsilon Y^\ell$ and $b = XY^p$ with $\varepsilon \in U(S)$ and $\ell, p > 0$ such that $(k-2)\ell = 2p-1$. Then a, b is a system of parameters of S. We set

$$\varphi = \begin{cases} -\varepsilon^{-1}XY & \text{if } k=3 \\ \displaystyle\sum_{i=0}^{k-4}(-1)^{i+k-4}(i+1)\varepsilon^{-(k-2)+i}X^{k-2-i}Y^{i\ell+1} & \text{if } k \geq 5 \end{cases},$$

$$\psi = \begin{cases} Y^p & \text{if } k=3 \\ -(k-2)\varepsilon^{-1}XY^{p-\ell} & \text{if } k \geq 5 \end{cases}, \text{and } \ \delta = (-1)^{k-4}\varepsilon^{-(k-2)}.$$

Then we have $a^2\varphi + ab\psi + b^2 = \delta X^kY$, thus $(\overline{a}, \overline{b}) \in \mathcal{X}_R$ by Proposition 4.4. The Assertion (3) follows from the same technique as in the proof of Proposition 4.18 (2). □

As a consequence, we get the following.

Theorem 4.30. *The following assertions hold true.*

(1) *Suppose that* $R = k[[X,Y]]/(X^3Y)$. *Then*

$$\mathcal{X}_R = \{(x^3, y)\} \cup \{(x+\overline{\varepsilon}y^{2p-1}, xy^p) \mid p > 0, \varepsilon \in U(S)\}.$$

(2) *Suppose that* $R = k[[X,Y]]/(X^4Y)$. *Then*

$$\mathcal{X}_R = \{(x^4, y)\} \cup \{(x^2+\overline{\varepsilon}y, xy) \mid \varepsilon \in U(S)\}.$$

Proof. These assertions readily follow from Corollary 4.15, Proposition 4.27, Proposition 4.28, and Proposition 4.29. □

References

[1] H. Bass, On the ubiquity of Gorenstein rings, *Math. Zeitschr.*, **82** (1963), 8–28.

[2] V. Barucci and R. Fröberg, One-dimensional almost Gorenstein rings, *J. Algebra*, **188** (1997), no. 2, 418–442.

[3] J. P. Brennan, J. Herzog, and B. Ulrich, Maximally generated maximal Cohen-Macaulay modules, *Math. Scand.*, **61** (1987), no. 2, 181–203.

[4] T. D. M. Chau, S. Goto, S. Kumashiro, and N. Matsuoka, Sally modules of canonical ideals in dimension one and 2-AGL rings, *J. Algebra*, **521** (2019), 299–330.

[5] M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, *J. Algebra Appl.*, **6** (2007), 443–459.

[6] S. Endo, On semi-hereditary rings, *J. Math. Soc. Japan*, **13**, (1961), 109–119.

[7] J. A. Eagon and D. G. Northcott, *Ideals defined by matrices and a certain complex associated with them*, Proc. Roy. Soc. Ser. A, **269** (1962), 188–204.

[8] D. Farkas, Self-injective group algebras. *J. Algebra*, **25** (1973), 313-315.

[9] L. Ghezzi, S. Goto, J. Hong, and W. V. Vasconcelos, Invariants of Cohen-Macaulay rings associated to their canonical ideals, *J. Algebra*, **489** (2017), 506–528.

[10] S. Goto, Ulrich ideals in $k[[X,Y]]/(Y^3)$ and $k[[X,Y]]/(X^2Y)$, *The Proceedings of the 40-th Symposium on Commutative Algebra*, in preparation.

[11] S. Goto and R. Isobe, Anti-stable rings, Preprint 2018.

[12] S. Goto, R. Isobe, and S. Kumashiro, The structure of chains of Ulrich ideals in Cohen-Macaulay local rings of dimension one, *Acta Mathematica Vietnamica*, **44** (2019), Issue 1, 65–82.

[13] S. Goto, R. Isobe, and S. Kumashiro, Correspondence between trace ideals and birational extensions with application to the analysis of the Gorenstein property of rings, *J. Pure Appl. Algebra*, **224** (2020), 747–767.

[14] S. Goto, R. Isobe, and N. Taniguchi, Ulrich ideals and 2-AGL rings, arXiv:1902.05335.

[15] S. Goto, S.-i. Iai, and K.-i. Watanabe, Good ideals in Gorenstein local ring, *Trans. Amer. Math. Soc.*, **353** (2000), 2309–2346.

[16] S. Goto and S. Kumashiro, On GGL rings, Preprint 2017.

[17] S. Goto, S. Kumashiro, and N. T. H. Loan, Residually faithful modules and the Cohen-Macaulay type of idealizations, *J. Math. Soc. Japan* (to appear).

[18] S. Goto, N. Matsuoka, and T. T. Phuong, Almost Gorenstein rings, *J. Algebra*, **379** (2013), 355–381.

[19] S. Goto, N. Matsuoka, N. Taniguchi, and K.-i. Yoshida, The almost Gorenstein Rees algebras of parameters, *J. Algebra*, **452** (2016), 263–278.

[20] S. Goto, N. Matsuoka, N. Taniguchi, and K.-i. Yoshida, The almost Gorenstein Rees algebras over two-dimensional regular local rings, *J. Pure Appl. Algebra*, **220** (2016), 3425–3436.

[21] S. Goto, N. Matsuoka, N. Taniguchi, and K.-i. Yoshida, On the almost Gorenstein property in Rees algebras of contracted ideals, *Kyoto J. Math.* (to appear).

[22] S. Goto, N. Matsuoka, N. Taniguchi, and K.-i. Yoshida, The almost Gorenstein Rees algebras of p_g-ideals, good ideals, and powers of the maximal ideals, *Michigan Math. J.*, **67** (2018), 159–174.

[23] S. Goto, K. Nishida, and K. Ozeki, Sally modules of rank one, *Michigan Math. J.*, **57** (2008), 359–381.

[24] S. Goto, K. Ozeki, R. Takahashi, K.-i. Yoshida, and K.-i. Watanabe, Ulrich ideals and modules, *Math. Proc. Camb. Phil. Soc.*, **156** (2014), 137–166.

[25] S. Goto, K. Ozeki, R. Takahashi, K.-i. Yoshida, and K.-i. Watanabe, Ulrich ideals and modules over two-dimensional rational singularities, *Nagoya Math. J.*, **221** (2016), 69–110.

[26] S. Goto, M. Rahimi, N. Taniguchi, and H. L. Truong, When are the Rees algebras of parameter ideals almost Gorenstein graded rings?, *Kyoto J. Math.*, **57** (2017), no.3, 655–666.

[27] S. Goto and Y. Shimoda, On the Rees algebras of Cohen-Macaulay local rings, *Lecture Notes in Pure and Appl. Math.*, **68** (1982), 201–231, Marcel Dekker.

[28] S. Goto, R. Takahashi, and N. Taniguchi, Almost Gorenstein rings - towards a theory of higher dimension, *J. Pure Appl. Algebra*, **219** (2015), 2666–2712.

[29] S. Goto, R. Takahashi, and N. Taniguchi, Ulrich ideals and almost Gorenstein rings, *Proc. Amer. Math. Soc.*, **144** (2016), 2811–2823.

[30] S. Goto and K. Watanabe, On graded rings, I, *J. Math. Soc. Japan*, **30** (1978), 179–213.

[31] J. Herzog, T. Hibi, D. I. Stamate, The trace of canonical module, arXiv:1612.02723.

[32] J. Herzog and E. Kunz, Der kanonische Modul eines Cohen-Macaulay-Rings, Lecture Notes in Mathematics, **238**, Springer-Verlag, 1971.

[33] J. Herzog and E. Kunz, Die Wertehalbgruppe eines lokalen Rings der Dimension 1, *S.-B. Heidelberger Akad. Wiss.*, 1971, 2 Abh. (1971).

[34] R. Isobe, The structure of Ulrich ideals in hypersurfaces, arXiv:1905.02048.

[35] A. Koura and N. Taniguchi, Bounds for the first Hilbert coefficients of $\mathfrak{m}$-primary ideals, *Tokyo J. Math.*, **38** (2015), 125–133.

[36] T. Kobayashi and R. Takahashi, Rings whose ideals are isomorphic to trace ideals, arXiv:1807.00291.

[37] H. Lindo, Trace ideals and centers of endomorphism rings of modules over commutative rings, J. Algebra, **482** (2017), 102–130.

[38] H. Lindo, Self-injective commutative rings have no nontrivial rigid ideals, arXiv:1710.01793, 2017.

[39] H. Lindo and N. Pande, Trace ideals and the Gorenstein property, arXiv:1802.06491.

[40] J. Lipman, Stable ideals and Arf rings, *Amer. J. Math.*, **93** (1971), 649–685.

[41] M. Nagata, Local Rings, Interscience, 1962.

[42] J. C. Rosales and P. A. García-Sánchez, Numerical Semigroups, Developments in Mathematics, **20**, Springer, New York, 2009.

[43] J. Sally, Cohen-Macaulay local rings of maximal embedding dimension, *J. Algebra*, **56** (1979), 168–183.

[44] R. Takahashi, On G-regular local rings, *Comm. Algebra*, **36** (2008), no. 12, 4472–4491.

[45] J. Sally and W. V. Vasconcelos, Stable rings, *J. Pure Appl. Algebra*, **4** (1974), 319–336.

[46] J. Tate, Homology of Noetherian rings and local rings, *Illinois J. Math.*, **1** (1957), 14–27.

[47] J. v. Neumann, On regular rings, *Proc. Nat. Acad. Sci. U. S. A.*, **22** (1936), 707–713.

[48] W. V. Vasconcelos, Hilbert functions, analytic spread, and Koszul homology, Commutative algebra: Syzygies, multiplicities, and birational algebra (South Hadley, 1992), *Contemp. Math.*, **159**, 410–422, *Amer. Math. Soc.*, Providence, RI, 1994.

[49] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, *London Mathematical Society Lecture Notes Series*, **146**, Cmabridge University Press, 1990.