On generalized Gorenstein rings

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Contents

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First of all, I would like to express my profound gratitude and respect to Professor Shiro Goto of Meiji University. He has taught me the theme of this research since I was an undergraduate student at Meiji University. I was greatly influenced by his sincere approach to research.

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Finally, I wish to appreciate my family's active support and reliance on me.

Preface

The notion of commutative rings is important in singularity theory, and it is deeply related to various algebraic areas, say, algebraic geometry, representation theory, invariant theory, and combinatorics. At the end of the 19th century, commutative ring theory was originally established by D. Hilbert throughout the study of invariant algebras. After the breakthrough of his work, E. Noether played a central role in the development of commutative algebra. In the middle of the 20th century, the notion of the homological method was innovated into commutative ring theory by many researchers, say, M. Auslander, D. A. Buchsbaum, D. Rees, D. G. Northcott, J. -P. Serre and others. Since then, and up to the present day, the study of Cohen-Macaulay rings and modules has been becoming central subject of commutative ring theory.

The purpose of this dissertation is to stratify Cohen-Macaulay rings. As is well-known, Cohen-Macaulay rings are stratified

$$\text{regular ring} \Rightarrow \text{complete intersection} \Rightarrow \text{Gorenstein ring} \Rightarrow \text{Cohen-Macaulay ring}$$

in terms of homological algebra. Among them, the notion of Gorenstein rings is defined by the local finiteness of the self-injective dimension. Gorenstein rings are known to have interesting properties such as total reflexivity and vanishing of cohomology. They appear with beautiful symmetry in not only commutative algebra but also combinatorics, algebraic geometry, invariant theory and so on. However, on the other hand, there is a huge gap in whether the self-injective dimension is finite or not, and many Cohen-Macaulay rings appearing in concrete examples are actually not Gorenstein rings. For instance, although any normal semigroup rings are Cohen-Macaulay, a normal semigroup ring is Gorenstein if and only if its interior coincides with itself after some shift (see [11, Theorem 6.3.5]). Furthermore, if R is a Cohen-Macaulay local ring and M is a maximal Cohen-Macaulay R-module, the idealization $R \ltimes M$ is always Cohen-Macaulay local ring again. However, $R \ltimes M$ is Gorenstein if and only if M is isomorphic to the canonical module of R (see, [35, 69]). Therefore, our problem is now stated as follows.

Problem. Give a new theory of rings between Gorenstein and Cohen-Macaulay.

One of the important results of the problem is about *almost Gorenstein rings*. The basic papers [7, 36, 46] revealed the properties of the non-Gorenstein almost Gorenstein rings such as G-Regularity and the Gorensteinness of the Blow-up algebra. On the other hand, one also comes to feel cramped for almost Gorenstein rings. For instance, the almost Gorenstein property is not preserved by flat base changes, that is, $R \to R[X]/(X^n)$ is no longer almost Gorenstein if R is an almost Gorenstein local ring and $n > 1$. Besides the almost Gorenstein theory, the study of non-Gorenstein Cohen-Macaulay rings has been carried out under intense competition. One can also find other stratifications of Cohen-Macaulay rings in [15, 55]. These theories have not been unified yet. The problem here is not only inconvenience but also some questions cannot be answered in the individual frameworks.

In this dissertation, we try to solve these problems by unifying these theories based on the almost Gorenstein theory. That is, we defined the notion of *generalized Gorenstein rings* as a generalization of almost Gorenstein rings, and solved the crippled problems of the almost Gorenstein theory (Theorems 1.2.3 and 1.1.6). Furthermore, we established theorems, which cannot be stated in the almost Gorenstein theory (Theorems 1.2.5, 2.1.2, and 3.5.5). We also clarified the relation between almost Gorensteinness and nearly Gorensteinness in the sense of [55] (see, Corollary 1.8.12).

The main themes of this dissertation are classified into the following three directions.

A. Formation of the theory of generalized Gorenstein rings

B. Ubiquity of Ulrich ideals and trace ideals

C. Auslander-Reiten conjecture

Direction A is written in Chapters 1 and 2. We further develop the almost Gorenstein ring theory and give a new structure of rings, that is, generalized Gorenstein rings. Chapter 1 and 2 are reproductions of the work [34] and [31], respectively. The final version of the papers [34] and [31] will be submitted elsewhere for publication.

Direction B is written in Chapters 3 and 4. Ulrich ideals are a special class of trace ideals. One can consult [67, 43] for basic properties of trace ideals and Ulrich ideals, respectively. In these chapters, we study the ubiquity of Ulrich ideals and describe the structure of rings from the ubiquity of trace ideals (Theorems 3.1.4 and 4.1.3). Chapter 3 and 4 are reproductions of the contents of [29] and [30], respectively. These papers [29] and [30] were already published on the journals.

Direction C is written in Chapter 5. Based on the results of Directions A and B, we embark on the Auslander-Reiten conjecture, which is a long-standing problem in the representation theory. As a result, we obtain a new result on determinantal rings although it is not a complete answer (Theorem 5.2.9). Chapter 5 is a reproduction of the work [62]. The final version will be submitted elsewhere for publication. All co-authors have already given reproduction permission.

February, 2020

Shinya Kumashiro

Chapter 1

On generalized Gorenstein rings

1.1 Introduction

Almost Gorenstein rings are one of the most interesting objects in the study of non-Gorenstein Cohen-Macaulay rings. The notion of almost Gorenstein rings was originated from V. Barucci and R. Fröberg [7] for one-dimensional analytically unramified local rings. After that, S. Goto, N. Matsuoka, and T. T. Phuong [36] developed the theory of almost Gorenstein ring of dimension one. Nowadays the notion of almost Gorenstein rings is defined in arbitrary Cohen-Macaulay local/graded rings by S. Goto, R. Takahashi, and N. Taniguchi [46], through the notion of Ulrich R-modules. Let us recall their definition of almost Gorenstein local rings.

Definition 1.1.1. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, possessing the canonical module K_R. Then we say that R is an *almost Gorenstein local ring*, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that either $C = (0)$ or C is an Ulrich R-module with respect to $\mathfrak{m}$.

Here, for a finitely generated R-module C and an $\mathfrak{m}$-primary ideal $\mathfrak{a}$, C is called an Ulrich R-module with respect to $\mathfrak{a}$ if the following three conditions are satisfied.

(1) C is a Cohen-Macaulay R-module (i.e. $\mathrm{depth}_R\, C = \dim_R C$),

(2) $\mathrm{e}_{\mathfrak{a}}^0(C) = \ell_R(C/\mathfrak{a}C)$, and

(3) $C/\mathfrak{a}C$ is an $R/\mathfrak{a}$-free module,

where $\ell_R(*)$ stands for the length and

$$\mathrm{e}_{\mathfrak{a}}^0(C) = \lim_{n\to\infty} (d-1)! \cdot \frac{\ell_R(C/\mathfrak{a}^{n+1}C)}{n^{d-1}}$$

denotes the multiplicity of C with respect to $\mathfrak{a}$.

Almost Gorenstein rings admit many interesting properties. For instance, let R be a non-Gorenstein almost Gorenstein local ring. Then, for a finitely generated R-module

M, $\mathrm{Ext}_R^i(M, R) = 0$ for all $i > 0$ implies that M is free. In particular, R is G-regular in the sense of [74], that is, every totally reflexive module is free. In addition, all the known Cohen-Macaulay local rings of finite representation type are almost Gorenstein local rings. It is also studied that problems of when Rees algebras of ideals/modules, determinantal rings, numerical semigroup rings are almost Gorenstein rings. The research on almost Gorenstein rings is still in progress. On the other hand, one also comes to feel cramped for almost Gorenstein local rings. For instance, let R be an almost Gorenstein local ring but not a Gorenstein ring. Then, for any positive integer $n > 1$, $R[x]/(x^n)$ is no longer an almost Gorenstein local ring, where $R[x]$ is a polynomial ring over R. Therefore it seems to be natural to ask what rings contain almost Gorenstein local rings naturally.

The purpose of this chapter is to introduce the notion of generalized Gorenstein rings. almost Gorenstein local rings are generalized Gorenstein rings and we regard theory of almost Gorenstein local rings as a part of theory of generalized Gorenstein rings. Through the notion of generalized Gorenstein rings, we not only solve some problems on almost Gorenstein local rings, but also find many propositions which cannot be obtained without the notion of generalized Gorenstein rings. Let us fix our notation to state the definition of generalized Gorenstein rings and our main results precisely. Throughout this section, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring possessing the canonical module K_R. Then generalized Gorenstein local rings are defined as follows.

Definition 1.1.2. We say that R is a *generalized Gorenstein local ring*, if there exist an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ and an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that

(i) C is an Ulrich R-module with respect to $\mathfrak{a}$ and

(ii) $\mathrm{K}_R/\mathfrak{a}\mathrm{K}_R$ is $R/\mathfrak{a}$-free.

If R is a Gorenstein ring, then R is a generalized Gorenstein local ring by taking a parameter ideal $\mathfrak{a} = (a_1, a_2, \dots, a_d)$ a and a natural exact sequence

$$0 \to R \xrightarrow{a_1} R \to R/(a_1) \to 0.$$

We say that R is a *generalized Gorenstein ring with respect to* $\mathfrak{a}$, if R is a non-Gorenstein generalized Gorenstein local ring possessing an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ which satisfies Definition 1.2.2, see Proposition 1.4.3. With this notation the notion of almost Gorenstein local rings is the same as the notion of generalized Gorenstein local rings with respect to $\mathfrak{m}$ if R is not a Gorenstein ring. Let us explain the utilities of the notion of generalized Gorenstein local rings. First of all, we have the following statement so-called characterizations of non-zerodivisor and flat base change.

Theorem 1.1.3 (Theorem 1.4.6 and 1.4.7)**.** *The following assertions hold true.*

(1) *Suppose that R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$. Suppose that $\dim R \geq 2$ and the residue field is infinite. Then we can choose a non-zerodivisor $f \in \mathfrak{a}$ of R so that $R/(f)$ is a generalized Gorenstein local ring with respect to $\mathfrak{a}/(f)$.*

(2) *Let* $f \in \mathfrak{m}$ *be a non-zerodivisor of* R *and suppose* $R/(f)$ *is a generalized Gorenstein local ring with respect to* $[\mathfrak{a} + (f)]/(f)$. *Then* R *is a generalized Gorenstein local ring with respect to* $\mathfrak{a} + (f)$ *and* $f \notin \mathfrak{m}\mathfrak{a}$.

(3) *Let* $\psi : R \to S$ *be a flat local homomorphism of Noetherian local rings such that* $S/\mathfrak{m}S$ *is a Cohen-Macaulay local ring. Let* $J \subseteq S$ *be a parameter ideal in* $S/\mathfrak{m}S$. *Consider the following two conditions.*

  (i) R *is a generalized Gorenstein local ring with respect to* $\mathfrak{a}$ *and* $S/\mathfrak{m}S$ *is a Gorenstein ring.*

  (ii) S *is a generalized Gorenstein local ring with respect to* $\mathfrak{a}S + J$.

  Then (i) $\Rightarrow$ (ii) *holds true.* (ii) $\Rightarrow$ (i) *also holds true if* $R/\mathfrak{m}$ *is infinite.*

In particular, if R is a generalized Gorenstein local ring, then so is $R[x]/(x^n)$ for every $n > 0$. One can find some other constructions of generalized Gorenstein local rings. For instance, every Cohen-Macaulay local ring whose multiplicity is at most three is a generalized Gorenstein local ring, if the residue field is infinite (see Proposition 1.5.14). We will construct generalized Gorenstein local rings from numerical semigroup rings, idealizations, and determinantal rings, see Section 1.5 and 1.7.

The notion of generalized Gorenstein local rings provides a deeper understanding for the trace of the canonical module. Here the trace of the canonical module is the image of the following R-linear map

$$t : \mathrm{Hom}_R(\mathrm{K}_R, R) \otimes_R \mathrm{K}_R \to R,$$

where $t(f \otimes x) = f(x)$ for all $f \in \mathrm{Hom}_R(\mathrm{K}_R, R)$ and $x \in \mathrm{K}_R$. One of the most important facts of the trace of the canonical module $\mathrm{tr}(\mathrm{K}_R)$ is to describe non-Gorenstein locus of R. Since many properties of generalized Gorenstein local rings are condensed into the one-dimensional case by Theorem 1.2.3, for a while, we focus on the case where $\dim R = 1$. Then, once R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$ for some ideal $\mathfrak{a}$, we have $\mathfrak{a} = \mathrm{tr}_R(\mathrm{K}_R)$. Furthermore, if R has maximal embedding dimension, we have the following.

Theorem 1.1.4 (Theorem 1.5.18 and 1.8.14)**.** *Suppose that* $\dim R = 1$ *and* R *has maximal embedding dimension. Assume that there exist a canonical ideal* $I \subsetneq R$ *and its minimal reduction* $(a) \subseteq I$. *Then the following conditions are equivalent.*

(1) R *is a generalized Gorenstein local ring but not an almost Gorenstein local ring.*

(2) $B = \mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m})$ *is a generalized Gorenstein local ring but not a Gorenstein ring and* $\mathrm{v}(B) = \mathrm{e}(B) = \mathrm{e}(R)$.

When this is the case, there exists an element $\alpha \in \mathfrak{m}$ *such that* $\mathfrak{m}^2 = \alpha\mathfrak{m}$ *and we have the following.*

  (i) $R/\mathfrak{m} \cong B/\mathfrak{n}$,

(ii) $\ell_B(B/\mathrm{tr}_B(\mathrm{K}_B)) = \ell_R(R/\mathrm{tr}_R(\mathrm{K}_R)) - 1$, *and*

(iii) $\mathfrak{n}^2 = \alpha\mathfrak{n}$.

Here $\mathfrak{n}$ *denotes the unique maximal ideal of* B *and* $\mathrm{v}(R)$ (resp. $\mathrm{e}(R)$) *denotes the embedding dimension of* R (resp. *the multiplicity of* R).

On the other hand, S. Goto, N. Matsuoka, and T. T. Phuong proved that R is an almost Gorenstein local ring possessing maximal embedding dimension if and only if $\mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m})$ is a Gorenstein ring (see [36, Theorem 5.1.]). Combining to these two results, generalized Gorenstein local rings possessing maximal embedding dimension finally reach Gorenstein rings by the action taking the endomorphism of the maximal ideal.

We can also find the relation between generalized Gorenstein local rings and Ulrich ideals. Here the notion of Ulrich ideals is introduced by the S. Goto, K. Ozeki, R. Takahashi, K. Watanabe, and K. Yoshida [43] and they showed that Ulrich ideals enjoy very interesting properties. One can consult [43, 47] and Theorem 1.8.1 for the basic properties of Ulrich ideals. Here let us note the definition of Ulrich ideals.

Definition 1.1.5. [43, Definition 2.1.] Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Let I be an $\mathfrak{m}$-primary ideal of R and assume that I contains a parameter ideal Q of R as a reduction. We say that I is an *Ulrich ideal* of R, if the following conditions are satisfied.

(1) $I \neq Q$, but $I^2 = QI$.

(2) I/I^2 is a free R/I-module.

Assume that R is a non-Gorenstein generically Gorenstein ring, that is, the total ring $\mathrm{Q}(R)$ of fractions of R is a Gorenstein ring. Then we see that there is no Ulrich ideal which proper contained in $\mathrm{tr}_R(\mathrm{K}_R)$ if $\dim R = 1$ (see Theorem 1.5.27). Conversely, every Ulrich ideal which can not be generated by $\dim R + 1$ elements contain $\mathrm{tr}_R(\mathrm{K}_R)$ (see Theorem 1.8.4). These observations provide the question of when $\mathrm{tr}_R(\mathrm{K}_R)$ is an Ulrich ideal. We will answer the question by using the notion of generalized Gorenstein local rings (see Theorem 1.8.7 and Corollary 1.8.10). As a Corollary, we give a generalization of the theorem of J. Herzog, T. Hibi, and D. I. Stamate [55, Theorem 7.4.]. Furthermore, we completely determined the set of all Ulrich ideals for one-dimensional generalized Gorenstein local rings possessing maximal embedding dimension as following.

Theorem 1.1.6 (Theorem 1.8.18)**.** *Suppose that* R *is not a Gorenstein ring and* $\dim R = 1$. *Set* $v = \mathrm{v}(R)$ *and* $N = \ell_R(R/\mathrm{tr}_R(\mathrm{K}_R)) > 0$. *Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring possessing maximal embedding dimension.*

(2) $\mathrm{tr}_R(\mathrm{K}_R)$ *and* $\mathfrak{m}$ *are Ulrich ideals.*

(3) *R is G-regular and a length of a maximal chain of Ulrich ideals is* $N - 1$.

(4) *There exist elements* $\alpha, x_2, x_3, \dots, x_v \in \mathfrak{m}$ *which satisfy the following two conditions.*

   (i) $\mathfrak{m} = (\alpha, x_2, x_3, \dots, x_v)$ *and*

   (ii) $\mathcal{X}_R = \{(\alpha^i, x_2, x_3, \dots, x_v) \mid 1 \le i \le N\}$.

Here $\mathcal{X}_R$ *denotes the set of all Ulrich ideals.*

Let us give one more result of generalized Gorenstein local rings.

Theorem 1.1.7 (Corollary 1.6.5)**.** *Let* $(S, \mathfrak{n})$ *be a Gorenstein local ring and* $(R, \mathfrak{m})$ *a one-dimensional Cohen-Macaulay local ring but not a Gorenstein ring. Let* $\varphi : S \to R$ *be a surjective ring homomorphism and suppose the projective dimension of* R *over* S *is finite. Let* $\mathfrak{a}$ *be an ideal of* S *such that* $\mathfrak{a} \supseteq \operatorname{Ker}\varphi$ *and set* $n = \mu_S(\mathfrak{a})$ *and* $\mathfrak{a} = (x_1, x_2, \dots, x_n)$. *Then the following conditions are equivalent.*

(1) R *is a generalized Gorenstein local ring with respect to* $\mathfrak{a}R$.

(2) *There exists a minimal* S*-free resolution*

$$0 \to S^{\oplus r} \xrightarrow{\mathbb{M}} S^{\oplus q} \to \cdots \to S \to R \to 0$$

of R *such that*

$$ {}^t\mathbb{M} = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{**} & * \\ \hline x_1\ x_2\ \dots\ x_n & & & \mathit{0} & \\ & x_1\ x_2\ \dots\ x_n & & & \mathit{0} \\ & & \ddots & & \\ \mathit{0} & & & x_1\ x_2\ \dots\ x_n & \end{array}\right), $$

where all components of $**$ *and* $*$ *are in* $\mathfrak{a}$.

Theorem 1.2.8 has applications for determinantal rings and the Rees algebras of parameter ideals over Gorenstein local rings (see Theorem 1.7.3 and Corollary 1.7.5).

We now explain how this chapter is organized. In Section 1.3 we give a brief survey on Ulrich modules with respect to $\mathfrak{a}$, which we need throughout this chapter. In Section 1.4 we explore basic properties of generalized Gorenstein local rings which contain non-zerodivisor characterization and flat base change. In Section 1.5 we focus our attention on the one-dimensional case. We see that the notion of generalized Gorenstein local rings is important to study that Ulrich ideals and the endomorphism of the maximal ideal. Numerous examples of generalized Gorenstein local rings are given via numerical semigroup rings and idealizations. Furthermore, we will see that many results in the case of dimension one can be extended for the higher-dimensional case, through Theorem 1.2.3. The purpose of Section 1.6 is to show Theorem 1.2.8. In Section 1.7 we construct the examples of higher-dimensional generalized Gorenstein local rings by using Theorem 1.2.8. In Section 1.8 we again consider Ulrich ideals by using the trace of the canonical module.

Let us fix our notation throughout this chapter. In what follows, unless otherwise specified, let R denote a Noetherian local ring with maximal ideal $\mathfrak{m}$. For each finitely generated R-module M, let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators of M (resp. the length of M). If M is a Cohen-Macaulay R-module, $\mathrm{r}_R(M)$ denotes the Cohen-Macaulay type of M.

Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R and set $s = \dim_R M$. Then we have the integers $\{\mathrm{e}^i_{\mathfrak{a}}(M)\}_{0 \le i \le s}$ such that the equality

$$\ell_R(M/\mathfrak{a}^{n+1}M) = \mathrm{e}^0_{\mathfrak{a}}(M) \cdot \binom{n+s}{s} - \mathrm{e}^1_{\mathfrak{a}}(M) \cdot \binom{n+s-1}{s-1} + \cdots + (-1)^s \mathrm{e}^s_{\mathfrak{a}}(M)$$

holds for all $n \gg 0$. $\mathrm{e}^i_{\mathfrak{a}}(M)$ is called the ith Hilbert coefficient of M with respect to $\mathfrak{a}$. Let $\mathrm{e}(R) = \mathrm{e}^0_{\mathfrak{m}}(R)$ (resp. $\mathrm{v}(R)$) denote the multiplicity of R (resp. the embedding dimension of R). $\mathrm{r}(R) = \mathrm{r}_R(R)$ denotes the Cohen-Macaulay type of R if R is a Cohen-Macaulay local ring.

Let $\mathrm{Q}(R)$ be the total ring of fractions of R. For R-submodules X and Y of $\mathrm{Q}(R)$, let

$$X : Y = \{a \in \mathrm{Q}(R) \mid aY \subseteq X\}.$$

For ideals I, J of R, we set $I :_R J = \{a \in R \mid aJ \subseteq I\}$, whence $I :_R J = (I : J) \cap R$.

1.2 Introduction

Almost Gorenstein rings are one of the most interesting objects in the study of non-Gorenstein Cohen-Macaulay rings. The notion of almost Gorenstein rings was originated from V. Barucci and R. Fröberg [7] for one-dimensional analytically unramified local rings. After that, the first author, N. Matsuoka, and T. T. Phuong [36] developed the theory of almost Gorenstein ring of dimension one. Nowadays the notion of almost Gorenstein rings is defined in arbitrary Cohen-Macaulay local/graded rings by the first author, R. Takahashi, and N. Taniguchi [46], through the notion of Ulrich R-modules. Let us recall their definition of almost Gorenstein local rings.

Definition 1.2.1. ([46, Definition 1.1]) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, possessing the canonical module K_R. Then we say that R is an *almost Gorenstein local ring*, if there exists an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that either $C = (0)$ or C is an Ulrich R-module with respect to $\mathfrak{m}$.

Here, for a finitely generated R-module C and an $\mathfrak{m}$-primary ideal $\mathfrak{a}$, C is called an Ulrich R-module with respect to $\mathfrak{a}$ if the following three conditions are satisfied (cf. [43, Definition 1.2]) .

(1) C is a Cohen-Macaulay R-module (i.e. $\operatorname{depth}_R C = \dim_R C$),

(2) $\mathrm{e}^0_{\mathfrak{a}}(C) = \ell_R(C/\mathfrak{a}C)$, and

(3) $C/\mathfrak{a}C$ is an $R/\mathfrak{a}$-free module,

where $\ell_R(*)$ stands for the length and

$$\mathrm{e}^0_{\mathfrak{a}}(C) = \lim_{n\to\infty} (d-1)!\cdot\frac{\ell_R(C/\mathfrak{a}^{n+1}C)}{n^{d-1}}$$

denotes the multiplicity of C with respect to $\mathfrak{a}$.

Almost Gorenstein rings admit many interesting properties. For instance, let R be a non-Gorenstein almost Gorenstein local ring. Then, for a finitely generated R-module M, $\mathrm{Ext}^i_R(M,R) = 0$ for all $i > 0$ implies that M is free ([46, Corollary 4.5], [3, Corollary 4.6]). In particular, R is G-regular in the sense of [74], that is, every totally reflexive module is free. In addition, all the known Cohen-Macaulay local rings of finite representation type are almost Gorenstein local rings ([46]). It is also studied that problems of when Rees algebras of ideals/modules, determinantal rings, and numerical semigroup rings are almost Gorenstein rings. On the other hand, one also comes to feel cramped for almost Gorenstein local rings. For instance, let R be an almost Gorenstein local ring but not a Gorenstein ring. Then, for any positive integer $n > 1$, $R[x]/(x^n)$ is no longer an almost Gorenstein local ring, where $R[x]$ is a polynomial ring over R.

Besides the almost Gorenstein theory, the study of non-Gorenstein Cohen-Macaulay rings has been carried out under intense competition. One can also find other stratifications of Cohen-Macaulay rings, say, nearly Gorenstein rings and 2-almost Gorenstein local rings (see [15, 55]). These theories have not been unified yet. The problem here is not only inconvenience but also some questions cannot be answered in the individual frameworks.

In this chapter, we try to solve these problems by unifying these theories based on the almost Gorenstein theory. That is, we introduce the notion of *generalized Gorenstein rings* and we regard the theory of almost Gorenstein local rings as a part of the theory of generalized Gorenstein rings. As a result, we solve the crippled problems of the almost Gorenstein theory (Theorems 1.4.6 and 1.4.7). Furthermore, we establish propositions, which cannot be stated in the almost Gorenstein theory (Corollary 1.5.12 and Theorems 1.8.14 and 1.8.18). These results make the relations among almost Gorenstein local rings, 2-almost Gorenstein local rings, and nearly Gorenstein rings easier to understand.

Let us fix our notation to state the definition of generalized Gorenstein rings and our main results precisely. Throughout this section, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring possessing the canonical module K_R. Then generalized Gorenstein local rings are defined as follows.

Definition 1.2.2. We say that R is a *generalized Gorenstein local ring*, if there exist an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ and an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that

(i) C is an Ulrich R-module with respect to $\mathfrak{a}$ and

(ii) $\mathrm{K}_R/\mathfrak{a}\mathrm{K}_R$ is $R/\mathfrak{a}$-free.

If R is a Gorenstein ring, then R is a generalized Gorenstein local ring by taking a parameter ideal $\mathfrak{a} = (a_1, a_2, \dots, a_d)$ and a natural exact sequence

$$0 \to R \xrightarrow{a_1} R \to R/(a_1) \to 0.$$

We say that R is a *GGL ring with respect to* $\mathfrak{a}$, if R is a non-Gorenstein generalized Gorenstein local ring possessing an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ which satisfies Definition 1.2.2, see Proposition 1.4.3. With this notation, the notion of non-Gorenstein almost Gorenstein local rings is the same as the notion of generalized Gorenstein local rings with respect to $\mathfrak{m}$. Let us explain the utilities of the notion of generalized Gorenstein local rings. First of all, we have the following statement so-called characterizations of non-zerodivisor and flat base change.

Theorem 1.2.3 (Theorem 1.4.6)**.** *The following assertions hold true.*

(1) *Suppose that R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$. Suppose that $\dim R \geq 2$ and the residue field is infinite. Then we can choose a non-zerodivisor $f \in \mathfrak{a}$ of R so that $R/(f)$ is a generalized Gorenstein local ring with respect to $\mathfrak{a}/(f)$.*

(2) *Let $f \in \mathfrak{m}$ be a non-zerodivisor of R and suppose $R/(f)$ is a generalized Gorenstein local ring with respect to $[\mathfrak{a} + (f)]/(f)$. Then R is a generalized Gorenstein local ring with respect to $\mathfrak{a} + (f)$ and $f \notin \mathfrak{m}\mathfrak{a}$.*

Theorem 1.2.4 (Theorem 1.4.7)**.** *Let $\psi : R \to S$ be a flat local homomorphism of Noetherian local rings such that $S/\mathfrak{m}S$ is a Cohen-Macaulay local ring. Let $J \subseteq S$ be a parameter ideal in $S/\mathfrak{m}S$. Consider the following two conditions.*

(1) *R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$ and $S/\mathfrak{m}S$ is a Gorenstein ring.*

(2) *S is a generalized Gorenstein local ring with respect to $\mathfrak{a}S + J$.*

Then (1) $\Rightarrow$ (2) *holds true.* (2) $\Rightarrow$ (1) *also holds true if $R/\mathfrak{m}$ is infinite.*

By Theorem 1.2.4, if R is a generalized Gorenstein local ring, then so is $R[x]/(x^n)$ for every $n > 0$. One can find some other constructions of generalized Gorenstein local rings. For instance, every Cohen-Macaulay local ring whose multiplicity is at most three is a generalized Gorenstein local ring, if the residue field is infinite (see Proposition 1.5.14). We will construct generalized Gorenstein local rings from numerical semigroup rings, idealizations, and determinantal rings, see Sections 1.5 and 1.7.

The notion of generalized Gorenstein local rings provides a deeper understanding for the trace of the canonical module. Here the trace of the canonical module is the image of the following R-linear map

$$t : \mathrm{Hom}_R(\mathrm{K}_R, R) \otimes_R \mathrm{K}_R \to R,$$

where $t(f \otimes x) = f(x)$ for all $f \in \mathrm{Hom}_R(\mathrm{K}_R, R)$ and $x \in \mathrm{K}_R$. One of the most important facts of the trace of the canonical module $\mathrm{tr}(\mathrm{K}_R)$ is to describe non-Gorenstein locus of R, see for instance [55, Lemma 2.1]. Since many properties of generalized Gorenstein local rings are condensed into the one-dimensional case by Theorem 1.2.3, for a while, we focus on the case where $\dim R = 1$. Then, once R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$ for some ideal $\mathfrak{a}$, we have $\mathfrak{a} = \mathrm{tr}_R(\mathrm{K}_R)$ (Theorem 1.5.8 and Remark 1.8.3(2)). Furthermore, if R has maximal embedding dimension, that is, the embedding dimension of R is the multiplicity of R, then we have the following. Let $\mathrm{v}(R)$ (resp. $\mathrm{e}(R)$) denote the embedding dimension of R (resp. the multiplicity of R).

Theorem 1.2.5 (Theorem 1.8.14). *Suppose that* $\dim R = 1$ *and* R *has maximal embedding dimension. Assume that there exist a canonical ideal* $I \subsetneq R$ *and its minimal reduction* $(a) \subseteq I$. *Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring but not an almost Gorenstein local ring.*

(2) *$B = \mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m})$ is a generalized Gorenstein local ring with $\mathrm{v}(B) = \mathrm{e}(B) = \mathrm{e}(R)$, but not a Gorenstein ring.*

In particular, B is a Cohen-Macaulay local ring with the maximal ideal $\mathfrak{n}$. When this is the case, there exists an element $\alpha \in \mathfrak{m}$ such that $\mathfrak{m}^2 = \alpha\mathfrak{m}$ and we have the following.

(i) $R/\mathfrak{m} \cong B/\mathfrak{n}$,

(ii) $\ell_B(B/\mathrm{tr}_B(\mathrm{K}_B)) = \ell_R(R/\mathrm{tr}_R(\mathrm{K}_R)) - 1$, *and*

(iii) $\mathfrak{n}^2 = \alpha\mathfrak{n}$.

On the other hand, the first author, N. Matsuoka, and T. T. Phuong proved that R is an almost Gorenstein local ring possessing maximal embedding dimension if and only if $\mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m})$ is a Gorenstein ring (see [36, Theorem 5.1.]). Combining to these two results, generalized Gorenstein local rings possessing maximal embedding dimension finally reach Gorenstein rings by the action taking the endomorphism of the maximal ideal.

We can also find the relation between generalized Gorenstein local rings and Ulrich ideals. Here the notion of Ulrich ideals is introduced by the first author, K. Ozeki, R. Takahashi, K. Watanabe, and K. Yoshida [43] and they showed that Ulrich ideals enjoy very interesting properties. One can consult [43, 47] for the basic properties of Ulrich ideals. Here let us note the definition of Ulrich ideals.

Definition 1.2.6. ([43, Definition 2.1.]) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Let I be an $\mathfrak{m}$-primary ideal of R and assume that I contains a parameter ideal Q of R as a reduction. We say that I is an *Ulrich ideal* of R, if the following conditions are satisfied.

(1) $I \neq Q$, but $I^2 = QI$.

(2) I/I^2 is a free R/I-module.

Let R be a non-Gorenstein generically Gorenstein ring, that is, $R_\mathfrak{p}$ is Gorenstein for all minimal prime $\mathfrak{p}$. Then we see that there is no Ulrich ideal which is strictly contained in $\mathrm{tr}_R(\mathrm{K}_R)$ if $\dim R = 1$ (see Theorem 1.5.27). Furthermore, every Ulrich ideal which can not be generated by $\dim R + 1$ elements contain $\mathrm{tr}_R(\mathrm{K}_R)$ (see Theorem 1.8.4). These observations provide the question of when $\mathrm{tr}_R(\mathrm{K}_R)$ is an Ulrich ideal. We will answer the question by using the notion of generalized Gorenstein local rings (see Theorem 1.8.7 and Corollary 1.8.10). As a corollary, we give a generalization of the result of J. Herzog, T. Hibi, and D. I. Stamate [55, Theorem 7.4.]. Furthermore, we completely determined the set of all Ulrich ideals for one-dimensional generalized Gorenstein local rings possessing maximal embedding dimension as follows.

Theorem 1.2.7 (Theorem 1.8.18)**.** *Suppose that R is not a Gorenstein ring and* $\dim R = 1$. *Set $v = \mathrm{v}(R)$. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring possessing maximal embedding dimension.*

(2) $\mathrm{tr}_R(\mathrm{K}_R)$ *and $\mathfrak{m}$ are Ulrich ideals.*

(3) *R is G-regular and a length of a maximal chain of Ulrich ideals is $\ell_R(R/\mathrm{tr}_R(\mathrm{K}_R)) - 1$.*

(4) *There exist elements $\alpha, x_2, x_3, \dots, x_v \in \mathfrak{m}$ which satisfy the following two conditions.*

   (i) $\mathfrak{m} = (\alpha, x_2, x_3, \dots, x_v)$ *and*

   (ii) $\{(\alpha^i, x_2, x_3, \dots, x_v) \mid 1 \le i \le \ell_R(R/\mathrm{tr}_R(\mathrm{K}_R))\}$ *is the set of all Ulrich ideals.*

Let us give one more result of generalized Gorenstein local rings.

Theorem 1.2.8 (Corollary 1.6.5)**.** *Let $(S, \mathfrak{n})$ be a Gorenstein local ring and $(R, \mathfrak{m})$ a one-dimensional Cohen-Macaulay local ring but not a Gorenstein ring. Let $\varphi : S \to R$ be a surjective ring homomorphism and suppose the projective dimension of R over S is finite. Let $\mathfrak{a}$ be an ideal of S such that $\mathfrak{a} \supseteq \operatorname{Ker}\varphi$ and set $n = \mu_S(\mathfrak{a})$ and $\mathfrak{a} = (x_1, x_2, \dots, x_n)$. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring with respect to $\mathfrak{a}R$.*

(2) *There exists a minimal S-free resolution*

$$0 \to S^{\oplus r} \xrightarrow{\mathbb{M}} S^{\oplus q} \to \cdots \to S \to R \to 0$$

of R such that

$$ {}^t\mathbb{M} = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{**} & * \\ \hline x_1\ x_2\ \dots\ x_n & & & & \\ & x_1\ x_2\ \dots\ x_n & & \mathit{0} & \\ & & \ddots & & \mathit{0} \\ \mathit{0} & & & x_1\ x_2\ \dots\ x_n & \end{array}\right), $$

where all components of $**$ and $*$ are in $\mathfrak{a}$.

Theorem 1.2.8 has applications for determinantal rings and the Rees algebras of parameter ideals over Gorenstein local rings (see Theorem 1.7.3 and Corollary 1.7.5).

We now explain how this chapter is organized. In Section 1.3 we give a brief survey on Ulrich modules with respect to $\mathfrak{a}$, which we need throughout this chapter. In Section 1.4 we explore basic properties of generalized Gorenstein local rings which contain non-zerodivisor characterization and flat base change. In Section 1.5 we focus our attention on the one-dimensional case. We see that the notion of generalized Gorenstein local rings is important to study that Ulrich ideals and the endomorphism algebra of the maximal ideal. Numerous examples of generalized Gorenstein local rings are given arising from numerical semigroup rings and idealizations. Furthermore, we will see that several results in the case of dimension one can be extended for the higher-dimensional case, through Theorem 1.2.3. The purpose of Section 1.6 is to show Theorem 1.2.8. In Section 1.7 we construct the examples of higher-dimensional generalized Gorenstein local rings by using Theorem 1.2.8. In Section 1.8 we again consider Ulrich ideals by using the trace of the canonical module.

Let us fix our notation throughout this chapter. In what follows, unless otherwise specified, let R denote a Noetherian local ring with maximal ideal $\mathfrak{m}$. For each finitely generated R-module M, let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators of M (resp. the length of M). If M is a Cohen-Macaulay R-module, $\mathrm{r}_R(M)$ denotes the Cohen-Macaulay type of M.

Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R and set $s = \dim_R M$. Then we have the integers $\{\mathrm{e}_{\mathfrak{a}}^i(M)\}_{0 \le i \le s}$ such that the equality

$$\ell_R(M/\mathfrak{a}^{n+1}M) = \mathrm{e}_{\mathfrak{a}}^0(M) \cdot \binom{n+s}{s} - \mathrm{e}_{\mathfrak{a}}^1(M) \cdot \binom{n+s-1}{s-1} + \cdots + (-1)^s \mathrm{e}_{\mathfrak{a}}^s(M)$$

holds for all $n \gg 0$. $\mathrm{e}_{\mathfrak{a}}^i(M)$ is called the ith Hilbert coefficient of M with respect to $\mathfrak{a}$. Let $\mathrm{e}(R) = \mathrm{e}_{\mathfrak{m}}^0(R)$ (resp. $\mathrm{v}(R)$) denote the multiplicity of R (resp. the embedding dimension of R). $\mathrm{r}(R) = \mathrm{r}_R(R)$ denotes the Cohen-Macaulay type of R if R is a Cohen-Macaulay local ring.

Let $\mathrm{Q}(R)$ be the total ring of fractions of R. For R-submodules X and Y of $\mathrm{Q}(R)$, let

$$X : Y = \{a \in \mathrm{Q}(R) \mid aY \subseteq X\}.$$

For ideals I, J of R, we set $I :_R J = \{a \in R \mid aJ \subseteq I\}$, whence $I :_R J = (I : J) \cap R$.

For a matrix $\mathbb{M}$ and positive integer t, $I_t(\mathbb{M})$ denotes the ideal of S generated by the $t \times t$ minors of the matrix $\mathbb{M}$.

1.3 Survey on Ulrich modules with respect to $\mathfrak{a}$

In this section, we summarize some preliminaries on Ulrich R-modules, which we need throughout this chapter. Suppose that $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are Noetherian local rings and M is a nonzero finitely generated R-module. For an ideal I of R and a finitely generated

R-module X, we denote

$$\mathrm{gr}_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1} \text{ and}$$
$$\mathrm{gr}_I(X) = \bigoplus_{n\geq 0} I^n X/I^{n+1}X.$$

Note that $\mathrm{gr}_I(X)$ is a $\mathbb{Z}$-graded $\mathrm{gr}_I(R)$-module. Our goal in this section is Proposition 1.3.5. We start with the following lemma.

Lemma 1.3.1. *Let $\varphi : R \to S$ be a flat local homomorphism such that $S/\mathfrak{m}S$ is a Cohen-Macaulay local ring of dimension ℓ. Let $g_1, g_2, \dots, g_\ell \in S$ be a system of parameters of $S/\mathfrak{m}S$ and set $J = (g_1, g_2, \dots, g_\ell)$. Then $g_1t, g_2t, \dots, g_\ell t$ is a $\mathrm{gr}_{IS+J}(S \otimes_R X)$-regular sequence for any ideal I of R.*

Proof. First of all, we show that $g_1, g_2, \dots, g_\ell$ is a $\mathrm{gr}_{IS}(S\otimes_R X)$-regular sequence. Actually, the exact sequence $0 \to S \xrightarrow{g_1} S \to S/g_1S \to 0$ of S-modules induces the exact sequence

$$0 \to S \otimes_R I^n X \xrightarrow{g_1} S \otimes_R I^n X \to (S/g_1S) \otimes_R I^n X \to 0$$

as S-modules for all $n \geq 0$ since $R \to S/g_1S$ is a flat local homomorphism. This induces the exact sequence

$$0 \to \mathrm{gr}_{IS}(L) \xrightarrow{g_1} \mathrm{gr}_{IS}(L) \to \mathrm{gr}_{IS}((S/g_1S) \otimes_R X) \to 0$$

as graded $\mathrm{gr}_{IS}(S)$-modules, where $L = S \otimes_R X$. Hence g_1 is a $\mathrm{gr}_{IS}(L)$-regular element. By induction on ℓ, $g_1, g_2, \dots, g_\ell$ is a $\mathrm{gr}_{IS}(L)$-regular sequence. Hence

$$\begin{aligned} JL \cap (IS+J)^{k+1}L &= JL \cap [I^{k+1}L + J(IS+J)^k L] = JL \cap I^{k+1}L + J(IS+J)^k L \\ &= JI^{k+1}L + J(IS+J)^k L = J(IS+J)^k L \end{aligned}$$

for all $k \geq 0$. Therefore $g_1t, g_2t, \dots, g_\ell t$ is a $\mathrm{gr}_{IS+J}(L)$-regular sequence by [76]. □

Definition 1.3.2. (cf. [43, Definition 1.2.]) Let M be a nonzero finitely generated R-module and $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. We say that M is an *Ulrich R-module with respect to $\mathfrak{a}$* if M satisfies the following three conditions.

(1) M is a Cohen-Macaulay R-module (i.e. $\mathrm{depth}_R M = \dim_R M$),

(2) $\mathrm{e}^0_{\mathfrak{a}}(M) = \ell_R(M/\mathfrak{a}M)$, and

(3) $M/\mathfrak{a}M$ is an $R/\mathfrak{a}$-free module.

Remark 1.3.3. (1) The notion of Ulrich R-modules with respect to $\mathfrak{a}$ is a natural generalization of the notion of *maximally generated maximal Cohen-Macaulay* (MGMCM for short) R-modules (see [10]). In fact, for an R-module M, M is a MGMCM R-module if and only if M is an Ulrich R-modules with respect to the maximal ideal $\mathfrak{m}$ with $\dim_R M = d$.

(2) We can replace the condition (3) of Definition 1.3.2 to the equality

$$\ell_R(M/\mathfrak{a}M) = \mu_R(M)\cdot\ell_R(R/\mathfrak{a})$$

since there is a surjection $(R/\mathfrak{a})^{\oplus \mu_R(M)} \to M/\mathfrak{a}M$.

We can also rephrase the condition (2) of Definition 1.3.2 as following.

Lemma 1.3.4. *Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Suppose that $R/\mathfrak{m}$ is infinite and M is a Cohen-Macaulay R-module. Set $s = \dim_R M$. Then the following conditions are equivalent.*

(1) $\mathrm{e}^0_{\mathfrak{a}}(M) = \ell_R(M/\mathfrak{a}M)$.

(2) $\mathfrak{a}M = (f_1, f_2, \dots, f_s)M$ *for some elements* $f_1, f_2, \dots, f_s \in \mathfrak{a}$.

When this is the case, $f_1, f_2, \dots, f_s$ form a system of parameters of M and a part of minimal system of generator of $\mathfrak{a}$.

Proof. Set $\overline{R} = R/[(0) :_R M]$ and $\overline{\mathfrak{a}} = \mathfrak{a}\overline{R}$. For $a \in R$, $\overline{a}$ denotes the image of a in $\overline{R}$.

(1) $\Rightarrow$ (2) Since $R/\mathfrak{m}$ is infinite, there exists a parameter ideal $\mathfrak{q} = (\overline{f_1}, \overline{f_2}, \dots, \overline{f_s})$ of $\overline{R}$ as a reduction of $\overline{\mathfrak{a}}$, where $f_1, f_2, \dots, f_s \in \mathfrak{a}$. Hence

$$\ell_R(M/\mathfrak{a}M) = \mathrm{e}^0_{\mathfrak{a}}(M) = \mathrm{e}^0_{\overline{\mathfrak{a}}}(M) = \mathrm{e}^0_{\mathfrak{q}}(M) = \ell_{\overline{R}}(M/\mathfrak{q}M) = \ell_R(M/(f_1, f_2, \dots, f_s)M).$$

We have $\mathfrak{a}M = (f_1, f_2, \dots, f_s)M$ since $(f_1, f_2, \dots, f_s) \subseteq \mathfrak{a}$.

(2) $\Rightarrow$ (1) Since $\overline{\mathfrak{a}}M = (\overline{f_1}, \overline{f_2}, \dots, \overline{f_s})M$ and M is a faithful $\overline{R}$-module, $\mathfrak{q} = (\overline{f_1}, \overline{f_2}, \dots, \overline{f_s})$ is a reduction of $\overline{\mathfrak{a}}$. Hence

$$\mathrm{e}^0_{\mathfrak{a}}(M) = \mathrm{e}^0_{\overline{\mathfrak{a}}}(M) = \mathrm{e}^0_{\mathfrak{q}}(M) = \ell_{\overline{R}}(M/\mathfrak{q}M) = \ell_R(M/\mathfrak{a}M).$$

When this is the case, $f_1, f_2, \dots, f_s$ form a system of parameters of M since $M/\mathfrak{a}M$ has finite length and $s = \dim_R M$. Assume that $f_1, f_2, \dots, f_s$ is not a part of minimal system of generator of $\mathfrak{a}$. Then we have $(f_1, f_2, \dots, f_s) + \mathfrak{a}\mathfrak{m} = (f_1, f_2, \dots, f_{s-1}) + \mathfrak{a}\mathfrak{m}$ after renumbering of $f_1, f_2, \dots, f_s$. Hence $(\overline{f_1}, \overline{f_2}, \dots, \overline{f_{s-1}})$ is a reduction of $\overline{\mathfrak{a}}$. This is a contradiction for $s = \dim \overline{R}$. □

Now we can prove the basic properties of Ulrich R-module with respect to $\mathfrak{a}$.

Proposition 1.3.5. (cf. [46, Proposition 2.2]) *Let M be a nonzero finitely generated R-module and set $s = \dim_R M$. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Then the following assertions hold true.*

(1) *Suppose $s = 0$. Then M is an Ulrich R-module with respect to $\mathfrak{a}$ if and only if $\mathfrak{a}M = 0$ and M is $R/\mathfrak{a}$-free.*

(2) *Suppose that $s > 0$ and M is a Cohen-Macaulay R-module. Assume that $f \in \mathfrak{a}$ is a superficial element for M with respect to $\mathfrak{a}$. Then M is an Ulrich R-module with respect to $\mathfrak{a}$ if and only if M/fM is an Ulrich $R/(f)$-module with respect to $\mathfrak{a}/(f)$.*

(3) *Let $\varphi : R \to S$ be a flat local homomorphism such that $S/\mathfrak{m}S$ is a Cohen-Macaulay local ring of dimension ℓ. Let $g_1, g_2, \ldots, g_\ell \in S$ be a system of parameters of $S/\mathfrak{m}S$. Then M is an Ulrich R-module with respect to $\mathfrak{a}$ if and only if $S \otimes_R M$ is an Ulrich S-module with respect to $\mathfrak{a}S + (g_1, g_2, \ldots, g_\ell)$.*

(4) *Suppose that $f \in \mathfrak{m}$ is M-regular and M/fM is an Ulrich $R/(f)$-module with respect to $[\mathfrak{a} + (f)]/(f)$. Then M is an Ulrich R-module with respect to $\mathfrak{a} + (f)$ and $f \notin \mathfrak{m}\mathfrak{a}$.*

Proof. (1) This follows from $\mathrm{e}^0_{\mathfrak{a}}(M) = \ell_R(M)$.

(2) Set $\overline{M} = M/fM$, $\overline{R} = R/fR$, and $\overline{\mathfrak{a}} = \mathfrak{a}\overline{R}$. Then we have the equalities $\mathrm{e}^0_{\mathfrak{a}}(M) = \mathrm{e}^0_{\overline{\mathfrak{a}}}(\overline{M})$, $\ell_R(M/\mathfrak{a}M) = \ell_{\overline{R}}(\overline{M}/\overline{\mathfrak{a}}\overline{M})$, $\mu_R(M) = \mu_{\overline{R}}(\overline{M})$, and $\ell_R(R/\mathfrak{a}) = \ell_{\overline{R}}(\overline{R}/\overline{\mathfrak{a}})$ since f is M-regular. Therefore we have the equivalence by Remark 1.3.3 (2).

(3) By Lemma 1.3.1 and (2), $S \otimes_R M$ is an Ulrich S-module with respect to $\mathfrak{a}S + (g_1, g_2, \ldots, g_\ell)$ if and only if $\overline{S} \otimes_R M$ is an Ulrich $\overline{S}$-module with respect to $\mathfrak{a}\overline{S}$, where $\overline{S} = S/(g_1, g_2, \ldots, g_\ell)$. Hence, by passing to $R \to S/(g_1, g_2, \ldots, g_\ell)$, we may assume $\ell = 0$. Then, since $\ell_S(S \otimes_R M/\mathfrak{a}(S \otimes_R M)) = \ell_S(S/\mathfrak{m}S) \cdot \ell_R(M/\mathfrak{a}M)$ and $\mu_S(S \otimes_R M) = \mu_R(M)$, M is an Ulrich R-module with respect to $\mathfrak{a}$ if and only if $S \otimes_R M$ is an Ulrich S-module with respect to $\mathfrak{a}S$ by Remark 1.3.3 (2).

(4) By passing to $R \to R[X]_{\mathfrak{m}R[X]}$ if necessary, we may assume that $R/\mathfrak{m}$ is infinite. To prove that M is an Ulrich R-module with respect to $\mathfrak{a} + (f)$, we have only to show that $\mathrm{e}^0_{\mathfrak{a}+(f)}(M) = \ell_R(M/[\mathfrak{a}+(f)]M)$. By Lemma 1.3.4, we have $[\mathfrak{a}+(f)] \cdot M/fM = (f_2, \ldots, f_s) \cdot M/fM$ for some elements $f_2, \ldots, f_s \in [\mathfrak{a} + (f)]$, whence $[\mathfrak{a} + (f)]M = (f, f_2, \ldots, f_s)M$. Therefore M is an Ulrich R-module with respect to $\mathfrak{a} + (f)$ and $f \notin \mathfrak{m}\mathfrak{a}$. □

1.4 generalized Gorenstein local rings

In this section, we introduce the notion of generalized Gorenstein local rings which is the main object of this chapter. We also study about basic properties of generalized Gorenstein local rings. Throughout this section, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 0$, possessing the canonical module K_R. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. First, we note some remarks which might be known.

Lemma 1.4.1. *The following assertions hold true.*

(1) *R is a generically Gorenstein ring, that is, $R_\mathfrak{p}$ is a Gorenstein ring for all $\mathfrak{p} \in \operatorname{Min} R$ if and only if there exists a short exact sequence $0 \to R \to \mathrm{K}_R \to C \to 0$.*

(2) *Suppose that there exists a short exact sequence*

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0.$$

Then the assertions hold true.

   (i) ([46, Lemma 3.1 (3)]) *If $d = 0$, then $C = (0)$. Hence R is a Gorenstein ring.*

   (ii) ([46, Lemma 3.1 (2)]) *C is a Cohen-Macaulay R-module of dimension $d - 1$ if $C \neq (0)$.*

(iii) (cf. [46, Corollary 3.10]) *Suppose that C is an Ulrich R-module with respect to $\mathfrak{a}$. If $\varphi(1) \in \mathfrak{a}\mathrm{K}_R$, then R is a Gorenstein ring and $\mathfrak{a}$ is a parameter ideal of R.*

Proof. (1) Suppose that R is a generically Gorenstein ring. Then, since K_R has a rank, there exists a canonical ideal I, that is, $I \cong \mathrm{K}_R$ and $I \subsetneq R$. Since I contains a non-zerodivisor of R, R can be embedded into I. Conversely, suppose that there exists a short exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0.$$

Then, $C_\mathfrak{p} = (0)$ for all $\mathfrak{p} \in \operatorname{Min} R$ since $\ell_{R_\mathfrak{p}}(R_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(\mathrm{K}_{R_\mathfrak{p}})$. Hence R is a generically Gorenstein ring.

(2) (i) Since $\ell_R(R) = \ell_R(\mathrm{K}_R)$, we have $C = (0)$.

(ii) Suppose $C \neq (0)$. Then, since $C_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Min} R$, $\dim_R C \leq d-1$. On the other hand, we have $\operatorname{depth} C \geq d-1$ by the depth lemma.

(iii) Thanks to Proposition 1.3.5(3), we may assume that the residue field $R/\mathfrak{m}$ is infinite. We know that $d > 0$ by (i). Assume $d = 1$ and take a canonical ideal $I \subsetneq R$. We may assume that the given exact sequence has the following form:

$$0 \to R \xrightarrow{\varphi} I \to C \to 0.$$

Set $a = \varphi(1) \in I$. Then $C \cong I/(a)$. Since C is an Ulrich R-module with respect to $\mathfrak{a}$, $\mathfrak{a}I \subseteq (a)$ because of Proposition 1.3.5(1), whence $\mathfrak{a}I = (a)$ by assumption. Hence $\mathfrak{a}$ and I are cyclic since R is a local ring. Therefore R is a Gorenstein ring and $\mathfrak{a}$ is a parameter ideal of R. Suppose $d \geq 2$ and take $f \in \mathfrak{a}$ so that f is R-regular and superficial for C with respect to $\mathfrak{a}$. We get

$$0 \to R/(f) \xrightarrow{\overline{\varphi}} \mathrm{K}_{R/(f)} \to C/fC \to 0,$$

where C/fC is an Ulrich $R/(f)$-module with respect to $\mathfrak{a}/(f)$ since Proposition 1.3.5(2). Therefore we have $R/(f)$ is a Gorenstein ring and $\mu_{R/(f)}(\mathfrak{a}/(f)) = d-1$, thus this completes the proof. □

We are now in a position to define generalized Gorenstein local rings.

Definition 1.4.2. We say that R is a *generalized Gorenstein local ring*, if there exist an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ and an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that

(i) C is an Ulrich R-module with respect to $\mathfrak{a}$ and

(ii) $\mathrm{K}_R/\mathfrak{a}\mathrm{K}_R$ is $R/\mathfrak{a}$-free.

By definition, the notion of generalized Gorenstein local rings is a natural generalization of the notion of almost Gorenstein local rings, that is, we have the implications:

Gorenstein ring $\Rightarrow$ almost Gorenstein local ring $\Rightarrow$ generalized Gorenstein local ring.

The notion of generalized Gorenstein local rings is rephrased as follows.

Proposition 1.4.3. *R is a generalized Gorenstein local ring if and only if one of the following conditions is satisfied.*

(1) *R is a Gorenstein ring.*

(2) *R is not a Gorenstein ring but there exist an exact sequence*

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules and $\mathfrak{m}$-primary ideal $\mathfrak{a}$ such that

(i) *C is an Ulrich R-module with respect to $\mathfrak{a}$ and*

(ii) *the induced homomorphism $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \to \mathrm{K}_R/\mathfrak{a}\mathrm{K}_R$ is injective.*

Proof. (if part) If R is a Gorenstein ring, then we regard R as a generalized Gorenstein local ring by a natural exact sequence

$$0 \to R \xrightarrow{a_1} R \to R/(a_1) \to 0,$$

where $(a_1, a_2, \ldots, a_d)$ is a parameter ideal of R. Suppose that Case (2) occurs. Then we have

$$0 \to R/\mathfrak{a} \xrightarrow{R/\mathfrak{a}\otimes_R\varphi} \mathrm{K}_R/\mathfrak{a}\mathrm{K}_R \to C/\mathfrak{a}C \to 0$$

and $C/\mathfrak{a}C$ is $R/\mathfrak{a}$-free. Hence so is $\mathrm{K}_R/\mathfrak{a}\mathrm{K}_R$.

(only if part) Let

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

be a defining exact sequence. If $\varphi(1) \notin \mathfrak{m}\mathrm{K}_R$, then $\mu_R(C) = \mu_R(\mathrm{K}_R) - 1 = \mathrm{r}(R) - 1$. We have

$$0 \to \mathrm{Ker}(R/\mathfrak{a} \otimes_R \varphi) \to R/\mathfrak{a} \xrightarrow{R/\mathfrak{a}\otimes_R\varphi} \mathrm{K}_R/\mathfrak{a}\mathrm{K}_R \to (R/\mathfrak{a})^{\oplus(\mathrm{r}(R)-1)} \to 0.$$

Hence $R/\mathfrak{a} \otimes_R \varphi$ is injective since $\ell_R(\mathrm{Ker}(R/\mathfrak{a} \otimes_R \varphi)) = 0$. Suppose that $\varphi(1) \in \mathfrak{m}\mathrm{K}_R$. Then we have

$$R/\mathfrak{a} \xrightarrow{R/\mathfrak{a}\otimes_R\varphi} \mathrm{K}_R/\mathfrak{a}\mathrm{K}_R \to (R/\mathfrak{a})^{\oplus\mathrm{r}(R)} \to 0,$$

whence $R/\mathfrak{a} \otimes_R \varphi$ is the zero map. Thus $\varphi(1) \in \mathfrak{a}\mathrm{K}_R$, and R is a Gorenstein ring and $\mathfrak{a}$ is a parameter ideal of R by Lemma 1.4.1(2)(iii). ☐

Definition 1.4.4. When Case (2) in Proposition 1.4.3 occurs, we especially say that R is a *GGL ring with respect to* $\mathfrak{a}$.

Let us give some examples of generalized Gorenstein local rings.

Example 1.4.5. Let k be a field. Then the following assertions hold true.

1. Let $k[[t]]$ be the formal power series ring over k. Then, $R_1 = k[[t^5, t^6, t^8]]$ is a generalized Gorenstein local ring of dimension one but not an almost Gorenstein local ring.

2. Let $S = k[[X_1, X_2, X_3, Y_1, Y_2, Y_3]]$ be the formal power series ring over k. Then, $R_2 = S/I$ is a generalized Gorenstein local ring of dimension one, where

$$I = (Y_1, Y_2, Y_3)^2 + I_2\left(\begin{smallmatrix} X_1^2 & X_2 & X_3 \\ Y_1 & Y_2 & Y_3 \end{smallmatrix}\right) + I_2\left(\begin{smallmatrix} X_1^2 & X_2 & X_3 \\ X_2 & X_3 & X_1^3 \end{smallmatrix}\right) + I_2\left(\begin{smallmatrix} X_1^2 & X_2 & X_3 \\ Y_2 & Y_3 & X_1^2Y_1 \end{smallmatrix}\right).$$

3. Let $R_3 = k[[X, Y, Z, W]]/I_2\left(\begin{smallmatrix} X^2 & Y^2 & Z^2 \\ Y^2 & Z^2 & X^2 \end{smallmatrix}\right)$, where $k[[X, Y, Z, W]]$ denotes the formal power series ring over k. Then R_3 is a generalized Gorenstein local ring of dimension two. However, R_3 is not an almost Gorenstein local ring.

4. If $R/\mathfrak{m}$ is infinite and $\mathrm{e}(R) \le 3$, then R is a generalized Gorenstein local ring.

5. If R is a generalized Gorenstein local ring, then $R[X]/(X^n)$ is also a generalized Gorenstein local ring for all $n > 1$, where $R[X]$ denotes the polynomial ring over R.

Proof. (1) See Corollary 1.5.35.

(2) Let $A = k[[t^3, t^7, t^8]]$ and set $I = (t^6, t^7, t^8)$. A is a generalized Gorenstein local ring with respect to I by Corollary 1.5.12. Furthermore, by computations, we have $R_2 \cong A \ltimes I$. We will see that $A \ltimes I$ is a generalized Gorenstein local ring due to Corollary 1.5.26.

(3) By Corollary 1.7.4, R_3 is a generalized Gorenstein local ring. If R_3 is an almost Gorenstein local ring, then so is $k[[X, Y, Z]]/I_2\left(\begin{smallmatrix} X^2 & Y^2 & Z^2 \\ Y^2 & Z^2 & X^2 \end{smallmatrix}\right)$ by Theorem 1.4.6. This is a contradiction for Proposition 1.8.8.

(4) See Proposition 1.5.14.

(5) $R \to R[X]/(X^n)$ is a flat local ring homomorphism such that the fiber is isomorphic to $(R/\mathfrak{m})[X]/(X^n)$. Hence $R[X]/(X^n)$ is a generalized Gorenstein local ring by Theorem 1.4.7. □

We note that the assertions (4) and (5) of Example 1.4.5 do not hold true for the case of almost Gorenstein local rings (see [46, Proposition 3.12.]). Thanks to the argument of Section 1.3, we have the following which called the non-zerodivisor characterization of generalized Gorenstein local rings.

Theorem 1.4.6. *Suppose R is not a Gorenstein ring. Then the following assertions hold true.*

(1) *Suppose that R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$ and $d \ge 2$. Take a defining exact sequence*

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules. If $f \in \mathfrak{a}$ is a superficial element for C with respect to $\mathfrak{a}$ and a non-zerodivisor of R, then $R/(f)$ is a generalized Gorenstein local ring with respect to $\mathfrak{a}/(f)$.

(2) *Let $f \in \mathfrak{m}$ be a non-zerodivisor of R and suppose $R/(f)$ is a generalized Gorenstein local ring with respect to $[\mathfrak{a} + (f)]/(f)$. Then R is a generalized Gorenstein local ring with respect to $\mathfrak{a} + (f)$ and $f \notin \mathfrak{m}\mathfrak{a}$.*

Proof. (1) Since f is a non-zerodivisor of R, we have the exact sequence

$$0 \to \overline{R} \xrightarrow{\varphi} \mathrm{K}_{\overline{R}} \to C/fC \to 0$$

as $\overline{R}$-modules, where $\overline{R} = R/(f)$. Since C/fC is an Ulrich $\overline{R}$-module by Proposition 1.3.5, $\overline{R}$ is a generalized Gorenstein local ring.

(2) Set $\overline{R} = R/(f)$ and $\overline{\mathfrak{a}} = \mathfrak{a}\overline{R}$. Choose an exact sequence $0 \to \overline{R} \xrightarrow{\psi} \mathrm{K}_{\overline{R}} \to D \to 0$ of $\overline{R}$-modules such that D is an Ulrich $\overline{R}$-module with respect to $\overline{\mathfrak{a}}$ and $0 \to R/[\mathfrak{a} + (f)] \xrightarrow{\overline{\varphi}} \mathrm{K}_R/[\mathfrak{a}+(f)]\mathrm{K}_R$ is exact. Take $\xi \in \mathrm{K}_R$ such that $\overline{\xi} = \psi(1)$, and consider the exact sequence

$$R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules, where $\varphi(1) = \xi$. Note that $C/fC \cong D$ and $\dim D = d - 2$. Thus φ is injective since $\mathrm{Ass}_R \mathrm{Ker}\, \varphi \subseteq \mathrm{Ass}\, R$ and $C_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Ass}\, R$. Therefore C is an Ulrich R-module with respect to $\mathfrak{a}$ and $f \notin \mathfrak{m}\mathfrak{a}$ by Proposition 1.3.5 and Lemma 1.4.1. □

Theorem 1.4.7. *Let $\psi : R \to S$ be a flat local homomorphism of Noetherian local rings such that $S/\mathfrak{m}S$ is a Cohen-Macaulay local ring of dimension ℓ. Let $J \subseteq S$ be a parameter ideal in $S/\mathfrak{m}S$. Consider the following two conditions.*

(1) *R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$ and $S/\mathfrak{m}S$ is a Gorenstein ring.*

(2) *S is a generalized Gorenstein local ring with respect to $\mathfrak{a}S + J$.*

Then (1) ⇒ (2) holds true. (2) ⇒ (1) also holds true if $R/\mathfrak{m}$ is infinite.

Proof. (1) ⇒ (2) Note that R is not a Gorenstein ring and $d > 0$ by the definition that R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$ and Lemma 1.4.1. Take a defining exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules. Then, we get the exact sequence

$$0 \to S \xrightarrow{S \otimes_R \varphi} \mathrm{K}_S \to S \otimes_R C \to 0$$

as S-modules. Here, $S \otimes_R C$ is an Ulrich S-module with respect to $\mathfrak{a}S + J$ by Proposition 1.3.5. Moreover, since $R \to S/J$ is a flat local homomorphism, we have the injection

$$S/(\mathfrak{a}S + J) \otimes_S (S \otimes_R \varphi) : S/(\mathfrak{a}S + J) \to \mathrm{K}_S/(\mathfrak{a}S + J)\mathrm{K}_S.$$

□

To prove the implication (2) ⇒ (1), we need more preliminaries for the case of dimension one. We later prove the implication (2) ⇒ (1), see after Proposition 1.5.13.

1.5 One-dimensional case

In this section we investigate the case of dimension one. By Theorem 1.4.6, some of the properties hold in arbitrary dimension. On the other hand, as specific arguments of dimension one, the analysis of the endomorphism algebra of the maximal ideal and relationship to Ulrich ideals are given. We also study generalized Gorenstein local rings arising from numerical semigroups and idealizations.

1.5.1 One-dimensional generalized Gorenstein local rings

Let us recall the definition of generalized Gorenstein local rings in dimension one.

Remark 1.5.1. Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring possessing the canonical module K_R. Then, by Proposition 1.3.5(1), R is a generalized Gorenstein local ring if and only if there exist an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ and an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that C and $\mathrm{K}_R/\mathfrak{a}\mathrm{K}_R$ are an $R/\mathfrak{a}$-free module.

Suppose that $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal and R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$ with $\dim R = 1$. Take a defining exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules. Then, there exists a canonical ideal $I \subsetneq R$ since Lemma 1.4.1. Replace K_R to I in the short exact sequence and set $a = \varphi(1) \in I$. Since $C \cong I/(a)$ is an $R/\mathfrak{a}$-free module, we have $\mathfrak{a}I \subseteq (a)$. On the other hand, since $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \to I/\mathfrak{a}I$ is injective by Proposition 1.4.3, we have $\mathfrak{a}I \cap (a) = \mathfrak{a}a$. Hence $\mathfrak{a}I = \mathfrak{a}a$. Therefore (a) is a minimal reduction of I, that is, $I^{n+1} = aI^n$ for some $n > 0$. Set

$$K = \frac{I}{a} = \left\{ \frac{x}{a} \mid x \in I \right\}$$

as an R-submodule of $\mathrm{Q}(R)$. Then K is a fractional ideal of R such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ since $K^{n+1} = K^n$, where $\overline{R}$ denotes the integral closure of R. It follows that the short exact sequence $0 \to R \xrightarrow{\varphi} I \to C \to 0$ induces the exact sequence $0 \to R \to K \to C \to 0$, where $R \to K$ is the embedding.

As a conclusion, we have the following.

Lemma 1.5.2. *Suppose that R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$ with $\dim R = 1$. Then there exists a fractional ideal K which satisfies the following conditions.*

(1) $R \subseteq K \subseteq \overline{R}$ *and* $K \cong \mathrm{K}_R$.

(2) $\mathfrak{a}K \subseteq R$ *and* K/R *is an* $R/\mathfrak{a}$*-free module.*

In particular, $\mathfrak{a} = R : K$.

From now on, throughout this section, we maintain the following setting unless otherwise noted.

Setting 1.5.3. Suppose $(R, \mathfrak{m})$ is a one-dimensional Cohen-Macaulay local ring possessing the canonical module K_R. We assume that there exists a fractional ideal K such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$, where $\overline{R}$ denotes the integral closure of R.

We denote $S = R[K]$ and $\mathfrak{c} = R : S$. Note that $S = K^n$ for all $n \gg 0$. Hence we have $\mathfrak{c} \subseteq R \subseteq K \subseteq S$.

Let us note some remarks on Setting 1.5.3.

Remark 1.5.4. (1) Suppose that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension one. If R is a generically Gorenstein ring and $R/\mathfrak{m}$ is infinite, then R satisfies Setting 1.5.3.

(2) Assume that Setting 1.5.3. Then the following assertion hold true.

- (i) $S = R[K]$ is independent of the choice of K.
- (ii) Let I and J be fractional ideals. If $J \subseteq I$, then $\ell_R(I/J) = \ell_R((K : J)/(K : I))$.

Proof. (1) Since there exists a canonical ideal I and we can choose minimal reduction of I as a parameter ideal (a), we can take $K = \frac{I}{a}$.

(2)(i) Although this is proven in [15, Theorem 2.5], let us give a brief proof for the sake of completeness. Take $R \subseteq L \subseteq \overline{R}$ so that $L \cong \mathrm{K}_R$. There is an element $\alpha \in \mathrm{Q}(R)$ such that $L = \alpha K$. Hence, since $R[K] = K^n = K^{n+1}$ for all $n \gg 0$, $\alpha{\cdot}R[K] = \alpha K{\cdot}R[K] = L{\cdot}R[K]$. Since $1 \in L{\cdot}R[K]$, there is an element $\beta \in R[K]$ such that $\alpha\beta = 1$, that is, α is a unit of $R[K]$. Therefore $R[L] = L^n = \alpha^n{\cdot}K^n = \alpha^n{\cdot}R[K] = R[K]$ for all $n \gg 0$.

(ii) Consider the exact sequence $0 \to J \xrightarrow{i} I \to I/J \to 0$, where i is the embedding. By applying the functor $\mathrm{Hom}_R(-, K)$, we have

$$0 \to K : I \xrightarrow{i} K : J \to \mathrm{Ext}^1_R(I/J, K) \to 0$$

since $K : I \cong \mathrm{Hom}_R(I, K)$ naturally. Hence $(K : J)/(K : I) \cong \mathrm{Ext}^1_R(I/J, K)$, whence we have the conclusion. $\square$

With this notation there are characterizations of the Gorenstein property.

Theorem 1.5.5. ([36, Theorem 3.7]) *The following conditions are equivalent.*

(1) *R is a Gorenstein ring.*	(2) $R = K$.	(3) $R = S$.
(4) $K = S$.	(5) $K = K^2$.	(6) $\mathfrak{c} = R$.
(7) $\mathrm{e}^1_I(R) = \ell_R(R/\mathfrak{c})$.	(8) $\mathrm{e}^1_I(R) = 0$.	

Here, $\mathrm{e}^1_I(R)$ denotes the first Hilbert coefficient of a canonical ideal I.

Proof. The implications (1) $\Leftrightarrow$ (2), (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5), and (3) $\Leftrightarrow$ (6) are trivial.

(5) $\Rightarrow$ (2) This follows from $K \subseteq K : K = R$.

Hence the conditions from (1) to (6) are equivalent. Note that $\mathrm{e}_I^1(R) = \ell_R(S/R)$ and $\ell_R(R/\mathfrak{c}) = \ell_R(S/K)$ by [36, Lemma 2.1 and Lemma 3.5(2)]. Hence we have (2) $\Leftrightarrow$ (7) and (3) $\Leftrightarrow$ (8). $\square$

Hence $\mathfrak{c} \subsetneq R \subsetneq K \subsetneq S$ if R is not a Gorenstein ring. It seems that the difference between R and S (or other inclusions) describes the distance of Gorensteinness of R. From this observation, let us consider such a kind of characterizations of generalized Gorenstein local rings in dimension one.

Lemma 1.5.6. *Suppose that R is not a Gorenstein ring. Then K/R and $K/\mathfrak{c} = K/\mathfrak{c}K$ are independent of the choice of K up to isomorphism.*

Proof. Take L such that $R \subseteq L \subseteq \overline{R}$ and $L \cong \mathrm{K}_R$. There is an element $\alpha \in \mathrm{Q}(R)$ such that $L = \alpha K$ since $L \cong K$. α is a unit of S because of $S = L^n = \alpha^n K^n = \alpha^n S$ for enough large $n > 0$ by Remark 1.5.4(2)(i). Therefore

$$L/\mathfrak{c} = \alpha K/\mathfrak{c} \cong K/\alpha^{-1}\mathfrak{c} = K/\mathfrak{c}$$

since $\mathfrak{c} = \mathfrak{c}S$. Hence $L/R \cong K/R$. $\square$

Lemma 1.5.7. *Set $\mathfrak{a} = R : K$. Then the following conditions are equivalent.*

(1) $K^2 = K^3$. (2) $\mathfrak{a} = \mathfrak{c}$. (3) $\mathfrak{a}K = \mathfrak{a}$.

Proof. (1) $\Leftrightarrow$ (2) Since $\mathfrak{a} = (K : K) : K = K : K^2$ and $\mathfrak{c} = (K : K) : S = K : S$, we have $\ell_R(R/\mathfrak{a}) = \ell_R(K^2/K)$ and $\ell_R(R/\mathfrak{c}) = \ell_R(S/K)$ by Remark 1.5.4. Hence, $\mathfrak{a} = \mathfrak{c}$ if and only if $K^2 = S$.

(2) $\Rightarrow$ (3) Since $\mathfrak{c}S = \mathfrak{c}$, we have $\mathfrak{a}K = \mathfrak{a}$.

(3) $\Rightarrow$ (2) Since $\mathfrak{a}K^n = \mathfrak{a}$ for all $n > 0$, $\mathfrak{a}S = \mathfrak{a} \subseteq R$. Hence $\mathfrak{a} = \mathfrak{c}$. $\square$

In Section 1.8 we see that $\mathfrak{a}K$ is exactly the trace of canonical ideal $\mathrm{tr}_R(\mathrm{K}_R)$, see Remark 1.8.3. Now let us give characterizations of generalized Gorenstein local rings.

Theorem 1.5.8. *Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring.*
(2) *K/R is an $R/\mathfrak{c}$-free module.*
(3) *$K/\mathfrak{c} = K/\mathfrak{c}K$ is an $R/\mathfrak{c}$-free module.*
(4) *S/R is an $R/\mathfrak{c}$-free module.*
(5) *$S/\mathfrak{c} = S/\mathfrak{c}S$ is an $R/\mathfrak{c}$-free module.*
(6) *$\mathrm{e}_I^1(R) = \ell_R(R/\mathfrak{c}){\cdot}\mathrm{r}(R)$.*

When this is the case, we have the following.

(i) $K^2 = K^3$.
(ii) *$R/\mathfrak{c}$ is a Gorenstein ring.*
(iii) $S/K \cong R/\mathfrak{c}$.
(iv) *R is a generalized Gorenstein local ring with respect to $\mathfrak{c}$.*

Proof. (2) ⇔ (3) and (4) ⇔ (5) follow from the exact sequences

$$0 \to R/\mathfrak{c} \to K/\mathfrak{c} \to K/R \to 0 \quad \text{and} \quad 0 \to R/\mathfrak{c} \to S/\mathfrak{c} \to S/R \to 0$$

since $R/\mathfrak{c}$ is an Artinian ring.

(1) ⇒ (2), (i), and (iv) Suppose that R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$. By the argument of the proof of Lemma 1.5.2, we may assume that a defining short exact sequence has the following form

$$0 \to R \to K \to K/R \to 0.$$

Hence K/R is $R/\mathfrak{a}$-free and $0 \to R/\mathfrak{a} \to K/\mathfrak{a}K$ is exact. The latter condition says that $\mathfrak{a}K = K$ since $\mathfrak{a}K \subseteq R$, thus $\mathfrak{a} = \mathfrak{c}$ and $K^2 = K^3$ by Lemma 1.5.7.

(2) ⇒ (1) Since K/R is an $R/\mathfrak{c}$-free module and $K/\mathfrak{c} = K/\mathfrak{c}K$,

$$0 \to R \to K \to K/R \to 0$$

is the exact sequence as desired.

Hence (1), (2), and (3) are equivalent. To prove (2) ⇒ (6), we need to show that (ii) and (iii).

(ii) Let $r = \mathrm{r}(R)$ be the Cohen-Macaulay type of R. Since $K/R \cong (R/\mathfrak{c})^{\oplus(r-1)}$, we have

$$0 \to R \to K \to (R/\mathfrak{c})^{\oplus(r-1)} \to 0.$$

By taking K-dual, we have

$$0 \to R \to K \to \mathrm{Ext}_R^1(R/\mathfrak{c}, K)^{\oplus(r-1)} \to 0.$$

Thus we have $R/\mathfrak{c} \cong \mathrm{Ext}_R^1(R/\mathfrak{c}, K) \cong \mathrm{K}_{R/\mathfrak{c}}$, whence $R/\mathfrak{c}$ is a Gorenstein ring.

(iii) We have

$$(0) :_{S/K} \mathfrak{m} = [K :_S \mathfrak{m}]/K \subseteq [K : \mathfrak{m}]/K \cong \mathrm{Ext}_R^1(R/\mathfrak{m}, K)$$

in general. Hence S/K is a Cohen-Macaulay faithful $R/\mathfrak{c}$-module with $\mathrm{r}_{R/\mathfrak{c}}(S/K) = 1$, that is, the canonical $R/\mathfrak{c}$-module $\mathrm{K}_{R/\mathfrak{c}}$. Therefore we have $S/K \cong R/\mathfrak{c}$ by (ii).

(3) ⇒ (5) Consider the exact sequence $0 \to K/\mathfrak{c} \to S/\mathfrak{c} \to S/K \to 0$. Since $K/\mathfrak{c}$ and S/K are $R/\mathfrak{c}$-free, $S/\mathfrak{c}$ is also $R/\mathfrak{c}$-free.

(2) ⇒ (6) Thanks to (iii), S/R is $R/\mathfrak{c}$-free of rank r. Hence $\mathrm{e}_I^1(R) = \ell_R(S/R) = \ell_R(R/\mathfrak{c}){\cdot}\mathrm{r}(R)$ by [36, Lemma 2.1].

(6) ⇒ (2) Suppose that $\mathrm{e}_I^1(R) = \ell_R(S/R) = \ell_R(R/\mathfrak{c}){\cdot}\mathrm{r}(R)$. Then, $\ell_R(K/R) = \ell_R(S/R) - \ell_R(S/K) = (r-1){\cdot}\ell_R(R/\mathfrak{c})$ since $S/K \cong \mathrm{K}_{R/\mathfrak{c}}$. Therefore K/R is an $R/\mathfrak{c}$-free module since there is a surjection $(R/\mathfrak{c})^{\oplus(r-1)} \to K/R$.

(5) ⇒ (3) Note that $\mathfrak{c}S_M \cong \mathrm{K}_{S_M}$ for all $M \in \operatorname{Max} S$ since $\mathfrak{c} = K : S \cong \mathrm{Hom}_R(S, \mathrm{K}_R)$. Hence $S/\mathfrak{c}$ is a Gorenstein ring. Since $R/\mathfrak{c} \to S/\mathfrak{c}$ is flat, $R/\mathfrak{c}$ is also a Gorenstein ring. Thus we have $S/K \cong \mathrm{K}_{R/\mathfrak{c}} \cong R/\mathfrak{c}$ and $K/\mathfrak{c}$ is an $R/\mathfrak{c}$-free module since the exact sequence $0 \to K/\mathfrak{c} \to S/\mathfrak{c} \to S/K \to 0$. □

Remark 1.5.9. The conditions (i), (ii), and (iii) of Theorem 1.5.8 do not imply that the ring is a generalized Gorenstein local ring. For example, set $R = k[[t^4, t^7, t^9, t^{10}]] \subseteq k[[t]]$, where $k[[t]]$ denotes a formal power series ring over a field k. Then R satisfies the conditions (i), (ii), (iii), however, R is not a generalized Gorenstein local ring (see [15, Example 3.5.]). This example also shows that 2-almost Gorenstein local rings need not be a generalized Gorenstein local ring in the sense of [15, Theorem 1.4], that is, $K^2 = K^3$ and $\ell_R(K^2/K) = 2$.

Corollary 1.5.10. *Let $(R_1, \mathfrak{m}_1)$ be a Cohen-Macaulay local ring of dimension one and let $\varphi : R \to R_1$ be a flat local homomorphism of local rings such that $R_1/\mathfrak{m}R_1$ is a Gorenstein ring. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring.*

(2) *R_1 is a generalized Gorenstein local ring.*

Proof. By [15, Proposition 3.8], for each $n \geq 0$ the following assertions hold true.

(1) $K_1^n = K_1^{n+1}$ if and only if $K^n = K^{n+1}$ and

(2) $\ell_{R_1}(K_1^{n+1}/K_1^n) = \ell_{R_1}(R_1/\mathfrak{m}R_1){\cdot}\ell_R(K^{n+1}/K^n)$,

where $K_1 = R_1{\cdot}K \cong \mathrm{K}_{R_1}$. Hence, $K^2 = K^3$ if and only if $K_1^2 = K_1^3$ and K/R is $R/(R:K)$-free if and only if K_1/R_1 is $R_1/(R_1 : K_1)$-free. Therefore, R is a generalized Gorenstein local ring if and only if R_1 is a generalized Gorenstein local ring by Lemma 1.5.7 and Theorem 1.5.8. $\square$

Corollary 1.5.11. *Suppose that* $\mathrm{r}(R) = 2$*. Then, R is a generalized Gorenstein local ring if and only if $K^2 = K^3$.*

Proof. Since $\mathrm{r}(R) = 2$, we have $K/R \cong R/(R : K)$. Hence, R is a generalized Gorenstein local ring if and only if $K^2 = K^3$ by Lemma 1.5.7. $\square$

Corollary 1.5.12. *If* $\mathrm{e}(R) \leq 3$*, then R is a generalized Gorenstein local ring.*

Proof. We may assume that R is not a Gorenstein ring and $R/\mathfrak{m}$ is infinite. Then, since $\mathrm{v}(R) = \mathrm{e}(R) = 3$, we have $K^2 = K^3$ and $\mathrm{r}(R) = 2$. $\square$

1.5.2 Applications for higher dimension

Thanks to the characterization of generalized Gorenstein local rings in dimension one, by induction on $\dim R$, we have some properties of generalized Gorenstein local rings in arbitrary dimension, which includes Theorem 1.4.7 (2) $\Rightarrow$ (1). For a while, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring possessing the canonical module K_R. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. Set $d = \dim R > 0$.

Corollary 1.5.13. *If R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$, then $R/\mathfrak{a}$ is a Gorenstein ring. In particular, $\mathfrak{a}$ is not a parameter ideal.*

Proof. By Theorem 1.4.7 (1) $\Rightarrow$ (2), we may assume that $R/\mathfrak{m}$ is infinite. The case where $d = 1$ is proven in Theorem 1.5.8. Let $d > 1$ and assume that our assertion holds true for $d - 1$. Then we can choose $f \in \mathfrak{a}$ such that $R/(f)$ is a generalized Gorenstein local ring with respect to $\mathfrak{a}/(f)$ by Theorem 1.4.6. Hence $R/\mathfrak{a}$ is a Gorenstein ring by induction hypothesis. $\square$

Now we reach to prove the rest of Theorem 1.4.7.

Proof of Theorem 1.4.7 (2) $\Rightarrow$ (1). Note that S is not a Gorenstein ring and $\dim S > 0$. Set $\mathfrak{b} = \mathfrak{a}S + J$. $S/\mathfrak{b}$ is a Gorenstein ring by Corollary 1.5.13 and so is $S/\mathfrak{a}S$. Hence $R/\mathfrak{a}$ and $S/\mathfrak{m}S$ are also Gorenstein rings since $\overline{\varphi} : R/\mathfrak{a} \to S/\mathfrak{a}S$ is a flat local homomorphism. It remains to show that R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$. Assume that $\ell = \dim S/\mathfrak{m}S > 0$. If $\dim S = 1$, then $d = 0$ and $\ell = 1$ since $d + \ell = 1$. Thus there exists $\mathfrak{p} \in \operatorname{Min} S$ such that $\mathfrak{p} \supseteq \mathfrak{m}S$. Since $\mathfrak{p} \cap R = \mathfrak{m}$, $R \to S_\mathfrak{p}$ is flat local homomorphism. Since $S_\mathfrak{p}$ is a Gorenstein ring, so is R. This concludes that S is also a Gorenstein ring. This is a contradiction. Hence $\dim S > 1$. Choose an exact sequence

$$0 \to S \xrightarrow{\psi} \mathrm{K}_S \to D \to 0$$

of S-modules such that D is an Ulrich S-module with respect to $\mathfrak{b}$ and $0 \to S/\mathfrak{b} \xrightarrow{\overline{\psi}} \mathrm{K}_S/\mathfrak{b}\mathrm{K}_S$ is exact. Set $A = S/((0) :_S D)$ and $B = S/\mathfrak{a}S$. Then, $\dim A = \dim S - 1 = d + \ell - 1$, $\dim B = \dim R/\mathfrak{a} + \ell = \ell$, and $\mathfrak{b}B = JB$. Hence we can choose $h_1, h_2, \dots, h_\ell, \xi_1, \xi_2, \dots, \xi_{d-1} \in \mathfrak{b}$ which satisfy the following conditions.

(1) $(h_1, h_2, \dots, h_\ell)B = \mathfrak{b}B$ and

(2) $(h_1, h_2, \dots, h_\ell, \xi_1, \xi_2, \dots, \xi_{d-1})A$ is a minimal reduction of $\mathfrak{b}A$.

Set $\mathfrak{q} = (h_1, h_2, \dots, h_\ell, \xi_1, \xi_2, \dots, \xi_{d-1})$ as an ideal of S. Then $\mathfrak{b}^n D = \mathfrak{q}^n D$ for all $n > 0$ since

$$\ell_S(D/\mathfrak{q}D) = \ell_A(D/\mathfrak{q}D) = \mathrm{e}^0_{\mathfrak{q}A}(D) = \mathrm{e}^0_{\mathfrak{b}A}(D) = \mathrm{e}^0_{\mathfrak{b}}(D) = \ell_S(D/\mathfrak{b}D).$$

Furthermore, $h_1, h_2, \dots, h_\ell, \xi_1, \xi_2, \dots, \xi_{d-1}$ is a D-regular sequence and $\mathfrak{q}D \cap \mathfrak{b}^{n+1}D = \mathfrak{q}\mathfrak{b}^n D$ for all $n > 0$, whence $h_1, h_2, \dots, h_\ell, \xi_1, \xi_2, \dots, \xi_{d-1}$ is a super regular sequence for D with respect to $\mathfrak{b}$ in the sense of [52], that is, $h_1t, h_2t, \dots, h_\ell t, \xi_1 t, \xi_2 t, \dots, \xi_{d-1}t$ is a $\mathrm{gr}_\mathfrak{b}(D)$-regular sequence. In particular, $h_1, h_2, \dots, h_\ell, \xi_1, \xi_2, \dots, \xi_{d-1}$ is a superficial sequence for D with respect to $\mathfrak{b}$. On the other hand, since $(h_1, h_2, \dots, h_\ell)B = \mathfrak{b}B$, $h_1, h_2, \dots, h_\ell$ is a system of parameter of $B = S/\mathfrak{a}S$ and $S/\mathfrak{m}S$. Hence, by passing to $R \to S \to S/(h_1, h_2, \dots, h_\ell)$, without loss of generality, we may assume that $\ell = 0$ by Theorem 1.4.6. Choose an exact sequence

$$0 \to S \xrightarrow{\psi} \mathrm{K}_S \to D \to 0$$

of S-modules such that D is an Ulrich S-module with respect to $\mathfrak{a}S$ and $0 \to S/\mathfrak{a}S \xrightarrow{\overline{\psi}} \mathrm{K}_S/\mathfrak{a}\mathrm{K}_S$ is exact. If $d > 1$, then we can take an element $f \in \mathfrak{a}$ such that f is R-regular and $\varphi(f)$ is superficial for D with respect to $\mathfrak{a}S$. Hence, by passing to $R/fR \to S/fS$, we may assume $d = 1$. However, the case where $\dim R = \dim S = 1$ are already proven in Corollary 1.5.10. Therefore we have proven Theorem 1.4.7. $\square$

Proposition 1.5.14. *Suppose that $R/\mathfrak{m}$ is infinite. If $\mathrm{e}(R) \leq 3$, then R is a generalized Gorenstein local ring.*

Proof. We may assume that R is not a Gorenstein ring. The case where $d = 1$ is proven in Theorem 1.5.12. Let $d > 1$ and assume that our assertion holds true for $d-1$. Take $f \in \mathfrak{m}$ so that f is superficial for R with respect to $\mathfrak{m}$. Then, since $\mathrm{e}(R/(f)) = \mathrm{e}(R) \leq 3$, $R/(f)$ is a generalized Gorenstein local ring by induction hypothesis. Since f is a non-zerodivisor of R, R is also a generalized Gorenstein local ring by Theorem 1.4.6. $\square$

1.5.3 The endomorphism algebra of the maximal ideal

Let us consider again the case where dimension one. Throughout this sebsection, suppose Setting 1.5.3. Set $B = \mathfrak{m} : \mathfrak{m} \cong \mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m})$. Let $\mathrm{J}(B)$ denote the Jacobson radical of B. Our next purpose is to explore the generalized Gorenstein local property of the algebra B in connection with that of R. Let us begin with the following.

Proposition 1.5.15. *Suppose that R is a generalized Gorenstein local ring but not an almost Gorenstein local ring. Then we have the following assertions.*

(1) *B is a local ring and $\mathrm{J}(B) = \mathfrak{m}S \cap B$.*

(2) *$R/\mathfrak{m} \cong B/\mathrm{J}(B)$.*

Proof. Set $r = \mathrm{r}(R)$. By Theorem 1.5.8, we can take elements $f_1, f_2, \dots, f_{r-1} \in K$, $g \in S$, and $v \in \mathfrak{m}$ such that $K = R + \langle f_1, f_2, \dots, f_{r-1} \rangle$, $S = K + \langle g \rangle$, and $\mathfrak{c} :_R \mathfrak{m} = \mathfrak{c} + (v)$. We show that

$$B = R + \langle vf_1, vf_2, \dots, vf_{r-1} \rangle + \langle vg \rangle .$$

In fact, since $\ell_R([(R : \mathfrak{m}) \cap K]/R) = \mathrm{r}_R(K/R) = r - 1$ and $K/R \cong (R/\mathfrak{c})^{\oplus (r-1)}$, we have

$$B \cap K = (R : \mathfrak{m}) \cap K = R + \langle vf_1, vf_2, \dots, vf_{r-1} \rangle .$$

On the other hand, since $R/\mathfrak{c} \cong S/K$ and $\overline{g}$ is a free basis of S/K, $vg \notin K$ and $vg \in K : \mathfrak{m} \subseteq B$. Hence we have the equality $B = R + \langle vf_1, vf_2, \dots, vf_{r-1} \rangle + \langle vg \rangle$ since $\ell_R((R : \mathfrak{m})/R) = r$. Therefore, since $\langle vf_1, vf_2, \dots, vf_{r-1} \rangle + \langle vg \rangle \subseteq \mathfrak{m}S$, we get $B = R + \mathfrak{m}S \cap B$. We have $\mathfrak{m}S \cap B \in \operatorname{Max} B$ since $R/\mathfrak{m} \cong B/(\mathfrak{m}S \cap B)$. Therefore, since $\mathfrak{m}S \cap B \subseteq \mathrm{J}(B)$, B is a local ring and the unique maximal ideal is $\mathrm{J}(B) = \mathfrak{m}S \cap B$. $\square$

Let us note that Proposition 1.5.15 does not hold true without the assumption that R is not an almost Gorenstein local ring. Let us give an example.

Example 1.5.16. ([15, Example 5.10]) Let $V = k[[t]]$ be the formal power series ring over an infinite field k. We consider the direct product $A = k[[t^3, t^4, t^5]] \times k[[t^3, t^4, t^5]]$ of rings and set $R = k{\cdot}(1, 1) + \mathrm{J}(A)$ where $\mathrm{J}(A)$ denotes the Jacobson radical of A. Then R is a subring of A and a one-dimensional Cohen-Macaulay complete local ring with the maximal ideal $\mathrm{J}(A)$. We have the ring R is an almost Gorenstein local ring and $\mathrm{v}(R) = \mathrm{e}(R)$. However

$$\mathfrak{m} : \mathfrak{m} = V \times V$$

which is not a local ring.

The following lemma is known. For a convenience to readers, we include the proof.

Lemma 1.5.17. *Let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$. Set $d = \dim A \geq 0$ and $\mathrm{e}(A) > 1$. Then $\mathrm{r}(A) \leq \mathrm{e}(A) - 1$ and equality holds true if and only if A has maximal embedding dimension, that is, $\mathrm{v}(A) = \mathrm{e}(A) + d - 1$.*

Proof. We may assume $A/\mathfrak{m}$ is infinite. Then we can take a parameter ideal Q as a reduction of $\mathfrak{m}$. Hence $\mathrm{e}(A) - 1 = \ell_A(A/Q) - \ell_A(A/\mathfrak{m}) \geq \ell_A(Q : \mathfrak{m}/Q) = \mathrm{r}(A)$. Hence the equality holds true if and only if $\mathfrak{m} = Q : \mathfrak{m}$. This is equivalent to saying that $\mathfrak{m}^2 = Q\mathfrak{m}$ by [16, Theorem 2.2.]. $\square$

Now we reach the theorem which is a goal of this subsection.

Theorem 1.5.18. *Suppose that there is an element $\alpha \in \mathfrak{m}$ such that $\mathfrak{m}^2 = \alpha\mathfrak{m}$. Set $\mathfrak{n} = \mathrm{J}(B)$ and $L = BK$. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring but not an almost Gorenstein local ring.*

(2) *B is a generalized Gorenstein local ring with $\mathrm{v}(B) = \mathrm{e}(B) = \mathrm{e}(R)$, but not a Gorenstein ring.*

When this is the case, we have the following.

(i) $R/\mathfrak{m} \cong B/\mathfrak{n}$.

(ii) $\ell_B(B/(B : B[L])) = \ell_R(R/\mathfrak{c}) - 1$.

(iii) $\mathfrak{n}^2 = \alpha\mathfrak{n}$.

Proof. Since $\mathfrak{m}^2 = \alpha\mathfrak{m}$ and R is not a regular local ring, by [15, Proposition 5.1], we have the following.

(a) $B = R : \mathfrak{m} = \frac{\mathfrak{m}}{\alpha}$.

(b) $B \subseteq L \subseteq \overline{B}$ and $L = K : \mathfrak{m} \cong \mathrm{K}_B$ as a B-module.

(c) $S = B[L]$.

Furthermore, we have $\ell_R(L/K) = 1$ since $(K : \mathfrak{m})/K \cong R/\mathfrak{m}$ by Remark 1.5.4. Set $\mathfrak{a} = B : B[L] = B : S$.

(1) $\Rightarrow$ (2) We have the natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B/R & \longrightarrow & S/R & \longrightarrow & S/B & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \| & & \\ 0 & \longrightarrow & (R/\mathfrak{m})^{\oplus r} & \xrightarrow{\varphi} & (R/\mathfrak{c})^{\oplus r} & \longrightarrow & S/B & \longrightarrow & 0 \end{array}$$

as R-modules. Since $R/\mathfrak{c}$ is an Artinian Gorenstein ring, $\mathrm{Im}\varphi = ([\mathfrak{c} :_R \mathfrak{m}]/\mathfrak{c})^{\oplus r}$. Hence $S/B \cong (R/[\mathfrak{c} :_R \mathfrak{m}])^{\oplus r}$ as R-modules. We show that $R/[\mathfrak{c} :_R \mathfrak{m}] \cong B/\mathfrak{a}$ as R-modules. Note that $\mathfrak{a} \cap R = \mathfrak{c} :_R \mathfrak{m}$ since

$$a \in \mathfrak{a} \Leftrightarrow a\mathfrak{m}S \subseteq R \Leftrightarrow a \in \mathfrak{c} :_R \mathfrak{m}$$

for an element $a \in R$. Therefore we have an exact sequence $0 \to R/[\mathfrak{c} :_R \mathfrak{m}] \to B/\mathfrak{a}$. On the other hand, we have

$$\ell_R(B/\mathfrak{a}) = \ell_R(S/L) = \ell_R(S/K) - 1 = \ell_R(R/\mathfrak{c}) - 1 = \ell_R(R/[\mathfrak{c} :_R \mathfrak{m}]),$$

where the first and third equalities follow by the facts $\mathrm{K}_{B/\mathfrak{a}} \cong S/L$ and $\mathrm{K}_{R/\mathfrak{c}} \cong S/K$ (see the proof of Theorem 1.5.8(iii)). Hence $S/B \cong (B/\mathfrak{a})^{\oplus r}$ as R-module and so as B-module since $(B/\mathfrak{a})^{\oplus r} \to S/B \to 0$ is exact as B-modules. Therefore B is a generalized Gorenstein local ring, $\ell_B(B/\mathfrak{a}) = \ell_R(R/\mathfrak{c}) - 1$, and $\mathrm{r}(R) = \mathrm{r}(B)$ by Theorem 1.5.8. We need to show that $\alpha\mathfrak{n} = \mathfrak{n}^2$. In fact, since αS is a reduction of $\mathfrak{m}S$, we have

$$B \subseteq \frac{\mathfrak{m}S \cap B}{\alpha} \subseteq \frac{\mathfrak{m}S}{\alpha} \subseteq \overline{S} = \overline{B},$$

where $\overline{*}$ stands for the integral closure of $*$. Hence $\alpha B \subseteq \mathfrak{m}S \cap B = \mathfrak{n}$ is a reduction and $\mathrm{r}(B) = \mathrm{r}(R) = \mathrm{e}(R) - 1 = \mathrm{e}(B) - 1$, where the last equality follows from

$$\mathrm{e}(R) = \mathrm{e}^0_{\mathfrak{m}}(R) = \mathrm{e}^0_{\mathfrak{m}}(B) = \ell_R(B/\alpha B) = \ell_B(B/\alpha B) = \mathrm{e}(B)$$

by Proposition 1.5.15. Therefore, by Lemma 1.5.17, we have $\mathrm{e}(B) = \mathrm{v}(B) = \mathrm{e}(R)$, whence $\alpha\mathfrak{n} = \mathfrak{n}^2$.

(2) $\Rightarrow$ (1) Consider the following exact sequences

$$\begin{aligned} &0 \to L/K \to S/K \to S/L \to 0, \\ &0 \to B/R \to S/R \to S/B \to 0, \text{ and} \\ &0 \to K/R \to S/R \to S/K \to 0. \end{aligned}$$

Note that $\mathrm{r}(R) = \mathrm{e}(R) - 1 = \mathrm{e}(B) - 1 = \mathrm{r}(B)$, $S/K \cong \mathrm{K}_{R/\mathfrak{c}}$, and $S/L \cong \mathrm{K}_{B/\mathfrak{a}}$. Hence

$$\ell_R(K/R) + \ell_R(R/\mathfrak{c}) = \ell_R(S/R) = \ell_R(B/R) + \ell_R(S/B) = \mathrm{r}(R)\cdot(\ell_R(B/\mathfrak{a}) + 1) = \mathrm{r}(R)\cdot\ell_R(R/\mathfrak{c}),$$

where the last equality follows from $\ell_R(B/\mathfrak{a}) = \ell_R(S/L) = \ell_R(S/K) - 1 = \ell_R(R/\mathfrak{c}) - 1$. Therefore K/R is an $R/\mathfrak{c}$-free module. □

On the other hand, it is known that R is an almost Gorenstein local ring possessing maximal embedding dimension if and only if $\mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m})$ is a Gorenstein ring (see [36, Theorem 5.1.]). Therefore, by combining to Theorem 1.5.18 and [36, Theorem 5.1.], generalized Gorenstein local rings possessing maximal embedding dimension finally reach Gorenstein rings by the action taking the endomorphism of the maximal ideal.

Corollary 1.5.19. *Suppose that R is a generalized Gorenstein local ring but not a Gorenstein ring and $\mathrm{v}(R) = \mathrm{e}(R)$. Then S is a Gorenstein ring.*

Proof. After enlarging the residue class field of R, we may assume that $R/\mathfrak{m}$ is infinite. We prove by induction on $N = \ell_R(R/\mathfrak{c})$. If $N = 1$, then R is an almost Gorenstein local ring. Therefore $S = B$ is a Gorenstein ring by [36, Theorem 5.1]. Suppose $N > 1$ and our assertion holds true for $N - 1$. Then B is a generalized Gorenstein local ring such that $\ell_B(B/(B : B[BK])) = N - 1$ by Theorem 1.5.18. Hence $S = B[BK]$ is a Gorenstein ring. □

1.5.4 Ulrich ideals and idealizations

Next we study about Ulrich ideals in connection with the generalized Gorenstein local property. The notion of Ulrich ideals is defined over arbitrary Cohen-Macaulay local rings, and enjoy many interesting properties (Definition 1.2.6 and [43, 47]). We will summarize and study more detail in Section 1.8. Here we focus on the one-dimensional case. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal. At this moment, we focus on the case where $\mathfrak{a} \subseteq \mathfrak{c}$. We start with the following.

Lemma 1.5.20. *Suppose that $\mathfrak{a}$ is an Ulrich ideal. Assume that $\mathfrak{a} \subseteq \mathfrak{c}$. Then $\mathfrak{a} = R : T$ and $K \subseteq T$, where $T = \mathfrak{a} : \mathfrak{a}$.*

Proof. Take a non-zerodivisor $\alpha \in \mathfrak{a}$ of R so that $\mathfrak{a}^2 = \alpha\mathfrak{a}$. Then, $T = \frac{\mathfrak{a}}{\alpha}$ and $\mathfrak{a} = (\alpha) :_R \mathfrak{a}$ since [43, Corollary 2.6.]. Therefore $\mathfrak{a} = R : \frac{\mathfrak{a}}{\alpha} = R : T$. Furthermore, we have the following equalities:

$$T = \frac{\mathfrak{a}}{\alpha} = \frac{(\alpha) :_R \mathfrak{a}}{\alpha} = R : \mathfrak{a} \supseteq R : \mathfrak{c} = (K : K) : \mathfrak{c} = K : \mathfrak{c} = S.$$

Here, the last equality follows from $\mathfrak{c} = K : S$. Hence $K \subseteq T$. $\square$

By Lemma 1.5.20, to investigate Ulrich ideals contained in $\mathfrak{c}$, we may assume that $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of R satisfying the following conditions:

(1) T is a subring of $\mathrm{Q}(R)$ and a finitely generated R-module,

(2) $K \subseteq T$, and

(3) $\mathfrak{a} = R : T$.

In this subsection, let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal satisfying the above conditions (1), (2), and (3).

Lemma 1.5.21. *The following assertions hold true.*

(1) *$\mathfrak{a}T_M$ is isomorphic to the canonical module K_{T_M} for all $M \in \operatorname{Max} T$.*

(2) *The following conditions are equivalent.*

- (i) *T is a Gorenstein ring.*
- (ii) *$\mathfrak{a} = \alpha T$ for some $\alpha \in \mathfrak{a}$.*
- (iii) *$\mathfrak{a}^2 = \alpha\mathfrak{a}$ for some $\alpha \in \mathfrak{a}$.*

Proof. (1) Since $\mathfrak{a} = K : S \cong \mathrm{Hom}_R(S, \mathrm{K}_R)$, $\mathfrak{a}T_M \cong \mathrm{K}_{T_M}$ for all $M \in \operatorname{Max} T$.

(2) (i) $\Rightarrow$ (ii) If T is a Gorenstein ring, then $\mathfrak{a}T_M \cong T_M$ for all $M \in \operatorname{Max} T$ by (1). Hence, since T is a semilocal ring, $\mathfrak{a} \cong T$ and $\mathfrak{a} = \alpha T$ for some $\alpha \in \mathfrak{a}$.

(ii) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (i) Assume that $\mathfrak{a}^2 = \alpha\mathfrak{a}$ for some $\alpha \in \mathfrak{a}$ and set $L = \frac{\mathfrak{a}}{\alpha}$. Then $T \subseteq L \subseteq \overline{T}$ because of $L^2 = L$. Hence $T_M \subseteq L_M \subseteq \overline{T}_M$ and $L_M \cong \mathrm{K}_{T_M}$ for all $M \in \operatorname{Max} T$. Therefore, since $L_M^2 = L_M$, T_M is a Gorenstein ring by Theorem 1.5.5. $\square$

Proposition 1.5.22. *Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) *$\mathfrak{a}$ is an Ulrich ideal of R.*

(2) *T is a Gorenstein ring and T/R is $R/\mathfrak{a}$-free.*

When this is the case, $R/\mathfrak{a}$ is a Gorenstein ring and $\mu_R(\mathfrak{a}) = \mu_R(T)$.

Proof. By Lemma 1.5.21, we have only to show that T/R is $R/\mathfrak{a}$-free if and only if so is $\mathfrak{a}/\mathfrak{a}^2$. In fact, since $\mathfrak{a} = \alpha T$ for some $\alpha \in \mathfrak{a}$, we have $\mathfrak{a}/\mathfrak{a}^2 \cong T/\mathfrak{a}$. Therefore we have the equivalence since the exact sequence $0 \to R/\mathfrak{a} \to T/\mathfrak{a} \to T/R \to 0$.

When this is the case, $T/\mathfrak{a} = T/\alpha T$ is a Gorenstein ring since α is a non-zerodivisor of T. Since $T/\mathfrak{a}$ is $R/\mathfrak{a}$-free, we have $R/\mathfrak{a} \to T/\mathfrak{a}$ is a flat homomorphism. Hence we get $R/\mathfrak{a}$ is a Gorenstein ring. The latter equality $\mu_R(\mathfrak{a}) = \mu_R(T)$ is now trivial since $\mathfrak{a} \cong T$. $\square$

Next we give a characterization of generalized Gorenstein local rings obtained by idealization. It will relate the notion of Ulrich ideals. First, let us recall basic properties of idealizations. For a moment let R be an arbitrary commutative ring and M an R-module. Let $A = R \ltimes M$ be the idealization of M over R, that is, $A = R \oplus M$ as an R-module and the multiplication in A is given by

$$(a, x)(b, y) = (ab, bx + ay)$$

where $a, b \in R$ and $x, y \in M$. Let K be an R-module and set $L = \mathrm{Hom}_R(M, K) \oplus K$. We consider L to be an A-module under the following action of A

$$(a, x) \circ (f, y) = (af, f(x) + ay),$$

where $(a, x) \in A$ and $(f, y) \in L$. Then it is standard to check that the map

$$\mathrm{Hom}_R(A, K) \to L, \ \alpha \mapsto (\alpha \circ j, \alpha(1))$$

is an isomorphism of A-modules, where $j : M \to A, \ x \mapsto (0, x)$ and $1 = (1, 0)$ denotes the identity of the ring A.

We are now back to our Setting 1.5.3. The problem here is, for an R-module M, when the idealization $R \ltimes M$ becomes a generalized Gorenstein local ring. It is known that $R \ltimes M$ is Gorenstein if and only if $M \cong \mathrm{K}_R$ ([69]). Since a generalized Gorenstein local ring is generically Gorenstein by Lemma 1.4.1(1), we may assume that, for all $\mathfrak{p} \in \operatorname{Ass} R$, either $M_\mathfrak{p} \cong R_\mathfrak{p}$ or $M_\mathfrak{p} \cong 0$. With this observation, we concentrate the case where M is an ideal $\mathfrak{a}$. Furthermore we assume that $\mathfrak{a}$ satisfies the conditions (1), (2), and (3) stated after Lemma 1.5.20.

Remark 1.5.23. The following assertions hold true.

(1) $\mathfrak{a} = K : T$.

(2) $S \subseteq T$ and $\mathfrak{a} \subseteq \mathfrak{c}$.

Proof. (1) $R : T = (K : K) : T = K : KT = K : T$.

(2) Since $K \subseteq T$, we have $S = R[K] \subseteq T$. By taking K-dual, $\mathfrak{a} \subseteq \mathfrak{c}$. □

Set $A = R \ltimes \mathfrak{a}$ and $L = T \times K$. Then A is a one-dimensional Cohen-Macaulay local ring and

$$\mathrm{K}_A \cong \mathrm{Hom}_R(A, K) \cong \mathrm{Hom}_R(\mathfrak{a}, K) \times K \cong L$$

as A-modules by Remark 1.5.23. Therefore

$$A = R \ltimes \mathfrak{a} \subseteq L = T \times K \subseteq \overline{R} \ltimes \mathrm{Q}(R),$$

where $\overline{R}$ denotes the integral closure of R. Since $\mathrm{Q}(A) = \mathrm{Q}(R) \ltimes \mathrm{Q}(R)$ and $\overline{A} = \overline{R} \ltimes \mathrm{Q}(R)$, our idealization $A = R \ltimes \mathfrak{a}$ satisfies the same assumption in Setting 1.5.3. We pose the question of when A is a generalized Gorenstein local ring.

Lemma 1.5.24. *The following assertions hold true.*

(1) $A[L] = L^2 = T \ltimes T$.

(2) $A : A[L] = \mathfrak{a} \times \mathfrak{a}$.

(3) $\mathrm{v}(A) = \mathrm{v}(R) + \mu_R(\mathfrak{a})$ *and* $\mathrm{e}(A) = 2{\cdot}\mathrm{e}(R)$.

Proof. (1) Since $L^n = (T \times K)^n = T^n \times T^{n-1}K$ for $n \geq 2$, we have $A[L] = L^2 = T \ltimes T$.

(2) This is straightforward, since $A[L] = T \ltimes T$.

(3) First equality follows from the facts that $\mathfrak{m} \times \mathfrak{a}$ is the maximal ideal of A and $(\mathfrak{m} \times \mathfrak{a})^2 = \mathfrak{m}^2 \times \mathfrak{m}\mathfrak{a}$. Second one follows from the facts that $\mathfrak{m}A$ is a reduction of $\mathfrak{m} \times \mathfrak{a}$ and $A = R \oplus \mathfrak{a}$ as an R-module. □

By Lemma 1.5.24 we have the following.

Proposition 1.5.25. *Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) *A is a generalized Gorenstein local ring.*

(2) *R is a generalized Gorenstein local ring and $S = T$.*

(3) *T/R is $R/\mathfrak{a}$-free.*

When this is the case, $\mathfrak{a} = \mathfrak{c}$.

Proof. (1) ⇔ (2) follows from

A is a generalized Gorenstein local ring ⇔ L/A is $A/(A : A[L])$-free by Theor
⇔ $T/R \times K/\mathfrak{a}$ is $R/\mathfrak{a}$-free
⇔ T/R and K/R are $R/\mathfrak{a}$-free
⇔ T/R and K/R are $R/\mathfrak{a}$-free and $\mathfrak{a} = \mathfrak{c}$
⇔ R is a generalized Gorenstein local ring and $S = T$.

Here, the forth equality follows from $\mathfrak{a} = \mathrm{Ann}(K/R) = R : K \supseteq \mathfrak{c} = R : S \supseteq \mathfrak{a}$ and the fifth equality follows from $K : T = \mathfrak{a} = \mathfrak{c} = K : S$.

(1) $\Leftrightarrow$ (3) follows from

$$\begin{aligned} A \text{ is a generalized Gorenstein local ring} &\Leftrightarrow A[L]/A \text{ is } A/(A : A[L])\text{-free} \quad \text{by Theorem 1.5.8} \\ &\Leftrightarrow T/R \times T/\mathfrak{a} \text{ is } R/\mathfrak{a}\text{-free} \\ &\Leftrightarrow T/R \text{ is } R/\mathfrak{a}\text{-free.} \end{aligned}$$

□

Corollary 1.5.26. *Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring.*

(2) *$R \ltimes \mathfrak{c}$ is a generalized Gorenstein local ring.*

Combining Propositions 1.5.22 and 1.5.25, we have the following.

Theorem 1.5.27. *Suppose that R is not a Gorenstein ring. Let T be a subring of $\mathrm{Q}(R)$ such that T is a finitely generated R-module. Set $\mathfrak{a} = R : T$ and assume that $\mathfrak{a} \subseteq \mathfrak{c}$. Then the following conditions are equivalent.*

(1) *$\mathfrak{a}$ is an Ulrich ideal of R.*

(2) *T is a Gorenstein ring and T/R is $R/\mathfrak{a}$-free.*

(3) *T is a Gorenstein ring and $A = R \ltimes \mathfrak{a}$ is a generalized Gorenstein local ring.*

(4) *T is a Gorenstein ring, R is a generalized Gorenstein local ring, and $S = T$.*

When this is the case, $\mathfrak{a} = \mathfrak{c}$ and $\mu_R(\mathfrak{c}) = \mu_R(S) = \mathrm{r}(R) + 1$. In particular, there is no Ulrich ideal which proper contained in $\mathfrak{c}$.

Proof. The equality $\mu_R(S) = \mathrm{r}(R) + 1$ follows from the fact that $S/\mathfrak{c}S$ is an $R/\mathfrak{c}$-free module of rank $\mathrm{r}(R) + 1$ by Theorem 1.5.8. □

The followings are the direct consequences of Theorem 1.5.27.

Corollary 1.5.28. *Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) *$\mathfrak{c}$ is an Ulrich ideal of R.*

(2) *S is a Gorenstein ring and R is a generalized Gorenstein local ring.*

Corollary 1.5.29. *Suppose that $\mathrm{v}(R) = \mathrm{e}(R) \geq 3$. Then R is a generalized Gorenstein local ring if and only if $\mathfrak{c}$ is an Ulrich ideal.*

Proof. Note that $\mathrm{v}(R) = \mathrm{e}(R)$ if and only if $\mathfrak{m}^2 = \alpha\mathfrak{m}$ for some $\alpha \in \mathfrak{m}$ by [67, Corollary 1.10]. Therefore, by Theorem 1.5.27 and Corollary 1.5.19, we come to the conclusion. □

This result is related to the result of J. Herzog, T. Hibi, and D. I. Stamate [55, Theorem 7.4.]. Thus, in Section 1.8, we again consider this result, see Theorem 1.8.7 and Corollary 1.8.12.

1.5.5 Numerical semigroup rings

Next we study about generalized Gorenstein local property of numerical semigroup rings. If R is a numerical semigroup ring, then the algebra $B = \mathfrak{m} : \mathfrak{m}$ is also a numerical semigroup ring. Hence we have the structure theorem for generalized Gorenstein local rings having maximal embedding dimension by Theorem 1.5.18. We also have numerous examples of generalized Gorenstein local rings via numerical semigroup rings. First of all, we fix some notations of numerical semigroup rings.

Setting 1.5.30. Let $0 < a_1, a_2, \ldots, a_\ell \in \mathbb{Z}$ $(\ell > 0)$ be positive integers such that $\mathrm{GCD}\ (a_1, a_2, \ldots, a_\ell) = 1$. We set

$$H = \langle a_1, a_2, \ldots, a_\ell \rangle = \left\{ \sum_{i=1}^{\ell} c_i a_i \mid 0 \le c_i \in \mathbb{Z} \text{ for all } 1 \le i \le \ell \right\}$$

and call it the *numerical semigroup generated by the numbers* $\{a_i\}_{1 \le i \le \ell}$. Let $V = k[[t]]$ be the formal power series ring over a field k. We set

$$R = k[[H]] = k[[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]]$$

in V and call it the *semigroup ring of H over k*. The ring R is a one-dimensional Cohen-Macaulay local domain with $\overline{R} = V$ and $\mathfrak{m} = (t^{a_1}, t^{a_2}, \ldots, t^{a_\ell})$.

Recall some basic notion on numerical semigroups. Let

$$\mathrm{c}(H) = \min\{n \in \mathbb{Z} \mid m \in H \text{ for all } m \in \mathbb{Z} \text{ such that } m \ge n\}$$

be the *conductor of H* and set $\mathrm{f}(H) = \mathrm{c}(H) - 1$. Hence $\mathrm{f}(H) = \max\ (\mathbb{Z} \setminus H)$, which is called the *Frobenius number of H*. Let

$$\mathrm{PF}(H) = \{n \in \mathbb{Z} \setminus H \mid n + a_i \in H \text{ for all } 1 \le i \le \ell\}$$

denote the set of pseudo-Frobenius numbers of H. Therefore $\mathrm{f}(H)$ equals the a-invariant of the graded k-algebra $k[t^{a_1}, t^{a_2}, \ldots, t^{a_\ell}]$ and $\sharp \mathrm{PF}(H) = \mathrm{r}(R)$ ([49, Example (2.1.9), Definition (3.1.4)]). We set $f = \mathrm{f}(H)$ and write $\mathrm{PF}(H) = \{c_1 < c_2 < \cdots < c_r = f\}$, where $r = \mathrm{r}(R)$. Set

$$K = \sum_{c \in \mathrm{PF}(H)} R t^{f-c}$$

in V. Then K is a fractional ideal of R such that $R \subseteq K \subseteq \overline{R}$ and

$$K \cong \mathrm{K}_R = \sum_{c \in \mathrm{PF}(H)} R t^{-c}$$

as an R-module ([49, Example (2.1.9)]). Hence R satisfies Setting 1.5.3. Note that $\mathfrak{m} : \mathfrak{m} = R : \mathfrak{m} = R + \sum_{c \in \mathrm{PF}(H)} R{\cdot}t^c$ if $R \subsetneq V$.

Proposition 1.5.31. *Suppose that R is not a Gorenstein ring. Set $r = \mathrm{r}(R)$. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring.*

(2) *$R/\mathfrak{c}$ is a Gorenstein ring and $f + b = c_i + c_{r-i}$ for all $1 \le i \le r-1$, where t^b is the element in $[\mathfrak{c} :_R \mathfrak{m}] \setminus \mathfrak{c}$.*

Proof. (1) ⇒ (2) By Theorem 1.5.8 $R/\mathfrak{c}$ is a Gorenstein ring and the R-linear map $(R/\mathfrak{c})^{\oplus(r-1)} \to K/R$, where $\mathbf{e}_i \mapsto \overline{t^{f-c_i}}$ is an isomorphism. Here, $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{r-1}$ denotes a free basis of $(R/\mathfrak{c})^{\oplus(r-1)}$. Hence $t^{f-c_i+b} \notin R$ and $\mathfrak{m} \cdot t^{f-c_i+b} \subseteq R$, whence $f - c_i + b \in \mathrm{PF}(H)$ for all $1 \le i \le r-1$. Thus $\mathrm{PF}(H) = \{f - c_{r-1} + b, f - c_{r-2} + b, \dots, f - c_1 + b, f\}$, and $f - c_{r-i} + b = c_i$ for all $1 \le i \le r-1$.

(2) ⇒ (1) Consider the surjection $\varphi : Y = (R/\mathfrak{c})^{\oplus(r-1)} \to K/R$, where $\mathbf{e}_i \mapsto \overline{t^{f-c_i}}$ for all $1 \le i \le r-1$. Then

$$R \text{ is a generalized Gorenstein local ring} \Leftrightarrow \mathrm{Soc}(\mathrm{Ker}\,\varphi) = (0) \Leftrightarrow \varphi\,|_{\mathrm{Soc}Y} \text{ is injective.}$$

Let us show that $\varphi\,|_{\mathrm{Soc}Y}$ is injective. In fact, if the map is not injective, we have $t^b \cdot \mathbf{e}_i \mapsto 0$ for some $1 \le i \le r-1$ since $\mathrm{Soc}Y = ([\mathfrak{c} :_R \mathfrak{m}]/\mathfrak{c})^{\oplus(r-1)}$. This implies that $c_{r-i} = f - c_i + b \in H$, a contradiction. □

Corollary 1.5.32. *Suppose that R is not a Gorenstein ring. Let $n_1, n_2, \dots, n_\ell$ be integers and set $J = (t^{(n_1+1)a_1}, t^{(n_2+1)a_2}, \dots, t^{(n_\ell+1)a_\ell})$. Suppose that*

(1) *$J \subseteq \mathfrak{c}$ and*

(2) *$f + b = c_i + c_{r-i}$ for all $1 \le i \le r-1$,*

where $b = \sum_{j=1}^{\ell} n_j a_j$. Then R is a generalized Gorenstein local ring, $J = \mathfrak{c}$, and $\mathfrak{c} :_R \mathfrak{m} = \mathfrak{c} + (t^b)$.

Proof. Since $b + f - c_i = c_{r-i}$ for all $1 \le i \le r-1$, we have $t^b \notin R : t^{f-c_i}$. On the other hand, $t^b \in J :_R \mathfrak{m}$ by the form of J. We show that R/J is a Gorenstein ring. In fact, take $t^h \in [J :_R \mathfrak{m}] \setminus J$ and write $h = m_1a_1 + m_2a_2 + \cdots + m_\ell a_\ell$ for some non negative integer $m_1, m_2, \dots, m_\ell$. Then $m_j \le n_j$ for all $1 \le j \le \ell$. Hence $t^b = t^h \cdot t^{b-h}$ and $b - h \in H$. Since $t^b, t^h \in [J :_R \mathfrak{m}] \setminus J$, we get $h = b$, that is, $\mathrm{r}(R/J) = 1$. We also have $J = \mathfrak{c}$. In fact, if $J \subsetneq \mathfrak{c}$, then $t^b \in J :_R \mathfrak{m} \subseteq \mathfrak{c}$. This is a contradiction since $t^b \notin R : t^{f-c_i} \supseteq R : K \supseteq \mathfrak{c}$. Hence R is a generalized Gorenstein local ring by Proposition 1.5.31. □

Corollary 1.5.33. *Assume $e \le h_1 \le h_2 \le \ldots \le h_{e-1}$ and let $H = \langle e, h_1, h_2, \dots, h_{e-1}\rangle$. Set $R = k[[H]]$. Suppose that $e = \mathrm{e}(R) = \mathrm{v}(R) \ge 3$. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring.*

(2) *There exists an integer $n_0 \ge 0$ such that $\mathfrak{c} = (t^{(n_0+1)e}, t^{h_1}, t^{h_2}, \dots, t^{h_{e-1}})$ and $h_{e-1} + (n_0+1)e = h_i + h_{e-1-i}$ for all $1 \le i \le e-2$.*

Proof. (2) $\Rightarrow$ (1) By Corollary 1.5.32.

(1) $\Rightarrow$ (2) Since K/R is $R/\mathfrak{c}$-free and $K = \sum_{i=1}^{e-1} Rt^{f-c_i}$, we have $\mathfrak{c} = R : t^{f-c_i}$ for all $1 \le i \le e-2$. Hence $t^{h_i} = t^{c_i+e} \in \mathfrak{c}$ for all $1 \le i \le e-2$. Furthermore, since $f - c_1 + h_{e-1} = f-(h_1-e)+h_{e-1} = f+e+(h_{e-1}-h_1) > f$, we have $t^{h_{e-1}} \in R : t^{f-c_1} = \mathfrak{c}$. Hence $\mathfrak{c} = (t^{(n_0+1)e}, t^{h_1}, t^{h_2}, \dots, t^{h_{e-1}})$ for some integer $n_0 \ge 0$. Therefore we have $n_0 e + f - c_i \in \mathrm{PF}(H)$ since $\mathfrak{c} = R : t^{f-c_i}$ for all $1 \le i \le e-2$. Thus $h_{e-1}+(n_0+1)e = h_i + h_{e-1-i}$ for all $1 \le i \le e-2$. $\square$

Let us give more concrete examples. With the notation of Setting 1.5.30 suppose that $\ell = 3$ and set $T = k[t^{a_1}, t^{a_2}, t^{a_3}]$ in the polynomial ring $k[t]$. Let $P = k[X_1, X_2, X_3]$ be the polynomial ring over k. We consider P to be a $\mathbb{Z}$-graded ring such that $P_0 = k$ and $\deg X_i = a_i$ for $i = 1, 2, 3$. Let

$$\varphi : P = k[X_1, X_2, X_3] \to T = k[t^{a_1}, t^{a_2}, t^{a_3}]$$

denote the homomorphism of graded k-algebras defined by $\varphi(X_i) = t^{a_i}$ for each $i = 1, 2, 3$. Let us write $X = X_1$, $Y = X_2$, and $Z = X_3$ for short. If T is not a Gorenstein ring, then by [51] it is known that $\mathrm{Ker}\,\varphi = I_2\begin{pmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{pmatrix}$ for some integers $\alpha, \beta, \gamma, \alpha', \beta', \gamma' > 0$.

We use a result of [36, Section 4]. Let $\Delta_1 = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta}$, $\Delta_2 = X^{\alpha+\alpha'} - Y^{\beta'}Z^{\gamma}$, and $\Delta_3 = Y^{\beta+\beta'} - X^{\alpha}Z^{\gamma'}$. Then $\mathrm{Ker}\,\varphi = (\Delta_1, \Delta_2, \Delta_3)$ and thanks to the theorem of Hilbert–Burch ([20, Theorem 20.15]), the graded ring T possesses a graded minimal P-free resolution of the form

$$0 \longrightarrow \begin{matrix} P(-m) \\ \oplus \\ P(-n) \end{matrix} \xrightarrow{\begin{bmatrix} X^{\alpha} & Y^{\beta'} \\ Y^{\beta} & Z^{\gamma'} \\ Z^{\gamma} & X^{\alpha'} \end{bmatrix}} \begin{matrix} P(-d_1) \\ \oplus \\ P(-d_2) \\ \oplus \\ P(-d_3) \end{matrix} \xrightarrow{[\Delta_1\ -\Delta_2\ \Delta_3]} P \xrightarrow{\varphi} T \longrightarrow 0,$$

where $d_1 = \deg \Delta_1 = a_3(\gamma + \gamma')$, $d_2 = \deg \Delta_2 = a_1(\alpha + \alpha')$, $d_3 = \deg \Delta_3 = a_2(\beta + \beta')$, $m = a_1\alpha + d_1 = a_2\beta + d_2 = a_3\gamma + d_3$, and $n = a_1\alpha' + d_3 = a_2\beta' + d_1 = a_3\gamma' + d_2$. Therefore

$$n - m = a_2\beta' - a_1\alpha = a_3\gamma' - a_2\beta = a_1\alpha' - a_3\gamma. \tag{1.5.33.1}$$

Let $\mathrm{K}_P = P(-d)$ denote the graded canonical module of P where $d = a_1 + a_2 + a_3$. Then, taking the K_P–dual of the above resolution, we get the minimal presentation

$$\begin{matrix} P(d_1 - d) \\ \oplus \\ P(d_2 - d) \\ \oplus \\ P(d_3 - d) \end{matrix} \xrightarrow{\begin{bmatrix} X^{\alpha} & Y^{\beta} & Z^{\gamma} \\ Y^{\beta'} & Z^{\gamma'} & X^{\alpha'} \end{bmatrix}} \begin{matrix} P(m - d) \\ \oplus \\ P(n - d) \end{matrix} \xrightarrow{\varepsilon} \mathrm{K}_T \longrightarrow 0 \tag{1.5.33.2}$$

of the graded canonical module $\mathrm{K}_T = \mathrm{Ext}_P^2(T, \mathrm{K}_P)$ of T. Therefore, because $\mathrm{K}_T = \sum_{c \in \mathrm{PF}(H)} Tt^{-c}$ ([49, Example (2.1.9)]), we have $\ell_k([\mathrm{K}_T]_i) \le 1$ for all $i \in \mathbb{Z}$, whence $m \ne n$. After the permutation of a_2 and a_3 if necessary, we may assume without loss of generality that $n > m$. Then the presentation (1.5.33.2) shows that $\mathrm{PF}(H) = \{m-d, n-d\}$ and $f = n - d$.

We set $a = n - m$. Hence $a > 0$, $f = a + (m - d)$, and $K = R + Rt^a$. With this notation we have the following. Remember that R is the MT_M-adic completion of the local ring T_M, where $M = (t^{a_i} \mid i = 1, 2, 3)$ denotes the graded maximal ideal of T. Then we have the following.

Theorem 1.5.34. *Suppose that H is 3-generated. Assume that $R = k[[H]]$ is not a Gorenstein ring and $a > 0$. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring.*

(2) *$3a \in H$.*

(3) *$\alpha \le \alpha'$, $\beta \le \beta'$, and $\gamma \le \gamma'$.*

When this is the case, $\mathfrak{c} = (t^{\alpha a_1}, t^{\beta a_2}, t^{\gamma a_3})$ and $\ell_R(R/\mathfrak{c}) = \alpha\beta\gamma$.

Proof. (1) $\Rightarrow$ (3) By (1.5.33.2), we get $\mathrm{K}_T/P\xi \cong P/(X^\alpha, Y^\beta, Z^\gamma)$, where $\xi = \varepsilon\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. By $K/R \cong R/\mathfrak{c}$ since $\mathrm{r}(R) = 2$, we have $\mathfrak{c} = (X^\alpha, Y^\beta, Z^\gamma)R = (t^{\alpha a_1}, t^{\beta a_2}, t^{\gamma a_3})$. On the other hand, by applying the functor $- \otimes_P P/(X^\alpha, Y^\beta, Z^\gamma)$ to (1.5.33.2), we get the exact sequence

$$\begin{array}{c} P/(X^\alpha, Y^\beta, Z^\gamma)(d_1 - d) \\ \oplus \\ P/(X^\alpha, Y^\beta, Z^\gamma)(d_2 - d) \\ \oplus \\ P/(X^\alpha, Y^\beta, Z^\gamma)(d_3 - d) \end{array} \xrightarrow{\left[\begin{smallmatrix}\overline{X^\alpha} & \overline{Y^\beta} & \overline{Z^\gamma} \\ \overline{Y^{\beta'}} & \overline{Z^{\gamma'}} & \overline{X^{\alpha'}}\end{smallmatrix}\right]} \begin{array}{c} P/(X^\alpha, Y^\beta, Z^\gamma)(m - d) \\ \oplus \\ P/(X^\alpha, Y^\beta, Z^\gamma)(n - d) \end{array} \xrightarrow{\varepsilon} \mathrm{K}_T/(X^\alpha, Y^\beta, Z^\gamma)\mathrm{K}_T \longrightarrow 0$$

as $P/(X^\alpha, Y^\beta, Z^\gamma)$-modules, where $\overline{*}$ denotes image of $* \in P$ in $P/(X^\alpha, Y^\beta, Z^\gamma)$. We get $\left[\begin{smallmatrix}\overline{X^\alpha} & \overline{Y^\beta} & \overline{Z^\gamma} \\ \overline{Y^{\beta'}} & \overline{Z^{\gamma'}} & \overline{X^{\alpha'}}\end{smallmatrix}\right] = 0$ since $K/\mathfrak{c}K \cong (R/\mathfrak{c})^{\oplus 2}$, whence $(Y^{\beta'}, Z^{\gamma'}, X^{\alpha'}) \subseteq (X^\alpha, Y^\beta, Z^\gamma)$, that is, $\alpha \le \alpha'$, $\beta \le \beta'$,and $\gamma \le \gamma'$.

(3) $\Rightarrow$ (2) Because $3a = (a_2\beta' - a_1\alpha) + (a_3\gamma' - a_2\beta) + (a_1\alpha' - a_3\gamma) = a_1(\alpha - \alpha') + a_2(\beta - \beta') + a_3(\gamma - \gamma') \in H$ by the equality (1.5.33.1).

(2) $\Rightarrow$ (1) By Corollary 1.5.11 and $K = R + Rt^a$. $\square$

When H is 3-generated and $\mathrm{e}(R) = \min\{a_1, a_2, a_3\}$ is small, we have the following structure theorem of H for R to be a generalized Gorenstein local ring.

Corollary 1.5.35. *Let $\ell = 3$. Then the following assertions are true.*

(1) *If $\min\{a_1, a_2, a_3\} = 3$, then R is a generalized Gorenstein local ring.*

(2) *Suppose that $\min\{a_1, a_2, a_3\} = 4$. Then the following conditions are equivalent.*

   (a) *R is a generalized Gorenstein local ring, but not an almost Gorenstein local ring.*

   (b) *$H = \langle 4, 3\alpha + 2\alpha', \alpha + 2\alpha' \rangle$ for some $\alpha' \ge \alpha \ge 3$ such that $\alpha \not\equiv 0 \bmod 2$.*

(3) *Suppose that $\min\{a_1, a_2, a_3\} = 5$. Then the following conditions are equivalent.*

(a) *R is a generalized Gorenstein local ring, but not an almost Gorenstein local ring.*

(b) (i) $H = \langle 5, 2\alpha + \alpha', \alpha + 3\alpha' \rangle$ *for some* $\alpha' \geq \alpha \geq 2$ *such that* $2\alpha + \alpha' \not\equiv 0 \bmod 5$ *or* (ii) $H = \langle 5, 4\alpha + 3\alpha', \alpha + 2\alpha' \rangle$ *for some* $\alpha' \geq \alpha \geq 2$ *such that* $\alpha + 2\alpha' \not\equiv 0 \bmod 5$.

Proof. (1) This is followed by Corollary 1.5.12.

Suppose that R is a generalized Gorenstein local ring but not an almost Gorenstein local ring. Then, by [36, Corollary 4.2], after a suitable permutation of a_1, a_2, a_3 we may assume that $\alpha \geq 2$. Note that

$$a_1 = \beta\gamma + \beta'\gamma' + \beta'\gamma,$$

since $a_1 = \ell_R(R/t^{a_1}R) = \ell_P(P/[(X) + \operatorname{Ker}\varphi]) = \ell_k(k[Y,Z]/(Y^{\beta'+1}, Y^{\beta'}Z, Z^{\gamma'+1})$. We similarly have that

$$a_2 = \alpha\gamma + \alpha\gamma' + \alpha'\gamma' \geq 6, \quad a_3 = \alpha'\beta' + \alpha'\beta + \alpha\beta \geq 6$$

since $\alpha' \geq \alpha \geq 2$. Therefore $\mathrm{e}(R) = a_1 = \beta\gamma + \beta'\gamma' + \beta'\gamma$ and $\beta = \gamma = 1$ if $\mathrm{e}(R) \leq 5$.

(2) (a) $\Rightarrow$ (b) Since $a_1 = \beta'\gamma' + \beta' + 1 = 4$, we have $\beta' = 1$ and $\gamma' = 2$. Hence $a_2 = 3\alpha + 2\alpha'$ and $a_3 = \alpha + 2\alpha'$. Note that $\alpha \not\equiv 0 \bmod 2$ since GCD $(a_1, a_2, a_3) = 1$.

(3) (a) $\Rightarrow$ (b) Since $a_1 = \beta'\gamma' + \beta' + 1 = 5$, we have either the case where $\beta' = 2$, $\gamma' = 1$ or the case where $\beta' = 1$, $\gamma' = 3$. For the former case, we get $H = \langle 5, 2\alpha + \alpha', \alpha + 3\alpha' \rangle$ for some $\alpha' \geq \alpha \geq 2$ such that $2\alpha + \alpha' \not\equiv 0 \bmod 5$ since GCD $(5, 2\alpha + \alpha', \alpha + 3\alpha') = 1$. For the latter case, we get $H = \langle 5, 4\alpha + 3\alpha', \alpha + 2\alpha' \rangle$ for some $\alpha' \geq \alpha \geq 2$ such that $\alpha + 2\alpha' \not\equiv 0 \bmod 5$ since GCD $(5, 4\alpha + 3\alpha', \alpha + 2\alpha') = 1$.

(2) (b) $\Rightarrow$ (a) Since we have $\mathfrak{a} = I_2\left(\begin{smallmatrix} X^\alpha & Y & Z \\ Y & Z^2 & X^{\alpha'} \end{smallmatrix}\right)$, this is a generalized Gorenstein local ring by Theorem 1.5.34. However, R is not an almost Gorenstein local ring by [36, Corollary 4.2].

(3) (b) $\Rightarrow$ (a) For the case (i) we have $\mathfrak{a} = I_2\left(\begin{smallmatrix} X^\alpha & Y & Z \\ Y^2 & Z & X^{\alpha'} \end{smallmatrix}\right)$, and for the case (ii) we have $\mathfrak{a} = I_2\left(\begin{smallmatrix} X^\alpha & Y & Z \\ Y & Z^3 & X^{\alpha'} \end{smallmatrix}\right)$. Therefore R is a generalized Gorenstein local ring but not an almost Gorenstein local ring. $\square$

Let $0 < e \in H$ and set $\alpha_i = \min\{h \in H \mid h \equiv i \bmod e\}$ for each $0 \leq i \leq e-1$. Then the set

$$Ap_e(H) = \{\alpha_i \mid 0 \leq i \leq e-1\} = \{h \in H \mid h - e \notin H\}$$

is called the *Apery set* of H mod e. With the notation, the following result is known.

Theorem 1.5.36. (cf.[15, Theorem 6.9.]) *Let H be a numerical semigroup and assume that H is symmetric, that is, $R = k[[H]]$ is a Gorenstein ring. Take an element $0 < e \in H$ and consider $Ap_e(H) = \{0 < h_1 < h_2 < \cdots < h_{e-1}\}$. Set*

$$H_n = \langle e, h_1 + ne, h_2 + ne, \ldots, h_{e-1} + ne \rangle \text{ and } R_n = k[[H_n]]$$

for all $n > 0$. Let K_n denote a fractional canonical ideal of R_n such that $R_n \subseteq K_n \subseteq \overline{R_n}$. Then we have the following.

(1) *R_n is a generalized Gorenstein local ring,*

(2) $\mathrm{v}(R_n) = \mathrm{e}(R_n)$, *and*

(3) $\ell_{R_n}(R_n/\mathfrak{c}_n) = n$,

where $\mathfrak{c}_n = R_n : R_n[K_n]$.

Due to Theorem 1.5.18, we get the following which is the converse of [15, Theorem 6.9.].

Proposition 1.5.37. *Let* $e \leq h_1 \leq h_2 \leq \ldots \leq h_{e-1}$ *be positive integers such that* GCD $(e, h_1, h_2, \ldots, h_{e-1}) = 1$. *Set* $H = \langle e, h_1, h_2, \ldots, h_{e-1} \rangle$ *and* $R = k[[H]]$. *Assume that* R *is a generalized Gorenstein local ring with* $e = \mathrm{e}(R) = \mathrm{v}(R) \geq 3$. *Set* $n = \ell_R(R/\mathfrak{c}) > 0$ *and* $H' = \langle e, h_1 - ne, h_2 - ne, \ldots, h_{e-1} - ne \rangle$. *Then the following assertions hold true.*

(1) $R' = k[[H']]$ *is a Gorenstein ring.*

(2) $Ap_e(H') = \{0, h_1 - ne, h_2 - ne, \ldots, h_{e-1} - ne\}$.

Hence R *is reconstructed by* $Ap_e(H')$ *and* $n > 0$.

Proof. Suppose that $n > 1$, that is, R is not an almost Gorenstein local ring. Set $R_0 = R$ and $\mathfrak{m}_0 = (t^e, t^{h_1}, t^{h_2}, \ldots, t^{h_{e-1}})$. Define that

$$R_j = \mathfrak{m}_{j-1} : \mathfrak{m}_{j-1} \text{ and } \mathfrak{m}_j = \mathfrak{m}_{j-1} S \cap R_{j-1}$$

for $0 < j \leq n$ inductively. Note that $R_j = k[[H_j]]$ for all $0 < j \leq n$, where $H_j = \langle e, h_1 - je, h_2 - je, \ldots, h_{e-1} - je \rangle$ since Theorem 1.5.18. Furthermore, we have

(i) R_j is a generalized Gorenstein local ring but not a Gorenstein ring,

(ii) $\ell_{R_j}(R_j/\mathfrak{c}_j) = n - j$,

(iii) $\mathrm{v}(R_j) = \mathrm{e}(R_j) = e$, and

(iv) $Ap_e(H_j) = \{0, h_1 - je, h_2 - je, \ldots, h_{e-1} - je\}$

for all $0 < j < n$, where $\mathfrak{c}_j = R_j : S$. Hence we may assume $n = 1$. This is the case where R is an almost Gorenstein local ring but not a Gorenstein ring, whence this is proven in [36, Theorem 5.1]. □

1.6 Minimal free resolutions of generalized Gorenstein local rings

We are now back to the arbitrary dimensional case. Throughout this section, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring possessing the canonical module K_R. Set $d = \dim R > 0$ and $r = \mathrm{r}(R)$. In this section, we consider the following condition.

Condition. *There exists an exact sequence*

$$0 \to R \to \mathrm{K}_R \to \bigoplus_{i=2}^{r} R/\mathfrak{a}_i \to 0$$

of R-modules, where $\mathfrak{a}_i$ is an ideal of R for all $2 \leq i \leq r$.

Remark 1.6.1. With the notation of above Condition the following assertions are true.

(1) Suppose that R has an exact sequence of Condition. Then $R/\mathfrak{a}_i$ is a Gorenstein local ring of dimension $d-1$ for all $2 \leq i \leq r$.

(2) If R is a generically Gorenstein ring with $r = 2$, then Condition holds true.

(3) If R is a generalized Gorenstein local ring with $d = 1$, then Condition holds true.

Proof. (1) By Lemma 1.4.1, $R/\mathfrak{a}_i$ is a Cohen-Macaulay local ring of dimension $d-1$. By applying the functor $\mathrm{Hom}_R(*, \mathrm{K}_R)$, we get the isomorphism $\bigoplus_{i=2}^{r} R/\mathfrak{a}_i \cong \bigoplus_{i=2}^{r} \mathrm{Ext}_R^1(R/\mathfrak{a}_i, \mathrm{K}_R)$. Since $\mathrm{Ext}_R^1(R/\mathfrak{a}_i, \mathrm{K}_R) \cong \mathrm{K}_{R/\mathfrak{a}_i}$, we conclude $R/\mathfrak{a}_i$ is a Gorenstein ring.

(2) By assumption, there exists a canonical ideal generated by two elements.

(3) This follows from Theorem 1.5.8. $\square$

We have the following which is a natural generalization of [46, Theorem 7.8].

Theorem 1.6.2. *Let $(S, \mathfrak{n})$ be a Gorenstein local ring and $I, \mathfrak{a}_2, \mathfrak{a}_3, \ldots, \mathfrak{a}_r$ be ideals of S. Suppose that $R \cong S/I \neq 0$ and R is a Cohen-Macaulay ring but not a Gorenstein ring. Assume that the projective dimension of R over S is finite. Then the following conditions are equivalent.*

(1) *There exists an exact sequence*

$$0 \to R \to \mathrm{K_R} \to \bigoplus_{\mathrm{i=2}}^{\mathrm{r}} \mathrm{S}/\mathfrak{a}_\mathrm{i} \to 0$$

of S-modules.

(2) *There exist a minimal S-free resolution*

$$0 \to S^{\oplus r} \xrightarrow{\mathbb{M}} S^{\oplus q} \to \cdots \to S \to R \to 0$$

of R and a non-negative integer m, such that

$${}^t\mathbb{M} = \left(\begin{array}{ccccccccccccc|cccc}
y_{21} & y_{22} & \cdots & y_{2u_2} & y_{31} & y_{32} & \cdots & y_{3u_3} & \cdots & y_{r1} & y_{r2} & \cdots & y_{ru_r} & z_1 & z_2 & \cdots & z_m \\
\hline
x_{21} & x_{22} & \cdots & x_{2u_2} & & & & & & & & & & & & & \\
& & & & x_{31} & x_{32} & \cdots & x_{3u_3} & & & 0 & & & & & & \\
& & & & & & & & \ddots & & & & & & 0 & & \\
& 0 & & & & & & & & x_{r1} & x_{r2} & \cdots & x_{ru_r} & & & &
\end{array}\right),$$

where $\mu_S(\mathfrak{a}_i) = u_i$, $\mathfrak{a}_i = (x_{i1}, x_{i2}, \ldots, x_{iu_i})$, and $\dim S/\mathfrak{a}_i = \dim R - 1$ for all $2 \leq i \leq r$.

Furthermore, if $x_{i1}, x_{i2}, \ldots, x_{iu_i}$ is an S-regular sequence for all $2 \le i \le r$, then we have the equality:

$$I = \sum_{i=2}^{r} I_2 \begin{pmatrix} y_{i1} & y_{i2} & \cdots & y_{iu_i} \\ x_{i1} & x_{i2} & \ldots & x_{iu_i} \end{pmatrix} + (z_1, z_2, \ldots, z_m). \tag{1.6.2.1}$$

Proof. This result is essentially proven in the paper [46, Theorem 7.8]. However, we include a proof for the sake of completeness.

(1) ⇒ (2) Choose the exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K_R} \to \bigoplus_{\mathrm{i=2}}^{\mathrm{r}} \mathrm{S}/\mathfrak{a}_\mathrm{i} \to 0$$

of S-modules and set $f_1 = \varphi(1)$. We have $\mathrm{K_R}/\mathrm{f_1S} \cong \bigoplus_{\mathrm{i=2}}^{\mathrm{r}} \mathrm{S}/\mathfrak{a}_\mathrm{i}$. Choose elements $f_2, \ldots, f_r \in \mathrm{K_R}$ such that $\overline{f_i}$ corresponds to $(0, \ldots, 0, 1_{S/\mathfrak{a}_i}, 0, \ldots, 0)$, where $\overline{f_i}$ denotes the image of f_i in $\mathrm{K_R}/\mathrm{f_1S}$. Then we have a surjective homomorphism

$$\psi : S^{\oplus r} \to \mathrm{K_R},\ \mathbf{e}_\mathrm{i} \mapsto \mathrm{f_i},$$

where $\{\mathbf{e}_i\}_{1 \le i \le r}$ denote the standard basis of $S^{\oplus r}$. Set $L = \operatorname{Ker} \psi$ and $u_i = \mu_S(\mathfrak{a}_i)$. Take $x_{ij} \in S$ so that $\mathfrak{a}_i = (x_{i1}, x_{i2}, \ldots, x_{iu_i})$ for all $2 \le i \le r$ and $1 \le j \le u_i$. We explore a minimal basis of L. Since $\mathrm{K_R}/\mathrm{f_1S} \cong \bigoplus_{\mathrm{i=2}}^{\mathrm{r}} \mathrm{S}/\mathfrak{a}_\mathrm{i}$, we have $x_{ij} f_i \in f_1 S$, that is, $x_{ij} f_i + y_{ij} f_1 = 0$ for some $y_{ij} \in S$. Therefore we get $x_{ij}\mathbf{e}_i + y_{ij}\mathbf{e}_1 \in L$. Set $\mathbf{a}_{ij} = x_{ij}\mathbf{e}_i + y_{ij}\mathbf{e}_1$ for all $2 \le i \le r$ and $1 \le j \le u_i$. Let $\mathbf{a} \in L$ and write $\mathbf{a} = \sum_{i=1}^{r} b_i \mathbf{e}_i$ with $b_i \in S$. Then $b_i \in \mathfrak{a}_i$ for all $2 \le i \le r$ since $\sum_{i=2}^{r} b_i \cdot (0, \ldots, 0, 1_{S/\mathfrak{a}_i}, 0, \ldots, 0) = 0$ in $\bigoplus_{i=2}^{r} S/\mathfrak{a}_i$. Write $b_i = \sum_{j=1}^{u_i} c_{ij} x_{ij}$ with $c_{ij} \in S$. Then we have

$$\begin{aligned} \mathbf{a} &= b_1\mathbf{e}_1 + \sum_{i=2}^{r} b_i \mathbf{e}_i = b_1\mathbf{e}_1 + \sum_{i=2}^{r}\sum_{j=1}^{u_i} c_{ij} x_{ij} \mathbf{e}_i \\ &= b_1\mathbf{e}_1 + \sum_{i=2}^{r}\sum_{j=1}^{u_i} c_{ij}(\mathbf{a}_{ij} - y_{ij}\mathbf{e}_i). \end{aligned}$$

Hence we have $\mathbf{a} - \sum_{i=2}^{r}\sum_{j=1}^{u_i} c_{ij}\mathbf{a}_{ij} \in L \cap S\mathbf{e}_1$, whence L is generated by $\{\mathbf{a}_{ij}\}_{2 \le i \le r,\ 1 \le j \le u_i} \cup \{z_k \mathbf{e}_1\}_{1 \le k \le m}$ for some integer $m \ge 0$ and $z_k \in S$ for $1 \le k \le m$. Thus, with $q = \sum_{i=2}^{r} u_i + m$, we have an S-free resolution

$$S^{\oplus q} \xrightarrow{{}^t\mathbb{M}} S^{\oplus r} \xrightarrow{\psi} \mathrm{K_R} \to 0 \tag{1.6.2.2}$$

of $\mathrm{K_R}$, where the matrix ${}^t\mathbb{M}$ has required form. We have to show that we may assume that (1.6.2.2) is minimal, that is, $\{\mathbf{a}_{ij}\}_{2 \le i \le r,\ 1 \le j \le u_i}$ is a part of minimal system of generators of L. Since $\{\mathbf{a}_{ij}\}_{2 \le i \le r,\ 1 \le j \le u_i} \cup \{z_k\mathbf{e}_1\}_{1 \le k \le m}$ generates L, we can choose minimal system of generator of L in them. Assume that $\mathbf{a}_{ij}$ is not a part of minimal system of generators of L for some $2 \le i \le r$ and $1 \le j \le u_i$. Then, we can construct an S-free resolution

$$S^{\oplus q} \xrightarrow{{}^t\mathbb{M}'} S^{\oplus r} \xrightarrow{\psi} \mathrm{K_R} \to 0$$

of $\mathrm{K_R}$, where ${}^t\mathbb{M}'$ is a matrix such that the columns corresponding to $\mathbf{a}_{ij}$ is excepted from ${}^t\mathbb{M}$. Hence we have

$$\begin{aligned}\bigoplus_{i=2}^{r} S/\mathfrak{a}_i &\cong \mathrm{K_R/f_1S} \cong \mathrm{S^{\oplus r}/(Im{}^t\mathbb{M}' + S\mathbf{e}_1)} \\ &\cong S/\mathfrak{a}_2 \oplus \cdots \oplus S/\mathfrak{a}_{i-1} \oplus S/\mathfrak{a}' \oplus S/\mathfrak{a}_{i+1} \oplus \cdots \oplus S/\mathfrak{a}_r,\end{aligned}$$

where $\mathfrak{a}' = (x_{i1}, x_{i2}, \ldots, x_{i\,j-1}, x_{i\,j+1}, \ldots, x_{iu_i})$. This is a contradiction for $u_i = \mu_S(\mathfrak{a}_i)$. Hence we may assume that (1.6.2.2) is minimal. Then the S-module K_R possesses a minimal free resolution

$$0 \to S \to \cdots \to S^{\oplus q} \to S^{\oplus r} \to \mathrm{K}_R \to 0$$

with $q = \sum_{i=2}^{r} u_i + m$. Therefore, by taking S-dual, Assertion (2) holds.

(2) $\Rightarrow$ (1) By taking S-dual, we have the exact sequence (1.6.2.2). Set $f_i = \psi(\mathbf{e}_i)$ for all $1 \leq i \leq r$, where $\{\mathbf{e}_i\}_{1 \leq i \leq r}$ denotes the standard basis of $S^{\oplus r}$. We then have

$$\mathrm{K_R/f_1S} \cong \mathrm{S^{\oplus r}/(Im{}^t\mathbb{M} + S\mathbf{e}_1)} \cong \bigoplus_{\mathrm{i=2}}^{\mathrm{r}} \mathrm{S}/\mathfrak{a}_{\mathrm{i}}.$$

Hence we have an exact sequence

$$R \xrightarrow{\varphi} \mathrm{K_R} \to \bigoplus_{\mathrm{i=2}}^{\mathrm{r}} \mathrm{S}/\mathfrak{a}_{\mathrm{i}} \to 0$$

of R-modules, where $\varphi(1) = f_1$. Since $\dim S/\mathfrak{a}_i = \dim R - 1$ for all $2 \leq i \leq r$, φ is injective.

Now we prove the equality (1.6.2.1). Suppose that $x_{i1}, x_{i2}, \ldots, x_{iu_i}$ is an S-regular sequence for all $2 \leq i \leq r$. Note that for $a \in S$ we get the equivalences

$$\begin{aligned} a \in I \quad &\Leftrightarrow \quad af_1 = 0 \quad \Leftrightarrow \quad a\mathbf{e}_1 \in L \\ &\Leftrightarrow \quad a\mathbf{e}_1 = \sum_{2 \leq i \leq r,\ 1 \leq j \leq u_i} c_{ij}\mathbf{a}_{ij} + \sum_{k=1}^{m} d_k z_k \mathbf{e}_1 \quad \text{for some } c_{ij}, d_k \in S \\ &\Leftrightarrow \quad a = \sum_{2 \leq i \leq r,\ 1 \leq j \leq u_i} c_{ij}y_{ij} + \sum_{k=1}^{m} d_k z_k \quad \text{and} \quad 0 = \sum_{2 \leq i \leq r,\ 1 \leq j \leq u_i} c_{ij}x_{ij}, \end{aligned} \tag{1.6.2.3}$$

where the first equivalence follows from φ is an injective map and the second equivalence follows from $L = \operatorname{Ker}\psi$. For the third equivalence, this follows from

$$L = \sum_{2 \leq i \leq r,\ 1 \leq j \leq u_i} S\mathbf{a}_{ij} + \sum_{k=1}^{m} S\mathbf{e}_1.$$

The fourth equivalence follows from $\mathbf{a}_{ij} = x_{ij}\mathbf{e}_i + y_{ij}\mathbf{e}_1$.

($\supseteq$) By (1.6.2.3), we obtain $I \supseteq (z_1, z_2, \dots, z_m)$. For all $3 \le i \le r$ and $1 \le \alpha < \beta \le u_i$, we get

$$(x_{i\beta}y_{i\alpha} - x_{i\alpha}y_{i\beta})\mathbf{e}_1 = x_{i\beta}(\mathbf{a}_{i\alpha} - x_{i\alpha}\mathbf{e}_i) - x_{i\alpha}(\mathbf{a}_{i\beta} - x_{i\beta}\mathbf{e}_i) = x_{i\beta}\mathbf{a}_{i\alpha} - x_{i\alpha}\mathbf{a}_{i\beta} \in L,$$

whence $x_{i\beta}y_{i\alpha} - x_{i\alpha}y_{i\beta} \in I$ by (1.6.2.3).

($\subseteq$) Let $a \in I$. By (1.6.2.3), it is enough to show that, for any $2 \le i \le r$,

$$\sum_{1 \le j \le u_i} c_{ij}y_{ij} \in I_2 \begin{pmatrix} y_{i1} & y_{i2} & \cdots & y_{iu_i} \\ x_{i1} & x_{i2} & \dots & x_{iu_i} \end{pmatrix}$$

if $\sum_{1 \le j \le u_i} c_{ij}x_{ij} = 0$. Let

$$K_2(\mathbf{x}) \xrightarrow{\partial_2} K_1(\mathbf{x}) \xrightarrow{\partial_1} K_0(\mathbf{x}) \quad \text{and} \quad K_2(\mathbf{y}) \xrightarrow{\partial_2'} K_1(\mathbf{y}) \xrightarrow{\partial_1'} K_0(\mathbf{y})$$

be the parts of Koszul complexes of the sequences $\mathbf{x} = x_{i1}, x_{i2}, \dots, x_{iu_i}$ and $\mathbf{y} = y_{i1}, y_{i2}, \dots, y_{iu_i}$, respectively. Let $T_1, \dots, T_{u_i}$ be a basis of $K_1(\mathbf{x}) = K_1(\mathbf{y})$. Then, since $\partial_1(\sum_{1 \le j \le u_i} c_{ij}T_j) = \sum_{1 \le j \le u_i} c_{ij}x_{ij} = 0$, $\sum_{1 \le j \le u_i} c_{ij}y_{ij} \in \partial_1'(\operatorname{Ker}\partial_1)$. On the other hand, since the sequence $\mathbf{x} = x_{i1}, x_{i2}, \dots, x_{iu_i}$ is an S-regular sequence, $\operatorname{Im}\partial_2 = \operatorname{Ker}\partial_1$. Hence $\sum_{1 \le j \le u_i} c_{ij}y_{ij} \in \operatorname{Im}(\partial_1' \circ \partial_2)$. It follows that $\sum_{1 \le j \le u_i} c_{ij}y_{ij} \in I_2 \begin{pmatrix} y_{i1} & y_{i2} & \cdots & y_{iu_i} \\ x_{i1} & x_{i2} & \dots & x_{iu_i} \end{pmatrix}$ since $\operatorname{Im}(\partial_1' \circ \partial_2)$ is generated by $\partial_1' \circ \partial_2(T_\alpha T_\beta) = x_{i\beta}y_{i\alpha} - x_{i\alpha}y_{i\beta}$ for all $1 \le \alpha < \beta \le u_i$. □

Corollary 1.6.3. *Let $(S, \mathfrak{n})$ be a regular local ring and $I, \mathfrak{a}_2, \mathfrak{a}_3, \dots, \mathfrak{a}_r$ be ideals of S. Suppose that $R \cong S/I \ne 0$ and R is a Cohen-Macaulay ring but not a Gorenstein ring. Assume that there exists an exact sequence*

$$0 \to R \to \mathrm{K_R} \to \bigoplus_{\mathrm{i=2}}^{\mathrm{r}} \mathrm{S}/\mathfrak{a}_\mathrm{i} \to 0$$

of R-modules. If $S/\mathfrak{a}_i$ is a complete intersection, then

$$I = \sum_{i=2}^{r} I_2 \begin{pmatrix} y_{i1} & y_{i2} & \cdots & y_{iu_i} \\ x_{i1} & x_{i2} & \dots & x_{iu_i} \end{pmatrix} + (z_1, z_2, \dots, z_m),$$

for some $y_{i1}, y_{i2}, \dots, y_{iu_i} \in S$ and $z_1, z_2, \dots, z_m \in S$, where $\mathfrak{a}_i = (x_{i1}, x_{i2}, \dots, x_{iu_i})$ and $\mu_S(\mathfrak{a}_i) = u_i$.

Proof. $\mathfrak{a}_i$ is generated by an S-regular sequence, see [11, Theorem 2.3.3.]. □

Corollary 1.6.4. *With the notation of* Theorem 1.6.2 *suppose that* Condition (1) *holds true. Set $n = \dim S - \dim R$. We then have the following.*

(1) *If $n = 2$, then $r = 2$, $q = 3$, and $m = 0$.*

(2) *Suppose that S is a regular local ring, $I \subseteq \mathfrak{n}^2$, and R has maximal embedding dimension. Then $r = n$, $q = n^2 - 1$, and $m = 0$.*

Proof. Note that

$$q = \sum_{i=2}^{r} u_i + m \geq (r-1)(n+1) + m$$

since $n+1 = \mathrm{ht}_S \mathfrak{a}_i \leq u_i$.

(1) Since a minimal S-free resolution of R has the form $0 \to S^{\oplus r} \to S^{\oplus (r+1)} \to S \to R \to 0$, we have $q = r+1 \geq (r-1)(2+1)+m$. Since R is not a Gorenstein ring, we get $r=2$, $q=3$ and $m=0$.

(2) Set $e = \mathrm{e(R)}$. By [71, Theorem 1.(iii)], $n = e-1$, $r = e-1$, and $q = (e-2)e$. Therefore we have $q = (e-2)e \geq (e-2)e + m$, whence $m = 0$. $\square$

As a corollary, we have a characterization of one-dimensional generalized Gorenstein local rings in terms of minimal free resolution, which will be used in Section 1.7.

Corollary 1.6.5. *Let $(S, \mathfrak{n})$ be a Gorenstein local ring and $(R, \mathfrak{m})$ a one-dimensional Cohen-Macaulay local ring but not a Gorenstein ring. Let $\varphi : S \to R$ be a surjective ring homomorphism and suppose the projective dimension of R over S is finite. Let $\mathfrak{a}$ be an ideal of S such that $\mathfrak{a} \supseteq \mathrm{Ker}\,\varphi$ and set $n = \mu_S(\mathfrak{a})$ and $\mathfrak{a} = (x_1, x_2, \ldots, x_n)$. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring with respect to $\mathfrak{a}R$.*

(2) *There exists a minimal S-free resolution*

$$0 \to S^{\oplus r} \xrightarrow{\mathbb{M}} S^{\oplus q} \to \cdots \to S \to R \to 0$$

of R such that

$${}^t\mathbb{M} = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{**} & * \\ \hline x_1\ x_2\ \ldots\ x_n & & & & \\ & x_1\ x_2\ \ldots\ x_n & & \mathit{0} & \\ & & \ddots & & \mathit{0} \\ \mathit{0} & & & x_1\ x_2\ \ldots\ x_n & \end{array}\right),$$

where all components of $**$ and $*$ are in $\mathfrak{a}$.

Proof. (1) $\Rightarrow$ (2) By definition of generalized Gorenstein local rings with respect to $\mathfrak{a}R$, there exists an exact sequence

$$0 \to R \to \mathrm{K_R} \to (\mathrm{S}/\mathfrak{a})^{\oplus(\mathrm{r}-1)} \to 0.$$

Hence we can apply Theorem 1.6.2 and get the minimal S-free resolution of R which stated in Assertion (2). We have only to show that all components of $**$ and $*$ are in $\mathfrak{a}$. In fact, by taking S-dual, we have $S^{\oplus q} \xrightarrow{{}^t\mathbb{M}} S^{\oplus r} \to \mathrm{K_R} \to 0$. Hence, by applying the functor $S/\mathfrak{a} \otimes_S -$, we get the following exact sequence

$$(S/\mathfrak{a})^{\oplus q} \xrightarrow{\overline{{}^t\mathbb{M}}} (S/\mathfrak{a})^{\oplus r} \to \mathrm{K_R}/\mathfrak{a}\mathrm{K_R} \to 0.$$

Therefore ${}^t\overline{\mathbb{M}}$ is forced to be the zero matrix since $(S/\mathfrak{a})^{\oplus r} \cong (R/\mathfrak{a}R)^{\oplus r} \cong \mathrm{K_R}/\mathfrak{a}\mathrm{K_R}$.

(2) ⇒ (1) Thanks to Theorem 1.6.2, there exists a exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K_R} \to (\mathrm{S}/\mathfrak{a})^{\oplus(\mathrm{r}-1)} \to 0.$$

Hence $\varphi(1) \notin \mathfrak{m}\mathrm{K}_R$. Furthermore, similarly to the above proof, we get the exact sequence

$$(S/\mathfrak{a})^{\oplus q} \xrightarrow{{}^t\overline{\mathbb{M}}} (S/\mathfrak{a})^{\oplus r} \to \mathrm{K_R}/\mathfrak{a}\mathrm{K_R} \to 0.$$

Since ${}^t\mathbb{M}$ is a zero matrix, we have $(S/\mathfrak{a})^{\oplus r} \cong \mathrm{K_R}/\mathfrak{a}\mathrm{K_R}$. Hence R is a generalized Gorenstein local ring with respect to $\mathfrak{a}R$. □

1.7 Construction of determinantal generalized Gorenstein local rings

In this section we study how to construct generalized Gorenstein local rings obtained by determinantal ideals. Throughout this section, let $(S, \mathfrak{n})$ be a Noetherian local ring of $d = \dim S > 0$. For an ideal I of S and a finitely generated S-module M, $\mathrm{grade}(I, M)$ denotes the *grade of M in I* in the sense of [11, Definition 1.2.6.], that is, the length of the maximal M-regular sequence in I. We start at the following lemma which is well-known.

Lemma 1.7.1. *Let I be an ideal of S and $x \in \mathfrak{n}$ be a non-zerodivisor of S. Then*

$$\mathrm{grade}(I, S) \le \mathrm{grade}([I + (x)]/(x), S/(x)) \le \mathrm{grade}(I, S).$$

Proof. Set $g = \mathrm{grade}(I, S)$. By applying the functor $\mathrm{Hom}_S(S/I, *)$ to the exact sequence $0 \to S \xrightarrow{x} S \to S/(x) \to 0$, we have $\mathrm{Ext}^i_S(S/I, S/(x)) = 0$ for all $i \le g - 2$ and

$$0 \to \mathrm{Ext}^{g-1}_S(S/I, S/(x)) \to \mathrm{Ext}^g_S(S/I, S) \xrightarrow{x} \mathrm{Ext}^g_S(S/I, S) \to \mathrm{Ext}^g_S(S/I, S/(x)).$$

By Nakayama's lemma, at least either $\mathrm{Ext}^{g-1}_S(S/I, S/(x))$ or $\mathrm{Ext}^g_S(S/I, S/(x))$ does not vanish. Therefore $g - 1 \le \mathrm{grade}(I, S/(x)) = \mathrm{grade}([I + (x)]/(x), S/(x)) \le g$. □

Lemma 1.7.2. *For a positive integer $n > 0$, the following assertions hold true.*

(1) *Let $0 < \alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n$ be positive integers and $x_1, x_2, \dots, x_n \in \mathfrak{n}$ an S-regular sequence. Set*

$$I = I_2 \begin{pmatrix} x_1^{\alpha_1} & x_2^{\alpha_2} & \dots & x_{n-1}^{\alpha_{n-1}} & x_n^{\alpha_n} \\ x_2^{\beta_2} & x_3^{\beta_3} & \dots & x_n^{\beta_n} & x_1^{\beta_1} \end{pmatrix}.$$

Then $\mathrm{grade}(I, S) = n - 1$.

(2) *Let $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in \mathfrak{n}$ be an S-regular sequence. Set*

$$J = I_2 \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}.$$

Then $\mathrm{grade}(J, S) = n - 1$.

Proof. The hight of I and J are at most $n-1$ in general, thus we have only to show that the converse inequality.

(1) Note that $I + (x_1) = I_2 \begin{pmatrix} 0 & x_2^{\alpha_2} & \dots & x_{n-1}^{\alpha_{n-1}} & x_n^{\alpha_n} \\ x_2^{\beta_2} & x_3^{\beta_3} & \dots & x_n^{\beta_n} & 0 \end{pmatrix} + (x_1)$.

Set $I' = I_2 \begin{pmatrix} 0 & x_2^{\alpha_2} & \dots & x_{n-1}^{\alpha_{n-1}} & x_n^{\alpha_n} \\ x_2^{\beta_2} & x_3^{\beta_3} & \dots & x_n^{\beta_n} & 0 \end{pmatrix}$. We show that $x_2, x_3, \dots, x_n \in \sqrt{I'}$ by induction on $2 \le i \le n$. The cases where $i = 2$ and $i = n$ are clear. For the case of $2 < i < n$, suppose the assertion holds true for $i-1$. Then since $\det \begin{pmatrix} x_{i-1}^{\alpha_{i-1}} & x_i^{\alpha_i} \\ x_i^{\beta_i} & x_{i+1}^{\beta_{i+1}} \end{pmatrix} \in I'$, we have $x_i^{\alpha_i+\beta_i} \in I'$. Hence $n-1 \le \operatorname{grade}([I' + (x_1)]/(x_1), S/(x_1)) \le \operatorname{grade}(I, S)$.

(2) Set $y_0 = y_n$ and $Q = (x_i - y_{i-1} \mid 1 \le i \le n) + (x_1)$. Then, we have

$$[J + Q]/Q = [(x_2, x_3, \dots, x_n)^2 + Q]/Q.$$

Hence $n-1 \le \operatorname{grade}([J+Q]/Q, S/Q) \le \operatorname{grade}(J, S)$. □

Theorem 1.7.3. *Let S be a Gorenstein local ring and n be a positive integer with $3 \le n \le \dim S = d$. Assume that $x_1, x_2, \dots, x_d$ is a system of parameters of S. Set $Q = (x_1, x_2, \dots, x_n)$ and take elements $y_1, y_2, \dots, y_n \in Q$. Set*

$$I = I_2 \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}.$$

If $\operatorname{grade}(I, S) = n-1$, then $R = S/I$ is a generalized Gorenstein local ring with respect to $(x_1, x_2, \dots, x_d)R$.

Proof. We reduced the case where $n = d$. Assume that $n < d$. Then $S/(x_n, x_{n+1}, \dots, x_d)$ and $R/(x_n, x_{n+1}, \dots, x_d)R$ satisfy the same assertions of S and R. Hence, thanks to Theorem 1.4.6, we may assume that $n = d$. Then, by hypothesis, the Eagon-Northcott complex [18] gives a minimal S-free resolution of R. Remember that the Eagon-Northcott complex has the form

$$0 \to S^{\oplus r} \xrightarrow{\mathbb{M}} S^{\oplus n(n-2)} \to \cdots \to S \to R \to 0$$

of R such that

$$ {}^t\mathbb{M} = \begin{pmatrix} Y & & & & \\ X & Y & & 0 & \\ & X & & & \\ & & \ddots & & \\ & 0 & & X & Y \\ & & & & X \end{pmatrix},$$

where $X = (x_1 \;\; -x_2 \;\; x_3 \;\; \dots \;\; (-1)^{n-1}x_n)$ and $Y = (y_1 \;\; -y_2 \;\; y_3 \;\; \dots \;\; (-1)^{n-1}y_n)$ are submatrices of ${}^t\mathbb{M}$. Since $y_i \in Q$ for all $1 \le i \le n$, by taking fundamental transformation, we have

$$
{}^t\mathbb{M} \sim \left(\begin{array}{ccccc} Y & & * & & \\ \hline X & & & & \\ & X & & & \\ & & \ddots & & \text{\huge 0} \\ & & & X & \\ \text{\huge 0} & & & & X \end{array}\right) = \mathbb{N},
$$

where all components in $*$ are in Q. After replacing the basis of $S^{\oplus n(n-2)}$, we may assume that ${}^t\mathbb{M} = \mathbb{N}$. Therefore R is a generalized Gorenstein local ring with respect to QR by Corollary 1.6.5 $\square$

Corollary 1.7.4. *Let S be a Gorenstein local ring and n be a positive integer with $3 \leq n \leq \dim S = d$. Then the following assertions hold true.*

(1) *Let $0 < \alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n$ be positive integers and $x_1, x_2, \dots, x_n, z_1, z_2, \dots, z_{d-n} \in \mathfrak{n}$ a system of parameters of S. Set*

$$
I = I_2 \begin{pmatrix} x_1^{\alpha_1} & x_2^{\alpha_2} & \dots & x_{n-1}^{\alpha_{n-1}} & x_n^{\alpha_n} \\ x_2^{\beta_2} & x_3^{\beta_3} & \dots & x_n^{\beta_n} & x_1^{\beta_1} \end{pmatrix}.
$$

Then, $R_1 = S/I$ is a generalized Gorenstein local ring with respect to $(x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_n^{\alpha_n}, z_1, z_2, \dots, z_{d-n})R_1$ if and only if $\alpha_i \leq \beta_i$ for all $1 \leq i \leq n$.

(2) *Let $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_{d-2n} \in \mathfrak{n}$ be a system of parameters of S. Set*

$$
J = I_2 \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \end{pmatrix}.
$$

Then, $R_2 = S/J$ is a generalized Gorenstein local ring with respect to

$$
(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_{d-2n})R_2.
$$

Proof. (1) Thanks to Theorem 1.7.3 and Lemma 1.7.2, we have only to show that the only if part. By Lemma 1.7.2, the Eagon-Northcott complex induces the exact sequence $S^{\oplus n(n-2)} \xrightarrow{{}^t\mathbb{M}} S^{\oplus r} \to \mathrm{K_R} \to 0$, where

$$
{}^t\mathbb{M} = \begin{pmatrix} Y & & & & \\ X & Y & & \text{\huge 0} & \\ & X & & & \\ & & \ddots & & \\ & \text{\huge 0} & & X & Y \\ & & & & X \end{pmatrix}.
$$

Here, X and Y denotes the submatrices of ${}^t\mathbb{M}$ such that

$$X = (x_1^{\alpha_1} \ \ -x_2^{\alpha_2} \ x_3^{\alpha_3} \ \ \ldots \ (-1)^{n-1}x_n^{\alpha_n}) \text{ and}$$
$$Y = (x_2^{\beta_2} \ \ -x_3^{\beta_3} \ x_4^{\beta_4} \ \ \ldots \ (-1)^{n-2}x_n^{\beta_n} \ (-1)^{n-1}x_1^{\beta_1}).$$

Set $\mathfrak{a} = (x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_n^{\alpha_n}, z_1, z_2, \ldots, z_{d-n})$. Take the functor $S/\mathfrak{a} \otimes_S -$. Then, since $\mathrm{K_R}/\mathfrak{a}\mathrm{K_R}$ is $S/\mathfrak{a}$-free, we get all components in ${}^t\mathbb{M}$ are in $\mathfrak{a}$. Thus we come to the conclusion.

(2) Set $y_0 = y_n$ and $\mathfrak{q} = (x_i - y_{i-1} \mid 1 \leq i \leq n)$. Note that $\{x_i - y_{i-1} \mid 1 \leq i \leq n\}$ is a regular sequence of S and R_2. Hence $S/(J + \mathfrak{q})$ is a generalized Gorenstein local ring with respect to $(x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_{d-n}){\cdot}S/(J + \mathfrak{q})$ by (1). This implies that R_2 is also a generalized Gorenstein local ring with respect to $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \ldots, z_{d-2n})R_2$ by Theorem 1.4.6. □

We close this section with an application for Rees algebras which is a generalization of [46, Theorem 8.3.].

Corollary 1.7.5. (cf. [46, Theorem 8.3.]) *Let $(S, \mathfrak{n})$ be a Gorenstein local ring and $1 \leq n \leq \dim S = d$. Let $a_1, a_2, \ldots, a_n$ be a subsystem of parameters of S. Set $Q = (a_1, a_2, \ldots, a_n)$. We denote that*

$$\mathcal{R} = \mathcal{R}(Q) = S[a_1t, a_2t, \ldots, a_dt] \subseteq S[t]$$

is the Rees algebra of Q, where $S[t]$ is the polynomial ring over S. Then $\mathcal{R}_M$ is a generalized Gorenstein local ring, where $M = \mathfrak{n}\mathcal{R} + \mathcal{R}_+$ is the unique graded maximal ideal.

Proof. By [6], $\mathcal{R} \cong S[T_1, T_2, \ldots, T_n]/I_2\left(\begin{smallmatrix} T_1 & T_2 & \cdots & T_n \\ a_1 & a_2 & \cdots & a_n \end{smallmatrix}\right)$. This shows that $\mathcal{R}_M$ is a generalized Gorenstein local ring. □

1.8 Ulrich ideals and generalized Gorenstein local rings

As we showed in Corollary 1.5.28, there are relations between the notion of Ulrich ideals and the notion of generalized Gorenstein local rings. The purpose of this section is to study about Ulrich ideals again and generalize previous results. Throughout this section, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 0$, possessing the canonical module K_R. Let us summarize the some basic properties of Ulrich ideals, as seen in [43, 47].

Theorem 1.8.1 ([43, 47])**.** *Let I be an Ulrich ideal of R and set $n = \mu_R(I)$. Let*

$$\cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 = R \to R/I \to 0$$

be a minimal free resolution of R/I. Then $\mathrm{r}(R) = (n - d){\cdot}\mathrm{r}(R/I)$ and the following assertions hold true.

(1) *$I(\partial_i) = I$ for $i \geq 1$.*

(2) *For* $i \geq 0$, $\beta_i = \begin{cases} (n-d)^{i-d}\cdot(n-d+1)^d & (i \geq d), \\ \binom{d}{i} + (n-d)\cdot\beta_{i-1} & (1 \leq i \leq d), \\ 1 & (i = 0). \end{cases}$

(3) *For* $i \in \mathbb{Z}$, $\operatorname{Ext}_R^i(R/I, R) \cong \begin{cases} (0) & (i < d), \\ (R/I)^{\oplus(n-d)} & (i = d), \\ (R/I)^{\oplus\{(n-d)^2-1\}\cdot(n-d)^{i-(d+1)}} & (i > d). \end{cases}$

Here $I(\partial_i)$ *denotes the ideal of* R *generated by the entries of the matrix* ∂_i, *and* $\beta_i = \mathrm{rank}_R F_i$.

Next, let us introduce the notion of trace ideals.

Definition 1.8.2. For an R-module M, let

$$t_M : \operatorname{Hom}_R(M, R) \otimes_R M \to R$$

denote the R-linear map defined by $t_M(f \otimes m) = f(m)$ for all $f \in \operatorname{Hom}_R(M, R)$ and $m \in M$. Then, $\mathrm{tr}_R(M) = \mathrm{Im} t_M$ is called the *trace of* M.

In this chapter, we focus on the trace of the canonical module $\mathrm{tr}_R(\mathrm{K}_R)$. Let us note some properties of the trace of the canonical module.

Remark 1.8.3. The following assertions are true.

(1) $\mathrm{tr}_R(\mathrm{K}_R)$ describes non-Gorenstein locus, that is, for $\mathfrak{p} \in \operatorname{Spec} R$, $R_\mathfrak{p}$ is not a Gorenstein ring if and only if $\mathrm{tr}_R(\mathrm{K}_R) \subseteq \mathfrak{p}$.

(2) If there exists a canonical ideal L, then $\mathrm{tr}_R(\mathrm{K}_R) = (R : L)L$. In particular, if R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$ of dimension one, then $\mathfrak{a} = \mathrm{tr}_R(\mathrm{K}_R)$.

Proof. (1) For instance, see [55, Lemma 2.1.].

(2) Since canonical ideal L contains a non-zerodivisor of R, we have a natural isomorphism $\operatorname{Hom}_R(L, R) \cong R : L$. Therefore $\mathrm{tr}_R(\mathrm{K}_R) = \mathrm{Im} t_L = (R : L)L$. Latter statement follows from Lemma 1.5.7 and Theorem 1.5.8. □

Let us give the relation between Ulrich ideals and the trace of the canonical module.

Theorem 1.8.4. *Suppose that* $(R, \mathfrak{m})$ *is a generically Gorenstein local ring but not a Gorenstein ring. Let* I *be an Ulrich ideal such that* $\mu_R(I) > d + 1$. *Then* $\mathrm{tr}_R(\mathrm{K}_R) \subseteq I$.

Proof. Since R is a generically Gorenstein local ring, there exists a canonical ideal $L \subsetneq R$. For any R-regular element $f \in L$, we get an exact sequence

$$0 \to R \to L \to L/(f) \to 0$$

of R-modules. Thanks to Theorem 1.8.1, by applying the functor $\operatorname{Hom}_R(R/I, *)$, we have

$$\operatorname{Ext}_R^i(R/I, L/(f)) \cong \operatorname{Ext}_R^{i+1}(R/I, R) \cong (R/I)^{\oplus u_i}$$

for $i > d$, where $u_i = \{(\mu_R(I) - d)^2 - 1\}(\mu_R(I) - d)^{i-d-1} > 0$. This implies that

$$\sum_{f \in L \text{ R-regular}} (f) :_R L \subseteq I.$$

We show that

$$\mathrm{tr}_R(\mathrm{K}_R) = \sum_{f \in L \text{ R-regular}} (f) :_R L.$$

In fact, by Davis's lemma, we can take non-zerodivisors $f_1, f_2, \dots, f_r$ of R such that $L = (f_1, f_2, \dots, f_r)$, where $r = \mathrm{r}(R)$. Therefore, since $(f) :_R L = f(R : L)$, we have

$$\begin{aligned}\sum_{f \in L \text{ R-regular}} (f) :_R L &= \sum_{f \in L \text{ R-regular}} f(R : L) \\ &= \sum_{i=1}^{r} f_i(R : L) \\ &= L(R : L) = \mathrm{tr}_R(\mathrm{K}_R).\end{aligned}$$

□

Combining [47, Theorem 2.8.] and Theorem 1.8.4 yields the following the result.

Corollary 1.8.5. *Suppose that R is G-regular, that is, every totally reflexive module is free. If I is an Ulrich ideal, then* $\mathrm{tr}_R(\mathrm{K}_R) \subseteq I$.

Corollary 1.8.5 provides the question when $\mathrm{tr}_R(\mathrm{K}_R)$ is an Ulrich ideal. Corollary 1.5.28 and Remark 1.8.3 say that $\mathrm{tr}_R(\mathrm{K}_R)$ is an Ulrich ideal if R is a non-Gorenstein generalized Gorenstein local ring of dimension one and S is a Gorenstein ring. For a while, we focus the case where $\dim R = 1$. We assume that Setting 1.5.3 unless otherwise noted. Note that $\mathrm{tr}_R(\mathrm{K}_R) = \mathfrak{a}K$ by Remark 1.8.3.

Lemma 1.8.6. *Suppose that $\mathrm{tr}_R(\mathrm{K}_R)$ is stable, that is, $\mathrm{tr}_R(\mathrm{K}_R)^2 = \alpha{\cdot}\mathrm{tr}_R(\mathrm{K}_R)$ for some non-zerodivisor $\alpha \in \mathrm{tr}_R(\mathrm{K}_R)$ of R. Then $\mathrm{tr}_R(\mathrm{K}_R) = \mathfrak{c}$.*

Proof. Since $\mathrm{tr}_R(\mathrm{K}_R) = \mathfrak{a}K$, $\mathrm{tr}_R(\mathrm{K}_R) : \mathrm{tr}_R(\mathrm{K}_R) = \mathfrak{a}K : \mathfrak{a}K = (\mathfrak{a}K)^n : (\mathfrak{a}K)^n$ for all $n > 0$. Since $\mathfrak{a}^n S = (\mathfrak{a}K)^n$ for $n \gg 0$, we have $\mathrm{tr}_R(\mathrm{K}_R)S \subseteq \mathrm{tr}_R(\mathrm{K}_R)$. This shows that $\mathrm{tr}_R(\mathrm{K}_R)$ is an ideal of S, whence $\mathrm{tr}_R(\mathrm{K}_R) \subseteq \mathfrak{c}$. The converse inclusion is clear. □

The following improves Corollary 1.5.28.

Theorem 1.8.7. *Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) $\mathrm{tr}_R(\mathrm{K}_R)$ *is an Ulrich ideal of R.*

(2) $\mathfrak{c}$ *is an Ulrich ideal of R.*

(3) *R is a generalized Gorenstein local ring and S is a Gorenstein ring.*

In particular, if $\mathrm{tr}_R(\mathrm{K}_R)$ is an Ulrich ideal of R, then R is a generalized Gorenstein local ring with respect to $\mathrm{tr}_R(\mathrm{K}_R)$.

Proof. (2) $\Leftrightarrow$ (3) is proven in Corollary 1.5.28.

(2) $\Rightarrow$ (1) This follows from the fact of $\mathrm{tr}_R(\mathrm{K}_R) = \mathfrak{c}$ by Lemma 1.5.7 and the equivalence (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2) This follows from Lemma 1.8.6. □

Let us give examples of Theorem 1.8.7.

Proposition 1.8.8. *Let $(S, \mathfrak{n})$ be a Gorenstein local ring and $3 \le n = \dim S$. Take $x_1, x_2, \ldots, x_n \in \mathfrak{n}$ a system of parameters of S and set*

$$I = I_2 \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_2 & x_3 & \cdots & x_n & x_1 \end{pmatrix}$$

and $R = S/I$. Then we have the following.

(1) *R is a one-dimensional generalized Gorenstein local ring with respect to $(x_1, x_2, \ldots, x_n)R$.*

(2) *$\mathrm{tr}_R(\mathrm{K}_R) = (x_1, x_2, \ldots, x_n)R$ is an Ulrich ideal of R.*

Proof. (1) This follows from Corollary 1.7.4.

(2) By (1), we have the equality $\mathrm{tr}_R(\mathrm{K}_R) = (x_1, x_2, \ldots, x_n)R$ since Lemma 1.5.7. Set $J = \mathrm{tr}_R(\mathrm{K}_R)$. Then, we have

$$J^2 = \overline{x_1}J + (\overline{x_2}, \overline{x_3}, \ldots, \overline{x_n})^2 = \overline{x_1}J$$

since $\overline{x_i x_j} = \overline{x_{i-1}x_{j+1}}$ for all $2 \le i \le j \le n$, where $\overline{x}$ denotes the image of $x \in S$ in R and $\overline{x_{n+1}} = \overline{x_1}$. Therefore, by Lemma 1.5.21 and Theorem 1.8.7, we have J is an Ulrich ideal of R. □

Proposition 1.8.9. *Let $R = k[[t^{a_1}, t^{a_2}, t^{a_3}]]$ be a numerical semigroup ring over a field k. With the notation of Theorem 1.5.34 suppose that $a > 0$. Then the following conditions are equivalent.*

(1) *$\mathrm{tr}_R(\mathrm{K}_R)$ is an Ulrich ideal.*

(2) *Two of the three pairs (α, α'), (β, β'), and (γ, γ') are equal.*

When this is the case, after renumbering, we have the equalities:

$$a_1 = 3\beta\gamma,\ a_2 = \gamma(2\alpha + \alpha'),\ \text{and}\ a_3 = \beta(2\alpha' + \alpha).$$

Proof. (2) ⇒ (1) After suitable permutation of a_1, a_2, and a_3 if necessary, we may assume that $\alpha < \alpha', \beta = \beta', \gamma = \gamma'$. Then R is a generalized Gorenstein local ring and $\mathrm{tr}_R(\mathrm{K}_R) = (t^{\alpha a_1}, t^{\beta a_2}, t^{\gamma a_3})$ by Theorem 1.5.34. It is straightforward that $\mathrm{tr}_R(\mathrm{K}_R)^2 = t^{a_1 \cdot \alpha} {\cdot} \mathrm{tr}_R(\mathrm{K}_R)$. Hence $\mathrm{tr}_R(\mathrm{K}_R) = R : R[K]$ by Lemma 1.8.6, where $K = R + Rt^a$. Hence $R[K]$ is a Gorenstein ring by Lemma 1.5.21. Therefore $\mathrm{tr}_R(\mathrm{K}_R)$ is an Ulrich ideal by Theorem 1.8.7.

(1) ⇒ (2) We have the equalities

$$\alpha \le \alpha',\ \beta \le \beta',\ \gamma \le \gamma', \text{ and } \mathrm{tr}_R(\mathrm{K}_R) = (t^{\alpha a_1}, t^{\beta a_2}, t^{\gamma a_3})$$

by Theorem 1.5.34. We may assume that $(t^{a_1 \cdot \alpha})$ is a reduction of $\mathrm{tr}_R(\mathrm{K}_R)$ after renumbering. Then,

$$R[K] = K^2 = \left\langle 1, t^a, t^{2a} \right\rangle = \frac{\mathrm{tr}_R(\mathrm{K}_R)}{t^{a_1 \cdot \alpha}} = \left\langle 1, t^{a_2 \cdot \beta - a_1 \cdot \alpha}, t^{a_3 \cdot \gamma - a_1 \cdot \alpha} \right\rangle,$$

by Theorem 1.5.8. Therefore $\begin{cases} a_2{\cdot}\beta - a_1{\cdot}\alpha = 2a \\ a_3{\cdot}\gamma - a_1{\cdot}\alpha = a \end{cases}$ or $\begin{cases} a_2{\cdot}\beta - a_1{\cdot}\alpha = a \\ a_3{\cdot}\gamma - a_1{\cdot}\alpha = 2a \end{cases}$.
Assume the former case. Then, since $a = n - m = a_1\alpha' - a_3\gamma$ by (1.5.33.1), we have

$$2{\cdot}a_3\gamma = a_1(\alpha + \alpha') = a_2\beta' + a_3\gamma,$$

where the last equality follows from $X^{\alpha+\alpha'} - Y^{\beta'}Z^{\gamma} \in \operatorname{Ker}\varphi$. Therefore $a_2\beta' = a_3\gamma$, whence $Y^{\beta'} - Z^{\gamma} \in \operatorname{Ker}\varphi$. This is a contradiction for the construction of β and γ since R is not a Gorenstein ring (see [51]). Hence the latter case holds. Then, since $a_2{\cdot}\beta - a_1{\cdot}\alpha = a = a_2\beta' - a_1\alpha$, $\beta = \beta'$. Similarly, we have $\gamma = \gamma'$ since

$$a_3{\cdot}\gamma - a_1{\cdot}\alpha = 2a = (a_2\beta' - a_1\alpha) + (a_3\gamma' - a_2\beta).$$

Equalities for a_1, a_2, and a_3 follow from the general equalities that

$$a_1 = \beta\gamma + \beta'\gamma' + \beta'\gamma,\ a_2 = \alpha\gamma + \alpha\gamma' + \alpha'\gamma', \text{ and } a_3 = \alpha'\beta' + \alpha'\beta + \alpha\beta,$$

see the proof of Corollary 1.5.35. □

Now we are back to the setting that $(R, \mathfrak{m})$ is an arbitrary Cohen-Macaulay local ring, possessing the canonical module K_R. Set $d = \dim R > 0$.

Corollary 1.8.10. *Suppose that the residue field $R/\mathfrak{m}$ is infinite. If $\mathrm{tr}_R(\mathrm{K}_R)$ is an Ulrich ideal of R with $\mu_R(\mathrm{tr}_R(\mathrm{K}_R)) > d + 1$, then R is a generalized Gorenstein local ring with respect to $\mathrm{tr}_R(\mathrm{K}_R)$. When this is the case, $R/\mathrm{tr}_R(\mathrm{K}_R)$ is a Gorenstein ring and $\mu_R(\mathrm{tr}_R(\mathrm{K}_R)) = d + \mathrm{r}(R)$.*

Proof. Set $J = \mathrm{tr}_R(\mathrm{K}_R)$. We prove by induction on d. The case where $d = 1$ is proven in Theorem 1.8.7. Let $d > 1$ and assume that our assertion holds true for $d - 1$. Since $R/\mathfrak{m}$ is infinite, we can choose a parameter ideal $Q = (f = f_1, f_2, \dots, f_d)$ as a minimal reduction of J. Set $\overline{*} = R/(f) \otimes_R *$. Then $t_{\mathrm{K}_R} : \mathrm{Hom}_R(\mathrm{K}_R, R) \otimes_R \mathrm{K}_R \to R$ induces

$$\overline{t_{\mathrm{K}_R}} : \overline{\mathrm{Hom}_R(\mathrm{K}_R, R)} \otimes_{\overline{R}} \mathrm{K}_{\overline{R}} \to \overline{R}.$$

Hence $\mathrm{tr}_R(\mathrm{K}_R)\overline{R} = \mathrm{Im}\overline{t_{\mathrm{K}_R}} \subseteq \sum_{f \in \mathrm{Hom}_{\overline{R}}(\mathrm{K}_{\overline{R}}, \overline{R})} \mathrm{Im} f = \mathrm{tr}_{\overline{R}}(\mathrm{K}_{\overline{R}})$. On the other hand, $J\overline{R}$ is an Ulrich ideal of $\overline{R}$ since [43, Lemma 3.3.] and $\mu_{\overline{R}}(J\overline{R}) > (d-1)-1$. Hence, thanks to Theorem 1.8.4, $J\overline{R} = \mathrm{tr}_{\overline{R}}(\mathrm{K}_{\overline{R}})$. Therefore $\overline{R}$ is a generalized Gorenstein local ring with respect to $\mathrm{tr}_{\overline{R}}(\mathrm{K}_{\overline{R}})$, whence R is also a generalized Gorenstein local ring with respect to $\mathrm{tr}_R(\mathrm{K}_R)$ by Theorem 1.4.6. Thus $R/\mathrm{tr}_R(\mathrm{K}_R)$ is a Gorenstein ring by Corollary 1.5.13 and $\mu_R(\mathrm{tr}_R(\mathrm{K}_R)) = d + \mathrm{r}(R)$ by Theorem 1.8.1. □

Let us give an application for nearly Gorenstein local rings which is defined by J. Herzog, T. Hibi, and D. I. Stamate.

Definition 1.8.11. ([55, Definition 2.2.]) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring possessing K_R. Then R is called a *nearly Gorenstein local ring* if $\mathrm{tr}_R(\mathrm{K}_R) \supseteq \mathfrak{m}$.

As a direct consequence of Corollary 1.8.10, we have the following which generalizes the result of J. Herzog, T. Hibi, and D. I. Stamate ([55, Theorem 7.4.]).

Corollary 1.8.12. *Suppose that R is a nearly Gorenstein ring and R has maximal embedding dimension. Then R is an almost Gorenstein local ring if the residue field $R/\mathfrak{m}$ is infinite.*

Proof. We may assume that R is not a Gorenstein ring. Then, $\mathrm{tr}_R(\mathrm{K}_R) = \mathfrak{m}$ is an Ulrich ideal and $\mathrm{v}(R) = \mathrm{e}(R) = d + \mathrm{r}(R) > d+1$ by Lemma 1.5.17. Therefore R is a generalized Gorenstein local ring with respect to $\mathfrak{m}$, that is, an almost Gorenstein local ring. □

We give examples of Corollary 1.8.10.

Example 1.8.13. Let $(S, \mathfrak{n})$ be a Gorenstein local ring of $\dim S = 4$ and $x_1, x_2, x_3, x_4 \in \mathfrak{n}$ a system of parameters of S. Set

$$I = \mathrm{I}_2 \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$$

and $R = S/I$. Then $\mathrm{tr}_R(\mathrm{K}_R) = (x_1, x_2, x_3, x_4)R$ is an Ulrich ideal of R. Therefore R is a generalized Gorenstein local ring with respect to $\mathrm{tr}_R(\mathrm{K}_R)$.

Proof. By [55, Corollary 3.4.] and Hilbert-Burch's theorem, we have $\mathrm{tr}_R(\mathrm{K}_R) = (x_1, x_2, x_3, x_4)R$. Set $J = (x_1, x_2, x_3, x_4)R$ and $\overline{*}$ denotes the image of $* \in S$ in R. Then

$$J^2 = (\overline{x_1}, \overline{x_4})J + (\overline{x_2}, \overline{x_3})^2 = (\overline{x_1}, \overline{x_4})J.$$

Furthermore, we have

$$\begin{aligned} \mathrm{e}_J^0(R) =& \ell_R(R/(x_1, x_4)R) \\ =& \ell_S(S/I + (x_1, x_4)) \\ =& \ell_S(S/[(x_2, x_3)^2 + (x_1, x_4)]) \\ =& \ell_{S'}(S'/(x_2, x_3)^2 S') \qquad \text{where } S' = S/(x_1, x_4) \\ =& 3 \cdot \ell_{S'}(S'/(x_2, x_3)S') \\ =& 3 \cdot \ell_R(R/JR), \end{aligned}$$

where the fifth equality follows from the fact that $(x_2, x_3)S'$ is a parameter ideal of S'. Hence I is an Ulrich ideal of R by [43, Lemma 2.3.]. □

Next purpose is to determine the set of all Ulrich ideals of generalized Gorenstein local rings of dimension one. Let us remember Theorem 1.5.18, which rephrased in terms of the trace of the canonical module.

Theorem 1.8.14 (Theorem 1.5.18)**.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring possessing the canonical module K_R. Assume that* $\dim R = 1$ *and* Setting 1.5.3*. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring but not an almost Gorenstein local ring and* $\mathrm{v}(R) = \mathrm{e}(R)$*.*

(2) *$B = \mathrm{Hom}_R(\mathfrak{m}, \mathfrak{m})$ is a generalized Gorenstein local ring with* $\mathrm{v}(B) = \mathrm{e}(B) = \mathrm{e}(R)$*, but not a Gorenstein ring.*

When this is the case, there exists an element such that $\mathfrak{m}^2 = \alpha\mathfrak{m}$ and we have the following.

(i) *$R/\mathfrak{m} \cong B/\mathfrak{n}$,*

(ii) *$\ell_B(B/\mathrm{tr}_B(\mathrm{K}_B)) = \ell_R(R/\mathrm{tr}_R(\mathrm{K}_R)) - 1$, and*

(iii) *$\mathfrak{n}^2 = \alpha\mathfrak{n}$.*

Here $\mathfrak{n}$ denotes the unique maximal ideal of B.

Proof. Note that the existence of a minimal reduction (α) of $\mathfrak{m}$ follows from [67, Corollary 1.10]. □

We are now back to the case of dimension one. In what follows, we assume Setting 1.5.3. Due to Theorem 1.8.14, we have the following which is the heart of Theorem 1.8.18.

Proposition 1.8.15. *Suppose that R is a generalized Gorenstein local ring but not a Gorenstein ring. Assume that $\mathfrak{m}^2 = \alpha\mathfrak{m}$ for some element $\alpha \in \mathfrak{m}$. Set $v = \mathrm{v}(R)$ and $N = \ell_R(R/\mathrm{tr}_R(\mathrm{K}_R)) > 0$. Then there exist elements $x_2, x_3, \ldots, x_v \in \mathfrak{m}$ which satisfies the following two conditions.*

(1) *$\mathfrak{m} = (\alpha, x_2, x_3, \ldots, x_v)$.*

(2) *$\mathrm{tr}_R(\mathrm{K}_R) = (\alpha^N, x_2, x_3, \ldots, x_v)$ and (α^N) is a minimal reduction of* $\mathrm{tr}_R(\mathrm{K}_R)$*.*

Proof. We prove by induction on $N > 0$. The case where $N = 1$ is trivial since $\mathrm{tr}_R(\mathrm{K}_R) = \mathfrak{m}$. Let $N > 1$ and assume that our assertion holds true for $N - 1$. Then, $B = \mathfrak{m} : \mathfrak{m}$ is also a generalized Gorenstein local ring but not a Gorenstein ring. Let $\mathfrak{n}$ be the unique maximal ideal of B. Then due to Theorem 1.8.14 and induction hypothesis, there exist elements $y_2, y_3, \ldots, y_v \in \mathfrak{n}$ which satisfy the following conditions:

(1) $\mathfrak{n} = B\alpha + \sum_{i=2}^{v} By_i$.

(2) $\mathrm{tr}_B(\mathrm{K}_B) = B\alpha^{N-1} + \sum_{i=2}^{v} By_i$ and $B\alpha^{N-1}$ is a minimal reduction of $\mathrm{tr}_B(\mathrm{K}_B)$.

We need to prove the following.

Claim. *The following assertions hold true.*

(i) $\mathrm{tr}_B(\mathrm{K}_B) = \frac{1}{\alpha}\mathrm{tr}_R(\mathrm{K}_R)$.

(ii) $\mathfrak{m} = R\alpha + \sum_{i=2}^{v} R\alpha y_i$.

Proof of Claim. (i) Since B is a generalized Gorenstein local ring and due to [15, Proposition 5.1.], we have $\mathrm{tr}_B(\mathrm{K}_B) = B : S = \frac{\mathfrak{m}}{\alpha} : S = \frac{1}{\alpha}(\mathfrak{m} : S)$. Note that $\mathfrak{c} = R : S = \mathfrak{c} : S$ since $\mathfrak{c} = \mathfrak{c}S$. Therefore $\mathrm{tr}_B(\mathrm{K}_B) = \frac{1}{\alpha}\mathfrak{c} = \frac{1}{\alpha}\mathrm{tr}_R(\mathrm{K}_R)$.

(ii) Since $\mathfrak{n}^2 = \alpha\mathfrak{n} \subseteq \alpha B = \mathfrak{m}$, $\mathfrak{n}/\mathfrak{m}$ is a $B/\mathfrak{n}$-vector space, whence $\mathfrak{n}/\mathfrak{m} = \sum_{i=2}^{v} B/\mathfrak{n}{\cdot}y_i$. Due to Theorem 1.8.14, we have a natural isomorphism $R/\mathfrak{m} \cong B/\mathfrak{n}$, thus $\mathfrak{n}/\mathfrak{m} = \sum_{i=2}^{v} R/\mathfrak{m}{\cdot}y_i$. We have $\mathfrak{n} = \sum_{i=2}^{v} Ry_i + \mathfrak{m}$, and $\alpha\mathfrak{n} = \sum_{i=2}^{v} R\alpha y_i + \alpha\mathfrak{m}$. On the other hand, since $\mathfrak{m}/\alpha\mathfrak{n} \cong B/\mathfrak{n} \cong R/\mathfrak{m}$ and $\alpha \in \mathfrak{m}\backslash\alpha\mathfrak{n}$, we have $\mathfrak{m} = \alpha\mathfrak{n} + \alpha R$. Therefore $\mathfrak{m} = \sum_{i=2}^{v} R\alpha y_i + \alpha\mathfrak{m} + R\alpha$. This concludes the claim by Nakayama's lemma. ☐

Set $J = R\alpha^N + \sum_{i=2}^{v} R\alpha y_i$. By Claim, $\mathrm{tr}_R(\mathrm{K}_R) = B\alpha^N + \sum_{i=2}^{v} B\alpha y_i \supseteq J$ and $\ell_R(R/J) \leq N$. Hence we have $\mathrm{tr}_R(\mathrm{K}_R) = J$. It remains to show that $\alpha^N R$ is a reduction of $\mathrm{tr}_R(\mathrm{K}_R)$. In fact, $\alpha^{N-1}R \subseteq \alpha^{N-1}B \subseteq \mathrm{tr}_B(\mathrm{K}_B) = \frac{1}{\alpha}\mathrm{tr}_R(\mathrm{K}_R) \subseteq \alpha^{N-1}\overline{B} = \alpha^{N-1}\overline{R}$. Hence $\alpha^N R \subseteq \mathrm{tr}_R(\mathrm{K}_R) \subseteq \alpha^N\overline{R} \cap R$, this implies that $\alpha^N R$ is a reduction of $\mathrm{tr}_R(\mathrm{K}_R)$. ☐

As a corollary, we have the following.

Corollary 1.8.16. *Suppose that R has a maximal embedding dimension. If R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$, then $\mathrm{v}(R/\mathfrak{a}) \leq 1$. In particular, $R/\mathfrak{a}$ is a complete intersection.*

Note that Corollary 1.8.16 not necessarily true without the assumption that R has maximal embedding dimension. For instance, see Proposition 1.8.8.

Lemma 1.8.17. *Let A be a commutative ring. Suppose that Q_0, Q, and J are ideals of A. Set*

$$I_0 = Q_0 + J \quad \textit{and} \quad I = Q + J.$$

Assume that $Q_0 \subseteq Q$. Then, $I_0^{m+1} = Q_0 I_0^m$ implies $I^{m+1} = QI^m$.

Proof. Since $I_0^{m+1} = Q_0{I_0}^m + J^{m+1} = Q_0{I_0}^m$, we have $J^{m+1} \subseteq Q_0{I_0}^m \subseteq QI^m$. Therefore $I^{m+1} = QI^m + J^{m+1} = QI^m$. ☐

We are now reach one of the main results of this chapter, which completely determines the set of Ulrich ideals via the notion of generalized Gorenstein local rings. Let $\mathcal{X}_R$ denote the set of all Ulrich ideals.

Theorem 1.8.18. *Suppose that R is not a Gorenstein ring. Set $v = \mathrm{v}(R)$ and $N = \ell_R(R/\mathrm{tr}_R(\mathrm{K}_R)) > 0$. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring possessing maximal embedding dimension.*

(2) *$\mathrm{tr}_R(\mathrm{K}_R)$ and $\mathfrak{m}$ are Ulrich ideals.*

(3) *R is G-regular and a length of a maximal chain of Ulrich ideals is $N-1$.*

(4) *There exist elements $\alpha, x_2, x_3, \ldots, x_v \in \mathfrak{m}$ which satisfy the following two conditions.*

- (i) $\mathfrak{m} = (\alpha, x_2, x_3, \ldots, x_v)$ *and*
- (ii) $\mathcal{X}_R = \{(\alpha^i, x_2, x_3, \ldots, x_v) \mid 1 \leq i \leq N\}$.

Proof. (4) $\Rightarrow$ (3) Set $I_i = (\alpha^i, x_2, x_3, \ldots, x_v)$ and $R_i = R/I_i$ for $1 \leq i \leq N$. We need to show that $I_{i+1} \subsetneq I_i$ for all $1 \leq i \leq N-1$. Assume that $I_i = I_{i+1}$ for some $1 \leq i \leq N-1$. Then, $\alpha^i = c_1\alpha^{i+1} + \sum_{j=2}^{v} c_j x_j$ for some $c_1, c_2, \ldots, c_v \in R$. This shows that $\alpha^i \in (x_2, x_3, \ldots, x_v)$ since $1 - c_1\alpha$ is an unit element in R. On the other hand, R_i is a complete intersection by Corollary 1.8.16. Therefore we have

$$\mu_R(I_i) = 1 + \mathrm{r}(R) = \mathrm{e}(R) = v$$

by Lemma 1.5.17 and Theorem 1.8.1. This is a contradiction. Therefore

$$I_N \subsetneq I_{N-1} \subsetneq \cdots \subsetneq I_1 = \mathfrak{m}$$

is a chain of Ulrich ideals, and R is G-regular since $\mathfrak{m} \in \mathcal{X}_R$ by [81, Corollary 2.5.].

(3) $\Rightarrow$ (2) Take $J_0, J_1, \ldots, J_{N-1} \in \mathcal{X}_R$ so that $R \supsetneq J_0 \supsetneq J_1 \supsetneq \cdots \supsetneq J_{N-1}$. Then $\ell_R(R/J_{N-1}) \geq N$. Remember that $J_{N-1} \supseteq \mathrm{tr}_R(\mathrm{K}_R)$ by Corollary 1.8.5, whence we have $J_{N-1} = \mathrm{tr}_R(\mathrm{K}_R)$ and $J_0 = \mathfrak{m}$.

(1) $\Leftrightarrow$ (2) This follows from Corollary 1.5.29 and Theorem 1.8.7.

(1) $\Rightarrow$ (4) By Proposition 1.8.15, there exist elements $x_2, x_3, \ldots, x_v \in \mathfrak{m}$ which satisfy the following conditions.

(a) $\mathfrak{m} = (\alpha, x_2, x_3, \ldots, x_v)$ and

(b) $\mathrm{tr}_R(\mathrm{K}_R) = (\alpha^N, x_2, x_3, \ldots, x_v)$ and (α^N) is a minimal reduction of $\mathrm{tr}_R(\mathrm{K}_R)$.

Therefore, since R has maximal embedding dimension, we have

$$\mathcal{X}_R \subseteq \left\{(\alpha^i, x_2, x_3, \ldots, x_v) \mid 1 \leq i \leq N\right\}$$

by Corollary 1.8.5 and [81, Corollary 2.5.]. Set $I_i = (\alpha^i, x_2, x_3, \ldots, x_v)$ for $1 \leq i \leq N$. Note that $I_{i+1} \subsetneq I_i$ and $\mu_R(I_i) = v$ for all $1 \leq i \leq N-1$ since $I_N = \mathrm{tr}_R(\mathrm{K}_R)$ and $\mu_R(\mathrm{tr}_R(\mathrm{K}_R)) = 1 + \mathrm{r}(R) = v$ by Theorem 1.5.27. By Lemma 1.8.17, $I_i^2 = \alpha^i I_i$. To show that I_i is an Ulrich ideal for all $1 \leq i \leq N$, we have only to show that I_i/I_i^2 is an R/I_i-free module for $1 \leq i \leq N$. This is equivalent to showing that $I_i/(\alpha^i)$ is an R/I_i-free module for $1 \leq i \leq N$ since the following exact sequence

$$0 \to (\alpha^i)/I_i^2 \to I_i/I_i^2 \to I_i/(\alpha^i) \to 0$$

and $(\alpha^i)/I_i^2 = (\alpha^i)/\alpha^i I_i \cong R/I_i$. We show R/I_i-freeness of $I_i/(\alpha^i)$ by descending induction on $1 \leq i \leq N$. The case where $i = N$ is trivial. Let $1 \leq i < N$ and assume that our assertion holds true for $i+1$. Then

$$\begin{aligned}\ell_R(I_i/(\alpha^i)) =& \ell_R(R/(\alpha^{i+1})) - \ell_R(R/I_i) - \ell_R((\alpha^i)/(\alpha^{i+1}))\\ =& [\ell_R(R/I_{i+1}) + \ell_R(I_{i+1}/(\alpha^{i+1}))] - \ell_R(R/I_i) - \ell_R(R/(\alpha))\\ =& [(i+1) + (i+1)(v-1)] - i - \mathrm{e}(R)\\ =& i(v-1).\end{aligned}$$

Thus $I_i/(\alpha^i)$ is an R/I_i-free module since there is a surjection $(R/I_i)^{\oplus(v-1)} \to I_i/(\alpha^i)$. □

Corollary 1.8.19. *Suppose that* $\mathrm{e}(R) = \mathrm{v}(R) = 3$ *and set* $v = \mathrm{v}(R)$. *Then there exist elements* $\alpha, x_2, x_3, \dots, x_v \in \mathfrak{m}$ *which satisfy the following two conditions.*

(i) $\mathfrak{m} = (\alpha, x_2, x_3, \dots, x_v)$ *and*

(ii) $\mathcal{X}_R = \{(\alpha^i, x_2, x_3, \dots, x_v) \mid 1 \le i \le \ell_R(R/\mathrm{tr}_R(\mathrm{K}_R))\}$.

Proof. By Corollary 1.5.12, R is a generalized Gorenstein local ring if $\mathrm{e}(R) = \mathrm{v}(R) = 3$. □

We close this chapter with the following examples.

Example 1.8.20. Let k be a field and $0 < a_1, a_2, \dots, a_\ell \in \mathbb{Z}$ ($\ell > 0$) be positive integers such that GCD $(a_1, a_2, \dots, a_\ell) = 1$. Then $R = k[[t^{a_1}, t^{a_2}, \dots, t^{a_\ell}]]$ is a Cohen-Macaulay local ring with $\dim R = 1$, and the following assertions hold true.

(1) Let $R_1 = k[[t^5, t^6, t^8]]$. Then $\mathrm{tr}_{R_1}(\mathrm{K}_{R_1}) = (t^{10}, t^6, t^8)$ is an Ulrich ideal of R_1. But R_1 does not have maximal embedding dimension.

(2) Let $R_2 = k[[t^5, t^{18}, t^{26}, t^{34}, t^{42}]]$. Then $\mathrm{tr}_{R_2}(\mathrm{K}_{R_2}) = (t^{10}, t^{18}, t^{26}, t^{34}, t^{42})$ is an Ulrich ideal of R_2. Moreover R_2 has maximal embedding dimension. Hence $\mathcal{X}_{R_2} = \{\mathrm{tr}_{R_2}(\mathrm{K}_{R_2}), \mathfrak{m}\}$.

Chapter 2

A Characterization of generalized Gorenstein rings

2.1 Introduction

The notion of generalized Gorenstein local ring is one of the generalizations of Gorenstein rings. Similarly for the almost Gorenstein local rings, the notion is given in terms of a certain specific embedding of the rings into their canonical modules (see the article [34]). However, the research of almost Gorenstein local rings developed by the article [36] of S. Goto, N. Matsuoka, and T. T. Phuong [36] for arbitrary one-dimensional Cohen-Macaulay local rings is based on the investigation of the relationship between the two invariants; the first Hilbert coefficient of canonical ideals and the Cohen-Macaulay type of the rings. Therefore, it seems natural to ask for a possible characterization of almost Gorenstein local rings of higher dimension, and also that of generalized Gorenstein local rings, in terms of their canonical ideals and some related invariants. As for almost Gorenstein local rings, it has been done by S. Goto, R. Takahashi, and N. Taniguchi. They have already given a satisfactory result [46, Theorem 5.1]. The present purpose is to perform the task for generalized Gorenstein local rings of higher dimension.

Originally, the series of researches [15, 25, 36, 39, 40, 41, 42, 45, 46, 47, 48, 74] aim to find a new class of Cohen-Macaulay local rings, which contains the class of Gorenstein rings. almost Gorenstein local rings are one of the candidates for such a class. Historically, the notion of almost Gorenstein ring in our sense has its root in the article [7] of V. Barucci and R. Fröberg in 1997, where they dealt with one-dimensional analytically unramified local rings. They explored also numerical semigroup rings, starting a very beautiful theory. In [36], S. Goto and N. Matsuoka and T. T. Phuong relaxed the notion to arbitrary Cohen-Macaulay local rings of dimension one, based on a different point of view. Repairing a gap in the proof of [7, Proposition 25], they opened frontiers in the study of one-dimensional Cohen-Macaulay local rings. Among various results of [36], the most striking achievement seems that their arguments have prepared for a possible definition [46, Definition 3.3] of almost Gorenstein rings of higher dimension. We now have two more notion; 2-almost Gorenstein local ring ([15]) and generalized Gorenstein local ring ([34]), both of which are candidates of reasonable generalizations of Gorenstein rings and almost Gorenstein rings as well.

As is stated above, the present purpose is to find a characterization of generalized Gorenstein local rings in terms of canonical ideals and related invariants. To state our motivation and the results more precisely, let us review on the definition of generalized Gorenstein local rings. Throughout this article, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R > 0$, possessing the canonical module K_R. For simplicity, let us always assume that the residue class field $R/\mathfrak{m}$ of R is infinite. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R. With this notation the definition of generalized Gorenstein local ring is stated as follows.

Definition 2.1.1 ([34, Definition 1.2])**.** We say that R is a generalized Gorenstein local ring, if one of the following conditions is satisfied.

(1) R is a Gorenstein ring.

(2) R is not a Gorenstein ring but there exists an exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules such that

(i) C is an Ulrich R-module with respect to $\mathfrak{a}$ and

(ii) the induced homomorphism $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \to \mathrm{K}_R/\mathfrak{a}\mathrm{K}_R$ is injective.

When Case (2) occurs, we especially say that R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$.

Let us explain a little about Definition 2.1.1. Let M be a finitely generated R-module of dimension $s \geq 0$. Then we say that M is an Ulrich R-module with respect to $\mathfrak{a}$, if the following three conditions are satisfied.

(i) M is a Cohen-Macaulay R-module,

(ii) $\mathrm{e}^0_{\mathfrak{a}}(M) = \ell_R(M/\mathfrak{a}M)$, and

(iii) $M/\mathfrak{a}M$ is a free $R/\mathfrak{a}$-module,

where $\ell_R(*)$ stand for the length and

$$\mathrm{e}^0_{\mathfrak{a}}(M) = \lim_{n\to\infty} s!{\cdot}\frac{\ell_R(M/\mathfrak{a}^{n+1}M)}{n^s}$$

denotes the multiplicity of M with respect to $\mathfrak{a}$ ([43]). The notion of Ulrich R-module with respect to an $\mathfrak{m}$-primary ideal is a generalization of maximally generated maximal Cohen-Macaulay R-module (that is, maximal Ulrich R-module with respect to $\mathfrak{m}$; see [10]). One can consult [34, 43, 44, 46] for basic properties of Ulrich modules in our sense. Here, let us note one thing. In the setting of Definition 2.1.1, suppose that there is an exact sequence

$$0 \to R \to \mathrm{K}_R \to C \to 0$$

of R-modules such that $C \neq (0)$. Then C is a Cohen-Macaulay R-module of dimension $d-1$ ([46, Lemma 3.1 (2)]), and C is an Ulrich R-module with respect $\mathfrak{a}$ if and only if

$$\mathfrak{a}C = (f_2, f_3, \ldots, f_d)C$$

for some elements $f_2, f_3, \ldots, f_d \in \mathfrak{a}$ ([34, Proof of Proposition 2.4, Claim]). Therefore, if $\mathfrak{a} = \mathfrak{m}$, Definition 2.1.1 is exactly the same as that of almost Gorenstein local rings given by [46, Definition 3.3]. In [34], S. Goto and the author investigate generalized Gorenstein local rings, and one can find a report of basic results on generalized Gorenstein local rings, which greatly generalizes several results in [46], clarifying what almost Gorenstein local rings are.

The present purpose is to give a characterization of generalized Gorenstein local rings. Let $\mathrm{r}(R)$ stand for the Cohen-Macaulay type of R. We then have the following, which is the main result of this article.

Theorem 2.1.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R > 0$ and infinite residue class field, possessing the canonical module K_R. Let $I \subsetneq R$ be an ideal of R such that $I \cong \mathrm{K}_R$ as an R-module. We choose a parameter ideal $Q = (f_1, f_2, \ldots, f_d)$ of R so that $f_1 \in I$ and set $J = I + Q$. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R. Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$.*

(2) *The following three conditions are satisfied.*

 (i) $\mathfrak{a} = Q :_R J$.
 (ii) $\mathfrak{a}J = \mathfrak{a}Q$.
 (iii) $\mathrm{e}_J^1(R) = \ell_R(R/\mathfrak{a}) \cdot \mathrm{r}(R)$.

 When this is the case, $R/\mathfrak{a}$ is a Gorenstein ring, and the following assertions hold true.

(a) *$J^3 = QJ^2$ but $J^2 \neq QJ$.*

(b) *$\mathcal{S}_Q(J) \cong (\mathcal{T}/\mathfrak{a}\mathcal{T})(-1)$ as a graded T-module, where $\mathcal{S}_Q(J)$ (resp. $\mathcal{T} = \mathcal{R}(Q)$) denotes the Sally module of J with respect to Q (see [77]) (resp. the Rees algebra of Q).*

(c) *$f_2, f_3, \ldots, f_d$ forms a super-regular sequence of R with respect to J, and $\operatorname{depth} \mathrm{gr}_J(R) = d-1$, where $\mathrm{gr}_J(R) = \bigoplus_{n \geq 0} J^n/J^{n+1}$ denotes the associated graded ring of J.*

(d) *The Hilbert function of R with respect to J is given by*

$$\ell_R(R/J^{n+1}) = \mathrm{e}_J^0(R) \cdot \binom{n+d}{d} - \left[\mathrm{e}_J^0(R) - \ell_R(R/J) + \ell_R(R/\mathfrak{a})\right] \cdot \binom{n+d-1}{d-1} + \ell_R(R/\mathfrak{a}) \cdot \binom{n+d-2}{d-2}$$

 for $n \geq 1$. Hence, $\mathrm{e}_J^2(R) = \ell_R(R/\mathfrak{a})$ if $d \geq 2$, and $\mathrm{e}_J^i(R) = 0$ for all $3 \leq i \leq d$ if $d \geq 3$.

The study of generalized Gorenstein local rings is still in progress, and our theorem 2.1.2 now completely generalizes the corresponding assertion [46, Theorem 5.1] of almost Gorenstein local rings to arbitrary generalized Gorenstein local rings of higher dimension, certifying not only that the notion of generalized Gorenstein local ring is a reasonable generalization of almost Gorenstein local rings but also that of generalized Gorenstein local ring is one of the candidates of broader notion which generalizes Gorenstein rings.

We now briefly explain how this chapter is organized. The proof of Theorem 2.1.2 shall be given in Sections 3 and 4. In Section 2 we summarize some of the known results given by [34], which we throughout need to prove Theorem 2.1.2. We will explore in Section 5 an example in order to illustrate Theorem 2.1.2.

2.2 Preliminaries

In this section we summarize some of the results in [34, Section 4] about one-dimensional generalized Gorenstein local rings, which we need to prove Theorem 2.1.2. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one, admitting the canonical module K_R. Let $I \subsetneq R$ be an ideal of R such that $I \cong \mathrm{K}_R$ as an R-module. We assume that I contains a parameter ideal (a) of R as a reduction. We set

$$K = \frac{I}{a} = \left\{ \frac{x}{a} \;\middle|\; x \in I \right\}$$

in the total ring $\mathrm{Q}(R)$ of fractions of R. Hence K is a fractional ideal of R such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$, where $\overline{R}$ denotes the integral closure of R in $\mathrm{Q}(R)$. We set $S = R[K]$ in $\mathrm{Q}(R)$. Hence S is a module-finite birational extension of R. Note that the ring $S = R[K]$ is independent of the choice of canonical ideals I and reductions (a) of I ([15, Theorem 2.5]). We set $\mathfrak{c} = R : S$. We then have the following, which shows the $\mathfrak{m}$-primary ideal $\mathfrak{a}$ which appears in Definition 2.1.1 of a generalized Gorenstein local ring R is uniquley determined, when $\dim R = 1$.

Proposition 2.2.1. *Suppose R is not a Gorenstein ring but R is a generalized Gorenstein local ring with respect an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R. Then $\mathfrak{a} = \mathfrak{c}$.*

Proof. We choose an exact sequence

$$0 \to R \xrightarrow{\varphi} I \to C \to 0$$

of R-module such that C is an Ulrich R-module with respect to $\mathfrak{a}$ and the induced homomorphism $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \to I/\mathfrak{a}I$ is injective. We set $f = \varphi(1)$ and identify $C = I/(f)$. Then $\mathfrak{a}I \subseteq (f)$ since $\mathfrak{a}{\cdot}(I/(f)) = (0)$, while $(f) \cap \mathfrak{a}I = \mathfrak{a}f$ since the homomorphism $R/\mathfrak{a} \otimes_R \varphi$ is injective. Consequently, $\mathfrak{a}I = \mathfrak{a}f$, whence (f) is a reduction of I. We consider $L = \frac{I}{f}$ and set $S = R[L]$. Then $\mathfrak{a}L = \mathfrak{a}$ since $\mathfrak{a}I = \mathfrak{a}f$, so that $\mathfrak{a}S = \mathfrak{a}$ since $S = L^n$ for all $n \gg 0$. Therefore, $\mathfrak{a} \subseteq \mathfrak{c} = R : S$, so that $\mathfrak{a} = \mathfrak{c}$, because $\mathfrak{c} \subseteq R : L = (f) :_R I = \mathfrak{a}$. $\square$

In general we have the following.

Fact 2.2.2 ([34, Lemma 4.4])**.** Let $\mathfrak{a} = R : K$. Then the following conditions are equivalent.

(1) $K^2 = K^3$.

(2) $\mathfrak{a} = \mathfrak{c}$.

(3) $\mathfrak{a}K = \mathfrak{a}$.

The key in the theory of one-dimensional generalized Gorenstein local rings is the following, which we shall freely use in the present article. See [34, Section 4] for the proof.

Theorem 2.2.3 ([34])**.** *Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring (necessarily with respect to $\mathfrak{c}$).*

(2) *K/R is a free $R/\mathfrak{c}$-module.*

(3) *$K/\mathfrak{c} = K/\mathfrak{c}K$ is a free $R/\mathfrak{c}$-module.*

(4) *S/R is a free $R/\mathfrak{c}$-module.*

(5) *$S/\mathfrak{c} = S/\mathfrak{c}S$ is a free $R/\mathfrak{c}$-module.*

(6) $\mathrm{e}_1(I) = \ell_R(R/\mathfrak{c}){\cdot}\mathrm{r}(R)$.

When this is the case, the following assertions hold true.

(i) $K^2 = K^3$.

(ii) *$R/\mathfrak{c}$ is a Gorenstein ring.*

(iii) $S/K \cong R/\mathfrak{c}$.

2.3 A Characterization of generalized Gorenstein local rings

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R > 0$ and infinite residue class field, possessing the canonical module K_R. Let $I \subsetneq R$ be an ideal of R such that $I \cong \mathrm{K}_R$ as an R-module. We choose a parameter ideal $Q = (f_1, f_2, \ldots, f_d)$ of R so that $f_1 \in I$. We set $\mathfrak{q} = (f_2, f_3, \ldots, f_d)$ and $J = I + \mathfrak{q}$. Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of R. In Sections 3 and 4 we throughout assume that R is not a Gorenstein ring. The purpose is to prove the equivalence between Conditions (1) and (2) in Theorem 2.1.2.

Let us begin with the following.

Proposition 2.3.1. $\mathfrak{q} \cap I = \mathfrak{q}I$ *and* $J \neq Q$.

Proof. We get $\mathfrak{q} \cap I = \mathfrak{q}I$, since $\mathfrak{q}$ is a parameter ideal of the Gorenstein ring R/I. If $J = Q$, then $I = Q \cap I = (f_1) + (\mathfrak{q} \cap I)$, whence $I = (f_1)$ by the first equlaity. Therefore, R is a Gorenstein ring, which is impossible. $\square$

Theorem 2.3.2. *The following conditions are equivalent.*

(1) $\mathfrak{a} = Q :_R J$, $\mathfrak{a}J = \mathfrak{a}Q$, *and* $\mathrm{e}^1_J(R) = \ell_R(R/\mathfrak{a}){\cdot}\mathrm{r}(R)$.

(2) $\mathfrak{a}J = \mathfrak{a}Q$ *and the* $R/\mathfrak{a}$*-module* J/Q *is free.*

When this is the case, $R/\mathfrak{q}$ *is a generalized Gorenstein local ring with respect to* $\mathfrak{a}/\mathfrak{q}$*, whence so is the ring* R *with respect to* $\mathfrak{a}$*.*

Proof. We may assume $\mathfrak{a}J = \mathfrak{a}Q$. Hence, Q is a reduction of J. Because $J/Q \neq (0)$ by Proposition 2.3.1, we get $\mathfrak{a} = Q :_R J$, once J/Q is $R/\mathfrak{a}$-free. Consequently, we may also assume that $\mathfrak{a} = Q :_R J$. First, suppose that $d = 1$. Hence $J = I$. We set $K = \frac{I}{f_1}$ in the total ring of fractions of R. Then, since $\mathfrak{a}K = \mathfrak{a}$, we have $\mathfrak{a} = R : R[K]$ by Fact 2.2.2. Consequently, by Theorem 2.2.3, R is a generalized Gorenstein local ring (necessarily with respect to $\mathfrak{c}$; see Proposition 2.2.1) if and only if $\mathrm{e}^1_I(R) = \ell_R(R/\mathfrak{a}){\cdot}\mathrm{r}(R)$. By Theorem 2.2.3, the former condition is also equivalent to saying that I/Q ($\cong K/R$) is a free $R/\mathfrak{a}$-module, whence the equivalence of Conditions (1) and (2) follows.

Let us consider the case where $d \geq 2$. Assume that Condition (2) is satisfied. Let us check that $\overline{R} = R/\mathfrak{q}$ is a generalized Gorenstein local ring. Set

$$\overline{R} = R/\mathfrak{q}, \quad \overline{Q} = Q/\mathfrak{q}, \quad \overline{J} = J/\mathfrak{q}, \quad \text{and} \quad \overline{\mathfrak{a}} = \mathfrak{a}/\mathfrak{q}.$$

We then have $\overline{J} = (I + \mathfrak{q})/\mathfrak{q} \cong I/\mathfrak{q}I = \mathrm{K}_{\overline{R}}$, since $I \cong \mathrm{K}_R$ and $f_2, f_3, \ldots, f_d$ is a regular sequence for the R-module I. Consequently, because $\overline{\mathfrak{a}}{\cdot}\overline{J} = \overline{\mathfrak{a}}{\cdot}\overline{Q}$ and $\overline{J}/\overline{Q}$ is $\overline{R}/\overline{\mathfrak{a}}$-free, from the case of $d = 1$ it follows that $\overline{R}$ is a generalized Gorenstein local ring (Fact 2.2.2 and Theorem 2.2.3), whence so is R with respect $\mathfrak{a}$ ([34, Theorem 3.3 (2)]).

We now assume that the implication (2) $\Rightarrow$ (1) holds true for $d - 1$. Since Q is a reduction of J and the field $R/\mathfrak{m}$ is infinite, there exist elements $h_1, h_2, \ldots, h_d \in Q$ such that (i) $h_1 \in I$, (ii) $Q = (h_1, h_2, \ldots, h_d)$, and (iii) h_2 is superficial for R with respect to J. This time, we consider the ring $\overline{R} = R/(h_2)$ and let $\overline{*}$ denote the reduction mod (h_2). Then $\overline{I} = [I + (h_2)]/(h_2) \cong I/h_2I = \mathrm{K}_{\overline{R}}$ and $\overline{h_1} \in \overline{I}$. Condition (2) is clearly satisfied for the ring $\overline{R}$ as for the ideals $\overline{\mathfrak{a}}$, $\overline{Q}$, and $\overline{J}$. Therefore, by the hypothesis of induction on d we get

$$\mathrm{e}^1_{\overline{J}}(\overline{R}) = \ell_{\overline{R}}\left(\overline{R}/\overline{\mathfrak{a}}\right) {\cdot}\mathrm{r}(\overline{R}).$$

Consequently, $\mathrm{e}^1_J(R) = \ell_R(R/\mathfrak{a}){\cdot}\mathrm{r}(R)$, because $\mathrm{e}^1_J(R) = \mathrm{e}^1_{\overline{J}}(\overline{R})$ (remember that h_2 is superficial for R with respect to J).

The reverse implication (1) $\Rightarrow$ (2) also follows by induction on d, chasing the above argument in the opposite direction. $\square$

With the same notation as Theorem 2.3.2, suppose that the equivalent conditions of Theorem 2.3.2 are satisfied. Then we have the following.

Proposition 2.3.3. $J \subseteq \mathfrak{a}$*. Hence* $J^2 \subseteq Q$*.*

Proof. Suppose $d = 1$. We set $K = \frac{I}{f_1}$, $S = R[K]$, and $\mathfrak{c} = R : S$. Then since R is a generalized Gorenstein local ring with respect $\mathfrak{a}$, we get $\mathfrak{a} = \mathfrak{c}$ by Proposition 2.2.1 and $\mathfrak{c} = R : K$ (see Fact 2.2.2 and Theorem 2.2.3). Therefore, $I \subseteq \mathfrak{c}$, because $\mathfrak{c}$ is an ideal of S and $f_1 \in \mathfrak{c}$ (note that $I = f_1 K \subseteq R$). If $d > 1$, then passing to $R/\mathfrak{q}$, we have $J/\mathfrak{q} \subseteq \mathfrak{a}/\mathfrak{q}$, whence $J \subseteq \mathfrak{a}$. Therefore $J^2 \subseteq Q$, because $\mathfrak{a} = Q :_R J$. □

We are now ready to prove the equivalence of Conditions (1) and (2) in Theorem 2.1.2.

Proof of the main part in Theorem 2.1.2. See Theorem 2.3.2 for the implication (2) ⇒ (1). To see the implication (1) ⇒ (2), we consider the exact sequence

$$0 \to R \xrightarrow{\varphi} I \to C \to 0$$

of R-modules such that C is an Ulrich R-module with respect to $\mathfrak{a}$ and the induced homomorphism $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \to I/\mathfrak{a}I$ is injective. Let $f_1 = \varphi(1) \in I$. Then f_1 is a non-zerodivisor of R. Choose elements $f_2, f_3, \ldots, f_d \in \mathfrak{a}$ so that in the ring $R' = R/(f_1)$ these elements generate a reduction of $\mathfrak{a}{\cdot}R'$. Then $f_1, f_2, \ldots, f_d$ is a system of parameters of R with $f_1 \in I$. Set $\mathfrak{q} = (f_2, f_3, \ldots, f_d)$, $Q = (f_1) + \mathfrak{q}$, and $J = I + \mathfrak{q}$. Then, $\mathfrak{q}$ is a parameter ideal of R/I, and because

$$\ell_{R'}(C/\mathfrak{q}C) = \mathrm{e}^0_{\mathfrak{a}\cdot R'}(C) = \mathrm{e}^0_{\mathfrak{a}}(C) = \ell_R(C/\mathfrak{a}C),$$

we get $\mathfrak{a}C = \mathfrak{q}C$. Therefore, since $\mathfrak{a}^n C = \mathfrak{q}^n C$ for all $n \in \mathbb{Z}$, $f_2, f_3, \ldots, f_d$ forms a super-regular sequence of C with respect to $\mathfrak{a}$, whence it is a superficial sequnece of C with respect to $\mathfrak{a}$. Consequently, by [34, Theorem 3.3 (1)] the ring $\overline{R} = R/\mathfrak{q}$ is a generalized Gorenstein local ring with respect to $\mathfrak{a}/\mathfrak{q}$, so that $\overline{\mathfrak{a}}{\cdot}\overline{J} = \overline{\mathfrak{a}}{\cdot}\overline{Q}$ and $\overline{J}/\overline{Q}$ is $\overline{R}/\overline{\mathfrak{a}}$-free, where

$$\overline{\mathfrak{a}} = \mathfrak{a}/\mathfrak{q}, \ \overline{J} = J/\mathfrak{q}, \text{ and } \overline{Q} = Q/\mathfrak{q}.$$

Hence J/Q is $R/\mathfrak{a}$-free, and $\mathfrak{a}J \subseteq \mathfrak{a}Q + \mathfrak{q}$. Therefore

$$\mathfrak{a}J = (\mathfrak{a}Q + \mathfrak{q}) \cap \mathfrak{a}J = \mathfrak{a}Q + [\mathfrak{q} \cap \mathfrak{a}J] = \mathfrak{a}Q + [\mathfrak{q} \cap \mathfrak{a}I] \subseteq \mathfrak{a}Q + [\mathfrak{q} \cap I] = \mathfrak{a}Q + \mathfrak{q}I,$$

where the third equality follows from the fact that $\mathfrak{a}J = \mathfrak{a}I + \mathfrak{a}\mathfrak{q}$. Hence $\mathfrak{a}J = \mathfrak{a}Q$, as $I \subseteq \mathfrak{a}$ by Proposition 2.3.3. Therefore, Theorem 2.3.2 certifies that Condition (2) in Theorem 2.1.2 is satisfied for the ideals $\mathfrak{a}$, Q, and J. This completes the proof of the equivalence of Conditions (1) and (2) in Theorem 2.1.2. □

2.4 The Sally modules of J in generalized Gorenstein local rings

Let us show the last assertions of Theorem 2.1.2. In what follows, assume that our ideals Q and J satisfy the equivalent conditions in Theorem 2.3.2. Hence R (resp. $R/\mathfrak{q}$) is a generalized Gorenstein local ring with respect to $\mathfrak{a}$ (resp. $\mathfrak{a}/\mathfrak{q}$), and $R/\mathfrak{a}$ is a Gorenstein ring by [34, Corollary 4.9]. To prove the last assertions of Theorem 2.1.2, we need some preliminaries. Let us maintain the same notation as in the proof of Theorem 2.3.2.

We begin with the following.

Lemma 2.4.1. $\mathfrak{q} \cap J^2 = \mathfrak{q}J$.

Proof. Remember that $J^2 = \mathfrak{q}J + I^2$, since $J = I + \mathfrak{q}$. We then have $\mathfrak{q} \cap J^2 = \mathfrak{q}J + (\mathfrak{q} \cap I^2)$, so that $\mathfrak{q} \cap J^2 = \mathfrak{q}J$, because $\mathfrak{q} \cap I^2 \subseteq \mathfrak{q} \cap I = \mathfrak{q}I$. □

Proposition 2.4.2. *We set $L = Q :_R \mathfrak{a}$. Then the following assertions hold true.*

(1) $L = J :_R \mathfrak{a}$ *and* $L^2 \subseteq Q$.

(2) $L/J \cong R/\mathfrak{a}$ *as an R-module.*

(3) $J^2/QJ \cong R/\mathfrak{a}$ *as an R-module.*

(4) $\mathfrak{a}L = \mathfrak{a}Q$.

(5) $L^2 = QL$.

(6) $J^3 = QJ^2$ *but* $J^2 \neq QJ$.

Proof. (1), (2), (3) First, consider the case where $d = 1$. Let us maintain the notation of the proof of Proposition 2.3.3. Then $Q :_R \mathfrak{c} = I :_R \mathfrak{c} = f_1S$. In fact, we have $\mathfrak{c} = K : S = R : K$ (see Fact 2.2.2 and Theorem 2.2.3) and hence $f_1 \in \mathfrak{c}$. Let $x \in R$. Then $x{\cdot}\mathfrak{c} \subseteq I$ if and only if $\frac{x}{f_1}{\cdot}\mathfrak{c} \subseteq K$. The latter condition is equivalent to saying that $\frac{x}{f_1} \in S$, since $K : \mathfrak{c} = K : (K : S) = S$. Thus $I :_R \mathfrak{c} = f_1S$. Because $f_1S{\cdot}\mathfrak{c} = f_1\mathfrak{c} \subseteq Q = (f_1)$, we get $I :_R \mathfrak{c} = f_1S \subseteq Q :_R \mathfrak{c}$. Hence $Q :_R \mathfrak{c} = I :_R \mathfrak{c} = f_1S$. Consequently

$$(Q :_R \mathfrak{c})^2 = f_1(f_1S) \subseteq Q = (f_1),$$

and $[Q :_R \mathfrak{c}]/I = f_1S/f_1K \cong S/K \cong R/\mathfrak{c}$ by Theorem 2.2.3, which proves Assertions (1) and (2), because $\mathfrak{a} = \mathfrak{c}$. Assertion (3) is now clear, since

$$I^2/f_1I \cong K^2/K \cong R/\mathfrak{c}$$

by Theorem 2.2.3. Now consider the case where $d \geq 2$. To show Assertions (1) and (2), passing to the ring $R/\mathfrak{q}$, we can safely assume that $d = 1$, and we have already done with the case. Consider Assertion (3). We set $\overline{R} = R/\mathfrak{q}$ and denote by $\overline{*}$ the reduction mod $\mathfrak{q}$. Let $\varphi : J^2/QJ \to \overline{J}^2/f_1\overline{J}$ be the canonical epimorphism. We then have $\operatorname{Ker}\varphi = [J^2 \cap (\mathfrak{q} + f_1J)]/QJ$. Hence, because $J^2 \cap \mathfrak{q} = \mathfrak{q}J$ by Lemma 2.4.1, we have

$$J^2 \cap (\mathfrak{q} + f_1J) = f_1J + (J^2 \cap \mathfrak{q}) = f_1J + \mathfrak{q}J = QJ,$$

whence the required isomorphism $J^2/QJ \cong R/\mathfrak{a}$ follows.

(4) Suppose $d = 1$. Then $\mathfrak{c}L = \mathfrak{c}{\cdot}f_1S = f_1{\cdot}\mathfrak{c}$, whence the assertion follows. Suppose that $d \geq 2$ and that Assertion (3) holds true for $d - 1$. Note that $Q = (f_1, f_1 + f_2) + (f_3, \ldots, f_d)$. Then, because $R/(f + f_2)$ and $R/(f_2)$ are generalized Gorenstein local rings with respect to $\mathfrak{a}/(f_1 + f_2)$ and $\mathfrak{a}/(f_2)$ respectively, thanks to the hypothesis of induction on d, we get

$$\mathfrak{a}L \subseteq [\mathfrak{a}Q + (f_1 + f_2)] \cap [\mathfrak{a}Q + (f_2)] = \mathfrak{a}Q + \{(f_1 + f_2) \cap [\mathfrak{a}Q + (f_2)]\}\,.$$

Since $\mathfrak{a}Q + (f_2) \subseteq \mathfrak{a}{\cdot}(f_1 + f_2) + (f_2, f_3, \ldots, f_d)$, we furthermore have that

$$\mathfrak{a}L \subseteq \mathfrak{a}Q + \{(f_1 + f_2) \cap [\mathfrak{a}{\cdot}(f_1 + f_2) + (f_2, f_3, \ldots, f_d)]\} = \mathfrak{a}Q + (f_1 + f_2){\cdot}(f_2, f_3, \ldots, f_d) = \mathfrak{a}Q.$$

Hence $\mathfrak{a}L = \mathfrak{a}Q$.

(5) Let $x \in L^2$. Then, $x \in Q$, since $L^2 \subseteq Q$ by Assertion (1). We write $x = \sum_{i=1}^{d} f_i x_i$ with $x_i \in R$. Let $\alpha \in \mathfrak{a}$. Then, because

$$\alpha x = \sum_{i=1}^{d} f_i(\alpha x_i) \in \mathfrak{a}L^2 \subseteq Q^2$$

by Assertion (3), we get $\alpha x_i \in Q$ for all $1 \leq i \leq d$, whence $x_i \in Q :_R \mathfrak{a} = L$. Thus $L^2 = QL$.

(6) The equality $J^3 = QJ^2$ is a direct consequence of [38, Proposition 2.6], since $\mu_R(L/J) = 1$ by Assertion (2). Suppose that $J^2 = QJ$ and let $\overline{*}$ denote the reduction mod $\mathfrak{q}$. Then since $\overline{J} \cong \mathrm{K}_{\overline{R}}$ and $\overline{J}^2 = \overline{Q}{\cdot}\overline{J}$, by [36, Theorem 3.7] $\overline{R}$ is a Gorenstein ring, which is impossible. Hence $J^2 \neq QJ$. □

Proposition 2.4.3. *The sequence $f_2, f_3, \ldots, f_d$ is a super-regular sequence of R with respect to J. Hence* $\operatorname{depth} \operatorname{gr}_J(R) = d - 1$.

Proof. To see the first assertion, it suffices to show that $\mathfrak{q} \cap J^{n+1} = \mathfrak{a}J^n$ for all $n \geq 1$. By Lemma 2.4.1 we may assume that $n \geq 2$ and that our assertion holds true for $n - 1$. Then, since $J^{n+1} = QJ^n = \mathfrak{q}J^n + f_1 J^n$ by Proposition 2.4.2 (5), we have

$$\mathfrak{q} \cap J^{n+1} = \mathfrak{q}J^n + (\mathfrak{q} \cap f_1 J^n).$$

Consequently, because $\mathfrak{q} \cap f_1 J^n = f_1{\cdot}(\mathfrak{q} \cap J^n)$ (remember that $f_1, f_2, \ldots, f_d$ is an R-regular sequence), by the hypothesis of induction on n we have $\mathfrak{q} \cap f_1 J^n \subseteq \mathfrak{q}J^n$. Hence $\mathfrak{q} \cap J^{n+1} = \mathfrak{q}J^n$. Consequently, $\operatorname{depth} \operatorname{gr}_J(R) \geq d - 1$. Suppose that $\operatorname{depth} \operatorname{gr}_J(R) = d$. Then, $f_1, f_2, \ldots, f_d$ is a super-regular sequence of R with respect to J, so that $Q \cap J^2 = QJ$. Therefore, $J^2 = QJ$, because $J^2 \subseteq Q$ by Proposition 2.4.2 (1), which contradicts Proposition 2.4.2 (6). Hence $\operatorname{depth} \operatorname{gr}_J(R) = d - 1$. □

Let $\mathcal{T} = \mathcal{R}(Q)$ and $\mathcal{R} = \mathcal{R}(J)$ be the Rees algebras of Q and J respectively. We now consider the Sally module $\mathcal{S}_Q(J) = J\mathcal{R}/J\mathcal{T}$ of J with respect to Q (see [77]).

Theorem 2.4.4. *$\mathcal{S}_Q(J) \cong (\mathcal{T}/\mathfrak{a}\mathcal{T})(-1)$ as a graded $\mathcal{T}$-module.*

Proof. We set $\mathcal{S} = \mathcal{S}_Q(J)$ and denote, for each $n \in \mathbb{Z}$, by $[\mathcal{S}]_n$ the homogeneous component of $\mathcal{S}$ of degree n. Then $[\mathcal{S}]_1 = J^2/QJ$ ([37, Lemma 2.1]) and $\mathcal{S} = \mathcal{T}{\cdot}[\mathcal{S}]_1$. Hence by Proposition 2.4.2 (3), we get an epimorpism $\varphi : (\mathcal{T}/\mathfrak{a}\mathcal{T})(-1) \to \mathcal{S}$ of graded $\mathcal{T}$-modules. Let $X = \operatorname{Ker} \varphi$ and assume that $X \neq (0)$. We choose an element $\mathfrak{p} \in \operatorname{Ass}_{\mathcal{T}} X$. Then, since $\mathfrak{p} \in \operatorname{Ass}_{\mathcal{T}} \mathcal{T}/\mathfrak{a}\mathcal{T}$, and $\mathcal{T}/\mathfrak{a}\mathcal{T} = (R/\mathfrak{a})[X_1, X_2, \ldots, X_d]$ is the polynomial ring over $R/\mathfrak{a}$ (remember that $\mathcal{T}$ is isomorphic to the symmetric algebra of Q over R), we have $\mathfrak{p} = \mathfrak{m}\mathcal{T}$. Then $\ell_{\mathcal{T}_\mathfrak{p}}((\mathcal{T}/\mathfrak{a}\mathcal{T})_\mathfrak{p}) = \ell_R(R/\mathfrak{a})$, while by [37, Proposition 2.2] we have

$$\ell_{\mathcal{T}_\mathfrak{p}}(\mathcal{S}_\mathfrak{p}) = \mathrm{e}_J^1(R) - \ell_R(J/Q).$$

Therefore, $\ell_{\mathcal{T}_\mathfrak{p}}(\mathcal{S}_\mathfrak{p}) = \ell_R(R/\mathfrak{a})$, because $\mathrm{e}_J^1(R) = \ell_R(R/\mathfrak{a})\cdot\mathrm{r}(R)$ by Theorem 2.3.2 and $\ell_R(J/Q) = \ell_R(R/\mathfrak{a})\cdot(\mathrm{r}(R)-1)$ by Theorem 2.2.3. Thus $\ell_{\mathcal{T}_\mathfrak{p}}((\mathcal{T}/\mathfrak{a}\mathcal{T})_\mathfrak{p}) = \ell_{\mathcal{T}_\mathfrak{p}}(\mathcal{S}_\mathfrak{p})$, which forces $X_\mathfrak{p} = (0)$. This is absurd. Thus $(\mathcal{T}/\mathfrak{a}\mathcal{T})(-1) \cong \mathcal{S}$ as a graded $\mathcal{T}$-module. $\square$

Because

$$\ell_R(R/J^{n+1}) = \mathrm{e}_J^0(R)\cdot\binom{n+d}{d} - \left[\mathrm{e}_J^0(R) - \ell_R(R/J)\right]\cdot\binom{n+d-1}{d-1} - \ell_R([\mathcal{S}]_n)$$

for all $n \in \mathbb{Z}$ (see [37, Proposition 2.2]), by Theorem 2.4.4 we readily get the following.

Corollary 2.4.5. *The Hilbert function of R with respect to J is given by*

$$\ell_R(R/J^{n+1}) = \mathrm{e}_J^0(R)\cdot\binom{n+d}{d} - \left[\mathrm{e}_J^0(R) - \ell_R(R/J) + \ell_R(R/\mathfrak{a})\right]\cdot\binom{n+d-1}{d-1} + \ell_R(R/\mathfrak{a})\cdot\binom{n+d-2}{d-2}$$

for $n \geq 1$. Hence, $\mathrm{e}_J^2(R) = \ell_R(R/\mathfrak{a})$ if $d \geq 2$, and $\mathrm{e}_J^i(R) = 0$ for all $3 \leq i \leq d$ if $d \geq 3$.

2.5 Example

Let $S = k[[X,Y,Z,V]]$ be the formal power series ring over an infinite field k and let $\mathfrak{b} = \mathbb{I}_2\left(\begin{smallmatrix} X^2 & Y+V & Z \\ Y & Z & X^3 \end{smallmatrix}\right)$ denote the ideal of S generated by 2×2 minors of the matrix $\left(\begin{smallmatrix} X^2 & Y+V & Z \\ Y & Z & X^3 \end{smallmatrix}\right)$. We set $R = S/\mathfrak{b}$. We denote by x, y, z, v the images of X, Y, Z, V in S, respectively. Then we have the following.

Example 2.5.1. The following assertions hold true.

(1) R is a two-dimensional generalized Gorenstein local ring with respect to $\mathfrak{a} = (x^2, y, z, v)$.

(2) $\mathrm{r}(R) = 2$ and $I = (x^2, y)$ is a canonical ideal of R.

(3) Set $Q = (x^2, v)$ and $J = I + Q$. Then Q is a parameter ideal of R with $x^2 \in I$.

(4) We have $\mathfrak{a} = Q :_R J$, $\mathfrak{a}J = \mathfrak{a}Q$, and $\mathrm{e}_J^1(R) = \ell_R(R/\mathfrak{a})\cdot\mathrm{r}(R) = 4$.

Proof. Since

$$R/(v) \cong k[[X,Y,Z]]/\mathbb{I}_2\left(\begin{smallmatrix} X^2 & Y & Z \\ Y & Z & X^3 \end{smallmatrix}\right) \cong k[[t^3, t^7, t^8]]$$

where t denotes an indeterminate over k, we have $\dim R/(v) = 1$. Hence $\mathrm{ht}_S\,\mathfrak{b} \geq 2$, so that R is a Cohen-Macaulay ring with $\dim R = 2$. Because $R/vR = k[[t^3, t^7, t^8]]$ is a generalized Gorenstein local ring with respect to (t^6, t^7, t^8) and v is a non-zerodivisor of R, by [34, Theorem 3.3] R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$. To see that $I \cong \mathrm{K}_R$, note that (t^6, t^7) is a canonical ideal of $k[[t^3, t^7, t^8]]$. Since $R/I = S/(X^2, Y, Z^2)$, the element v acts on R/I as a non-zerodivisor, so that $(v) \cap I = vI$. Hence $J/(v) = [I + (v)]/(v) \cong I/vI$. Because the ideal $J/(v)$ corresponds to (t^6, t^7) under the identification

$$R/(v) = k[[X,Y,Z]]/\mathbb{I}_2\left(\begin{smallmatrix} X^2 & Y & Z \\ Y & Z & X^3 \end{smallmatrix}\right) = k[[t^3, t^7, t^8]],$$

we see $\mathrm{r}_R(I/vI) = 1$, where $\mathrm{r}_R(*)$ stands for the Cohen-Macaulay type. Therefore $\mathrm{r}_R(I) = 1$, whence $I \cong \mathrm{K}_R$ because $(0) :_R I = (0)$. Since

$$R/Q \cong k[X,Y,Z,V]/(X^2, Y^2, Z^2, YZ, V),$$

we get $Q :_R J = Q :_R y = (x^2, y, z, v) = \mathfrak{a} \supseteq J$. It is direct to check that $\mathfrak{a}J = \mathfrak{a}Q$. The equality $\mathrm{e}_J^1(R) = \ell_R(R/\mathfrak{a}){\cdot}\mathrm{r}(R) = 4$ follows from the fact that $\ell_R(R/(x^2, y, z, v)) = \mathrm{r}(R) = 2$. $\square$

Chapter 3

The structure of chains of Ulrich ideals in Cohen-Macaulay local rings of dimension one

3.1 Introduction

The purpose of this chapter is to investigate the behavior of chains of Ulrich ideals, in a one-dimensional Cohen-Macaulay local ring, in connection with the structure of birational finite extensions of the base ring.

The notion of Ulrich ideals is a generalization of stable maximal ideals, which dates back to 1971, when the monumental paper [67] of J. Lipman was published. The modern treatment of Ulrich ideals was started by [43, 44] in 2014, and has been explored in connection with the representation theory of rings. In [43], the basic properties of Ulrich ideals are summarized, whereas in [44], Ulrich ideals in two-dimensional Gorenstein rational singularities are closely studied with a concrete classification. However, in contrast to the existing research on Ulrich ideals, the theory pertaining to the one-dimensional case does not seem capable of growth. Some part of the theory, including research on the ubiquity as well as the structure of the chains of Ulrich ideals, seems to have been left unchallenged. In the current chapter, we focus our attention on the one-dimensional case, clarifying the relationship between Ulrich ideals and the birational finite extensions of the base ring. The main objective is to understand the behavior of chains of Ulrich ideals in one-dimensional Cohen-Macaulay local rings.

To explain our objective as well as our main results, let us begin with the definition of Ulrich ideals. Although we shall focus our attention on the one-dimensional case, we would like to state the general definition, in the case of any arbitrary dimension. Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $d = \dim R \geq 0$.

Definition 3.1.1 ([43]). Let I be an $\mathfrak{m}$-primary ideal of R and assume that I contains a parameter ideal $Q = (a_1, a_2, \ldots, a_d)$ of R as a reduction. We say that I is an *Ulrich ideal* of R, if the following conditions are satisfied.

(1) $I \neq Q$,

(2) $I^2 = QI$, and

(3) I/I^2 is a free R/I-module.

We notice that Condition (2) together with Condition (1) are equivalent to saying that the associated graded ring $\mathrm{gr}_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ of I is a Cohen-Macaulay ring and $\mathrm{a}(\mathrm{gr}_I(R)) = 1 - d$, where $\mathrm{a}(\mathrm{gr}_I(R))$ denotes the a-invariant of $\mathrm{gr}_I(R)$. Therefore, these two conditions are independent of the choice of reductions Q of I. In addition, assuming Condition (2) is satisfied, Condition (3) is equivalent to saying that I/Q is a free R/I-module ([43, Lemma 2.3]). We also notice that Condition (3) is automatically satisfied if $I = \mathfrak{m}$, so that the maximal ideal $\mathfrak{m}$ is an Ulrich ideal of R if and only if R is not a regular local ring, possessing minimal multiplicity ([70]). From this perspective, Ulrich ideals are a kind of generalization of stable maximal ideals, which Lipman [67] started to analyze in 1971.

Here, let us briefly summarize some basic properties of Ulrich ideals, as seen in [43, 47]. Although we need only a part of them, let us also include some superfluity in order to show what specific properties Ulrich ideals enjoy. Throughout this chapter, let $\mathrm{r}(R)$ denote the Cohen-Macaulay type of R, and let $\mathrm{Syz}_R^i(M)$ denote, for each integer $i \geq 0$ and for each finitely generated R-module M, the i-th syzygy module of M in its minimal free resolution.

Theorem 3.1.2 ([43, 47])**.** *Let I be an Ulrich ideal of a Cohen-Macaulay local ring R of dimension $d \geq 0$ and set $t = n - d$ (> 0), where n denotes the number of elements in a minimal system of generators of I. Let*

$$\cdots \to F_i \overset{\partial_i}{\to} F_{i-1} \to \cdots \to F_1 \overset{\partial_1}{\to} F_0 = R \to R/I \to 0$$

be a minimal free resolution of R/I. Then $\mathrm{r}(R) = t{\cdot}\mathrm{r}(R/I)$ and the following assertions hold true.

(1) $\mathbf{I}(\partial_i) = I$ *for* $i \geq 1$.

(2) *For* $i \geq 0$, $\beta_i = \begin{cases} t^{i-d}{\cdot}(t+1)^d & (i \geq d), \\ \binom{d}{i} + t{\cdot}\beta_{i-1} & (1 \leq i \leq d), \\ 1 & (i = 0). \end{cases}$

(3) $\mathrm{Syz}_R^{i+1}(R/I) \cong [\mathrm{Syz}_R^i(R/I)]^{\oplus t}$ *for* $i \geq d$.

(4) *For* $i \in \mathbb{Z}$, $\mathrm{Ext}_R^i(R/I, R) \cong \begin{cases} (0) & (i < d), \\ (R/I)^{\oplus t} & (i = d), \\ (R/I)^{\oplus (t^2-1)\cdot t^{i-(d+1)}} & (i > d). \end{cases}$

Here $\mathbf{I}(\partial_i)$ denotes the ideal of R generated by the entries of the matrix ∂_i, and $\beta_i = \mathrm{rank}_R F_i$.

Because Ulrich ideals are a very special kind of ideals, it seems natural to expect that, in the behavior of Ulrich ideals, there might be contained ample information on base rings, once they exist. As stated above, this is the case of two-dimensional Gorenstein rational singularities, and the present objects of study are rings of dimension one.

In what follows, unless otherwise specified, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$. Our main targets are chains $I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1$ $(n \geq 2)$ of Ulrich ideals in R. Let I be an Ulrich ideal of R with a reduction $Q = (a)$. We set $A = I : I$ in the total ring of fractions of R. Hence, A is a birational finite extension of R, and $I = aA$. Firstly, we study the close connection between the structure of the ideal I and the R-algebra A. Secondly, let J be an Ulrich ideal of R and assume that $I \subsetneq J$. Then, we will show that $\mu_R(J) = \mu_R(I)$, where $\mu_R(*)$ denotes the number of elements in a minimal system of generators, and that $J = (b) + I$ for some $a, b \in \mathfrak{m}$ with $I = abA$. Consequently, we have the following, which is one of the main results of this chapter.

Theorem 3.1.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$. Then the following assertions hold true.*

(1) *Let I be an Ulrich ideal of R and $A = I : I$. Let $a_1, a_2, \ldots, a_n \in \mathfrak{m}$ $(n \geq 2)$ and assume that $I = a_1 a_2 \cdots a_n A$. For $1 \leq i \leq n$, let $I_i = (a_1 a_2 \cdots a_i) + I$. Then each I_i is an Ulrich ideal of R and*

$$I = I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1.$$

(2) *Conversely, let $I_1, I_2, \ldots, I_n$ $(n \geq 2)$ be Ulrich ideals of R and suppose that*

$$I_n \subsetneq I_{n-1} \subsetneq \cdots \subsetneq I_1.$$

We set $I = I_n$ and $A = I : I$. Then there exist elements $a_1, a_2, \ldots, a_n \in \mathfrak{m}$ such that $I = a_1 a_2 \cdots a_n A$ and $I_i = (a_1 a_2 \cdots a_i) + I$ for all $1 \leq i \leq n - 1$.

Let I and J be Ulrich ideals of R and assume that $I \subsetneq J$. We set $B = J : J$. Let us write $J = (b) + I$ for some $b \in \mathfrak{m}$. We then have that $J^2 = bJ$ and that B is a local ring with the maximal ideal $\mathfrak{n} = \mathfrak{m} + \frac{I}{b}$, where $\frac{I}{b} = \left\{ \frac{i}{b} \mid i \in I \right\}$ $(= b^{-1}I)$. We furthermore have the following.

Theorem 3.1.4. *$\frac{I}{b}$ is an Ulrich ideal of the Cohen-Macaulay local ring B of dimension one and there is a one-to-one correspondence $\mathfrak{a} \mapsto \frac{\mathfrak{a}}{b}$ between the Ulrich ideals $\mathfrak{a}$ of R such that $I \subseteq \mathfrak{a} \subsetneq J$ and the Ulrich ideals $\mathfrak{b}$ of B such that $\frac{I}{b} \subseteq \mathfrak{b}$.*

These two theorems convey to us that the behavior of chains of Ulrich ideals in a given one-dimensional Cohen-Macaulay local ring could be understood via the correspondence, and the relationship between the structure of Cohen-Macaulay local rings R and B could be grasped through the correspondence, which we shall closely discuss in this chapter.

We now explain how this chapter is organized. In Section 2, we will summarize some preliminaries, which we shall need later to prove the main results. The proof of Theorems 3.1.3 and 3.1.4 will be given in Section 3. In Section 4, we shall study the case where the base rings R are not regular but possess minimal multiplicity ([70]), and show that the set of Ulrich ideals of R are totally ordered with respect to inclusion. In Section 5, we explore the case where R is a generalized Gorenstein local ring ([34]).

In what follows, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$. Let $\mathrm{Q}(R)$ (resp. $\mathcal{X}_R$) stand for the total ring of fractions of R (resp. the set of all the Ulrich ideals in R). We denote by $\overline{R}$, the integral closure of R in $\mathrm{Q}(R)$. For a finitely generated R-module M, let $\mu_R(M)$ (resp. $\ell_R(M)$) be the number of elements in a minimal system of generators (resp. the length) of M. For each $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R, let

$$\mathrm{e}^0_{\mathfrak{a}}(R) = \lim_{n\to\infty} \frac{\ell_R(R/\mathfrak{a}^n)}{n}$$

stand for the multiplicity of R with respect to $\mathfrak{a}$. By $\mathrm{v}(R)$ (resp. $\mathrm{e}(R)$) we denote the embedding dimension $\mu_R(\mathfrak{m})$ of R (resp. $\mathrm{e}^0_{\mathfrak{m}}(R)$). Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R.

3.2 Preliminaries

Let us summarize preliminary facts on $\mathfrak{m}$-primary ideals of R, which we need throughout this chapter.

In this section, let I be an $\mathfrak{m}$-primary ideal of R, for which we will assume Condition (C) in Definition 3.2.2 to be satisfied. This condition is a partial extraction from Definition 3.1.1 of Ulrich ideals; hence every Ulrich ideal satisfies it (see Remark 3.2.3).

Firstly, we assume that I contains an element $a \in I$ with $I^2 = aI$. We set $A = I : I$ and

$$\frac{I}{a} = \left\{\frac{x}{a} \mid x \in I\right\} = a^{-1}I$$

in $\mathrm{Q}(R)$. Therefore, A is a birational finite extension of R such that $R \subseteq A \subseteq \overline{R}$, and $A = \frac{I}{a}$, because $I^2 = aI$; hence $I = aA$. We then have the following.

Proposition 3.2.1. *If $I = (a) :_R I$, then $A = R : I$ and $I = R : A$, whence $R : (R : I) = I$.*

Proof. Notice that $I = (a) :_R I = (a) : I = a[R : I]$ and we have $A = R : I$, because $I = aA$. We get $R : A = I$, since $R : A = R : \frac{I}{a} = a[R : I] = aA$. □

Let us now give the following.

Definition 3.2.2. Let I be an $\mathfrak{m}$-primary ideal of R and set $A = I : I$. We say that I satisfies Condition (C), if

(i) $A/R \cong (R/I)^{\oplus t}$ as an R-module for some $t > 0$, and

(ii) $A = R : I$.

Consequently, $I = R : A$ by Condition (i), when I satisfies Condition (C).

Remark 3.2.3. Let $I \in \mathcal{X}_R$. Then I satisfies Condition (C). In fact, choose $a \in I$ so that $I^2 = aI$. Then, $I/(a) \cong (R/I)^{\oplus t}$ as an R/I-module, where $t = \mu_R(I) - 1 > 0$ ([43, Lemma 2.3]). Therefore, $I = (a) :_R I$, so that I satisfies the hypothesis in Proposition 3.2.1, whence $A = R : I$. Notice that $A/R \cong I/(a) \cong (R/I)^{\oplus t}$, because $I = aA$.

We assume, throughout this section, that our $\mathfrak{m}$-primary ideal I satisfies Condition (C). We choose elements $\{f_i\}_{1 \le i \le t}$ of A so that

$$A = R + \sum_{i=1}^{t} Rf_i.$$

Therefore, the images $\{\overline{f_i}\}_{1 \le i \le t}$ of $\{f_i\}_{1 \le i \le t}$ in A/R form a free basis of the R/I-module A/R. We then have the following.

Lemma 3.2.4. *$aA \cap R \subseteq (a) + I$ for all $a \in R$.*

Proof. Let $x \in aA \cap R$ and write $x = ay$ with $y \in A$. We write $y = c_0 + \sum_{i=1}^{t} c_i f_i$ with $c_i \in R$. Then, $ac_i \in I$ for $1 \le i \le t$, since $x = ac_0 + \sum_{i=1}^{t} (ac_i) f_i \in R$. Therefore, $(ac_i) f_i \in IA = I$ for all $1 \le i \le t$, so that $x \in (a) + I$ as claimed. $\square$

Corollary 3.2.5. *Let J be an $\mathfrak{m}$-primary ideal of R and assume that J contains an element $b \in J$ such that $J^2 = bJ$ and $J = (b) :_R J$. If $I \subseteq J$, then $J = (b) + I$.*

Proof. We set $B = J : J$. Then $B = R : J$ and $J = bB$ by Proposition 3.2.1, so that $B = R : J \subseteq A = R : I$, since $I \subseteq J$. Consequently, $J = bB \subseteq bA \cap R \subseteq (b) + I$ by Lemma 3.2.4, whence $J = (b) + I$. $\square$

In what follows, let J be an $\mathfrak{m}$-primary ideal of R and assume that J contains an element $b \in J$ such that $J^2 = bJ$ and $J = (b) :_R J$. We set $B = J : J$. Then $B = R : J = \frac{J}{b}$ by Proposition 3.2.1. Throughout, suppose that $I \subsetneq J$. Therefore, since $J = (b) + I$ by Corollary 3.2.5, we get

$$B = \frac{J}{b} = R + \frac{I}{b}.$$

Let $\mathfrak{a} = \frac{I}{b}$. Therefore, $\mathfrak{a}$ is an ideal of A containing I, so that $\mathfrak{a}$ is also an ideal of B with

$$R/(\mathfrak{a} \cap R) \cong B/\mathfrak{a}.$$

With this setting, we have the following.

Lemma 3.2.6. *The following assertions hold true.*

(1) *$A/B \cong (B/\mathfrak{a})^{\oplus t}$ as a B-module.*

(2) $\mathfrak{a} \cap R = I :_R J$.

(3) $\ell_R([I :_R J]/I) = \ell_R(R/J)$.

(4) $I = [b{\cdot}(I :_R J)]A$.

Proof. (1) Since $A = R + \sum_{i=1}^{t} Rf_i$, we get $A/B = \sum_{i=1}^{t} B\overline{f_i}$ where $\overline{f_i}$ denotes the image of f_i in A/B. Let $\{b_i\}_{1 \le i \le t}$ be elements of $B = \dfrac{J}{b}$ and assume that $\sum_{i=1}^{t} b_i f_i \in B$. Then, since $\sum_{i=1}^{t}(bb_i)f_i \in R$ and $bb_i \in R$ for all $1 \le i \le t$, we have $bb_i \in I$, so that $b_i \in \dfrac{I}{b} = \mathfrak{a}$. Hence $A/B \cong (B/\mathfrak{a})^{\oplus t}$ as a B-module.

(2) This is standard, because $J = (b) + I$ and $\mathfrak{a} = \dfrac{I}{b}$.

(3) Since $J/I = [(b) + I]/I \cong R/[I :_R J]$, we get

$$\ell_R([I :_R J]/I) = \ell_R(R/I) - \ell_R(R/[I :_R J]) = \ell_R(R/I) - \ell_R(J/I) = \ell_R(R/J).$$

(4) We have $[b{\cdot}(I :_R J)]A \subseteq I$, since $b{\cdot}(I :_R J) \subseteq I$ and $IA = I$. To see the reverse inclusion, let $x \in I$. Then $x \in J = bB \subseteq bA$. We write $x = b[c_0 + \sum_{i=1}^{t} c_i f_i]$ with $c_i \in R$. Then $bc_i \in I$ for $1 \le i \le t$ since $x \in R$, so that $(bc_i)f_i \in I$ for all $1 \le i \le t$, because I is an ideal of A. Therefore, $bc_0 \in I$, since $x = bc_0 + \sum_{i=1}^{t}(bc_i)f_i \in I$. Consequently, $c_i \in I :_R b = I :_R J$ for all $0 \le i \le t$, so that $x \in [b{\cdot}(I :_R J)]A$ as wanted. □

Corollary 3.2.7. *$J/(b) \cong ([I :_R J]/I)^{\oplus t}$ as an R-module. Hence $\ell_R(J/(b)) = t{\cdot}\ell_R(R/J)$.*

Proof. We consider the exact sequence

$$0 \to B/R \to A/R \to A/B \to 0$$

of R-modules. By Lemma 3.2.6 (1), A/B is a free $B/\mathfrak{a}$-module of rank t, possessing the images of $\{f_i\}_{1 \le i \le t}$ in A/B as a free basis. Because A/R is a free R/I-module of rank t, also possessing the images of $\{f_i\}_{1 \le i \le t}$ in A/R as a free basis, we naturally get an isomorphism between the following two canonical exact sequences;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B/R & \xrightarrow{\ i\ } & A/R & \longrightarrow & A/B & \longrightarrow & 0 \\
 & & \downarrow \wr & \circlearrowleft & \downarrow \wr & \circlearrowleft & \downarrow \wr & & \\
0 & \longrightarrow & ([\mathfrak{a} \cap R]/I)^{\oplus t} & \xrightarrow{\ i\ } & (R/I)^{\oplus t} & \longrightarrow & (B/\mathfrak{a})^{\oplus t} & \longrightarrow & 0
\end{array}$$

Since $B/R = \dfrac{J}{b}/R \cong J/(b)$ and $\mathfrak{a} \cap R = I :_R J$ by Lemma 3.2.6 (2), we get

$$J/(b) \cong ([I :_R J]/I)^{\oplus t}.$$

The second assertion now follows from Lemma 3.2.6 (3). □

The following is the heart of this section.

Proposition 3.2.8. *The following conditions are equivalent.*

(1) $J \in \mathcal{X}_R$.

(2) $\mu_R([I :_R J]/I) = 1$.

(3) $[I :_R J]/I \cong R/J$ *as an* R*-module.*

When this is the case, $\mu_R(J) = t + 1$.

Proof. The implication (3) ⇒ (2) is clear, and the reverse implication follows from the equality $\ell_R([I :_R J]/I) = \ell_R(R/J)$ of Lemma 3.2.6 (3).

(1) ⇒ (3) Suppose that $J \in \mathcal{X}_R$. Then $J/(b)$ is R/J-free, so that by Corollary 3.2.7, $[I :_R J]/I$ is a free R/J-module, whence $[I :_R J]/I \cong R/J$ by Lemma 3.2.6 (3).

(3) ⇒ (1) We have $J/(b) \cong ([I :_R J]/I)^{\oplus t} \cong (R/J)^{\oplus t}$ by Corollary 3.2.7, so that by Definition 3.1.1, $J \in \mathcal{X}_R$ with $\mu_R(J) = t + 1$. □

We now come to the main result of this section, which plays a key role in Section 5.

Theorem 3.2.9. *The following assertions hold true.*

(1) *Suppose that* $J \in \mathcal{X}_R$. *Then there exists an element* $c \in \mathfrak{m}$ *such that* $I = bcA$. *Consequently,* $I \in \mathcal{X}_R$ *and* $\mu_R(I) = \mu_R(J) = t + 1$.

(2) *Suppose that* $t \geq 2$. *Then* $I \in \mathcal{X}_R$ *if and only if* $J \in \mathcal{X}_R$.

Proof. (1) Since $J \in \mathcal{X}_R$, by Proposition 3.2.8 we get an element $c \in \mathfrak{m}$ such that $I :_R J = (c) + I$. Therefore, by Lemma 3.2.6 (4) we have

$$I = [b{\cdot}(I :_R J)]A = [b{\cdot}((c) + I)]A = bcA + bIA = bcA + bI,$$

whence $I = bcA$ by Nakayama's lemma. Let $a = bc$. Then $I^2 = (aA)^2 = a{\cdot}aA = aI$, so that (a) is a reduction of I; hence $A = \frac{I}{a}$. Consequently, $I/(a) \cong A/R \cong (R/I)^{\oplus t}$, so that $I \in \mathcal{X}_R$ with $\mu_R(I) = t + 1$. Therefore, $\mu_R(I) = \mu_R(J)$, because $\mu_R(J) = t + 1$ by Proposition 3.2.8.

(2) We have only to show the *only if* part. Suppose that $I \in \mathcal{X}_R$ and choose $a \in I$ so that $I^2 = aI$; hence $A = \frac{I}{a}$. We then have $\mu_R(I) = t + 1$, since $I/(a) \cong A/R \cong (R/I)^{\oplus t}$. Consequently, since $J = (b) + I$, we get

$$\mu_R(J/(b)) = \mu_R([(b) + I]/(b)) \leq \mu_R(I) = t + 1.$$

On the other hand, we have $\mu_R(J/(b)) = t{\cdot}\mu_R([I :_R J]/I)$, because $J/(b) \cong ([I :_R J]/I)^{\oplus t}$ by Corollary 3.2.7. Hence

$$t{\cdot}(\mu_R([I :_R J]/I) - 1) \leq 1,$$

so that $\mu_R([I :_R J]/I) = 1$ because $t \geq 2$. Thus by Proposition 3.2.8, $J \in \mathcal{X}_R$ as claimed. □

3.3 Chains of Ulrich ideals

In this section, we study the structure of chains of Ulrich ideals in R. First of all, remember that all the Ulrich ideals of R satisfy Condition (C) stated in Definition 3.2.2 (see Remark 3.2.3), and summarizing the arguments in Section 2, we readily get the following.

Theorem 3.3.1. *Let $I, J \in \mathcal{X}_R$ and suppose that $I \subsetneq J$. Choose $b \in J$ so that $J^2 = bJ$. Then the following assertions hold true.*

(1) $J = (b) + I$.

(2) $\mu_R(J) = \mu_R(I)$.

(3) *There exists an element $c \in \mathfrak{m}$ such that $I = bcA$, so that (bc) is a reduction of I, where $A = I : I$.*

We begin with the following, which shows that Ulrich ideals behave well, if R possesses minimal multiplicity. We shall discuss this phenomenon more closely in Section 4.

Corollary 3.3.2. *Suppose that $\mathrm{v}(R) = \mathrm{e}(R) > 1$ and let $I \in \mathcal{X}_R$. Then $\mu_R(I) = \mathrm{v}(R)$ and R/I is a Gorenstein ring.*

Proof. We have $\mathfrak{m} \in \mathcal{X}_R$ and $\mathrm{r}(R) = \mathrm{v}(R) - 1$, because $\mathrm{v}(R) = \mathrm{e}(R) > 1$. Hence by Theorem 3.3.1 (2), $\mu_R(I) = \mu_R(\mathfrak{m}) = \mathrm{v}(R)$. The second assertion follows from the equality $\mathrm{r}(R) = [\mu_R(I) - 1]{\cdot}\mathrm{r}(R/I)$ (see [47, Theorem 2.5]). $\square$

For each $I \in \mathcal{X}_R$, Assertion (3) in Theorem 3.3.1 characterizes those ideals $J \in \mathcal{X}_R$ such that $I \subsetneq J$. Namely, we have the following.

Corollary 3.3.3. *Let $I \in \mathcal{X}_R$. Then*

$$\{J \in \mathcal{X}_R \mid I \subsetneq J\} = \{(b) + I \mid b \in \mathfrak{m} \text{ such that } (bc) \text{ is a reduction of } I \text{ for some } c \in \mathfrak{m}\}.$$

Proof. Let $b, c \in \mathfrak{m}$ and suppose that (bc) is a reduction of I. We set $J = (b) + I$. We shall show that $J \in \mathcal{X}_R$ and $I \subsetneq J$. Because $bc \notin \mathfrak{m}I$, we have $b, c \notin I$, whence $I \subsetneq J$. If $J = (b)$, we then have $I = bcA \subseteq J = (b)$ where $A = I : I$, so that $cA \subseteq R$. This is impossible, because $c \notin R : A = I$ (see Lemma 3.2.1). Hence, $(b) \subsetneq J$. Because $I^2 = bcI$, we have $J^2 = bJ + I^2 = bJ + bcI = bJ$. Let us check that $J/(b)$ is a free R/J-module. Let $\{f_i\}_{1 \le i \le t}$ ($t = \mu_R(I) - 1 > 0$) be elements of A such that $A = R + \sum_{i=1}^t Rf_i$, so that their images $\{\overline{f_i}\}_{1 \le i \le t}$ in A/R form a free basis of the R/I-module A/R (remember that I satisfies Condition (C) of Definition 3.2.2). We then have

$$J = (b) + I = (b) + bcA = (b) + \sum_{i=1}^{t} R{\cdot}(bc)f_i.$$

Let $\{c_i\}_{1 \le i \le t}$ be elements of R and assume that $\sum_{i=1}^t c_i{\cdot}(bcf_i) \in (b)$. Then, since $\sum_{i=1}^t c_i c{\cdot}f_i \in R$, we have $c_i c \in I = bcA$, so that $c_i \in bA \cap R$ for all $1 \le i \le t$. Therefore, because $bA \cap R \subseteq (b) + I = J$ by Lemma 3.2.4, we get $c_i \in J$, whence $J/(b) \cong (R/J)^{\oplus t}$. Thus, $J = (b) + I \in \mathcal{X}_R$. $\square$

The equality $\mu_R(I) = \mu_R(J)$ does not hold true in general, if I and J are incomparable, as we show in the following.

Example 3.3.4. Let $S = k[[X_1, X_2, X_3, X_4]]$ be the formal power series ring over a field k and consider the matrix $\mathbb{M} = \left(\begin{smallmatrix} X_1 & X_2 & X_3 \\ X_2 & X_3 & X_1 \end{smallmatrix}\right)$. We set $R = S/[\mathfrak{a} + (X_4^2)]$, where $\mathfrak{a}$ denotes the ideal of S generated by the 2×2 minors of $\mathbb{M}$. Let x_i denote the image of X_i in R for each $i = 1, 2, 3, 4$. Then, (x_1, x_2, x_3) and (x_1, x_4) are Ulrich ideals of R with different numbers of generators, and they are incomparable with respect to inclusion.

We are now ready to prove Theorem 3.1.3.

Proof of Theorem 3.1.3. (1) This is a direct consequence of Corollary 3.3.3.

(2) By Theorem 3.3.1, we may assume that $n > 2$ and that our assertion holds true for $n-1$. Therefore, there exist elements $a_1, a_2, \ldots, a_{n-1} \in \mathfrak{m}$ such that $(a_1 a_2 \cdots a_{n-1})$ is a reduction of I_{n-1} and $I_i = (a_1 a_2 \cdots a_i) + I_{n-1}$ for all $1 \le i \le n-2$. Now apply Theorem 3.3.1 to the chain $I_n \subsetneq I_{n-1}$. We then have $I_{n-1} = (a_1 a_2 \cdots a_{n-1}) + I_n$ together with one more element $a_n \in \mathfrak{m}$ so that $(a_1 a_2 \cdots a_{n-1}) \cdot a_n A = I_n$. Hence

$$I_i = (a_1 a_2 \cdots a_i) + I_{n-1} = (a_1 a_2 \cdots a_i) + I_n$$

for all $1 \le i \le n-1$. ☐

In order to prove Theorem 3.1.4, we need more preliminaries. Let us begin with the following.

Theorem 3.3.5. *Suppose that $I, J \in \mathcal{X}_R$ and $I \subsetneq J$. Let $b \in J$ such that $J^2 = bJ$ and $B = J : J$. Then the following assertions hold true.*

(1) $B = R + \frac{I}{b}$ *and* $\frac{I}{b} = I : J$.

(2) *B is a Cohen-Macaulay local ring with* $\dim B = 1$ *and* $\mathfrak{n} = \mathfrak{m} + \frac{I}{b}$ *the maximal ideal. Hence* $R/\mathfrak{m} \cong B/\mathfrak{n}$.

(3) $\frac{I}{b} \in \mathcal{X}_B$ *and* $\mu_B(\frac{I}{b}) = \mu_R(I)$.

(4) $\mathrm{r}(B) = \mathrm{r}(R)$ *and* $\mathrm{e}(B) = \mathrm{e}(R)$. *Therefore,* $\mathrm{v}(B) = \mathrm{e}(B)$ *if and only if* $\mathrm{v}(R) = \mathrm{e}(R)$.

Proof. We set $A = I : I$. Hence $R \subsetneq B \subsetneq A$ by Proposition 3.2.1. Let $t = \mu_R(I) - 1$.

(1) Because $J = (b) + I$ and $B = \frac{J}{b}$, we get $B = R + \frac{I}{b}$. We have $I : J \subseteq \frac{I}{b}$, since $b \in J$. Therefore, $\frac{I}{b} = I : J$, because

$$J \cdot \frac{I}{b} = I \cdot \frac{J}{b} = IB \subseteq IA = I.$$

(2) It suffices to show that B is a local ring with maximal ideal $\mathfrak{n} = \mathfrak{m} + \dfrac{I}{b}$. Let $\mathfrak{a} = \dfrac{I}{b}$. Choose $c \in \mathfrak{m}$ so that $I = bcA$. We then have $\mathfrak{a} = cA \subseteq \mathfrak{m}A \subseteq \mathrm{J}(A)$, where $\mathrm{J}(A)$ denotes the Jacobson radical of A. Therefore, $\mathfrak{n} = \mathfrak{m} + cA$ is an ideal of $B = R + cA$, and $\mathfrak{n} \subseteq \mathrm{J}(B)$, because A is a finite extension of B. On the other hand, because $R/\mathfrak{m} \cong B/\mathfrak{n}$, $\mathfrak{n}$ is a maximal ideal of B, so that $(B, \mathfrak{n})$ is a local ring.

(3) We have $\mathfrak{a}^2 = c\mathfrak{a}$, since $\mathfrak{a} = cA$. Notice that $\mathfrak{a} \neq cB$, since $A \neq B$. Then, because $\mathfrak{a}/cB \cong A/B \cong (B/\mathfrak{a})^{\oplus t}$ by Lemma 3.2.6 (1), we get $\mathfrak{a} \in \mathcal{X}_B$ and $\mu_B(\mathfrak{a}) = t + 1 = \mu_R(I)$.

(4) We set $L = (c) + I$. Then, since $bcA = I$, $L \in \mathcal{X}_R$ and $\mu_R(L) = \mu_R(I) = t + 1$ by Corollary 3.3.3 and Theorem 3.3.1 (2). Therefore, $\mathrm{r}(R) = t{\cdot}\mathrm{r}(R/L)$ by [47, Theorem 2.5], while $\mathrm{r}(B) = t{\cdot}\mathrm{r}(B/\mathfrak{a})$ for the same reason, because $\mathfrak{a} \in \mathcal{X}_B$ by Assertion (3). Remember that the element c is chosen so that $I :_R J = (c) + I$ (see the proof of Theorem 3.2.9 (1)). We then have $\mathrm{r}(B/\mathfrak{a}) = \mathrm{r}(R/[I :_R J])$, because $B = R + \mathfrak{a}$ and

$$R/L = R/[\mathfrak{a} \cap R] \cong B/\mathfrak{a}$$

where the first equality follows from Lemma 3.2.6 (2). Thus

$$\mathrm{r}(B) = t{\cdot}\mathrm{r}(B/\mathfrak{a}) = t{\cdot}\mathrm{r}(R/L) = \mathrm{r}(R),$$

as is claimed. To see the equality $\mathrm{e}(B) = \mathrm{e}(R)$, enlarging the residue class field of R, we may assume that $R/\mathfrak{m}$ is infinite. Choose an element $\alpha \in \mathfrak{m}$ so that (α) is a reduction of $\mathfrak{m}$. Hence αB is a reduction of $\mathfrak{m}B$, while $\mathfrak{m}B$ is a reduction of $\mathfrak{n}$, because

$$\mathfrak{n}A = (\mathfrak{m} + cA)A = \mathfrak{m}A = (\mathfrak{m}B)A.$$

Therefore, αB is a reduction of $\mathfrak{n}$, so that

$$\mathrm{e}(B) = \ell_B(B/\alpha B) = \ell_R(B/\alpha B) = \mathrm{e}^0_{\alpha R}(B) = \mathrm{e}^0_{\alpha R}(R) = \mathrm{e}(R),$$

where the second equality follows from the fact that $R/\mathfrak{m} \cong B/\mathfrak{n}$ and the fourth equality follows from the fact that $\ell_R(B/R) < \infty$. Hence $\mathrm{e}(B) = \mathrm{e}(R)$ and $\mathrm{r}(B) = \mathrm{r}(R)$. Because $\mathrm{v}(R) = \mathrm{e}(R) > 1$ if and only if $\mathrm{r}(R) = \mathrm{e}(R) - 1$, the assertion that $\mathrm{v}(B) = \mathrm{e}(B)$ if and only if $\mathrm{v}(R) = \mathrm{e}(R)$ now follows. □

We need one more lemma.

Lemma 3.3.6. *Suppose that $I, J \in \mathcal{X}_R$ and $I \subsetneq J$. Let $\alpha \in J$. Then $J = (\alpha) + I$ if and only if $J^2 = \alpha J$.*

Proof. It suffices to show the *only if* part. Suppose $J = (\alpha) + I$. We set $A = I : I$, $B = J : J$, and choose $b \in J$ so that $J^2 = bJ$. Then $J = bB$ and $B \subseteq A$, whence $JA = bA$, while $JA = [(\alpha) + I]A = \alpha A + I$. We now choose $c \in \mathfrak{m}$ so that $I = bcA$ (see Theorem 3.3.1 (3)). We then have $bA = JA = \alpha A + bcA$, whence $bA = \alpha A$ by Nakayama's lemma. Therefore, $JA = \alpha A$, whence (α) is a reduction of J, so that $J^2 = \alpha J$. □

We are now ready to prove Theorem 3.1.4.

Proof of Theorem 3.1.4. Let $I, J \in \mathcal{X}_R$ such that $I \subsetneq J$. We set $A = I : I$ and $B = J : J$. Let $b \in J$ such that $J = (b) + I$. Then $J^2 = bJ$ by Lemma 3.3.6 and B is a local ring with $\mathfrak{n} = \mathfrak{m} + \frac{I}{b}$ the maximal ideal by Theorem 3.3.5.

Let $\mathfrak{a} \in \mathcal{X}_R$ such that $I \subseteq \mathfrak{a} \subsetneq J$. First of all, let us check the following.

Claim 1. *$\frac{\mathfrak{a}}{b} \in \mathcal{X}_B$ and $\frac{\mathfrak{a}}{b} = \mathfrak{a} : J$.*

Proof of Claim 1. Since $b \in J$, $\mathfrak{a} : J \subseteq \frac{\mathfrak{a}}{b}$. On the other hand, since

$$B = R : J \subseteq R : \mathfrak{a} = \mathfrak{a} : \mathfrak{a}$$

by Lemma 3.2.1, we get

$$J \cdot \frac{\mathfrak{a}}{b} = \mathfrak{a} \cdot \frac{J}{b} = \mathfrak{a}B \subseteq \mathfrak{a}{\cdot}(\mathfrak{a} : \mathfrak{a}) = \mathfrak{a},$$

so that $\frac{\mathfrak{a}}{b}$ is an ideal of $B = \frac{J}{b}$ and $\mathfrak{a} : J = \frac{\mathfrak{a}}{b}$. Since $\frac{I}{b} \in \mathcal{X}_B$ by Theorem 3.3.5 (3), to show that $\frac{\mathfrak{a}}{b} \in \mathcal{X}_B$, we may assume $I \subsetneq \mathfrak{a}$. We then have, by Theorem 3.1.3 (2), elements $a_1, a_2 \in \mathfrak{m}$ such that $I = ba_1a_2A$ and $\mathfrak{a} = (ba_1) + I$; hence $\frac{\mathfrak{a}}{b} = a_1R + \frac{I}{b}$. We get $\frac{\mathfrak{a}}{b} = a_1B + \frac{I}{b}$, since $\frac{\mathfrak{a}}{b}$ is an ideal of B. Therefore, $\frac{\mathfrak{a}}{b} \in \mathcal{X}_B$ by Corollary 3.3.3, because a_1a_2B is a reduction of $\frac{I}{b} = a_1a_2A$. □

We now have the correspondence φ defined by $\mathfrak{a} \mapsto \frac{\mathfrak{a}}{b}$, and it is certainly injective. Suppose that $\mathfrak{b} \in \mathcal{X}_B$ and $\frac{I}{b} \subsetneq \mathfrak{b}$. We take $\alpha \in \mathfrak{b}$ so that $\mathfrak{b}^2 = \alpha\mathfrak{b}$. Then, since B is a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m} + \frac{I}{b}$, we have $\mathfrak{b} = \alpha B + \frac{I}{b}$ by Theorem 3.3.1. Let us write $\alpha = a + x$ with $a \in \mathfrak{m}$ and $x \in \frac{I}{b}$. We then have $\mathfrak{b} = aB + \frac{I}{b}$, so that $\mathfrak{b}^2 = a\mathfrak{b}$ by Lemma 3.3.6. Set $L = \frac{I}{b}$. Then, since $A = I : I = L : L$, by Theorem 3.3.1 we have an element $\beta \in \mathfrak{n} = \mathfrak{m} + L$ such that $L = a\beta A$; hence $a\beta \in L$. Let us write $\beta = c + y$ with $c \in \mathfrak{m}$ and $y \in L$. We then have $ac = a\beta - ay \in L$ and $yA \subseteq L$, so that because

$$L = a\beta A \subseteq acA + a{\cdot}yA \subseteq acA + \mathfrak{m}L,$$

we get $L = acA$ by Nakayama's lemma. Therefore, $I = abcA$. On the other hand, since $aB = aR + a{\cdot}\frac{I}{b}$, we get $\mathfrak{b} = aB + \frac{I}{b} = aR + \frac{I}{b}$. Hence, because $b\mathfrak{b} = (ab) + I$ and $I = (ab)cA$, we finally have that $b\mathfrak{b} \in \mathcal{X}_R$ and

$$I = abcA \subsetneq b\mathfrak{b} = (ab) + I \subsetneq J$$

by Theorem 3.1.3 (1). Thus, the correspondence φ is bijective, which completes the proof of Theorem 3.1.4. □

3.4 The case where R possesses minimal multiplicity

In this section, we focus our attention on the case where R possesses minimal multiplicity. Throughout, we assume that $\mathrm{v}(R) = \mathrm{e}(R) > 1$. Hence, $\mathfrak{m} \in \mathcal{X}_R$ and $\mu_R(I) = v$ for all $I \in \mathcal{X}_R$ by Corollary 3.3.2, where $v = \mathrm{v}(R)$. We choose an element $\alpha \in \mathfrak{m}$ so that $\mathfrak{m}^2 = \alpha\mathfrak{m}$.

Let $I, J \in \mathcal{X}_R$ such that $I \subsetneq J$ and assume that there are no Ulrich ideals contained strictly between I and J. Let $b \in J$ with $J^2 = bJ$ and set $B = J : J$. Hence $B = \frac{J}{b}$, and $J = (b) + I$ by Theorem 3.3.1. Remember that by Theorem 3.3.5, B is a local ring and $\mathrm{v}(B) = \mathrm{e}(B) = \mathrm{e}(R) > 1$. We have $\mathfrak{n}^2 = \alpha\mathfrak{n}$ by the proof of Theorem 3.3.5 (4), where $\mathfrak{n}$ denotes the maximal ideal of B.

We furthermore have the following.

Lemma 3.4.1. *The following assertions hold true.*

(1) $\ell_R(J/I) = 1$.

(2) $I = b\mathfrak{n} = J\mathfrak{n}$. *Hence, the ideal I is uniquely determined by J, and $I : I = \mathfrak{n} : \mathfrak{n}$.*

(3) $(b\alpha)$ *is a reduction of I. If $I = (b\alpha) + (x_2, x_3, \ldots, x_v)$, then $J = (b, x_2, x_3, \ldots, x_v)$.*

Proof. By Theorem 3.1.4, we have the one-to-one correspondence

$$\{\mathfrak{a} \in \mathcal{X}_R \mid I \subseteq \mathfrak{a} \subsetneq J\} \xrightarrow{\varphi} \{\mathfrak{b} \in \mathcal{X}_B \mid \frac{I}{b} \subseteq \mathfrak{b}\}, \quad \mathfrak{a} \mapsto \frac{\mathfrak{a}}{b},$$

where the set of the left hand side is a singleton consisting of I, and the set of the right hand side contains $\mathfrak{n}$. Hence $\mathfrak{n} = \frac{I}{b}$, that is $I = b\mathfrak{n} = J\mathfrak{n}$, because $J = bB$. Therefore, $I^2 = b^2\mathfrak{n}^2 = b\alpha{\cdot}b\mathfrak{n} = b\alpha{\cdot}I$, so that $(b\alpha)$ is a reduction of I. Because

$$J/I = bB/b\mathfrak{n} \cong B/\mathfrak{n}$$

and $R/\mathfrak{m} \cong B/\mathfrak{n}$ by Theorem 3.3.5 (2), we get $\ell_R(J/I) = 1$. Assertion (3) is clear, since $J = (b) + I$. □

Since $\ell_R(R/I) < \infty$ for all $I \in \mathcal{X}_R$, we get the following.

Corollary 3.4.2. *Suppose that $I, J \in \mathcal{X}_R$ and $I \subsetneq J$. Then there exists a composition series $I = I_\ell \subsetneq I_{\ell-1} \subsetneq \cdots \subsetneq I_1 = J$ such that $I_i \in \mathcal{X}_R$ for all $1 \leq i \leq \ell$, where $\ell = \ell_R(J/I) + 1$.*

The following is the heart of this section.

Theorem 3.4.3. *The set $\mathcal{X}_R$ is totally ordered with respect to inclusion.*

Proof. Suppose that there exist $I, J \in \mathcal{X}_R$ such that $I \nsubseteq J$ and $J \nsubseteq I$. Since $I \subsetneq \mathfrak{m}$ and $J \subsetneq \mathfrak{m}$, thanks to Corollary 3.4.2, we get composition series

$$I = I_\ell \subsetneq I_{\ell-1} \subsetneq \cdots \subsetneq I_1 = \mathfrak{m} \quad \text{and} \quad J = J_n \subsetneq J_{n-1} \subsetneq \cdots \subsetneq J_1 = \mathfrak{m}$$

such that $I_i, J_j \in \mathcal{X}_R$ for all $1 \leq i \leq \ell$ and $1 \leq j \leq n$. We may assume $\ell \leq n$. Then Lemma 3.4.1 (2) shows that $I_i = J_i$ for all $1 \leq i \leq \ell$, whence $J \subseteq J_\ell = I_\ell \subseteq I$. This is a contradiction. □

Remark 3.4.4. Theorem 3.4.3 is no longer true, unless R possesses minimal multiplicity. For example, let k be a field and consider $R = k[[t^3, t^7]]$ in the formal power series ring $k[[t]]$. Then, $\mathcal{X}_R = \{(t^6 - ct^7, t^{10}) \mid 0 \neq c \in k\}$, which is not totally ordered, if $\sharp k > 2$. See Example 3.5.7 (3) also.

Let us now summarize the results in the case where R possesses minimal multiplicity.

Theorem 3.4.5. *Let $I \in \mathcal{X}_R$ and take a composition series*

$$(E) \quad I = I_\ell \subsetneq I_{\ell-1} \subsetneq \cdots \subsetneq I_1 = \mathfrak{m}$$

so that $I_i \in \mathcal{X}_R$ for every $1 \leq i \leq \ell = \ell_R(R/I)$. We set $B_0 = R$ and $B_i = I_i : I_i$ for $1 \leq i \leq \ell$ and let $\mathfrak{n}_i = \mathrm{J}(B_i)$ denote the Jacobson radical of B_i for each $0 \leq i \leq \ell$. Then we obtain a tower

$$R = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_{\ell-1} \subsetneq B_\ell \subseteq \overline{R}$$

of birational finite extensions of R and furthermore have the following.

(1) *(α^i) is a reduction of I_i for every $1 \leq i \leq \ell$.*

(2) *$B_i = \mathfrak{n}_{i-1} : \mathfrak{n}_{i-1}$ for every $1 \leq i \leq \ell$.*

(3) *For $0 \leq i \leq \ell - 1$, $(B_i, \mathfrak{n}_i)$ is a local ring with $\mathrm{v}(B_i) = \mathrm{e}(B_i) = \mathrm{e}(R) > 1$ and $\mathfrak{n}_i^2 = \alpha\mathfrak{n}_i$.*

(4) *Choose $x_2, x_3, \ldots, x_v \in I$ so that $I = (\alpha^\ell, x_2, \ldots, x_v)$. Then $I_i = (\alpha^i, x_2, x_3, \ldots, x_v)$ for every $1 \leq i \leq \ell$. In particular, $\mathfrak{m} = (\alpha, x_2, x_3, \ldots, x_v)$, so that the series (E) is a unique composition series of ideals in R which connects I and $\mathfrak{m}$.*

(5) *Let J be an ideal of R and assume that $I \subseteq J \subseteq \mathfrak{m}$. Then $J = I_i$ for some $1 \leq i \leq \ell$.*

Proof. The uniqueness of composition series in Assertion (4) follows from the fact that the maximal ideal $\mathfrak{m}/I$ of R/I is cyclic, and then, Assertion (5) readily follows from the uniqueness. Assertions (1), (2), (3), and the first part of Assertion (4) follow by standard induction on ℓ. □

Corollary 3.4.6. *Suppose that there exists a minimal element I in $\mathcal{X}_R$. Then $\sharp\mathcal{X}_R = \ell < \infty$ with $\ell = \ell_R(R/I)$.*

Proof. Since $\mathcal{X}_R$ is totally ordered by Theorem 3.4.3, I is the smallest element in $\mathcal{X}_R$, so that $I \subseteq J$ for all $J \in \mathcal{X}_R$. Therefore, by Theorem 3.4.5 (5), J is one of the I_i's in the compoosition series $I = I_\ell \subsetneq I_{\ell-1} \subsetneq \cdots \subsetneq I_1 = \mathfrak{m}$. □

Corollary 3.4.7. *If $\widehat{R}$ is a reduced ring, then $\mathcal{X}_R$ is a finite set.*

Proof. Since by Theorem 3.4.5 $\ell_R(R/I) \leq \ell_R(\overline{R}/R) < \infty$ for every $I \in \mathcal{X}_R$, the set $\mathcal{X}_R$ contains a minimal element, so that $\mathcal{X}_R$ is a finite set. $\square$

Here let us note the following.

Example 3.4.8. Let $(S, \mathfrak{n})$ be a two-dimensional regular local ring. Let $\mathfrak{n} = (X, Y)$ and consider the ring $A = S/(Y^2)$. Then $\mathrm{v}(A) = \mathrm{e}(A) = 2$ and

$$\mathcal{X}_A = \{(x^n, y) \mid n \geq 1\}$$

where x, y denote the images of X, Y in A, respectively. Hence $\sharp\mathcal{X}_A = \infty$.

Proof. Let $I_n = (x^n, y)$ for each $n \geq 1$. Then $(x^n) \subsetneq I_n$ and $I_n^2 = x^n I_n$. Let $\mathrm{J}(A) = (x, y)$ be the maximal ideal of A. We then have $J(A)^2 = xJ(A)$, whence $\mathrm{v}(A) = \mathrm{e}(A) = 2$. Because $I_n = (x^n) :_A y$, we get $I_n/(x^n) \cong A/I_n$. Therefore, $I_n \in \mathcal{X}_A$ for all $n \geq 1$. To see that $\mathcal{X}_A$ consists of these ideals I_n's, let $I \in \mathcal{X}_A$ and set $\ell = \ell_A(A/I)$. Then $I \subseteq I_\ell$ or $I \supseteq I_\ell$, since $\mathcal{X}_A$ is totally ordered. In any case, $I = I_\ell$, because $\ell_A(A/I_\ell) = \ell$. Hence $\mathcal{X}_A = \{(x^n, y) \mid n \geq 1\}$. $\square$

We close this section with the following. Here, the hypothesis about the existence of a fractional canonical ideal K is equivalent to saying that R contains an $\mathfrak{m}$-primary ideal I such that $I \cong \mathrm{K}_R$ as an R-module and such that I possesses a reduction $Q = (a)$ generated by a single element a of R ([36, Corollary 2.8]). The latter condition is satisfied, once $\mathrm{Q}(\widehat{R})$ is a Gorenstein ring and the field $R/\mathfrak{m}$ is infinite.

Theorem 3.4.9. *Suppose that there exists a fractional ideal K of R such that $R \subseteq K \subseteq \overline{R}$ and $K \cong \mathrm{K}_R$ as an R-module. Then the following conditions are equivalent.*

(1) $\sharp\mathcal{X}_R = \infty$.

(2) $\mathrm{e}(R) = 2$ *and $\widehat{R}$ is not a reduced ring.*

(3) *The ring $\widehat{R}$ has the form $\widehat{R} \cong S/(Y^2)$ for some regular local ring $(S, \mathfrak{n})$ of dimension two with $Y \in \mathfrak{n} \setminus \mathfrak{n}^2$.*

Proof. (1) $\Rightarrow$ (2) The ring $\widehat{R}$ is not reduced by Corollary 3.4.7. Suppose R is not a Gorenstein ring; hence $R \subsetneq K$ and $\mathrm{e}(R) > 2$. We set $\mathfrak{a} = R : K$. Let $I \in \mathcal{X}_R$. Then, since $\mu_R(I) = v = \mathrm{e}(R) > 2$ by Corollary 3.3.2, we have $\mathfrak{a} \subseteq I$ by [47, Corollary 2.12], so that $\ell_R(R/I) \leq \ell_R(R/\mathfrak{a}) < \infty$. Therefore, the set $\mathcal{X}_R$ contains a minimal element, which is a contradiction.

(3) $\Rightarrow$ (1) See Example 3.4.8 and use the fact that there is a one-to-one correspondence $I \mapsto I\widehat{R}$ between Ulrich ideals of R and $\widehat{R}$, respectively.

(2) $\Rightarrow$ (3) Since $\mathrm{v}(R) = \mathrm{e}(R) = 2$, the completion $\widehat{R}$ has the form $\widehat{R} = S/I$, where $(S, \mathfrak{n})$ is a two-dimensional regular local ring and $I = (f)$ a principal ideal of S. Notice that $\mathrm{e}(S/(f)) = 2$ and $\sqrt{(f)} \neq (f)$. We then have $(f) = (Y^2)$ for some $Y \in \mathfrak{n} \setminus \mathfrak{n}^2$, because $f \in \mathfrak{n}^2 \setminus \mathfrak{n}^3$. $\square$

Remark 3.4.10. In Theorem 3.4.9, the hypothesis on the existence of fractional canonical ideals K is not superfluous. In fact, let V denote a discrete valuation ring and consider the idealization $R = V \ltimes F$ of the free V-module $F = V^{\oplus n}$ $(n \geq 2)$. Let t be a regular parameter of V. Then for each $n \geq 1$, $I_n = (t^n) \times F$ is an Ulrich ideal of R ([43, Example 2.2]). Hence $\mathcal{X}_R$ is infinite, but $\mathrm{v}(R) = \mathrm{e}(R) = n + 1 \geq 3$.

Higher dimensional cases are much wilder. Even though $(R, \mathfrak{m})$ is a two-dimensional Cohen-Macaulay local ring possessing minimal multiplicity, the set $\mathcal{X}_R$ is not necessarily totally ordered. Before closing this section, let us note examples.

Example 3.4.11. We consider two examples.

(1) Let $S = k[[X_0, X_1, \ldots, X_n]]$ $(n \geq 3)$ be the formal power series ring over a field k. Let $\ell \geq 1$ be an integer and consider the $2 \times n$ matrix

$$\mathbb{M} = \begin{pmatrix} X_1 & X_2 & \cdots & X_n \\ X_0^\ell & X_1 & \cdots & X_{n-1} \end{pmatrix}.$$

We set $R = S/\mathbb{I}_2(\mathbb{M})$, where $\mathbb{I}_2(\mathbb{M})$ denotes the ideal of S generated by the 2×2 minors of the matrix $\mathbb{M}$. Then, R is a Cohen-Macaulay local ring of dimension two, possessing minimal multiplicity. For this ring, we have

$$\mathcal{X}_R = \{(x_0^i, x_1, x_2, \ldots, x_n) \mid 1 \leq i \leq \ell\},$$

where x_i denotes the image of X_i in R for each $0 \leq i \leq n$. Therefore, the set $\mathcal{X}_R$ is totally ordered with respect to inclusion.

(2) Let $(S, \mathfrak{n})$ be a regular local ring of dimension three. Let $F, G, H, Z \in \mathfrak{n}$ and assume that $\mathfrak{n} = (F, G, Z) = (G, H, Z) = (H, F, Z)$. (For instance, let $S = k[[X, Y, Z]]$ be the formal power series ring over a field k with $\operatorname{ch} k \neq 2$, and choose $F = X, G = X + Y, H = X - Y$.) We consider the ring $R = S/(Z^2 - FGH)$. Then R is a two-dimensional Cohen-Macaulay local ring of minimal multiplicity two. Let f, g, h, z denote, respectively, the images of F, G, H, Z in R. Then, (f, gh, z), (g, fh, z), (h, fg, z) are Ulrich ideals of R, but any two of them are incomparable.

3.5 The case where R is a generalized Gorenstein local ring

In this section, we study the case where R is a generalized Gorenstein local ring. The notion of generalized Gorenstein local rings is given by [34]. Let us briefly review the definition.

Definition 3.5.1 ([34]). Suppose that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring with $d = \dim R \geq 0$, possessing the canonical module K_R. We say that R is a *generalized Gorenstein local* ring, if one of the following conditions is satisfied.

(1) R is a Gorenstein ring.

(2) R is not a Gorenstein ring, but there exists an exact sequence

$$0 \to R \xrightarrow{\varphi} \mathrm{K}_R \to C \to 0$$

of R-modules and an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R such that

(i) C is an Ulrich R-module with respect to $\mathfrak{a}$ and

(ii) the induced homomorphism $R/\mathfrak{a} \otimes_R \varphi : R/\mathfrak{a} \to \mathrm{K}_R/\mathfrak{a}\mathrm{K}_R$ is injective.

When Case (2) occurs, we especially say that R is a generalized Gorenstein local ring with respect to $\mathfrak{a}$.

Since our attention is focused on the one-dimensional case, here let us summarize a few results on generalized Gorenstein local rings of dimension one. Suppose that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension one, admitting a fractional canonical ideal K. Hence, K is an R-submodule of $\overline{R}$ such that $K \cong \mathrm{K}_R$ as an R-module and $R \subseteq K \subseteq \overline{R}$. One can consult [36, Sections 2, 3] and [54, Vortrag 2] for basic properties of K. We set $S = R[K]$ in $\mathrm{Q}(R)$. Therefore, S is a birational finite extension of R with $S = K^n$ for all $n \gg 0$, and the ring $S = R[K]$ is independent of the choice of K ([15, Theorem 2.5]). We set $\mathfrak{c} = R : S$. First of all, let us note the following.

Lemma 3.5.2 (cf. [36, Lemma 3.5]). *$\mathfrak{c} = K : S$ and $S = \mathfrak{c} : \mathfrak{c} = R : \mathfrak{c}$.*

Proof. Since $R = K : K$ ([54, Bemerkung 2.5 a)]), we have $\mathfrak{c} = (K : K) : S = K : KS = K : S$, while $R : \mathfrak{c} = (K : K) : \mathfrak{c} = K : K\mathfrak{c} = K : \mathfrak{c}$. Hence $R : \mathfrak{c} = K : \mathfrak{c} = K : (K : S) = S$ ([54, Definition 2.4]). Therefore, $\mathfrak{c} : \mathfrak{c} = (K : S) : \mathfrak{c} = K : S\mathfrak{c} = K : \mathfrak{c} = S$. $\square$

We then have the characterization of generalized Gorenstein local rings.

Theorem 3.5.3 ([34]). *Suppose that R is not a Gorenstein ring. Then the following conditions are equivalent.*

(1) *R is a generalized Gorenstein local ring with respect to some $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R.*

(2) *K/R is a free $R/\mathfrak{c}$-module.*

(3) *S/R is a free $R/\mathfrak{c}$-module.*

When this is the case, one necessarily has $\mathfrak{a} = \mathfrak{c}$, and the following assertions hold true.

(i) *$R/\mathfrak{c}$ is a Gorenstein ring.*

(ii) *$S/R \cong (R/\mathfrak{c})^{\oplus \mathrm{r}(R)}$ as an R-module.*

The following result is due to [34, 47]. Let us include a brief proof of Assertion (1) for the sake of completeness.

Theorem 3.5.4 ([34, 47])**.** *Suppose that R is not a Gorenstein ring. Let $I \in \mathcal{X}_R$. Then the following assertions hold true.*

(1) *If $I \subseteq \mathfrak{c}$, then $I = \mathfrak{c}$.*

(2) *If $\mu_R(I) \neq 2$, then $\mathfrak{c} \subseteq I$.*

(3) *$\mathfrak{c} \in \mathcal{X}_R$ if and only if R is a generalized Gorenstein local ring and S is a Gorenstein ring.*

Proof. (1) Let $I \in \mathcal{X}_R$ and assume that $I \subseteq \mathfrak{c}$. We choose an element $a \in I$ so that $I^2 = aI$. We then have $I \neq (a)$ and I/I^2 is a free R/I-module. Let $A = I : I$; hence $I = aA$. On the other hand, because $\mathfrak{c} \subseteq I$, by Lemmata 3.2.1 and 3.5.2 we have

$$A = R : I \supseteq R : \mathfrak{c} = S \supseteq K.$$

Claim 2. *A is a Gorenstein ring and A/K is the canonical module of R/I.*

Proof of Claim 2. Taking the K-dual of the canonical exact sequence $0 \to I \to R \to R/I \to 0$, we get the exact sequence

$$0 \to K \xrightarrow{\iota} K : I \to \mathrm{Ext}^1_R(R/I, K) \to 0,$$

where $\iota : K \to K : I$ denotes the embedding. On the other hand, $K : I = A$, because

$$I = R : A = (K : K) : A = K : KA = K : A$$

(remember that $K \subseteq A$). Therefore, since $I = K : A$ is a canonical ideal of A ([54, Korollar 5.14]) and $I = aA \cong A$, A is a Gorenstein ring, and $A/K \cong \mathrm{Ext}^1_R(R/I, K)$. $\square$

We consider the exact sequence $0 \to (a)/aI \to I/aI \to I/(a) \to 0$ of R/I-modules. Then, because $I = aA$, we get the canonical isomorphism between the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/I & \xrightarrow{i} & A/I & \longrightarrow & A/R & \longrightarrow & 0 \\ & & \downarrow \wr & \circlearrowleft & \downarrow \wr & \circlearrowleft & \downarrow \wr & & \\ 0 & \longrightarrow & (a)/aI & \xrightarrow{i} & I/aI & \longrightarrow & I/(a) & \longrightarrow & 0 \end{array}$$

of R/I-modules, where A/I is a Gorenstein ring, since A is a Gorenstein ring and $I = aA$. Therefore, since A/I ($\cong I/aI$) is a flat extension of R/I, R/I is a Gorenstein ring, so that $A/K \cong R/I$ by Claim 2. Consequently, the exact sequence

$$0 \to K/R \to A/R \to A/K \to 0$$

of R/I-modules is split, whence K/R is a non-zero free R/I-module, because so is A/R ($\cong I/(a)$). Hence, $\mathfrak{c} = R : S \subseteq R : K = R :_R K = I$, so that $I = \mathfrak{c}$. $\square$

Thanks to Theorem 3.5.4, we get the following.

Theorem 3.5.5. *Let R be a generalized Gorenstein local ring and assume that R is not a Gorenstein ring. Then the following assertions hold true.*

(1) $\{I \in \mathcal{X}_R | \ \mathfrak{c} \subsetneq I\} = \{(a) + \mathfrak{c} \ | \ a \in \mathfrak{m}$ *such that* $\mathfrak{c} = abS$ *for some* $b \in \mathfrak{m}\}$.
In particular, $\mathfrak{c} \in \mathcal{X}_R$*, once the set* $\{I \in \mathcal{X}_R | \ \mathfrak{c} \subsetneq I\}$ *is non-empty.*

(2) $\mu_R(I) = \mathrm{r}(R) + 1$ *for all* $I \in \mathcal{X}_R$ *such that* $\mathfrak{c} \subseteq I$.

(3) $\{I \in \mathcal{X}_R | \ \mathfrak{c} \subseteq I\} = \{I \in \mathcal{X}_R | \ \mu_R(I) \neq 2\}$.

Therefore, if R possesses minimal multiplicity, then the set $\mathcal{X}_R$ is totally ordered, and $\mathfrak{c}$ is the smallest element of $\mathcal{X}_R$.

Proof. (1) Let us show the first equality. First of all, assume that $\mathfrak{c} \in \mathcal{X}_R$. Then since $S = \mathfrak{c} : \mathfrak{c}$, for each $\alpha \in \mathfrak{c}$, (α) is a reduction of $\mathfrak{c}$ if and only if $\mathfrak{c} = \alpha S$, so that the required equality follows from Corollary 3.3.3. Assume that $\mathfrak{c} \notin \mathcal{X}_R$. Hence, by Theorem 3.5.4 (3), S is not a Gorenstein ring, because R is a generalized Gorenstein local ring. Therefore, since $\mathfrak{c} = K : S$ is a canonical module of S (Lemma 3.5.2 and [54, Korollar 5.14]), we have $\mathfrak{c} \neq \alpha S$ for any $\alpha \in \mathfrak{c}$, whence the set $\{(a) + \mathfrak{c} \mid a \in \mathfrak{m}$ such that $abS = \mathfrak{c}$ for some $b \in \mathfrak{m}\}$ is empty. On the other hand, since $S = \mathfrak{c} : \mathfrak{c} = R : \mathfrak{c}$ and $S/R \cong (R/\mathfrak{c})^{\oplus \mathrm{r}(R)}$ (see Theorem 3.5.3 (ii)), the $\mathfrak{m}$-primary ideal $\mathfrak{c}$ of R satisfies Condition (C) in Definition 3.2.2. Therefore, if the set $\{I \in \mathcal{X}_R \mid \mathfrak{c} \subsetneq I\}$ is non-empty, then $\mathfrak{c} \in \mathcal{X}_R$ by Theorem 3.2.9 (2), because $\mathrm{r}(R) \geq 2$. Thus, $\{I \in \mathcal{X}_R \mid \mathfrak{c} \subsetneq I\} = \emptyset$.

(2) By Assertion (1), we may assume $\mathfrak{c} \in \mathcal{X}_R$. Then, $\mathfrak{c} = \alpha S$ for some $\alpha \in \mathfrak{c}$, and therefore, $\mu_R(\mathfrak{c}) = \mathrm{r}(R) + 1$, since $\mathfrak{c}/(\alpha) \cong S/R \cong (R/\mathfrak{c})^{\oplus \mathrm{r}(R)}$. Thus, by Theorem 3.3.1, $\mu_R(I) = \mu_R(\mathfrak{c}) = \mathrm{r}(R) + 1$ for every $I \in \mathcal{X}_R$ with $\mathfrak{c} \subseteq I$.

(3) The assertion follows from Assertion (2) and Theorem 3.5.4 (3).

The last assertion follows from Assertion (3), since $\mu_R(I) = \mathrm{v}(R) > 2$ for every $I \in \mathcal{X}_R$ (see Corollary 3.3.2). ☐

Combining Theorems 3.1.3 and 3.5.5, we have the following.

Corollary 3.5.6. *Let R be a generalized Gorenstein local ring and assume that R is not a Gorenstein ring. Then the following assertions hold true.*

(1) *Let* $a_1, a_2, \ldots, a_n, b \in \mathfrak{m}$ $(n \geq 1)$ *and assume that* $\mathfrak{c} = a_1 a_2 \cdots a_n bS$. *We set* $I_i = (a_1 a_2 \cdots a_i) + \mathfrak{c}$ *for each* $1 \leq i \leq n$. *Then* $\mathfrak{c} \in \mathcal{X}_R$ *and* $I_i \in \mathcal{X}_R$ *for all* $1 \leq i \leq n$, *forming a chain* $\mathfrak{c} \subsetneq I_n \subsetneq I_{n-1} \subsetneq \ldots \subsetneq I_1$ *in* $\mathcal{X}_R$.

(2) *Conversely, let* $I_1, I_2, \ldots, I_n \in \mathcal{X}_R$ $(n \geq 1)$ *and assume that* $\mathfrak{c} \subsetneq I_n \subsetneq I_{n-1} \subsetneq \ldots \subsetneq I_1$. *Then* $\mathfrak{c} \in \mathcal{X}_R$ *and there exist elements* $a_1, a_2, \ldots, a_n, b \in \mathfrak{m}$ *such that* $\mathfrak{c} = a_1 a_2 \cdots a_n bS$ *and* $I_i = (a_1 a_2 \cdots a_i) + \mathfrak{c}$ *for all* $1 \leq i \leq n$.

Concluding this chapter, let us note a few examples of generalized Gorenstein local rings.

Example 3.5.7. Let $k[[t]]$ be the formal power series ring over a field k.

(1) Let $H = \langle 5, 7, 9, 13 \rangle$ denote the numerical semigroup generated by $5, 7, 9, 13$ and $R = k[[t^5, t^7, t^9, t^{13}]]$ the semigroup ring of H over k. Then, R is a generalized Gorenstein local ring, possessing $S = k[[t^3, t^5, t^7]]$ and $\mathfrak{c} = (t^7, t^9, t^{10}, t^{13})$. For this ring R, S is not a Gorenstein ring, and $\mathcal{X}_R = \emptyset$.

(2) Let $R = k[[t^4, t^9, t^{15}]]$. Then, R is a generalized Gorenstein local ring, possessing $S = k[[t^3, t^4]]$ and $\mathfrak{c} = (t^9, t^{12}, t^{15}) = t^9 S$. For this ring R, $\mathcal{X}_R = \{\mathfrak{c}\}$.

(3) Let $R = k[[t^6, t^{13}, t^{28}]]$. Then, R is a generalized Gorenstein local ring, possessing $S = k[[t^2, t^{13}]]$ and $\mathfrak{c} = (t^{24}, t^{26}, t^{28}) = t^{24} S$. For this ring R, the set $\{I \in \mathcal{X}_R \mid \mathfrak{c} \subsetneq I\}$ consists of the following families.

   (i) $\{(t^6 + at^{13}) + \mathfrak{c} \mid a \in k\}$,

   (ii) $\{(t^{12} + at^{13} + bt^{19}) + \mathfrak{c} \mid a, b \in k\}$, and

   (iii) $\{(t^{18} + at^{25}) + \mathfrak{c} \mid a \in k\}$.

   For each $a \in k$, we have a maximal chain

$$\mathfrak{c} \subsetneq (t^{18} + at^{25}) + \mathfrak{c} \subsetneq (t^{12} + at^{19}) + \mathfrak{c} \subsetneq (t^6 + at^{13}) + \mathfrak{c}$$

   in $\mathcal{X}_R$. On the other hand, for $a, b \in k$ such that $a \neq 0$,

$$\mathfrak{c} \subsetneq (t^{12} + at^{13} + bt^{19}) + \mathfrak{c}$$

   is also a maximal chain in $\mathcal{X}_R$.

(4) Let $H = \langle 6, 13, 28 \rangle$. Choose integers $0 < \alpha \in H$ and $1 < \beta \in \mathbb{Z}$ so that $\alpha \notin \{6, 13, 28\}$ and $\mathrm{GCD}(\alpha, \beta) = 1$. We consider $R = k[[t^\alpha, t^{6\beta}, t^{13\beta}, t^{28\beta}]]$. Then, R is a generalized Gorenstein local ring with $\mathrm{v}(R) = 4$ and $\mathrm{r}(R) = 2$. For this ring R, $S = k[[t^\alpha, t^{2\beta}, t^{13\beta}]]$, and $\mathfrak{c} = t^{24\beta} S$. For instance, take $\alpha = 12$ and $\beta = 5n$, where $n > 0$ and $\mathrm{GCD}(2, n) = \mathrm{GCD}(3, n) = 1$. Then, $\mathfrak{c} = t^{120n} S = (t^{12})^{10n} S$, so that the set $\{I \in \mathcal{X}_R \mid \mathfrak{c} \subsetneq I\}$ seems rather wild, containing chains of large length.

Chapter 4

Correspondence between trace ideals and birational extensions with application to the analysis of the Gorenstein property of rings

4.1 Introduction

This chapter aims to explore the structure of (not necessarily Noetherian) commutative rings in connection with their trace ideals. Let R be a commutative ring. For R-modules M and X, let

$$\tau_{M,X} : \mathrm{Hom}_R(M, X) \otimes_R M \to X$$

denote the R-linear map defined by $\tau_{M,X}(f \otimes m) = f(m)$ for all $f \in \mathrm{Hom}_R(M, X)$ and $m \in M$. We set $\tau_X(M) = \mathrm{Im}\, \tau_{M,X}$. Then, $\tau_X(M)$ is an R-submodule of X, and we say that an R-submodule Y of X is a *trace module* in X, if $Y = \tau_X(M)$ for some R-module M. When $X = R$, we call trace modules in R, simply, *trace ideals* in R. There is a recent movement in the theory of trace ideals, raised by H. Lindo and N. Pande [64, 65, 66]. Besides, J. Herzog, T. Hibi, and D. I. Stamate [55] studied the traces of canonical modules, and gave interesting results. We explain below our motivation for the present researches and how this chapter is organized, claiming the main results in it.

The present researches are strongly inspired by [64, 65, 66]. In [65] Lindo asked when every ideal of a given ring R is a trace ideal in it, and noted that this is the case when R is a self-injective ring. Subsequently, Lindo and Pande [66] proved that the converse is also true if R is a Noetherian local ring. Our researches have started from the following complete answer to their prediction, which we shall prove in Section 4.

Theorem 4.1.1 (Theorem 4.4.1)**.** *Suppose that R is a Noetherian ring and let X be an R-module. Then the following conditions are equivalent.*

(1) *Every R-submodule of X is a trace module in X.*

(2) *Every cyclic R-submodule of X is a trace module in X.*

(3) *There is an embedding*

$$0 \to X \to \bigoplus_{\mathfrak{m} \in \operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{m})$$

of R-modules, where for each $\mathfrak{m} \in \operatorname{Max} R$, $\mathrm{E}_R(R/\mathfrak{m})$ denotes the injective envelope of the cyclic R-module $R/\mathfrak{m}$.

However, the main activity in the present chapter is focused on the study of the structure of the set of regular trace ideals in R. Let I be an ideal of a commutative ring R and suppose that I is *regular*, that is I contains a non-zerodivisor of R. Then, as is essentially shown by [65, Lemma 2.3], I is a trace ideal in R if and only if $R : I = I : I$, where the colon is considered inside the total ring $\mathrm{Q}(R)$ of fractions of R. We denote by $\mathcal{X}_R$ the set of regular trace ideals in R, and explore the structure of $\mathcal{X}_R$ in connection with the structure of $\mathcal{Y}_R$, where $\mathcal{Y}_R$ denotes the set of birational extensions A of R such that $aA \subseteq R$ for some non-zerodivisor a of R. We also consider the set $\mathcal{Z}_R$ of regular ideals I of R such that $I^2 = aI$ for some $a \in I$. We then have the following natural correspondences

$$\xi : \mathcal{Z}_R \to \mathcal{Y}_R, \quad \xi(I) = I : I,$$

$$\eta : \mathcal{Y}_R \to \mathcal{X}_R, \quad \eta(A) = R : A,$$

$$\rho : \mathcal{X}_R \to \mathcal{Y}_R, \quad \rho(I) = I : I$$

among these sets. The basic framework is the following.

Proposition 4.1.2 (Proposition 4.2.9, Lemma 4.2.6 (1))**.** *The correspondence $\xi : \mathcal{Z}_R \to \mathcal{Y}_R$ is surjective, and the following conditions are equivalent.*

(1) $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ *is surjective.*

(2) $\eta : \mathcal{Y}_R \to \mathcal{X}_R$ *is injective.*

(3) $A = R : (R : A)$ *for every* $A \in \mathcal{Y}_R$*.*

Our strategy is to make use of these correspondences in order to analyze the structure of commutative rings R which are not necessarily Noetherian (see, e.g., [28]). This approach is partially inspired by and originated in [27], where certain specific ideals (called *good ideals*) in Gorenstein local rings are closely studied. Similarly, as in [27] and as is shown later in Sections 2 and 3, the above correspondences behave very well, especially in the case where R is a Gorenstein ring of dimension one. We actually have $\eta \circ \rho = 1_{\mathcal{X}_R}$ and $\rho \circ \eta = 1_{\mathcal{Y}_R}$ in that case (Lemma 4.2.6). Nevertheless, being different from [27], our present interest is in the question of when the correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is bijective. As is shown in Section 2 (Example 4.2.10), in general there is no hope for the surjectivity of ρ in the case where $\dim R \geq 2$, even if R is a Noetherian integral domain of dimension two. On the other hand, with very specific, so to speak extremal exceptions (Proposition 4.5.1), the surjectivity of ρ guarantees the Gorenstein property of R, provided R is a Cohen-Macaulay local ring of dimension one. In fact, we will prove in Section 5 the following, in which let us refer to [36] for the notion of almost Gorenstein local ring.

Theorem 4.1.3 (Theorem 4.5.2)**.** *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension one. Let* $B = \mathfrak{m} : \mathfrak{m}$ *and let* $\mathrm{J}(B)$ *denote the Jacobson radical of* B*. Then the following assertions are equivalent.*

(1) $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ *is bijective.*

(2) $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ *is surjective.*

(3) *Either* R *is a Gorenstein ring, or* R *satisfies the following two conditions.*

  (i) B *is a* DVR *and* $\mathrm{J}(B) = \mathfrak{m}$.

  (ii) *There is no proper intermediate field between* $R/\mathfrak{m}$ *and* $B/\mathrm{J}(B)$.

When this is the case, R *is an almost Gorenstein local ring in the sense of [36].*

Therefore, ρ is surjective if and only if R is a Gorenstein ring, provided R is the semigroup ring of a numerical semigroup over a field.

In Section 6, we introduce the notion of anti-stable and strongly anti-stable rings. We say that a commutative ring R is *anti-stable* (resp. *strongly anti-stable*), if $\mathrm{Hom}_R(I, R)$ is an invertible module over the ring $\mathrm{End}_R I$ (resp. $\mathrm{Hom}_R(I, R) \cong \mathrm{End}_R I$ as an $\mathrm{End}_R I$-module), for every *regular* ideal I of R. The purpose of Section 6 is to provide some basic properties of anti-stable rings and strongly anti-stable rings, mainly in dimension one.

Here, let us remind the reader that R is said to be a *stable* ring, if every ideal I of R is *stable*, that is I is projective over $\mathrm{End}_R I$ ([73]). The notion of stable ideals and rings is originated in the famous articles [6] and [67] of H. Bass and J. Lipman, respectively, and there are known many deep results about them ([73]). Our definition of anti-stable rings is, of course, different from that of stable rings. It requires the projectivity of the dual module $\mathrm{Hom}_R(I, R)$ of I, only for regular ideals I of R, claiming nothing about the projectivity of I itself. Nevertheless, with some additional conditions in dimension one, R is also a stable ring, once it is anti-stable, as we shall show in the following.

Theorem 4.1.4 (Theorem 4.6.10)**.** *Let* R *be a Cohen-Macaulay ring with* $\dim R_M = 1$ *for every* $M \in \mathrm{Max}\, R$*. If* R *is an anti-stable ring, then* R *is a stable ring.*

The results of Section 6 are obtained as applications of the observations developed in Sections 2, 3, and 5. One can also find, in the forthcoming paper [28], further developments of the theory of anti-stable rings of higher dimension.

Similarly as [61], our research is motivated by the works [64, 65, 66] of Lindo and Pande, so that the topics of Section 6 are similar to those of [61], but these two researches were done with entire independence of each other. In [66], Lindo and Pande posed a problem what kind of properties a Noetherian ring R enjoys, if *every* ideal of R is isomorphic to a trace ideal in it. In [61], T. Kobayashi and R. Takahashi have given complete answers to the problem. We were also interested in the problem, and thereafter, came to the notion of anti-stable ring. If the ideal I considered is *regular*, the condition (C) that I is isomorphic to a trace ideal is equivalent to saying that $\mathrm{Hom}_R(I, R) \cong \mathrm{End}_R I$ as an $\mathrm{End}_R I$-module (Lemma 4.6.2). Therefore, if we restrict our attention, say on integral domains R, the condition that every regular ideal satifies condition (C) is equivalent to saying that R is a strongly anti-stable ring. However, in general, these two conditions are apparently different (e.g., consider the case where every non-zerodivisor of the ring is invertible in it, and see [61, Theorem 3.2]). It must be necessary, and might have some significance, to start a basic theory of anti-stable and strongly anti-stable rings in our context, with a different viewpoint from [61], which we have performed in Section 6.

In what follows, unless otherwise specified, R denotes a commutative ring. Let $\mathrm{Q}(R)$ be the total ring of fractions of R. For R-submodules X and Y of $\mathrm{Q}(R)$, let

$$X : Y = \{a \in \mathrm{Q}(R) \mid aY \subseteq X\}.$$

If we consider ideals I, J of R, we set $I :_R J = \{a \in R \mid aJ \subseteq I\}$; hence

$$I :_R J = (I : J) \cap R.$$

When $(R, \mathfrak{m})$ is a Noetherian local ring of dimension d, for each finitely generated R-module M, let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators (resp. the length) of M. We denote by

$$\mathrm{e}(M) = \lim_{n\to\infty} d! \cdot \frac{\ell_R(M/\mathfrak{m}^{n+1}M)}{n^d}$$

the multiplicity of M. Let $\mathrm{r}(R) = \ell_R(\mathrm{Ext}^d_R(R/\mathfrak{m}, R))$ stand for the Cohen-Macaulay type of R, where we assume the local ring R is Cohen-Macaulay.

4.2 Correspondence between trace ideals and birational extensions of the base ring

Let R be a commutative ring and let M, X be R-modules. We denote by $\tau_{M,X} : \mathrm{Hom}_R(M, X) \otimes_R M \to X$ the R-linear map such that $\tau_{M,X}(f \otimes m) = f(m)$ for all $f \in \mathrm{Hom}_R(M, X)$ and $m \in M$. Let $\tau_X(M) = \mathrm{Im}\, \tau_{M,X}$. Then, $\tau_X(M)$ is an R-submodule of X, and we say that an R-submodule Y of X is a trace module in X, if $Y = \tau_X(M)$ for some R-module M. When $X = R$, we simply say that Y is a trace ideal in R. With this notation we have the following.

Proposition 4.2.1 ([65, Lemma 2.3]). *For an R-submodule Y of X, the following conditions are equivalent.*

(1) *Y is a trace module in X.*

(2) *$Y = \tau_X(Y)$.*

(3) *The embedding $\iota : Y \to X$ induces the isomorphism $\iota_* : \mathrm{Hom}_R(Y, Y) \to \mathrm{Hom}_R(Y, X)$ of R-modules.*

(4) *$f(Y) \subseteq Y$ for all $f \in \mathrm{Hom}_R(Y, X)$.*

We denote by W the set of non-zerodivisors of R. Let $\mathcal{F}_R$ be the set of *regular* ideals of R, that is the ideals I of R with $I \cap W \neq \emptyset$. We then have the following, characterizing trace ideals.

Corollary 4.2.2. *Let $I \in \mathcal{F}_R$. Then the following conditions are equivalent.*

(1) *I is a trace ideal in R.*

(2) *$I = (R : I)I$.*

(3) *$I : I = R : I$.*

Proof. Since $I \cap W \neq \emptyset$, we have natural identifications $R : I = \mathrm{Hom}_R(I, R)$ and $I : I = \mathrm{Hom}_R(I, I)$, so that the equivalence of conditions (1) and (3) follows from Proposition 4.2.1. Suppose that $I = (R : I)I$. Then $R : I \subseteq I : I$, whence $R : I = I : I$. Conversely, if $I : I = R : I$, then $(R : I)I = (I : I)I \subseteq I$, while $I \subseteq (R : I)I$, since $1 \in R : I$. Thus $(R : I)I = I$. □

We now consider the following sets:

$$\mathcal{X}_R = \{I \in \mathcal{F}_R \mid I \text{ is a trace ideal in } R\},$$
$$\mathcal{Y}_R = \{A \mid R \subseteq A \subseteq \mathrm{Q}(R),\ A \text{ is a subring of } \mathrm{Q}(R) \text{ such that } aA \subseteq R \text{ for some } a \in W\},$$
$$\mathcal{Z}_R = \left\{I \in \mathcal{F}_R \mid I^2 = aI \text{ for some } a \in I\right\}.$$

If R is a Noetherian ring, then $\mathcal{Y}_R$ is the set of birational finite extensions of R. In what follows, we shall clarify the relationship among these sets. We begin with the following.

Proposition 4.2.3. *The following assertions hold true.*

(1) *Let X be an R-submodule of $\mathrm{Q}(R)$ and set $Y = R : X$. Then $Y = R : (R : Y)$.*

(2) *Let $I \in \mathcal{Z}_R$ and assume that $I^2 = aI$ with $a \in I$. Then $a \in W$ and $I : I = a^{-1}I$.*

Proof. (1) Since $X \subseteq R : Y$, $Y = R : X \supseteq R : (R : Y)$, so that $Y = R : (R : Y)$.

(2) We have $a \in W$, because $I \in \mathcal{F}_R$. Since $a \in I$, $I : I \subseteq a^{-1}I$, while $a^{-1}I \subseteq I : I$, because $a^{-1}I{\cdot}I = a^{-1}I^2 = a^{-1}(aI) = I$. Hence $I : I = a^{-1}I$. □

Lemma 4.2.4. *The following assertions hold true.*

(1) *Let $I \in \mathcal{X}_R$ and $a \in I \cap W$. We set $J = (a) :_R I$. Then, $J \subseteq I$ and $J^2 = aJ$, so that $J \in \mathcal{Z}_R$.*

(2) *Let $I \in \mathcal{Z}_R$ and write $I^2 = aI$ with $a \in I$. We set $J = (a) :_R I$. Then, $I \subseteq J$ and $J \in \mathcal{X}_R$.*

Proof. (1) We set $A = I : I$. Then, $A = R : I$ by Corollary 4.2.2. Hence, $J = (a) :_R I = (a) : I = a(R : I) = aA$, where the second equality follows from the fact that $a \in I \cap W$. Therefore, $J^2 = aJ$ and $J = a(I : I) \subseteq I$.

(2) Notice that $J = (a) : I = a(R : I)$. Let $A = I : I$. Then, $I = aA$, since $A = a^{-1}I$ by Proposition 4.2.3 (2), so that $R : I = R : aA = a^{-1}(R : A)$. Therefore, $J = R : A$, whence

$$J : J = (R : A) : (R : A) = R : A(R : A) = R : (R : A) = R : J.$$

Thus, $J \in \mathcal{X}_R$. □

Let $I \in \mathcal{F}_R$. We say that I is a *good ideal* of R, if $I^2 = aI$ and $I = (a) :_R I$ for some $a \in I$ (cf. [27]). Let $\mathcal{G}_R$ denote the set of good ideals in R. We then have the following, characterizing good ideals.

Proposition 4.2.5. $\mathcal{X}_R \cap \mathcal{Z}_R = \mathcal{G}_R = \{I \in \mathcal{X}_R \mid (a) :_R I \in \mathcal{X}_R$ *for some* $a \in I \cap W\}$.

Proof. Let $I \in \mathcal{X}_R \cap \mathcal{Z}_R$ and set $A = I : I$. We write $I^2 = aI$ with $a \in I$. Then, since $I = aA$ and $A = R : I$ (see Proposition 4.2.3 and Corollary 4.2.2), $(a) :_R I = (a) : I = a(R : I) = aA = I$, so that I is a good ideal of R. Conversely, suppose that I is a good ideal of R and assume that $I^2 = aI$ and $I = (a) :_R I$ with $a \in I$. Then $I \in \mathcal{Z}_R$, while $(a) :_R I \in \mathcal{X}_R$ by Lemma 4.2.4 (2). Hence $I \in \mathcal{X}_R \cap \mathcal{Z}_R$.

Assume that $I \in \mathcal{X}_R$ and that $(a) :_R I \in \mathcal{X}_R$ for some $a \in I \cap W$. We set $J = (a) :_R I$. Then, $J^2 = aJ$ and $J \subseteq I$, by Lemma 4.2.4 (1). For the same reason, we get $(a) :_R J \subseteq J$, because $J \in \mathcal{X}_R$ and $a \in J$. Therefore, $I \subseteq (a) :_R J \subseteq J \subseteq I$; hence $I = J$. Thus, $I^2 = aI$ and $I = (a) :_R I$, that is $I \in \mathcal{G}_R$. □

Let us consider three correspondences

$$\xi : \mathcal{Z}_R \to \mathcal{Y}_R, \quad \xi(I) = I : I,$$

$$\eta : \mathcal{Y}_R \to \mathcal{X}_R, \quad \eta(A) = R : A,$$

$$\rho : \mathcal{X}_R \to \mathcal{Y}_R, \quad \rho(I) = I : I.$$

Here, we briefly confirm the well-definedness of η. Let $A \in \mathcal{Y}_R$ and set $I = R : A$. Since I is an ideal of A, we get $I : I = (R : A) : I = R : AI = R : I$. Therefore, $I \in \mathcal{X}_R$, which shows η is well-defined.

With this notation, we have the following, which plays a key role in this chapter.

Lemma 4.2.6. *The following assertions hold true.*

(1) *The correspondence* $\xi : \mathcal{Z}_R \to \mathcal{Y}_R$ *is surjective. For each* $I, J \in \mathcal{Z}_R$, $\xi(I) = \xi(J)$ *if and only if* $I \cong J$ *as an* R*-module, so that* $\mathcal{Y}_R = \mathcal{Z}_R/\cong$, *that is the set of the isomorphism classes in* $\mathcal{Z}_R$.

(2) $\eta(\rho(I)) = R : (R : I)$ *for every* $I \in \mathcal{X}_R$.

(3) $\rho(\eta(A)) = R : (R : A)$ *for every* $A \in \mathcal{Y}_R$.

Consequently, $\rho(\mathcal{X}_R) = \{A \in \mathcal{Y}_R \mid A = R : (R : A)\}$, $\eta(\mathcal{Y}_R) = \{I \in \mathcal{X}_R \mid I = R : (R : I)\}$, *and we have a bijective correspondence* $\eta(\mathcal{Y}_R) \to \rho(\mathcal{X}_R)$, $I \mapsto I : I$.

Proof. (1) Let $A \in \mathcal{Y}_R$ and choose $a \in W$ so that $aA \subseteq R$. We set $I = aA$. We then have $I^2 = aI$ and $I : I = aA : aA = A : A = A$, whence $I \in \mathcal{Z}_R$, and ξ is surjective, because $\xi(I) = A$. Let $I, J \in \mathcal{Z}_R$ and choose $a \in I, b \in J$ so that $I^2 = aI$ and $J^2 = bJ$. Then, $I : I = a^{-1}I$ and $J : J = b^{-1}J$. Hence, if $\xi(I) = \xi(J)$, then $a^{-1}I = b^{-1}J$, so that $I \cong J$ as an R-module. Conversely, suppose that $I, J \in \mathcal{Z}_R$ and $I \cong J$. Then $J = \alpha I$ for some invertible element α of $Q(R)$, whence $\xi(J) = J : J = \alpha I : \alpha I = I : I = \xi(I)$.

(2) (3) Notice that $\eta(\rho(I)) = R : (I : I) = R : (R : I)$ for every $I \in \mathcal{X}_R$ and

$$\rho(\eta(A)) = (R : A) : (R : A) = R : A(R : A) = R : (R : A)$$

for every $A \in \mathcal{Y}_R$.

The last assertions follow from the fact that $R : (R : Y) = Y$ for every R-submodule Y of Q(R), once $Y = R : X$ for some R-submodule X of Q(R) (see Proposition 4.2.3 (1)). $\square$

Corollary 4.2.7. *The correspondence ρ induces a bijection*

$$\mathcal{G}_R \to \{A \in \mathcal{Y}_R \mid aA = R : A \text{ for some } a \in W\}, \quad I \mapsto I : I.$$

Proof. Let $I \in \mathcal{G}_R$. We then have, by Proposition 4.2.5, $I^2 = aI$ and $I = (a) :_R I$ for some $a \in I$. Since $I = (a) : I = R : a^{-1}I$, $I = R : (R : I)$ by Proposition 4.2.3 (1). Therefore, setting $A = I : I$ $(= a^{-1}I)$, because $A = R : I$ by Corollary 4.2.2, we get $R : A = R : (R : I) = I = aA$. Hence, $\rho(I) = A$ belongs to the set of the right hand side. By Lemma 4.2.6, the induced correspondence is automatically injective, since $I = R : (R : I)$ for every $I \in \mathcal{G}_R = \mathcal{X}_R \cap \mathcal{Z}_R$.

To see the induced correspondence is surjective, let $A \in \mathcal{Y}_R$ and assume that $aA = R : A$ for some $a \in W$. Let $I = aA$; hence $I = \eta(A) \in \mathcal{X}_R$. We then have $I^2 = aI$ and $I : I = aA : aA = A$, so that $I \in \mathcal{X}_R \cap \mathcal{Z}_R$ and $\rho(I) = A$. $\square$

If R is a Gorenstein ring of dimension one, $L = R : (R : L)$ for every finitely generated R-submodule L of Q(R) such that Q(R)$\cdot L$ = Q(R). Therefore, by Lemma 4.2.6 we readily get the following.

Corollary 4.2.8. *Suppose that R is a Gorenstein ring of dimension one. Then, $\eta \circ \rho = 1_{\mathcal{X}_R}$ and $\rho \circ \eta = 1_{\mathcal{Y}_R}$, so that the correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is bijective.*

We note the following.

Proposition 4.2.9. *The following conditions are equivalent.*

(1) *$\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is surjective.*

(2) *$\eta : \mathcal{Y}_R \to \mathcal{X}_R$ is injective.*

(3) *$A = R : (R : A)$ for every $A \in \mathcal{Y}_R$.*

Proof. (1) ⇔ (3) See Lemma 4.2.6.

(2) ⇒ (3) Let $A \in \mathcal{Y}_R$ and set $L = R : (R : A)$. Therefore, $L = \rho(\eta(A)) \in \mathcal{Y}_R$, while $\eta(A) = R : A = R : L = \eta(L)$, where the second equality follows from Proposition 4.2.3 (1). Hence, $A = L$, because η is injective.

(3) ⇒ (2) We have $\rho \circ \eta = 1_{\mathcal{Y}_R}$ by Lemma 4.2.6, so that ρ is surjective. □

We explore one example, which shows that when $\dim R \geq 2$, in general we cannot expect the bijectivity of the correspondence ρ.

Example 4.2.10. Let $S = k[X, Y]$ be the polynomial ring over a field k. We set $R = k[X^4, X^3Y, XY^3, Y^4]$ and $T = k[X^4, X^3Y, X^2Y^2, XY^3, Y^4]$ in S. Let $\mathfrak{m} = (X^4, X^3Y, XY^3, Y^4)R$. Then $T = \overline{R}$ and $\mathfrak{m} = R : T$. We have $\dim R = 2$ and $\operatorname{depth} R_{\mathfrak{m}} = 1$, whence $R_{\mathfrak{m}}$ is not Cohen-Macaulay. With this setting the following assertions hold true.

(1) $\mathcal{X}_R = \{I \mid I \text{ is an ideal of } R \text{ with } \operatorname{ht}_R I \geq 2, \text{ and } I \not\subseteq \mathfrak{m} \text{ or } IT = I\}$. Hence, $\mathfrak{m}^\ell \in \mathcal{X}_R$ for all $\ell > 0$.

(2) $\mathcal{Y}_R = \{T, R\}$, and the correspondence $\eta : \mathcal{Y}_R \to \mathcal{X}_R$ is injective.

(3) The isomorphism classes in $\mathcal{Z}_R$ are $[(X^4, X^6Y^2)]$ and $[R]$, where for each $I \in \mathcal{Z}_R$, $[I]$ denotes the isomorphism class of I in $\mathcal{Z}_R$.

Proof. $T = \sum_{n \geq 0} S_{4n}$ is the Veronesean subring of S with order 4, whence T is a normal ring with $\dim T = 2$. We get $\mathfrak{m} = T_+ \cap R$, where T_+ is the maximal ideal $(X^4, X^3Y, X^2Y^2, XY^3, Y^4)T$ of T. Because $T = R + kX^2Y^2$ and $\mathfrak{m}{\cdot}X^2Y^2 \subseteq \mathfrak{m}$, $T = \overline{R}$, the normalization of R, and $\mathfrak{m}T = \mathfrak{m}$. Hence, $R : T = \mathfrak{m}$, and $\dim R = \dim R_{\mathfrak{m}} = 2$. However, because $T/R \cong R/\mathfrak{m}$, $\operatorname{depth} R_{\mathfrak{m}} = 1$, whence $R_{\mathfrak{m}}$ is not Cohen-Macaulay. We get $\mathcal{Y}_R = \{T, R\}$, since $\ell_R(T/R) = 1$. Therefore, since $\mathfrak{m} = R : T$, the correspondence $\eta : \mathcal{Y}_R \to \mathcal{X}_R$ is injective, and by Lemma 4.2.6 (1) the isomorphism classes in $\mathcal{Z}_R$ are exactly $[(X^4, X^6Y^2)]$ (notice that $(X^4, X^6Y^2) = X^4T$) and $[R]$.

Let us check Assertion (1). Firstly, let I be an ideal of R with $\operatorname{ht}_R I \geq 2$ such that $I \not\subseteq \mathfrak{m}$ or $IT = I$. We will show that $I \in \mathcal{X}_R$. We may assume $I \neq R$. Suppose that $I \not\subseteq \mathfrak{m}$ and let $\mathfrak{p} \in \operatorname{Spec} R$ such that $I \subseteq \mathfrak{p}$. Then, $R_{\mathfrak{p}} = T_{\mathfrak{p}}$, since $\mathfrak{p} \neq \mathfrak{m}$, so that $R_{\mathfrak{p}}$ is a Cohen-Macaulay ring with $\dim R_{\mathfrak{p}} = 2$. Therefore, $\operatorname{grade}_R I = 2$, and hence $I \in \mathcal{X}_R$ by Proposition 4.2.1.

Suppose that $IT = I$ and let $f \in \operatorname{Hom}_R(I, R)$. Let $\iota : R \to T$ denote the embedding. Then, the composite map $g : I \xrightarrow{f} R \xrightarrow{\iota} T$ is T-linear, because it is the restriction of the homothety of some element of $\mathrm{Q}(R) = \mathrm{Q}(T)$, while $\operatorname{grade}_T I = \operatorname{ht}_T I = 2$, since $\operatorname{ht}_R I = 2$. Consequently, because $T : I = T$, we have $g(I) \subseteq I$, so that $f(I) \subseteq I$. Thus, $I \in \mathcal{X}_R$ by Proposition 4.2.1.

Conversely, let $I \in \mathcal{X}_R$. Therefore, I is a non-zero ideal of R with $R : I = I : I$. Hence, $R : I = R$ or $R : I = T$, because $\mathcal{Y}_R = \{R, T\}$. If $R : I = R$, then $\operatorname{grade}_R I \geq 2$. Therefore, $\operatorname{ht}_R I \geq 2$, and $I \not\subseteq \mathfrak{m}$, because $\operatorname{depth} R_{\mathfrak{m}} = 1$. Suppose that $R : I = T$. Then, I

is an ideal of T. We have to show $\mathrm{ht}_R I \geq 2$. Assume the contrary and choose $\mathfrak{p} \in \operatorname{Spec} R$ so that $I \subseteq \mathfrak{p}$ and $\mathrm{ht}_R \mathfrak{p} = 1$. We then have $R_\mathfrak{p} = T_\mathfrak{p}$, and

$$R_\mathfrak{p} : IR_\mathfrak{p} = [R : I]_\mathfrak{p} = [I : I]_\mathfrak{p} = T_\mathfrak{p} = R_\mathfrak{p}.$$

This is impossible, because $IR_\mathfrak{p}$ is a proper ideal in the DVR $R_\mathfrak{p} = T_\mathfrak{p}$. Therefore, $\mathrm{ht}_R I \geq 2$, which completes the proof of Assertion (1). □

4.3 The case where R is a Gorenstein ring of dimension one

We now concentrate our attention on the case where R is a Gorenstein ring of dimension one.

Proposition 4.3.1. *Assume that R is a Gorenstein ring of dimension one. We then have the following.*

(1) *$I : I$ is a Gorenstein ring for every $I \in \mathcal{G}_R$.*

(2) *Let $A \in \mathcal{Y}_R$ and suppose that A is a Gorenstein ring. Then, $A = I : I$ for some $I \in \mathcal{G}_R$, if R is semi-local.*

Consequently, when R is semi-local, the correspondence ρ induces the bijection

$$\mathcal{G}_R \to \{A \in \mathcal{Y}_R \mid A \text{ is a Gorenstein ring}\}.$$

Proof. (1) Let $A = I : I$. Then, by Corollary 4.2.7, $R : A = aA$ for some $a \in W$, so that A is a Gorenstein ring (see [54, Satz 5.12]; remember that $\mathrm{Hom}_R(A, R) \cong R : A$.)

(2) We have $R : A = aA$ for some $a \in W$, because $R : A$ is a canonical ideal of A and A is semi-local. Hence, by Corollary 4.2.7, $A = I : I$ for some $I \in \mathcal{G}_R$. □

When $(R, \mathfrak{m})$ is a Gorenstein local ring of dimension one, we furthermore have the following, which characterizes Gorenstein local rings of dimension one, in which every regular trace ideal is a good ideal. The problem of when A is a Gorenstein ring for every $A \in \mathcal{Y}_R$ is originated in the paper of H. Bass [6], where one can find many deep observations related to the problem. The equivalence of Assertions (1) and (3) in the following theorem is essentially due to [6, (7.7) Theorem].

Theorem 4.3.2. *Let R be a semi-local Gorenstein ring of dimension one. Then the following conditions are equivalent.*

(1) *Every $A \in \mathcal{Y}_R$ is a Gorenstein ring.*

(2) *$\mathcal{X}_R = \mathcal{G}_R$.*

When $(R, \mathfrak{m})$ is a local ring, one can add the following.

(3) $\mathrm{e}(R) \leq 2$.

Proof. (2) $\Rightarrow$ (1) We have by Lemma 4.2.6 $A = I : I$ for some $I \in \mathcal{X}_R$, so that by Proposition 4.3.1 (1) A is a Gorenstein ring.

(1) $\Rightarrow$ (2) Every good ideal of R belongs to $\mathcal{X}_R$ by Proposition 4.2.5. Conversely, let $I \in \mathcal{X}_R$ and set $A = I : I$. Then, by Proposition 4.3.1 (2), $A = J : J$ for some $J \in \mathcal{G}_R$, since A is a Gorenstein ring. Therefore, $I = J$, because $I, J \in \mathcal{X}_R$ and the correspondence ρ is bijective (Corollary 4.2.8).

Suppose that $(R, \mathfrak{m})$ is a local ring.

(3) $\Rightarrow$ (1) See [46, Lemma 12.2].

(2) $\Rightarrow$ (3) We may assume that $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$; otherwise, R is a DVR, since $x\mathfrak{m} = R$ for some $x \in R : \mathfrak{m}$. Therefore, $\mathfrak{m} \in \mathcal{X}_R = \mathcal{G}_R$, whence $\mathfrak{m}^2 = a\mathfrak{m}$ for some $a \in \mathfrak{m}$. Thus, $\mathrm{e}(R) = 2$, because R is a Gorenstein ring. $\square$

We close this section with a few examples. To state Example 4.3.3, we need the notion of idealization, which we now briefly explain. Let R be a commutative ring and M an R-module. We set $A = R \oplus M$ as an additive group, and define the multiplication in A by $(a, x)\cdot(b, y) = (ab, ay + bx)$ for $(a, x), (b, y) \in A$. Then, A forms a commutative ring, which is denoted by $A = R \ltimes M$, and called the idealization of M over R.

Example 4.3.3. Let V be a DVR with t a regular parameter. Let $R = V \ltimes V$ denote the idealization of V over itself. Then, R is a Gorenstein local ring with $\dim R = 1$, $\mathrm{e}(R) = 2$, and $\mathcal{X}_R = \{t^nV \times V \mid n \geq 0\}$.

Proof. Because $R \cong V[X]/(X^2)$ where X denotes an indeterminate, R is a Gorenstein local ring with $\dim R = 1$, $\mathrm{e}(R) = 2$. Let $K = \mathrm{Q}(V)$. Then, $\mathrm{Q}(R) = K \ltimes K$, and $\overline{R} = V \ltimes K$. Consequently

$$\mathcal{Y}_R = \{V \ltimes L \mid L \text{ is a finitely generated } V\text{-submodule of } K \text{ such that } V \subseteq L\}.$$

Therefore, $\mathcal{X}_R = \{t^nV \times V \mid n \geq 0\}$ by Corollary 4.2.8, because R is a Gorenstein local ring with $\dim R = 1$ and $R : [V \ltimes L] = \mathrm{Ann}_V(L/V) \times V$ for every finitely generated V-submodule L of K such that $V \subseteq L$. $\square$

Example 4.3.4. Let k be a field.

(1) Let $R = k[[t^4, t^5, t^6]]$. Then R is a Gorenstein ring, possessing

$$\mathcal{X}_R = \left\{(t^8, t^9, t^{10}, t^{11}), (t^6, t^8, t^9), (t^5, t^6, t^8), (t^4, t^5, t^6), R\right\} \cup \left\{(t^4 - at^5, t^6) \mid a \in k\right\} \text{ and}$$

$$\mathcal{Y}_R = \left\{k[[t]], k[[t^2, t^3]], k[[t^3, t^4, t^5]], k[[t^4, t^5, t^6, t^7]], R\right\} \cup \left\{k[[t^2 + at^3, t^5]] \mid a \in k\right\},$$

and the correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is bijective.

(2) Let $R = k[[t^3, t^4, t^5]]$. Then R is not a Gorenstein ring, possessing

$$\mathcal{X}_R = \left\{(t^3, t^4, t^5), R\right\} \text{ and } \mathcal{Y}_R = \left\{k[[t]], k[[t^2, t^3]], R\right\},$$

and the correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is not surjective.

Proof. (1) We set $V = k[[t]]$ (the formal power series ring) and $S = k[[t^4, t^5, t^6, t^7]]$. We will show the set $\mathcal{Y}_R$ consists of the rings in the list. Let $\mathfrak{m}$ and $\mathfrak{m}_S$ denote the maximal ideals of R and S, respectively. We begin with the following.

Claim 3. *The following assertions hold true.*

(1) *Set $B_a = k[[t^2 + at^3, t^5]]$ for each $a \in k$. Then $S \subsetneq B_a \subseteq k[[t^2, t^3]]$, $B_a = S + S{\cdot}(t^2 + at^3)$, and $\ell_S(B_a/S) = 1$.*

(2) *Let $a, b \in k$. Then $B_a = B_b$ if and only if $a = b$.*

Proof. (1) We set $T = k[[(t^2 + at^3)^2, t^5, (t^2 + at^3)^3, (t^2 + at^3){\cdot}t^5]]$. Then, $T \subseteq B_a$, and $T \subseteq S$, since $S = k + t^4V$. Because

$$\mathfrak{m}_T S + \mathfrak{m}_S^2 \supseteq (t^4, t^5, t^6, t^7)S = \mathfrak{m}_S = t^4V,$$

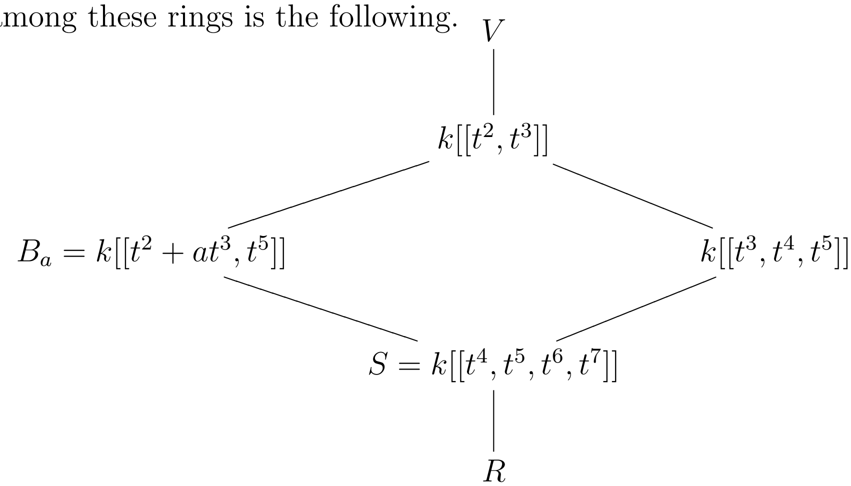
we get $\mathfrak{m}_T S = \mathfrak{m}_S$, whence $T = S$ (remember that $T/\mathfrak{m}_T = S/\mathfrak{m}_S = k$). Consequently, $T = S \subsetneq k[[t^2, t^3]]$, and $B_a = S + S{\cdot}(t^2 + at^3)$, because $t^5 \in \mathfrak{m}_S$. Therefore, $\mu_S(B_a) = 2$, and $\ell_S(B_a/S) = 1$, since $\mathfrak{m}_S B_a = \mathfrak{m}_S \subseteq S$.

(2) Suppose $B_a = B_b$. Then, since the k-space $B_a/\mathfrak{m}_S B_a$ (resp. $B_b/\mathfrak{m}_S B_b$) is spanned by the images of 1 and $t^2 + at^3$ (resp. 1 and $t^2 + bt^3$), we have

$$t^2 + at^3 = \alpha + \beta(t^2 + bt^3) + \gamma$$

for some $\alpha, \beta \in k$ and $\gamma \in t^4V$. Hence, $\alpha = 0$, $\beta = 1$, and $a = b\beta$, so that $a = b$. ☐

By this claim, we see $R, S, k[[t^3, t^4, t^5]], B_a\ (a \in k), k[[t^2, t^3]], V \in \mathcal{Y}_R$. The relation of embedding among these rings is the following.



We have to show that $\mathcal{Y}_R$ consists of these rings. To see it, let $A \in \mathcal{Y}_R$ and assume that $R \subsetneq A \subsetneq V$. Then, because R is a Gorenstein local ring with $R : \mathfrak{m} = R + kt^7$ and $R \subsetneq A$, we get $S = R + kt^7 \subseteq A$. Let us assume that $S \subsetneq A$ and set $\ell = \ell_S(A/S)$. Then $\ell = 1, 2$, since $\ell_S(V/S) = 3$. We write $\mathfrak{m}_A V = t^n V$ with an integer $n > 0$. We then have $n \leq 4$, since $t^4 \in \mathfrak{m}_A$. Because $A = k + \mathfrak{m}_A \not\subseteq S = k + t^4V$ and $A \neq V$, we furthermore have $n = 2$ or 3.

Suppose that $\ell = 1$. If $n = 3$, then choosing an element $f = t^3 + g$ with $g \in t^4V = \mathfrak{m}_S$, we see $t^3 \in A$, so that $k[[t^3, t^4, t^5]] \subseteq A$. Therefore, $k[[t^3, t^4, t^5]] = A$, because $\ell_S(A/S) = 1$ and $S \subsetneq k[[t^3, t^4, t^5]] \subseteq A$. Let $n = 2$ and choose an element $f = t^2 + at^3 \in A$ with $a \in k$. Then, $B_a \subseteq A$, and $\ell_S(B_a/S) = 1$ by Claim (1), whence $A = B_a$. Suppose now that $\ell = 2$. Then $\ell_A(V/A) = 1$, whence $\mathfrak{m}_A = A : V = t^nV$, so that

$$A = k + t^nV = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$$

with $n = 2$ or 3. This proves that $\mathcal{Y}_R = \{R, S, k[[t^3, t^4, t^5]], B_a\ (a \in k), k[[t^2, t^3]], V\}$.

Because $\mathcal{X}_R = \{R : A \mid A \in \mathcal{Y}_R\}$ by Corollary 4.2.8 it is direct to show that $\mathcal{X}_R$ consists of the following ideals $R : V = (t^8, t^9, t^{10}, t^{11}), R : k[[t^2, t^3]] = (t^6, t^8, t^9), R : k[[t^3, t^4, t^5]] = (t^5, t^6, t^8), R : S = (t^4, t^5, t^6) = \mathfrak{m}, R$, and $R : B_a = (t^4 - at^5, t^6)$ with $a \in k$. Let us note a proof for the fact that $R : B_a = (t^4 - at^5, t^6)$. We set $I = R : B_a$. Firstly, notice that $B_a = R + R{\cdot}(t^2 + at^3)$, since $t^5, (t^2 + at^3)^2 \in \mathfrak{m}$. We then have $t^6 \in I$, since $R : V = t^8V \subseteq I$. Let $\varphi \in I$ and write $\varphi = \alpha t^4 + \beta t^5 + \gamma t^6 + \delta$ with $\alpha, \beta, \gamma \in k$ and $\delta \in R : V$. Then, $\alpha t^4 + \beta t^5 \in I$, and $(\alpha t^4 + \beta t^5)(t^2 + at^3) \in R$ if and only if $(\alpha a + \beta)t^7 \in R$ if and only if $\beta = -\alpha a$, which shows $I = (t^4 - at^5, t^6)$.

(2) The fact $\mathcal{Y}_R = \{k[[t^3, t^4, t^5], k[[t^2, t^3]], V\}$ readily follows from Assertion (1). The assertion on $\mathcal{X}_R$ is a special case of the following. □

Proposition 4.3.5. *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring and let $V = \overline{R}$ denote the integral closure of R in $\mathrm{Q}(R)$. Assume that $R \neq V$ but $\mathfrak{m}V \subseteq R$. Then $\mathcal{X}_R = \{\mathfrak{m}, R\}$.*

Proof. Because $R \neq V$, we have $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$, whence $\mathfrak{m} \in \mathcal{X}_R$, so that $\{\mathfrak{m}, R\} \subseteq \mathcal{X}_R$. Let $I \in \mathcal{X}_R$ and set $A = I : I\ (= R : I)$. If $R = A$, then $\mathrm{grade}_R I \geq 2$, and $I = R$. Suppose that $R \subsetneq A$. Then, $I \subseteq \mathfrak{m}$, whence $V \subseteq R : \mathfrak{m} \subseteq A = R : I = I : I \subseteq V$. Therefore, $A = V$. Consequently, I is an ideal of V, whence $I \cong V \cong \mathfrak{m}$ as V-modules (remember that V is a direct product of finitely many principal ideal domains). Therefore, $\tau_R(I) = \tau_R(\mathfrak{m}) = \mathfrak{m}$, because $I \cong \mathfrak{m}$ as an R-module. Hence $\mathcal{X}_R = \{\mathfrak{m}, R\}$. □

We will use Proposition 4.3.5 later in Section 5, in order to prove Proposition 4.5.1.

4.4 Modules in which every submodule is a trace module

In this section, we are interested in the question of, for a given R-module X, when every R-submodule of X is a trace module in it. As is shown in [65], this is the case when $X = R$ and R is a self-injective ring. Our goal is the following, which is known by [66, Theorem 3.5] in the case where R is a Noetherian local ring and $X = R$.

Theorem 4.4.1. *Suppose that R is a Noetherian ring and let X be an R-module. Then the following conditions are equivalent.*

(1) *Every R-submodule of X is a trace module in X.*

(2) *Every cyclic R-submodule of X is a trace module in X.*

(3) *There is an embedding*

$$0 \to X \to \bigoplus_{\mathfrak{m} \in \operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{m})$$

of R-modules, where for each $\mathfrak{m} \in \operatorname{Max} R$, $\mathrm{E}_R(R/\mathfrak{m})$ denotes the injective envelope of the cyclic R-module $R/\mathfrak{m}$.

To prove Theorem 4.4.1, we need some preliminaries. The following is a direct consequence of Proposition 4.2.1.

Proposition 4.4.2. *The following assertions hold true.*

(1) *Let Y be an R-submodule of X. If every cyclic R-submodule of Y is a trace module in X, then Y is a trace module in X.*

(2) *Let Z and Y be R-submodules of X and assume that $Z \subseteq Y$. If Z is a trace module in X, then Z is a trace module in Y.*

(3) *([65]) If R is a self-injective ring, then every ideal of R is a trace ideal in R.*

We begin with the following.

Lemma 4.4.3. *Let Y be an R-submodule of X and assume that Y is a finitely presented R-module. Then the following conditions are equivalent.*

(1) *Y is a trace module in X.*

(2) *$Y_\mathfrak{m}$ is a trace module in $X_\mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Max} R$.*

(3) *$Y_\mathfrak{p}$ is a trace module in $X_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. Let $\iota : Y \to X$ denote the embedding and let

$$\iota_* : \operatorname{Hom}_R(Y, Y) \to \operatorname{Hom}_R(Y, X)$$

be the induced homomorphism. We set $C = \operatorname{Coker} \iota_*$. By Proposition 4.2.1, Y is a trace module in X, if and only if $C = (0)$, that is $C_\mathfrak{p} = (0)$ for all $\mathfrak{p} \in \operatorname{Spec} R$. On the other hand, since Y is finitely presented, we have

$$S^{-1}\left[\operatorname{Hom}_R(Y, Z)\right] = \operatorname{Hom}_{S^{-1}R}(S^{-1}Y, S^{-1}Z)$$

for every R-module Z and for every multiplicatively closed subset S in R. Hence, the condition that Y is a trace module in X is a local condition. $\square$

For each R-module X, let $\mathrm{E}_R(X)$ stand for the injective envelope of X. We firstly consider the case where R is a local ring.

Theorem 4.4.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and set $E = \mathrm{E}_R(R/\mathfrak{m})$. Let X be an R-module. Then the following conditions are equivalent.*

(1) *Every R-submodule of X is a trace module in X.*

(2) *There is an embedding $0 \to X \to E$ of R-modules.*

Proof. (1) $\Rightarrow$ (2) We may assume that $X \neq (0)$. Let $V = (0) :_X \mathfrak{m}$. We want to show that $\mathrm{E}_R(X) \cong E$, that is, $\ell_R(V) = 1$ and V is an essential R-submodule of X. To do this, it suffices to show that for every non-zero finitely generated R-submodule M of X, $\ell_R(M) < \infty$ and $\ell_R((0) :_M \mathfrak{m}) = 1$. First of all, we show $\operatorname{depth}_R M = 0$. In fact, suppose that $\operatorname{depth}_R M > 0$, and let $a \in \mathfrak{m}$ be a non-zerodivisor on M. We then have by Proposition 4.2.1 $aM = \tau_X(aM)$ and $M = \tau_X(M)$, since both aM and M are trace modules in X, while $\tau_X(aM) = \tau_X(M)$, because $aM \cong M$. Hence, $aM = M$, which is impossible because $M \neq (0)$. We now fix one socle element $0 \neq x \in (0) :_M \mathfrak{m}$ of M. Let N be an arbitrary non-zero R-submodule of M. Then, since $R/\mathfrak{m}$ is a homomorphic image of $N/\mathfrak{m}N$ and since $R/\mathfrak{m} \cong Rx$, we get a homomorphism $f : N \to M$ such that $f(N) = Rx$, which implies $x \in N$, because N is a trace module in X (see Proposition 4.2.1). Therefore, if $\dim_R M > 0$, then $x \in \mathfrak{m}^n M$ for all $n > 0$, because $\mathfrak{m}^n M \neq (0)$, so that $x \in \bigcap_{n>0} \mathfrak{m}^n M = (0)$, which is a contradiction. Hence, $\dim_R M = 0$, that is $\ell_R(M) < \infty$. The above observation also shows that $x \in Ry$ for every $0 \neq y \in (0) :_M \mathfrak{m}$, whence $\ell_R((0) :_M \mathfrak{m}) = 1$, and therefore, X is an R-submodule of E.

(2) $\Rightarrow$ (1) By Proposition 4.4.2 (2), we may assume $X = E$. Let Y be an R-submodule of E. It suffices to show that $f(Y) \subseteq Y$ for all $f \in \operatorname{Hom}_R(Y, E)$. We take a homomorphism $g : E \to E$ so that $f = g \circ \iota$, where $\iota : Y \to E$ denotes the embedding. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R, and remember that E is an $\widehat{R}$-module such that

$$\operatorname{Hom}_R(E, E) = \operatorname{Hom}_{\widehat{R}}(E, E) = \widehat{R}.$$

Choose $\alpha \in \widehat{R}$ so that g is the homothety by α. We then have $\alpha Y \subseteq Y$, because every R-submodule of E is actually an $\widehat{R}$-submodule of E. Therefore

$$f(Y) = g(Y) = \alpha Y \subseteq Y,$$

and hence Y is a trace module in E. ☐

We are now ready to prove Theorem 4.4.1.

Proof of Theorem 4.4.1. (1) $\Leftrightarrow$ (2) See Proposition 4.4.2 (1).

(3) $\Rightarrow$ (1) Let $\mathfrak{m} \in \operatorname{Max} R$. We then have the embedding $0 \to X_\mathfrak{m} \to \mathrm{E}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m})$, since

$$[\bigoplus_{\mathfrak{n} \in \operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{n})]_\mathfrak{m} = \mathrm{E}_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}).$$

Therefore, by Theorem 4.4.4, for every cyclic R-submodule Y of X, $Y_\mathfrak{m}$ is a trace module in $X_\mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Max} R$, so that Lemma 4.4.3 guarantees that Y is a trace module in X. Hence, by Proposition 4.4.2 (2), every R-submodule of X is a trace module in X.

(1) $\Rightarrow$ (3) Let $\mathfrak{m} \in \operatorname{Max} R$. Since every cyclic $R_\mathfrak{m}$-submodule of $X_\mathfrak{m}$ is a localization of a cyclic R-submodule of X, by Lemma 4.4.3 every $R_\mathfrak{m}$-submodule of $X_\mathfrak{m}$ is a trace module in $X_\mathfrak{m}$. Therefore, by Theorem 4.4.4, for every $\mathfrak{m} \in \operatorname{Max} R$ we have

$$\operatorname{Ass}_{R_\mathfrak{m}} X_\mathfrak{m} \subseteq \{\mathfrak{m}R_\mathfrak{m}\} \text{ and } \ell_{R_\mathfrak{m}}((0) :_{X_\mathfrak{m}} \mathfrak{m}R_\mathfrak{m}) \leq 1.$$

Consequently, $\operatorname{Ass}_R X \subseteq \operatorname{Max} R$ and $\ell_R\left((0):_X \mathfrak{m}\right) \le 1$ for all $\mathfrak{m} \in \operatorname{Max} R$, so that

$$\mathrm{E}_R(X) \cong \bigoplus_{\mathfrak{m} \in \operatorname{Max} R} \mathrm{E}_R(R/\mathfrak{m})^{\oplus \mu(\mathfrak{m})}$$

with $\mu(\mathfrak{m}) \in \{0, 1\}$ for each $\mathfrak{m} \in \operatorname{Max} R$. □

The following is a direct consequence of Theorem 4.4.1.

Corollary 4.4.5 (cf. [66, Theorem 3.5])**.** *For a Noetherian ring R, the following conditions are equivalent.*

(1) *Every ideal of R is a trace ideal in R.*

(2) *R is a self-injective ring.*

For the implication (1) ⇒ (2) in Corollary 4.4.5, we cannot remove the assumption that R is a Noetherian ring. To explain more precisely about this phenomenon, let R be a commutative ring. We say that R is a *Von Neumann regular* ring, if for each $a \in R$, there exists an element $b \in R$ such that $a = aba$ (cf. [68]). Here, we need only the definition, but interested readers can find in [22] a basic characterization of Von Neumann regular rings.

Lemma 4.4.6. *Let R be a Von Neumann regular ring. Then $\tau_R(I) = I$ for every ideal I of R.*

Proof. Let $\varphi : I \to R$ be an R-linear map and $a \in I$. Then, $a = aba$ for some $b \in R$, so that $\varphi(a) = a\varphi(ba) \in I$. Thus, $\varphi(I) \subseteq I$. □

We have learned the following example from M. Hashimoto.

Example 4.4.7. Let K be a commutative ring and assume that there exists an integer $p \geq 2$ such that $a^p = a$ for every $a \in K$. We consider the direct product $S = \prod_{i \in \Lambda} K_i$ of infinitely many copies $\{K_i = K\}_{i \in \Lambda}$ of K, and set $R = \mathbb{Z}{\cdot}1 + \bigoplus_{i \in \Lambda} K_i$ in S. Then, R is a subring of S, and R is Von Neumann regular, since $a^p = a$ for every $a \in S$. We have that S is an essential extension of R, but $R \neq S$, because Λ is infinite. Therefore, R is not a self-injective ring.

Let us note one more example. The following fact is known, when $\mathrm{ch} k = 2$ and $\alpha_i = 1$ for every $i \in \Lambda$. Indeed, with the same notation as Example 4.4.8, if $\mathrm{ch} k = 2$ and $\alpha_i = 1$ for all $i \in \Lambda$, then $R = k[\{T_i\}_{i \in \Lambda}]/(T_i^2 - 1 \mid i \in \Lambda)$ where $T_i = X_i - 1$ for each $i \in \Lambda$, so that $R = k[G]$, the group algebra of the direct sum $G = \bigoplus_{i \in \Lambda} C_i$ of infinitely many copies of the cyclic group $C_i = \mathbb{Z}/(2)$. Therefore, thanks to [24, Theorem], R is not self-injective. We have learned this result from K. Kurano, and we are grateful to him, since the method of proof given in [24] works also in the setting of Example 4.4.8, as we will briefly confirm below.

Example 4.4.8. Let $\Lambda = \{1, 2, 3, \ldots\}$ be the set of positive integers. Let $\{X_i\}_{i \in \Lambda}$ be a family of indeterminates and $\{\alpha_i\}_{i \in \Lambda}$ a family of positive integers. We set $S = k[\{X_i\}_{i \in \Lambda}]$ over a field k, $\mathfrak{a} = (X_i^{\alpha_i+1} \mid i \in \Lambda)$, and consider the ring $R = S/\mathfrak{a}$. Then, R is not a self-injective ring, but $\tau_R(I) = I$ for every ideal I of R.

Proof. Let x_i denote, for each $i \in \Lambda$, the image of X_i in R. For each $n \in \Lambda$, we set $R_n = k[x_1, x_2, \ldots, x_n]$ in R. Then, $R = \bigcup_{n \in \Lambda} R_n$, and

$$R_n = k[X_1, X_2, \ldots, X_n]/(X_1^{\alpha_1+1}, X_2^{\alpha_2+1}, \ldots, X_n^{\alpha_n+1}),$$

so that R_n is a self-injective ring for every $n \in \Lambda$. Let $a \in R$ and assume that $a \in R_n$. Then

$$(0) :_R [(0) :_R a] \subseteq \bigcup_{\ell \geq n} \{(0) :_{R_\ell} [(0) :_{R_\ell} a]\},$$

whence $(0) :_R [(0) :_R a] = (a)$, because $(0) :_{R_\ell} [(0) :_{R_\ell} a] = a{\cdot}R_\ell$ for all $\ell \geq n$ (here we use the fact that R_ℓ is a self-injective ring). Therefore, $\tau_R(I) = I$ for every ideal I of R, because $\tau_R((a)) = (0) :_R [(0) :_R a] = (a)$ for each $a \in R$.

To see that R is not self-injective, we set for each $n \in \Lambda$

$$a_n = \begin{cases} 1, & \text{if } n = 1 \\ 1 + x_1 + x_1x_2 + x_1x_2x_3 + \ldots + x_1x_2 \cdots x_{n-1}, & \text{if } n > 1 \end{cases}$$

and set $I_n = (x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_n^{\alpha_n})$. Then, $I_n \subseteq I_{n+1}$, and $I = \bigcup_{n \in \Lambda} I_n$, where $I = (x_i^{\alpha_i} \mid i \in \Lambda)$. We then have $a_{n+1}x = a_nx$ for every $x \in I_n$, which one can show by a simple use of induction on n, since $x_i^{\alpha_i+1} = 0$ for all $i \in \Lambda$. Therefore, we may define the R-linear map $\varphi : I \to R$ so that $\varphi(x) = a_nx$ if $x \in I_n$. We now assume that R is a self-injective ring. Then, there must exist an element $a \in R$ such that $ax = \varphi(x)$ for every $x \in I$, namely $ax = a_nx$ for every $x \in I_n$. Choose $n \in \Lambda$ so that $a \in R_n$. Then, because $(a - a_{n+2})x_{n+2}^{\alpha_{n+2}} = 0$, we get $a - a_{n+2} \in (0) :_R x_{n+2}^{\alpha_{n+2}} = (x_{n+2})$. Let $f \in k[X_1, X_2, \ldots, X_n]$ such that a is the image of f in R. Then

$$f = 1 + X_1 + X_1X_2 + \ldots + X_1X_2 \cdots X_{n+1} + X_{n+2}g + h$$

for some $g \in S$ and $h \in \mathfrak{a}$. Substituting X_i by 0 for all $i \geq n+2$, we may assume that $g = 0$ and $h \in (X_1^{\alpha_1+1}, X_2^{\alpha_2+1}, \ldots, X_{n+1}^{\alpha_{n+1}+1})T$, where $T = k[X_1, X_2, \ldots, X_{n+1}]$, that is

$$f = 1 + X_1 + X_1X_2 + \ldots + X_1X_2 \ldots X_{n+1} + \sum_{i=1}^{n+1} X_i^{\alpha_i+1}h_i$$

with $h_i \in T$. This is, however, impossible, because $f \in k[X_1, X_2, \ldots, X_n]$ and the monomial $X_1X_2 \cdots X_{n+1}$ is not involved in the polynomial $\sum_{i=1}^{n+1} X_i^{\alpha_i+1}h_i$. Thus, R is not a self-injective ring. $\square$

It seems interesting, but hard, to ask for a complete characterization of (not necessarily Noetherian) commutative rings, in which every ideal is a trace ideal.

4.5 Surjectivity of the correspondence ρ in dimension one

In this section, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one. We are interested in the question of when the correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is bijective. The second example in Example 4.3.4 seems to suggest that R is a Gorenstein ring, if $\dim R = 1$ and ρ is bijective. Unfortunately, this is still not the case, as we show in the following. Here, we say that a one-dimensional Cohen-Macaulay local ring $(R, \mathfrak{m})$ has maximal embedding dimension, if $\mathfrak{m}^2 = a\mathfrak{m}$ for some $a \in \mathfrak{m}$ ([67]). We refer to [36, 46] for the notion of almost Gorenstein local ring.

Proposition 4.5.1 (cf. [60, Example 4.7]). *Let K/k be a finite extension of fields. Assume that $k \neq K$ and there is no intermediate field F such that $k \subsetneq F \subsetneq K$. Let $B = K[[t]]$ be the formal power series ring over K and set $R = k[[Kt]]$ in B. Set $n = [K : k]$. We then have the following.*

(1) *R is a Noetherian local ring with $B = \overline{R}$ and $\mathfrak{m} = tB$, where $\mathfrak{m}$ denotes the maximal ideal of R. Hence $B = \mathfrak{m} : \mathfrak{m} = R : \mathfrak{m}$.*

(2) *R is an almost Gorenstein local ring, possessing maximal embedding dimension $n \geq 2$.*

(3) *R is not a Gorenstein ring, if $n \geq 3$.*

(4) *$\mathcal{X}_R = \{\mathfrak{m}, R\}$ and $\mathcal{Y}_R = \{B, R\}$, so that $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is a bijection.*

Proof. Let $\omega_1 = 1, \omega_2, \dots, \omega_n$ be a k-basis of K. Then $R = k[[\omega_1 t, \omega_2 t, \dots, \omega_n t]]$, whence R is a Noetherian complete local ring. Since $B/\mathfrak{m}B \cong K$, $B = \sum_{i=1}^{n} R\omega_i$, so that $tB = \mathfrak{m}$. Hence, $\mathfrak{m}$ is also an ideal of B, $\mathfrak{m} = \mathfrak{m}B = tB$, and $\mathfrak{m}^2 = t\mathfrak{m}$. Because B is a module-finite extension of R and $\omega_i = \frac{\omega_i t}{\omega_1 t} \in \mathrm{Q}(R)$ for all $1 \leq i \leq n$, we have $B = \overline{R}$. Therefore, R is an almost Gorenstein ring by [36, Corollary 3.12], possessing maximal embedding dimension $\mathrm{e}(R) = n$. Consequently, R is not a Gorenstein ring, if $n \geq 3$. We get $\mathcal{X}_R = \{\mathfrak{m}, R\}$ by Proposition 4.3.5, because $R \neq B$ but $\mathfrak{m}B \subseteq R$. The assertion that $\mathcal{Y}_R = \{B, R\}$ is due to [60, Example 4.7]. Let us note a brief proof for the sake of completeness. Let $A \in \mathcal{Y}_R$ and let $\mathfrak{n}$ denote the maximal ideal of A. We then have $\mathfrak{n} = \mathfrak{m}$, because $\mathfrak{n} = \mathfrak{m}_B \cap A = \mathfrak{m} \cap A = \mathfrak{m}$. Consequently, we have an extension $k = R/\mathfrak{m} \subseteq A/\mathfrak{m} \subseteq K = B/\mathfrak{m}$ of fields, so that $R/\mathfrak{m} = A/\mathfrak{m}$, or $A/\mathfrak{m} = B/\mathfrak{m}$ by the choice of the extension K/k. Hence, $R = A$ or $A = B$, and thus $\mathcal{Y}_R = \{R, B\}$. Therefore, because $\mathfrak{m} : \mathfrak{m} = tB : tB = B$ and $R : R = R$, the correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is a bijection. $\square$

In what follows, we intensively explore the question of when the correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is bijective. The goal is the following, which essentially shows that except the case of Proposition 4.5.1, the surjectivity of ρ implies the Gorenstein property of the ring R.

Theorem 4.5.2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one. We set $B = \mathfrak{m} : \mathfrak{m}$ and let $\mathrm{J}(B)$ denote the Jacobson radical of B. Then the following assertions are equivalent.*

(1) $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ *is bijective.*

(2) $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ *is surjective.*

(3) *Either R is a Gorenstein ring, or the following two conditions are satisfied.*

  (i) *B is a* DVR *and* $\mathrm{J}(B) = \mathfrak{m}$.

  (ii) *There is no proper intermediate field between $R/\mathfrak{m}$ and $B/\mathrm{J}(B)$.*

When this is the case, R is an almost Gorenstein local ring.

We set $B = \mathfrak{m} : \mathfrak{m}$. Let $\mathrm{J}(B)$ be the Jacobson radical of B. To prove Theorem 4.5.2, we need some preliminaries. Let us begin with the following.

Lemma 4.5.3. *Suppose that R is not a* DVR. *Then $R \neq B$ and $\ell_R(B/R) = \mathrm{r}(R)$.*

Proof. We have $R : \mathfrak{m} = \mathfrak{m} : \mathfrak{m}$, since R is not a DVR. The second assertion is clear, since $\ell_R((R : \mathfrak{m})/R) = \mathrm{r}(R)$. □

Proposition 4.5.4. *Suppose that $R \neq B$ and that $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is surjective. Then there is no proper intermediate ring between R and B.*

Proof. We have $\mathfrak{m}, R \in \mathcal{X}_R$ and $B, R \in \mathcal{Y}_R$. Let A be an extension of R such that $R \subsetneq A \subseteq B$. We write $A = \rho(I) = R : I$ with $I \in \mathcal{X}_R$. Then $I \subseteq \mathfrak{m}$, since $A \neq R$. Therefore, $A = R : I \supseteq R : \mathfrak{m} = B$, so that $A = B$. □

The following is the heart of our argument.

Theorem 4.5.5. *Let $(R, \mathfrak{m})$ be a non-Gorenstein Cohen-Macaulay local ring of dimension one. Assume that R is $\mathfrak{m}$-adically complete and there is no proper intermediate ring between R and B. Then the following assertions hold true.*

(1) *$B = \overline{R}$, and B is a* DVR *with* $\mathrm{J}(B) = \mathfrak{m}$.

(2) $[B/\mathfrak{m} : R/\mathfrak{m}] = \mathrm{r}(R) + 1 \geq 3$.

(3) *There is no proper intermediate field between $R/\mathfrak{m}$ and $B/\mathfrak{m}$.*

(4) *$\mathcal{X}_R = \{\mathfrak{m}, R\}$ and the correspondence ρ is bijective.*

Proof. We have $\mathfrak{m}B = \mathfrak{m}$, and $R \neq B$, since R is not a DVR (Lemma 4.5.3). Let $x \in B \setminus R$. Then $B = R[x]$ and $B/\mathfrak{m} = k[\overline{x}]$, where $k = R/\mathfrak{m}$ and $\overline{x}$ denotes the image of x in $B/\mathfrak{m}$. Let n (> 0) be the degree of the minimal polynomial of $\overline{x}$ over k. We then have

$$B = R + Rx + Rx^2 + \cdots + Rx^{n-1}$$

and $n = \mu_R(B)$, so that $n - 1 = \mathrm{r}(R)$ by Lemma 4.5.3. Therefore, $n \geq 3$ since R is not a Gorenstein ring, so that $x^2 \notin R$ since the elements $1, x, \ldots, x^{n-1}$ form a minimal system of generators of the R-module B. Hence

$$B = R[x^2] = R + Rx^2 + Rx^4 + \cdots + Rx^{2(n-1)}.$$

Let us write $x = \sum_{i=0}^{n-1} c_i x^{2i}$ with $c_i \in R$. We then have $x(1 - ax) = c_0$, where $a = \sum_{i=1}^{n-1} c_i x^{2i-2}$. We will show that $x \notin \mathrm{J}(B)$. If $c_0 \notin \mathfrak{m}$, then x is a unit of B, whence $x \notin \mathrm{J}(B)$. Assume that $c_0 \in \mathfrak{m}$. Then, if $x \in \mathrm{J}(B)$, $1 - ax$ is a unit of B, so that $x = (1 - ax)^{-1}c_0 \in \mathfrak{m}B = \mathfrak{m}$, which is a contradiction. Therefore, $x \notin \mathrm{J}(B)$ for all $x \in B \setminus R$, which shows $\mathrm{J}(B) \subseteq R$, whence $\mathrm{J}(B) = \mathfrak{m}$. Therefore, we have $B = \mathfrak{m} : \mathfrak{m} = \mathrm{J}(B) : \mathrm{J}(B)$. Hence, $B_M = MB_M : MB_M$ for all $M \in \operatorname{Max} B$, which implies the local ring B_M is a DVR (see Lemma 4.5.3). Therefore, because B is integrally closed in $\mathrm{Q}(B) = \mathrm{Q}(R)$, we get $B = \overline{B} = \overline{R}$.

Since R is $\mathfrak{m}$-adically complete, we have a decomposition

$$B = B_1 \times B_2 \times \cdots \times B_\ell$$

of B into a finite product of DVR's $\{B_j\}_{1 \leq j \leq \ell}$. We want to show that $\ell = 1$. Let $\mathbf{e}_j = (0, \ldots, 0, 1_{B_j}, 0, \ldots, 0)$ in B and set $\mathbf{e} = \sum_{j=1}^{\ell} \mathbf{e}_j$. Assume now that $\ell \geq 2$. We then have $B = R[\mathbf{e}_1]$, since $\mathbf{e}_1 \notin R$ and since there is no proper intermediate ring between R and B. Hence $B = R\mathbf{e} + R\mathbf{e}_1$, since $\mathbf{e}_1^2 = \mathbf{e}_1$. This is however impossible, because

$$\mu_R(B) = \ell_R(B/\mathfrak{m}B) = 1 + \mathrm{r}(R) > 2.$$

Thus, $\ell = 1$, that is $B = \overline{R}$ is a DVR with the maximal ideal $\mathrm{J}(B) = \mathfrak{m}$. It remains the proof of Assertions (3) and (4). Assume that there is contained a field F such that $R/\mathfrak{m} \subseteq F \subseteq B/\mathfrak{m}$. We consider the natural epimorphism $\varepsilon : B \to B/\mathfrak{m}$ of rings. Then, since $\varepsilon^{-1}(F)$ is an intermediate ring between R and B, either $\varepsilon^{-1}(F) = R$, or $\varepsilon^{-1}(F) = B$, which shows either $F = R/\mathfrak{m}$, or $F = B/\mathfrak{m}$.

Let $I \in \mathcal{X}_R$ and assume that $I \neq R$. Then, since $I \subseteq \mathfrak{m}$, we have

$$B = \mathfrak{m} : \mathfrak{m} = R : \mathfrak{m} \subseteq R : I = I : I \subseteq \overline{R} = B,$$

whence $I : I = B$, so that I is an ideal of B. Let us write $I = aB$ with $0 \neq a \in B$. We then have

$$B = R : I = R : aB = a^{-1}(R : B) = a^{-1}\mathfrak{m},$$

since $\mathfrak{m} = R : B$, so that $\mathfrak{m} = aB = I$. Thus, $\mathcal{X}_R = \{\mathfrak{m}, R\}$, which shows the correspondence ρ is bijective. This completes the proof of Theorem 4.5.5. □

We are now ready to prove Theorem 4.5.2.

Proof of Theorem 4.5.2. (1) ⇒ (2) This is clear.

(3) ⇒ (1) See Lemma 4.2.6, Proposition 4.5.4, and Theorem 4.5.5 (4).

(2) ⇒ (3) We may assume that R is not a Gorenstein ring. Passing to the $\mathfrak{m}$-adic completion $\widehat{R}$ of R, without loss of generality we may also assume that R is $\mathfrak{m}$-adically

complete. Then by Proposition 4.5.4, there is no proper intermediate ring between R and B, so that the assertion follows from Theorem 4.5.5.

If ρ is bijective but R is not a Gorenstein ring, we then have $B = \mathfrak{m} : \mathfrak{m}$ is a DVR, so that R is an almost Gorenstein ring by [36, Theorem 5.1]. □

We note the following, which is a direct consequence of Theorem 4.5.2.

Corollary 4.5.6. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension one. Suppose that one of the following conditions is satisfied.*

(i) *The field $R/\mathfrak{m}$ is algebraically closed.*

(ii) *$\overline{R}$ is a local ring, and $R/\mathfrak{m} \cong \overline{R}/\mathfrak{n}$, where $\mathfrak{n}$ denotes the maximal ideal of $\overline{R}$.*

Then the following assertions are equivalent.

(1) *R is a Gorenstein ring.*

(2) *The correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is bijective.*

(3) *The correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is surjective.*

When R is a numerical semigroup ring over a field, Condition (ii) of Corollary 4.5.6 is always satisfied.

4.6 Anti-stable rings

Let R be a commutative ring and let $\mathcal{F}_R$ denote the set of regular ideals of R. Then, because $(R : I){\cdot}(I : I) \subseteq R : I$, for every $I \in \mathcal{F}_R$ the R-module $R : I$ has also the structure of an $(I : I)$-module. Keeping this fact together with the natural identifications $R : I = \mathrm{Hom}_R(I, R)$ and $I : I = \mathrm{End}_R I$ in our mind, we give the following.

Definition 4.6.1. We say that R is an *anti-stable* (resp. *strongly anti-stable*) ring, if $R : I$ is an invertible $I : I$-module (resp. $R : I \cong I : I$ as an $(I : I)$-module) for every $I \in \mathcal{F}_R$.

Therefore, every Dedekind domain is anti-stable, and every UFD is a strongly anti-stable ring. Notice that when R is a Noetherian semi-local ring, R is anti-stable if and only if it is strongly anti-stable. Indeed, let $I \in \mathcal{F}_R$, and set $A = I : I$, $M = R : I$. Then, A is also a Noetherian semi-local ring, and therefore, because M has rank one as an A-module, M must be cyclic and free, once it is an invertible module over A.

Let us recall here that R is said to be a *stable* ring, if every ideal I of R is *stable*, that is projective over its endomorphism ring $\mathrm{End}_R I$ ([73]). When R is a Noetherian semi-local ring and $I \in \mathcal{F}_R$, I is a stable ideal of R if and only if $I \in \mathcal{Z}_R$, that is $I^2 = aI$ for some $a \in I$ ([67], [73, Proposition 2.2]). Our definition of anti-stable rings is, of course, different from that of stable rings. However, we shall later show in Corollary 4.6.10 that the anti-stability of rings implies the stability of rings, under suitable conditions.

First of all, we will show that R is a strongly anti-stable ring if and only if every $I \in \mathcal{F}_R$ is isomorphic to a trace ideal in R.

Lemma 4.6.2. *Let $I \in \mathcal{F}_R$ and set $A = I : I$. Then the following conditions are equivalent.*

(1) *$I \cong J$ as an R-module for some $J \in \mathcal{X}_R$.*

(2) *$I \cong \tau_R(I)$ as an R-module.*

(3) *$R : I \cong A$ as an R-module.*

(4) *$R : I \cong A$ as an A-module.*

(5) *$R : I = aA$ for some unit a of $\mathrm{Q}(R)$.*

Proof. (1) $\Leftrightarrow$ (2) Since $\tau_R(I) \in \mathcal{X}_R$, the implication (2) $\Rightarrow$ (1) is clear. Since $J = \tau_R(J)$ for every $J \in \mathcal{X}_R$ (Proposition 4.2.1), we have $\tau_R(I) = J$, if $J \in \mathcal{X}_R$ and $I \cong J$ as an R-module, whence the implication (1) $\Rightarrow$ (2) follows.

(4) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (4) Because the given isomorphsim $R : I \to A$ of R-modules is the restriction of the homothety of some unit a of $\mathrm{Q}(R)$, it must be also a homomorphism of A-modules, whence $R : I \cong A$ as an A-module.

(4) $\Leftrightarrow$ (5) This is now clear.

(1) $\Rightarrow$ (3) We have $I = aJ$ for some unit a of $\mathrm{Q}(R)$, whence $R : I = R : aJ = a^{-1}(R : J)$, and $I : I = aJ : aJ = J : J$. Thus, $R : I \cong I : I$ as an R-module, because $R : J = J : J$.

(5) $\Rightarrow$ (2) We have $\tau_R(I) = (R : I)I = aA{\cdot}I = aI$, whence $\tau_R(I) \cong I$ as an R-module. □

For a Noetherian ring R, we set $\mathrm{Ht}_1(R) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{ht}_R\mathfrak{p} = 1\}$. Let us note the following example of strongly anti-stable rings. We include a brief proof.

Example 4.6.3 ([61, Corollary 3.10])**.** For a Noetherian normal domain R, R is a strongly anti-stable ring if and only if R is a UFD.

Proof. Suppose that R is a strongly anti-stable ring and let $\mathfrak{p} \in \mathrm{Ht}_1(R)$. Then, since R is normal, the R-module $\mathfrak{p}$ is reflexive with $\mathfrak{p} : \mathfrak{p} = R$, while $R : \mathfrak{p} \cong \mathfrak{p} : \mathfrak{p}$ by Lemma 4.6.2. Hence, $\mathfrak{p} \cong R$, so that R is a UFD. Conversely, suppose that R is a UFD and let $I \in \mathcal{X}_R$. Then, $I \cong J$ for some ideal J of R with $\mathrm{grade}_R J \geq 2$, so that $I \cong \tau_R(I)$, since $J \in \mathcal{X}_R$ by Corollary 4.2.2. Thus, R is a strongly anti-stable ring. □

We explore one example of anti-stable rings which are not strongly anti-stable.

Example 4.6.4. Let k be a field and $S = k[t]$ the polynomial ring. Let $\ell \geq 2$ be an integer and set $R = k[t^2, t^{2\ell+1}]$. We consider the maximal ideal $I = (t^2 - 1, t^{2\ell+1} - 1)$ in R. Then, $\tau_R(I) = R$, and $I \ncong J$ as an R-module for any $J \in \mathcal{X}_R$. Therefore, R is not a strongly anti-stable ring, while R is an anti-stable ring, because $\dim R = 1$ and for every $M \in \operatorname{Max} R$, R_M is an anti-stable local ring. See Theorem 4.6.9 for details.

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$. If $I \nsubseteq \mathfrak{p}$, then $IR_\mathfrak{p} = R_\mathfrak{p}$. If $I \subseteq \mathfrak{p}$, then $t^2 \notin \mathfrak{p} = I$, whence $R_\mathfrak{p} = S_\mathfrak{p}$ is a DVR, because $R : S = (t^2, t^{2\ell+1})R$, so that $IR_\mathfrak{p} \cong R_\mathfrak{p}$. We now notice that $I \subseteq \tau_R(I) \subseteq R$. Hence, either $I = \tau_R(I)$ or $\tau_R(I) = R$. If $I = \tau_R(I)$, then setting $\mathfrak{p} = I$, we get $R_\mathfrak{p}$ is a DVR and $IR_\mathfrak{p} = \tau_{R_\mathfrak{p}}(IR_\mathfrak{p}) \subsetneq R_\mathfrak{p}$, while $\tau_{R_\mathfrak{p}}(IR_\mathfrak{p}) = R_\mathfrak{p}$, because $IR_\mathfrak{p} \cong R_\mathfrak{p}$. This is absurd. Hence $\tau_R(I) = R$. Consequently, $I \ncong J$ for any $J \in \mathcal{X}_R$. In fact, if $I \cong J$ for some $J \in \mathcal{X}_R$, then $J = \tau_R(J) = \tau_R(I) = R$, so that $\mu_R(I) = 1$. We write $I = fR$ with some monic polynomial $f \in R$. Let $\overline{k}$ denote the algebraic closure of k and choose $a \in \overline{k}$ so that $f(a) = 0$. Then, since $a^2 = a^{2\ell+1} = 1$, we get $a = 1$, whence $f = (t-1)^n$ with $0 < n \in H$, where $H = \langle 2, 2\ell+1 \rangle$ denotes the numerical semigroup generated by $2, 2\ell+1$. Therefore, $2-n, (2\ell+1)-n \in H$, because $t^2-1, t^{2\ell+1}-1 \in fR$. Hence, $n = 2$, and $2\ell+1 \in 2+H$, which is impossible. Thus, I is not a principal ideal of R, and $I \ncong J$ for any $J \in \mathcal{X}_R$. $\square$

The key in our argument is the following, which plays a key role also in [28].

Lemma 4.6.5. *Let R be a strongly anti-stable ring. Then the correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is surjective. More precisely, let $A \in \mathcal{Y}_R$ and set $J = R : A$. Then $J \in \mathcal{G}_R = \mathcal{X}_R \cap \mathcal{Z}_R$.*

Proof. Let $A \in \mathcal{Y}_R$ and choose $b \in W$ so that $bA \subseteq R$. Then, since $bA \in \mathcal{F}_R$, by Lemma 4.6.2 $bA \cong J$ as an R-module for some $J \in \mathcal{X}_R$. Let us write $J = aA$ with a a unit of $\mathrm{Q}(R)$ (hence $a \in J \cap W$). We then have $J : J = aA : aA = A : A = A$, whence $A = J : J = R : J = R : aA = a^{-1}(R : A)$, so that $R : A = aA = J \in \mathcal{X}_R \cap \mathcal{Z}_R$. Therefore, $\rho(J) = J : J = A$, and the correspondence $\rho : \mathcal{X}_R \to \mathcal{Y}_R$ is surjective. $\square$

Let us recall one of the fundamental results on stable rings, which we need to prove Theorem 4.6.7.

Proposition 4.6.6 ([73, Lemma 3.2, Theorem 3.4])**.** *Let R be a Cohen-Macaulay semi-local ring and assume that $\dim R_M = 1$ for every $M \in \operatorname{Max} R$. If $\mathrm{e}(R_M) \le 2$ for every $M \in \operatorname{Max} R$, then R is a stable ring.*

We should compare the following theorem with [11, Theorem 3.6].

Theorem 4.6.7. *Let R be a Cohen-Macaulay local ring of dimension one. Then, R is an anti-stable ring, if and only if $\mathrm{e}(R) \le 2$.*

Proof. Suppose that $\mathrm{e}(R) \le 2$. Let $I \in \mathcal{F}_R$ and set $A = I : I$. Then, by Proposition 4.6.6 R is a stable ring. Hence, $I^2 = aI$ for some $a \in I$, whence $A = a^{-1}I$. Therefore, $I \cong A$ as an R-module. We now consider $J = (R : I)I$. Then, $J = \tau_R(I) \in \mathcal{X}_R$, whence

$$J : J = R : J = R : (R : I)I = [R : (R : I)] : I = I : I,$$

where the last equality follows from the fact that R is a Gorenstein ring. Consequently, $A = J : J \cong J$ (since $J \in \mathcal{F}_R$), so that $I \cong J = \tau_R(J)$. Thus, R is an anti-stable ring.

Conversely, suppose that R is an anti-stable ring. First of all, we will show that R is a Gorenstein ring. Assume the contrary. Then, passing to the $\mathfrak{m}$-adic completion of R, by Proposition 4.5.4 and Theorem 4.5.5 we get $\mathcal{X}_R = \{\mathfrak{m}, R\}$. Consequently, either

$I \cong \mathfrak{m}$ or $I \cong R$, for every ideal $I \in \mathcal{F}_R$. We set $n = \mu_R(\mathfrak{m})$ and write $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ with non-zerodivisors x_i of R. Then, $n > 2$ since R is not a Gorenstein ring, and setting $I = (x_1, x_2, \ldots, x_{n-1})$, we have either $I \cong \mathfrak{m}$ or $I \cong R$, both of which violates the fact that $n = \mu_R(\mathfrak{m}) > 2$. Thus R is a Gorenstein ring. We want to show $\mathrm{e}(R) \leq 2$. Assume that $\mathrm{e}(R) \geq 2$ and consider $B = \mathfrak{m} : \mathfrak{m}$. Then, $B \in \mathcal{Y}_R$ and $R \neq B$, because R is not a DVR. Consequently, because $\mathfrak{m} = R : B$, by Lemma 4.6.5 $\mathfrak{m}^2 = a\mathfrak{m}$ for some $a \in \mathfrak{m}$, which implies $\mathrm{e}(R) = 2$, since R is a Gorenstein ring. □

We say that a Noetherian ring R satisfies the condition (S_1) of Serre, if $\operatorname{depth} R_\mathfrak{p} \geq \inf\{1, \dim R_\mathfrak{p}\}$ for every $\mathfrak{p} \in \operatorname{Spec} R$.

Corollary 4.6.8. *Let R be a Noetherian ring and suppose that R satisfies (S_1). Then, $\mathrm{e}(R_\mathfrak{p}) \leq 2$ for every $\mathfrak{p} \in \mathrm{Ht}_1(R)$, if R is an anti-stable ring.*

Proof. Let $\mathfrak{p} \in \mathrm{Ht}_1(R)$ and set $A = R_\mathfrak{p}$. Hence A is a Cohen-Macaulay local ring of dimension one. Let $I \in \mathcal{F}_A$ and set $J = I \cap R$. We will show that $A : I$ is a cyclic $(I : I)$-module. We may assume that $I \neq A$. Hence, J is a $\mathfrak{p}$-primary ideal of R, since I is a $\mathfrak{p}R_\mathfrak{p}$-primary ideal of $A = R_\mathfrak{p}$. Hence, because $J \in \mathcal{F}_R$ (remember that R satisfies (S_1)), $R : J$ is a projective $(J : J)$-module. Therefore, $A : I = [R : J]_\mathfrak{p}$ is a cyclic module over $I : I = [J : J]_\mathfrak{p}$, since it has rank one over the semi-local ring $I : I$. Thus, $\mathrm{e}(A) \leq 2$ by Theorem 4.6.7. □

We now come to the main results of this section.

Theorem 4.6.9. *Let R be a Noetherian ring and suppose that R satisfies (S_1). Let us consider the following four conditions.*

(1) *R is anti-stable.*

(2) *R is strongly anti-stable.*

(3) *Every $I \in \mathcal{F}_R$ is isomorphic to $\tau_R(I)$.*

(4) *$\mathrm{e}(R_\mathfrak{p}) \leq 2$ for every $\mathfrak{p} \in \mathrm{Ht}_1(R)$.*

Then, we have the implications $(3) \Leftrightarrow (2) \Rightarrow (1) \Rightarrow (4)$. If R is semi-local (resp. $\dim R = 1$), then the implication $(1) \Rightarrow (2)$ (resp. $(4) \Rightarrow (1)$) holds true.

Proof. $(3) \Leftrightarrow (2) \Rightarrow (1) \Rightarrow (4)$ See Lemma 4.6.2 and Theorem 4.6.8.

If R is semi-local, then every birational module -finite extension of R is also semi-local, so that the implication $(1) \Rightarrow (2)$ follows.

Suppose that $\dim R = 1$. Let $I \in \mathcal{F}_R$ and set $A = I : I$. Then, by Theorem 4.6.7 $R_M : IR_M = [R : I]_M$ is a cyclic A_M-module for every $M \in \operatorname{Max} R$, so that $R : I$ is an invertible A-module. Hence, the implication $(4) \Rightarrow (1)$ follows. □

Theorem 4.6.10. *Let R be a Cohen-Macaulay ring with $\dim R_M = 1$ for every $M \in \operatorname{Max} R$. If R is an anti-stable ring, then R is a stable ring.*

Proof. For every $M \in \operatorname{Max} R$, $\mathrm{e}(R_M) \leq 2$ by Corollary 4.6.8. Let I be an arbitrary ideal of R and set $A = \mathrm{End}_R I$. Then, because R_M is a stable ring by Proposition 4.6.6, for every $M \in \operatorname{Max} R$ IR_M is a projective A_M-module, so that I is a projective A-module. Thus, R is a stable ring. □

Chapter 5

The Auslander-Reiten conjecture for non-Gorenstein Cohen-Maaulay rings

5.1 Introduction

The purpose of this chapter is to study the vanishing of Ext modules. The vanishing of homology plays a very important role in the study of rings and modules. The Auslander-Reiten conjecture and several related conjectures are problems about the vanishing. For a guide to these conjectures, one can consult [17, Appendix A] and [14, 50, 79, 80]. These conjectures originate from the representation theory of rings. However, interesting results also have been developed from the theory of commutative rings; see, for examples, [2, 57, 58, 59]. Let us recall the Auslander-Reiten conjecture over a commutative Noetherian ring R.

Conjecture 5.1.1. [5] Let M be a finitely generated R-module. If $\operatorname{Ext}_R^i(M, M \oplus R) = 0$ for all $i > 0$, then M is a projective R-module.

Although a lot of partial results on the Auslander-Reiten conjecture are known, in this chapter, we are especially interested in the following one; see [2, Theorem 3.], [4, Proposition 1.9.], and [57, Theorem 0.1].

Theorem 5.1.2. (Araya, Auslander-Ding-Solberg, Huneke-Leuschke) *Suppose that R is a Gorenstein ring which is a complete intersection in codimension one. Then the Auslander-Reiten conjecture holds for R. In particular, the Auslander-Reiten conjecture holds for complete intersections and Gorenstein normal domains.*

As is well-known, non-zerodivisors preserve the Auslander-Reiten conjecture (Proposition 5.2.1). Hence, through Theorem 5.1.2, many Gorenstein rings which satisfy the Auslander-Reiten conjecture are given. Even if a given local ring is not Gorenstein, the conjecture still holds if the ring is Golod or almost Gorenstein ([59, Proposition 1.4.] and [46, proof of Corollary 4.5.]). However, Golod rings and almost Gorenstein rings do not form the best possible classes of rings that satisfy the Auslander-Reiten conjecture. In fact, if a local ring $(R, \mathfrak{m})$ is a Golod ring (resp. a non-Gorenstein almost Gorenstein ring) and $x \in \mathfrak{m}$ is a non-zerodivisor of R, then the ring $R/(x^n)$ is no longer Golod (resp. almost Gorenstein), where $n > 1$ ([46, Theorem 3.7.] and [74, Proposition 4.6.]). Motivated by these results, in this chapter, we study the Auslander-Reiten conjecture for non-Gorenstein rings.

In Section 5.2, we study the Auslander-Reiten conjecture for the residue ring R/Q^ℓ in connection with that for R, where Q is an ideal of R generated by a regular sequence on R and ℓ is a positive integer. As a result, we have the following which is one of the main results of this chapter.

Theorem 5.1.3. (Theorem 5.2.2) *Suppose that R is a Gorenstein local ring. Let $Q = (x_1, x_2, \dots, x_n)$ be an ideal of R generated by a regular sequence on R. Then the following conditions are equivalent.*

(1) *The Auslander-Reiten conjecture holds for R.*

(2) *There is an integer $\ell > 0$ such that the Auslander-Reiten conjecture holds for R/Q^ℓ.*

(3) *For all integers $1 \le \ell \le n$, the Auslander-Reiten conjecture holds for R/Q^ℓ.*

As is well-known, unlike localizations and dividing by non-zerodivisors, homological properties do not necessarily preserve through dividing by the powers of parameter ideals. In fact, letting R be a Gorenstein ring and $Q = (x_1, x_2, \dots, x_n)$ be an ideal of R generated by a regular sequence on R, R/Q^ℓ is no longer Gorenstein if $n \ge 2$ and $\ell \ge 2$. Therefore Theorem 5.1.3 gives a new class of rings which satisfy the Auslander-Reiten conjecture.

The powers of parameter ideals are related to determinantal rings. Let s, t be positive integers and $A[\mathbf{X}] = A[X_{ij}]_{1 \le i \le s, 1 \le j \le t}$ be a polynomial ring over a commutative ring A. Assume $s \le t$ and let $\mathbb{I}_s(\mathbf{X})$ denote the ideal of $A[\mathbf{X}]$ generated by the maximal minors of the $s \times t$ matrix (X_{ij}). With these assumptions and notations, we have the following.

Theorem 5.1.4. (Theorem 5.2.9) *Suppose that $2s \le t + 1$ and A is a Gorenstein ring which is a complete intersection in codimension one. Then the Auslander-Reiten conjecture holds for the determinantal ring $A[\mathbf{X}]/\mathbb{I}_s(\mathbf{X})$.*

In Section 5.3, we study a new class of rings arising from Theorem 5.1.3, that is, the class of rings R that there exist a parameter ideal $\mathfrak{q}$ of R, a complete intersection S, and a parameter ideal Q of S such that $R/\mathfrak{q} \cong S/Q^2$ as rings. We will see that the condition is characterized by an ideal condition and strongly related to the existence of Ulrich ideals. Here the notion of Ulrich ideals is given by [43] and a generalization of maximal ideals of rings possessing maximal embedding dimension. It is known that Ulrich ideals enjoy many good properties, see [43, 47] and [29, Theorem 1.2]. The ubiquity and existence of Ulrich ideals are also studied ([29, 44, 47]). In the current chapter, we study the existence of Ulrich ideals whose residue rings are complete intersections in connection with a new class of rings. As a goal of this chapter, we have the following.

Theorem 5.1.5. (Corollary 5.3.8) *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Suppose that there exists an Ulrich ideal of R whose residue ring is a complete intersection. Let v* (resp. *r*) *denotes the embedding dimension of R* (resp. *the Cohen-Macaulay type of R*). *Then the following assertions are true.*

(1) *The Auslander-Reiten conjecture holds for R.*

(2) $r + d \le v$.

(3) *There exist a parameter ideal $\mathfrak{q}$ of R, a local complete intersection S of dimension r, and a parameter ideal Q of S such that $R/\mathfrak{q} \cong S/Q^2$ as rings.*

(4) *Assume that there are a regular local ring T of dimension v and a surjective ring homomorphism $T \to R$. Let $0 \to F_{v-d} \to \cdots \to F_1 \to F_0 \to R \to 0$ be a minimal T-free resolution of R. Then*

$$\mathrm{rank}_T F_0 = 1 \quad \textit{and} \quad \mathrm{rank}_T F_i = \sum_{j=0}^{v-r-d} \beta_{i-j} \cdot \binom{v-r-d}{j}$$

for $1 \le i \le v-d$, *where* $\beta_k = \begin{cases} 1 & \textit{if } k = 0 \\ k \cdot \binom{r+1}{k+1} & \textit{if } 1 \le k \le r \\ 0 & \textit{otherwise.} \end{cases}$

Theorem 5.1.5 (3) claims that an Ulrich ideal determines the structure of the ring. Furthermore, Theorem 5.1.5 (4) recovers the result of J. Sally [71, Theorem 1.] by taking the maximal ideal $\mathfrak{m}$ as an Ulrich ideal.

Let us fix our notations throughout this chapter. In what follows, unless otherwise specified, let R denote a Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$. For each finitely generated R-module M, let $\mu_R(M)$ (resp. $\ell_R(M)$) denote the number of elements in a minimal system of generators of M (resp. the length of M). If M is a Cohen-Macaulay R-module, $\mathrm{r}_R(M)$ denotes the Cohen-Macaulay type of M. Let $\mathrm{v}(R)$ (resp. $\mathrm{r}(R)$) denote the embedding dimension of R (resp. the Cohen-Macaulay type of R). For convenience, letting M and N be R-modules, $\mathrm{Ext}_R^{>0}(M, N) = 0$ (resp. $\mathrm{Tor}^R_{>0}(M, N) = 0$) denotes $\mathrm{Ext}_R^i(M, N) = 0$ for all $i > 0$ (resp. $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$).

5.2 Powers of parameter ideals and determinantal rings

The purpose of this section is to prove Theorem 5.1.3. First of all, let us sketch a brief proof that non-zerodivisors preserve the Auslander-Reiten conjecture since Theorem 5.1.3 is based on the fact.

Proposition 5.2.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $a \in \mathfrak{m}$ be a non-zerodivisor of R. Then the Auslander-Reiten conjecture holds for R if and only if it holds for the residue ring $R/(a)$.*

Proof. (if part) Let M be a finitely generated R-module such that $\mathrm{Ext}_R^{>0}(M, M \oplus R) = 0$. Take an exact sequence $0 \to X \to F \to M \to 0$, where F is a free R-module of rank $\mu_R(M)$. By applying the functors

$$\mathrm{Hom}_R(-, R),\ \mathrm{Hom}_R(M, -),\ \text{and}\ \mathrm{Hom}_R(-, X)$$

to the above short exact sequence, we have $\mathrm{Ext}_R^{>0}(X, X \oplus R) = 0$. Thus $\mathrm{Ext}_{\overline{R}}^{>0}(\overline{X}, \overline{X} \oplus \overline{R}) = 0$ since $a \in \mathfrak{m}$ is a non-zerodivisor of X and R, where $\overline{*}$ denotes $R/(a) \otimes_R *$. Therefore the $\overline{R}$-module $\overline{X}$ is free and so is the R-module X. Hence the R-module $\mathrm{Hom}_R(M, R)$ is free and so is the R-module $M \cong \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$.

(only if part) Let N be a finitely generated $\overline{R}$-module such that $\mathrm{Ext}_{\overline{R}}^{>0}(N, N \oplus \overline{R}) = 0$. Then there exists a finitely generated R-module M such that $M/aM \cong N$ and $\mathrm{Tor}_{>0}^R(M, \overline{R}) = 0$ by [4, Proposition 1.7.]. Hence, by the exact sequence $0 \to M \xrightarrow{a} M \to N \to 0$, $\mathrm{Ext}_R^{>0}(M, M \oplus R) = 0$. Therefore the R-module M is free and so is the $\overline{R}$-module N. □

Theorem 5.2.2. *Let $(S, \mathfrak{n})$ be a Gorenstein local ring and $x_1, x_2, \ldots, x_n$ be a regular sequence on S. Set $Q = (x_1, x_2, \ldots, x_n)$. Then the following conditions are equivalent.*

(1) *The Auslander-Reiten conjecture holds for S.*

(2) *The Auslander-Reiten conjecture holds for S/Q.*

(3) *There is an integer $\ell > 0$ such that the Auslander-Reiten conjecture holds for S/Q^ℓ.*

(4) *For all integers $1 \leq \ell \leq n$, the Auslander-Reiten conjecture holds for S/Q^ℓ.*

Proof. The implications (1) ⇔ (2) follow from Proposition 5.2.1 and the implication (4) ⇒ (3) is trivial. Hence we have only to show that (1) ⇒ (4) and (3) ⇒ (1). First of all, we reduce our assertions to the case where Q is a parameter ideal. Set $R = S/Q^\ell$. Note that R is a Cohen-Macaulay local ring with $\dim R = \dim S - n$ since Q^ℓ is perfect. In fact, Q^ℓ is generated by $\ell \times \ell$-minors of the $\ell \times (n + \ell - 1)$ matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n & & & \\ & x_1 & x_2 & \cdots & \cdots & x_{n-1} & x_n & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix},$$

whence the projective dimension of S/Q^ℓ over S is n, see [18] or [13, (2.14) Proposition]. Suppose $\dim R > 0$. Then we can take $a \in S$ so that a is a non-zerodivisor of R and S/Q. By replacing R and S by R/aR and S/aS, we finally may assume that R is an Artinian local ring, that is, Q is a parameter ideal of S. We may also assume that $n \geq 2$ and $\ell \geq 2$ by Proposition 5.2.1.

(1) ⇒ (4) Assume $1 \leq \ell \leq n$. Suppose that M is a finitely generated R-module such that $\mathrm{Ext}_R^{>0}(M, M \oplus R) = 0$. We will show that $\mathrm{Ext}_{S/Q}^{>0}(M/QM, M/QM \oplus S/Q) = 0$ in several steps. Note that we have an exact sequence

$$0 \to Q^{i-1}/Q^i \to S/Q^i \to S/Q^{i-1} \to 0 \qquad (*)$$

of R-modules for all $2 \le i \le \ell$ and Q^i/Q^{i+1} is an S/Q-free module of rank $\binom{i+n-1}{n-1}$ for all $i > 0$.

Claim 4. $\mathrm{Ext}_R^{>0}(M, S/Q) = 0$.

Proof of Claim 4. By applying the functor $\mathrm{Hom}_R(M, -)$ to the exact sequence $(*)$, we have the following exact sequence and isomorphism

$$\mathrm{Ext}_R^j(M, S/Q)^{\oplus\binom{i-1+n-1}{n-1}} \to \mathrm{Ext}_R^j(M, S/Q^i) \to \mathrm{Ext}_R^j(M, S/Q^{i-1}) \to \mathrm{Ext}_R^{j+1}(M, S/Q)^{\oplus\binom{i-1+n-1}{n-1}}$$

$$\mathrm{Ext}_R^j(M, S/Q^{\ell-1}) \cong \mathrm{Ext}_R^{j+1}(M, S/Q)^{\oplus\binom{\ell-1+n-1}{n-1}} \qquad \text{(i)}$$

of R-modules for all $2 \le i \le \ell - 1$ and $j > 0$. Set $E_j = \ell_R(\mathrm{Ext}_R^j(M, S/Q))$ for $j > 0$. Then, by (i),

$$\begin{aligned} E_{j+1}\binom{\ell+n-2}{n-1} &= \ell_R(\mathrm{Ext}_R^j(M, S/Q^{\ell-1})) \le E_j\cdot\binom{\ell-2+n-1}{n-1} + \ell_R(\mathrm{Ext}_R^j(M, S/Q^{\ell-2})) \\ &\le E_j\left\{\binom{\ell-2+n-1}{n-1} + \binom{\ell-3+n-1}{n-1}\right\} + \ell_R(\mathrm{Ext}_R^j(M, S/Q^{\ell-3})) \le \cdots \\ &\le E_j\cdot\sum_{i=0}^{\ell-2}\binom{i+n-1}{n-1} = E_j\cdot\binom{\ell-2+n}{n}, \end{aligned}$$

that is,

$$E_{j+1} \le \frac{\binom{\ell+n-2}{n}}{\binom{\ell+n-2}{n-1}}\cdot E_j = \frac{\ell-1}{n}\cdot E_j$$

for all $j > 0$. Hence, for enough large integer $m \ge 0$,

$$E_{m+1} \le \left(\frac{\ell-1}{n}\right)^m E_1 < 1$$

since $\ell \le n$. Hence $\mathrm{Ext}_R^j(M, S/Q) = 0$ for all $j > m$. On the other hand, for $j > 0$, $\mathrm{Ext}_R^{j+1}(M, S/Q) = 0$ implies that $\mathrm{Ext}_R^j(M, S/Q^i) = 0$ for all $1 \le i \le \ell-1$ by above isomorphism and exact sequence (i). Hence, by using descending induction, $\mathrm{Ext}_R^j(M, S/Q) = 0$ for all $j > 0$. $\square$

Let $\cdots \to F_1 \to F_0 \to M \to 0$ be a minimal R-free resolution of M. Then, by applying the functor $\mathrm{Hom}_R(-, S/Q)$ to the minimal free resolution, we have the following commutative diagram;

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}_R(M, S/Q) & \to & \mathrm{Hom}_R(F_0, S/Q) & \to & \mathrm{Hom}_R(F_1, S/Q) & \to & \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \to & \mathrm{Hom}_{S/Q}(M/QM, S/Q) & \to & \mathrm{Hom}_{S/Q}(F_0/QF_0, S/Q) & \to & \mathrm{Hom}_{S/Q}(F_1/QF_1, S/Q) & \to & \end{array}$$

The upper row is exact by Claim 4 and so is the lower row. Since S/Q is self-injective, the sequence $\cdots \to F_1/QF_1 \to F_0/QF_0 \to M/QM \to 0$ is a minimal S/Q-free resolution of M/QM. Hence $\mathrm{Tor}^R_{>0}(M, S/Q) = 0$. Moreover, by applying the functor $M \otimes_R -$ to the sequence $(*)$,

$$\mathrm{Tor}^R_{>0}(M, S/Q^i) = 0 \quad \text{for all } 1 \le i \le \ell - 1. \tag{ii}$$

Apply the functor $M \otimes_R -$ to $(*)$ again. Then we get the exact sequence

$$0 \to (M/QM)^{\oplus \binom{i-1+n-1}{n-1}} \to M/Q^iM \to M/Q^{i-1}M \to 0 \tag{$**$}$$

of R-modules for all $2 \le i \le \ell$ by (ii). Therefore, by applying the functor $\mathrm{Hom}_R(M, -)$ to $(**)$, we get the following exact sequence and isomorphism

$$\begin{aligned} &\mathrm{Ext}^j_R(M, M/QM)^{\oplus \binom{i-1+n-1}{n-1}} \to \mathrm{Ext}^j_R(M, M/Q^iM) \to \mathrm{Ext}^j_R(M, M/Q^{i-1}M) \\ &\to \mathrm{Ext}^{j+1}_R(M, M/QM)^{\oplus \binom{i-1+n-1}{n-1}} \end{aligned}$$

$$\mathrm{Ext}^j_R(M, M/Q^{\ell-1}M) \cong \mathrm{Ext}^{j+1}_R(M, M/QM)^{\oplus \binom{\ell-1+n-1}{n-1}}$$

of R-modules for all $2 \le i \le \ell - 1$ and $j > 0$. By setting $E'_j = \ell_R(\mathrm{Ext}^j_R(M, M/QM))$ for all $j > 0$ and calculation as the proof of Claim 4, we have $\mathrm{Ext}^{>0}_R(M, M/QM) = 0$. This induces that $\mathrm{Ext}^{>0}_{S/Q}(M/QM, M/QM) = 0$. In fact, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(M, \overline{M} & \longrightarrow & \mathrm{Hom}_R(F_0, \overline{M}) & \longrightarrow & \mathrm{Hom}_R(F_1, \overline{M}) & \longrightarrow & \cdots \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \mathrm{Hom}_{S/Q}(\overline{M}, \overline{M}) & \longrightarrow & \mathrm{Hom}_{S/Q}(F_0/QF_0, \overline{M}) & \longrightarrow & \mathrm{Hom}_{S/Q}(F_1/QF_1, \overline{M}) & \longrightarrow & \cdots, \end{array}$$

where $\overline{M} = M/QM$, and both of rows are exact. Since $\cdots \to F_1/QF_1 \to F_0/QF_0 \to M/QM \to 0$ is a minimal S/Q-free resolution of M/QM by (ii), $\mathrm{Ext}^{>0}_{S/Q}(M/QM, M/QM) = 0$. Thus we have

$$\mathrm{Ext}^{>0}_{S/Q}(M/QM, M/QM \oplus S/Q) = 0$$

since S/Q is self-injective, whence M/QM is S/Q-free by Proposition 5.2.1. This shows that M is R-free because $\mathrm{Tor}^R_1(M, S/Q) = 0$ by (ii).

(3) $\Rightarrow$ (1) Let N be a finitely generated S-module and suppose that $\mathrm{Ext}^{>0}_S(N, N \oplus S) = 0$. Then, by applying the functor $\mathrm{Hom}_S(N, -)$ to the exact sequence $0 \to S \xrightarrow{x_1} S \to S/x_1S \to 0$ of S-modules, $\mathrm{Ext}^{>0}_S(N, S/x_1S) = 0$. Hence

$$\mathrm{Ext}^{>0}_S(N, S/Q) = 0 \tag{iii}$$

by induction on n. Similarly, $\mathrm{Ext}^{>0}_S(N, N/QN) = 0$ since N is a maximal Cohen-Macaulay S-module by $\mathrm{Ext}^{>0}_S(N, S) = 0$.

Let $\cdots \to G_1 \to G_0 \to N \to 0$ be a minimal S-free resolution of N. Then, by applying the functor $\mathrm{Hom}_S(-, S/Q)$ to the minimal free resolution, we see that the sequence $\cdots \to G_1/QG_1 \to G_0/QG_0 \to N/QN \to 0$ is a minimal S/Q-free resolution of N/QN since (iii) and S/Q is self-injective. Hence $\mathrm{Tor}^S_{>0}(N, S/Q) = 0$. Moreover, by applying the functor $N \otimes_S -$ to the sequence $(*)$,

$$\mathrm{Tor}^S_{>0}(N, S/Q^i) = 0 \quad \text{for all } 1 \le i \le \ell. \tag{iv}$$

Therefore, for all $1 \le i \le \ell$, the sequence $\cdots \to G_1/Q^iG_1 \to G_0/Q^iG_0 \to N/Q^iN \to 0$ is a minimal S/Q^i-free resolution of N/Q^iN and

$$0 \to (N/QN)^{\oplus\binom{i-1+n-1}{n-1}} \to N/Q^iN \to N/Q^{i-1}N \to 0 \tag{v}$$

is exact as S-modules. Hence, by applying the functor $\mathrm{Hom}_S(N, -)$ to (v),

$$\mathrm{Ext}_S^{>0}(N, N/Q^iN) = 0 \quad \text{for all } 1 \le i \le \ell$$

since $\mathrm{Ext}_S^{>0}(N, N/QN) = 0$. Thus $\mathrm{Ext}_{S/Q^\ell}^{>0}(N/Q^\ell N, N/Q^\ell N) = 0$. Similarly, $\mathrm{Ext}_S^{>0}(N, S/Q^\ell) = 0$, whence $\mathrm{Ext}_{S/Q^\ell}^{>0}(N/Q^\ell N, S/Q^\ell \oplus N/Q^\ell N) = 0$. Hence $N/Q^\ell N$ is S/Q^ℓ-free, whence N is S-free by (iv). $\square$

The following assertions are direct consequences of Theorem 5.2.2.

Corollary 5.2.3. *Let S be a Gorenstein local ring and Q be a parameter ideal of S generated by a regular sequence on S. Then the Auslander-Reiten conjecture holds for S if and only if it holds for S/Q^2.*

Corollary 5.2.4. *Let S be either a complete intersection or a Gorenstein normal domain. Let $x_1, x_2, \ldots, x_n$ be regular sequence on S and set $Q = (x_1, x_2, \ldots, x_n)$. Then the Auslander-Reiten conjecture holds for S/Q^ℓ for all $1 \le \ell \le n$.*

Corollary 5.2.5. *Let R be a Cohen-Macaulay local ring. Suppose that there exist a parameter ideal $\mathfrak{q}$ of R, a local complete intersection S, and a parameter ideal Q of S such that $R/\mathfrak{q} \cong S/Q^2$ as rings. Then the Auslander-Reiten conjecture holds for R.*

In Section 5.3, we will characterize rings obtained in Corollary 5.2.5 by the existence of ideals in R. In the remainder of this section, we explore the Auslander-Reiten conjecture for determinantal rings. We start with the following.

Proposition 5.2.6. *Let s, t be positive integers and assume that $2s \le t+1$. Suppose that S is a Gorenstein local ring and $\{x_{ij}\}_{1 \le i \le s, 1 \le j \le t}$ forms a regular sequence on S. Let I be an ideal of S generated by $s \times s$ minors of the $s \times t$ matrix $(x_{ij}^{\alpha_{ij}})$, where α_{ij} is a positive integer for all $1 \le i \le s$ and $1 \le j \le t$. Set $R = S/I$. Then the Auslander-Reiten conjecture holds for S if and only if it holds for R.*

Proof. First of all, we show the case where $\alpha_{ij} = 1$ for all $1 \le i \le s$ and $1 \le j \le t$. Set

$$A = \{(i,j) \in \mathbb{Z} \oplus \mathbb{Z} \mid 1 \le i \le s, 1 \le j \le t\},$$
$$B = \bigcup_{1 \le i \le s-1} \{(i, i+k) \in A \mid 0 \le k \le t-s\}, \text{ and}$$
$$C = B \cup \{(s, s+k) \in A \mid 0 \le k \le t-s\}.$$

Then

$$\{x_{ij} - x_{i+1\,j+1} \mid (i,j) \in B\} \cup \{x_{ij} \mid (i,j) \in A \setminus C\}$$

forms a regular sequence on $S/(x_{11}, x_{12}, \ldots, x_{1\,t-s+1})$ and R, whence our assertion reduces to the case of Theorem 5.2.2. In fact, letting Q be an ideal of S generated by the above regular sequence,

$$R/QR \cong S/(I+Q) = S/[(x_{11}, x_{12}, \ldots, x_{1\,t-s+1})^s + Q] \text{ and } s \leq t-s+1.$$

Set $D = \{(i,j) \in A \mid \alpha_{ij} > 1\}$. We prove our assertion by induction on $N = \sharp D$. Assume that $N > 0$ and our assertion holds for $N-1$. Take $(i,j) \in D$. We may assume that $(i,j) = (1,1)$. Let J be an ideal of S generated by $s \times s$ minors of the $s \times t$ matrix $(x_{ij}^{\beta_{ij}})$, where $\beta_{ij} = \alpha_{ij}$ for all $(i,j) \in A \setminus \{(1,1)\}$ and $\beta_{11} = 1$. Noting that x_{11} is a non-zerodivisor of R, S/J, and S, we have the conclusion since $R/x_{11}R \cong S/(x_{11}S+I) = S/(x_{11}S+J)$. $\square$

Let $H \subseteq \mathbb{Z}$ be a numerical semigroup and k be a field. Then the numerical semigroup ring $k[[H]]$ often have the form obtained in Proposition 5.2.6; see, for examples, [32, 52]. In particular, the Auslander-Reiten conjecture holds for all three generated numerical semigroup rings. Let us note another concrete example.

Example 5.2.7. Let n be a positive integer. Let $k[[t]]$ and $S = k[[X,Y,Z,W]]$ be formal power series rings over a field k. Set $R = k[[t^{10}, t^{14}, t^{16}, t^{2n+1}]]$ and assume that $n \geq 6$. Then there exists an element $f \in (X)$ such that

$$R \cong S/[\mathbb{I}_2\left(\begin{smallmatrix} X & Y^2 & Z \\ Y & Z^2 & X^2 \end{smallmatrix}\right) + (W^2 - f)],$$

where $\mathbb{I}_2(\mathbb{M})$ denote the ideal of S generated by 2×2-minors of the matrix $\mathbb{M}$. In particular, the Auslander-Reiten conjecture holds for R.

Proof. Let $\varphi : S \to R$ be a ring homomorphism such that

$$X \mapsto t^{10},\ Y \mapsto t^{14},\ Z \mapsto t^{16}, \text{ and } W \mapsto t^{2n+1}.$$

Then, by a standard argument, $\operatorname{Ker}\varphi = \mathbb{I}_2\left(\begin{smallmatrix} X & Y^2 & Z \\ Y & Z^2 & X^2 \end{smallmatrix}\right) + (W^2 - f)$, where

$$f = \begin{cases} X^m{\cdot}Y^{m-1}{\cdot}Z & \text{if } n = 6m \\ X^{m+2}{\cdot}Z^{m-1} & \text{if } n = 6m+1 \\ X^{m+1}{\cdot}Y^m & \text{if } n = 6m+2 \\ X^m{\cdot}Y^{m-1} & \text{if } n = 6m+3 \\ X^m{\cdot}Y^{m-1}{\cdot}Z^2 & \text{if } n = 6m+4 \\ X^{m+2}{\cdot}Y^{m-1}{\cdot}Z & \text{if } n = 6m+5 \end{cases}$$

for some positive integer m. $\square$

Let us consider determinantal rings, which are not local rings. From now on until the end of this section, let s, t be positive integers. Let A be a commutative ring and $A[\mathbf{X}] = A[X_{ij}]_{1\leq i\leq s, 1\leq j\leq t}$ be a polynomial ring over A. Suppose that $s \leq t$ and $\mathbb{I}_s(\mathbf{X})$ is an ideal of $A[\mathbf{X}]$ generated by $s \times s$ minors of the $s \times t$ matrix $\mathbf{X} = (X_{ij})$. The following lemma is well-known.

Lemma 5.2.8. *With the above assumptions and notations, suppose that A is a Gorenstein ring which is a complete intersection in codimension one. Then $A[\mathbf{X}]$ is also a Gorenstein ring which is a complete intersection in codimension one.*

Theorem 5.2.9. *Suppose that $2s \leq t+1$ and A is a Gorenstein ring which is a complete intersection in codimension one. Then the Auslander-Reiten conjecture holds for the determinantal ring $A[\mathbf{X}]/\mathbb{I}_s(\mathbf{X})$.*

Proof. The case where $s = 1$ is trivial. Suppose that $s > 1$. Let $\mathfrak{N}$ be a maximal ideal of $A[\mathbf{X}]$ such that $\mathfrak{N} \supseteq \mathbb{I}_s(\mathbf{X})$. It is sufficient to show that the Auslander-Reiten conjecture holds for $(A[\mathbf{X}]/\mathbb{I}_s(\mathbf{X}))_{\mathfrak{N}}$. For integers $1 \leq p \leq s$ and $1 \leq q \leq t$, let

$$\mathfrak{M}_{pq} = (X_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q)$$

denote a monomial ideal of $A[\mathbf{X}]$. The case where $\mathfrak{M}_{st} \subseteq \mathfrak{N}$ follows from Proposition 5.2.6 and Lemma 5.2.8. Suppose that $\mathfrak{M}_{st} \not\subseteq \mathfrak{N}$ and take a variable X_{ij} so that $X_{ij} \notin \mathfrak{N}$. We may assume that $X_{ij} = X_{st}$. Then the matrix $\mathbf{X} = (X_{ij})$ is transformed to

$$\left(\begin{array}{ccc|c} & & & 0 \\ & X_{ij} - \frac{X_{it} \cdot X_{sj}}{X_{st}} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)$$

by elementary transformation in $A[\mathbf{X}]_{\mathfrak{N}}$. By [13, (2.4) Proposition], we have the isomorphism $\varphi : A[\mathbf{X}][X_{st}^{-1}] \to A[\mathbf{X}][X_{st}^{-1}]$ of A-algebras, where

$$\varphi(X_{ij}) = \begin{cases} X_{ij} - \frac{X_{it} \cdot X_{sj}}{X_{st}} & \text{if } X_{ij} \in \mathfrak{M}_{s-1\,t-1} \\ X_{ij} & \text{otherwise.} \end{cases}$$

Therefore we have the commutative diagram

$$\begin{array}{ccc} A[\mathbf{X}]_{\mathfrak{N}} & \xrightarrow{\varphi_{\mathfrak{N}}} & A[\mathbf{X}]_{\mathfrak{N}} \\ \cup\!\mid & \circlearrowleft & \cup\!\mid \\ \mathbb{I}_s(\mathbf{X})_{\mathfrak{N}} & \longrightarrow & \mathbb{I}_{s-1}(\mathbf{X}_{st})_{\mathfrak{N}}, \end{array}$$

where $\mathbf{X}_{st}$ is the $(s-1) \times (t-1)$ matrix that results from deleting the s-th row and the t-th column of $\mathbf{X}$ and $\mathbb{I}_{s-1}(\mathbf{X}_{st})$ is an ideal of $A[\mathbf{X}]$ generated by $(s-1) \times (s-1)$ minors of $\mathbf{X}_{st}$. Hence $(A[\mathbf{X}]/\mathbb{I}_s(\mathbf{X}))_{\mathfrak{N}} \cong (A[\mathbf{X}]/\mathbb{I}_{s-1}(\mathbf{X}_{st}))_{\mathfrak{N}}$ as rings. If $\mathfrak{M}_{s-1\,t-1} \subseteq \mathfrak{N}$, the Auslander-Reiten conjecture holds for $(A[\mathbf{X}]/\mathbb{I}_{s-1}(\mathbf{X}_{st}))_{\mathfrak{N}}$ by Proposition 5.2.6 and Lemma 5.2.8 since $2(s-1) \leq (t-1)+1$. If $\mathfrak{M}_{s-1\,t-1} \not\subseteq \mathfrak{N}$, repeat the above argument. Then, after finite steps, we finally see that the Auslander-Reiten conjecture holds for $(A[\mathbf{X}]/\mathbb{I}_{s-1}(\mathbf{X}_{st}))_{\mathfrak{N}}$. $\square$

5.3 Ulrich ideals whose residue rings are complete intersections

In this section, we study rings obtained in Corollary 5.2.5 in connection with the existence of ideals. Throughout this section, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d.

Lemma 5.3.1. *Let I be an $\mathfrak{m}$-primary ideal of R and $\mathfrak{q} = (x_1, x_2, \ldots, x_d)$ be a parameter ideal of R. Set $n = \mu_R(I)$. Suppose the following two conditions.*

(1) $\mathfrak{q} \subseteq I$ *and* $x_1, x_2, \ldots, x_d$ *is a part of minimal generators of* I.

(2) $I^2 \subseteq \mathfrak{q}$ *and* $I/\mathfrak{q}$ *is* R/I*-free.*

Then $\mathrm{r}(R) = (n-d){\cdot}\mathrm{r}(R/I)$. *In particular,* $n = d + \mathrm{r}(R)$ *if* R/I *is a Gorenstein ring.*

Proof. Since $I/\mathfrak{q} \cong (R/I)^{\oplus(n-d)}$, $I = \mathfrak{q} :_R I$. Hence $I/\mathfrak{q} = (\mathfrak{q} :_R I)/\mathfrak{q} \cong \mathrm{Hom}_R(R/I, R/\mathfrak{q})$. Therefore

$$\mathrm{Hom}_R(R/\mathfrak{m}, (R/I)^{\oplus(n-d)}) \cong \mathrm{Hom}_R(R/\mathfrak{m}, \mathrm{Hom}_R(R/I, R/\mathfrak{q})) \cong \mathrm{Hom}_R(R/\mathfrak{m} \otimes_R R/I, R/\mathfrak{q}),$$

whence we have the conclusion by comparing the lengths of them. □

For a moment, let $(R, \mathfrak{m})$ be an Artinian local ring. Then there are a regular local ring $(S, \mathfrak{n})$ and a surjective local ring homomorphism $\varphi : S \to R$. We can take S so that the dimension of S is equal to the embedding dimension of R. Set $v = \mathrm{v}(R) = \dim S$ and $r = \mathrm{r}(R)$. With these assumptions and notations, we have the following.

Proposition 5.3.2. *Let* $(R, \mathfrak{m})$ *be an Artinian local ring and* S *be as above. The following conditions are equivalent.*

(1) $r \leq v$ *and there exists a regular sequence* $X_1, X_2, \ldots, X_v \in \mathfrak{n}$ *on* S *such that*

$$R \cong S/\left[(X_1, X_2, \ldots, X_r)^2 + (X_{r+1}, X_{r+2}, \ldots, X_v)\right]$$

as rings.

(2) *There exists a nonzero ideal* I *of* R *such that*

   (i) $I^2 = 0$ *and* I *is* R/I*-free.*

   (ii) R/I *is a complete intersection.*

Proof. (2) ⇒ (1) Let $\overline{\varphi} : S \xrightarrow{\varphi} R \to R/I$ be a surjective local ring homomorphism. Set $\mathfrak{a} = \mathrm{Ker}\,\varphi$ and $J = \mathrm{Ker}\,\overline{\varphi}$. Since $R/I \cong S/J$ is a complete intersection, J is generated by a regular sequence $x_1, x_2, \ldots, x_v \in \mathfrak{n}$ on S, see [11, Theorem 2.3.3.(c)]. Hence, after renumbering of $x_1, x_2, \ldots, x_v$,

$$I = JR = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_v}) = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_r})$$

by Lemma 5.3.1, where $\overline{x}$ denotes the image of $x \in S$ to R. Thus $J = (x_1, x_2, \ldots, x_r) + \mathfrak{a}$. For all $r+1 \leq i \leq v$, take $y_i \in \mathfrak{a}$ and $c_{j_1}, c_{j_2}, \ldots, c_{j_r} \in R$ so that $x_i = y_i + \sum_{j=1}^{r} c_{j_i} x_j$. Then $J = (x_1, x_2, \ldots, x_r) + (y_{r+1}, y_{r+2}, \ldots, y_v)$. Set $X = (x_1, x_2, \ldots, x_r)$ and $Y = (y_{r+1}, y_{r+2}, \ldots, y_v)$, where Y denotes (0) if $r = v$. We then have inclusions

$$J^2 + Y \subseteq \mathfrak{a} \subseteq J,$$

where the first inclusion follows from $I^2 = 0$. On the other hand, setting $S' = S/Y$,

$$\ell_S(J/\left[J^2 + Y\right]) = \ell_{S'}(XS'/X^2S') = \ell_{S'}(S'/XS'){\cdot}r = \ell_R(R/I){\cdot}r = \ell_R(I) = \ell_S(J/\mathfrak{a}),$$

where the forth equality follows from the fact that I is an R/I-free module. Thus $\mathfrak{a} = J^2 + Y = (x_1, x_2, \ldots, x_r)^2 + (y_{r+1}, y_{r+2}, \ldots, y_v)$.

(1) ⇒ (2) Let $X = (X_1, X_2, \ldots, X_r)$ and $Y = (X_{r+1}, X_{r+2}, \ldots, X_v)$ be ideals of S. Set $I = XR$. Then $I^2 = 0$ and $I = [X + Y]/[X^2 + Y] \cong [S/(X+Y)]^{\oplus r} \cong (R/I)^{\oplus r}$. □

We are now back to the Setting that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring. Let us generalize Proposition 5.3.2 to arbitrary Cohen-Macaulay local rings.

Theorem 5.3.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring. Then the following conditions are equivalent.*

(1) *There exist a parameter ideal $\mathfrak{q}$ of R, a local complete intersection S of positive dimension, and a parameter ideal Q of S such that $R/\mathfrak{q} \cong S/Q^2$ as rings.*

(2) *There exist a parameter ideal $\mathfrak{q}$ of R, a local complete intersection S of dimension $\mathrm{r}(R)$, and a parameter ideal Q of S such that $R/\mathfrak{q} \cong S/Q^2$ as rings.*

(3) *There exist an $\mathfrak{m}$-primary ideal I and a parameter ideal $\mathfrak{q}$ of R such that*

 (i) *$I^2 \subseteq \mathfrak{q} \subsetneq I$ and $I/\mathfrak{q}$ is R/I-free.*

 (ii) *R/I is a complete intersection.*

Proof. (1) ⇒ (3) Set $\ell = \dim S$ and $Q = (x_1, x_2, \ldots, x_\ell)$. Then, since $R/\mathfrak{q} \cong S/Q^2$, we can choose $y_1, y_2, \ldots, y_\ell \in R$ so that $\overline{y_i}$ corresponds to $\overline{x_i}$ for all $1 \leq i \leq \ell$, where $\overline{x_i}$ denotes the image of x_i in S/Q^2 and $\overline{y_i}$ denotes the image of y_i in $R/\mathfrak{q}$. Set $I = (y_1, y_2, \ldots, y_\ell) + \mathfrak{q}$. Then $R/I \cong S/Q$ is a complete intersection and $I/\mathfrak{q} \cong Q/Q^2$ is R/I-free. Furthermore $I^2 = \mathfrak{q}I + (y_1, y_2, \ldots, y_\ell)^2 \subseteq \mathfrak{q}$.

(3) ⇒ (2) Because $I/\mathfrak{q}$ is an ideal of $R/\mathfrak{q}$ which satisfies the assumption of Proposition 5.3.2(2). □

Theorem 5.3.3 is applicable to Ulrich ideals. Here the definition of Ulrich ideals is stated as follows.

Definition 5.3.4. ([43, Definition 2.1.]) Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and I be an $\mathfrak{m}$-primary ideal of R. Assume that I contains a parameter ideal $\mathfrak{q}$ of R as a reduction. We say that I is an *Ulrich ideal* of R if the following conditions are satisfied.

(1) $I \neq \mathfrak{q}$, but $I^2 = \mathfrak{q}I$.

(2) I/I^2 is a free R/I-module.

Note that the condition (1) of Definition 5.3.4 is independent of the choice of $\mathfrak{q}$; see, for example, [51, Theorem 2.1.]. The following assertions say that the notion of Ulrich ideals is closely related to the condition (3)(i) of Theorem 5.3.3.

Proposition 5.3.5. [43, Lemma 2.3. and Proposition 2.3.]

(1) *If I is an Ulrich ideal of R, then $I^2 = \mathfrak{q}I \subseteq \mathfrak{q}$ and $I/\mathfrak{q}$ is R/I-free for every parameter ideal $\mathfrak{q}$ of R such that $\mathfrak{q}$ is a reduction of I.*

(2) *Assume that $R/\mathfrak{m}$ is infinite. If $I^2 \subseteq \mathfrak{q}$ and $I/\mathfrak{q}$ is R/I-free for all minimal reductions $\mathfrak{q}$ of I, then I is an Ulrich ideal of R.*

The following result recovers the result of J. Sally [71] by taking the maximal ideal $\mathfrak{m}$ as an Ulrich ideal. For convenience, set $d = \dim R$, $r = \mathrm{r}(R)$, and $v = \mathrm{v}(R)$.

Theorem 5.3.6. (cf. [71, Theorem 1.]) *Suppose that there are a regular local ring $(T, \mathfrak{n})$ of dimension v and a surjective local ring homomorphism $\varphi : T \to R$. If there exists an Ulrich ideal I of R such that R/I is a complete intersection, then $\mu_R(I) = d + r \leq v$ and there exists a regular sequence $x_1, x_2, \ldots, x_v$ on T such that*

(1) *$x_1, x_2, \ldots, x_d$ is a regular sequence on R.*

(2) $R/(x_1, x_2, \ldots, x_d)R \cong T/\left[(x_1, \ldots, x_d) + (x_{d+1}, \ldots, x_{d+r})^2 + (x_{d+r+1}, \ldots, x_v)\right]$.

Therefore, letting $0 \to F_{v-d} \to \cdots \to F_1 \to F_0 \to R \to 0$ be a minimal T-free resolution of R,

$$\mathrm{rank}_T F_0 = 1 \quad \textit{and} \quad \mathrm{rank}_T F_i = \sum_{j=0}^{v-r-d} \beta_{i-j} \cdot \binom{v-r-d}{j}$$

for $1 \leq i \leq v-d$, where $\beta_k = \begin{cases} 1 & \textit{if } k = 0 \\ k \cdot \binom{r+1}{k+1} & \textit{if } 1 \leq k \leq r \\ 0 & \textit{otherwise.} \end{cases}$

In particular, $\mathrm{rank}_T F_i = i \cdot \binom{r+1}{i+1}$ for $1 \leq i \leq r$ if $\mu_R(I) = v$.

Proof. Let $\overline{\varphi} : T \xrightarrow{\varphi} R \to R/I$ be a surjective local ring homomorphism. Set $\mathfrak{a} = \mathrm{Ker}\,\varphi$ and $J = \mathrm{Ker}\,\overline{\varphi}$. Since $R/I \cong T/J$ is a complete intersection, J is generated by a regular sequence $x_1, x_2, \ldots, x_v \in \mathfrak{n}$ on T. Hence, after renumbering of $x_1, x_2, \ldots, x_v$,

$$I = JR = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_v}) = (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_{d+r}})$$

by Lemma 5.3.1, where $\overline{x}$ denotes the image of $x \in T$ to R. Let $(\overline{x'_1}, \overline{x'_2}, \ldots, \overline{x'_d}) \subseteq I$ be a minimal reduction of I. Then, after renumbering of $x_1, x_2, \ldots, x_{d+r}$,

$$I = (\overline{x'_1}, \overline{x'_2}, \ldots, \overline{x'_d}, \overline{x_{d+1}}, \ldots, \overline{x_{d+r}}).$$

Thus $J = (x'_1, \ldots, x'_d) + (x_{d+1}, \ldots, x_{d+r}) + \mathfrak{a}$. Since $\mu_T(J) = v$, we can choose v elements in $\{x'_1, \ldots, x'_d, x_{d+1}, \ldots, x_{d+r}\} \cup \{a \mid a \in \mathfrak{a}\}$ as a minimal system of generators. Assume that x'_i cannot be chosen as a part of minimal system of generators. Then

$$I = (\overline{x'_1}, \ldots, \overline{x'_{i-1}}, \overline{x'_{i+1}}, \cdots, \overline{x'_d}, \overline{x_{d+1}}, \ldots, \overline{x_{d+r}}).$$

This is contradiction for $\mu_R(I) = r + d$ by Lemma 5.3.1. Hence

$$J = (x'_1, \ldots, x'_d) + (x_{d+1}, \ldots, x_{d+r}) + (y_{d+r+1}, \ldots, y_v)$$

for some $y_{d+r+1}, \ldots, y_v \in \mathfrak{a}$. Set $X_1 = (x'_1, \ldots, x'_d)$, $X_2 = (x_{d+1}, \ldots, x_{d+r})$, and $Y = (y_{d+r+1}, \ldots, y_v)$. Then $X_1 + X_2^2 + Y \subseteq \mathfrak{a} + X_1 \subseteq J$, whence $\mathfrak{a} + X_1 = X_1 + X_2^2 + Y$ since $\ell_T(J/[\mathfrak{a} + X_1]) = \ell_T(J/[X_1 + X_2^2 + Y]) = r \cdot \ell_T(T/J)$.

Let $0 \to F_{v-d} \to \cdots \to F_1 \to F_0 \to R \to 0$ be a minimal T-free resolution of R. Then

$$0 \to F_{v-d}/X_1F_{v-d} \to \cdots \to F_1/X_1F_1 \to F_0/X_1F_0 \to R/X_1R \to 0$$

is a minimal T/X_1-free resolution of R/X_1R and $R/X_1R \cong T/[X_1 + X_2^2 + Y]$ since $\mathfrak{a} + X_1 = X_1 + X_2^2 + Y$. On the other hand, the Eagon-Northcott complex [18] gives the minimal T/X_1-free resolution

$$0 \to G_r \xrightarrow{\partial_r} G_{r-1} \xrightarrow{\partial_{r-1}} \cdots \xrightarrow{\partial_2} G_1 \xrightarrow{\partial_1} G_0 \xrightarrow{\partial_0} T/[X_1 + X_2^2] \to 0$$

of $T/[X_1 + X_2^2]$, thus $\mathrm{rank}_{T/X_1} G_k = \beta_k$ for all $k \in \mathbb{Z}$. Therefore, as is well-known,

$$0 \to G_r \xrightarrow{\partial'_{r+1}} \begin{matrix} G_{r-1} \\ \oplus \\ G_r \end{matrix} \xrightarrow{\partial'_r} \begin{matrix} G_{r-2} \\ \oplus \\ G_{r-1} \end{matrix} \xrightarrow{\partial'_{r-1}} \cdots \xrightarrow{\partial'_2} \begin{matrix} G_0 \\ \oplus \\ G_1 \end{matrix} \xrightarrow{\partial'_1} G_0 \xrightarrow{\partial'_0} T/[X_1 + X_2^2 + (y_{d+r+1})] \to 0$$

becomes a minimal T/X_1-free resolution, where $\partial'_i = \begin{pmatrix} \partial_{i-1} & 0 \\ (-1)^{i-1} \cdot y_{d+r+1} & \partial_i \end{pmatrix}$. This show inductively $\mathrm{rank}_T F_i = \sum_{j=0}^{v-r-d} \beta_{i-j} \cdot \binom{v-r-d}{j}$ as desired. □

Remark 5.3.7. With the assumption of Theorem 5.3.6, the equality $\mu_R(I) = v$ does not necessarily hold in general; see Example 5.3.11 (1). On the other hand, if R is a one-dimensional Cohen-Macaulay local ring possessing maximal embedding dimension, every Ulrich ideal I satisfy that R/I is a complete intersection and $\mu_R(I) = v$; see [29, Theorem 4.5].

Combining Theorem 5.2.2, 5.3.3, and 5.3.6, we have the following which is a goal of this chapter.

Corollary 5.3.8. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d. Suppose that there exists an Ulrich ideal of R whose residue ring is a complete intersection. Then the following assertions are true.*

(1) *The Auslander-Reiten conjecture holds for R.*

(2) *$r + d \leq v$.*

(3) *There exist a parameter ideal $\mathfrak{q}$ of R, a local complete intersection S of dimension r, and a parameter ideal Q of S such that $R/\mathfrak{q} \cong S/Q^2$ as rings.*

(4) *Assume that there are a regular local ring T of dimension v and a surjective ring homomorphism $T \to R$. Let $0 \to F_{v-d} \to \cdots \to F_1 \to F_0 \to R \to 0$ be a minimal T-free resolution of R. Then*

$$\mathrm{rank}_T F_0 = 1 \quad \textit{and} \quad \mathrm{rank}_T F_i = \sum_{j=0}^{v-r-d} \beta_{i-j} \cdot \binom{v-r-d}{j}$$

for $1 \leq i \leq v - d$, where $\beta_k = \begin{cases} 1 & \textit{if } k = 0 \\ k \cdot \binom{r+1}{k+1} & \textit{if } 1 \leq k \leq r \\ 0 & \textit{otherwise.} \end{cases}$

Proof. (1) This follows from (3) and Corollary 5.2.5.

(2) Passing to the completion of R, we may assume that there exist a regular local ring T of dimension v and a surjective ring homomorphism $T \to R$. Then the assertion follows from Theorem 5.3.6.

(3) This follows from Theorem 5.3.3 and Proposition 5.3.5. □

Let us note that it is not necessarily unique for a given ring that an Ulrich ideal whose residue ring is a complete intersection.

Proposition 5.3.9. *Let $(S, \mathfrak{n})$ be a local complete intersection of dimension three and $f, g, h \in \mathfrak{n}$ be a regular sequence on S. Set*

$$R = S/(f^2 - gh, g^2 - hf, h^2 - fg).$$

Then R is a Cohen-Macaulay local ring of dimension one and $I = (f, g, h)R$ is an Ulrich ideal of R such that R/I is a complete intersection. Furthermore, if $f = f_1 {\cdot} f_2$ for $f_1, f_2 \in \mathfrak{n}$, then $I_1 = (f_1, g, h)R$ is also an Ulrich ideal of R such that R/I_1 is a complete intersection.

Proof. By direct calculation,

$$I^2 = fI,\ \ell_R(R/I) = \ell_S(S/(f, g, h)), \text{ and}$$
$$\ell_R(I/fR) = \ell_S((f, g, h)/[(f) + (g, h)^2]) = 2{\cdot}\ell_S(S/(f, g, h)).$$

Hence a surjection $(R/I)^{\oplus 2} \to I/fR$ must be an isomorphism, that is, I is an Ulrich ideal of R and $R/I \cong S/(f, g, h)$ is a complete intersection.

Assume that $f = f_1 {\cdot} f_2$. Then, similarly to the above,

$$I_1^2 = f_1 I_1,\ \ell_R(R/I_1) = \ell_S(S/(f_1, g, h)), \text{ and}$$
$$\ell_R(I_1/f_1R) = \ell_S((f_1, g, h)/[(f_1) + (g, h)^2]) = 2{\cdot}\ell_S(S/(f_1, g, h)).$$

Hence I_1 is an Ulrich ideal of R and $R/I_1 \cong S/(f_1, g, h)$ is a complete intersection. □

Here are some examples arising from Proposition 5.3.9.

Example 5.3.10. With the same notations of Proposition 5.3.9, let $S = k[[X, Y, Z]]$ be a formal power series ring over a field k. Let ℓ, m, n be positive integers such that $(\ell, m, n) \neq (0, 0, 0)$. Then we have the following examples.

(1) Take (f, g, h) so that (X^ℓ, Y^m, Z^n). Then

$$(X^i, Y^m, Z^n)R,\ (X^\ell, Y^j, Z^n)R,\ (X^\ell, Y^m, Z^k)R$$

are Ulrich ideals for all $0 \le i \le \ell$, $0 \le j \le m$, $0 \le k \le n$.

(2) Take (f, g, h) so that $(X^\ell {\cdot} Y^m {\cdot} Z^n, X^2 + Y^2, Y^2 + Z^2)$. Then

$$(X^i {\cdot} Y^j {\cdot} Z^k, X^2 + Y^2, Y^2 + Z^2)R$$

is an Ulrich ideal for all $0 \le i \le \ell$, $0 \le j \le m$, $0 \le k \le n$. Furthermore $(X^\ell{\cdot}Y^m{\cdot}Z^n, X+Y, Y^2+Z^2)R$ is also an Ulrich ideal if k is a field of characteristic two or an algebraically closed field.

We close this chapter and the dissertation with several examples.

Example 5.3.11. Let $k[[t]]$ and $S = k[[X,Y,Z,W]]$ be formal power series rings over a field k. The following assertions are true.

(1) Let $R_1 = k[[t^6, t^{11}, t^{16}, t^{26}]]$ and $I = (t^6, t^{16}, t^{26})$ be an ideal of R_1. Then (t^6) is a reduction of I, I is an Ulrich ideal of R_1, and R_1/I is a complete intersection. Therefore the minimal S-free resolution of R_1 has the following form

$$0 \to S^{\oplus 2} \to S^{\oplus 5} \to S^{\oplus 4} \to S \to R_1 \to 0.$$

On the other hand, $R_1 \cong S/\left(X^7 - ZW, Y^2 - XZ, Z^2 - XW, W^2 - X^6 Z\right)$ as rings, thus R_1 does not have the form obtained in Proposition 5.2.6.

(2) (cf. [59, Proposition 1.4.]) Set

$$R_2 = S/(X^2 - YZ, Y^2 - ZX, Z^2 - XY, W^2).$$

Then X is a non-zerodivisor of R_2 and $R_2/XR_2 \cong k[[Y,Z,W]]/\left[(Y,Z)^2 + (W^2)\right]$. Hence the Auslander-Reiten conjecture holds for R_2. On the other hand, $I = (X,W)R_2$ is an Ulrich ideal, whence I is a non-free totally reflexive R_2-module ([47, Theorem 2.8.]). Hence R_2 is not G-regular in the sense of [74]. In particular, R_2 is not a Golod ring.

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