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Article

# Searching for the 'Missing Equations' of the U.S.A., the U.K., and Japan: Quarterly Estimations, 1980–2015

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## 1. Introduction

In a paper of 1970, M. Friedman put forward the concept of the 'missing equation,' that determines the proportion of output changes and price changes that are due to nominal income changes, which occur over time-series development of national economies. Friedman considered, as a key element of the missing equation, the gap between actual nominal income changes and expected counterparts.

Since around the time of his paper's publication, at least several works were put forth which are concerned with the missing equations, such as Laidler (1995), Gordon (1974, 2009). One also should note Keynes' (1936, Chs.20 and 21) elasticities of output and prices in response to nominal income changes (*eo* and *ep*), which are closely related to the missing equation. It is interesting to note Friedman's characterization of extreme (caricaturized) quantity theory where $eo = 0$ and $ep = 1$ in responses to external disturbances to the economy (particularly to changes in money supply); and caricaturized Keynesian theory where $eo = 1$ and $ep = 0$. In summary our search for missing equation is to look for the equations in intermediate situations between extreme forms of quantity theory and Keynesian theory.

I derived the missing equation (Amano 2012), starting with firms' optimal price and output decisions in dynamic contexts, and estimated the equations for the U.S.A., the U.K., Japan, some other OECD countries, and developing economies, using annual data.

In this paper I will estimate the equations for the three countries, using quarterly data, for the period 1980 through 2015, and compare among the three country results, and between annual estimation and quarterly estimation.

The next section describes representative firm's pricing and output decisions, and derives the estimable forms of missing equations, making clear the proportions of output and price changes following changes in nominal income.

Section 3 estimates the equations using quarlterly data of the three countries, the U.S.A., the U.K., and Japan. Section 4 concludes.

## 2. Firm Behavior and Price and Output Changes

The economy's nominal income ($Y$) is given as a product of output ($Q$) and the price level ($P$): $Y = Q \cdot P$. Writing the growth rate of $Y$, $Q$, $P$ as $gY$, $gQ$, $gP$, respectively, we have

$$gY = gQ + gP.$$

If we write the elasticity of output regarding nominal income as $eo$, and the elasticity of the price regarding nominal income as $ep$, they can be written as

$$eo = \frac{dQ}{dY}\frac{Y}{Q} = \frac{dQ/Q}{dY/Y},$$

where $d$ is a differential operator. If we approximate $dY = Y_t - Y_{t-1}$, etc., where the subscripts refer quarterly periods, we have

$$eo=\frac{gQ}{gQ+gP},\quad ep=\frac{gP}{gQ+gP},\quad eo+ep=1. \qquad (1)$$

*eo*'s of the three countries are shown in Figs. 1 and 2, where only Japan's *eo* is a five-period moving average, because its original *eo* exhibits erratic behavior from around 2000 through 2005. This is presumably because around 2000 onward, Japan's nominal income is declining.

(1) How the rate of inflation is generated

We consider the rate of inflation of finite magnitude because price changes entail costs to the firm, and the costs are the larger, the faster the firm adjusts the price. In line with Calvo (1983), Rotemberg (1996), and Gali and Gertler (1999), we start with the firm's costs involved in its pricing policy:

**Fig. 1　U.S. and U.K.'s *eo***

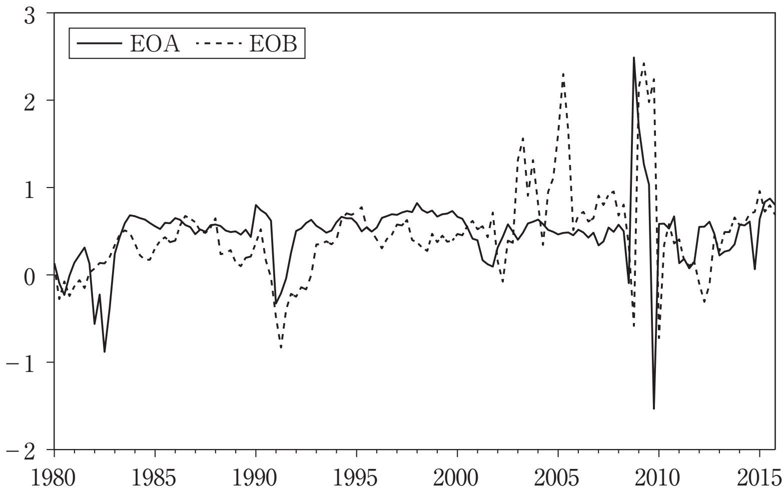
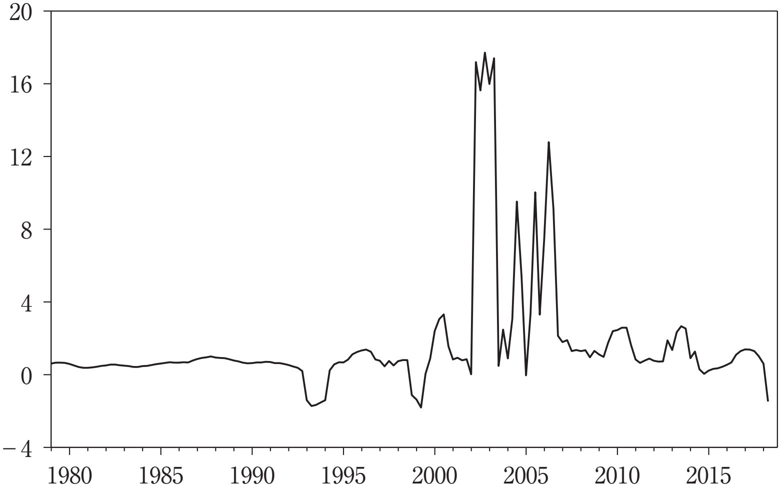


**Fig. 2 Japan's *eo***




$$C_t = E_t \sum_{s=t}^{\infty} \sigma^{s-t}[(p_s - p_s^*)^2 + k(p_s - p_{s-1})^2].$$

where $p_s = \ln(P_s)$, and the first squared term is the out-of-equilibrium cost born by the firm, and $p_s^*$ is the equilibrium price at period $s$; see equation (3) for the determinants of $p_s^*$. The second squared term is the price adjustment cost from period $s$-1 to $s$. $E_t$ is an expectation operator at period $t$ for random variables dated $t+j$ ($j \geq 1$). A constant $k$ measures a relative weight between the price adjustment cost and the out-of-equilibrium cost. A constant δ is a discount factor taking a value between zero and unity but close to unity.

Since $p_t^*$ does not depend on $p_t$ (see Eq.(3)) the firm's optimization can be achieved when

$$\frac{\partial C_t}{\partial p_t} = 0 = 2E_t[p_t - p_t^* + k(p_t - p_{t-1}) - \delta k(p_{t+1} - p_t)].$$

The second order condition is easily seen to be met. Noting that the discount factor δ is close to unity, and writing $gP_t = p_t - p_{t-1}$, hence $E_t gP_{t+1} = E_t(p_{t+1} - p_t)$, where 'g' stands for 'the growth rate of,' the above relation (second equality) can be approximated by

$$p_t - p_t^* + k(gp_t - E_t gp_{t+1}) = 0 \quad (2)$$

We here suppose that the ratio of the firm's optimal price at period $t$ to the competitor's $p_t^c$ is equal to the log of firm's marginal cost to its trend level, $mc_t$ (see Gali and Gertler 1999 and Gali 2008 for similar assumptions). Then one can write this relationship as

$$p_t^* = mc_t + pc_t + e_t, \quad (3)$$

where $e_t$ is an i.i.d. (independent and identically distributed) error term with mean zero and constant variance (that is white noise; this characterization is applied to other error terms in the following). Also, $mc_t$ can be written as

$$mc_t = x_t + h(Q_t) \quad (4)$$

where $x_t$ is a log of the nominal wage level relative to its trend, and $h(Q_t)$, a function of output $Q_t$, is a log of a reciprocal of marginal productivity of labor relative to its trend level. Also, it can be supposed that $h'(Q_t)$ will increase in the short-run when some components of aggregate demand increase, because if they happen, they will lead to temporary increases in output, which will result in increases in the reciprocal of marginal productivity of labor (on the assumption of decreasing marginal productivity of labor in the short-run; i.e., $h'(Q_t) > 0$).

Now, substituting (3) and (4) into (2) and solving for $gP_t$, we obtain

$$gP_t = E_t gP_{t+1} + \frac{1}{k}(P_t^* - P_t) = E_t gP_{t+1} + \frac{1}{k}[x_t + h(Q_t) + p_t^c + e_t - p_t].$$

When the above relationship is applied to the macro-economy, it will be natural to suppose that the firm's actual price can be approximated to the competitors' prices, i.e. $p_t = p^c_t$. Then the above equation becomes

$$gP_t = E_t gP_{t+1} + \frac{1}{k}[x_t + h(Q_t) + e_t],$$

where $k$ is a positive constant. Changing the notation slightly such that $x_t/k = w_t$, $h(Q_t)/k = l_t$, and $e_t/k = u_t$, the above equation becomes

$$gP_t = E_t gP_{t+1} + w_t + l_t + u_t \qquad (5)$$

where $l_t$ is a function of some aggregate demand components: their higher values will raise the price at least in the short-run. In other contexts where I dealt with OECD countries using annual data, I used the rate of change of unemployment rates as determinants of wages relative to their trend levels. In this paper I use the business cycle components of Hodrick-Prescott filter (Hodrick and Prescott 1989), where business cycle components are the difference between actual logged output and the trend factor of output, as well as the unemployment rate, as determinants of $w_t$ and $l_t$. In other words, both the wage levels (relative to the trend) and aggregate demand movements are represented by the rate of change of unemployment rates and business cycle components in the Hodrick-Prescott filter for logged output (see the following discussion for not using the level of the unemployment).

Denoting the cyclical components of Hodrick-Prescott filter in $lq_t$ by $qhp_t$, and attaching coefficients on independent variables, we can rewrite equation (5) as

$$gP_t = a_0 + a_1 E_t gP_{t+1} + a_2 qhp_s + gur_s + v_t, \qquad (6)$$

where subscript $s$ refers either $t$ (the current year) or $t-j$ ($j>1$)

(some past year). $v_t$ is also a zero mean *i.i.d.* error term.

(2) The determinants of short-run output growth

We next turn to what determine short-run output growth as a function of past and current macroeconomic variables. Since we are concerned with short-run determinants, they mainly consist of demand factors, as well as a supply factor.

As demand factors, we consider (a) current account surpluses in ratio form, exports/imports, written as *em*, (b) the central government's budget deficits, also in ratio form, expenditure/revenue, written as *er*, (c) broader money supply *m*2, consisting of cash in circulation and demand as well as time deposits. It is now well recognized that broader money supply has a closer relationship with real economic activity, (d) the business cycle component of the Hodrick-Prescott filter in logged output, where the business cycle components is actual logged output minus the trend factor. This component is generally regarded as demand pressure in output markets, and also may represent the speed of technical progress, and (e) the rate of unemployment which works in a negative direction on *eo*.

When I looked for the missing equation for the U.S.A., the U.K., and Japan using annual data, I represented the speed of technical progress as the growth rate of total factor productivity. But in a quarterly context we are currently facing, compiling capital stock series was quite a difficult task, so I included the technical progress in the form of business cycle components of the Hodrick-Prescott filter.

The levels of the above five factors will affect the output level by way of its demand and supply sides, hence it would be their growth rates that affect the growth of output, *gQ*; note that in current quarterly estimation, growth and rates of change are measured relative to the value of the four period past, that is, if $z$ is some dynamic variable, $gz(t) = (z(t) - z(t-4))/z(t-4)$. Measuring rates of change by $(z(t) - z(t-1))/z(t-1)$ gives less clear-cut changing patterns in $z(t)$. Then output growth will be represented by

$$gQ = c + b_1 geis + b_2 gers + b_3 gm2s + b_4 qhps + b_5 gur_s + v_t \tag{7}$$

where $g$ represents as before 'the growth rate of.' The subscript $s$ refers to either $t$ (the current year) or $t-j$ ($j>1$).

Substituting the determinants of $gP$ and $gQ$, (6) and (7), into the definition of $eo$ (1) and then linearizing around zero values of independent variables, we find an equation for $eo$ as a function of explanatory variables, which is a concrete form of the missing equation, and which is to be estimated in the next section,

$$eo_t = c_0 + c_1 gei_s + c_2 gem_s + c_3 gm2_s + c_4 qhp_s + c_5 gur_s + c_6 E_t gP_{t+1} + \varepsilon_t \tag{8}$$

where $c_k$ ($k=0$ through 6) are the parameters to be estimated.

Before moving on to estimating $eo$'s of the three countries, we will discuss expectation formation on inflation $E_t gP_{t+1}$. The rational expectations hypothesis implies

$$gP_{t+1} = E_t gP_{t+1} + \lambda_{t+1}, \tag{9}$$

where $gP_{t+1}$ is the actual inflation rate, $\lambda_{t+1}$ is an i.i.d. error term with mean zero, and the two terms on the right-hand side are not correlated. See, e.g., Maddala (2001, Ch.10). Substituting $E_t gP_{t+1}$ from (9) into (8) yields

$$eo_t = f(s \leq t) + c_6 (gP_{t+1} - \lambda_{t+1}) + \varepsilon_t \tag{10}$$

where $f(s \leq t)$ is the first six terms on the right-hand side of (8). If the OLS is applied to the above, the resulting estimate is not consistent because $gP_{t+1}$ and $\lambda_{t+1}$ are correlated because of (9). Hence we need to estimate $gP_{t+1}$ in (10) by the instrumental variable method. Actually, however, since people cannot know $gP_{t+1}$ in period $t$, we regress $gP_t$ on past four $gP_t$'s, and shift the estimate one period ahead to regard it as $E_t gP_{t+1}$. The estimated expected inflation rate in $t+1$, $E_t gP_{t+1}$, was then used in (10) in place of $gP_{t+1}$ , and we next estimated (10). In the next section $E_t gP_{t+1}$ is written as *gpe* (1) for notational convenience.

## 3. Estimating *eo*'s for the Three Countries

We first deal with the U.S. case. The U.S.'s *eo*, *eoa*, is estimated as follows:

| *eoa* | *geia*(−2) | *gera*(−2) | *gm2a*(−2) | *lqcya* | *gpea*(1) | *gura*(−2) | *ua* |
|---|---|---|---|---|---|---|---|
| *coeff.* | −0.346 | 0.862 | −0.381 | 9.214 | −8.578 | −0.641 | 1.729 |
| *p* | 0.346 | 0.008 | 0.689 | 0.000 | 0.000 | 0.004 | 0.000 |

*smpl* = 1981: 3-2015: 1 (138), $r^2$ = 0.512, *ser* = 0.268, *aver* = 0.495,
*DW* = 1.795, *SCLM* (lag = 4) = 0.064.

The last letter *a* in each variable implies it belongs to the U.S.A., and the first *g* is a growth rate operator, which is a rate of change as compared with the value of four period past. A constant is present in the estimation but is not shown in the above table. Number 138 is an effective sample number. $r^2$ is a coefficient of determination adjusted for the degree of freedom, and *p* implies the *p*-value and marginal significance level, showing the probability of an error if one abandons the hypothesis that the coefficient is zero. *ser* is a standard error of regression, *aver* is the sample average of *eoa*, *DW* is the Durbin-Watson ratio, and *SCLM* is a *p*-value for the Breush-Godfrey serial correlation *LM* test, with the null of no serial correlation. The *DW* ratio for sample 138 is around 1. 80, which is the maximum number of the region for which the ratio is indeterminate as for the existence of serial correlation; note that the *DW* ratio can be applied only for serial correlation of two neighboring variables; but *SCLM* is concerned with more remote components (the lag number is four), and in the current case, the estimate involves no serial correlation for 5% significance level.

*geia* is the growth rate of export/import ratio, *gera* is the growth rate of government expenditure/revenue ratio, *gm2a* is the growth rate of money supply *m2*, *lqcya* is the cyclical component of the Hodrick-Prescott filter, where the cyclical component = *lqa*-smoothed series (trend component) after the filtering, with a smoothing parameter 1600. *gpea* (1) is the expected inflation rate one period forward. *gura* is the growth rate of U.S. unemployment rate, and, finally, *ua* is a dummy variable taking on 1 in periods 2008: 4 and 2009: 1, and 0 in all other periods. In the $gP_t$ equation (6) I suggested that it may be the level *ura*, not the growth rate, of unemployment rates, but the level did not turn out significant.

The minus numbers attached to explanatory variables are lag length. Lag lengths are generally hard to explain based on the first (rationality) principle, for they involve adjustments in variables and habit formation. Here I supposed the lengths that provide the better fit. The four coefficients except the one of the dummy exhibit the anticipated signs but I will postpone further discussions until I compare the three country results.

The corresponding U.K. results are shown as follows:

| *eob* | *geib*(−2) | *gerb*(−2) | *gm2b*(−2) | *lqcyb*(−2) | *gpeb*(1) | *gurb*(−2) | *ub* |
|---|---|---|---|---|---|---|---|
| *coeff.* | −0. 777 | 0. 381 | 1. 944 | −0. 424 | −18. 024 | −0. 278 | 1. 880 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| *p* | 0.288 | 0.772 | 0.000 | 0.052 | 0.000 | 0.033 | 0.000 |

*smpl*: 1981: 4-2015: 3 (136), $r^2 = 0.794$, *ser* = 0.234, *aver* = 0.493.

The estimation used the limited information maximum likelihood method, because the *OLS* is accompanied by a very low *DW* ratio. The last letters *b* is a reminder that the variables are British ones.

We now turn to Japan, which yielded the following results:

| *eoj* | *geij*(−1) | *gerj*(−1) | *gm2j*(−1) | *lqcyj*(−4) | *gpej*(1) | *gurj*(−2) | *uj* |
|---|---|---|---|---|---|---|---|
| *coeff.* | −2.329 | −0.686 | 0.253 | −15.341 | −29.697 | −2.554 | 12.888 |
| *p* | 0.040 | 0.373 | 0.951 | 0.066 | 0.001 | 0.035 | 0.000 |

*smpl*: 1980: 3-2015: 4 (142), $r^2 = 0.843$, *ser* = 1.381, *aver* = 1.741, *DW* = 1.690.

The estimation was conducted using the *OLS*. In the Japanese case, the original *eoj* exhibited abnormal values probably because *lyj* (logged nominal income) was declining during the period around 2000 through 2010. Hence I used in the estimation the 5-period moving average series of original *eoj*, and call the moving averaged series *eoj* (I made another series in which the period 1991: 1 through 2004: 4 (or 2006: 3) was moving averaged four periods, and other parts are original *eoj*, but this partially moving averaged series could not yield significant or efficient estimates). The *DW* ratio is not very low but it belongs to an indeterminate region, so that this statistics cannot eliminate the serial correlation.

Having finished the individual estimation of the three countries, we can now compare those results.

(1) The effect of the growth of export/import ratio on *eo* is estimated significantly only in Japan. An increase in the ratio lowers Japan's *eo*, meaning that it raised *ep* significantly ($eo + ep = 1$); in other words, in

Japan, output pressure in a form of more exports over imports raised the nominal income elasticity of prices and lowers the elasticity of output.

(2) A higher growth in the government expenditure/revenue ratio raises the elasticity of output in the U.S.A. only, and it reduces the price elasticity of nominal income in the U.S.

(3) An increasing growth in broader money supply raises *eo* only in the U.K. and lowers its *ep*. This increase did not affect *eo* nor *ep* in the other countries.

(4) Higher business cycle components in the Hodrick-Prescott filter raised U.S.'s *eo* but lowered *eo*'s in other two countries. More vibrant economic activity in the form of larger business cycle components was reflected in larger output responses in the U.S., but in larger price responses in the U.K. and Japan.

(5) In the three countries uniformly, higher inflation expectation one period ahead lower *eo* and raises *ep*. In other words, higher inflation expectations were absorbed in price increases rather than output rises.

(6) Finally, in all the three countries, higher growth in unemployment rates lowered *eo*, and raised *ep*; in the face of higher unemployment growth, higher nominal income induced smaller output responses and larger price responses.

## 4. Conclusions

In my previous book (Amano 2012) I estimated the *eo*'s and *ep*'s of other OECD countries and some developing countries, in addition to the three countries we have focused upon in this paper, using annual data. Quarterly exploration with larger number of countries necessarily involves much more difficulties in data collection and processing.

In the current work, the Japanese *eo* and *ep* exhibited quite erratic behavior around 2000 and after, which probably was much to do with the fact that during this period, nominal income and the price level had been declining. Hence quarterly estimation and comparison of some few countries, restricting the periods prior to around 2000, i.e. when main macro variables exhibit more normal time patterns, will be of some interest and worthwhile.

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# Summary

## Searching for the 'Missing Equations' of the U.S.A., the U.K., and Japan: Quarterly Estimations, 1980–2015

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This paper sets up a relationship representing an elasticity of output and the price level regarding nominal income. The concept dates back at least to Keynes (1936), and was taken up by Friedman (1970) and Laidler (1995). The name 'missing equation' comes from the fact that it makes clear the proportion of output changes due to changes in nominal income. We estimatedel this proportion for periods around 1980 through 2015 using quarterly data of the U.S.A., the U.K., and Japan. Some comparisons are made among the estimated results for those three countries.