THE CHARACTERIZATIONS OF A SEMICIRCLE LAW BY THE CERTAIN FREENESS IN A C^*- PROBABILITY SPACE

Osamu Hiwatashi*, Masaru Nagisa* and Hiroaki Yoshida†

* Department of Mathematics and Informatics
Faculty of Science
Chiba University

1-33, Yayoi-Cho, Inage-Ku, Chiba 263 Japan

† Department of Information Sciences
Faculty of Science
Ochanomizu University

2-1-1, Otsuka, Bunkyou, Tokyo 112 Japan

Abstract. In usual probability theory, various characterizations of the Gaussian law have been obtained. For instance, independence of the sample mean and the sample variance of independently identically distributed random variables characterizes the Gaussian law and the property of remaining independent under rotations characterizes the Gaussian random variables. In this paper, we consider the free anologue of such a kind of characterizations replacing independence by freeness. We show that freeness of the certain pair of the linear form and the quadratic form in freely identically distributed noncommutative random variables, which covers the case for the sample mean and the sample variance, characterizes the semicircle law. Moreover we give the alternative proof for Nica's result that the property of remaining free under rotations characterizes a semicircular system. Our proof is more direct and straightforward one.

0 Introduction

In [Vo1], D. Voiculescu began studying operator algebra free products from the probabilistic point of view. The idea is to look at free products as an analogue of tensor products and to develop a corresponding noncommutative probabilitic framework where freeness is given a treatment similar to independence. The analogy between freeness and independence is that around freeness, several concepts can be developed similar to those around independence, free random variables, the central limit theorem for free random variables, the addition of free random variables, process with free increments etc..

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Especially the central limit theorem for free random variables holds with limit distribution equal to a semicircle law. And semicircle laws play in many respects the role of the Gaussian laws, when independence is replaced by freeness in a noncommutative probability space, consequently, semicircle laws are in some sense the free analogue of the Gaussian ones. The explanations for occurrence of the semicircle law, both in the free central limit theorem and Wigner's works (see [Wi]) on asymptotics of large random matrices, are found in [Vo3].

In usual probability theory, various characterizations of the Gaussian laws have been obtained, for instance see [KLR]. Thus it is natural to consider the free analogue of such characterization problems of the semicircle law in noncommutative case too. In [BV], H. Bercovici and D. Voiculescu, however, showed that the free analogue of the Cramér theorem on the decomposition of a semicircle law is failure. Nevertheless, A. Nica showed that the property of remaining free under rotations characterizes a semicircular system in [Ni], which is the free analogue of the well-known fact in usual probability theory, that the property of remaining independent under rotations characterizes the Gaussian random variables, see [Fe], Section III.4.

In this paper, we establish some characterizations of the semicircle law by the freeness of noncommutative random variables replacing independence by freeness. We have three sections in this paper. Section 1 is devoted to preliminary materials for noncommutative probability spaces and free random variables.

In section 2, we consider the characterization of the semicircle law by freeness of the certain pair of the linear from and the quadratic form in freely identically distributed noncommutative random variables, which covers the free analogue of the well-known fact in usual probability theory that the independence of the sample mean and the sample variance of independently identically distributed random variables characterizes the Gaussian laws [KS]. We show the semicircleness of the distribution by calculating the higher moments with induction because our distribution has a compact support.

In section 3, we give the alternative proof for the above Nica's work. He showed this characterization as an application of the multidimensional $R-$transform introduced in his paper [Ni], that is the multivariate version of Voiculescu's $R-$transform [Vo2] for a noncommutative random variable, the free analogue of the cumulants generating series. However, we show it by using the same technique that we use in section 2 and our poof is rather direct and straightforward one.

1 Noncommutative probability spaces and a semicircular system

This section contains preliminaries concerning with noncommutative probability spaces and free random variables. Recall that a usual probability space is (Ω, Σ, μ), where Ω is a base space, Σ is a $\sigma-$ algebra and μ is a probability measure (i.e. positive and satisfying $\mu(\Omega) = 1$). A random variable is a mesurable function $f : \Omega \to \mathbb{C}$, and if f is integrable then its expectation $E(f)$ is given by

$$E(f) = \int_{\omega \in \Omega} f d\mu(\omega). \tag{1.1}$$

We can consider a noncommutative probability space in a purely algebraic frame as an analogue of the above usual probability space.

Definition 1.1. A noncommutative probability space is $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital algebra and $\phi : \mathcal{A} \to \mathbb{C}$ is a linear functional with $\phi(1) = 1$. We say that $(\mathcal{A}, \phi)$ is a C^*- probability space when, in addition, $\mathcal{A}$ is a C^*- algebra and ϕ is a state, and it is a W^*- probability space when $\mathcal{A}$ is a von Neumann algebra and ϕ is a normal state.

One can define independence in a noncommutative probability space, generalizing the usual definition. Note that independence of usual random variables was actually express in terms of the subalgebra of $L^\infty(\Omega)$, so it makes sense to speak of independence of subalgebras in a noncommutative context.

Definition 1.2. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space, and $\mathcal{A}_i \subset \mathcal{A}$ be subalgebra $(i \in I)$, for the index set I. We say that the family $(\mathcal{A}_i)_{i \in I}$ is *independent* if

(i) for every $i_1, i_2 \in I$ $(i_1 \neq i_2)$,

$$x_1 x_2 = x_2 x_1 \quad \text{for } x_1 \in \mathcal{A}_{i_1} \text{ and } x_2 \in \mathcal{A}_{i_2},$$

(ii)

$$\phi(x_1 x_2 \cdots x_n) = \phi(x_1)\phi(x_2) \cdots \phi(x_n)$$

whenever $x_j \in \mathcal{A}_{i_j}$ and $i_1, i_2, \cdots, i_n$ are distinct.

Note that if $1 \in \mathcal{A}_i$ (ii) is equivalent to (ii)$'$.

(ii)$'$

$$\phi(x_1 x_2 \cdots x_n) = 0$$

whenever $x_j \in \mathcal{A}_{i_j}$ and $i_1, i_2, \cdots, i_n$ are distinct and $\phi(x_j) = 0$ for all j.

This form of independence is based on the tensor product of algebras. We will replace this concept by a more noncommutative one which takes instead of the tensor product the reduced free product. This concept is due to D. Vioculescu and explained below.

Definition 1.3. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space and $\mathcal{A}_i$ be subalgebra of $\mathcal{A}$ containing the identity element of $\mathcal{A}$, $1 \in \mathcal{A}_i \subset \mathcal{A}$, for $i \in I$. We say that the family $(\mathcal{A}_i)_{i\in I}$ is *free* if

$$\phi(x_1 x_2 \cdots x_n) = 0$$

whenever $x_j \in \mathcal{A}_{i_j}$ and $i_1 \neq i_2 \neq \cdots \neq i_n$ and $\phi(x_j) = 0$ for all j.

A family of subsets $X_i \subset \mathcal{A}$ (resp. elements $x_i \in \mathcal{A}$) will be called *free* if the family of subalgebras $\mathcal{A}_i$ generated by $\{1\} \cup X_i$ (resp. $\{1, x_i\}$) is free.

Freeness is related to independence only by analogy, since independence is about mutually commuting noncommutative subalgebras and freeness is highly noncommutative. The definition of freeness allows us to calculate all moments, the expectation of a product element. This calculation is done according the following recursive procedure.

Let $i_1 \neq i_2 \neq \cdots \neq i_n$ and $x_j \in \mathcal{A}_{i_j}$ be given, where the family of subalgebras $(\mathcal{A}_i)$ is free. We put

$$x_j^\circ = x_j - \phi(x_j)1 \tag{1.2}$$

then $x_j^\circ \in \mathcal{A}_{i_j}$ and $\phi(x_j^\circ) = 0$. It follows that

$$\begin{aligned} \phi(x_1 x_2 \cdots x_n) &= \phi\big((x_1^\circ + \phi(x_1)1)(x_2^\circ + \phi(x_2)1) \cdots (x_n^\circ + \phi(x_n)1)\big) \\ &= \sum_{\pi,\sigma} \phi(x_{\pi(1)})\phi(x_{\pi(2)}) \cdots \phi(x_{\pi(m)})\phi(x_{\sigma(1)}^\circ x_{\sigma(2)}^\circ \cdots x_{\sigma(n-m)}^\circ), \end{aligned} \tag{1.3}$$

where the sum runs over all partitions of $\{1, 2, \cdots, n\}$ into two ordered sets, $\{\pi(1), \pi(2), \cdots, \pi(m)\}$ and $\{\sigma(1), \sigma(2), \cdots, \sigma(n-m)\}$ with $m \geq 1$. From the definition, the term for $m = 0$ vanishes. After combining neighboring elements of the same algebra, the terms $x_{\sigma(1)}^\circ x_{\sigma(2)}^\circ \cdots x_{\sigma(n-m)}^\circ$ can be written as a product $y_1 y_2 \cdots y_{n'}$ with $y_j \in \mathcal{A}_{l_j}$ and $l_1 \neq l_2 \neq \cdots \neq l_{n'}$. Now we can repeat this procedure and because of $n' < n$, this procedure will be stopped after finitely many steps.

In general, the calculation of a moment becomes much more combinatorially complicated for longer one. We will, however, have the next useful property of the calculation procedure which will be sometimes used in this paper.

Lemma 1.4. *Let $(\mathcal{A}_i)_{i\in I}$ be a free family of subalgebras in a noncommutative probability space $(\mathcal{A}, \phi)$. Consider a product $x_1x_2\cdots x_n$ with $x_j \in \mathcal{A}_{i_j}$ for $j = 1, 2, \cdots, n$. If there is one algebra which appears only once, i.e. these exists the number k with $i_k \neq i_j$ for all $k \neq j$, then we have*

$$\phi(x_1x_2\cdots x_n) = \phi(x_k)\phi(x_1\cdots x_{k-1}x_{k+1}\cdots x_n). \tag{1.4}$$

This lemma follows from the definition of freeness. Of course, if $x_1\cdots x_{k-1}x_{k+1}\cdots x_n$ fulfils again the requirements of this lemma, we may repeart this procedure.

We call an expectation of the form $\phi(x^n)$ the simple moment of x of order n.

Definition 1.5. Let $(\mathcal{A}, \phi)$ be a noncommutative probability space. A *random variable* is an element $x \in \mathcal{A}$. The *distribution* of x is the linear functional μ_x on $\mathbb{C}[X]$ (the algebra of complex polynomials in the variable X), defined by

$$\mu_x(P(X)) = \phi(P(x)), \qquad \text{for all } P \in \mathbb{C}[X]. \tag{1.5}$$

Note that the distribution of a random variable $x \in \mathcal{A}$ is nothing more than a way of discribing the simple moments.

Remark 1.6. In a C^*- probability space $(\mathcal{A}, \phi)$, if x is a self-adjoint element of $\mathcal{A}$ then the distribution of x, μ_x, extends to a compactly supported measure on $\mathbb{R}$, namely there exists a unique measure, $d\mu_x$ on $\mathbb{R}$ such that

$$\int P(t)d\mu_x(t) = \phi(P(x)). \tag{1.6}$$

An important distribution in a noncommutative probability space is the semicircle law, which plays in free probability theory a role analogous to one of Gaussian law in usual probability theory.

Definition 1.7. The *semicircle law* centered at $m \in \mathbb{R}$ and of radius $r > 0$ is distribution $\gamma_{m,r} : \mathbb{C}[X] \to \mathbb{C}$ definied by

$$\gamma_{m,r}(P(X)) = \frac{2}{\pi r^2}\int_{m-r}^{m+r} P(t)\sqrt{r^2 - (t-m)^2}dt. \tag{1.7}$$

Suppose $(\mathcal{A}, \phi)$ is a noncommutative probability space, an element $x \in \mathcal{A}$ will be called *semicircular* centered at $m \in \mathbb{R}$ and of radius $r > 0$ if its distribution μ_x is $\gamma_{m,r}$.

It is known that the semicircle law play in many respects the role of the Gaussian one, when independence is replaced by freeness. And similary to the Gaussian case, the semicircle law is determined by its moments of orders one and two; $\gamma_{m,r}(X) = m$ and $\gamma_{m,r}(X^2) = m^2 + \frac{r^2}{4}$.

Remark 1.8. Concerning with the moments of a centered ($m = 0$) semicircular element x of radius r, it is easily done via integration by parts and induction that

$$\phi(x^k) = \int_{-r}^{r} t^k \sqrt{r^2 - t^2}\,dt \tag{1.8}$$
$$= \begin{cases} 0 & \text{if } k = 2m+1, \\ \dfrac{(2m)!}{m!(m+1)!}\left(\dfrac{r^2}{4}\right)^m & \text{if } k = 2m, \quad \text{for } m \geq 0. \end{cases}$$

The number $C_m = \frac{(2m)!}{m!(m+1)!}$ is known as the Catalan number, which will appear in various fields of combinatorics. Moreover, the second moment of the centered semicircular element of radius r is given by $\frac{r^2}{4}$ so the relations (1.8) can be read as

$$\begin{cases} \phi(x^{2m+1}) = 0, & \\ \phi(x^{2m}) = C_m \phi(x^2)^m, & \text{for } m \geq 0. \end{cases} \tag{1.9}$$

As we noted before, the distribution of a self-adjoint element in a C^*- probability space extends to a compactly supported probability measure on $\mathbb{R}$ and if it is absolutely continuous with respect to the Lebesgue measure and has the continuous probability density function, then it can be completely determined by its moments. Hence we can assert that, for a self-adjoint element x in a C^*- probability space $(\mathcal{A}, \phi)$, x is a centered semicircular element if and only if the moments of x satisfy the relations (1.9).

Definition 1.9. In a C^*- probability space $(\mathcal{A}, \phi)$, a family $(x_i)_{i=1,2,\cdots,n}$ of random variables is called *freely identically distributed* if it is a free family of the self-adjoint elements and each x_i has the same distribution. We call a family $(x_i)_{i=1,2,\cdots,n}$ of freely identically distributed random variables *semicircular system* if the identical distribution of each x_i is given by the centered semicircle law $\gamma_{0,r}$ for some $r > 0$.

Speaking in the terms of its moments, they satisfy the following relations:

$$\begin{cases} \phi(x_i^{2m+1}) = 0, & \\ \phi(x_i^{2m}) = C_m \phi(x_i^2)^m & \text{for } 1 \leq i \leq n \text{ and } m \geq 0, \\ \phi(x_i^2) = \phi(x_j^2) > 0 & \text{for } 1 \leq i, j \leq n, \end{cases} \tag{1.10}$$

where C_m are the Catalan numbers.

2 Freeness of the linear form and the quadratic form

In this section, we will give the characterization of a semicircular system by the freeness of the linear form and the quadratic form in noncommutative random variables.

In commutative case, for a given family of independently identically distributed random variables, the most important linear form and quadratic form are the sample mean and the sample variance, respectively.

It is known that if the identical distribution is Gaussian then the sample mean and the sample variance are independent. Conversely, independence of the sample mean and the sample variance implies the Gaussianity of the identical distribution. That is, independence of the sample mean and the sample variance characterizes the Gaussian law [KS].

Having the above fact in mind, we establish the noncommutative version of such a characterization replacing independence by freeness.

We shall begin with the key result which is found in [VDN], Proposition 5.1.2.

Proposition 2.1. *Let $(x_i)_{i=1,2,\cdots,n}$ be a semicircular system in a C^*- probability space $(\mathcal{A},\phi)$ and let $T=(t_{ij})\in O(n)$ be an $n\times n$ real orthogonal matrix. We put*

$$y_i=\sum_{j=1}^{n}t_{ij}x_j,\quad (i=1,2,\cdots,n). \tag{2.1}$$

Then the family $(y_i)_{i=1,2,\cdots,n}$ is also a semicircular system.

In Proposition 5.1.2 in [VDN], the traciality of the state ϕ was imposed, but in order to have the above statement, it is no matter to regard the C^*- algebra $\mathcal{A}$ as the C^*- free product $\prod *_{i=1}^{n}\mathcal{A}_i$ of the algebras $\mathcal{A}_i$ generated by $\{1,x_i\}$. Since each $\mathcal{A}_i$ is abelian and the state on $\prod *_{i=1}^{n}\mathcal{A}_i$ is determined by its restriction on each $\mathcal{A}_i$, see Proposition 2.5.5 in [VDN], so the traciality of the state is automatically derived in this case.

We consider the certain pair of a linear form and a quadratic form which covers the case of the sample mean and the sample variance.

Proposition 2.2. *Let $(x_i)_{i=1,2,\cdots,n}$ be a semicircular system in a C^*- probability space $(\mathcal{A},\phi)$. Let $A=(a_{ij})\in M_n(\mathbb{R})$ be an $n\times n$ real matrix and $\mathbf{b}={}^t(b_1,b_2,\cdots,b_n)\in\mathbb{R}^n$ a non-zero $n-$ dimensional real vector satisfying with*

$$A\mathbf{b}=\mathbf{0}\quad \textit{and}\quad {}^t\mathbf{b}A=\mathbf{0}. \tag{2.2}$$

Then we have that the linear form $\ell = \sum_{i=1}^n b_i x_i$ *and the quadratic form* $q = \sum_{i,j=1}^n a_{ij} x_i x_j$ *are free.*

Proof. Without loss of generality, we may asuume that $||\mathbf{b}|| = \left(\sum_{i=1}^n b_i^2\right)^{1/2} = 1$. Take the orthonormal system of $\mathbb{R}^n$, $\{\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_{n-1}, \mathbf{c}_n = \mathbf{b}\}$ and denote $\mathbf{c}_i = {}^t(c^i{}_1, c^i{}_2, \cdots, c^i{}_n)$. We make the orthogonal matrix S by $S\mathbf{c}_i = \mathbf{e}_i$, where $\mathbf{e}_i$ is the i-th canonical unit vector ${}^t(0, \cdots, \underbrace{1}_{i}, \cdots, 0)$. By the assumption (2.2), we have $SA^tSS\mathbf{b} = \mathbf{0}$ and ${}^t\mathbf{b}^tSSA^tS = \mathbf{0}$, which implies that the matrix SA^tS can be written of the form

$$SA^tS = \begin{pmatrix} & & & 0 \\ & \alpha_{ij} & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix} = \sum_{i,j=1}^{n-1} \alpha_{ij} E_{ij}, \tag{2.3}$$

where E_{ij} means the $(i,j)-$th matrix unit and α_{ij} is a constant. Thus we obtain that

$$A = \sum_{i,j=1}^{n-1} \alpha_{ij} \mathbf{c}_i {}^t\mathbf{c}_j \tag{2.4}$$

and the quadratic form q turns out

$$q = \sum_{i,j=1}^{n-1} \alpha_{ij} y_i y_j, \tag{2.5}$$

where $y_i = \sum_{k=1}^n c^i{}_k x_k$, $i = 1, 2, \cdots, n$. From Proposition 2.1, the elements $y_1, y_2, \cdots, y_{n-1}$ and $\ell = b_1 x_1 + b_2 x_2 + \cdots + b_n x_n$ are free. Hence we obtain that ℓ is free from the C^*- subalgebra generated by $\{y_1, y_2, \cdots, y_{n-1}\}$, namely ℓ and $q = \sum_{i,j=1}^{n-1} \alpha_{ij} y_i y_j = \sum_{i,j=1}^n a_{ij} x_i x_j$ are free. Because we know that, in general, if x, y, z are free in $(\mathcal{A}, \phi)$ then x is free from the algebra $C^*(\{y, z\}) \subset \mathcal{A}$, see Proposition 2.5.5 in [VDN].

□

Theorem 2.3. *Let* $x_1, x_2, \cdots, x_n$ *be freely identically distributed random variables with zero expectations,* $\phi(x_i) = 0$*, in a* C^*-*probability space* $(\mathcal{A}, \phi)$*. Let* $A = (a_{ij}) \in M_n(\mathbb{R})$ *be an* $n \times n$ *non-negative definite real matrix and* $\mathbf{b} = {}^t(b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$ *be an* $n-$ *dimensional non-negative vector satisfying with the conditions*

$$A\mathbf{b} = \mathbf{0} \quad \textit{and} \quad \sum_{i=1}^n b_i a_{ii} \neq 0. \tag{2.6}$$

We put the linear form $\ell = \sum_{i=1}^n b_i x_i$ and the quadratic form $q = \sum_{i,j=1}^n a_{ij} x_i x_j$. Then the forms ℓ and q are free if and only if the family $(x_i)_{i=1,2,\cdots,n}$ is a semicircular system.

Proof. First, we show that $\phi(x_1^{2m+1}) = 0$ for $m \geq 0$ by induction on m.

For $m = 0$, it is nothing but our assumption. For $m \geq 1$, we consider the expectation $\phi(\ell q^m)$. Using the freeness of ℓ and q, we have $\phi(\ell q^m) = \phi(\ell)\phi(q^m) = 0$. On the other hand, we have the following expansion:

$$
\begin{aligned}
\phi(\ell q^m) &= \phi\Big(\big(\sum_{i=1}^n b_i x_i\big)\big(\sum_{i,j=1}^n a_{ij} x_i x_j\big)^m \Big) \\
&= \sum_{i=1}^n b_i a_{ii}^m \phi(x_i^{2m+1}) \\
&\qquad + \mathrm{lin}\Big\{ \phi(x_1^{\iota_{1,1}})\phi(x_1^{\iota_{1,2}})\cdots\phi(x_1^{\iota_{1,k(1)}})\phi(x_2^{\iota_{2,1}})\phi(x_2^{\iota_{2,2}})\cdots \\
&\qquad\qquad \cdots\phi(x_2^{\iota_{2,k(2)}})\cdots\phi(x_n^{\iota_{n,1}})\phi(x_n^{\iota_{n,2}})\cdots\phi(x_n^{\iota_{n,k(n)}})\ ; \\
&\qquad\qquad \sum_{j=1}^n \sum_{l=1}^{k(j)} \iota_{j,l} = 2m+1, \quad \#\{j : k(j) > 0\} \geq 2 \Big\}.
\end{aligned} \tag{2.7}
$$

Here the second term of the above expression means that the linear combination of the products of the simple moments. And this linear combination is homogeneous of dgree $2m+1$ in the sense that $\sum_{j=1}^n \sum_{l=1}^{k(j)} \iota_{j,l} = 2m+1$. If all $\iota_{j,l}$ are even integers then the summation $\sum \iota_{j,l}$ should be even. Nevertheless $\sum \iota_{j,l}$ is odd number so we can find a moment of some odd order among the factors of each summand, and its order is strictly smaller than $2m+1$. Thus the induction hypothesis implies that the term of this linear combination must be 0. Hence we have that

$$
\phi(\ell q^m) = \sum_{i=1}^n b_i a_{ii}^m \phi(x_i^{2m+1}) = 0 \tag{2.8}
$$

and then

$$
\Big(\sum_{i=1}^n b_i a_{ii}^m\Big)\phi(x_1^{2m+1}) = 0 \tag{2.9}
$$

since all x_i's have the same distribution. From the assumption, it follows that $\sum_{i=1}^n b_i a_{ii}^m > 0$, which implies that $\phi(x_1^{2m+1}) = 0$.

Next using induction on m, we shall show that $\phi(x_1^{2m})$ can be expressed as

$$
\phi(x_1^{2m}) = \alpha_m \phi(x_1^2)^m, \tag{2.10}
$$

where α_m is the constant which depends only on the matrix A and the vector $\mathbf{b}$. For $m = 1$, it is clear that $\alpha_1 = 1$. For $m \geq 2$, we assume that the constants $\alpha_1, \alpha_2, \cdots, \alpha_{m-1}$ have been determined by the matrix A and the vector $\mathbf{b}$, as the induction hypothesis. Now we consider the expectation $\phi(\ell^2 q^{m-1})$ and expand it in two different ways.

The first way is as follows:

$$\phi(\ell^2 q^{m-1}) = \phi(\ell^2)\phi(q^{m-1}) \qquad \text{by the freeness of } \ell \text{ and } q$$

$$(2.11) \qquad = \phi\Big(\sum_{i,j=1}^{n} b_i b_j x_i x_j\Big)\phi\Big(\big(\sum_{i,j=1}^{n} a_{ij} x_i x_j\big)^{m-1}\Big).$$

The freeness of (x_i) implies that $\phi(x_i x_j) = 0$ if $i \neq j$. Thus we have, on the first factor of (2.11),

$$(2.12) \qquad \phi\Big(\sum_{i,j=1}^{n} b_i b_j x_i x_j\Big) = \phi\Big(\sum_{i=1}^{n} b_i^2 x_i^2\Big) = \sum_{i=1}^{n} b_i^2 \phi(x_i^2) = \Big(\sum_{i=1}^{n} b_i^2\Big)\phi(x_1^2),$$

because x_i's have the same distribution. The second factor of (2.11) can be expanded into $2(m-1)$-homogeneous linear combination of the products of the simple moments. And only the summands constituted from the simple moments of even orders could appear because the moments of odd orders will vanish as we have shown just above. Using the induction hypothesis and the assumption that x_i's have the same distribution, the second factor can be written as

$$(2.13) \qquad \phi\Big(\big(\sum_{i,j=1}^{n} a_{jk} x_i x_j\big)^{m-1}\Big) = F_1(A, \mathbf{b}, \alpha_1, \alpha_2, \cdots, \alpha_{m-1})\phi(x_1^2)^{m-1},$$

where F_1 is the constant determined by the matrix A, the vector $\mathbf{b}$ and the constants $\alpha_1, \alpha_2, \cdots, \alpha_{m-1}$. Combinding the equalities (2.11), (2.12) and (2.13), we have

$$(2.14) \qquad \phi(\ell^2 q^{m-1}) = \Big(\sum_{i=1}^{n} b_i^2\Big) F_1(A, \mathbf{b}, \alpha_1, \alpha_2, \cdots, \alpha_{m-1})\phi(x_1^2)^m.$$

The other way for the expansion is as follows:

$$\phi(\ell^2 q^{m-1}) = \phi\Big(\big(\sum_{i,j=1}^{n} b_i b_j x_i x_j\big)\big(\sum_{k,l=1}^{n} a_{kl} x_k x_l\big)^{m-1}\Big)$$

$$= \Big(\sum_{i=1}^{n} b_i^2 a_{ii}^{m-1}\Big)\phi(x_1^{2m})$$

$$(2.15) \qquad + \operatorname{lin}\Big\{\phi(x_1^{\iota_{1,1}})\phi(x_1^{\iota_{1,2}})\cdots\phi(x_1^{\iota_{1,k(1)}})\phi(x_2^{\iota_{2,1}})\phi(x_2^{\iota_{2,2}})\cdots$$

$$\cdots\phi(x_2^{\iota_{2,k(2)}})\cdots\phi(x_n^{\iota_{n,1}})\phi(x_n^{\iota_{n,2}})\cdots\phi(x_n^{\iota_{n,k(n)}})\ ;$$

$$\sum_{j=1}^{n}\sum_{l=1}^{k(j)} \iota_{j,l} = 2m, \quad \#\{j : k(j) > 0\} \geq 2\Big\}$$

As we used in (2.7), the second term of the above expression means that the linear combination of the products of the simple moments. Moreover this linear combination is homogeneous of dgree $2m$ in the sense that $\sum_{j=1}^{n}\sum_{l=1}^{k(j)} \iota_{j,l} = 2m$. Then only the products of the moments of even orders, which are strictly smaller than $2m$, could appear because we know that the moments of odd orders will vanish. Again use the induction hypothesis and the assumption that x_i's have the same distribution, this linear combination can be written as

$$F_2(A, \mathbf{b}, \alpha_1, \alpha_2, \cdots, \alpha_{m-1})\phi(x_1^2)^m, \tag{2.16}$$

where F_2 means the constant determined by the matrix A, the vector $\mathbf{b}$ and $\alpha_1, \alpha_2, \cdots, \alpha_{m-1}$. Consequently, we obtain that

$$\phi(\ell^2 q^{m-1}) = \Big(\sum_{i=1}^{n} b_i^2 a_{ii}^{m-1}\Big)\phi(x_1^{2m}) + F_2(A, \mathbf{b}, \alpha_1, \alpha_2, \cdots, \alpha_{m-1})\phi(x_1^2)^m. \tag{2.17}$$

From the equalities (2.14) and (2.17), it follows that

$$\begin{aligned}\phi(x_1^{2m}) = \Big(\sum_{i=1}^{n} b_i^2 a_{ii}^{m-1}\Big)^{-1} \Big\{ &\big(\sum_{i=1}^{n} b_i^2\big) F_1(A, \mathbf{b}, \alpha_1, \alpha_2, \cdots, \alpha_{m-1}) \\ &- F_2(A, \mathbf{b}, \alpha_1, \alpha_2, \cdots, \alpha_{m-1})\Big\}\phi(x_1^2)^m,\end{aligned} \tag{2.18}$$

where the assumption implies that $\sum_{i=1}^{n} b_i^2 a_{ii}^{m-1} > 0$. Hence we obtain that α_m is determined by the matix A and the vector $\mathbf{b}$, which ends induction.

By the way, since the matrix A and the vector $\mathbf{b}$ satisfy the conditions of Proposition 2.2, if we set $(x_1, x_2, \cdots, x_n)$ be a semicircular system then ℓ and q become a free pair. So the constants α_i's are the universal constants determined only by the freeness of ℓ and q. This implies that α_m is the Catalan number $C_m = \dfrac{(2m)!}{(m+1)!m!}$.

□

Corollary 2.4. *Let $x_1, x_2, \cdots, x_n$ be freely identically distributed random variables in a C^*- probability space $(\mathcal{A}, \phi)$. We put $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ (the sample mean) and $v = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ (the sample variance) are free if and only if (x_i) is the semicircular system.*

Proof. As similar to the commutative case, v is given by putting the coefficient matrix $A = (a_{ij})$ as $a_{ij} = \frac{1}{n}\delta_{ij} - \frac{1}{n^2}$, where δ_{ij} is the Kronecker's delta and $\bar{x}$ is given by the vector $\mathbf{b} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. Then the matrix A and the vector $\mathbf{b}$ satisfy the requirements of Theorem 2.3.

□

3 Remaining free under rotations and semicircle laws

In this section, we give the alternative proof for Nica's work [Ni], that the property of remaining free under rotations characterizes semicircular elements. This is the free analogue of the well-known fact in usual probability theory that property of remaining independent under rotations characterizes Gaussian random varibles, see for instance Section III.4 in [Fe].

In [Ni], A. Nica introduced the multidimensional $R-$transform (free analogue of the cumulants generating series) and investigated its behavior under linear transformations of the coodinates. Using these results, he proved the above characterization. We will, however, apply the same method which we have used in the previous section, that is a direct calculation of the higher moments of noncommutative random variables, in order to give the alternative proof.

We shall begin with the definition of the some property for a matrix which will be required for an orthogonal transformation of random variables.

Definition 3.1. Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be an $n \times n$ real matrix. For the pair of integers $(i, j) \in \{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\}$, we consider the set $K_{(i,j)} = \{k \mid 1 \leq k \leq n, a_{ki}a_{kj} \neq 0\}$. If the set $K_{(i,j)}$ has at least two elements, i.e. $\# K_{(i,j)} \geq 2$, then we write $i \overset{A}{\sim} j$ or simply write $i \sim j$ if no confusion.

Moreover, we write $i \overset{A}{\approx} j$ if there exist numbers $j_1, j_2, \cdots, j_l$ such that $i = j_1$, $j_1 \sim j_2, \cdots, j_{l-1} \sim j_l, j_l = j$.

Definition 3.2. A matrix A is called *indecomposable* if $i \overset{A}{\approx} j$ for any i and j.

Proposition 3.3. *Let $A \in M_n(\mathbb{R})$ be an orthogonal matrix. Then A is indecomposable if and only if C_1AC_2 does not have the form*

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \tag{3.1}$$

for any permutation matrices C_1, C_2, where A_1, A_2 are orthogonal matrices and their dimensions are strictly smaller than n.

Proof The part of "if" is trivial. Now, we assume that A is not indecomposable. Then without loss of generality, we may assume that $1 \overset{A}{\approx} 2 \overset{A}{\approx} \cdots \overset{A}{\approx} k$ and $i \overset{A}{\not\approx} j$ for every $1 \leq i \leq k$, $k+1 \leq j \leq n$.

Note first that, in general, if row vectors $(b_1, b_2, \cdots, b_n)$ and $(c_1, c_2, \cdots, c_n)$ in $\mathbb{R}^n$satisfy

$$b_1c_1 + b_2c_2 + \cdots + b_nc_n = 0 \tag{3.2}$$

and

$$\#\{k \,:\, b_kc_k \neq 0\} \leq 1 \tag{3.3}$$

then $b_l \neq 0$ implies $c_l = 0$. This means that $a_{ki} \neq 0$ implies $a_{kj} = 0$ if $i \nsim j$.

Exchanging column vectors of A, we may assume that there exist $1 = i_0 \leq i_1 \leq \cdots \leq i_k \leq n$ such that

$$\begin{aligned} a_{st} &\neq 0, \qquad i_{s-1} \leq t \leq i_s \\ a_{st} &= 0, \qquad i_s < t \leq n \end{aligned} \tag{3.4}$$

for $1 \leq s \leq k$. In the case $1 \leq i \leq k < j \leq n$, $i \overset{A}{\not\approx} j$. So we have $i \nsim j$. By the above note and our assumption, we have

$$a_{st} = 0, \tag{3.5}$$

if $k < s \leq n$, $1 \leq t \leq i_k$ or $1 \leq s \leq k$, $i_k < t \leq n$. The linear independence of row (resp. column) vectors of A implies $k \leq i_k$ (resp. $k \geq i_k$). So we get $k = i_k$. That is, A is decomposed to a direct sum of two orthogonal matrices.

□

Remark 3.4. The condition that A is indecomposable coincides with the stronger condition which can be found in Theorem 5.3′ in [Ni].

Theorem 3.5. *Let $A = (a_{ij}) \in O(n)$ be an $n \times n$ - real orthogonal matrix and indecomposable in the sense of Defnition 3.2. Let $x_1, x_2, \cdots, x_n$ be non-scalar self-adjoint elements of a C^*- probability space $(\mathcal{A}, \phi)$ with zero expectation, $\phi(x_i) = 0$. If the family $(x_1, x_2, \cdots, x_n)$ is free and the family $(y_1, y_2, \cdots, y_n)$ is still free, where $y_i = \sum_{j=1}^{n} a_{ij}x_j$, then the family of noncommutative random variables $(x_1, x_2, \cdots, x_n)$ is a semicircular system.*

Proof. It is enough to show the relations (1.10).

First we shall show that $\phi(x_i^{2m+1}) = 0$ by induction on m. For $m = 0$, it is nothing but our assumption. For $m \geq 1$, all we have to prove is that

$\phi(x_{i_0}^{2m+1}) = \phi(x_{j_0}^{2m+1})$ only for a pair (i_0, j_0) with $i_0 \sim j_0$, because the matrix A is indecomposable in the sense of Definition 3.2.

By the definition, given a pair (i_0, j_0) with $i_0 \sim j_0$, we can find the integers k_1 and k_2 such that $a_{k_1 i_0} a_{k_1 j_0} \neq 0$ and $a_{k_2 i_0} a_{k_2 j_0} \neq 0$. Here we consider the n expectations, $\phi(y_{k_1}^{2m-1} y_{k_2} y_\ell)$ for $\ell = 1, 2, \cdots, n$.

If $\ell \neq k_2$ then,using the freeness of y_k's and applying the Lemma 1.4, it follows that

$$\phi(y_{k_1}^{2m-1} y_{k_2} y_\ell) = \phi(y_{k_2}) \phi(y_{k_1}^{2m-1} y_\ell) = 0 \tag{3.6}$$

since $\phi(y_{k_2}) = 0$. And if $\ell = k_2$ then we have

$$\phi(y_{k_1}^{2m-1} y_{k_2} y_\ell) = \phi(y_{k_1}^{2m-1}) \phi(y_{k_2}^2). \tag{3.7}$$

Apply the same method as we used in the proof of Thorem 2.3, to the simple moment $\phi(y_{k_1}^{2m-1})$. That is, we expand $\phi(y_{k_1}^{2m-1})$ to the $(2m-1)$ homogeneous linear combination of the products of the simple moments of x_i's and the induction hypothesis makes each summand vanish, i.e.

$$\phi(y_{k_1}^{2m-1}) = 0. \tag{3.8}$$

Consequently, we obtain that

$$\phi(y_{k_1}^{2m-1} y_{k_2} y_\ell) = 0 \quad \text{for all } \ell = 1, 2, \cdots, n. \tag{3.9}$$

On the other hand, $\phi(y_{k_1}^{2m-1} y_{k_2} y_\ell)$ can be expanded as follows:

$$\begin{aligned}
\phi(y_{k_1}^{2m-1} y_{k_2} y_\ell) &= \phi\Big(\big(\sum_{j=1}^{n} a_{k_1 j} x_j\big)^{2m-1} \big(\sum_{j=1}^{n} a_{k_2 j} x_j\big) \big(\sum_{j=1}^{n} a_{\ell j} x_j\big) \Big) \\
&= \phi\big(\sum_{j=1}^{n} a_{k_1 j}^{2m-1} a_{k_2 j} a_{\ell j} x_j^{2m+1}\big) \\
&\qquad + \mathrm{lin}\Big\{ \phi(x_1^{\iota_{1,1}}) \phi(x_1^{\iota_{1,2}}) \cdots \phi(x_1^{\iota_{1,k(1)}}) \cdots \\
&\qquad\qquad \cdots \phi(x_n^{\iota_{n,1}}) \phi(x_n^{\iota_{n,2}}) \cdots \phi(x_n^{\iota_{n,k(n)}}) \ ; \\
&\qquad\qquad \sum_{j=1}^{n} \sum_{l=1}^{k(j)} \iota_{j,l} = 2m+1, \quad \#\{j : k(j) > 0\} \geq 2 \Big\} \\
&= \sum_{j=1}^{n} a_{k_1 j}^{2m-1} a_{k_2 j} a_{\ell j} \phi(x_j^{2m+1}).
\end{aligned} \tag{3.10}$$

Of course, the part of the linear combination of the products of the simple moments will vanish by the induction hypothesis and only the highest simple moments can survive. Thus we have the linear equations,

$$\sum_{j=1}^{n} a_{k_1 j}^{2m-1} a_{k_2 j} a_{\ell j} \phi(x_j^{2m+1}) = 0 \qquad (\ell = 1, 2, \cdots, n), \tag{3.11}$$

which can be written in the matrix form as

$$A \begin{pmatrix} a_{k_1 1}^{2m-1} a_{k_2 1} \phi(x_1^{2m+1}) \\ a_{k_1 2}^{2m-1} a_{k_2 2} \phi(x_2^{2m+1}) \\ \vdots \\ a_{k_1 n}^{2m-1} a_{k_2 n} \phi(x_n^{2m+1}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.12}$$

The invertibility of the matrix A implies that

$$a_{k_1 j}^{2m-1} a_{k_2 j} \phi(x_j^{2m+1}) = 0 \qquad (j = 1, 2, \cdots, n).$$

Since the integers k_1 and k_2 have been chosen such that $a_{k_1 i_0} a_{k_1 j_0} \neq 0$ and $a_{k_2 i_0} a_{k_2 j_0} \neq 0$, it follows that $\phi(x_{i_0}^{2m+1}) = \phi(x_{j_0}^{2m+1}) = 0$.

Next we shall show the sameness of the variances that

$$\phi(x_1^2) = \phi(x_2^2) = \cdots = \phi(x_n^2). \tag{3.13}$$

Because of the indecomposablity of the matrix A, it is enough to show that $\phi(x_{i_0}^2) = \phi(x_{j_0}^2)$ only for a pair (i_0, j_0) of integers with $i_0 \sim j_0$. From the definition of $i_0 \sim j_0$, we can take the integer k_1 such that $a_{k_1 i_0} a_{k_1 j_0} \neq 0$. Here we consider the n expectations, $\phi(y_{k_1} y_\ell), \quad \ell = 1, 2, \cdots, n$. It is direct consequence of the freeness of families (x_i) and (y_j) that

$$\phi(y_{k_1} y_\ell) = \sum_{i=1}^{n} a_{k_1 i} a_{\ell i} \phi(x_i^2) = \begin{cases} 0 & \text{if } \ell \neq k_1 \\ \phi(y_{k_1}^2) & \text{if } \ell = k_1 \end{cases}, \tag{3.14}$$

which can be written in the matrix form that

$$A \begin{pmatrix} a_{k_1 1} \phi(x_1^2) \\ \vdots \\ a_{k_1 k_1} \phi(x_{k_1}^2) \\ \vdots \\ a_{k_1 n} \phi(x_n^2) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \phi(y_{k_1}^2) \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{the } k_1\text{th row.} \tag{3.15}$$

The orthogonality of the matrix A imples that

$$\begin{pmatrix} a_{k_1 1}\phi(x_1^2) \\ a_{k_1 2}\phi(x_2^2) \\ \vdots \\ a_{k_1 n}\phi(x_n^2) \end{pmatrix} = \begin{pmatrix} a_{k_1 1} \\ a_{k_1 2} \\ \vdots \\ a_{k_1 n} \end{pmatrix} \phi(y_{k_1}^2), \tag{3.16}$$

because A^{-1} is given by ${}^tA$. As the integer k_1 has been chosen such that $a_{k_1 i_0}a_{k_1 j_0} \neq 0$, it is clear that $a_{k_1 i_0} \neq 0$ and $a_{k_1 j_0} \neq 0$. Hence we obtain that $\phi(x_{i_0}^2) = \phi(x_{j_0}^2)$.

Finally, we shall show with induction that $\phi(x_i^{2m})$ can be written in the form

$$\phi(x_i^{2m}) = \alpha_m^{(i)}\phi(x_1^2)^m, \tag{3.17}$$

where $\alpha_m^{(i)}$ is the constant determined by the matrix A.

For $m = 1$, $\alpha_1^{(i)} = 1$ $(i = 1, 2, \cdots, n)$ as we have shown just above. For $m \geq 2$, we assume that the constants $\alpha_k^{(i)}$ $(i = 1, 2, \cdots, n,\ k = 1, 2, \cdots, m-1)$ have been determined by the matrix A. Here take a pair (i_0, j_0) of integers with $i_0 \sim j_0$ and find integers k_1 and k_2 such that $a_{k_1 i_0}a_{k_1 j_0} \neq 0$ and $a_{k_2 i_0}a_{k_2 j_0} \neq 0$. Then consider the n expectations, $\phi(y_{k_1}^{2m-2}y_{k_2}y_\ell)$, $\quad \ell = 1, 2, \cdots, n$. By freeness of (y_i), we have

$$\phi(y_{k_1}^{2m-2}y_{k_2}^2) = \begin{cases} 0 & \text{if } \ell \neq k_2 \\ \phi(y_{k_1}^{2m-2})\phi(y_{k_2}^2) & \text{if } \ell = k_2 \end{cases}. \tag{3.18}$$

Then $\phi(y_{k_1}^{2m-2})$ can be expanded as the $(2m-2)$ homogeneous linear combination of products of the simple moments, in which only ones of even orders will survive. Hence the induction hypothesis implies that

$$\phi(y_{k_1}^{2m-2}) = F(A, \alpha_1^{(1)}, \alpha_2^{(1)}, \cdots, \alpha_{m-1}^{(1)}, \alpha_1^{(2)}, \cdots, \alpha_{m-1}^{(n)})\phi(x_1^2)^{m-1}, \tag{3.19}$$

where F means the constant determined by the matrix A and the constants $\alpha_1^{(1)}$, $\alpha_2^{(1)}$, $\cdots$, $\alpha_{m-1}^{(1)}$. On the second factor of (3.18), we have

$$\begin{aligned} \phi(y_{k_2}^2) &= \phi(\sum_{i,j=1}^{n} a_{k_2 i}a_{k_2 j}x_i x_j) \\ &= \phi(\sum_{i=1}^{n} a_{k_2 i}^2 x_i^2) = \left(\sum_{i=1}^{n} a_{k_2 i}^2\right)\phi(x_1^2), \end{aligned} \tag{3.20}$$

by the knowledge that all $\phi(x_i^2)$ are equal. Combining (3.18), (3.19) and (3.20), we obtain that

$$\phi(y_{k_1}^{2m-2}y_{k_2}^2) = F(A)\left(\sum_{i=1}^{n} a_{k_2 i}^2\right)\phi(x_1^2)^m, \tag{3.21}$$

which means that $\phi(y_{k_1}^{2m-2}y_{k_2}^2)$ can be expressed as the product of the constant $F(A)$ determined by the matrix A and the power of the variance, $\phi(x_1^2)^m$.

On the other hand, the n expectations $\phi(y_{k_1}^{2m-2}y_{k_2}y_\ell)$ could be expaned as follows:

$$
\begin{aligned}
\phi(y_{k_1}^{2m-2}y_{k_2}y_\ell) &= \phi\Big(\big(\sum_{j=1}^n a_{k_1j}x_j\big)^{2m-2}\big(\sum_{j=1}^n a_{k_2j}x_j\big)\big(\sum_{j=1}^n a_{\ell j}x_j\big)\Big) \\
&= \phi\big(\sum_{j=1}^n a_{k_1j}^{2m-2}a_{k_2j}a_{\ell j}x_j^{2m}\big) \\
&\quad + \mathrm{lin}\Big\{ \phi(x_1^{\iota_{1,1}})\phi(x_1^{\iota_{1,2}})\cdots\phi(x_1^{\iota_{1,k(1)}})\cdots \\
&\qquad \cdots\phi(x_n^{\iota_{n,1}})\phi(x_n^{\iota_{n,2}})\cdots\phi(x_n^{\iota_{n,k(n)}})\ ; \\
&\qquad \sum_{j=1}^n\sum_{l=1}^{k(j)}\iota_{j,l} = 2m, \quad \#\{j : k(j) > 0\} \geq 2\Big\}.
\end{aligned} \tag{3.22}
$$

Again we apply the similar argument as we used in the proof of Theorem 2.3. That is, the part of the linear combination in the above equality is $2m$ - homogeneous and only the products of moments of even orders, which are strictly smaller than $2m$, could appear. Now the induction hypothesis implies that this linear combination can be written as the product of the constant determined by the matrix A and the power of the variance $\phi(x_1^2)^m$, i.e.

$$
\phi(y_{k_1}^{2m-2}y_{k_2}y_\ell) = \sum_{j=1}^n a_{k_1j}^{2m-2}a_{k_2j}a_{\ell j}\phi(x_j^{2m}) + F_\ell(A)\phi(x_1^2)^m, \tag{3.23}
$$

where $F_\ell(A)$ means some constant determined by the matrix A.

Having in mind the relations (3.18), (3.21) and (3.23), it turns out that

$$
\sum_{j=1}^n a_{k_1j}^{2m-2}a_{k_2j}a_{\ell j}\phi(x_j^{2m}) = G_\ell(A)\phi(x_1^2)^m \quad \text{for } \ell = 1, 2, \cdots, n, \tag{3.24}
$$

where we denote $G_\ell(A)$ the constants determined by the matrix A as

$$
G_\ell(A) = \begin{cases} -F_\ell(A) & \text{if } \ell \neq k_2, \\ F(A)\big(\sum_{i=1}^n a_{k_2i}^2\big) - F_{k_2}(A) & \text{if } \ell = k_2. \end{cases} \tag{3.25}
$$

In the matrix form, we obtain that

$$
A\begin{pmatrix} a_{k_11}^{2m-2}a_{k_21}\phi(x_1^{2m}) \\ a_{k_12}^{2m-2}a_{k_22}\phi(x_2^{2m}) \\ \vdots \\ a_{k_1n}^{2m-2}a_{k_2n}\phi(x_n^{2m}) \end{pmatrix} = \begin{pmatrix} G_1(A) \\ G_2(A) \\ \vdots \\ G_n(A) \end{pmatrix}\phi(x_1^2)^m. \tag{3.26}
$$

The invertibility of the matrix A (in this case A^{-1} is given by ${}^tA$) implies that

$$a_{k_1 j}^{2m-2} a_{k_2 j} \phi(x_j^{2m}) = \Big(\sum_{i=1}^{n} a_{ji} G_i(A)\Big) \phi(x_1^2)^m \quad \text{for } j = 1, 2, \cdots, n. \tag{3.27}$$

Since we chose the integers k_1 and k_2 such that $a_{k_1 i_0} a_{k_1 j_0} \neq 0$ and $a_{k_2 i_0} a_{k_2 j_0} \neq 0$, it follows that $\phi(x_{i_0}^{2m})$ and $\phi(x_{j_0}^{2m})$ can be written as $\phi(x_{i_0}^{2m}) = \alpha_m^{(i_0)} \phi(x_1^2)^m$ and $\phi(x_{j_0}^{2m}) = \alpha_m^{(j_0)} \phi(x_1^2)^m$, where $\alpha_m^{(i_0)}, \alpha_m^{(j_0)}$ are the constants determined by the matrix A. Since the pair (i_0, j_0) has been taken arbitrary, this ends induction.

By the way, if we set $(x_1, x_2, \cdots, x_n)$ be a semicircular system then the family $(y_1, y_2, \cdots, y_n)$ is still free system by Proposition 2.1. The scalar $\alpha_m^{(i)}$ must coincide with the Catalan number $C_m = \dfrac{(2m)!}{(m+1)!m!}$, since the constants $\alpha_m^{(i)}$'s are the universal constants determined only by the freeness.

□

If x is a centered semicircular element of some radius then the random variable λx, any scalar multiple of x, is still a centered semicircular element. That is, the relations (1.9) are invariant under dilations. It should be noted that, in the proof of Theorem 3.5, we have shown the relations (1.9) directly for the pair of random variables x_{i_0} and x_{j_0} with $i_0 \sim j_0$.

Moreover our assumption that each x_i has zero expectation $\phi(x_i) = 0$ is not essential requirement. Because freeness will be kept in translations. Thus we can give the following slight extended characterization of the above theorem as corollary.

Corollary 3.6. *Let $A = (a_{ij}) \in O(n)$ be an $n \times n$ - real orthogonal matrix and we put $B = (b_{ij})$ as*

$$B = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} A \begin{pmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_n \end{pmatrix}, \tag{3.28}$$

where $\lambda_1 \lambda_2 \cdots \lambda_n \neq 0$ and $\mu_1 \mu_2 \cdots \mu_n \neq 0$. Let $x_1, x_2, \cdots, x_n$ be non-scalar self-adjoint elements of a C^*- probability space $(\mathcal{A}, \phi)$. We assume that the family $(x_1, x_2, \cdots, x_n)$ is free and the family $(y_1, y_2, \cdots, y_n)$ is still free, where $y_i = \sum_{j=1}^n b_{ij} x_j$. If $i \neq j$ and $i \overset{B}{\sim} j$ then we have that x_i and x_j are semicircular elements.

Finally we comment that we can have futhermore characterizations of a semicircle law which are related to quadratic forms or the free analogue of noncentral chi-square distributions. They will be discussed in the sequent paper [HKNY].

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