SOME SUBSEMIGROUP OF AN EXTENSION SEMIGROUP OF C*-ALGEBRAS

YUTAKA KATABAMI AND MASARU NAGISA

ABSTRACT. In this paper we investigate the structure of the subsemigroup generated by the inner automorphisms in $\mathbf{Ext}(\mathbb{K}, \mathcal{Q})$. As an application, we give a new point of view to the example of J. Plastiras, which are two C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ satisfying $\mathfrak{A} \not\cong \mathfrak{B}$ and $\mathbb{M}_2 \otimes \mathfrak{A} \cong \mathbb{M}_2 \otimes \mathfrak{B}$.

0. INTRODUCTION

J. Plastiras exhibited an example which is a pair of C*-algebras such that $\mathfrak{A} \not\cong \mathfrak{B}$ and $\mathbb{M}_2 \otimes \mathfrak{A} \cong \mathbb{M}_2 \otimes \mathfrak{B}$ ([5], [6]). They are constructed as extensions of $\mathbb{K}$ by $\mathcal{Q}$, where $\mathbb{K}$ is the C*-algebra of compact operators and $\mathcal{Q}$ is the quotient C*-algebra of all the bounded linear operators $\mathbb{B}$ by $\mathbb{K}$. So they are not nuclear. For a class of nuclear C*-algebras, we can construct such a pair of C*-algebras using the classification result for them by K-theory([2], [3]). In [7], T. Sakamoto construct such a pair of non-nuclear C*-algebras.

In this paper, we consider a family of special extensions of $\mathbb{K}$ by $\mathcal{Q}$ which contains Plastiras' examples. Our aim is to investigate their semigroup structure and to show that the datum for this semigroup is the useful invariant for them as like as K-thoretic datum for some nuclear C*-algebras.

1. PRELIMINARIES AND MAIN RESULT

Here we give fundamental facts of extension theory along [1] and [8]. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. We denote by $\mathbb{B}$ (resp. $\mathbb{K}$) a C*-algebra $\mathbb{B}(\mathcal{H})$ (resp. $\mathbb{K}(\mathcal{H})$) of bounded linear operators (resp. compact operators) on $\mathcal{H}$. We also denote by $\mathcal{Q}$ a C*-algebra $\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$. Let A, B and C be C*-algebras and α (resp. β) a *-homomorphism from A to B (resp. from B to C). We call a short exact sequence E as below an extension of A by C:

$$E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \quad ,$$

that is, α is injective, β is surjective and $\mathrm{Im}\alpha = \mathrm{Ker}\beta$. Then there exists a *-homomorphism σ from B to the multiplier C*-algebra $M(A)$

of A with $\sigma \circ \alpha = \iota$, i.e.,

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \| & & \downarrow{\scriptstyle \sigma} \\ A & \xrightarrow[\iota]{} & M(A) \end{array},$$

where ι is the canonical inclusion map from A to $M(A)$. The Busby invariant for this extension E is defined as the *-homomorphism τ_E from C to $M(A)/A$ given by

$$\tau_E(c) = \pi \circ \sigma(b) \ ,$$

where b is a lift of c through β and π is the quotient map from $M(A)$ to $M(A)/A$. It is known that τ_E is characterized by the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau_E} & & \\ 0 & \longrightarrow & A & \xrightarrow{\iota} & M(A) & \xrightarrow{\pi} & M(A)/A & \longrightarrow & 0 \end{array}.$$

We remark that, if we define the pull-back C*-algebra PB and the map ψ as follows:

$$PB = \{(x, c) \in M(A) \oplus C \mid \pi(x) = \tau_E(c)\}$$
$$\psi : B \ni b \longmapsto (\sigma(b), \beta(b)) \in PB,$$

then B is isomorphic to PB for the isomorphism ψ making the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & PB & \longrightarrow & C & \longrightarrow & 0 \end{array}.$$

Let

$$E_1 : 0 \longrightarrow A \longrightarrow B_1 \longrightarrow C \longrightarrow 0$$
$$E_2 : 0 \longrightarrow A \longrightarrow B_2 \longrightarrow C \longrightarrow 0$$

be extensions and τ_i the Busby invariant for E_i $(i = 1, 2)$. We call E_1 and E_2 strongly equivalent when there is a unitary $u \in M(A)$ such that $\tau_2(c) = \pi(u)\tau_1(c)\pi(u)^*$ for all $c \in C$, equivalently there are a unitary

$v \in M(A)$ and a *-isomorphism γ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B_1 & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Ad(v)} & & \downarrow{\scriptstyle \gamma} & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B_2 & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is commutative. Then we denote $E_1 \approx E_2$ or $\tau_1 \approx \tau_2$. Let $\mathrm{Ext}(A, C)$ be a set of extensions of A by C. We denote by $\mathbf{Ext}(A, C)$ the set $\mathrm{Ext}(A, C)/\approx$ of strongly equivalent classes of $\mathrm{Ext}(A, C)$. When C is stable (i.e., $C \cong \mathbb{M}_2(C)$), $\mathbf{Ext}(A, C)$ becomes an abelian semigroup. The addition of $[E_1]$ and $[E_2] \in \mathbf{Ext}(A, C)$ is defined by the equivalent class of the extension which is corresponding to the Busby invariant

$$\tau_1 \oplus \tau_2 : C \longrightarrow C \oplus C \hookrightarrow C.$$

In this paper, we consider the extension semigroup $\mathbf{Ext}(\mathbb{K}, \mathcal{Q})$. We denote by π the canonical quotient map from $\mathbb{B}$ onto $\mathcal{Q}$. Let α be an inner *-automorphism of $\mathcal{Q}$. Then we can see that α is the Busby invariant for an extension $E \in \mathrm{Ext}(\mathbb{K}, \mathcal{Q})$. We denote by $\mathcal{G}$ a subsemigroup of $\mathbf{Ext}(\mathbb{K}, \mathcal{Q})$ generated by extensions corresponding to all the inner *-automorphisms of $\mathcal{Q}$.

The inner automorphism α has the form $\alpha(\cdot) = u^* \cdot u$ for some unitary $u \in \mathcal{Q}$. Let $V \in \mathbb{B}$ be a lift of u, that is, $\pi(V) = u$. Then V is a Fredholm operator, and we put $n = \mathrm{Index}\, V \in \mathbb{Z}$. Let $S (\in \mathbb{B})$ be a unilateral shift. We remark $\mathrm{Index}\, S = -1$. We define a *-automorphism $\tau(n)$ of $\mathcal{Q}$ by

$$\tau(n)(x) = \pi(S)^n x \pi(S^*)^n, \quad x \in \mathcal{Q}.$$

Then there exists a unitary $U \in \mathbb{B}$ such that $VS^n = U|VS^n|$, i.e., $u\pi(S)^n = \pi(U)$. So we have that α is strongly equivalent to $\tau(n)$, that is, $[\alpha] = [\tau(n)]$.

Let G be a restricted direct product of non-negative integers $\mathbb{Z}_{\geq 0}$ except $\mathbf{0}$, i.e.,

$$\begin{aligned} G = &\coprod_{\mathbb{Z}} \mathbb{Z}_{\geq 0} \setminus \{\mathbf{0}\} \\ =&\{g = (m(k))_{k \in \mathbb{Z}} \mid m(k) \in \mathbb{Z}_{\geq 0},\ 0 < \sharp\{k \in \mathbb{Z} \mid m(k) \neq 0\} < \infty\}, \end{aligned}$$

where $\sharp$ denotes the cardinal number of set. By the above fact, we can define the surjective semigroup homomorphism τ from G to $\mathcal{G}$ as follows:

$$\tau(g) = [\bigoplus_{k \in \mathbb{Z}} \bigoplus_{m(k)} \tau(k)] = \sum_{k \in \mathbb{Z}} m(k)[\tau(k)],$$

where $g = (m(k))_{k \in \mathbb{Z}} \in G$.

We define a map φ from G to $\mathbb{N}$ and a map ψ from G to $\mathbb{Z}$ as follows: for $g = (m(k))_{k\in\mathbb{Z}} \in G$,

$$\varphi(g) = \sum_{k\in\mathbb{Z}} m(k),$$
$$\psi(g) = \sum_{k\in\mathbb{Z}} km(k).$$

We introduce two notations as follows: for $l \in \mathbb{Z}$ and $g = (m(k))_{k\in\mathbb{Z}} \in G$,

$$l \cdot g = (m(lk))_{k\in\mathbb{Z}},$$
$$l + g = (m(l+k))_{k\in\mathbb{Z}}.$$

Then we can easily get

$$\varphi(l \cdot g) = \varphi(g),\ \psi(l \cdot g) = l\psi(g),$$
$$\varphi(l+g) = \varphi(g) \text{ and } \psi(l+g) = \psi(g) + l\varphi(g).$$

For $g = (m(k))_{k\in\mathbb{Z}} \in G$, we define a C*-subalgebra $\mathcal{A}(g)$ of $\mathbb{B}(\oplus_{\varphi(g)}\mathcal{H}) \cong \mathbb{M}_{\varphi(g)}(\mathbb{B})$ as follows:

$$\mathcal{A}(g) = \left\{ \bigoplus_{k\in\mathbb{Z}} (\overbrace{S^kTS^{*k} \oplus \cdots \oplus S^kTS^{*k}}^{m(k)}) \mid T \in \mathbb{B} \right\} + \mathbb{K}(\oplus_{\varphi(g)}\mathcal{H}),$$

where S^k (resp. $(S^*)^k$) means $(S^*)^{-k}$ (resp. S^{-k}) for a negative integer k. Let $\iota(g)$ be a injective *-homomorphism from $\mathbb{K}$ to $\mathcal{A}(g)$ which is obtained by a composition of a natural isomorphism of $\mathbb{K}$ to $\mathbb{K}(\oplus_{\varphi(g)}\mathcal{H})$ and the canonical inclusion map of $\mathbb{K}(\oplus_{\varphi(g)}\mathcal{H})$ into $\mathcal{A}(g)$. We define a surjective *-homomorphism $\pi(g)$ from $\mathcal{A}(g)$ to $\mathcal{Q}$ as follows:

$$\pi(g)(\bigoplus_{k\in\mathbb{Z}} (\overbrace{S^kTS^{*k} \oplus \cdots \oplus S^kTS^{*k}}^{m(k)}) + K) = \pi(T),$$

where $K \in \mathbb{K}(\oplus_{\varphi(g)}\mathcal{H})$. Then we have the following extension:

$$E(g) : 0 \longrightarrow \mathbb{K} \xrightarrow{\iota(g)} \mathcal{A}(g) \xrightarrow{\pi(g)} \mathcal{Q} \longrightarrow 0,$$

and its Busby invariant coincides with

$$\bigoplus_{k\in\mathbb{Z}} \bigoplus_{m(k)} \tau(k).$$

Then we have the following statement and this is our main result:

Theorem 1.1. *For $g, h \in G$ and $n \in \mathbb{N}$, we have the following:*

(1) $\tau(g) = \tau(h) \Longleftrightarrow \varphi(g) = \varphi(h)$ *and* $\psi(g) = \psi(h)$, *that is,*

$$\mathcal{G} \ni \tau(g) \longmapsto (\varphi(g), \psi(g)) \in \mathbb{N} \times \mathbb{Z}$$

gives a semigroup isomorphism from $\mathcal{G}$ *onto* $\mathbb{N} \times \mathbb{Z}$.

(2) $\mathcal{A}(g) \cong \mathcal{A}(h) \Longleftrightarrow \varphi(g) = \varphi(h)$ *and* $\psi(g) \equiv \psi(h) \bmod \varphi(g)$.

(3) $\mathcal{A}(g) \otimes \mathbb{M}_n \cong \mathcal{A}(n \cdot g)$.

We give the proof of theorem in the next section.

Corollary 1.2. *For any* $n \in \mathbb{N}$ *and* $n \geq 2$, *there exist* $g, h \in G$ *such that* $\mathcal{A}(g) \otimes \mathbb{M}_k$ *is not isomorphic to* $\mathcal{A}(h) \otimes \mathbb{M}_k$ *for any* $1 \leq k \leq n-1$ *and* $\mathcal{A}(g) \otimes \mathbb{M}_n$ *is isomorphic to* $\mathcal{A}(h) \otimes \mathbb{M}_n$.

Proof. We choose g and h such that

$$\varphi(g) = \varphi(h) = n, \ \phi(g) = 0 \text{ and } \psi(h) = 1.$$

Then we have $\psi(k \cdot g) = 0 < \psi(k \cdot h) = k < n$ for any $k = 1, 2, \ldots, n-1$ and $\varphi(n \cdot g) = \varphi(n \cdot h) = n$, $\psi(n \cdot g) \equiv \psi(n \cdot h) \equiv 0 \bmod n$. This implies that $\mathcal{A}(g)$ and $\mathcal{A}(h)$ satisfy the required property. □

For $g = (m(k))_{k \in \mathbb{Z}}, h = (n(k))_{k \in \mathbb{Z}} \in G$ with

$$m(k) = \begin{cases} 2 & k = 0 \\ 0 & \text{otherwise} \end{cases} \text{ and } n(k) = \begin{cases} 1 & k = 0, 1 \\ 0 & \text{otherwise} \end{cases},$$

we have $\varphi(g) = \varphi(h) = 2$, $\psi(g) = 0, \psi(h) = 1$. It follows that $\mathcal{A}(g) \otimes \mathbb{M}_2 \cong \mathcal{A}(h) \otimes \mathbb{M}_2$, but $\mathcal{A}(g)$ is not isomorphic to $\mathcal{A}(h)$. This example is the same one given by J. Plastiras.

2. Proof of Theorem

Lemma 2.1. *The* K_0*-group* $K_0(\mathcal{A}(g))$ *for* $\mathcal{A}(g)$ *is isomorphic to* $\mathbb{Z}/\varphi(g)\mathbb{Z}$.

Proof. Let $g = (m(k))_{k \in \mathbb{Z}} \in G$. From the short exact sequence of C*-algebras

$$0 \longrightarrow \mathbb{K} \xrightarrow{\iota(g)} \mathcal{A}(g) \xrightarrow{\pi(g)} \mathcal{Q} \longrightarrow 0,$$

we can get the exact sequence of K-groups of C*-algebras as follows:

$$\begin{array}{ccccc} K_0(\mathbb{K}) & \xrightarrow{\iota(g)_*} & K_0(\mathcal{A}(g)) & \xrightarrow{\pi(g)_*} & K_0(\mathcal{Q}) \\ {\scriptstyle\delta_1}\uparrow & & & & \downarrow \\ K_1(\mathcal{Q}) & \underset{\pi(g)_*}{\longleftarrow} & K_1(\mathcal{A}(g)) & \underset{\iota(g)_*}{\longleftarrow} & K_1(\mathbb{K}). \end{array}$$

Since $K_0(\mathcal{Q}) = \{0\}$, we have

$$K_0(\mathcal{A}(g)) \cong K_0(\mathbb{K})/\delta_1(K_1(\mathcal{Q})).$$

It is known that $K_0(\mathbb{K}) \cong K_1(\mathcal{Q}) \cong \mathbb{Z}$ and the class of P (resp. $\pi(S)$) is a generator of $K_0(\mathbb{K})$ (resp. $K_1(\mathcal{Q})$), where $P \in \mathbb{B}$ is a projection of rank one and $S \in \mathbb{B}$ is a unilateral shift.

We put $P_n = 1 - S^n S^{*n}$ $(n = 1, 2, \dots)$ and define a unitary $W(k) \in \mathbb{M}_2(\mathbb{B})$ as follows: for $k \geq 0$,

$$\begin{aligned} W(k) &= \begin{pmatrix} S(1-P_k) & P_{k+1} \\ -P_k & (1-P_k)S^* \end{pmatrix} \\ &= \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}^k \begin{pmatrix} S & P_1 \\ 0 & S^* \end{pmatrix} \begin{pmatrix} S^* & 0 \\ 0 & S^* \end{pmatrix}^k + \begin{pmatrix} 0 & P_k \\ -P_k & 0 \end{pmatrix}, \end{aligned}$$

and for $k < 0$,

$$W(k) = \begin{pmatrix} S & P_1 \\ 0 & S^* \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}^k \begin{pmatrix} S & P_1 \\ 0 & S^* \end{pmatrix} \begin{pmatrix} S^* & 0 \\ 0 & S^* \end{pmatrix}^k + \begin{pmatrix} 0 & P_1 \\ 0 & 0 \end{pmatrix}.$$

Then we have

$$W = \bigoplus_{k \in \mathbb{Z}} \overbrace{W(k) \oplus \cdots \oplus W(k)}^{m(k)}$$

is unitary in $\mathbb{M}_2(\mathcal{A}(g))$ and

$$\pi(g) \otimes \mathrm{id}_2(W) = \begin{pmatrix} \pi(S) & 0 \\ 0 & \pi(S^*) \end{pmatrix}.$$

By definition of δ_1, we have

$$\delta_1([\pi(S)]) = [W^*(1_{\mathcal{A}(g)} \oplus 0_{\mathcal{A}(g)})W] - [1_{\mathcal{A}(g)} \oplus 0_{\mathcal{A}(g)}].$$

By the calculation

$$[\begin{pmatrix} S(1-P_k) & P_{k+1} \\ -P_k & (1-P_k)S^* \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S(1-P_k) & P_{k+1} \\ -P_k & (1-P_k)S^* \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$$
$$= [\begin{pmatrix} 1-P_k & 0 \\ 0 & P_{k+1} \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] = [P]$$

and

$$[\begin{pmatrix} S & P_1 \\ 0 & S^* \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S & P_1 \\ 0 & S^* \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$$
$$=[\begin{pmatrix} 1 & 0 \\ 0 & P_1 \end{pmatrix}] - [\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}] = [P],$$

it follows that

$$\delta_1([\pi(S)]) = \varphi(g)[P].$$

This means that

$$K_0(\mathcal{A}(g)) \cong \mathbb{Z}/\varphi(g)\mathbb{Z}.$$

□

For $g = (m(k))_{k\in\mathbb{Z}}$, we can choose integers $k_1 < k_2 < \cdots < k_l$ such that

$$\{k \in \mathbb{Z} \mid m(k) \neq 0\} = \{k_1, k_2, \ldots, k_l\}.$$

We remark that, if we put

$$m_1 = \cdots = m_{m(k_1)} = k_1,\ m_{m(k_1)+1} = \cdots = m_{m(k_1)+m(k_2)} = k_2,$$
$$\ldots,\ m_{m(k_1)+\cdots+m(k_{l-1})+1} = \cdots = m_{\varphi(g)} = k_l,$$

then we have $\psi(g) = \sum_{j=1}^{\varphi(g)} m_j$ and

$$\mathcal{A}(g) = \{\bigoplus_{k\in\mathbb{Z}} (\overbrace{S^kTS^{*k} \oplus \cdots \oplus S^kTS^{*k}}^{m(k)}) \mid T \in \mathbb{B}\} + \mathbb{K}(\oplus_{\varphi(g)}\mathcal{H})$$
$$= \{\oplus_{j=1}^{\varphi(g)} S^{m_j}T(S^*)^{m_j} \mid T \in \mathbb{B}\} + \mathbb{K}(\oplus_{\varphi(g)}\mathcal{H}).$$

Lemma 2.2. *For any $n \in \mathbb{Z}$ and $g \in G$, we have*

$$\mathcal{A}(g) \cong \mathcal{A}(n+g).$$

Proof. It is sufficient to show that $\mathcal{A}(g) \cong \mathcal{A}(1+g)$. Using the above notation and $\varphi(g) = \varphi(1+g)$ and $\psi(1+g) = \psi(g) + \varphi(g)$, we have

$$\begin{aligned}\mathcal{A}(1+g) &= \{\oplus_{j=1}^{\varphi(g)} S^{m_j+1} T (S^*)^{m_j+1} \mid T \in \mathbb{B}\} + \mathbb{K}(\oplus_{\varphi(g)}\mathcal{H}) \\ &= \{\oplus_{j=1}^{\varphi(g)} S^{m_j} STS^* (S^*)^{m_j} \mid T \in \mathbb{B}\} + \mathbb{K}(\oplus_{\varphi(g)}\mathcal{H}).\end{aligned}$$

Clearly $\mathcal{A}(1+g) \subset \mathcal{A}(g)$. Remarking the fact $\mathbb{B} \subset S\mathbb{B}S^* + \mathbb{K}$, we have $\mathcal{A}(1+g) = \mathcal{A}(g)$. □

Lemma 2.3. *The class of the unit of $\mathcal{A}(g)$ is equal to $\psi(g)[P]$ in $K_0(\mathcal{A}(g))$, where P is a minimal projection of $\mathcal{A}(g)$.*

Proof. By the above lemma, we can see

$$\begin{aligned}\mathcal{A}(g) &= \{\oplus_{j=1}^{\varphi(g)} S^{m_j} T (S^*)^{m_j} \mid T \in \mathbb{B}\} + \mathbb{K}(\oplus_{\varphi(g)}\mathcal{H}) \\ &= \{\oplus_{j=1}^{\varphi(g)} S^{n+m_j} T (S^*)^{n+m_j} \mid T \in \mathbb{B}\} + \mathbb{K}(\oplus_{\varphi(g)}\mathcal{H}),\end{aligned}$$

for $n \in \mathbb{N}$ with $n+m_1 > 0$. Since $1 \in \mathbb{B}$ is equivalent to some orthogonal projections $Q_1, Q_2, \dots, Q_{\varphi(g)}$ such that $1 = Q_1 + Q_2 + \cdots + Q_{\varphi(g)}$, $S^k S^{*k}$ is equivalent to $S^k Q_i S^{*k}$ for positive integer k. So we have

$$\begin{aligned}&[1_{\mathcal{A}(g)}] \in K_0(\mathcal{A}(g)) \\ =&[\oplus_{j=1}^{\varphi(g)} S^{n+m_j}(S^*)^{n+m_j}] + \sum_{j=1}^{\varphi(g)} (n+m_j)[P] \\ =&\varphi(g)[\oplus_{j=1}^{\varphi(g)} S^{n+m_j}(S^*)^{n+m_j}] + (n\varphi(g) + \psi(g))[P]\end{aligned}$$

This implies $[1_{\mathcal{A}(g)}] = \psi(g)[P]$ in $K_0(\mathcal{A}(g))$. □

Lemma 2.4. *The commutant $(\mathcal{Q} \otimes 1_n)'$ of $\mathcal{Q} \otimes 1_n$ in $\mathcal{Q} \otimes \mathbb{M}_n$ coincides with $1_{\mathcal{Q}} \otimes \mathbb{M}_n$. In particular, any unitary element in $(\mathcal{Q} \otimes 1_n)'$ has a unitary lift in $\mathbb{B} \otimes \mathbb{M}_n$.*

Proof. It is sufficient to show that $\mathcal{Q}' \cap \mathcal{Q} = \mathbb{C}1_{\mathcal{Q}}$. This fact is well known as the following form:

$$\{T \in \mathbb{B} \mid TX - XT \in \mathbb{K} \text{ for all } X \in \mathbb{B}\} = \mathbb{C}1 + \mathbb{K}.$$

For the convenience of readers, we give its proof.

We denote by $\mathrm{EC}(\mathbb{B})$ the essential commutant for $\mathbb{B}$

$$\{T \in \mathbb{B} \mid TX - XT \in \mathbb{K} \text{ for all } X \in \mathbb{B}\}.$$

It is trivial that $\mathrm{EC}(\mathbb{B}) \supset \mathbb{C}1 + \mathbb{K}$. So we have to show that the reverse inclusion holds. Since $\mathrm{EC}(\mathbb{B})$ is a closed *-subalgebra of $\mathbb{B}$, any element in $\mathrm{EC}(\mathbb{B})$ is represented by a linear combination of self-adjoint elements. Let T be a self-adjoint element in $\mathrm{EC}(\mathbb{B})$ and its spectral decomposition

$$T = \int_{-||T||}^{||T||} \lambda de(\lambda),$$

where $\{e(\lambda)\}$ is the right continuous spectral family of projections for T.

For $-||T|| < a < b < ||T||$, we assume that two projections

$$\int_{-||T||}^{a} de(\lambda) \text{ and } \int_{b}^{||T||} de(\lambda)$$

are infinitely dimensional. Since $\mathcal{H}$ is separable ($\mathbb{B} = \mathbb{B}(\mathcal{H})$), there exists a partial isometry V such that

$$V^*V = \int_{b}^{||T||} de(\lambda), \quad VV^* = \int_{-||T||}^{a} de(\lambda).$$

Then we have

$$\begin{aligned}(VT - TV)V^* &= V\int_{b}^{||T||} \lambda de(\lambda)V^* - \int_{-||T||}^{a} \lambda de(\lambda) \\ &\geq b\int_{-||T||}^{a} de(\lambda) - \int_{-||T||}^{a} \lambda de(\lambda) \\ &\geq (b-a)\int_{-||T||}^{a} de(\lambda) \notin \mathbb{K}.\end{aligned}$$

This means that $VT - TV \notin \mathbb{K}$, i.e., $T \notin EC(\mathbb{B})$.

This fact implies that $\sigma(T)$ has at most one accumulation point. If an accumulation point c exists, then each $\lambda \in \sigma(T)\backslash\{c\}$ is an eigenvalue for T and its eigenprojection is finite dimensional. So we have

$$T - c1 \in \mathbb{K}.$$

If an accumulation point does not exist, then $\sigma(T)$ is a finite set of eigenvalues for T and their eigenprojections are finite dimensional except for one point c. Also we have

$$T - c1 \in \mathbb{K}.$$

□

Proof of Theorem 1.1. (1) First we assume that $\tau_g = \tau_h$. Then the fact $\mathcal{A}(g) \cong \mathcal{A}(h)$ implies $\varphi(g) = \varphi(h)$ by lemma2.1. We use the notation $\tau(g) = [\oplus_{k\in\mathbb{Z}} \oplus_{m(k)} \tau_k]$, $\tau(h) = [\oplus_{k\in\mathbb{Z}} \oplus_{n(k)} \tau_k]$ and $S_g = \oplus_{k\in\mathbb{Z}} \oplus_{m(k)} \pi(S)^k$, $S_h = \oplus_{k\in\mathbb{Z}} \oplus_{n(k)} \pi(S)^k \in \mathcal{Q} \otimes \mathbb{M}_{\varphi(g)}$. Then $\tau(g) = \tau(h)$ means that there exist a unitary U in $\mathbb{B} \otimes \mathbb{M}_{\varphi(g)}$ such that

$$S_g x \otimes 1_{\varphi(g)} S_g^* = \pi \otimes id_{\varphi(g)}(U)^* S_h x \otimes 1_{\varphi(g)} S_h^* \pi \otimes id_{\varphi(g)}(U)$$

for all $x \in \mathcal{Q}$. Using lemma2.4, we have $S_h^*(\pi \otimes id_{\varphi(g)})(U) S_g \in 1_{\mathcal{Q}} \otimes \mathbb{M}_{\varphi(g)}$. So $S_h^*(\pi \otimes id_{\varphi(g)})(U) S_g$ have a unitary lift in $1_{\mathbb{B}} \otimes \mathbb{M}_{\varphi(g)}$. This means $0 = \mathrm{Index}(\oplus_{k\in\mathbb{Z}} \oplus_{n(k)} S^k)^* U (\oplus_{k\in\mathbb{Z}} \oplus_{m(k)} S^k) = -\psi(h) + \psi(g)$, that is, $\psi(g) = \psi(h)$. Conversely we assume that $\varphi(g) = \varphi(h)$ and $\psi(g) = \psi(h)$. Then we have $\mathrm{Index}(\oplus_{k\in\mathbb{Z}} \oplus_{n(k)} S^k)(\oplus_{k\in\mathbb{Z}} \oplus_{m(k)} S^k)^* = 0$. So there exists a unitary U in $\mathbb{B} \otimes \mathbb{M}_{\varphi(g)}$ such that $S_h S_g^* = \pi \otimes id_{\varphi(g)}(U)$. This implies that $\tau(g) = \tau(h)$.

(2) First we assume that $\mathcal{A}_g \cong \mathcal{A}_h$. By lemma2.1 and lemma2.3, it is immediately found that $\varphi(g) = \varphi(h)$ and $\psi(g) \equiv \psi(h) \mod \varphi(g)$. Conversely we assume that $\varphi(g) = \varphi(h)$ and $\psi(g) = \psi(h) + n\varphi(g)$ for $n \in \mathbb{Z}$. Then we have $\tau(g) = \tau(n+g)$. This implies $\mathcal{A}(g) \cong \mathcal{A}(n+g) \cong \mathcal{A}(h)$ by lemma2.2.

(3) Suppose that $\mathcal{A}(g)$ is the following form:

$$\mathcal{A}(g) = \left\{ \oplus_{k\in\mathbb{Z}} \oplus_{m(k)} S^k T S^{*k} \mid T \in \mathbb{B} \right\} + \mathbb{K}(\oplus_{\varphi(g)} \mathcal{H}).$$

Therefore we can regard $\mathcal{A}(g) \otimes \mathbb{M}_n$ as the following:

$$\mathcal{A}(g) \otimes \mathbb{M}_n = \Big\{ \oplus_{k\in\mathbb{Z}} \oplus_{m(k)} (S^k \otimes \mathbf{1}_n) T' (S^{*k} \otimes \mathbf{1}_n) | \\ T' \in \mathbb{B}(\oplus_n \mathcal{H}) \Big\} + \mathbb{K}(\oplus_{n\varphi(g)} \mathcal{H}).$$

This means that $\mathcal{A}(g) \otimes \mathbb{M}_n \cong \mathcal{A}(n \cdot g)$. □

For $g \in G$, we define the C*-algebra

$$\mathcal{A}(g) = \left\{ \oplus_{k\in\mathbb{Z}} \oplus_{m(k)} S^k T S^{*k} \mid T \in \mathbb{B}(\mathcal{H}) \right\} + \mathbb{K}(\oplus_{\varphi(g)} \mathcal{H}).$$

Then we can see that the essential commutant $\mathrm{EC}(\mathcal{A}(g))$ of $\mathcal{A}(g)$ becomes an AF-algebra and $\pi \otimes id_{\varphi(g)}(\mathrm{EC}(\mathcal{A}(g))$ is isomorphic to $\mathbb{M}_{\varphi(g)}(\mathbb{C})$. Since $\mathcal{A}(g)$ and $\mathcal{A}(h)$ contain the algebra of compact operators, the isomorphism from $\mathcal{A}(g)$ to $\mathcal{A}(h)$ deduces the isomorphism from $\mathrm{EC}(\mathcal{A}(g))$ to $\mathrm{EC}(\mathcal{A}(h))$. It is known that isomorphism classes of AF-algebras are classified up by the K-theoretic datum. In this case, we can see

$$(K_0(\mathrm{EC}(\mathcal{A}(g))),\ K_0(\mathrm{EC}(\mathcal{A}(g)))_+,\ [1]_{K_0}) \\ = (\mathbb{Z} \oplus \mathbb{Z},\ (\{0\} \oplus \mathbb{Z}_{\geq 0}) \cup (\mathbb{Z}_{>0} \oplus \mathbb{Z}),\ (\varphi(g), \psi(g))).$$

We remark that, for any integer $k \in \mathbb{Z}$, the following groups are order isomorphic (and preserving the order unit):

$$(\mathbb{Z} \oplus \mathbb{Z},\ (\{0\} \oplus \mathbb{Z}_{\geq 0}) \cup (\mathbb{Z}_{>0} \oplus \mathbb{Z}),\ (\varphi(g), \psi(g))),$$
$$(\mathbb{Z} \oplus \mathbb{Z},\ (\{0\} \oplus \mathbb{Z}_{\geq 0}) \cup (\mathbb{Z}_{>0} \oplus \mathbb{Z}),\ (\varphi(g), \psi(g) + k\varphi(g))).$$

This means that the K-theoretic datum is complete invariant for a family $\{\mathcal{A}(g) | g \in G\}$ of non-nuclear C*-algebras.

Acknowledgement. The authors would like to express their thanks to Professor K. Kodaka for the meaningful discussion using the Space Collaboration System of National Institute of Multimedia Education in Japan.

References

[1] B. Blackadar, *K-theory for operator algebras, Second edition,* M. S. R. I. Publications 5. Cambridge University Press., (1998).
[2] J. Cuntz, *Simple C*-algebras generated by isometries,* Comm. Math. Phys. 57 (1977), pp. 173–185.
[3] M. Enomoto, M. Fujii and Y. Watatani, *K_0-groups and classifications of Cuntz-Krieger algebras,* Math. Japon. 26 (1981), pp. 443–460.
[4] Y. Katabami *Isomorphisms of C*-algebras after tensoring,* Technical Reports of Mathematical Sciences, Chiba University, vol 17 (2001), No. 2.
[5] J. Plastiras, *C*-algebras isomorphic after tensoring,* Proc. Amer. Math. Soc. 66 (1977), pp. 276–278.
[6] B. V. Rajarama Bhat, G. A. Elliott and P. A. Fillmore, *Lectures on Operator Theory, Fields Institute Monographs 13*, Amer. Math. Soc., (1999).
[7] T. Sakamoto, *Certain reduced free products with amalgamation of C*-algebras,* Scientiae Math. 3 (2000), pp. 37–48.
[8] N. E. Wegge-Olsen, *K-theory and C*-algebras,* Oxford. (1994).

*Department of Mathematics and Information, Graduate School of Science and Technology Chiba University, 1-33, Yayoi-Cho, Inage-Ku, Chiba 263-8522 Japan

E-mail address: 99um0102@g.math.s.chiba-u.ac.jp

E-mail address: nagisa@math.s.chiba-u.ac.jp