Hilbert Coefficients and Buchsbaumness of Associated Graded Rings

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Abstract

Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and I an $\mathfrak{m}$-primary ideal. The purpose of this paper is to generalize Northcott's inequality on Hilbert coefficients of I given in [8], without assuming that A is a Cohen-Macaulay ring. We will investigate when our inequality turns into an equality. It is related to the Buchsbaumness of the associated graded ring of I.

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1 Introduction

Let A be a d-dimensional Noetherian local ring with the maximal ideal $\mathfrak{m}$ and I an $\mathfrak{m}$-primary ideal of A. Then there exist integers $\mathrm{e}_0(I), \mathrm{e}_1(I), \cdots, \mathrm{e}_d(I)$ such that

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d\mathrm{e}_d(I)$$

for $n \gg 0$. These integers are called the Hilbert coefficients of I and a lot of results are known on them in the case where A is a Cohen-Macaulay ring. For example, as was proved by Northcott [8], we always have $\mathrm{e}_0(I) - \ell_A(A/I) \leq \mathrm{e}_1(I)$. Moreover, provided $A/\mathfrak{m}$ is infinite, Huneke and Ooishi [6, 9] proved that $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I)$ if and only if $I^2 = QI$ for some (any) minimal reduction Q of I, and when this is the case, by [11], the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a Cohen-Macaulay ring. The purpose of this paper is to extend their results without assuming that A is a Cohen-Macaulay ring.

Suppose that I contains a parameter ideal Q as a reduction. Then, from Northcott's inequality, one can easily deduce that $\mathrm{e}_0(I) - \ell_A(A/I) \leq \mathrm{e}_1(I) - \mathrm{e}_1(Q)$ (See 3.1). Assuming that Q is a standard ideal in the sense of [10, Definition 19 of Appendix], we will investigate

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when the equality $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I) - \mathrm{e}_1(Q)$ holds. In order to state our result, let us fix some notation. For an ideal $\mathfrak{q}$ of A which is minimally generated by $a_1, \cdots, a_s$, we set

$$\Sigma(\mathfrak{q}) = \mathfrak{q} + \sum_{i=1}^{s} [(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_s) :_A a_i] .$$

It is easy to see that $\Sigma(\mathfrak{q})$ does not depend on the choice of the minimal system of generators. For a module M over a ring R, we denote by $\mathrm{H}^i_{\mathfrak{a}}(M)$ the i-th local cohomology module of M with respect to $\mathfrak{a}$. In particular, we set $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. Then we have the following.

Theorem 1.1 *Suppose that I contains a standard parameter ideal Q as a reduction. Then* $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I) - \mathrm{e}_1(Q)$ *if and only if* $I^2 \subseteq QI + W$ *and* $\Sigma(Q) \subseteq I$.

If the length of $\mathrm{H}^i_{\mathfrak{m}}(A)$, which is denoted by $\mathrm{h}^i(A)$, is finite for any $0 \leq i < d$, we have that

$$-\mathrm{e}_1(Q) \leq \sum_{i=0}^{d-1} \binom{d-2}{i-1} \mathrm{h}^i(A)$$

with equality when Q is a standard ideal (See 2.4). Therefore, as a consequence of 1.1 and [3], we get the next result.

Corollary 1.2 *If A is a quasi-Buchsbaum ring, then*

$$\sup_{\sqrt{I}=\mathfrak{m}} \{\mathrm{e}_0(I) - \ell_A(A/I) - \mathrm{e}_1(I)\} = \sum_{i=0}^{d-1} \binom{d-2}{i-1} \mathrm{h}^i(A) .$$

Moreover, assuming that A is a Buchsbaum ring or a slightly different condition, for ideals I which enjoy the property stated in 1.1, we will study the Buchsbaumness of $\mathrm{G}(I)$ together with $\mathrm{I}(\mathrm{G}(I))$ and $\mathrm{a}(\mathrm{G}(I))$, where $\mathrm{I}(*)$ and $\mathrm{a}(*)$ denote the I-invariant (cf. [10, p. 254]) and a-invariant (cf. [4]) respectively.

Theorem 1.3 *Suppose that either (i) A is a Buchsbaum ring or (ii) A is a quasi-Buchsbaum ring and $I \subseteq \mathfrak{m}^2$. If I contains a parameter ideal Q such that $I^2 \subseteq QI + W$ and $\Sigma(Q) \subseteq I$, then $\mathrm{G}(I)$ is a Buchsbaum ring with $\mathrm{I}(\mathrm{G}(I)) = \mathrm{I}(A)$ and $\mathrm{a}(\mathrm{G}(I)) \leq 2 - d$.*

Throughout this paper $(A, \mathfrak{m})$ denotes a commutative Noetherian local ring with $d = \dim A > 0$ and I an $\mathfrak{m}$-primary ideal of A. The Rees algebra $\mathrm{R}(\mathfrak{a})$ of an ideal $\mathfrak{a}$ of a ring R is the subring $R[It]$ of $R[t]$, where t is an indeterminate. The associated graded ring $\mathrm{G}(\mathfrak{a})$ is the quotient ring $\mathrm{R}(\mathfrak{a})/\mathfrak{a}\mathrm{R}(\mathfrak{a})$. For $f \in \mathrm{R}(\mathfrak{a})$, we denote it's image in $\mathrm{G}(\mathfrak{a})$ by $\overline{f}$.

2 Preliminaries

We begin with the following result of one dimensional case.

Lemma 2.1 *Let $d = 1$. If I contains a parameter ideal Q as a reduction, then we have that* $\mathrm{e}_0(I) - \ell_A(A/I) \leq \mathrm{e}_1(I) + \ell_A(I \cap W)$ *with equality if and only if* $I^2 \subseteq QI + W$.

Proof. Let $B = A/W$. Then B is a Cohen-Macaulay ring with $\dim B = 1$ and QB is a parameter ideal of B contained in IB as a reduction. Hence, by Northcott's inequality and the result of Huneke and Ooishi stated in Introduction, we have that $\mathrm{e}_0(IB) - \ell_B(B/IB) \leq \mathrm{e}_1(IB)$ with equality if and only if $I^2B = QIB$. On the other hand, as $\ell_B(B/I^{n+1}B) = \ell_A(A/I^{n+1}) - \ell_A(W)$ for $n \gg 0$, we have $\mathrm{e}_0(IB) = \mathrm{e}_0(I)$ and $\mathrm{e}_1(IB) = \mathrm{e}_1(I) + \ell_A(W)$. Moreover, $\ell_B(B/IB) = \ell_A(A/I) - \ell_A(W) + \ell_A(I \cap W)$. Therefore we get the required assertion as $I^2B = QIB$ if and only if $I^2 \subseteq QI + W$.

When we investigate higher dimensional case, we reduce the dimension using a superficial element (cf. [7, Section 22]), and the next result, which may be well known, plays a key role.

Lemma 2.2 *Let $d \geq 2$ and a be a superficial element of I. We set $B = A/aA$. Then* $\dim B = d - 1$ *and*

$$\mathrm{e}_i(IB) = \begin{cases} \mathrm{e}_i(I) & \text{if } 0 \leq i < d-1 \\ \mathrm{e}_{d-1}(I) + (-1)^{d-1}\ell_A(0 :_A a) & \text{if } i = d-1\,. \end{cases}$$

Proof. Let $n \gg 0$. Then $I^{n+1} \cap aA = aI^n$ and $I^n \cap (0 :_A a) = 0$. Hence we have an exact sequence

$$0 \longrightarrow 0 :_A a \longrightarrow A/I^n \xrightarrow{a} (aA + I^{n+1})/I^{n+1} \longrightarrow 0\,,$$

so that

$$\begin{aligned} & \ell_B(B/I^{n+1}B) \\ = \ & \ell_A(A/I^{n+1}) - \ell_A(A/I^n) + \ell_A(0 :_A a) \\ = \ & \sum_{i=0}^{d}(-1)^i\mathrm{e}_i(I)\binom{n+d-i}{d-i} - \sum_{i=0}^{d}(-1)^i\mathrm{e}_i(I)\binom{n-1+d-i}{d-i} + \ell_A(0 :_A a) \\ = \ & \sum_{i=0}^{d-2}(-1)^i\mathrm{e}_i(I)\binom{n+d-1-i}{d-1-i} + (-1)^{d-1}\{\mathrm{e}_{d-1}(I) + (-1)^{d-1}\ell_A(0 :_A a)\}\,. \end{aligned}$$

Thus we get the required assertion.

Lemma 2.3 *Suppose that $A/\mathfrak{m}$ is infinite and J is a reduction of I. Then there exists an element $a \in J$ which is superficial for both of I and J. Moreover, for such element $a \in J$, setting $B = A/aA$, we have $\mathrm{e}_1(I) - \mathrm{e}_1(J) = \mathrm{e}_1(IB) - \mathrm{e}_1(JB)$ provided $d \geq 2$.*

Proof. By taking a general linear form in $\mathrm{G}(J)/\mathfrak{m}\mathrm{G}(J)$, we see the existence of $a \in J$ satisfying the required condition. If $d \geq 3$, we get the equality since $\mathrm{e}_1(IB) = \mathrm{e}_1(I)$ and $\mathrm{e}_1(JB) = \mathrm{e}_1(J)$. Even if $d = 2$, we have

$$\begin{aligned}\mathrm{e}_1(IB) - \mathrm{e}_1(JB) &= \{\mathrm{e}_1(I) - \ell_A(0 :_A a)\} - \{\mathrm{e}_1(J) - \ell_A(0 :_A a)\} \\ &= \mathrm{e}_1(I) - \mathrm{e}_1(J)\,.\end{aligned}$$

Lemma 2.4 *Let Q be a parameter ideal of A. We have the following statements provided $\mathrm{h}^i(A)$ is finite for any $0 \leq i < d$.*

(1) *Let $d = 1$. Then $-\mathrm{e}_1(Q) = \mathrm{h}^0(A)$.*

(2) *Let $d \geq 2$. Then we have that*

$$-\mathrm{e}_1(Q) \leq \sum_{i=1}^{d-1} \binom{d-2}{i-1} \mathrm{h}^i(A)$$

with equality if Q is a standard ideal.

Proof. Let $d = 1$. Then, taking $n \gg 0$ such that $W = 0 :_A Q^n$ and $\ell_A(A/Q^n) = \mathrm{e}_0(Q)\cdot n - \mathrm{e}_1(Q)$, we see that $-\mathrm{e}_1(Q) = \ell_A(W)$ since $\mathrm{e}_0(Q)\cdot n = \mathrm{e}_0(Q^n) = \ell_A(A/Q^n) - \ell_A(0 :_A Q^n)$. Thus we get the assertion (1).

Next we assume that $d \geq 2$. Moreover, in order to prove the assertion (2), we may assume that $A/\mathfrak{m}$ is infinite. Then we can choose $a \in Q \setminus \mathfrak{m}Q$ which is a superficial element of Q. Let $B = A/aA$ and $0 \leq i < d-1$. Considering the exact sequence

$$0 \longrightarrow 0 :_A a \longrightarrow A \stackrel{a}{\longrightarrow} A \longrightarrow B \longrightarrow 0\,,$$

we get the exact sequence

$$(\#) \quad \mathrm{H}^i_{\mathfrak{m}}(A) \stackrel{a}{\longrightarrow} \mathrm{H}^i_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(B) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \stackrel{a}{\longrightarrow} \mathrm{H}^{i+1}_{\mathfrak{m}}(A)\,.$$

Hence it follows that $\mathrm{h}^i(B) \leq \mathrm{h}^i(A) + \mathrm{h}^{i+1}(A)$ with equality when Q is a standard ideal.

Let $d = 2$. Then $-\mathrm{e}_1(Q) = -\{\mathrm{e}_1(QB) + \ell_A(0 :_A a)\} = \mathrm{h}^0(B) - \ell_A(0 :_A a)$. Because the exact sequence $(\#)$ implies $\mathrm{h}^0(B) \leq \ell_A(0 :_W a) + \mathrm{h}^1(A)$, we have $-\mathrm{e}_1(Q) \leq \mathrm{h}^1(A)$.

Furthermore, if Q is standard, then $\mathrm{h}^0(B) = \mathrm{h}^0(A) + \mathrm{h}^1(A)$ and $0 :_A a = W$, so that $-\mathrm{e}_1(Q) = \mathrm{h}^1(A)$.

Let $d \geq 3$. Then $\mathrm{e}_1(QB) = \mathrm{e}_1(Q)$. Hence we can easily verify the assertion (2) by induction on d.

3 General case

As a result in general case, we give the following assertion, which is a generalization of Northcott's inequality.

Theorem 3.1 *If I contains a parameter ideal Q as a reduction, then* $\mathrm{e}_0(I) - \ell_A(A/I) \leq \mathrm{e}_1(I) - \mathrm{e}_1(Q)$.

Proof. We prove by induction on d. If $d = 1$, the assertion follows from 2.1 and 2.4. Suppose that $d \geq 2$. We may assume that $A/\mathfrak{m}$ is infinite, so that there exists $a \in Q \setminus \mathfrak{m}Q$ which is superficial for both of I and Q. Then, setting $B = A/aA$, we have

$$\begin{aligned} \mathrm{e}_0(I) - \ell_A(A/I) &= \mathrm{e}_0(IB) - \ell_B(B/IB) \quad \text{by 2.2} \\ &\leq \mathrm{e}_1(IB) - \mathrm{e}_1(QB) \quad \text{by the inductive hypothesis} \\ &= \mathrm{e}_1(I) - \mathrm{e}_1(Q) \quad \text{by 2.3}. \end{aligned}$$

Thus we get the required inequality.

The next result gives a sufficient condition under which the inequality of 3.1 turns into an equality in the case where $I = \mathfrak{m}$.

Proposition 3.2 *Let Q be a parameter ideal which is a reduction of $\mathfrak{m}$. If there exists an ideal V of A such that $\dim_A V < d$ and $\mathfrak{m}^2 \subseteq Q\mathfrak{m} + V$, then* $\mathrm{e}_0(\mathfrak{m}) - 1 = \mathrm{e}_1(\mathfrak{m}) - \mathrm{e}_1(Q)$.

Proof. We prove by induction on d. If $d = 1$, then $V \subseteq W \subseteq \mathfrak{m}$, so that by 2.1 we have $\mathrm{e}_0(\mathfrak{m}) - 1 = \mathrm{e}_1(\mathfrak{m}) + \ell_A(W)$, which yields the required equality since $-\mathrm{e}_1(Q) = \ell_A(W)$ by 2.4. Suppose that $d \geq 2$. As we may assume that $A/\mathfrak{m}$ is infinite, it is possible to take an element $a \in Q \setminus \mathfrak{m}Q$ such that $\dim_A V/aV < d - 1$ and a is a superficial element for both of $\mathfrak{m}$ and Q. Let $B = A/aA$. Then $\dim_B VB < \dim B$ as VB is a homomorphic image of V/aV, so that by the inductive hypothesis we have $\mathrm{e}_0(\mathfrak{m}B) - 1 = \mathrm{e}_1(\mathfrak{m}B) - \mathrm{e}_1(QB)$, from which the required equality follows since $\mathrm{e}_0(\mathfrak{m}B) = \mathrm{e}_0(\mathfrak{m})$ and $\mathrm{e}_1(\mathfrak{m}B) - \mathrm{e}_1(QB) = \mathrm{e}_1(\mathfrak{m}) - \mathrm{e}_1(Q)$.

Corollary 3.3 *Let Q be a parameter ideal which is a reduction of $\mathfrak{m}$. Then* $\mathrm{e}_0(\mathfrak{m}) = 1$ *if and only if* $\mathrm{e}_1(\mathfrak{m}) = \mathrm{e}_1(Q)$.

Proof. Because $0 \leq \mathrm{e}_0(\mathfrak{m}) - 1 = \mathrm{e}_0(\mathfrak{m}) - \ell_A(A/\mathfrak{m}) \leq \mathrm{e}_1(\mathfrak{m}) - \mathrm{e}_1(Q)$, we get $\mathrm{e}_0(\mathfrak{m}) = 1$ if $\mathrm{e}_1(\mathfrak{m}) = \mathrm{e}_1(Q)$. In order to prove the converse implication, we may assume that A is complete. Now suppose that $\mathrm{e}_0(\mathfrak{m}) = 1$. Let $\mathfrak{a}(\mathfrak{p})$ be the $\mathfrak{p}$-primary component of a primary decomposition of 0. We set $V = \bigcap_{\mathfrak{p} \in \mathrm{Assh}\, A} \mathfrak{a}(\mathfrak{p})$, where $\mathrm{Assh}\, A$ denotes the set of associated primes of A whose coheight is d, and $B = A/V$. Then $\dim_A V < d$ and $\mathrm{e}_0(\mathfrak{m}B) = \mathrm{e}_0(\mathfrak{m}) = 1$, which implies that B is a regular local ring. Hence we have $\mathfrak{m} = Q + V$, so that $\mathfrak{m}^2 \subseteq Q\mathfrak{m} + V$. Therefore, by 3.2 it follows that $\mathrm{e}_1(\mathfrak{m}) = \mathrm{e}_1(Q)$.

4 The case where Q is a standard ideal

Lemma 4.1 *Let $d \geq 2$ and $Q = (a_1, a_2, \cdots, a_d)$ be a standard parameter ideal of A. We set $a = a_1, b = a_d, J = (a_1, a_2, \cdots, a_{d-1})$ and $K = (a_2, a_3, \cdots, a_d)$. Then we have the following.*

(1) $aJ :_A b^2 = aJ :_A b$.

(2) $aJ \cap bA \subseteq aJI$ *provided* $\Sigma(Q) \subseteq I$.

(3) $I^2 \subseteq QI + W$ *provided* $\Sigma(Q) \subseteq I$, $I^2 \subseteq JI + [bA :_A a]$ *and* $I^2 \subseteq KI + [aA :_A b]$.

Proof. (1) Let us take any $x \in aJ :_A b^2$ and write $b^2x = ay$, with $y \in J$. Then, as $y \in [b^2A :_A a] \cap (b^2, a_1, \cdots, a_{d-1})$, there exists $z \in A$ such that $y = b^2z$. Here we notice that $bz \in J$ since $z \in J :_A b^2 = J :_A b$. On the other hand, as $b^2x = ab^2z$, we have $bx - abz \in [0 :_A b] \cap bA = 0$, so that $bx = a \cdot bz \in aJ$. Thus we get $aJ :_A b^2 \subseteq aJ :_A b$ and the converse inclusion is obvious.

(2) Let us take any $\xi \in aJ \cap bA$ and write $\xi = ay = bz$, with $y \in J$ and $z \in A$. Moreover, we write $y = a_1y_1 + \cdots + a_{d-1}y_{d-1}$, with $y_1, \cdots, y_{d-1} \in A$. It is enough to show $y_i \in I$ for any $1 \leq i \leq d-1$. However, as $y_1 \in K :_A a^2 = K :_A a \subseteq \Sigma(Q) \subseteq I$, we may consider only the case that $d \geq 3$ and $2 \leq i \leq d-1$. Because $ay_1 \in K$, we can express $ay_1 = a_2z_2 + \cdots + a_dz_d$, with $z_2, \cdots, z_d \in A$. Then $z_i \in (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) :_A a_i \subseteq I$ for any $2 \leq i \leq d$. On the other hand, as

$$\begin{aligned} bz &= a(a_2z_2 + \cdots + a_dz_d) + aa_2y_2 + \cdots + aa_{d-1}y_{d-1} \\ &= aa_2(y_2 + z_2) + \cdots + aa_{d-1}(y_{d-1} + z_{d-1}) + aa_dz_d\,, \end{aligned}$$

it follows that

$$\begin{aligned} y_i + z_i &\in (a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) :_A aa_i \\ &\subseteq (a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) :_A a_i \subseteq I \end{aligned}$$

for $2 \leq i \leq d-1$, and hence we get $y_i \in I$.

(3) It is enough to show $[aA :_A b] \cap I^2 \subseteq JI + W$. Let us take any $x \in [aA :_A b] \cap I^2$. Then, $bx = ay$ for some $y \in A$, and $ax = a\xi + bz$ for some $\xi \in JI$ and $z \in A$. From these equalities we get $a^2 y = ab\xi + b^2 z$. Hence $z \in aJ :_A b^2 = aJ :_A b$, so that $bz = a\eta$ for some $\eta \in JI$. Then it follows that $ax = a\xi + a\eta$, which implies $x - \xi - \eta \in 0 :_A a = W$. Thus we have $x \in JI + W$ and the proof is completed.

Proof of Theorem 1.1. We prove by induction on d. By 2.1 and 2.4 we get the assertion when $d = 1$. Suppose that $d \geq 2$. As we may assume that $A/\mathfrak{m}$ is infinite, it is possible to choose a minimal system of generators $a_1, \cdots, a_d$ of Q such that a_1 and a_d are superficial for both of I and Q. We set $a = a_1, b = a_d, B = A/aA, J = (a_1, \cdots, a_{d-1}), K = (a_2, \cdots, a_d)$ and $Q_i = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d)$ for $1 \leq i \leq d$. Because $\mathrm{e}_0(I) = \mathrm{e}_0(IB), \ell_A(A/I) = \ell_B(B/IB)$ and $\mathrm{e}_1(I) - \mathrm{e}_1(Q) = \mathrm{e}_1(IB) - \mathrm{e}_1(KB)$, by the inductive hypothesis we have $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I) - \mathrm{e}_1(Q)$ if and only if $I^2 B \subseteq KIB + \mathrm{H}^0_{\mathfrak{m}}(B)$ and $\Sigma(KB) \subseteq IB$, which holds if $I^2 \subseteq QI + W$ and $\Sigma(Q) \subseteq I$ since $WB \subseteq \mathrm{H}^0_{\mathfrak{m}}(B)$ and $\Sigma(KB) \subseteq \Sigma(Q)B$. Now we assume that $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I) - \mathrm{e}_1(Q)$. Then it follows that $I^2 \subseteq KI + [aA :_A b]$ and $Q_i :_A a_i \subseteq I$ for $2 \leq i \leq d$. Moreover, by passing A/bA we get $I^2 \subseteq JI + [bA :_A a]$ and $Q_i :_A a_i \subseteq I$ for $1 \leq i \leq d-1$. Therefore, as $\Sigma(Q) \subseteq I$, we have $I^2 \subseteq QI + W$ by 4.1 and the proof is completed.

Proof of Corollary 1.2. We may assume that $A/\mathfrak{m}$ is infinite. Then any ideal of A has a minimal reduction, so that by 2.4 and 3.1 we have

$$\mathrm{e}_0(I) - \ell_A(A/I) - \mathrm{e}_1(I) \leq \sum_{i=1}^{d-2} \binom{d-2}{i-1} \mathrm{h}^i(A)$$

for any $\mathfrak{m}$-primary ideal I. Hence it is enough to find an $\mathfrak{m}$-primary ideal for which the equality holds. Let $x_1, \cdots, x_d$ be an sop for A contained in $\mathfrak{m}^2$ and $n_1, \cdots, n_d$ be integers not less than 2. We set $Q = ({x_1}^{n_1}, \cdots, {x_d}^{n_d})$ and $I = Q :_A \mathfrak{m}$. Then Q is a standard parameter ideal by [10, Proposition 2.1] and $I^2 = QI$ by [3]. Because we obviously have $\Sigma(Q) \subseteq I$, by 1.1 and 2.4 it follows that

$$\mathrm{e}_0(I) - \ell_A(A/I) - \mathrm{e}_1(I) = \sum_{i=1}^{d-2} \binom{d-2}{i-1} \mathrm{h}^i(A)\,,$$

and the proof is completed.

Example 4.2 *Let $R = k[[X,Y,Z,W]]$ be the formal power series ring with variables X,Y,Z and W over an infinite field k. Let $\mathfrak{a} = (X^2,Y)R$, $\mathfrak{b} = (Z,W)R$ and $A = R/\mathfrak{a}\cap\mathfrak{b}$. Let x,y,z and w respectively denote the images of X,Y,Z and W in A. We set $Q = (x-z, y-w)A$ and $\mathfrak{m} = (x,y,z,w)A$. Then we have the following assertion.*

(1) $\dim A = 2, \operatorname{depth} A = 1, \mathrm{h}^1(A) = 2$ *and A is not a quasi-Buchsbaum ring.*

(2) $\mathfrak{m}^3 = Q\mathfrak{m}^2$, *but* $\mathfrak{m}^2 \neq Q\mathfrak{m}$.

(3) *If V is an ideal of A with* $\dim_A V < 2$, *then $V = 0$, so that* $\mathfrak{m}^2 \not\subseteq Q\mathfrak{m} + V$.

(4) $\mathrm{e}_0(\mathfrak{m}) = 3, \mathrm{e}_1(\mathfrak{m}) = 1$ *and* $\mathrm{e}_1(Q) = -1$, *so that* $\mathrm{e}_0(\mathfrak{m}) - \ell_A(A/\mathfrak{m}) = \mathrm{e}_1(\mathfrak{m}) - \mathrm{e}_1(Q)$.

Proof. From the exact sequence $0 \longrightarrow A \longrightarrow R/\mathfrak{a} \oplus R/\mathfrak{b} \longrightarrow R/\mathfrak{a}+\mathfrak{b} \longrightarrow 0$, we get the assertion (1). One can directly check the assertion (2). Because $\dim A/\mathfrak{p} = 2$ for any $\mathfrak{p} \in \operatorname{Ass} A$, we have the assertion (3). The associated graded ring $\mathrm{G}(\mathfrak{m})$ of $\mathfrak{m}$ is isomorphic to

$$k[X,Y,Z,W]/(X^2,Y)\cap(Z,W),$$

so that we have the exact sequence

$$0 \longrightarrow \mathrm{G}(\mathfrak{m}) \longrightarrow k[X,Z,W]/(X^2) \oplus k[X,Y] \longrightarrow k[X]/(X^2) \longrightarrow 0.$$

This implies that the Poincaré series $\mathrm{P}(\mathrm{G}(\mathfrak{m}),\lambda)$ of $\mathrm{G}(\mathfrak{m})$ is

$$\frac{1+\lambda}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} - (1+\lambda),$$

from which it follows that

$$\ell_A(A/\mathfrak{m}^{n+1}) = \frac{3}{2}n^2 + \frac{7}{2}n$$

for $n \geq 2$. Hence $\mathrm{e}_0(\mathfrak{m}) = 3$ and $\mathrm{e}_1(\mathfrak{m}) = 1$. Because k is infinite, there exists $\mu \in k$ such that $c = (x-z) + \mu(y-w)$ is a superficial element of Q. Let $B = A/cA$. Then $\mathrm{e}_1(Q) = \mathrm{e}_1(QB) = -\mathrm{h}^0(B)$ and the exact sequence $0 \longrightarrow A \xrightarrow{c} A \longrightarrow B \longrightarrow 0$ yields the exact sequence

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(B) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(A) \xrightarrow{c} \mathrm{H}^1_{\mathfrak{m}}(A).$$

Because $\mathrm{H}^1_{\mathfrak{m}}(A) \cong R/\mathfrak{a}+\mathfrak{b} \cong k[[X]]/(X^2)$ and $(X-Z)+\mu(Y-W) \equiv X \bmod \mathfrak{a}+\mathfrak{b}$, we have $\mathrm{H}^0_{\mathfrak{m}}(B) \cong [(X^2) :_{k[[X]]} X]/(X^2) = (X)/(X^2)$. Thus we get $\mathrm{e}_1(Q) = -1$ and the proof is completed.

5 Buchsbaumness of $\mathrm{G}(I)$

Throughout this section we assume that I contains a parameter ideal $Q = (a_1, \cdots, a_d)$ as a reduction. We set $R = \mathrm{R}(I)$ and $G = \mathrm{G}(I)$. The graded maximal ideal of G is denoted by M. Furthermore, we set $f_i = a_i t \in R$ for $1 \leq i \leq d$. For certain elements $x_1, \cdots, x_n$ of a ring S and an S-module L, we denote by $\mathrm{e}(x_1, \cdots, x_n; L)$ the multiplicity symbol of $x_1, \cdots, x_n$ with respect to L (cf. [10, p. 24]).

Lemma 5.1 $\mathrm{e}({f_1}^{n_1}, \cdots, {f_d}^{n_d}; G_M) = \mathrm{e}({a_1}^{n_1}, \cdots, {a_d}^{n_d}; A)$ *for any* $n_1, \cdots, n_d > 0$.

Proof. Let G_+ be the ideal of G generated by homogeneous elements of positive degree. As $(f_1, \cdots, f_d)G$ is a reduction of G_+, we have $\mathrm{e}(f_1, \cdots, f_d; G_M) = \mathrm{e}_0((G_+)_M)$. On the other hand, as $\ell_{G_M}(G/(G_+)^n) = \ell_A(A/I^n)$ for any $n > 0$, we have $\mathrm{e}_0((G_+)_M) = \mathrm{e}_0(I)$. Hence it follows that $\mathrm{e}(f_1, \cdots, f_d; G_M) = \mathrm{e}(a_1, \cdots, a_d; A)$. Therefore, for any $n_1, \cdots, n_d > 0$

$$\begin{aligned}\mathrm{e}({f_1}^{n_1}, \cdots, {f_d}^{n_d}; G_M) &= n_1 n_2 \cdots n_d \cdot \mathrm{e}(f_1, \cdots, f_d; G_M) \\ &= n_1 n_2 \cdots n_d \cdot \mathrm{e}(a_1, \cdots, a_d; A) = \mathrm{e}({a_1}^{n_1}, \cdots, {a_d}^{n_d}; A)\,.\end{aligned}$$

Thus we get the required equality.

In the rest of this section, we furthermore assume that Q is a standard ideal such that $I^2 \subseteq QI + W$, $I^3 \subseteq Q$ and $\Sigma(Q) \subseteq I$.

Lemma 5.2 *Let* $n_1, \cdots, n_d$ *be positive integers. Then*

$$({a_1}^{n_1}, \cdots, {a_i}^{n_i}) \cap I^n = \sum_{j=1}^{i} {a_j}^{n_j} I^{n-n_j}$$

for any $n \in \mathbb{Z}$ *and* $1 \leq i \leq d$. *Hence we have*

$$G/({f_1}^{n_1}, \cdots, {f_i}^{n_i})G \cong \mathrm{G}(IB)\,,$$

where $B = A/({a_1}^{n_1}, \cdots, {a_i}^{n_i})$.

Proof. We may assume that $n > n_j$ for any $1 \leq j \leq i$. Let $x \in ({a_1}^{n_1}, \cdots, {a_i}^{n_i}) \cap I^n$. Then, as $x \in Q \cap (Q^{n-1}I + W) = Q^{n-1}I$, we can express

$$x = \sum_{\lambda \in \Lambda} y_\lambda a^\lambda \quad (y_\lambda \in I)\,,$$

where Λ is the set of $\lambda = (\lambda_1, \cdots, \lambda_d) \in \mathbb{Z}^d$ such that $\lambda_1 + \cdots + \lambda_d = n-1$ and $a^\lambda = {a_1}^{\lambda_1}{a_2}^{\lambda_2}\cdots{a_d}^{\lambda_d}$. On the other hand, as

$$x \in ({a_1}^{n_1}, \cdots, {a_i}^{n_i}) \cap Q^{n-1} = \sum_{j=1}^{i} {a_j}^{n_j} Q^{n-1-n_j},$$

we can write

$$x = \sum_{\gamma\in\Gamma} z_\gamma a^\gamma \quad (z_\gamma \in A),$$

where $\Gamma = \{\gamma \in \Lambda \mid \gamma_j \geq n_j \text{ for some } 1 \leq j \leq i\}$. It is enough to show that $z_\gamma \in I$ for any $\gamma \in \Gamma$.

Let $B = A[T_1, \cdots, T_d]$ be the polynomial ring with variables $T_1, \cdots, T_d$ over A and $\varphi : B \longrightarrow \mathrm{R}(Q)$ be the homomorphism of A-algebras such that $\varphi(T_j) = f_j$ for $1 \leq j \leq d$. Because $a_1, \cdots, a_d$ is a d-sequence, $\ker\varphi$ is generated by homogeneous elements of degree one (cf. [5]), so that $\ker\varphi \subseteq IB$ as $\Sigma(Q) \subseteq I$. Now we set

$$f = \sum_{\lambda\in\Lambda\setminus\Gamma} y_\lambda T^\lambda + \sum_{\gamma\in\Gamma}(y_\gamma - z_\gamma)T^\gamma.$$

Then $f \in \ker\varphi$. Hence we get $z_\gamma \in I$ for any $\gamma \in \Gamma$.

Lemma 5.3 *We have*

(1) $[0 :_G f_1]_n = \{\overline{wt^n} \mid w \in W \cap I^n\}$,

(2) $0 :_G f_1 = [0 :_G f_1]_1 \oplus [0 :_G f_1]_2$,

(3) $\ell_{G_M}(0 :_G f_1) = \ell_A(W)$, *and hence* $\operatorname{depth} G > 0$ *if* $\operatorname{depth} A > 0$.

Proof. (1) Let $x \in I^n$ and $\overline{xt^n} \in 0 :_G f_1$. Then $a_1 x \in I^{n+2}$, so that by 5.2 we have $a_1 x = a_1 y$ for some $y \in I^{n+1}$, which implies $x \in I^{n+1} + W$ since $x - y \in 0 :_A a_1 = W$. Hence $\overline{xt^n} = \overline{wt^n}$ for some $w \in W \cap I^n$. Thus we get $[0 :_G f_1]_n \subseteq \{\overline{wt^n} \mid w \in W \cap I^n\}$, and the converse inclusion is obvious.

(2) This follow from the assertion (1) as $W \cap I^n \subseteq W \cap Q = 0$ for $n \geq 3$.

(3) We get this assertion since $[0 :_G f_1]_1 \cong W/W \cap I^2$ and $[0 :_G f_1]_2 \cong W \cap I^2$.

Lemma 5.4 $f_1, \cdots, f_d$ *is a standard system of parameters for* G_M. *In particular, it follows that* $\mathrm{H}^0_M(G) = 0 :_G f_1$, *so that* $\mathrm{h}^0(G_M) = \mathrm{h}^0(A)$. *Moreover, we have* $\mathrm{I}(G_M) = \mathrm{I}(A)$.

Proof. By 5.2 we have $G/(f_1, \cdots, f_d)G \cong \mathrm{G}(I/Q)$, so that

$$\ell_{G_M}(G/(f_1, \cdots, f_d)G) = \ell_A(A/Q)\,.$$

Similarly, setting $\mathfrak{a} = ({a_1}^2, \cdots, {a_d}^2)$, we have

$$\ell_{G_M}(G/({f_1}^2, \cdots, {f_d}^2)G) = \ell_A(A/\mathfrak{a})\,.$$

Then, using 5.1 and that $a_1, \cdots, a_d$ is a standard system of parameters for A, we get

$$\begin{aligned} & \ell_{G_M}(G/(f_1, \cdots, f_d)G) - \mathrm{e}(f_1, \cdots, f_d; G_M) \\ = \; & \ell_A(A/Q) - \mathrm{e}(a_1, \cdots, a_d; A) \\ = \; & \ell_A(A/\mathfrak{a}) - \mathrm{e}({a_1}^2, \cdots, {a_d}^2; A) \\ = \; & \ell_{G_M}(G/({f_1}^2, \cdots, {f_d}^2)G) - \mathrm{e}({f_1}^2, \cdots, {f_d}^2; G_M)\,. \end{aligned}$$

Therefore by [10, Theorem and Definition 17 in Appendix], we have the required assertion.

Lemma 5.5 *We have the following.*

(1) *If* $0 < i < d$, *then* $\mathrm{H}^i_M(G)$ *is concentrated in degree* $1 - i$.

(2) $\mathrm{a}(G) \leq 2 - d$.

Proof. We prove by induction on d. Let $d = 1$. In this case, the assertion (1) insists nothing. In order to prove the assertion (2), let us consider the exact sequence

$$0 \longrightarrow \mathrm{H}^0_M(G)(-1) \longrightarrow G(-1) \xrightarrow{f_1} G \longrightarrow G/f_1G \longrightarrow 0\,.$$

This sequence yields the exact sequence

$$\mathrm{H}^0_M(G/f_1G) \longrightarrow \mathrm{H}^1_M(G)(-1) \xrightarrow{f_1} \mathrm{H}^1_M(G) \longrightarrow 0\,,$$

which implies $[\mathrm{H}^1_M(G)]_{n-1} \cong [\mathrm{H}^1_M(G)]_n$ for $n \geq 3$ since $[G/f_1G]_n \cong I^n/QI^{n-1} + I^{n+1} = 0$ for $n \geq 3$. Hence we get $[\mathrm{H}^1_M(G)]_n = 0$ for $n \geq 2$, so that $\mathrm{a}(G) \leq 1$.

Now we assume that $d \geq 2$. Let $B = A/W$. Then the kernel of the graded homomorphism $G \longrightarrow \mathrm{G}(IB)$ of A-algebras induced from the canonical surjection $A \longrightarrow B$ has finite length, so that we have $\mathrm{H}^i_M(G) \cong \mathrm{H}^i_M(\mathrm{G}(IB))$ for $i > 0$. On the other hand, QB is a standard parameter ideal of B such that $I^2B = QIB$ and $\Sigma(QB) \subseteq IB$. Hence by 5.3 and 5.4 we have that f_1 is $\mathrm{G}(IB)$-regular and $f_1 \cdot \mathrm{H}^i_M(\mathrm{G}(IB)) = 0$ for any $0 \leq i < d$.

Furthermore, setting $C = B/a_1B$, we have $\mathrm{G}(IB)/f_1\mathrm{G}(IB) \cong \mathrm{G}(IC)$ by 5.2. Therefore we get the exact sequence

$$0 \longrightarrow \mathrm{G}(IB)(-1) \xrightarrow{f_1} \mathrm{G}(IB) \longrightarrow \mathrm{G}(IC) \longrightarrow 0,$$

from which we see that $\mathrm{H}^i_M(\mathrm{G}(IB)) \hookrightarrow \mathrm{H}^i_M(\mathrm{G}(IC))$ for $0 \le i < d$ and $\mathrm{H}^{d-1}_M(\mathrm{G}(IB))$ is a homomorphic image of $\mathrm{H}^{d-2}_M(\mathrm{G}(IC))(1)$. Because $QC = (a_2, \cdots, a_d)C$ is a standard parameter ideal of C such that $I^2C = QIC$ and $\Sigma(QC) \subseteq IC$, the inductive hypothesis insists that $\mathrm{H}^i_M(\mathrm{G}(IC)) = [\mathrm{H}^i_M(\mathrm{G}(IC))]_{1-i}$ for any $0 \le i < d-1$ and $\mathrm{a}(\mathrm{G}(IC)) \le 3-d$. Now the assertion (1) can be verified easily. In order to see the assertion (2), let us consider the exact sequence

$$\mathrm{H}^{d-1}_M(\mathrm{G}(IC)) \longrightarrow \mathrm{H}^d_M(\mathrm{G}(IB))(-1) \xrightarrow{f_1} \mathrm{H}^d_M(\mathrm{G}(IB)) \longrightarrow 0.$$

If $n > 3-d$, then $[\mathrm{H}^{d-1}_M(\mathrm{G}(IC))]_n = 0$, so that $[\mathrm{H}^d_M(\mathrm{G}(IB))]_{n-1} \cong [\mathrm{H}^d_M(\mathrm{G}(IB))]_n$. Hence we have $[\mathrm{H}^d_M(G)]_n \cong [\mathrm{H}^d_M(\mathrm{G}(IB))]_n = 0$ for any $n \ge 3-d$. Therefore we get the assertion (2) and the proof is completed.

Lemma 5.6 *Suppose that $a_1, \cdots, a_d$ form a weak sequence (cf. [10, Definition 1.1]) in any order. We arbitrary take $x_i \in \mathfrak{m}$ for $1 \le i \le d$ and set $\xi_i = x_i - a_it$. Then*

$$(\xi_1, \cdots, \xi_d)G \cap \mathrm{H}^0_M(G) = 0.$$

Proof. Let us take any $\varphi \in (\xi_1, \cdots, \xi_d)G \cap \mathrm{H}^0_M(G)$. As $\mathrm{H}^0_M(G) = 0 :_G f_1$ by 5.4, we can express $\varphi = \overline{w_1t + w_2t^2}$, with $w_j \in W \cap I^j$ for $j = 1, 2$. We would like to show that $w_j \in I^{j+1}$ for $j = 1, 2$. For that, we write $\varphi = \sum_{i=1}^d \overline{\xi_i} \cdot \overline{\eta_i}$, with $\eta_i \in R$ for $1 \le i \le d$. Taking $N \gg 0$, we can express $\eta_i = \sum_{j=1}^N \eta_{ij}t^j$ ($\eta_{ij} \in I^j$) for $1 \le i \le d$. Our assumption implies $\mathfrak{m}W = 0$, so that $\mathfrak{m}I^2 \subseteq \mathfrak{m}QI$. Hence $I^j \subseteq Q$ for $j \ge 3$. Then, by 5.1 we have $\eta_{ij} \in QI^{j-1}$ for $j \ge 3$. Furthermore, we can choose η_{i2} in QI since $\xi_i \in \mathfrak{m}A[t]$, $I^2 \subseteq QI + W$ and $\mathfrak{m}W = 0$. Because

$$w_1t + w_2t^2 \equiv \sum_{i=1}^{d} \xi_i\eta_i \mod IR,$$

we get the following congruence equations:

$$
\begin{aligned}
0 &\equiv \sum_{i=1}^{d} x_i \eta_{i0} \mod I\,, \\
w_1 &\equiv \sum_{i=1}^{d} (x_i \eta_{i1} - a_i \eta_{i0}) \mod I^2\,, \\
w_2 &\equiv \sum_{i=1}^{d} (x_i \eta_{i2} - a_i \eta_{i1}) \mod I^3\,, \\
0 &\equiv \sum_{i=1}^{d} (x_i \eta_{ij} - a_i \eta_{i,j-1}) \mod I^{j+1} \quad \text{for } 3 \leq j \leq N \text{ and} \\
0 &\equiv \sum_{i=1}^{d} a_i \eta_{iN} \mod I^{N+2}\,.
\end{aligned}
$$

The third equation implies $w_2 \in Q$, so that $w_2 = 0$. Hence it is enough to show $w_1 \in I^2$. We need the following.

Claim *There exist elements $y_{\alpha\beta}^{(j)} \in I^j$ for any $1 \leq j \leq N$ and $1 \leq \alpha < \beta \leq d$ such that*

$$
\sum_{i=1}^{d} a_i (\eta_{ij} + \sum_{\alpha < i} x_\alpha y_{\alpha i}^{(j)} - \sum_{i < \beta} x_\beta y_{i\beta}^{(j)}) \in I^{j+2}\,.
$$

If this is true, setting

$$
v_i = \eta_{i1} + \sum_{\alpha < i} x_\alpha y_{\alpha i}^{(1)} - \sum_{i < \beta} x_\beta y_{i\beta}^{(1)}\,,
$$

we have $\sum_{i=1}^{d} a_i v_i \in I^3 = QI^2$. Hence there exist $v_i' \in I^2$ for $1 \leq i \leq d$ such that $\sum_{i=1}^{d} a_i (v_i - v_i') = 0$. Then, for any $1 \leq i \leq d$ we get

$$
\begin{aligned}
v_i - v_i' &\in (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) :_A a_i \\
&= (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) :_A \mathfrak{m}\,,
\end{aligned}
$$

so that $x_i(v_i - v_i') \in Q$, which implies $x_i v_i \in Q$ as $x_i v_i' \in \mathfrak{m} I^2 \subseteq Q$. On the other hand, we have $\sum_{i=1}^{d} x_i v_i = \sum_{i=1}^{d} x_i \eta_{i1}$, so that $q \in Q$, where $q = \sum_{i=1}^{d} (x_i \eta_{i1} - a_i \eta_{i0})$. Because $w_1 - q \in I^2$, we have $w_1 - q = q' + w'$ for some $q' \in QI$ and $w' \in W$. Then, as $w_1 - w' = q + q' \in Q \cap W = 0$, we get $w_1 \in I^2$.

Proof of Claim. We prove by descending induction on j. First, we set $y_{\alpha\beta}^{(N)} = 0$ for any $1 \leq \alpha < \beta \leq d$. Next, we assume that $2 \leq j \leq N$ and we have the required elements $y_{\alpha\beta}^{(j)}$. Of course, $y_{\alpha\beta}^{(j)} \in QI^{j-1}$ if $j \geq 3$. However, even if $j = 2$ we can choose $y_{\alpha\beta}^{(j)}$ in QI^{j-1} since $I^2 \subseteq QI + W$ and $\mathfrak{m}W = 0$. Now we set

$$v_{ij} = \eta_{ij} + \sum_{\alpha < i} x_\alpha y_{\alpha i}^{(j)} - \sum_{i < \beta} x_\beta y_{i\beta}^{(j)} .$$

Let $K_\bullet = \mathrm{K}_\bullet(f_1, \cdots, f_d; G)$ be the Koszul complex with the differential maps $\partial_p : K_p \longrightarrow K_{p-1}$ and let $T_1, T_2, \cdots, T_d$ be the free bases of K_1. We set

$$\sigma = \sum_{i=1}^{d} \overline{v_{ij} t^j} \cdot T_i .$$

Then $\sigma \in (f_1, \cdots, f_d)K_1$ as $v_{ij} \in QI^{j-1}$ for any $1 \leq i \leq d$. On the other hand,

$$\partial_1(\sigma) = \sum_{i=1}^{d} f_i \cdot \overline{v_{ij} t^j} = \overline{(\sum_{i=1}^{d} a_i v_{ij}) t^{j+1}} = 0$$

in G, so that $\sigma \in \mathrm{Z}_1(K_\bullet)$. Because $f_1, \cdots, f_d$ is a d-sequence on G, we have

$$(f_1, \cdots, f_d)K_1 \cap \mathrm{Z}_1(K_\bullet) = \mathrm{B}_1(K_\bullet) .$$

As a consequence, it follows that there exist elements $y_{\alpha\beta}^{(j-1)} \in I^{j-1}$ for any $1 \leq \alpha < \beta \leq d$ such that

$$\partial_2(\sum_{\alpha < \beta} \overline{y_{\alpha\beta}^{(j-1)} t^{j-1}} \cdot T_\alpha \wedge T_\beta) = \sigma .$$

The left hand side is equal to

$$\sum_{i=1}^{d} \overline{(\sum_{\alpha < i} a_\alpha y_{\alpha i}^{(j-1)} - \sum_{i < \beta} a_\beta y_{i\beta}^{(j-1)}) t^j} \cdot T_i ,$$

so that we have

$$v_{ij} \equiv \sum_{\alpha < i} a_\alpha y_{\alpha i}^{(j-1)} - \sum_{i < \beta} a_\beta y_{i\beta}^{(j-1)} \mod \quad I^{j+1}$$

for any $1 \leq i \leq d$. This implies

$$\sum_{i=1}^{d} x_i v_{ij} \equiv \sum_{\alpha < \beta} a_\alpha x_\beta y_{\alpha\beta}^{(j-1)} - \sum_{\alpha < \beta} x_\alpha a_\beta y_{\alpha\beta}^{(j-1)} \mod \quad I^{j+1} .$$

On the other hand,

$$\sum_{i=1}^{d} x_i v_{ij} = \sum_{i=1}^{d} x_i \eta_{ij} \equiv \sum_{i=1}^{d} a_i \eta_{i,j-1} \mod \quad I^{j+1}.$$

Therefore we get

$$\sum_{i=1}^{d} a_i(\eta_{i,j-1} + \sum_{\alpha<i} x_\alpha y_{\alpha i}^{(j-1)} - \sum_{i<\beta} x_\beta y_{i\beta}^{(j-1)}) \in I^{j+1}$$

and the proof is completed.

Proof of Theorem 1.3. Only the Buchsbaumness of G is left to show. We prove by induction on d. Because $\mathrm{H}_M^0(G) = \{\overline{w_1 t + w_2 t^2} \mid w_1 \in W, w_2 \in W \cap I^2\}$ and $\mathfrak{m}W = 0$, we have $M \cdot \mathrm{H}_M^0(G) = 0$. Hence G is a Buchsbaum ring if $d = 1$.

Suppose that $d \geq 2$. Let $B = A/W$ and $C = B/a_1 B$. Then C and IC inherits the assumption on A and I in 1.3 (cf. Proof of 5.5). Therefore the inductive hypothesis implies that $\mathrm{G}(IC)$ is a Buchsbaum ring, so that $\mathrm{G}(IB)$ is also a Buchsbaum ring since $\mathrm{G}(IB)/f_1\mathrm{G}(IB) \cong \mathrm{G}(IC)$, f_1 is $\mathrm{G}(IB)$-regular and $f_1 \cdot \mathrm{H}_M^i(\mathrm{G}(IB)) = 0$ for any $i < d$ (cf. [10, Proposition 2.19]). Furthermore, it is easy to see that the kernel of the graded homomorphism $G \longrightarrow \mathrm{G}(IB)$ coincides with $\mathrm{H}_M^0(G)$. Thus we get that $G/\mathrm{H}_M^0(G)$ is a Buchsbaum ring.

Let $V = \mathfrak{m} + It \subseteq R$. Because we may assume that $A/\mathfrak{m}$ is infinite, we can choose a system of generators $\xi_1, \cdots, \xi_\ell$ of V such that $\{\xi_i\}_{i\in\Lambda}$ form an sop for G_M for any subset $\Lambda \subseteq \{1, 2, \cdots, \ell\}$ with d-elements. In order to prove the Buchsbaumness of G, it is enough to show that

$$(\{\xi_i\}_{i\in\Lambda})G \cap \mathrm{H}_M^0(G) = 0$$

for any Λ stated above (cf. [10, Proposition 2.22]). Let $\Lambda = \{i_1 < i_2 < \cdots < i_d\}$ and $\xi_{i_k} = x_k - b_k t$ ($x_k \in \mathfrak{m}, b_k \in I$) for $1 \leq k \leq d$. Because $(b_1 t, \cdots, b_d t)G + \mathfrak{m}G$ coincides with the M-primary ideal $(\xi_{i_1}, \cdots, \xi_{i_d})G + \mathfrak{m}G$, we have that $b_1 t, \cdots, b_d t$ is an sop for $G/\mathfrak{m}G$. Hence $Q' = (b_1, \cdots, b_d)$ is a reduction of I. Then, by our assumption that (i) A is a Buchsbaum ring or (ii) A is a quasi-Buchsbaum ring and $I \subseteq \mathfrak{m}^2$, we have that Q' is a standard parameter ideal of A, and hence by 1.1 we get $I^2 \subseteq Q'I + W$ and $\Sigma(Q') \subseteq I$. Therefore, by 5.6 we have $(\xi_{i_1}, \cdots, \xi_{i_d}) \cap \mathrm{H}_M^0(G) = 0$ and the proof is completed.

The next example insists that the assumption of 1.3 that $I \subseteq \mathfrak{m}^2$ is necessary when A is a quasi-Buchsbaum ring but not a Buchsbaum ring.

Example 5.7 *Let $F = k[[X,Y,Z,W]]$ be the formal power series ring with variables X,Y,Z and W over a field k. Let $\mathfrak{a} = (X,Y)F \cap (Z,W)F \cap (X^2,Y,Z^2,W)F$ and $A = F/\mathfrak{a}$. Let x,y,z and w respectively denote the images of X,Y,Z and W in A. We set $\mathfrak{m} = (x,y,z,w)A, a = x-z, b = y-w$ and $Q = (a,b)A$. Then we have the following.*

(1) *A is a 2-dimensional quasi-Buchsbaum ring but not a Buchsbaum ring.*

(2) *Q is a standard parameter ideal of A such that $\mathfrak{m}^2 = Q\mathfrak{m} + W$. We obviously have $\Sigma(Q) \subseteq \mathfrak{m}$.*

(3) $\mathrm{G}(\mathfrak{m})$ *is not a Buchsbaum ring.*

Proof. Let $\mathfrak{n} = (X,Y,Z,W)F$ and $\mathfrak{b} = (X,Y)F \cap (Z,W)F$. Then we have the exact sequence $0 \longrightarrow F/\mathfrak{b} \longrightarrow F/(X,Y)F \oplus F/(Z,W)F \longrightarrow F/\mathfrak{n} \longrightarrow 0$, which implies that $F/\mathfrak{b}$ is a 2-dimensional Buchsbaum ring such that $\operatorname{depth} F/\mathfrak{b} = 1, \mathrm{H}^1_{\mathfrak{n}}(F/\mathfrak{b}) \cong k$ and $\mathrm{e}_0(\mathfrak{n}/\mathfrak{b}) = 2$. Because $\mathfrak{b} = \mathfrak{a} + XZF$ and $XZ\mathfrak{n} \subseteq \mathfrak{a}$, considering the exact sequence $0 \longrightarrow \mathfrak{b}/\mathfrak{a} \longrightarrow A \longrightarrow F/\mathfrak{b} \longrightarrow 0$, we get

$$W = \mathrm{H}^0_{\mathfrak{m}}(A) = \mathfrak{b}/\mathfrak{a} = xzA \cong k\,,$$
$$\mathrm{H}^1_{\mathfrak{m}}(A) \cong \mathrm{H}^1_{\mathfrak{n}}(F/\mathfrak{b}) \cong k\,,$$
$$\mathrm{e}_0(\mathfrak{m}) = \mathrm{e}_0(\mathfrak{n}/\mathfrak{b}) = 2\,.$$

Hence A is a 2-dimensional quasi-Buchsbaum ring with $\mathrm{I}(A) = \mathrm{h}^0(A) + \mathrm{h}^1(A) = 2$.

On the other hand, It is easy to see that $A/Q \cong k[[X,Y]]/(X^3,XY,Y^2)$ and Q is a reduction of $\mathfrak{m}$. Then $\ell_A(A/Q) = 4$ and $\mathrm{e}(a,b;A) = \mathrm{e}_0(\mathfrak{m}) = 2$, so that $\ell_A(A/Q) - \mathrm{e}(a,b;A) = \mathrm{I}(A)$, which implies that Q is a standard ideal of A. Because $F/\mathfrak{b}$ is a Buchsbaum ring with $\mathrm{e}_0(\mathfrak{n}/\mathfrak{b}) = 2$ and $\operatorname{depth} F/\mathfrak{b} > 0$, by [1] and [2] it follows that $F/\mathfrak{b}$ has maximal embedding dimension, so that we have $\mathfrak{n}^2 = (X-Z, Y-W)\mathfrak{n} + \mathfrak{b}$. Hence we get $\mathfrak{m}^2 = Q\mathfrak{m} + W$.

Let $a' = x - w$ and $b' = y - z$. Then $A/(a',b')A \cong k[[X,Y]]/(X^2,XY,Y^2)$ and $(a',b')A$ is a reduction of $\mathfrak{m}$. Hence $\ell_A(A/(a',b')A) = 3$ and $\mathrm{e}(a',b';A) = 2$, so that $\ell_A(A/(a',b')A) - \mathrm{e}(a',b';A) \neq \mathrm{I}(A)$. Therefore a',b' is not a standard sop for A, which implies that A is not a Buchsbaum ring. Then $\mathrm{G}(\mathfrak{m})$ is also not a Buchsbaum ring since

$$\mathrm{G}(\mathfrak{m}) \cong S/\{(X,Y)S \cap (Z,W)S \cap (X^2,Y,Z^2,W)S\}\,,$$

where $S = k[X,Y,Z,W]$, and the proof is completed.

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