Exposita Note

Additional Remarks on Duality-Based Microeconomics

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Assembled in this short note are some helpful remarks to complement my previous *Lecture Notes* (1996). Additions are kept minimal, and restricted to those that are warranted by the need to rectify the shortcomings I could not but feel when I test-taught the *Lecture Notes* in the last couple of academic years.

The Reference Format: The first double or triple digits in gothics refer to the corresponding §§'s of the *Lecture Notes*, with the parenthesized pagination followed by the line count, a minus if from the bottom, indicating the exact location of intended insertion.

Introduction (p. 3, *l*. 5).

Footnote: Resort to the second-order concavity sufficiency condition in terms of the negative semidefiniteness of the relevant Hessian matrix is an essential step in deducing the comparative statics results.

0.1.1 (p. 7, *l*. 5).

Footnote: $u(x,y)$ in Example 0.1 may be reckoned as *ordinal* if it is expressed in the form of $u(x,y)=\varphi U(x,y)$ for some *cardinal* utility function $U(x,y)$ in monetary terms by way of a monotone transform φ which incorporates the variable marginal utility of money. Solve

$$\begin{aligned}&\frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy \\ &= \varphi'\frac{\partial U(x,y)}{\partial x}dx + \varphi'\frac{\partial U(x,y)}{\partial y}dy \\ &= du = 0\end{aligned}$$

for

$$MRS(x,y) = \frac{dy}{dx}\bigg|_{du=0} = \frac{\dfrac{\partial u(x,y)}{\partial x}}{\dfrac{\partial u(x,y)}{\partial y}} = \frac{\dfrac{\partial U(x,y)}{\partial x}}{\dfrac{\partial U(x,y)}{\partial y}},$$

in which the more problematic cardinal-ordinal conversion terms φ' cancel out.

0.1.4 (p. 11, *l*. 3).

Sign Convention of the Constraint Inequalities: In constrained *maximization* [or *minimization*] problems, the constraint inequalities should be of *nonnegative* [*nonpositive*, resp.] sign in order for the Lagrangian multiplier λ to be nonnegative. This sign convention guarantees that the constrained maximization [or minimization] may be converted to the *maxmin* [*minmax*, resp.] problem where the corresponding Lagrangian $\mathcal{L}(\mathbf{x},\lambda)$ is maximized [minimized, resp.] with respect to the variable **x** and minimized [maximized, resp.] w.r.t. λ.

0.1.5 (p. 13, *l*. 5 before ■)

In short, the conclusion may be written as

$$\frac{\partial v(p,q,M)}{\partial M}=\frac{du(x(M),y(M))}{dM}=\lambda(M),$$

in terms of the *indirect utility function* $v(p,q,M) = u(\tilde{x}(p,q,M), \tilde{y}(p,q,M)) = u(x(M),y(M))$ to be introduced in §§1.1.

0.1.5 (p. 13, *l*. 6).

Historical Remark (*Consumer's Surplus Analysis*): Consider a single-commodity consumption **max** $\mu(u(x))-px$, where $\mu(u(x))$ measures the worth of x in dollars, certainly cardinal, after a monotone transform $\mu(.)$ of the ordinal utility $u(x)$. Note that $\mu(.)$ is notionally the inverse of the usual monotone transforms employed in the utility theory that transform the cardinal into ordinal utilities [see eg. φ in the preceding footnote to §§0.1.1]. Then, the first-order condition $\mu'u'(x)-p=0$ yields $u'(x)=p/\mu'$.

Next, incorporate the *purchasing power of money* in the preceding consumer's surplus analysis by denoting by y the purchasing power measured in terms of the number of units of the *composite good* other than x, with the composite good serving as the *numeraire*, i.e., priced 1. Then, the consumer's surplus maximization is modified as: **max** $\mu(u(x,y)) - (px+y)$. Thus, the first-order conditions read

$$\mu' \cdot \frac{\partial u}{\partial x} - p = 0$$

$$\mu' \cdot \frac{\partial u}{\partial y} - 1 = 0.$$

The second equation reflects the fact that the composite good is the numeraire.

Recalling that an *affine function* takes the form $f(x) = a + bx$, choose μ to be an affine function and define the *quasilinear utility* $\mu(u(x,y)) = U(x,y) = y + u(x)$ as a monotonic affine transformation of $u(x)$, linear in y and possibly nonlinear in x. The quasilinear utility function $U(x,y) = y + u(x)$ automatically satisfies the above marginal equations, and emerges as a natural functional form that incorporates constancy of a dollar's worth, the *unitary marginal utility of money* to be specific, and henceforth manages to (1) eliminate the income effect on the individual demand, and consequently (2) eliminate the effect of the income (wealth) distribution on the community aggregate demand.

Indeed, for (1) it suffices to remark that, given the quasilinear utility function $U(x,y) = y + u(x)$, the *MRS* of x for y reduces to $MRS(x,y) = u'(x)$ which remains constant for any given x, no matter which value y might take. Geometrically, the corresponding indifference curves are vertically parallel.

Pertaining to the above observations (1) and (2), the subsequent remark in §§1.8 explicates that the specification of consumers in terms of the quasilinear utility functions assures the EV and CV to be the *exact* measure of consumer's surpluses, and measurability of the deadweight loss (gain) as the *exact* sum of changes in consumer's surpluses.

More remarks are in order in §§1.4 on the special case of integrability problem for the quasilinear utility function, and in §§1.3 on the Gorman form of the indirect utility functions that subsumes the quasilinear utility functions from the dual point of view.

0.1.6 (p. 13, *l*. -4).

Equality Constriant: Suppose the jth constraint of the constrained maximization problem takes an equation form, i.e., $g^j(\mathbf{x})=0$. Then, rewrite the equation in two inequalities, $g^j(\mathbf{x})\leq 0$ and $-g^j(\mathbf{x})\leq 0$, in accordance with the sign convention prescribed in §§0.1.4. The Kuhn-Tucker procedure is applicable to the rewritten C.M.P. that includes these two among k constraint inequalities.

0.2.3 (p. 19, *l*. 5).

Start by an obvious observation that any positive [or negative] definite matrix is automatically positive [negative, resp.] semidefinite, but *not vice versa*. This observation warrants the following strengthening:

Necessary and Sufficient Condition for Semidefinite Matrices: *Let A be a symmetric square matrix of order n. Given any permuta-*

tion π of the indices $\{1, \ldots, n\}$, denote by A^{π} the matrix obtained by permuting the corresponding rows and columns of A in accordance to the specification π. Then,

(1) *A is positive semidefinite if and only if, for all permutations π of the indices $\{1, \ldots, n\}$, leading principal minors of A^{π} are positive.*

(2) *A is negative semidefinite if and only if, for all permutations π of the indices $\{1, \ldots, n\}$, leading principal minors of A^{π} alternate in sign from nonpositive to nonnegative, i.e., the k-th order leading principal minor is of sign $(-1)^k$ or 0 for $k=1, \ldots, n$.*

Thus, the semidefinite version strengthens the comparable requirements for definite matrices by stipulating their validity not only for the original matrices in question but also for all of their permuted matrices.

0.3 [New] (p. 19, *l*.-7).

Mathematics of Duality: Given a nonempty closed set $A \subset \mathbf{R}^l$, define for any $\mathbf{p} \in \mathbf{R}^l$ the *support function* of A by $\Pi_A(\mathbf{p}) = \mathbf{inf}\ \{\mathbf{p}.\mathbf{x} : \mathbf{x} \in A\}$.

$\Pi_A(\mathbf{p})$ serves as a dual description of A in that A is recoverable from $\Pi_A(\mathbf{p})$. Indeed, if A is convex, then $A = \cap\ \{\{\mathbf{x} \in \mathbf{R}^l : \mathbf{p}.\mathbf{x} \geq \Pi_A(\mathbf{p})\} : \mathbf{p} \in \mathbf{R}^l\}$, and if A is nonconvex, then its *convex hull* $\mathbf{con}\ A = \cap\ \{\{\mathbf{x} \in \mathbf{R}^l : \mathbf{p}.\mathbf{x} \geq \Pi_A(\mathbf{p})\} : \mathbf{p} \in \mathbf{R}^l\}$.

The following properties of $\Pi_A(\mathbf{p})$ are immediate. [For formal proofs, refer to the subsequent §§1.2, where $A = \{\mathbf{x} \in \mathbf{R}^l_+ : u(\mathbf{x}) \geq u\}$,

$\Pi_A(\mathbf{p}) = e(\mathbf{p}, u)$, and (3.5) reduces to *Shephard's Lemma*.]

(3.1)　$\mathbf{p}' \geq \mathbf{p} \Rightarrow \Pi_A(\mathbf{p}') \geq \Pi_A(\mathbf{p})$.

(3.2)　$\Pi_A(\mathbf{p})$ *is homogeneous of degree* 1 *in* $\mathbf{p}$.

(3.3)　$\Pi_A(\mathbf{p})$ *is concave*.

(3.4)　$\Pi_A(\mathbf{p})$ *is continuous*.

(3.5)　**Duality Theorem**: *Let* $A \subset \mathbf{R}^l$ *be a nonenempty and closed set and* $\Pi_A(\mathbf{p}) = \mathbf{inf}\{\mathbf{p.x}: \mathbf{x} \in A\}$ *its support function. Then,* (*a*) *there is a unique* $\mathbf{y} \in \mathbf{R}^l$ *such that* $\mathbf{p.y} = \Pi_A(\mathbf{p})$ *iff* $\Pi_A(.)$ *is differentiable at* $\mathbf{p}$, *and* (*b*) *for such* $\mathbf{y}$, $\mathbf{D}\Pi_A(\mathbf{p}) = \mathbf{y}$.

1.2(p. 21, *l*. -4).

An Elaboration on Monotonicity of Expenditure Function in Utility Levels: $u' > u \Rightarrow e(\mathbf{p}, u') > e(\mathbf{p}, u)$. [Refer to the *Alternative Proof* of (2.1).] Suppose otherwise, i.e., $\mathbf{p.h}(\mathbf{p}, u) \geq \mathbf{p.h}(\mathbf{p}, u') > 0$. Consider $\mathbf{h} = \mathbf{h}(\mathbf{p}, u') - \varepsilon\mathbf{u}$, where $\mathbf{u} = (1, \ldots, 1) \in \mathbf{R}^l$. For sufficiently small $\varepsilon > 0$, $u(\mathbf{h}) > u(\mathbf{h}(\mathbf{p}, u))$ by continuity of $u(.)$, and $\mathbf{p.h}(\mathbf{p}, u) > \mathbf{p.h}$. A contradiction to the definition of $\mathbf{h}(\mathbf{p}, u)$. ■

1.2a(p. 22, *l*. -8).

Typo: The last term of the equation in the statement of *Envelope Theorem* should read

$$\left.\frac{\partial g(\mathbf{x}, a)}{\partial a}\right|_{\mathbf{x}=\mathbf{x}(a)}.$$

1.2a (p. 23, *l*. 11).

Evelope Theorem with Multiple Inequality Constraints: Envelope Theorem may be strengthened to the maximization with k inequality constraints as specified in §§0.1.6. *Let* $M(a)=$ $\mathbf{max}\ \{F(\mathbf{x},a)\colon \mathbf{g}(\mathbf{x},a)\leq 0\}$. *Then*,

$$\frac{dM(a)}{da}=\frac{\partial F(\mathbf{x},a)}{\partial a}\bigg|_{\mathbf{x}=\mathbf{x}(a)}-\sum_{j=1}^{k}\lambda^{j}\cdot\frac{\partial g^{j}(\mathbf{x},a)}{\partial a}\bigg|_{\mathbf{x}=\mathbf{x}(a)},$$

where $(\lambda^{j})\in\mathbf{R}_{+}^{k}$ *are the corresponding Lagrangian multipliers*.

1.3(p. 24, *l*. 1).

Remark on (3.4): Consider $\mathbf{h}(\mathbf{p}+\mathbf{dp},u)$ for the case where $u=u(\mathbf{x}(\mathbf{p},m))$, i.e., $m=e(\mathbf{p},u)$. Then, $\mathbf{dp}=0$ implies $\mathbf{h}(\mathbf{p},u)=\mathbf{x}(\mathbf{p},m)$ by (3.4). ■

1.3 (p. 24, *l*. 10).

Alternative Proof of Roy's Identity via the Envelope Theorem: Let $M(p^{i})=\mathbf{max}\ \{u(\mathbf{x})\colon\ \mathbf{p}.\mathbf{x}\leq m\}=u(\mathbf{x}(\mathbf{p},m))=v(\mathbf{p},m)$. Then, $\mathscr{L}(\mathbf{x},p^{i},\lambda)=u(\mathbf{x})-\lambda(p^{i}.x^{i}+\sum_{j\neq i}p^{j}.x^{j}-m)$ and

$$\frac{dM(p^{i})}{dp^{i}}=\frac{\partial v(\mathbf{p},m)}{\partial p^{i}}=-\lambda.x^{i}(\mathbf{p},m).$$

Dividing this by $\partial v(\mathbf{p},m)/\partial m=\lambda$ yields the Roy's Identity. ■

An Implication of Roy's Identity (*Gorman Form Indirect Utility Functions*): Let there be N consumers. Indirect utility functions for N consumers are said to be of the *Gorman form* when all consumers $i=1, \ldots, N$ are characterized by indirect utility functions of the form $v_i(\mathbf{p}, m_i) = a_i(\mathbf{p}) + b(\mathbf{p}) m_i$, where $b(\mathbf{p})$ is notably common for all i's . Indeed, homotheticity and quasilinearity are two immediate candidate functional forms of utility for which the indirect counterparts are of the Gorman form.

Example 1 (*Homothetic Utility Functions*): Suppose $u_i(\mathbf{x}) = \mu_i(U_i(\mathbf{x}))$ for all $i=1, \ldots, N$, where μ_i is a monotone transform and U_i is homogeneous of degree 1. Then, $\mathbf{x}_i(\mathbf{p}, m_i) = \mathbf{x}_i(\mathbf{p}, 1) m_i$. Therefore, the corresponding indirect utility function takes a special case of the Gorman form $v_i(\mathbf{p}, m_i) = v_i(\mathbf{p}) m_i = b(\mathbf{p}) m_i$ where $v_i(\mathbf{p}) = v_i(\mathbf{p}, 1)$ is further assumed to be identical across all i's and equal to $b(\mathbf{p})$.

Example 2 (*Quasilinear Utility Functions*): Suppose $u_i(y, \mathbf{x}) = y + u_i(\mathbf{x})$ for every i, where y is the amount of the common *numeraire*, i.e., whose price always remains to be 1. After solving: **max** $u_i(y, \mathbf{x})$ **s.t.** $y + \mathbf{p}.\mathbf{x} \leq m_i$, the consumer i's demand is expressed as $\mathbf{x} = \mathbf{x}_i(\mathbf{p})$ and $y = m_i - \mathbf{p}.\mathbf{x}_i(\mathbf{p})$. Therefore, the consumer i's indirect utility function reduces to yet another special case of the Gorman form: $v_i(\mathbf{p}, m_i) = u_i(m_i - \mathbf{p}.\mathbf{x}_i(\mathbf{p}), \mathbf{x}_i(\mathbf{p})) = m_i - \mathbf{p}.\mathbf{x}_i(\mathbf{p}) + u_i(\mathbf{x}_i(\mathbf{p})) = a_i(\mathbf{p}) + m_i$, where $a_i(\mathbf{p}) = u_i(\mathbf{x}_i(\mathbf{p})) - \mathbf{p}.\mathbf{x}_i(\mathbf{p})$.

Suppose that N consumers' indirect utility functions are of the

Gorman form. Then, by Roy's Identity, the ith consumer's demand function for the jth good, $i=1, \ldots, N$ and $j=1, \ldots, l$ takes the form $x_i^j(\mathbf{p}, m_i) = \alpha_i^j(\mathbf{p}) + \beta^j(\mathbf{p}) m_i$, where

$$\alpha_i^j(\mathbf{p}) = -\frac{\frac{\partial a_i(\mathbf{p})}{\partial p^j}}{b(\mathbf{p})}; \ \beta^j(\mathbf{p}) = -\frac{\frac{\partial b(\mathbf{p})}{\partial p^j}}{b(\mathbf{p})}.$$

Summaries. (1) *Straight Line Engel Curves*: Since $\partial x_i^j(\mathbf{p}, m_i)/\partial m_i = \beta^j(\mathbf{p})$, which is independent of any consumer's money income m_i, the Engel curves for different consumers facing the identical $\mathbf{p}$ are parallel straight lines with the common slope $(\beta^1(\mathbf{p}), \beta^2(\mathbf{p}), \ldots, \beta^l(\mathbf{p}))$.

(2) *Representative Consumer*: Let $x^j(\mathbf{p}, (m_i)) = \sum_{i=1}^N x_i^j(\mathbf{p}, m_i)$ be the aggregate demand for the jth good. That is, $x^j(\mathbf{p}, (m_i)) = A^j(\mathbf{p}) + \beta^j(\mathbf{p}) M$, where $A^j(\mathbf{p}) = \sum_{i=1}^N \alpha_i^j(\mathbf{p})$ and $M = \sum_{i=1}^N m_i$. Define a representative consumer by her indirect utility function $v(\mathbf{p}, M) = \sum_{i=1}^N a_i(\mathbf{p}) + b(\mathbf{p}) M$ and her income M. Then, Roy's Identity again gurantees that $x^j(\mathbf{p}, M)$ is generated from the demand behavior of the representative consumer. ■

1.4 (p. 24, *l*. -5).

Corollary: *Suppose a consumer is characterized by her utility function* $u(\mathbf{x})$ *and initial endowment* $\mathbf{e}$, *instead of the fixed money income* m. *Then, the Slutsky Equation takes the following form:*

$$\mathbf{D_p x}(\mathbf{p}, \mathbf{p}.\mathbf{e}) = \mathbf{D_p h}(\mathbf{p}, u) - \mathbf{D}_m \mathbf{x}(\mathbf{p}, \mathbf{p}.\mathbf{e})(\mathbf{x}(\mathbf{p}, \mathbf{p}.\mathbf{e}) - \mathbf{e}).$$

Proof: Start by noting

$$\mathbf{D_p x(p,p.e)} = \mathbf{D_p x(p,p.e)}\mid_{\mathbf{p.e}=const} + \mathbf{D}_m \mathbf{x(p,p.e).e},$$

where $m = \mathbf{p.e}$ and $\mathbf{e} = \mathbf{D_p}\, m$. By identifying

$$\mathbf{D_p x(p,p.e)}\mid_{\mathbf{p.e}=const} = \mathbf{D_p x(p,}m\mathbf{)},$$

the R.H.S of which is nothing but the *gross* substitution term, one gets the desired equation after substituting this gross term for the Slutsky decomposition in $\mathbf{D_p x(p,p.e)}$. ■

Backgroud Remark: The following three examples all attest to the negative slope of $\mathbf{x(p,}m\mathbf{)}$ as vindicated by the consumer's surplus analysis on the basis of decreasing MU.

Example 1 (*Perfect Substitutes*): Let $u(x,y)=x+y$. Then, $x(p,q,m)=m/p$ if $p<q$; $[0,m/p]$ if $p=q$; 0 if $p>q$, and $y(p,q,m)=0$ if $p<q$; $[0,m/q]$ if $p=q$; m/q if $p>q$.

Example 2 (*Perfect Complements*): Let $u(x,y)=\mathbf{min}\ \{x,y\}$. Then, $x(p,q,m)=y(p,q,m)=m/(p+q)$.

Example 3 (*Cobb-Douglas Utility Function*): Let $u(x,y)=x^{\alpha}y^{1-\alpha}$. Then, $x(p,q,m)=\alpha m/p$ and $y(p,q,m)=(1-\alpha)m/q$. ■

Remark (*'Hicksian' Slutsky Equation*): $\mathbf{D_p x(p,}m\mathbf{)}$ expresses the *observable* change in the ordinary (Marshallian) demand consequent upon the price change **dp**, and could be of either sign, i.e., the ordinary demand could be either negatively or positively sloped. In order to explain the cause of the anomaly, the positively sloped demand for the Giffen's good, $\mathbf{D_p x}(\mathbf{p}, m)$ may be decomposed into two parts, the definitive part and the part dependent on the nature of the commodity in question, the dominant of which determines the slope. The first term $\mathbf{D_p h}(\mathbf{p}, u)$ corresponds to the (*Hicksian*) *substitution effect*, and the second $\mathbf{D}_m\mathbf{x}(\mathbf{p}, m) . \mathbf{x}(\mathbf{p}, m)$ to the (*Hicksian*) *income effect*.

The (Hicksian) substitution effect originates from the utility-constrained expenditure minimimization, and reflects the natural intuition that in order to achieve the same utility level u, one may effectively minimize the expenditure cost by substituting the relatively more expensive commodity for the relatively cheaper commodity under the new price $\mathbf{p}+\mathbf{dp}$. Thus, the substitution effect is definitive, and in particular in the two-good case is dictated by the diminishing *MRS*, the property *along* the identical indifference curve.

The (*Hicksian*) *income compensation* is interpreted as the *minimum* amount of subsidy [or the *maximum* amount of taxation] that compensates for the loss of the purchasing power of the initial money income m when one is worse off [that absorbs the extra purchasing power of m when one is better off, resp.] due to the price change $\mathbf{dp}$. More specifically, when $\mathbf{dp}$ is reduced to $\mathbf{dp}=(0, \ldots, 0, dp^i, 0, \ldots, 0)$ for some i, $\mathbf{dp}>0$ necessitates an introduction of the income compensation in the form of subsidy, while $\mathbf{dp}<0$ some form of the income absorption. Importantly, this *fictitious* income compensation, introduced purely for the explanatory sake and need not be actually implemented, is computed by stipulating the relevant utility level at the *ex ante* u and adopting the *ex post* prevalent price $\mathbf{p}+\mathbf{dp}$.

The (Hicksian) income effect corresponds to the revision of the ordinary demand necessitated by the absence of the above income compensation in actuality. The income effect is a composite of two factors: $\mathbf{D}_m\mathbf{x}(\mathbf{p}, m)$, i.e., whether the commodity in question is a *normal* or *inferior* good, and $\mathbf{x}(\mathbf{p}, m)$ being the weight, i.e., whether it is *staple* and commands a *high budget share*. In short, if the commodity in question is normal, then the substitution and income effects are in the same direction and reinforce each other. On the other hand, if inferior, then the two effects are in the opposite direction and there is even a possibility where the adverse income effect is so strong as to dominate the substition effect. The last possibility, known as the Giffen's good, tends to arise for inferior goods with high budget share. ■

Example (*Tax Reform* or *Consumption Tax Controversy*): Consider a *representative consumer* who plans ahead her/his expenditure of her disposable income $Y-T$, after income tax T, over the present and future consumptions (C, I) according to her *time preference* $u(C, I)$, given (p, q), the respective prices of (C, I). Recall that the residual of the intertemporal *MRS* of C for I over the unity represents the *subjective discount rate* $\rho(C, I)$. Thus, her subjective equilibrium requires $MRS(C, I) = 1 + \rho(C, I) = p/q$. [Naturally, $p = (1+r)q$ in a stationary economy, where r is the interest rate, and the equilibrium condition reduces to $\rho(C, I) = r$.]

Question: Design a tax reform of simultaneously lowering the income tax T and introducing the consumption tax by the rate t that

(1) secures the tax revenue no less than the current level T per capita, (2) accompanies no additional tax burden, (3) induces an increase in I, thus improving the productivity of the future generation, and (4) augments the *per capita* $GDP = pC + qI$, possibly helping to alleviate the international trade frictions.

Prescription: Lower the income tax T to the extent equivalent to the Hicksian income effect in the presence of the consumption tax t that is incorporated in the higher price of the present consumption as $p+t$.

Discussions: (1) Compare with the budget line, passing through the original consumption bundle (C_0, I_0) and evaluated by the prices $(p+t, q)$ inclusive of the consumption tax t, that is associated with the reform that pays back to tax payers all the revenue from the consumption tax.

(2) Consider the Hicksian income compensation.

(3) True by the Hicksian substitution effect.

(4) Compare the per capita GDP, as represented by the budget line through (C_1, I_1), the consumption bundle after the proposed tax reform, with that through (C_0, I_0), both evaluated by the pre-tax prices (p, q). ■

1.4(p. 26, *l*. 1).

Integrability Problem: *Given the observed demand data* $\mathbf{x}(\mathbf{p}, m)$ *whose substitution matrix is symmetric, it is possible to recover* *preferences that 'rationalize'* $\mathbf{x}(\mathbf{p}, m)$, *i.e., yield* $\mathbf{x}(\mathbf{p}, m)$ *as a result of consumer optimization.*

The proof consists of two steps.

Assertion 1 (*Recoverability of the expenditure function from the demand data*): ¶1, p. 26.

Assertion 2 (*Recoverability of preferences from the expenditure function*): *Suppose* $e(\mathbf{p}, u)$ *satisfies* (2.1)-(2.4) *and is differentiable. Given any* $\mathbf{p} \gg 0$, *define for each* u *a 'no-worse-than set'* $\Omega(u) \subset \mathbf{R}^l_+$ *by* $\Omega(u) = \{\mathbf{x} \in \mathbf{R}^l_+ : \mathbf{p}.\mathbf{x} \geq e(\mathbf{p}, u)\}$. *Then, for every* u, $e(\mathbf{p}, u)$ *is the expenditure function associated with* $\Omega(u)$, *i.e.,* $e(\mathbf{p}, u) = \mathbf{min}\ \{\mathbf{p}.\mathbf{x}: \mathbf{x} \in \Omega(u)\}$.

Proof of Assertion 2: By construction, $\Omega(u)$ is nonempty, closed and bounded from below. Therefore, for any $\mathbf{p} \gg 0$, $\mathbf{min}\ \{\mathbf{p}.\mathbf{x}: \mathbf{x} \in \Omega(u)\}$ exists and $e(\mathbf{p}, u) \leq \mathbf{min}\ \{\mathbf{p}.\mathbf{x}: \mathbf{x} \in \Omega(u)\}$ follows from the definition of $\Omega(u)$.

It remains to show $e(\mathbf{p}, u) \geq \mathbf{min}\ \{\mathbf{p}.\mathbf{x}: \mathbf{x} \in \Omega(u)\}$. By (2.3), concavity of $e(\mathbf{p}, u)$, $e(\mathbf{p}', u) \leq e(\mathbf{p}, u) + \mathbf{D_p} e(\mathbf{p}, u).(\mathbf{p}' - \mathbf{p})$. By (2.2), homogeneity of $e(\mathbf{p}, u)$, apply Euler's formula and express $e(\mathbf{p}, u) = \mathbf{p}.\mathbf{D_p}\, e(\mathbf{p}, u)$. Thus, $e(\mathbf{p}', u) \leq \mathbf{p}'.\mathbf{D_p}\, e(\mathbf{p}, u)$, which, together with $\mathbf{D_p} e(\mathbf{p}, u) = \mathbf{h}(\mathbf{p}, u)$ by Shephard's Lemma (2.5), implies $\mathbf{h}(\mathbf{p}, u) \in \Omega(u)$. Therefore, $\mathbf{min}\ \{\mathbf{p}.\mathbf{x}: \mathbf{x} \in \Omega(u)\} \leq \mathbf{p}.\mathbf{h}(\mathbf{p}, u) = e(\mathbf{p}, u)$. ■

Example (*Quasilinear Utility*): Let $u(y,x)=y+u(x)$ with the numeraire y always priced 1 and p the price of x. Recall from §§1.3 that the consumer optimization yields $x=u'^{-1}(p)=x(p)$ and $y=m-px(p)$.

(1) *Recoverability of* $u(y,x)$ *from the inverse demand* $p(x)=u'(x)$: $u(m-px(p),x(p))=m-px(p)+u(x(p))=m-px(p)+\int_0^x u'(t)\,dt+u(0)=\int_0^x p(t)\,dt-px(p)+u(m,0)$. Therefore, $u(m-px(p),x(p))-u(m,0)$ is equal to the area under the inverse demand curve $p(x)$ minus the expenditure on x, or alternatively the area to the left of $x(p)$.

(2) *Recoverability of* $u(y,x)$ *from the indirect utility* $v(p,m)=v(p)+m$, *where* $v(p)=u(x(p))-px(p)$: By Roy's Identity, $x(p)=-v'(p)$. Therefore, $v(p)=\int_p^\infty x(t)\,dt$, which represents the area to the left of $x(p)$ down to p, and is yet another way of measuring the same area that was described in (1). ■

1.4c (p. 35, l. 8).

Remark (Also an Elaboration on the *Proof of* (4.3) (B)): The *Slutsky income compensation* measures the exact amount of income compensation that guarantees the purchase of the *ex ante* consumption bundle $\mathbf{x}(\mathbf{p},m)$ under the *ex post* price $\mathbf{p}+\mathbf{dp}$, i.e., $dm=\mathbf{x}(\mathbf{p},m).\,\mathbf{dp}$. Note that the computation involves only observed data, $\mathbf{x}(\mathbf{p},m)$ and $\mathbf{dp}$, and more importantly dispenses with the exact specification of the utility function $u(\mathbf{x})$.

By a slight notational abuse, let $\mathbf{d_p x}(\mathbf{p},m)$ denote the *partial differential* w.r.t. $\mathbf{p}$, and $\mathbf{dx}(\mathbf{p},m)\,|_{dm=\mathbf{x}(\mathbf{p},m).\mathbf{dp}}$ the *total differential* when dm is restricted to $dm = \mathbf{x}(\mathbf{p}, m).\ \mathbf{dp}$. Then, the *Slutsky Identity* or the original *Slutsky Equation ã la Slutsky* is nothing but the identity of the form $\mathbf{d_p x}(\mathbf{p}, m) = \mathbf{dx}(\mathbf{p}, m) \mid_{dm=\mathbf{x}(\mathbf{p},m).\mathbf{dp}} - \mathbf{D}_m\mathbf{x}(\mathbf{p}+\mathbf{dp}, m).\ dm = \mathbf{dx}(\mathbf{p}, m) \mid_{dm=\mathbf{x}(\mathbf{p},m).\mathbf{dp}} - \mathbf{D}_m\mathbf{x}(\mathbf{p}+\mathbf{dp}, m).\ \mathbf{x}(\mathbf{p}, m).\mathbf{dp}$. Therefore, $\mathbf{D_p x}(\mathbf{p}, m) = \mathbf{D_p x}(\mathbf{p}, m) \mid_{dm=\mathbf{x}(\mathbf{p},m).\mathbf{dp}} - \mathbf{D}_m\mathbf{x}(\mathbf{p}+\mathbf{dp}, m).\mathbf{x}(\mathbf{p}, m)$.

The first term of the decomposition measures the (Slutsky) substitution effect in the presence of the Slutsky income compensation $dm = \mathbf{x}(\mathbf{p}, m).\mathbf{dp}$ that enables the consumer to recover the original consumption $\mathbf{x}(\mathbf{p}, m)$ under the new price $\mathbf{p}+\mathbf{dp}$. The Slutsky substitution effect is definitive in that $\{\mathbf{x}(\mathbf{p}+\mathbf{dp}, m+dm) - \mathbf{x}(\mathbf{p}, m)\}.\ \mathbf{dp} < 0$, or more specifically for the special case with $\mathbf{dp} = (0, \ldots, 0, dp^i, 0, \ldots, 0)$ for some i, dp^i and $x^i(\mathbf{p}+\mathbf{dp}, m+dm) - x^i(\mathbf{p}, m)$ are of the opposite sign. This property follows from two observations: $\mathbf{x}(\mathbf{p}+\mathbf{dp}, m+dm).\mathbf{p} > \mathbf{x}(\mathbf{p}, m).\mathbf{p}$ since $\mathbf{x}(\mathbf{p}, m)$ is the demand, and $\mathbf{x}(\mathbf{p}+\mathbf{dp}, m+dm).\ (\mathbf{p}+\mathbf{dp}) = \mathbf{x}(\mathbf{p}, m).\ (\mathbf{p}+\mathbf{dp})$ by the definition of dm. Subtracting the first inequality from the second equation yields the desired result.

The second term corresponds to the (Slutsky) income effect whose subtle nature, depending on the sign of $\mathbf{D}_m\mathbf{x}(\mathbf{p}+\mathbf{dp}, m)$, is similar to the preceding remark in §§1.4 on the Hicksian income effect. ■

Example (*Index Problem*, or the *Purchasing Power Parities* that tend to overvalue the key currency): Consider two situations, 0 and 1, where the preferences are unchaged, and compare therein the costs of purchasing the commodity bundles chosen, given the prevalent prices and income in the respective situations. Let $\mathbf{x}_0 = \mathbf{x}(\mathbf{p}_0, m_0)$

and $\mathbf{x}_1 = \mathbf{x}(\mathbf{p}_1, m_1)$ where $\mathbf{p}_1 = \mathbf{p}_0 + \mathbf{dp}$. Choose $dm = m_1 - m_0 = \mathbf{dp}.\ \mathbf{x}_0$. Then, $m_1 > e(\mathbf{p}_1, u(\mathbf{x}_0)) = \mathbf{p}_1.\ \mathbf{h}(\mathbf{p}_1,\ u(\mathbf{x}_0))$, or $u(\mathbf{x}_1) > u(\mathbf{x}_0)$ by comparison of the Slutsky and Hicksian substitution effects, and such dm is an overcompensation. Thus, the meaningful conclusions should be drawn by keeping the utility level at $u(\mathbf{x}_0)$, and consequently in terms of $e(\mathbf{p}_1, u(\mathbf{x}_0))$, *not* m_1, and m_0.

Important examples that call for caution against such overcompensation include: price indices and living cost indices over different points of time within the same nation where over-time changes in tastes are negligible (with the base year being chosen as the situation 0 in the above for the *Laspeyres indices* while the comparison year for the *Paasche indices*); and the Purchasing Power Parities (P.P.P.) of currencies between two countries with the identical tastes (where 0 designates the country whose currency serves as the key currency, and consequently *the key currency tends to be overvalued, whereas non-key currencies undervalued, according to the P.P.P.*). ■

1.8 (p. 38, *l*. 10).

Expenditure Function as Money Metric Indirect Utility Function: Start by remarking that $e(\mathbf{q}, v(\mathbf{p}, m))$ is monotone in $v(\mathbf{p}, m)$ for any arbitrary $\mathbf{q} \gg 0$ [*Alternative Proof of* (2.1) in §§1.2, and its elaboration in the above]. Thus, when looked upon as a function of $(\mathbf{p}, m)$, $e(\mathbf{q}, v(\mathbf{p}, m))$ is an indirect utility function in its own right, satisfying the properties (1.1)-(1.4) of §§1.1. The nomenclature *money metric* is an obvious reference to the welfare gain or loss in dollar terms as measured by $e(\mathbf{q}, v(\mathbf{p}_1, m_1)) - e(\mathbf{q}, v(\mathbf{p}_0, m_0))$. Note that this measure of the welfate change is uniquely determined for the given preferences since it is independent of a particular specification $v(\mathbf{p}, m)$ of the preferences that results in turn from their particular representation $u(x)$.

1.8 (p. 38, *l*. -4).

Interpretations of EV and CV: The consumer would be indifferent between the *receipt* of EV and the anticipated favorable price change. In short, $v(p_0, m+EV) = u_1$. When the proposed price change would make her worse off, $-EV$ is the *maximum* buy-out bid she is willing to offer in advance instead of experiencing this aggravating price change.

CV is the *maximum net revenue* of a fictitious compensator whose obligation is to compensate the consumer *ex post* for the price change so that she may be as well off as before the price change, i.e., $v(p_1, m - CV) = u_0$. Similar sign convention applies to CV when the price change makes the consumer worse off.

Consider a specific case where $\mathbf{p}_0 \leq \mathbf{p}_1$ and $m_0 = m_1$. Then, $-CV$ measures the Hicksian compensation evaluated at the current new prices $\mathbf{p}_1$, while $-EV$ measures the muximum buy-out cost, evaluated at the original prices $\mathbf{p}_0$, the consumer is willing to spend in order to recover the original utility level $v(\mathbf{p}_0, m_0)$. In this specific case, the Slutsky compensation amounts to $\mathbf{p}_1 . \mathbf{x}(\mathbf{p}_0, m_0) -- m_0 > -CV$, thus signifying an overcompensation.

1.8 (p. 39, *l*. 8)

Example (*Quasilinear Utility and the Use of EV and CV as Exact Measure of Gain or Loss in Consumer's Surplus*): Let $u(y, x) = y + u(x)$. Observe from the recoverability argument (2) for the example of quasilinear utility in §§1.4, that $e(p, v(q) + m) = \int_p^q x(t)\, dt + m$ measures the change in consumer's surplus associated with a price change from p to q. This measure of welfare change is called the *area variation* AV defined formally as

$$AV(p_0, p_1) = \int_{p_1}^{p_0} x(p)\, dp = v(p_1) - v(p_0),$$

the second equality of which follows from Roy's Identity. [For general utility functions other than the quasilinear type, since $x(p, m)$ is not free from the income effect, neither is $AV(p_0, p_1, m)$.] Then, $EV = e(p_0, v(p_1) + m_1) - m_0 = AV(p_1, p_0) + m_1 - m_0$ and $CV = m_1 - e(p_1, v(p_0) + m_0) = m_1 - AV(p_0, p_1) - m_0$ coincide, and measure the gain or loss in cosumer's surplus *independently of the prices at which they are evaluated*. [**Intuition**: Since EV and CV are both reduced to linear functions in m, in their evaluation the marginal utility of m is constant at every p. Certainly, this is a consequence from $x(p, m) = h(p, u)$ for all p when the income effect is assumed away by way of a quasilinear utility function.] ■

Remark (*Deadweight Gain or Loss When All Consumers' Utility Functions are Quasilinear*): Start by recalling from Example 2 and the statement (2) of Summaries on the representative consumer given in §§1.3 that when all N consumers have quasilinear utility function $u_i(y,x)=y+u_i(x)$, $i=1,\ldots, N$, their indirect utility function takes a Gorman form $v_i(p,m_i)=v_i(p)+m_i$, where $v_i(p)=u_i(x_i(p))-px_i(p)$. The aggregate indirect utility function, i.e., the one for the representative consumer $v(p,M)=v(p)+M$ where $v(p)=\sum_{i=1}^N v_i(p)$ and $M=\sum_{i=1}^N m_i$ will rationalize the aggregate demand $x(p,M)=\sum_{i=1}^N x_i(p,m_i)=-\ \{\sum_{i=1}^N \mathbf{D}v_i(p)+M\}$. Recall also from the previous example, $AV(p_0,p_1)=v(p_1)-v(p_0)$. Denote by $DW(p_0,p_1)$ the deadweight gain or loss associated with the price change from p_0 to p_1. Then, $DW(p_0,p_1)=v(p_1)-v(p_0)=\sum_{i=1}^N v_i(p_1)-\sum_{i=1}^N v_i(p_0)=\sum_{i=1}^N AV_i(p_0,p_1)$, i.e., the deadweight gain or loss is the aggregate of area variations. ■

2.2 (p. 48, *l*. -2).

Proof of (2.1): Choose the output level y as the parameter, and write the long-run cost function as $M(y)=\mathbf{min}\ \{\mathbf{w}.\mathbf{x}{:}f(\mathbf{x})\geq y\}$. Denote by $\mathbf{x}(y)$ the minimizer of $M(y)$. Given the short run as characterized by $x^i=\bar{x}^i$, $i=k+1,\ldots, n$, the corresponding short-run cost function may be written as $m(y)=\mathbf{min}\ \{\mathbf{w}.\mathbf{x}=\check{\mathbf{w}}_k.\check{\mathbf{x}}_k+\sum_{i=k+1}^n w^i.\bar{x}^i{:}\ f(\check{\mathbf{x}}_k,\ \bar{x}^{k+1},\ldots,\ \bar{x}^n)\geq y\}$, where $\check{\mathbf{z}}_k$ denotes the truncation of $\mathbf{z}\in\mathbf{R}^n$ up to the kth coordinate $(k<n)$. Also, find y^* such that $\bar{x}^i=x^i(y^*)$, $i=k+1,\ldots, n$. Then, at $y=y^*$, $(\check{\mathbf{x}}_k(y^*),\ \bar{x}^{k+1},\ldots,\ \bar{x}^n)=\mathbf{x}(y^*)$, the minimizer of $M(y^*)$, is also the minimizer of $m(y^*)$ by construction, and consequently $M(y^*)=m(y^*)$. Otherwise, $M(y)<m(y)$. ■

Additional References:

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