有限群の指標環の構造について

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研究代表者:　　山　内　憲　一
(千葉大学　教育学部　数学科)

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はじめに

この報告書は文部省科学研究費補助金(基盤研究(c)(2))の交付を受けて、平成11年度から平成12年度の2年間に実施された研究「有限群の指標環の構造について」に関するものである。

有限群Gの指標環$R(G)$について、研究代表者・山内を中心に次の点に焦点を当てて研究を進めた。

(i) $R(G)$の単数群の構造を調べること、とくに$R(G)$の単数(無限位数を持つ)を具体的に構成すること。

(ii) Induction Theoremについて、Brauer、Artin、Green等の研究結果があるが、もっと証明を簡易化できないか、または別証明が得られないかということについて考えること。色々な証明について見直しをしてみること。

(iii) $R(G)$についてはWeidman、Saksonov の定理があるが、これらについてBrauer指標環$BR(G)$に対して一般化できないかを考えること。

(iv) 有限群の表現の拡大について。Isaacs はこのことについていくつか結果をだしているが、もっと一般化できないかを考えること。

2年間の研究の結果いくつかの成果を得ることができた。成果の詳細については、本文の「研究成果」の項を参照して頂きたい。

各研究分担者には、それぞれの専門分野からの情報の提供と共に、共同研究を進め、必要に応じて外部からの研究協力を得ることになった。とくに研究代表者・山内が、2000年2月(9日－20日)及び2000年9月(9日－17日)の2回に渡り、Birmingham大学(英国)を訪問し、G.R.Robinson教授と意見交換し、助言を得ることができた。この研究に際して大きな役割を果たされました。

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(3) 研究代表者 山内　憲一　(千葉大学・教育学部・助教授)

(4) 研究分担者

野澤　宗平　(千葉大学・理学部・教授)

越谷　重夫　(千葉大学・理学部・教授)

北詰　正顕　(千葉大学・理学部・助教授)

越川　浩明　(千葉大学・教育学部・教授)

丸山　研一　(千葉大学・教育学部・助教授)

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(6) 研究発表

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(7) 研究成果

有限群 G の指標環 $R(G)$ の構造について、主として次の 4 項目について研究を進めた。得られた結果について順に述べることにする。

(i)　$R(G)$ の単数群の構造について

$R(G)$ の位数有限の単数については既に知られているので、ここでは無限位数の単数について考えることにした。今までには n 次交代群 $A_n(n \geq 5)$ に対し、$R(A_n)$ の単数群の構造について研究されている。これは A_n の既約指標がわかっているので考え易かったのである。

既約指標が一般に知られていない有限群 G の $R(G)$ に対して次のように考えた。

G を有限可解群とする。このとき G の可解性により、$H \lhd G, |G/H| = p$ (素数) を満たす G の部分群 H が存在する。今 $p(\geq 5)$ と仮定する。また C_p を位数 p の巡回群とする。剰余群 G/H の既約指標は自然に G の既約指標とみなせる。従って $R(C_p)$ の無限位数の単数を見つければ、それは自然に $R(G)$ の無限位数をもつ単数となる。実際に $R(C_p)$ の無元位数をもつ単数を構成することに成功した。これらについてはいずれ論文としてまとめる予定である。

(ii)　Induction Theorem について

J.A.Green は Brauer の Induction Theorem の逆が成り立つことを 1955 年に証明した。証明の方法は、誘導指標に関する Frobenius の公式を用いるものであった。この Green の定理の別証明として、有限群の特性関数 (characteristic class function) を用いる方法もある。(K.Yamauchi, "On a Theorem of J.A.Green", J.Algebra **209**, (1998), 708-712)

今回の研究では Mackey の分解定理を用いて上の 2 つの証明とは異なる方法で Green の定理を証明した。([**1**] K.Yamauchi,"Another proof of a Theorem of J.A. Green",J.Algebra **235**, (2001) , 829-832)

(iii)　有限群 G の Brauer 指標環 $BR(G)$ に対して Weidman,Saksonov の定理の一般化を試みること

このことについては色々大事な結果が得られ,論文としてまとめられた。([**2**] K.Yamauchi, "On isomorphisms of a Brauer character ring onto another", to appear in J.Algebra) 以下に主な結果を述べる。

(1)　$\lambda : R(G) \longrightarrow R(H)$ を同型写像とするとき、χ_i, χ_j が G の同じ block に属すれば,$\chi'_{i'}, \chi'_{j'}$ も H の同じ block に属する。但し,$\chi_t \xrightarrow{\lambda} \chi'_{t'}$ $(t = 1, ...r), \chi_i, \chi_j \in Irr(G)$, $\chi'_{i'}, \chi'_{j'} \in Irr(H)$.

以下に,次の記号を定める。

$p =$ 素数, $G_\circ = G$ の p-正則な元の集合、$c_1, ..., c_r$ を p-正則な共役類の代表元の集合、$\lambda : \overline{Z}BR(G) \longrightarrow \overline{Z}BR(H)$ を同型写像とし、$c_i \xrightarrow{\lambda} c'_{i'}, (i = 1, ...r)$ であるとする。但し,$c'_{1'}, ..., c'_{r'}$

は H の p-正則な共役類の代表元の集合。$|C_G(c_i)|_{p'}$ を $|C_G(c_i)|$ の p'-part とするとき、

$\mathbf{m}_{p'} = (|C_G(c_1)|_{p'}, ..., |C_G(c_r)|_{p'}), \qquad \mathbf{m}'_{p'} = (|C_H(c'_{1'})|_{p'}, ..., |C_H(c'_{r'})|_{p'})$

とおく。このとき次が成り立つ。

(2) $\mathbf{m}_{p'} = \mathbf{m}'_{p'}$

さらに次の記号を定める。

$\mathbf{m} = (|C_G(c_1)|, ..., |C_G(c_r)|), \qquad \mathbf{m}' = (|C_H(c'_{1'})|, ..., |C_H(c'_{r'})|)$

$IBr(G) = \{\varphi_1, ..., \varphi_r\}$: G の Brauer 既約指標

C, C' をそれぞれ G, H の Cartan matrix とし、A を λ を表す行列とする。このとき次が成り立つ。

(3) 次の三つは同値である。

(i) $\mathbf{m} = \mathbf{m}'$

(ii) $A^*CA = C'$

(iii) $(\varphi_i, \varphi_j)'_G = (\lambda(\varphi_i), \lambda(\varphi_j))'_H$

但し,$(f, g)'_G = \dfrac{1}{|G|} \displaystyle\sum_{x \in G_0} f(x)\overline{g(x)}$

さらに $CA = AC'$ ならば次が成り立つ。

(4) 上の仮定の下で次が成り立つ。

(i) $\lambda(\varphi_i) = \varepsilon_i \varphi'_{i'}$ (i=1,...,r) ここに $IBr(H) = \{\varphi'_{1'}, ..., \varphi'_{r'}\}, \varepsilon_i$ は 1 のベキ根である。

(ii) G と H の Brauer character table は同じである。

(iii) 行と列を適当に入れ替えれば $C = C'$ が得られる。

(iv) η_i と η_j が G の同じ block に属すれば、$\eta'_{i'}$ と $\eta'_{j'}$ も H の同じ block に属する。ここに $\eta_t \overset{\lambda}{\to} \eta'_{t'}$ (t=1,...,r), $\{\eta_i, \eta_j\}$ 及び $\{\eta'_{i'}, \eta'_{j'}\}$ はそれぞれ G, H の principal indecomposable character である。

(v) φ_i と φ_j が G の同じ block に属すれば $\varphi'_{i'}$ と $\varphi'_{j'}$ も H の同じ block に属する。ここに $\varphi_t \overset{\lambda}{\to} \varphi'_{t'}$ (t=1,...,r)。

(iv) 有限群 G の表現の拡大について

Isaacs は標数 0 の体 L に関して次のより一般化された定理を証明した。

定理 E を標数 0 の代数閉体とし、$L \subset E$ とする。$N \lhd G$ で χ は G-invariant な N の L-表現であるとする。α を χ の E-既約成分とし、$(|G:N|, \alpha(1)o(\alpha)) = 1$ と仮定する。このとき χ は G の L-表現に拡張される。

本研究では $(|G:N|, o(\alpha)) = 1$ を仮定しないで議論を進め, 種々の結果を得た。これらをまとめて論文として出版した。(**[3]** K.Yamauchi, "On the Extensions of Group Representations over Arbitrary Fields II" The Bulletin of the Faculty of Education,Chiba University,Vol.48 Part III (2000), 11-19.)

詳細な内容を述べるには、沢山の記号の説明や長い仮定を書かねばなりませんので、ここでは省略することにします。この報告書の後半にこの論文の全文を載せますので、それをご覧下さい。

最後に上の主要な結果を述べた3篇の論文 [**1**],[**2**],[**3**] を以下につづることにする。また研究代表者・山内が本研究で得られた成果について、日本数学会代数学分科会 (2000年9月、京都大学) で発表した原稿も載せることにする。

Another proof of a theorem of J.A.Green

Kenichi Yamauchi *

Department of Mathematics,Faculty of Education,Chiba University,Yayoicho
Chiba 263-8522,Japan
E-mail: yamauchi@math.e.chiba-u.ac.jp

Dedicated to Professors Eiichi Bannai and Etsuko Bannai

J.A.Green proved a theorem which is the converse to a theorem of R.Brauer (Proc.Camb.Philos.Soc.**51**(1955),237-239).The present author gived another proof of the theorem by making an application of the characteristic class functions of a finite group. In this article we give another proof of the theorem which is easier than the two previous proofs of the theorem, by using Mackey decomposition theorem.

1. Introduction

Throughout this article, G, Z and C denote a finite group, the ring of rational integers and the field of complex numbers respectively. Let $\{\chi_1 = 1_G(the\ principal\ character), ..., \chi_h\}$ be the full set of nonisomorphic irreducible complex characters of G. Let $char(G)$ be the character ring of G.

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That is, $char(G) = \{\sum_{i=1}^{h} a_i\chi_i \quad | \quad a_i \in Z \quad (i = 1, ..., h)\}$. Then $char(G)$ is a subring of the ring $cf(G)$ of all complex- valued class functions on G.

From now on,for any subring R of C we denote by $char_R(G)$ the set of $R-$linear combinations of the complex characters of G for simplicity. For $\theta \in cf(G)$ and a subgroup H of G,we denote the restriction of θ to H by $\theta|_H$ or $Res_H^G(\theta)$.

Let A be any subring of C consisting of algebraic integers such that $Z \subseteq A$ and let $\mathcal{H}$ be a family of subgroups of G.Then we consider the following four statements with respect to $\mathcal{H}$.

(i) *If for any $\theta \in cf(G), \theta|_H \in char(H)$ for all $H \in \mathcal{H}$,then $\theta \in char(G)$.*

(ii) *If for any $\theta \in cf(G), \theta|_H \in char_A(H)$ for all $H \in \mathcal{H}$,then $\theta \in char_A(G)$.*

(iii) *$\sum_{H\in\mathcal{H}}\{char(H)\}^G = char(G)$ where $\{char(H)\}^G$ is the set of all generalized characters of G of the form ϕ^*(the generalized character of G induced by ϕ) with $\phi \in char(H)$.*

(iv) *Each elementary subgroup of G is contained in some cojugate of some subgroup belonging to $\mathcal{H}$.*

These statements are equivalent(See the Introduction in [7]). Proofs of (i) $\Longleftrightarrow$(iii) and (ii)$\Longleftrightarrow$(iii) are obtained by Brauer's proof of Theorem 3 in [1] and by using two formulas (i) of (38.5) Theorem in [2] and $1_G = \sum a_H \lambda_H^*$ where $a_H \in Z, \lambda_H \in char(H)$ and $H \in \mathcal{H}$. In [4] J.A.Green gives a proof of (iii) $\Longrightarrow$ (iv) by using Frobenius's formula for induced characters, in order to prove (i)$\Longrightarrow$(iv) (that is, the converse to a theorem of R.Brauer). In [7] the present author gives a proof of (ii)$\Longrightarrow$(iv) in case $\epsilon \in A$ where ϵ is a primitive $|G|$th root of unity, by making an application of the characteristic class functions of G. We want to get a direct proof of (i)$\Longrightarrow$ (iv) but it seems to be difficult to get its proof.

In this article we intend to give a proof of (iii)$\Longrightarrow$(iv) by using Mackey decomposition theorem ((44.2)Theorem in [2]).

2. Proof of (iii)$\Longrightarrow$(iv)

Let ϵ be a $|G|$th root of unity and $A = Z[\epsilon]$ be the subring of C generated by ϵ over Z. Then we have

Lemma 2.1 *Let $E = \langle y \rangle \times P$ be a $p-$elementary subgroup of G where P is a $p-$group and $\langle y \rangle$ is a $p'-$group and $E_\circ$ be a proper subgroup of E. Let θ be any generalized character of $E_\circ$. Then we have*

$$Ind_{E_\circ}^{E}(\theta)(y) = \theta^*(y) \in pA$$

where $Ind_{E_\circ}^{E}(\theta)$ denotes the generalized character of E induced by θ.

Proof. If $y \notin E_\circ$,then by the definition of an induced character, we can easily show that $\theta^*(y) = 0 \in pA$. Hence we may assume $y \in E_\circ$. Then we can write $E_\circ = \langle y \rangle \times P_\circ$ where $P_\circ = P \cap E_\circ$ and $P_\circ$ is a proper subgroup of P. Let $P = \bigcup_{i=1}^{n} t_i P_\circ$ be a decomposition of P into disjoint left cosets with respect to $P_\circ$. Then $E = \bigcup_{i=1}^{n} t_i E_\circ$ is a decomposition of E into disjoint left cosets with respect to $E_\circ$. Hence we have

$$\theta^*(y) = \sum_{i=1}^{n} \dot{\theta}(t_i^{-1} y t_i) = n\theta(y)$$

Since $n = [P : P_\circ]$ and $p|n$, we have $\theta^*(y) \in pA$. Thus the proof is complete. ■

Remark. The above lemma may be similar to the lemmas which are stated in [3] and [6] (See (15.29) Lemma in [3] and Lemma 11 at page 85 in [6]).But it is essentially different from those lemmas because in Lemma 2.1 we only consider a generalized character of a proper subgroup of an elementary subgroup of G, instead of a generalized character of a subgroup H of G where H does not contain any conjugate of a given elementary subgroup of G.It seems that a proof of Lemma 2.1 is easier than the proofs of the two previous stated lemmas.

Proof of (iii)$\Longrightarrow$(iv) Let $\mathcal{H}$ be a family of subgroups of a finite group G which satisfies (iii) and let $E =< y > \times P$ be a $p-$elementary subgroup of G for a $p'-$element y and a $p-$ group P. By assumption (iii) we have

$$1_G = \sum_{H \in \mathcal{H}} \sum_{\lambda} \ a_{H,\lambda} \ Ind_H^G(\lambda)$$

where $a_{H,\lambda} \in Z$ and $\lambda \in char(H)$.

Assume by way of contradiction that E is contained in no conjugate of a subgroup belonging to $\mathcal{H}$. By Mackey decomposition theorem ((44.2) Theorem in [2]) we can write

$$Res_E^G(1_G) = 1_E = \sum_{H \in \mathcal{H}} \sum_{\lambda} \ a_{H,\lambda} \ Res_E^G(Ind_H^G(\lambda))$$
$$= \sum_{H \in \mathcal{H}} \sum_{\lambda} \ a_{H,\lambda} \sum_{t \in T} \ Ind_{H^t \cap E}^E \ (\lambda^t|_{H^t \cap E})$$

where T is a full set of the representatives of all (H, E) double cosets in G.

Since E is contained in no conjugate of a subgroup belonging to $\mathcal{H}$, $H^t \cap E$ is a proper subgroup of E. Therefore by Lemma 2.1 we have

$$Ind_{H^t \cap E}^E \ (\lambda^t|_{H^t \cap E})(y) \in pA.$$

Hence $1_E(y) \in pA$. This is contrary to $1_E(y) = 1$. Hence the result follows. ∎

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On isomorphisms of a Brauer character ring onto another II

Kenichi Yamauchi *

Department of Mathematics,Faculty of Education,Chiba University,Yayoicho

Chiba 263-8522,Japan

E-mail: yamauchi@math.e.chiba-u.ac.jp

Dedicated to Professor Eiichi Abe

This article is a continuation of my article "On isomorphisms of a Brauer character ring onto another ",(Tsukuba J.Math.20(1996),207-212).In this article we state a necessary and sufficient condition under which an isomorphism λ of a Brauer character ring onto another preserves an inner product. We also state the relations between λ and blocks of group algebras of finite groups.

1. Introduction

Throughout this article G, Z and Q denote a finite group,the ring of rational integers and the rational field respectively. Moreover we write $\overline{Z}$ to denote the ring of all algebraic integers in the complex numbers and $\overline{Q}$ to denote the algebraic closure of Q in the field of complex numbers. For a finite set S, we denote by $|S|$ the number of elements in S.

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Let $Irr(G) = \{\chi_1, ..., \chi_h\}$ be the complete set of absolutely irreducible complex characters of G. Then we write $\overline{Z}R(G)$ to denote the $\overline{Z}-$algebra spanned by $\chi_1, ..., \chi_h$.That is, $\overline{Z}R(G) = \{\sum_{i=1}^{h} a_i\chi_i \mid a_i \in \overline{Z}, (i = 1, ..., h)\}$.

For two finite groups G and H, let λ be a $\overline{Z}-$algebra isomorphism of $\overline{Z}R(G)$ onto $\overline{Z}R(H)$. Then we can write

$$\lambda(\chi_i) = \sum_{j=1}^{h} a_{ij}\chi_j^{'}, \qquad (i = 1, ..., h)$$

where $a_{ij} \in \overline{Z}$ and $Irr(H) = \{\chi_1^{'}, ..., \chi_h^{'}\}$. In this case we write A to denote the $h \times h$ matrix with $(i, j)-$ entry equal to a_{ij} and say that A is afforded by λ with respect to $Irr(G)$ and $Irr(H)$.

As is well known , concerning the isomorphism λ the following statements hold. These results seem to be most important.

(i) $|C_G(c_i)| = |C_H(c_{i'}^{'})|$, $(i = 1, ..., h)$ *where* $\{c_1, ..., c_h\}$ *and* $\{c_{1'}^{'}, ..., c_{h'}^{'}\}$ *are complete sets of representatives of the cojugate classes in G and H respectively and* $c_i \xrightarrow{\lambda} c_{i'}^{'}$, $(i = 1, ..., h)$. (The definition of $c_i \xrightarrow{\lambda} c_{i'}^{'}$ will be stated in the case of modular representations of finite groups in section 2.)

(ii) *A is unitary where A is the matrix afforded by* λ *with respect to* $Irr(G)$ *and* $Irr(H)$.

By using this result Weidman and Saksonov proved independently that if $\overline{Z}R(G)$ is isomorphic to $\overline{Z}R(H)$ for two finite groups G and H, then the character tables of G and H are the same.

(iii) *With respect to an inner product,* $(\chi_i, \chi_j)_G = (\lambda(\chi_i), \lambda(\chi_j))_H$ *for* $\chi_i, \chi_j \in Irr(G)$.

(iv) *Concerning the blocks of modular representation theory, if* χ_i *and* χ_j *are in the same block of G, then* $\chi_{i'}^{'}$ *and* $\chi_{j'}^{'}$ *are in the same block of H where* $\chi_t \xrightarrow{\lambda} \chi_{t'}^{'}$, $(i = 1, ..., h)$. (The definition of $\chi_t \xrightarrow{\lambda} \chi_{t'}^{'}$ will be stated and this result will be proved in section 2.)

In general, concerning an isomorphism λ of a Brauer character ring onto another, the above statements not always hold.

In this article our main objective is to give a necessary and sufficient condition under which some of the above statements hold and a sufficient condition under which the above statements (ii) and (iv) hold, concerning an isomorphism λ of a Brauer character ring onto another.

From now on, when we consider homomorphisms from an algebra to another, unless otherwise specified, we shall only deal with algebra homomorphisms.

2. Preliminaries

We fix a rational prime p and use the following notation with respect to a finite group G.

$G_\circ$: the set of all $p-$regular elements of G

$Cl(G_\circ) = \{\mathfrak{C}_1 = \{1\}, ..., \mathfrak{C}_r\}$:the complete set of $p-$regular conjugate classes in G

$\{c_1, ..., c_r\}$: a complete set of representatives of $\mathfrak{C}_1, ..., \mathfrak{C}_r$ respectively

$IBr(G) = \{\varphi_1 = 1, ..., \varphi_r\}$: the complete set of irreducible Brauer characters of G which can be viewed as functions from $G_\circ$ into the complex numbers.

For any subring R of the field of complex numbers such that $1 \in R$, we write $RBR(G)$ to denote the ring of linear combinations of $\varphi_1, ..., \varphi_r$ over R.That is , $RBR(G) = \{\sum_{i=1}^{r} a_i\varphi_i \mid a_i \in R, (i = 1, ..., r)\}$. In particular we use the notation $BR(G)$ instead of $ZBR(G)$ and say that $BR(G)$ is the Brauer character ring of G.

We are given two finite groups G and H. For G and H we assume that there exists an isomorphism λ of $\overline{Z}BR(G)$ onto $\overline{Z}BR(H)$. Then it follows that the rank of $BR(G)$= the rank of $BR(H)$ and $|Cl(G_\circ)| = |Cl(H_\circ)|$.

We also can extend λ to an isomorphism $\hat{\lambda}$ of $\overline{Q}BR(G)$ onto $\overline{Q}BR(H)$ by linearity.

Here we use the following additional notation.

$Cl(H_\circ) = \{\mathfrak{C}'_1 = \{1\}, ..., \mathfrak{C}'_r\}$

$\{c'_1 = 1, ..., c'_r\}$: a complete set of representatives of $\mathfrak{C}'_1, ..., \mathfrak{C}'_r$ respectively.

$IBr(H) = \{\varphi'_1, ..., \varphi'_r\}$

$\{f_1, ..., f_r\}$: the complete set of characteristic class functions on $G_\circ$ where f_i corresponds to $\mathfrak{C}_i$, $(i = 1, ..., r)$ (see Definition 2.1 in [4]).

$\{f'_1, ..., f'_r\}$: the complete set of characteristic class functions on $H_\circ$ where f'_i corresponds to $\mathfrak{C}'_i$, $(i = 1, ..., r)$.

By Lemma 2.2 and Lemma 2.3 in [4], it follows that $f_i \in \overline{Q}BR(G)$ and $\hat{\lambda}(f_i)$ is a characteristic class function on $H_\circ$, $(i = 1, ..., r)$.

Now we define a bijection from $Cl(G_\circ)$ to $Cl(H_\circ)$ through the isomorphism λ as follows. For a p-regular conjugate class $\mathfrak{C}_i$ of G, $\mathfrak{C}_i$ corresponds to a characteristic class function f_i on $G_\circ$ and $\hat{\lambda}(f_i)$ is also a characteristic class function $f'_{i'}$ on $H_\circ$ which corresponds to a p-regular cojugate class $\mathfrak{C}'_{i'}$ of H. Here we assign $\mathfrak{C}'_{i'}$ to $\mathfrak{C}_i$,$(i = 1, ..., r)$. Thus we get one-to-one correspondence between $Cl(G_\circ)$ and $Cl(H_\circ)$:

$$c_i \in \mathfrak{C}_i \longrightarrow f_i \longrightarrow \hat{\lambda}(f_i) = f'_{i'} \longrightarrow \mathfrak{C}'_{i'} \ni c'_{i'}$$

where $i \longrightarrow i'$, $(i = 1, ..., r)$ is a permutation. In this case we write $c_i \xrightarrow{\lambda} c'_{i'}$, $(i = 1, ..., r)$.

Keeping the above notation we give the following lemma concerning the Brauer character table of G. This lemma plays a fundamental role in proofs of Theorems 2.2 and 3.2. But a proof of this lemma is not given in [4] and [5] and so we give a proof.

Lemma 2.1. $(\varphi_i(c_j)) = (\lambda(\varphi_i)(c'_{j'}))$ $(r \times r$ *matrices) where* $c_j \xrightarrow{\lambda} c'_{j'}, (j = 1, ..., r)$.

Proof. Since we can write $f_i = \sum_{j=1}^{r} b_{ij}\varphi_j, b_{ij} \in \overline{Q}, (i = 1, ..., r)$, we have $f'_{i'} = \hat{\lambda}(f_i) = \sum_{j=1}^{r} b_{ij}\lambda(\varphi_j)$. Hence we have

$$\begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix} = B \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix}, \quad \begin{pmatrix} f'_{1'} \\ \vdots \\ f'_{r'} \end{pmatrix} = B \begin{pmatrix} \lambda(\varphi_1) \\ \vdots \\ \lambda(\varphi_r) \end{pmatrix}$$

where $B = (b_{ij})$ (an $r \times r$ matrix).Since $c_j \xrightarrow{\lambda} c'_{j'}, (j = 1, ..., r)$,we have $f_i(c_j) = \delta_{ij}$ and $f'_{i'}(c'_{j'}) = \delta_{i'j'}$.Hence

$$\begin{pmatrix} f_1(c_i) \\ \vdots \\ f_r(c_i) \end{pmatrix} = \begin{pmatrix} f'_{1'}(c'_{i'}) \\ \vdots \\ f'_{r'}(c'_{i'}) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$ is the vector with only i−th entry equal to 1 and others equal to 0.

Therefore $B \begin{pmatrix} \varphi_1(c_i) \\ \vdots \\ \varphi_r(c_i) \end{pmatrix} = B \begin{pmatrix} \lambda(\varphi_1)(c'_{i'}) \\ \vdots \\ \lambda(\varphi_r)(c'_{i'}) \end{pmatrix}$. Since B is regular, we get $\varphi_1(c_i) = \lambda(\varphi_1)(c'_{i'}), ..., \varphi_r(c_i) = \lambda(\varphi_r)(c'_{i'})$, $(i = 1, ..., r)$. That is, $(\varphi_i(c_j)) = (\lambda(\varphi_i)(c'_{j'}))$.Thus the proof is complete. ■

Now we return to an isomorphism λ of $\overline{Z}R(G)$ onto $\overline{Z}R(H)$. Then $|G| = |H|$ and by Saksonov's theorem we have $\lambda(\chi_i) = \epsilon_i\chi'_{i'}, (i = 1, ..., h)$ where $Irr(G) = \{\chi_1, ..., \chi_h\}, Irr(H) = \{\chi'_1, ..., \chi'_h\}$ and the ϵ_i are roots of unity. In this case we write $\chi_i \xrightarrow{\lambda} \chi'_{i'}, (i = 1, ..., h)$. Then we have

Theorem 2.2. *If χ_i and χ_j are in the same block of G for $\chi_i, \chi_j \in Irr(G)$, then $\chi'_{i'}$ and $\chi'_{j'}$ are in the same block of H where $\chi_t \overset{\lambda}{\to} \chi'_{t'}$, $(t = 1, ..., h)$.*

Proof. If we set $\theta(\chi_i) = \chi'_{i'}$, $(i = 1, ..., h)$, then θ is also an isomorphism of $\overline{Z}R(G)$ onto $\overline{Z}R(H)$, because $\lambda(\chi_i) = \epsilon_i \chi'_{i'}$, $(i = 1, ..., h)$ (see the proof of Theorem 1.2 (ii) in [5]).

Let $Cl(G) = \{\mathfrak{C}_1 = \{1\}, ..., \mathfrak{C}_h\}$ and $Cl(H) = \{\mathfrak{C}'_1, ..., \mathfrak{C}'_h\}$ be complete sets of conjugate classes in G and H respectively. Then by the lemma which corresponds to Lemma 2.1 in ordinary representations of finite groups, we have $\chi_i(c_k) = \chi'_{i'}(c'_{k''})$ where $c_k \in \mathfrak{C}_k$, $c'_{k''} \in \mathfrak{C}'_{k''}$ and $c_k \overset{\theta}{\to} c'_{k''}$, $(k = 1, ..., h)$. Since $\chi_i(1) = \chi_i(c_1) = \chi'_{i'}(c'_{1''}) \leqq \chi'_{i'}(1)$ and $\chi'_{i'}(1) = \chi_i(c_{1'}) \leqq \chi_i(1)$ where $\mathfrak{C}'_1 \ni 1 \overset{\theta^{-1}}{\to} c_{1'} \in \mathfrak{C}_{1'}$, we have $\chi_i(1) = \chi'_{i'}(1)$. Therefore

$$\frac{|\mathfrak{C}_k|\chi_i(c_k)}{\chi_i(1)} = \frac{|\mathfrak{C}'_{k''}|\chi'_{i'}(c'_{k''})}{\chi'_{i'}(1)} \quad , \quad (k = 1, ..., h) .$$

In a similar way we have

$$\frac{|\mathfrak{C}_k|\chi_j(c_k)}{\chi_j(1)} = \frac{|\mathfrak{C}'_{k''}|\chi'_{j'}(c'_{k''})}{\chi'_{j'}(1)} \quad , \quad (k = 1, ..., h) .$$

Since χ_i and χ_j are in the same block of G, by (85.12) Corollary in [1]

$$\frac{|\mathfrak{C}_k|\chi_i(c_k)}{\chi_i(1)} \equiv \frac{|\mathfrak{C}_k|\chi_j(c_k)}{\chi_j(1)} \quad (mod(\pi)) \quad , \quad (k = 1, ..., h)$$

where (π) is a maximal ideal of a complete discrete valuation ring in a $p-$modular system of G. Therefore we have

$$\frac{|\mathfrak{C}'_{k''}|\chi'_{i'}(c'_{k''})}{\chi'_{i'}(1)} \equiv \frac{|\mathfrak{C}'_{k''}|\chi'_{j'}(c'_{k''})}{\chi'_{j'}(1)} \quad (mod(\pi)) \quad , \quad (k = 1, ..., h) .$$

By (85.12) Corollary in [1], $\chi'_{i'}$ and $\chi'_{j'}$ are in the same block of H. Thus the result follows. ■

3. Main theorems

We keep the notation in section 2. Let G and H be two finite groups and let λ be an isomorphism of $\overline{Z}BR(G)$ onto $\overline{Z}BR(H)$ such that $\mathfrak{C}_i \ni c_i \xrightarrow{\lambda} c'_{i'} \in \mathfrak{C}'_{i'}$, $(i = 1, ..., r)$.

We write $\mathbf{m}_{p'}$ to denote the vector with i–th entry equal to $|C_G(c_i)|_{p'}$ (the p'–part of $|C_G(c_i)|$) and $\mathbf{m}'_{p'}$ to denote the vector with i–th entry equal to $|C_H(c'_{i'})|_{p'}$ $(i = 1, ..., r)$. Then we prove

Theorem 3.1. *In the above situation we have* $\mathbf{m}_{p'} = \mathbf{m}'_{p'}$.

Proof. Let f_i be the characteristic class function on $G_\circ$ which corresponds to $\mathfrak{C}_i$, (That is, $f_i(c_j) = \delta_{ij}$). Then f_i is written as a $\overline{Q}$–linear combination of $\eta_1, ..., \eta_r$ where $\eta_1, ..., \eta_r$ are the principal indecompoable characters of G which corresponds to $\varphi_1, ..., \varphi_r$ respectively. That is,

$$f_i = \frac{1}{|C_G(c_i)|} \sum_{j=1}^{r} \overline{\varphi_j(c_i)}\eta_j \quad , \quad (i = 1, ..., r).$$

By Theorem 61.4(2) in [2] we can see that $p^a|C_G(c_i)|\hat{\lambda}(f_i)$ is a linear combination of $\eta'_1, ..., \eta'_r$ with coefficients of algebraic integers where $\eta'_1, ..., \eta'_r$ are the principal indecomposable characters of H which corresponds to $\varphi'_1, ..., \varphi'_r$ respectively and p^a is the order of a Sylow p–subgroup of H.

On the other hand, since $c_i \xrightarrow{\lambda} c'_{i'}$ it follows that $\hat{\lambda}(f_i)$ is the characteristic class function $f'_{i'}$ on $H_\circ$ which corresponds to $\mathfrak{C}'_{i'}$. Hence we have

$$p^a|C_G(c_i)|\hat{\lambda}(f_i) = \frac{p^a|C_G(c_i)|}{|C_H(c'_{i'})|} \sum_{j=1}^{r} \overline{\varphi'_j(c'_{i'})}\eta'_j.$$

The coefficient of η'_1 in the above formula is equal to $\dfrac{p^a|C_G(c_i)|}{|C_H(c'_{i'})|}$ and is an algebraic integer. Hence we have $|C_H(c'_{i'})|_{p'} \mid |C_G(c_i)|_{p'}$. By considering $\lambda^{-1} : \overline{Z}BR(H) \longrightarrow$

$\overline{Z}BR(G)$ (the inverse of λ), in a similar way we can obtain $|C_G(c_i)|_{p'} \mid |C_H(c'_{i'})|_{p'}$. Hence we have $|C_G(c_i)|_{p'} = |C_H(c'_{i'})|_{p'}$. This completes the proof. ∎

Let G and H be two finite groups with Cartan matrices C and C' respectively. Let λ be an isomorphism of $\overline{Z}BR(G)$ onto $\overline{Z}BR(H)$ and $A = (a_{ij})$ be the matrix afforded by λ with respect to $IBr(G) = \{\varphi_1, ..., \varphi_r\}$ and $IBr(H) = \{\varphi'_1, ..., \varphi'_r\}$. Let $\eta_1, ..., \eta_r$ be the principal indecomposable characters of G which correspond to $\varphi_1, ..., \varphi_r$ respectively and let $\eta'_1, ..., \eta'_r$ be the principal indecomposable characters of H which correspond to $\varphi'_1, ..., \varphi'_r$ respectively.

We set $Cl(G_\circ) = \{\mathfrak{C}_1, ..., \mathfrak{C}_r\}$ and $Cl(H_\circ) = \{\mathfrak{C}'_1, ..., \mathfrak{C}'_r\}$ and assume that $c_i \xrightarrow{\lambda} c'_{i'}$ where $c_i \in \mathfrak{C}_i, c'_{i'} \in \mathfrak{C}'_{i'}$, $(i = 1, ..., r)$. We write $\mathbf{m}$ to denote the vector with i−th entry equal to $|C_G(c_i)|$ and $\mathbf{m}'$ to denote the vector with i−th entry equal to $|C_H(c'_{i'})|$, $(i = 1, ..., r)$.

We use the common notation X^* for the conjugate transpose of a matrix X.

For $\overline{Q}$−valued class functions f and g on G or $G_\circ$, we define an inner product $(f, g)'_G$ as follows

$$(f, g)'_G = \frac{1}{|G|} \sum_{x \in G_\circ} f(x)\overline{g(x)}.$$

Then we have the following two Theorems 3.2 and 3.4.

Theorem 3.2. *With the above notation the following conditions are equivalent:*

(i) $\mathbf{m} = \mathbf{m}'$

(ii) $A^*CA = C'$

(iii) $(\varphi_i, \varphi_j)'_G = (\lambda(\varphi_i), \lambda(\varphi_j))'_H$, $(i, j = 1, ..., r)$.

Proof. A proof of (i)$\Longleftrightarrow$ (ii) is given in [4].

A proof of (i) $\Longrightarrow$ (iii). Since $\mathbf{m} = \mathbf{m}'$, we have $|G| = |H|$ and $|\mathfrak{C}_i| = |\mathfrak{C}'_{i'}|$,$(i = 1, ..., r)$. By Lemma 2.1 $\varphi_i(c_k) = \lambda(\varphi_i)(c'_{k'})$,$(k = 1, ..., r)$.Therefore we have

$$(\varphi_i, \varphi_j)'_G = \frac{1}{|G|} \sum_{k=1}^r |\mathfrak{C}_k| \varphi_i(c_k)\overline{\varphi_j(c_k)}$$
$$= \frac{1}{|H|} \sum_{k=1}^r |\mathfrak{C}'_{k'}| \lambda(\varphi_i)(c'_{k'})\overline{\lambda(\varphi_j)(c'_{k'})} = (\lambda(\varphi_i), \lambda(\varphi_j))'_H.$$

A proof of (iii)$\Longrightarrow$(ii). Since C' is the Cartan matrix of H, we have

$$\begin{pmatrix} \eta'_1 \\ \vdots \\ \eta'_r \end{pmatrix} = C' \begin{pmatrix} \varphi'_1 \\ \vdots \\ \varphi'_r \end{pmatrix} \therefore (C')^{-1} \begin{pmatrix} \eta'_1 \\ \vdots \\ \eta'_r \end{pmatrix} = \begin{pmatrix} \varphi'_1 \\ \vdots \\ \varphi'_r \end{pmatrix}.$$

Here we set $C = (c_{ij})$ and $(C')^{-1} = (c''_{ij})$. By assumption $(\varphi_i, \varphi_j)'_G = (\lambda(\varphi_i), \lambda(\varphi_j))'_H$, we have

$$(\varphi_i, \eta_j)'_G = (\varphi_i, \sum_k c_{jk}\varphi_k)'_G = (\lambda(\varphi_i), \sum_k c_{jk}\lambda(\varphi_k))'_H =$$
$$(\sum_{j'} a_{ij'}\varphi'_{j'}, \sum_{k,l} c_{jk}a_{kl}\varphi'_l)'_H = (\sum_{j'} a_{ij'}\varphi'_{j'}, \sum_{k,l} c_{jk}a_{kl}(\sum_m c''_{lm}\eta'_m))'_H =$$
$$\sum_{j',k,l,m} a_{ij'}c_{jk}\overline{a_{kl}}c''_{lm}(\varphi'_{j'}, \eta'_m)'_H \tag{3.1}$$

Since $(\varphi_i, \eta_j)'_G = \delta_{ij}$ and $(\varphi'_{j'}, \eta'_m)'_H = \delta_{j'm}$, by the formula (3.1) we have

$$(\varphi_i, \eta_i)'_G = \sum_{k,l,m} c_{ik}\overline{a_{kl}}c''_{lm}a_{im} = 1$$
$$(\varphi_i, \eta_j)'_G = \sum_{k,l,m} c_{jk}\overline{a_{kl}}c''_{lm}a_{im} = 0 \quad , (i \neq j).$$

Therefore we have $C\overline{A}(C')^{-1}({}^tA) = I$ (an identity $r \times r$ matrix).Hence $C\overline{A} = ({}^tA)^{-1}C' \therefore {}^tAC\overline{A} = C'$. That is, $A^*CA = C'$. This completes the proof. ■

We don't know any necessary condition under which A is unitary. It seems to be difficult to find its condition because there is an isomorphisms of $\overline{Z}BR(G)$ onto $\overline{Z}BR(H)$ even if $|G| \neq |H|$ (see Remark in the end of this section). But we can give a sufficient condition under which A is unitary. For example if $CA = AC'$, then by Theorem 3.2 in [4] it follows that $\mathbf{m} = \mathbf{m}'$ and A is unitary.

We can easily prove the following corollary.

Corollary 3.3. *If* $\mathbf{m} = \mathbf{m}'$, *then the following conditions are equivalent:*

(i) $CA = AC'$

(ii) *A is unitary.*

Theorem 3.4. *If* $CA = AC'$, *then we have*

(i) $\lambda(\varphi_i) = \epsilon_i \varphi'_{i'}$ *where the* ϵ_i *are roots of unity and* $i \longrightarrow i'$ $(i = 1, ..., r)$ *is a permutation.* (In this case we write $\varphi_i \xrightarrow{\lambda} \varphi'_{i'}$, $(i = 1, ..., r)$.)

(ii) *The Brauer character tables of G and H are the same.*

(iii) *With a suitable arrangement of rows and columns,* $C = C'$.

(iv) $\lambda(\eta_i) = \epsilon_i \eta'_{i'}$ *where the* ϵ_i *are roots of unity and* $i \longrightarrow i'$ $(i = 1, ..., r)$ *is the permutation in* (i). (In this case we write $\eta_i \xrightarrow{\lambda} \eta'_{i'}$, $(i = 1, ..., r)$.)

(v) *If* η_i *and* η_j *are in the same block of G, then* $\eta'_{i'}$ *and* $\eta'_{j'}$ *are in the same block of H where* $\eta_t \xrightarrow{\lambda} \eta'_{t'}$, $(t = 1, ..., r)$.

(vi) *If* φ_i *and* φ_j *are in the same block of G , then* $\varphi'_{i'}$ *and* $\varphi'_{j'}$ *are in the same block of H where* $\varphi_t \xrightarrow{\lambda} \varphi'_{t'}$, $(t = 1, ..., r)$.

Proof. Proofs of (i) and (ii) are stated in [4].

Proofs of (iii) and (iv). By (i) of this theorem we have

$$\lambda(\varphi_i) = \epsilon_i \varphi'_{i'}, \quad (i = 1, ..., r).$$

Since $p^a \lambda(\eta_i)$ is a linear combination of $\eta'_1, ..., \eta'_r$ with coefficients of algebraic integers where p^a is the order of a Sylow p-subgroup of H, by renumbering $\eta'_1, ..., \eta'_r$ we may write

$$p^a \lambda(\eta_i) = a_{1'} \eta'_{1'} + \cdots + a_{r'} \eta'_{r'}, \qquad a_{i'} \in \overline{Z} \quad , (i = 1, ..., r)$$

where $i \longrightarrow i'$, $(i = 1, ..., r)$ is the above permutation. Then we have

$$p^a = (\varphi_i, p^a \eta_i)'_G = (\lambda(\varphi_i), p^a \lambda(\eta_i))'_H = \epsilon_i \overline{a_{i'}}.$$

$$0 = (\varphi_j, p^a \eta_i)'_G = (\lambda(\varphi_j), p^a \lambda(\eta_i))'_H = \epsilon_j \overline{a_{j'}}, \qquad (i \neq j)$$

Hence we have $a_{i'} = \overline{\epsilon_i^{-1}} p^a$, $a_{j'} = 0$ $(i \neq j)$.Therefore we can see that $\lambda(\eta_i) = \epsilon'_i \eta'_{i'}$ where $\epsilon'_i = \overline{\epsilon_i^{-1}}$ is a root of unity $(i = 1, ..., r)$.

Next we prove that $C = C'$ with a suitable arrangement of rows and columns. If we set $C = (c_{ij})$ and $C' = (c'_{ij})$, then

$$c_{ij} = (\eta_i, \eta_j)'_G = (\lambda(\eta_i), \lambda(\eta_j))'_H = (\epsilon'_i \eta'_{i'}, \epsilon'_j \eta'_{j'})'_H = \epsilon'_i \overline{\epsilon'_j} c'_{i'j'}.$$

If $c_{ij} \neq 0$, then c_{ij} and $c'_{i'j'}$ are positive integers and $\epsilon'_i \overline{\epsilon'_j}$ is a root of unity. Therefore $c_{ij} = c'_{i'j'}$. If $c_{ij} = 0$,then $c'_{i'j'} = 0$.

We set $C'' = (c'_{i'j'})$ where C'' has an entry $c'_{i'j'}$ at position (i, j). Then $C = C''$ and C'' is the Cartan matrix of H.

Proof of (v). Since $C = C'$ by (iii) of this theorem and η_i and η_j are in the same block of G by assumption, we can see by Theorem 46.2 in [2] that $\eta'_{i'}$ and $\eta'_{j'}$ are in the same block of H where $\eta_t \xrightarrow{\lambda} \eta'_{t'}$ $(t = 1, ..., r)$.

Proof of (vi). Since η_i and φ_i are in the same block of G and η_j and φ_j are in the same block of G, η_i and η_j are in the same block of G because φ_i and φ_j are in the same block of G. By (v) of this theorem $\eta'_{i'}$ and $\eta'_{j'}$ are in the same block of H where $\eta_t \xrightarrow{\lambda} \eta'_{t'}$ $(t = 1, ..., r)$. Therefore $\varphi'_{i'}$ and $\varphi'_{j'}$ are in the same block of H. Thus the proof is complete. ■

Remark. If $CA \neq AC'$, (v) and (vi) of Theorem 3.4 do not hold. We can give a counterexample. We consider the case $p = 2$. Let $G = S_4$ be a symmetric group on 4 symbols and $H = D_6$ be a dihedral group of order 12. D_6 is generated by two elements a, b such that $a^6 = 1, b^{-1}ab = a^{-1}, b^2 = 1$. Then

$Cl(G_\circ) = \{\mathfrak{C}_1 = \{1\}, \mathfrak{C}_2 = 3 - cycles\}$

$Cl(H_\circ) = \{\mathfrak{C}_1^{'} = \{1\}, \mathfrak{C}_2^{'} = \{a^2, a^4\}\}$

and we have the following Brauer character tables of G and H(see the examples of §91A and §91B in [1]).

	$\mathfrak{C}_1$	$\mathfrak{C}_2$
φ_1	1	1
φ_2	2	-1

	$\mathfrak{C}_1^{'}$	$\mathfrak{C}_2^{'}$
$\varphi_1^{'}$	1	1
$\varphi_2^{'}$	2	-1

where $IBr(G) = \{\varphi_1, \varphi_2\}$ and $IBr(H) = \{\varphi_1^{'}, \varphi_2^{'}\}$.

We set $\lambda(\varphi_i) = \varphi_i^{'}$ $(i = 1, 2)$. Then λ is an isomorphism of $\overline{Z}BR(G)$ onto $\overline{Z}BR(H)$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (the matrix afforded by λ).Then $CA \neq AC'$ because C and C' are different.

There is only one block B with respect to S_4 and there are exactly two blocks $B_1^{'}, B_2^{'}$ with respect to D_6.Therefre we can see that $\varphi_1, \varphi_2 \in B$ but $\varphi_1^{'} \in B_1^{'}, \varphi_2^{'} \in B_2^{'}$ and $\eta_1, \eta_2 \in B$ but $\eta_1^{'} \in B_1^{'}, \eta_2^{'} \in B_2^{'}$.

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On the Extensions of Group Representations over Arbitrary Fields II

Kenichi Yamauchi

1. Introduction

Let G be a finite group with $N \lhd G$, and let L be any field. An L-representation $\mathcal{X}$ of N is said to be invariant in G, if for every $g \in G$, the representation $\mathcal{X}^g$ defined by $\mathcal{X}^g(n) = \mathcal{X}(gng^{-1})$ is similar to $\mathcal{X}$.

I.M.Isaacs proved the following theorem which is a generalization of Gallagher's theorem. (See Theorem A of [4]).

Let $N \lhd G$ be a Hall subgroup and L be an arbitrary field. Then every invariant irreducile L-representation of N is extendible to an L-representation of G.

And further, in the case that char(L)=0, I.M.Isaacs proved the following theorem which is the strengthen version of the above theorem. (See Corollary 6.4 of [4]).

Theorem 1.1.(I.M.Isaacs) *Let $N \lhd G$ and let $\mathcal{X}$ be an invariant irreducible L-representation of N, where L is an arbitrary subfield of an algebraically closed field E of characteristic zero. Let α be an irreducible E-constituent of $\mathcal{X}$ and assume that $(|G : N|, \alpha(1)o(\alpha)) = 1$, where $o(\alpha)$ is the determinantal order of α. Then $\mathcal{X}$ extends to an L-representation of G which has γ as an E-constituent, where $\gamma = (\hat{\alpha})^G$ and $\hat{\alpha}$ is the canonical extension of α to the inertia group $I_G(\alpha)$.*

If we don't assume that $(|G : N|, o(\alpha)) = 1$, then we wonder how Isaacs's theorem changes. In this paper we intend to consider the extensions of invariant irreducible L-representations of normal subgroups,when $(|G : N|, \alpha(1)) = 1$ and L is an arbitrary field.

In section 2,we will consider the following problem.That is, let $N \lhd G$ and $\mathcal{X}$ be an invariant irreducible L-representation of N.Suppose that $\mathcal{X}$ extends to an L-representation $\mathcal{X}'$ of G.Let α be an irreducible E-constituent of $\mathcal{X}$ and β be an irreducible E-constituent of $\mathcal{X}'|_T$ (the restriction of $\mathcal{X}'$ to T),where $E \supseteq L$ is an algebraically closed field and $T = I_G(\alpha)$ is the inertia group of α in G. Then we will study the relation between α and β. (See Theorem 2.3.)

In [5], we gave the necessary and sufficient condition on which an invariant irreducible L-representation $\mathcal{X}$ of N extends to to an L-representation of G , in the case that the Schur index of an irreducible E-constituent of $\mathcal{X}$ is equal to 1.

In section 3, we will consider the the removal of the assumption that the Schur index of an irreducible E-constituent of $\mathcal{X}$ is equal to 1.(See Theorem 3.1.)

Throughout this paper G denotes always a finite group with $N \lhd G$,Z the ring of rational integers, E an algebraically closed field with $E \supseteq L$ being an arbitrary field. Finally we note that the definitions and the notations in this paper are the same as those in Isaacs's paper [4].

2. Semi-standard extensions and crossed representations

Let E be an algebraically closed field. We write $Irr_E(G)$ to denote the set of irreducible E-characters of G.Suppose $L \subseteq E$ is an arbitrary field.Then $G(E/L)$ permutes $Irr_E(G)$ into finite orbits.

Now we define an L-semi-invariance.

Definition 2.1. Let $N \lhd G$ and $\alpha \in Irr_E(N)$. Then we say that α is L-semi-invariant in G if its Galois orbit over L is G-invariant.

Definition 2.2. Let $N \lhd G$ and let $\alpha \in Irr_E(N)$ be L-semi-invariant in G,where E is an algebraically closed field and $L \subseteq E$ is an arbitrary field.Let $T = I_G(\alpha)$ be the inertia group of α in G.Then we say that $\beta \in Irr_E(T)$ is a semi-standard extension of α provided that

(i) β is an extension of α

(ii) β is L-semi-invariant in G.

In particular we say that a semi-standard extension $\beta \in Irr_E(T)$ of α is a standard extension provided that $L(\alpha) = L(\beta)$, where $L(\alpha)$ and $L(\beta)$ are the fields generated over L by the values of α on N and by the values of β on T respectively.

Let $N \lhd G$ and assume that $\alpha \in Irr_E(N)$ is L-semi-invariant in G and $(|G : N|, m_L(\alpha)) = 1$ where E is algebraically closed ,$L \subseteq E$,and $m_L(\alpha)$ is the Schur index of α over L. Let $T = I_G(\alpha)$ and let $\mathcal{X}$ be an irreducible L-representationof N having α as an E-constituent so that $\mathcal{X}$ is invariant in G.Suppose that $\mathcal{X}$ extends to an L-representation $\mathcal{X}'$ of G.Let $\beta \in Irr_E(T)$ be an E-constituent of $\mathcal{X}'|_T$ (the restriction of $\mathcal{X}'$ to T) such that $\beta|_N$ has α as a constituent.

Then we have the following theorem.

Theorem 2.3. *In the above situation we have*

(i) *If char(L)=0,then β is a semi-standard extension of α and we have $m_L(\alpha)|L(\alpha) : L| = m_L(\beta)|L(\beta) : L|$.*

In particular β is a standard extension of α if and only if $m_L(\alpha) = m_L(\beta)$.

(ii) *If char(L)=0 and L contains a primitive m-th root of unity where m is the exponent of $(T/N)/D(T/N)$ and $D(T/N)$ is the commutator subgroup of T/N,then β is a standard extension of α.*

(iii) *If char(L)$\neq$ 0,then β is a standard extension of α.*

Proof. (i) We write $tr\mathcal{X}$ and $tr\mathcal{X}'|_T$ to denote the characters of $\mathcal{X}$ and $\mathcal{X}'|_T$ respectively. Since $\mathcal{X}$ and $\mathcal{X}'|_T$ are irreducible L-representations of N and T respectively, $tr\mathcal{X}$ and $tr\mathcal{X}'|_T$ decompose as follows

$$tr\mathcal{X} = m(\alpha_1 + \cdots + \alpha_r), m = m_L(\alpha), \alpha_1 = \alpha, r = |L(\alpha) : L| \qquad \cdots (2.1)$$

$$tr\mathcal{X}'|_T = n(\beta_1 + \cdots + \beta_s), n = m_L(\beta), \beta_1 = \beta, s = |L(\beta) : L| \qquad \cdots (2.2)$$

where $\alpha_i \in Irr_E(N)(i = 1, \cdots, r)$ and $\beta_j \in Irr_E(T)(j = 1, \cdots, s)$ are distinct and constitute orbits under $G(L(\alpha)/L)$ and $G(L(\beta)/L)$ respectively. (See Lemma 2.1 of [4].)

Since $T = I_G(\alpha)$ is the inertia group of α in G, $\beta|_N$ can be written as follows

$$\beta|_N = e\alpha, e \geq 1, e \in Z \qquad \cdots (2.3)$$

Here we will show that $e = 1$. By the formulas (2.1) and (2.2) we get

$$ens\alpha(1) = mr\alpha(1) \text{ and so } ens = mr \qquad \cdots (2.4)$$

By Corollary 11.29 of [3] we get $e \mid |T : N|$.

On the other hand,by the formula (2.3),the assumption that char(L)=0,yields that $L(\alpha) \subseteq L(\beta)$ and so we have

$$s = |L(\beta) : L| = |L(\beta) : L(\alpha)||L(\alpha) : L| = |L(\beta) : L(\alpha)|r$$

Hence r divides s. Therefore by the formula (2.4), we can see that e divides $m = m_L(\alpha)$. The fact that $e \mid |T : N|$ and the assumption that $(|G : N|, m_L(\alpha)) = 1$ imply that $e = 1$ as claimed.By the formula (2.4) we get $m_L(\alpha)|L(\alpha) : L| = m_L(\beta)|L(\beta) : L|$.

Since $\mathcal{X}'|_T$ is the restriction of $\mathcal{X}'$ to T, it follows that β is L-semi-invariant in G. Hence β is a semi-standard extension of α.

In particular if we assume that β is a standard extension of α, then $L(\alpha) = L(\beta)$ and so we have $r = s$. By the formula (2.4) and $e = 1$, we have $m_L(\alpha) = m_L(\beta)$.

Conversely if we assume that $m_L(\alpha) = m_L(\beta)$, then we have $s = r$ by the formula (2.4) and $e = 1$. Hence we obtain $L(\alpha) = L(\beta)$ because $L(\alpha) \subseteq L(\beta)$ holds. Therefore β is a standard extension of α.

(ii) Keeping the notations in (i), we will show that $|L(\beta) : L(\alpha)|$ divides $m_L(\alpha)$. By the formula (2.4) and $e = 1$, we have $ns = mr$. In the proof of (i) we also showed that $s = |L(\beta) : L(\alpha)|r$ and so we have $n|L(\beta) : L(\alpha)| = m$. Hence $|L(\beta) : L(\alpha)|$ divides $m = m_L(\alpha)$ as claimed.

On the other hand we will show that $|L(\beta) : L(\alpha)|$ divides $|T : N|$. For any $\sigma \in G(L(\beta)/L(\alpha))$, β^σ is an extension of α and so by Corollary of [2](p 225),there is a unique linear character μ_σ of T/N such that $\beta^\sigma = \mu_\sigma \beta$. Here we set $H = \{\mu_\sigma | \sigma \in G(L(\beta)/L(\alpha))\}$.

Then H forms a subgroup of the group $\widehat{T/N}$ consisting of all linear characters of T/N. In fact for $\sigma, \tau \in G(L(\beta)/L(\alpha))$, we get

$$\beta^{\sigma\tau} = (\mu_\sigma \beta)^\tau = \mu_\sigma^\tau \beta^\tau = \mu_\sigma(\mu_\tau \beta) = \mu_\sigma \mu_\tau \beta$$

because L contains a primitive m-th root of unity. Hence we have $\mu_\sigma \mu_\tau = \mu_{\sigma\tau} \in H$ and so H is a subgroup of $\widehat{T/N}$. Therefore we can see that $|L(\beta) : L(\alpha)| = |H|$ divides $|T : N|$ as required.By the assumption that $(|G : N|, m_L(\alpha)) = 1$,it follows that$|L(\beta) : L(\alpha)| = 1$. Consequently β is a standard extension of α.

(iii) Since char(L)$\neq$0,we get $m_L(\alpha) = m_L(\beta) = 1$ by Theorem 9.21 (b) of [3],and so $tr\mathcal{X}$ and $tr\mathcal{X}''|_T$ decompose as follows

$$tr\mathcal{X} = \alpha_1 + \cdots + \alpha_r, \alpha_1 = \alpha, r = |L(\alpha) : L|$$

$$tr\mathcal{X}'|_T = \beta_1 + \cdots + \beta_s, \beta_1 = \beta, s = |L(\beta) : L|.$$

Since $\beta|_N = e\alpha, e \geq 1, e \in Z$ and $\mathcal{X}'|_N = \mathcal{X}$, we get

$$1 = \text{the multiplicity of } \alpha \text{ in } \mathcal{X} \geq e.$$

Hence $e = 1$ and so we have $r = s$. Therefore $L(\alpha) = L(\beta)$ holds because $\beta|_N = \alpha$. Consequently β is a standard extension of α.

This completes the proof of Theorem 2.3. Q.E.D.

Remark. Let $N \lhd G$ and let $\alpha \in Irr_E(N)$ be L-semi-invariant in G.Assume that α has a standard extension $\beta \in Irr_E(T)$ where $T = I_G(\alpha)$. Then we note that $m_L(\alpha) = m_L(\beta)$ holds. In fact by setting $T = G$ in Lemma 2.3 of [4],we can see that $m_L(\beta)$ divides $m_L(\alpha)$. Hence $m_L(\beta) \leq m_L(\alpha)$. On the other hand, since $L(\alpha) = L(\beta)$ and $\beta|_N = \alpha$, we obtain $m_L(\alpha) \leq m_L(\beta)$. Therefore $m_L(\alpha) = m_L(\beta)$ holds.

Let $N \lhd G$ and assume that $\alpha \in Irr_E(N)$ is L-semi-invariant in G where E is algebraically closed and $L \subseteq E$. Let $T = I_G(\alpha)$ and let $\mathcal{X}$ be an irreducible L-representation of G having α as an E-constituent. Then we have the following theorem.

Theorem 2.4. *In the above situation, if α has a semi-standard extension $\beta \in Irr_E(T)$ such that $m_L(\alpha)|L(\alpha) : L| = m_L(\beta)|L(\beta) : L|$, then $\mathcal{X}$ extends to an L-representation of T having β as an E-constituent.*

Proof. Let $\hat{\mathcal{X}}$ be an irreducible L-representation of T having β as an E-constituent.Then $tr\hat{\mathcal{X}}$ can be written as follows

$$tr\hat{\mathcal{X}} = m_L(\beta)(\beta_1 + \cdots + \beta_s), \beta_1 = \beta, s = |L(\beta) : L|$$

where the $\beta_i \in Irr_E(T) \quad (i = 1, ..., s)$ are distinct and constitute an orbit under $G(L(\beta)/L)$. Hence we have

$$tr\hat{\mathcal{X}}|_N = m_L(\beta)(\beta_1|_N + \cdots + \beta_s|_N).$$

For any $\sigma \in G(L(\beta)/L(\alpha)), {\beta_1}^\sigma|_N = (\beta_1|_N)^\sigma = \alpha^\sigma = \alpha$ and so it follows that the multiplicity of α in $\hat{\mathcal{X}}|_N$ is equal to

$m_L(\beta)|L(\beta) : L(\alpha)| = m_L(\beta)|L(\beta) : L|/|L(\alpha) : L| = m_L(\alpha)|L(\alpha) : L|/|L(\alpha) : L| = m_L(\alpha)$.

Hence we have $tr\mathcal{X} = tr\hat{\mathcal{X}}|_N$ and so $\mathcal{X}$ is similar to $\hat{\mathcal{X}}|_N$.

The proof is complete. Q.E.D.

Remark. Hereafter we will treat the case that $(|G : N|, \alpha(1)) = 1$. Since $m_L(\alpha)$ divides $\alpha(1)$ by Corollary 10.2 (h) of [3],we note that the condition that $(|G : N|, m_L(\alpha)) = 1$ is automatically satisfied and so we can always apply Theorem 2.3 to our extendibility problems.

Now we define an F-crossed representation of G where F is a field, which is an important technique for extending representations.

Definition 2.5. Let F be an arbitrary field and let G act on F via field automorphisms. This action induces an action of G on $GL(r, F)$ for positive integer r. Then we say that a map $\mathcal{Z} : G \longrightarrow GL(r, F)$ is an F-crossed representation of G, provided that

$$\mathcal{Z}(gh) = \mathcal{Z}(g)^h \mathcal{Z}(h) \text{ for all } g, h \in G.$$

For $\alpha \in Irr_E(N)$ we write α_d to denote the determinant of α.

Let $N \lhd G$ and let $\mathcal{X}$ be an irreducible L-representation of N which is invariant in G where E is algebraically closed and $L \subseteq E$. Suppose $\alpha \in Irr_E(N)$ is an irreducible E-constituent of $\mathcal{X}$. Then α is L-semi-invariant in G and thus G acts on $L(\alpha)$. Let $\mathcal{X}^{L(\alpha)}$ denote the representation $\mathcal{X}$ when viewed as an $L(\alpha)$-representation of N and let $\mathcal{Y}$ be an irreducible constituent of $\mathcal{X}^{L(\alpha)}$ which has α as an E-constituent. Let $\mathcal{X}_\circ$ be an irreducible L-representation of N having α_d as an E-constituent. Then we have the following theorem.

Theorem 2.6. *In the above situation we assume further that the Schur index* $m_L(\alpha) = 1$ *so that* α *is afforded by an* $L(\alpha)$*-representation* $\mathcal{Y}$*. Suppose that* $(|G : N|, \alpha(1)) = 1$*. Then we have*

(I) *The following conditions are equivalent.*

(i) $\mathcal{X}$ *extends to an* L*-representation of* G*.*

(ii) *The* $L(\alpha)$*-representation* α_d *of* N *extends to an* $L(\alpha)$*-crossed representation of* G *with respect to the given action of* G *on* $L(\alpha)$*.*

(II) *If* $\mathcal{X}_\circ$ *extends to an* L*-representation of* G*, then* $\mathcal{X}$ *extends to an* L*-representation of* G*.*

Proof. (I) It is obvious by Theorem 2.3 of [5].

(II) Since $\mathcal{X}_\circ$ extends to an L-representation of G, we may apply Theorem 3.1 of [4] and conclude that α_d extends to an $L(\alpha_d)$-crossed representation ω_α of G. Since α is afforded by an $L(\alpha)$-representation $\mathcal{Y}$, we see that $L(\alpha_d) \subseteq L(\alpha)$. Because $(\alpha^g)_d = (\alpha_d)^g$ (See the proof of Theorem 3.2 (i)) and $(\alpha^\sigma)_d = (\alpha_d)^\sigma$ for $\sigma \in G(L(\alpha)/L)$, it is clear that the action of G on $L(\alpha_d)$with respect to which ω_α is a crossed representation, is just the restriction of the original action on $L(\alpha)$

to $L(\alpha_d)$. Therefore α_d extends to an $L(\alpha)$-crossed representation of G and so we may apply (I) and conclude that $\mathcal{X}$ extends to an L-representation of G. The proof is complete. Q.E.D.

If char(L)> 0, the Schur index $m_L(\alpha) = 1$ holds by Theorem 9.21 (b) of [3] and so we can always apply Theorem 2.6 to the field L of prime characteristic. As an application of Theorem 2.6 we will prove the following theorem. (See Theorem of [1])

Theorem 2.7.(B.Fein) *Let N be a normal Hall subgroup of G and let L be an arbitrary field with char(L)> 0. Let $\mathcal{X}$ be an invariant irreducible L-representation of N.Suppose that $(|G : N|, deg\mathcal{X}) = 1$. Then $\mathcal{X}$ is extendible to an L-representation of G.*

Proof. (See the proof of Lemma 6.2 of [4]) Let $\alpha \in Irr_E(N)$ be an E-constituent of $\mathcal{X}$ where $E \supseteq L$ is algebraically closed. Since char(L)> 0, $m_L(\alpha) = 1$ holds and so we have $(|G : N|, \alpha(1)) = 1$ by the assumption that $(|G : N|, deg\mathcal{X}) = 1$. Let $\mathcal{X}_\circ$ be an irreducible L-representation of N having α_d as an E-constituent. Then we will show that $\mathcal{X}_\circ$ extends to an L-representation of G. Hence we may apply Theorem 2.6 (II) and conclude that $\mathcal{X}$ extends to an L-representation of G. In order to prove that $\mathcal{X}_\circ$ is extendible to an L-representation of G, by Theorem 2.4 of [4] it suffices to prove that α_d has a standard extension because $m_L(\alpha) = 1$. Since α_d is L-semi-invariant in G, $\mathcal{X}_\circ$ is invariant in G and so it follows that the kernel of $\mathcal{X}_\circ$ is a normal subgroup of G. Hence we may assume that $\mathcal{X}_\circ$ is faithful. Since all of the E-constituent of $\mathcal{X}_\circ$ are Galois conjugate, they all have the same kernel and it follows that α_d is faithful and so N is cyclic of order equal to $o(\alpha_d)$. Let $T = I_G(\alpha_d)$ be the inertia group of α_d in G. Then N is central in T because

α_d is faithful. Since N is a normal Hall subgroup of G, there is a subgroup K of G such that $T = N \times K$ (a direct product). We set $\beta = \alpha_d \times 1_K \in Irr_E(T)$. Then β is the canonical extension of α_d to T. This completes the proof of Theorem 2.7.

Q.E.D.

3. Characteristic zero

In Theorem 2.6 we assumed that the Schur index $m_L(\alpha) = 1$. To remove this assumption we will state some variation of Theorem 6.3 of [4].

We fix an algebraically closed field E of characteristic zero and all other fields considered in this section will be subfields of E. Let T be a normal subgroup of G and assume that $\beta \in Irr_E(T)$ is L-semi-invariant in G where L is a subfield of E. Suppose that $(|G : T|, \beta(1)) = 1$ and $I_G(\beta)$ (the inertia group of β in G) is equal to T. Let ϵ be a primitive n-th root of unity in E where n is the exponent of G. Then there is a unique minimal field K, $L \subseteq K \subseteq L(\epsilon)$ such that $|L(\epsilon) : K|$ involves no prime dividing the Schur index $m_L(\beta)$ because $G(L(\epsilon)/L)$ is abelian and $m_L(\beta)$ divides $|L(\epsilon) : L|$. We fix K as above and set $\gamma = \beta^G$. Let $\mathcal{X}$ be an irreducible L-representation of T having β as an E-constituent and let $\mathcal{Y}$ be an irreducible K-representation of T having β as an E-constituent. Then we have the following theorem.

Theorem 3.1. *In the above situation we have*

(I) (i) *The schur index $m_K(\beta) = 1$.*

(ii) *β is K-semi-invariant in G.*

(II) *The following conditions are equivalent*

(i) *$\mathcal{X}$ extends to an L-representation of G.*

(ii) $m_L(\beta) = m_L(\gamma)$.

(iii) $\mathcal{Y}$ extends to a K-representation of G.

(iv) $(m_K(\beta) =)m_K(\gamma) = 1$.

Proof. (See the proofs of Theorem 6.3 of [4] and Theorem 4.1 of [5]) (I) (i) Suppose that p is a prime divisor of $m_K(\beta)$. Then p divides $m_L(\beta)$ by Corollary 10.2 (f) of [3] and hence $p \nmid |K(\epsilon) : K| (= |L(\epsilon) : K|)$ by the choice of a field K. Thus the Sylow p-subgroup of $G(K(\epsilon)/K(\beta))$ is trivial and Theorem 10.12 of [3] yields that $p \nmid m_K(\beta)$. This contradiction implies that $m_K(\beta) = 1$ as claimed.

(ii) Since β is L-semi-invariant in G, by Lemma 2.1 of [4] $G(L(\beta)/L)$ contains a subgroup H which is isomorphic to G/T. Since $m_L(\beta)$ divides $\beta(1)$ and $(|G : T|, \beta(1)) = 1$, we have that $(|G : T|, m_L(\beta)) = 1$. Since $G(K(\beta)/K)$ is isomorphic to $G(L(\beta)/L(\beta) \cap K)$, it follows that $|K(\beta) : K| = |L(\beta) : L(\beta) \cap K|$. Let M be a fixed field of H. Then we have $|L(\beta) : M| = |G : T|$. If a prime p divides $|L(\beta) \cap K : L|$, then $p | m_L(\beta)$ and if a prime p divides $|L(\beta) : L(\beta) \cap K|$, then $p \nmid m_L(\beta)$. These facts yields that M is a subfield of $L(\beta)$ which contains $L(\beta) \cap K$ as a subfield. Therefore $G(K(\beta)/K)$ contains a subgroup whose restriction to $L(\beta)$ is equal to H. This implies that β is K-semi-invariant in G.

(II) (i)$\Longrightarrow$(ii) Let γ' be the irreducible E-character of G whose restriction to T has β as a constituent. Then we get

$$1 \leq (\beta, \gamma'|_T)_T = (\beta^G, \gamma')_G = (\gamma, \gamma')_G \qquad \cdots (3.1)$$

Since $\gamma = \beta^G \in Irr_E(G)$, we have $\gamma = \gamma'$ by the formula (3.1). Therefore γ is the only irreducible E-character of G whose restriction to T has β as a constituent. If $\mathcal{X}$ extends to an L-representation $\mathcal{X}'$ of G, then it follows that γ is an E-constituent of $\mathcal{X}'$ and $tr\mathcal{X}$ and $tr\mathcal{X}'$ decompose as follows

$$tr\mathcal{X} = m_L(\beta)(\beta + \cdots), \quad tr\mathcal{X}' = m_L(\gamma)(\gamma + \cdots).$$

Since $(\beta, \gamma|_T)_T = 1$, the above equations yield that $m_L(\beta)$ is equal to $m_L(\gamma)$.

(ii)$\Longrightarrow$(i) Assume that $m_L(\beta) = m_L(\gamma)$. Since T is equal to $I_G(\beta)$, we can consider β as a standard extension and so by Theorem 2.4 of [4], it is obvious that $\mathcal{X}$ extends to an L-representation of G.

(iii)$\Longleftrightarrow$(iv) The proof is quite similar to that of (i)$\Longleftrightarrow$(ii).

(ii)$\Longrightarrow$(iv) Assume that $m_L(\beta) = m_L(\gamma)$. Then we claim that $m_K(\gamma) = 1$, for suppose that p is a prime divisor of $m_K(\gamma)$. Hence p divides $m_L(\gamma) = m_L(\beta)$ by Corollary 10.2 (f) of [3] and so $p \nmid |K(\epsilon) : K|$. Thus the Sylow p-subgroup of $G(K(\epsilon)/K(\gamma))$ is trivial and Theorem 10.12 of [3] yields that $p \nmid m_K(\gamma)$. This contradiction implies that $m_K(\gamma) = 1$ as claimed.

(iv)$\Longrightarrow$(ii) The proof that $m_L(\beta)$ divides $m_L(\gamma)$ is similar to the beginning of the proof of Theorem 6.3 of [4] and so we omit its proof.

Conversely we will show that $m_L(\gamma)$ divides $m_L(\beta)$. Since $m_K(\gamma) = 1$, $m_L(\gamma)$ divides $|K : L|$ by Corollary 10.2 (g) of [3]. By the choice of K it follows that all prime divisors of $|K : L|$ are divisors of $m_L(\beta)$ and thus divide $\beta(1)$. These primes do not divide $|G : T|$ by the assumption that $(|G : T|, \beta(1)) = 1$ and so $m_L(\gamma)$ divides $m_L(\beta)$ by Lemma 2.3 of [4] and the fact that β is a standard extension. Therefore we have $m_L(\beta) = m_L(\gamma)$. This completes the proof of Theorem 3.1.

Q.E.D.

As an application of Theorem 3.1 we will prove Theorem 1.1 (I.M.Isaacs).

Proof of Theorem 1.1. To begin with we note that α has a standard extension $\hat{\alpha}$ such that $o(\alpha) = o(\hat{\alpha})$. (See the note below Definition 2.2 of [4]). By Theorem 2.4 it follows that $\mathcal{X}$ extendsto an L-representation $\hat{\mathcal{X}}$ of $T = I_G(\alpha)$

having $\hat{\alpha}$ as an E-constituent. To prove that $\hat{\mathcal{X}}$ extends to an L-representation of G, by Theorem 3.1 (II) (iii) it is no loss to assume that the Schur index $m_L(\hat{\alpha}) = 1$. Let $\hat{\mathcal{X}}_\circ$ be an irreducible L-representation of T having $\hat{\alpha}_d$ as an E-constituent. Then by Lemma 6.2 of [4] it follows that $\hat{\mathcal{X}}_\circ$ extends to an L-representation of G because $(|G : N|, o(\alpha)) = 1$ and $o(\alpha) = o(\hat{\alpha})$. By Theorem 2.6 we see that $\hat{\mathcal{X}}$ extends to an L-representation of G. The proof is complete. Q.E.D.

Let $N \lhd G$ and let $\mathcal{X}$ be an invariant irreducible L-representation of N where $L \subseteq E$ is an arbitrary field. Suppose that $\mathcal{X}$ extends to an L-representation of G. Let $\alpha \in Irr_E(N)$ be an E-constituent of $\mathcal{X}$. Then by Theorem 3.1 of [4] we can see that α_d^m extends to an $L(\alpha)$-crossed representation of G with respect to the given action of G on $L(\alpha)$ where $m = m_L(\alpha)$ is the Schur index of α over L. (See also the proof of Theorem 2.3 of [5])

Conversely we will consider the following situation. Let $\alpha \in Irr_E(N)$ be L-semi-invariant in G where $L \subseteq E$ so that G acts on $L(\alpha)$ and assume that $(|G : N|, \alpha(1)) = 1$. Suppose that α_d extends to an $L(\alpha)$-crossed representation ω_α of G with respect to the given action of G on $L(\alpha)$. We set $T = I_G(\alpha)$. Since each element of T acts trivially on $L(\alpha)$, for any $x, y \in T$ an equation $\omega_\alpha(xy) = \omega_\alpha(x)\omega_\alpha(y)$ holds. That is, $\omega_\alpha|_T$ (the restriction of ω_α to T) is an extension of α_d. Then we have the following theorem.

Theorem 3.2. *In the above situation we have*

(i) *There is a unique character $\beta \in Irr_E(T)$ such that $\beta|_N = \alpha$ and $\beta_d = \omega_\alpha|_T$. (In this case we say that β is determined by α and ω_α.) In addition β is L-semi-invariant inG.*

(ii) *Let $\hat{\mathcal{X}}$ be an irreducible L-representation of T having β as an E-constituent where β is determined by α and ω_α. Then $\hat{\mathcal{X}}$ extends to an L-representation of G.*

(iii) *Let $\beta \in Irr_E(T)$ be the character which is determined by α and ω_α such that $m_L(\alpha)|L(\alpha) : L| = m_L(\beta)|L(\beta) : L|$ and let $\mathcal{X}$ be an irreducible L-representation of N having α as an E-constituent. Then $\mathcal{X}$ extends to an L-representation of G.*

Proof. By Theorem 5 of [2], it is obvious that there is a unique character $\beta \in Irr_E(T)$ such that $\beta|_N = \alpha$ and $\beta_d = \omega_\alpha|_T$. Next we will prove that β is L-semi-invariant in G. Since α is L-semi-invariant in G, for $g \in G$ we can write $\alpha^g = \alpha^\sigma$ for some $\sigma \in G(L(\alpha)/L)$. And there is an automorphism $\hat{\sigma} \in G(L(\beta)/L)$ such that $\hat{\sigma}|_{L(\alpha)} = \sigma$. It follows that

$$\beta^{\hat{\sigma}}|_N = (\beta|_N)^{\hat{\sigma}} = \alpha^{\hat{\sigma}} = \alpha^\sigma \text{ and } (\beta^{\hat{\sigma}})_d = (\beta_d)^{\hat{\sigma}} = (\omega_\alpha|_T)^{\hat{\sigma}} = (\beta_d)^\sigma$$

because $\omega_\alpha(x) \in L(\alpha)$ for every $x \in G$.

On the other hand we get

$$\beta^g|_N = (\beta|_N)^g = \alpha^g = \alpha^\sigma \text{ and } (\beta^g)_d = (\beta_d)^g = (\omega_\alpha|_T)^g = (\omega_\alpha|_T)^\sigma = (\beta_d)^\sigma$$

because for every $x \in T$

$(\beta^g)_d(x) = detX^g(x) = detX(gxg^{-1}) = \beta_d(gxg^{-1}) = (\beta_d)^g(x)$ and so we have $(\beta^g)_d = (\beta_d)^g$ where X is an E-representation of T which affords β.

By Theorem 5 of [2] we have $\beta^{\hat{\sigma}} = \beta^g$. Consequently β is L-semi-invariant in G.

(ii) Since β is L-semi-invariant in G and $I_G(\beta)$ is equal to T, for β we can take a field $K, L \subseteq K \subseteq L(\epsilon)$ which we determined in Theorem 3.1 where ϵ is a primitive n-th root of unity in E and n is the exponent of G. Since $\omega_\alpha|_T = \beta_d$ and ω_α is an $L(\alpha)$-crossed representation of G, β_d extends to an $L(\alpha)$-crossed representation of G. Consequently it follows that β_d extends to an $K(\beta)$-crossed representation of

G because $L(\alpha) \subseteq L(\beta) \subseteq K(\beta)$ and β is K-semi-invariant in G by Theorem 3.1 (I) (ii). Since the Schur index $m_K(\beta) = 1$ by Theorem 3.1 (I) (i), we can see by Theorem 2.6 and Theorem 3.1 (II) (iii) that $\hat{\mathcal{X}}$ extends to an L-representation of G.

(iii) Let $\hat{\mathcal{X}}$ be an irreducible L-representation of T having β as an E-constituent. Then we showed in (ii) that $\hat{\mathcal{X}}$ extends to an L-representation of G. By the assumption that $m_L(\alpha)|L(\alpha) : L| = m_L(\beta)|L(\beta) : L|$, we can see that $\hat{\mathcal{X}}|_N$ is similar to $\mathcal{X}$. (See the proof of Theorem 2.4) Hence $\mathcal{X}$ extends to an L-representation of G. This completes the proof of Theorem 3.2. Q.E.D.

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Department of Mathematics

Faculty of Education
Chiba University
Chiba 263-8522
Japan

J.A.Greenの定理の別証明

山内 憲 一　　　　　千葉大学 教育学部

Gを有限群とし、Gの表現はすべて複素数体C上で考えることにする。Gの既約指標の全体を$\chi_1, \chi_2, ..., \chi_h$とする。$Z$は有理整数全体のなす環で、$A(\supseteq Z)$を代数的整数から成る$C$の任意の部分環とする。さらに次の記号を定義する。

$char(G) :=$ Gの指標環,　$char_A(G) := \{\sum_{i=1}^{h} a_i\chi_i | a_i \in A, i = 1, 2, ..., h\}$,
$cf(G) :=$ 複素数値を取るG上の類関数全体の集合,　$\mathcal{H} :=$ Gの或る部分群の集合

このとき次に述べる定理はよく知られているものである。

定理 1　次の(i),(ii),(iii),(iv)は同値である。

(i) 任意の$\theta \in cf(G)$に対し、$\theta|_H$(Hへの制限)$\in char(H)$ for all $H \in \mathcal{H}$ならば、$\theta \in char(G)$である。

(ii) 任意の$\theta \in cf(G)$に対し、$\theta|_H \in char_A(H)$ for all $H \in \mathcal{H}$ならば、$\theta \in char_A(G)$である。

(iii) $\sum_{H\in\mathcal{H}}\{char(H)\}^G = char(G)$

(iv) Gの各基本部分群は$\mathcal{H}$に属する或る部分群の共役に含まれる。

上の定理で(i)$\Longleftrightarrow$(iii)及び(ii)$\Longleftrightarrow$(iii)の証明はBrauerの定理3 in[**1**]の証明を見習えば出来る。Greenは(i)$\Longrightarrow$(iv) (i.e. BrauerのInduction theoremの逆)を証明するのに、誘導指標に関するFrobeniuusの公式を用いて(iii)$\Longrightarrow$(iv)の証明を与えている。我々はまた$A \ni \epsilon$(ここにϵは1の原始$|G|$乗根)の場合に、Gの特性関数を用いて、(ii)$\Longrightarrow$(iv)の証明を与えた。本当は(i)$\Longrightarrow$(iv)の直接の証明が欲しいのですが、これは相当に難しいと思われます。

我々の目標はMackey decomposition theoremを用いて(iii)$\Longrightarrow$(iv)の(Greenの証明とは異なる)別証明を与えることである。

これからはAは1の原始$|G|$乗根を含むものとする。

Lemma 2　$E =< y > \times P$はGの$p-$基本部分群で$E_\circ$はEの真の部分群であるとする。このときθを$E_\circ$の一般指標とすれば

$$Ind_{E_\circ}^{E}(\theta)(y) \in pA$$

がなりたつ。

(iii)$\Longrightarrow$(iv)の証明の要点

(iii)の仮定より

$$1_G = \sum_{H\in\mathcal{H}} \sum_{\lambda} \ a_{H,\lambda} \ Ind_H^G(\lambda)$$

Makey decomposition theorem より

$$Res_E^G(1_G) = 1_E = \sum_{H\in\mathcal{H}} \sum_\lambda \ a_{H,\lambda}\ Res_E^G(Ind_H^G(\lambda))$$
$$= \sum_{H\in\mathcal{H}} \sum_\lambda \ a_{H,\lambda} \sum_{t\in T}\ Ind_{H^t\cap E}^E\ (\lambda^t|_{H^t\cap E})$$

$p-$ 基本群 E が $\mathcal{H}$ に属する部分群のどの共役にも含まれないならば $H^t\cap E$ は E の真の部分群である。よって Lemma 2 より

$$Ind_{H^t\cap E}^E(\lambda^t|_{H^t\cap E})(y) \in pA$$

よって $1_E(y) \in pA$ を得る。これは $1_E(y) = 1$ に反する。

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