A three terms Arithmetic-Geometric mean

Kenji Koike
Faculty of Education
University of Yamanashi
Takeda 4-4-37,Kofu
Yamanashi 400-8510, Japan
kkoike@yamanashi.ac.jp

Hironori Shiga
Inst. of Math. and Physics
Chiba University
Yayoi–cho 1–33, Inage–ku
Chiba 263–8522, Japan
shiga@math.s.chiba-u.ac.jp

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0 Introduction

After the discovery of arithmetic geometric mean of Gauss in 1796, there has been proposed several its variants and generalizations. There are some interesting studies among them, like the cubic-AGM of Borweins in two terms case [B-B]. Also it has been aimed to find a new three terms AGM, for example the trial of Richelot ([R], also [B-M]). But up to present time there was no nicely settled theory for it. In this article we show a new AGM of three terms that has an expression by the Appell hypergeometric function. Our AGM is related to the Picard modular form appeared in [S], this story will be published elsewhere.

1 Definition

Definition 1.1 *Let a, b, c be real positive numbers satisfying $a \geq b \geq c$. Set*

$$\Psi(a,b,c) = (\alpha,\beta,\gamma) = (\frac{a+b+c}{3}, \sqrt[3]{A}, \sqrt[3]{B}), \tag{1.1}$$

with

$$\begin{cases} A = \frac{1}{6}(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2) + \frac{\sqrt{-1}}{6\sqrt{3}}(a-c)(a-b)(b-c) \\ B = \frac{1}{6}(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2) - \frac{\sqrt{-1}}{6\sqrt{3}}(a-c)(a-b)(b-c). \end{cases}$$

Here we choose the areguments of $\beta = \sqrt[3]{A}$ and $\gamma = \sqrt[3]{B}$ such that

$$0 \leq \arg \sqrt[3]{A} < \frac{\pi}{6},\ 0 < \beta + \gamma.$$

Remark 1.1 *If we make a permutation of a, b, c, it causes only the difference of the choice of complex conjugates.*

Lemma 1.1 *For a triple $(\alpha, \beta, \gamma) = \Psi(a, b, c)$, we find uniquely determined real triple $(x, y, z) = T(\alpha, \beta, \gamma)$ such that*

$$(\alpha,\beta,\gamma) = (x,y,z)\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}, \quad (\omega = \exp(2\pi i/3))$$

and

$$x > |y|, x > |z|.$$

proof].

We obtain the assertion by an easy direct calculation.

q.e.d.

For a general triple (a,b,c) of complex numbers and $(x,y,z)=T(a,b,c)$, we have

$$x=\alpha,\ x^3-y^3=\beta^3,\ x^3-z^3=\gamma^3$$

with $(\alpha,\beta,\gamma)=\Psi(a,b,c)$, and we don't mind the branches of β,γ.

By using the above notations, we have

Proposition 1.1 *For a real triple (a,b,c) with $a\geq b\geq c$ we can determine*

$$\Psi(\Psi(a,b,c))=(a',b',c')=(x,\sqrt[3]{x^3-y^3},\sqrt[3]{x^3-z^3}). \tag{1.2}$$

Especially the twice composite of Ψ induces a real positive triple again.

Lemma 1.2 *We have*

$$|\beta^3-\gamma^3|\geq|\beta-\gamma|^3,$$
$$|\beta^3-\alpha^3|\geq|\beta-\alpha|^3$$

proof].

Put $\beta=s+it$. Then
$$|\beta^2+\beta\gamma+\gamma^2|\geq|\beta-\gamma|^2\Longleftrightarrow s^2\geq t^2/3.$$
Because of our choice of the argument the first inequality holds.

We have
$$|\alpha^3-\beta^3||\alpha^3-\gamma^3|=27(x^2+xy+y^2)(x^2+xz+z^2)(y^2+yz+z^2)$$
and
$$|\alpha-\beta|^3|\alpha-\gamma|^3=27(y^2+yz+z^2)^3.$$
It holds
$$\begin{cases}(x^2+xy+y^2)-(y^2+yz+z^2)=(x-z)(x+y+z)=\alpha(x-z)>0\\(x^2+xz+z^2)-(y^2+yz+z^2)=(x-y)(x+y+z)=\alpha(x-z)>0.\end{cases}$$
So we obtain the second inequality.

q.e.d.

Proposition 1.2 *We have*

$$\begin{cases}|\beta^3-\alpha^3|^2=\frac{1}{18^3}((a-b)^2+(b-c)^2+(a-c)^2)^3,\\ \\ |\beta^3-\gamma^3|=\frac{1}{3\sqrt{3}}(a-c)(b-c)(a-b)\end{cases}$$

and

$$\begin{cases}|\beta-\gamma|\leq|\leq\frac{1}{\sqrt[3]{4}\sqrt{3}}(a-c),\\ \\ |\alpha-\beta|\leq\frac{1}{3}(a-c),\ |\alpha-\gamma|\leq\frac{1}{3}(a-c).\end{cases}$$

proof].

We have

$$|\beta^3-\alpha^3|^2=\frac{1}{3^6}\left(a^2-a\,b+b^2-a\,c-b\,c+c^2\right)^3=\frac{1}{3^6 2^3}((a-b)^2+(b-c)^2+(a-c)^2)^3.$$

So we get the equalities. By using the equalities and the above Lemma together with the inequalities

$$(a-b)^2+(b-c)^2\leq(a-c)^2,\ (b-c)(a-b)\leq\frac{1}{4}(a-c)^2,$$

we obtain the required inequalities.

q.e.d.

Proposition 1.3 *We have*

$$|b' - c'| < 0.1982(a - c), \ |a' - b'| < 0.2255(a - c), \ |a' - c'| < 0.2255(a - c). \tag{1.3}$$

[proof].

Acoording to the above Proposition we have:

$$|b' - c'|^3 \le |b'^3 - c'^3| = |y^3 - z^3| = \frac{1}{3\sqrt{3}}|\beta - \gamma||\alpha - \beta|^2$$
$$\le \frac{1}{3\sqrt{3}}\frac{1}{\sqrt[3]{4}\sqrt{3}}(a - c)\frac{1}{9}(a - c)^2.$$

So it holds

$$|y' - z'| \le 2^{(-2/9)}3^{(-4/3)}(a - c) < 0.1982(a - c), |y - z| \le 2^{(-2/9)}3^{(-4/3)}(a - c) < 0.1982(a - c).$$

$$\sqrt{3}|y - \omega^2 z| = |\alpha - \beta| \le \frac{1}{3}(a - c).$$

Hence

$$|b' - a'|^3 \le |b'^3 - a'^3| = |y|^3 \le (\frac{1}{\sqrt{3}}(|y - z| + |z - \omega y|))^3 \le [\frac{1}{\sqrt{3}}\{2^{(-2/9)}3^{(-4/3)} + 3^{(-3/2)}\}]^3(a - c)^3.$$

So we have

$$|a' - b'| < 0.2255(a - c).$$

By changing the roles of β and γ we have

$$|a' - c'| < 0.2255(a - c).$$

q.e.d.

By Proposition 1.2 and Proposition 1.3 we have the following.

Theorem 1.1 *Let a, b, c be real positive numbers satisfying the condition $a \ge b \ge c$. Set $\Psi^n(a, b, c) = (a_n, b_n, c_n)(= \Psi(a_{n-1}, b_{n-1}, c_{n-1}))$. Then there is a common limit*

$$M3(a, b, c) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

2 Functional equation and Appell's HGDE

2.1 Functional equation

Let x, y be real positive numbers. By the limit process of

$$\Psi(1, x, y) = (\frac{1 + x + y}{3}, \beta(1, x, y), \gamma(1, x, y)) = \frac{1 + x + y}{3}(1, \frac{3\beta}{1 + x + y}, \frac{3\gamma}{1 + x + y})$$

we have

$$M3(1, x, y) = \frac{1 + x + y}{3}M3(1, \frac{3\beta}{1 + x + y}, \frac{3\gamma}{1 + x + y}). \tag{2.1}$$

Set

$$H(x, y) = \frac{1}{M3(1, x, y)}.$$

Proposition 2.1 *We have*

$$H(x, y) = \frac{3}{1 + x + y}H(\frac{3\beta}{1 + x + y}, \frac{3\gamma}{1 + x + y}).$$

2.2 Appell's Hypergeometric function F_1

According to Appell [App] we define the hypergeometric function F_1 with parameters a, b, b', c (see also [Y]):

Definition 2.1

$$F_1(a,b,b',c;x,y) = \sum_{m,n\geq 0} \frac{(a,m+n)(b,m)(b',n)}{(c,m+n)m!n!} x^m y^n, \tag{2.2}$$

,where we use the conventional notation

$$(\lambda,k) = \begin{cases} \lambda(\lambda+1)\cdots(\lambda+k-1), & (k\geq 1) \\ 0, & (k=0) \end{cases}$$

$f = F_1(a,b,b',c;x,y)$ satisfies the following linear partial differential equation

$$\begin{cases} x(1-x)f_{xx} + (1-x)yf_{xy} + (c-(a+b+1)x)f_x - byf_y - abf = 0 \\ y(1-y)f_{yy} + (1-y)xf_{xy} + (c-(a+b'+1)y)f_y - b'xf_x - ab'f = 0 \\ (x-y)f_{xy} - b'f_x + bf_y = 0. \end{cases} \tag{2.3}$$

It has a three-dimensional space of solutions at a regular point, and it has unique holomorphic solution up to constant at the origin. The third equation is derived from the first and the second.

3 Differential equation

Proposition 3.1 *We have*

$$F_1(1/3,1/3,1/3,1;1-x^3,1-y^3) = \frac{3}{1+x+y} F_1\left(1/3,1/3,1/3,1;\left(\frac{1+\omega x+\omega^2 y}{1+x+y}\right)^3,\left(\frac{1+\omega^2 x+\omega y}{1+x+y}\right)^3\right)$$

.

We use the abreviation

$$F_1(x,y) = F_1(1/3,1/3,1/3,1;x,y),$$

and set

$$z(x,y) = F_1(1/3,1/3,1/3,1;1-x^3,1-y^3).$$

We describe the differental equation for $z(x,y)$ and deform it to an equivalent system under a change of variables. At first we obtain the system

$$\begin{cases} \delta_1 = \frac{1}{x}(1-x^3)\partial_{xx} + \frac{x}{y^2}(1-y^3)\partial_{xy} - 3x\partial_x + \frac{1}{y^2}(1-y^3)\partial_y - 1 \\ \delta_2 = \frac{1}{y}(1-y^3)\partial_{yy} + \frac{y}{x^2}(1-x^3)\partial_{xy} - 3y\partial_y + \frac{1}{x^2}(1-x^3)\partial_x - 1 \\ \delta_3 = (x^3-y^3)\partial_{xy} - y^2\partial_x + x^2\partial_y \end{cases} \tag{3.1}$$

for $z(x,y)$, where we use the convention $\partial_x = \frac{\partial}{\partial x}$ etc. Put

$$\begin{cases} X = \frac{1+\omega x+\omega^2 y}{1+x+y}, \\ Y = \frac{1+\omega^2 x+\omega y}{1+x+y}, \end{cases}$$

$$\begin{cases} P = 1+X+Y \\ Q = 1+\omega^2 X+\omega Y \\ R = 1+\omega X+\omega^2 Y \end{cases}$$

and

$$\begin{cases} D_X = (\omega - X)\partial_X + (\omega^2 - Y)\partial_Y \\ D_Y = (\omega^2 - X)\partial_X + (\omega - Y)\partial_Y. \end{cases}$$

Then we have

$$(1+x+y)(1+X+Y) = 3$$

and

$$\begin{cases} \partial_x = \frac{1+X+Y}{3} D_X \\ \partial_y = \frac{1+X+Y}{3} D_Y. \end{cases}$$

We can rewrite the system (3.1) in terms of X, Y, D_X, D_Y:

$$\begin{cases} \delta_1^* = \frac{1}{9Q}(P^3 - Q^3)(D_X D_X - D_Y) + \frac{Q}{9R^2}(P^3 - R^3)(D_X D_Y - D_Y) - QD_X + \frac{1}{3R^2}(P^3 - R^3)D_Y - 1 \\ \delta_2^* = \frac{1}{9R}(P^3 - R^3)(D_Y D_Y - D_X) + \frac{R}{9Q^2}(P^3 - Q^3)(D_Y D_X - D_X) - RD_Y + \frac{1}{3Q^2}(P^3 - Q^3)D_X - 1 \end{cases}$$

Namely, we have

$$\delta_1 z = \delta_2 z = \delta_3 z = 0 \Longleftrightarrow \delta_1^* Z = \delta_2^* Z = 0,$$

where $z = z(x, y) = Z(X, Y) = Z$. We obtain the system

$$\begin{cases} \delta_1^\sharp = \frac{1}{9Q}(P^3 - Q^3)(D_X D_X - 3D_X + 2) + \frac{Q}{9R^2}(P^3 - R^3)(D_X D_Y - D_X - 2D_Y + 2) \\ -Q(D_X - 1) + \frac{1}{3R^2}(P^3 - R^3)(D_Y - 1) - 1 \\ \delta_2^\sharp = \frac{1}{9QR}(P^3 - R^3)(D_Y D_Y - 3D_Y + 2) + \frac{R}{9Q^2}(P^3 - Q^3)(D_Y D_X - D_Y - 2D_X + 2) \\ -R(D_Y - 1) + \frac{1}{3Q^2}(P^3 - Q^3)(D_X - 1) - 1, \end{cases}$$

for $\tilde{Z} = \frac{Z}{1+X+Y} (= \frac{1+x+y}{3} z(x, y))$. Then we have

$$\delta_1^* Z(X, Y) = \delta_2^* Z(X, Y) = 0 \Longleftrightarrow \delta_1^\sharp \tilde{Z}(X, Y) = \delta_2^\sharp \tilde{Z}(X, Y) = 0.$$

On the other hand, we can rewrite the system (2.3) for the function $F_1(X^3, Y^3)$ that is coming from the right hand side. Namely we get

$$\begin{cases} \delta_1^\flat = \frac{1}{X}(1 - X^3)\partial_{XX} + \frac{Y}{X^2}(1 - X^3)\partial_{XY} + \frac{1}{X^2}(1 - 3X^3)\partial_X - Y\partial_Y - 1 \\ \delta_2^\flat = \frac{1}{Y}(1 - Y^3)\partial_{YY} + \frac{X}{Y^2}(1 - Y^3)\partial_{XY} + \frac{1}{Y^2}(1 - 3Y^3)\partial_Y - X\partial_X - 1 \\ \delta_3^\flat = \frac{1}{X^2Y^2}(Y^3 - X^3)\partial_{XY} + \frac{1}{X^2}\partial_X - \frac{1}{Y^2}\partial_Y. \end{cases}$$

By direct calculation we get the following equality of the differential operators:

$$\begin{cases} \frac{QR}{P}(R\delta_1^\sharp + Q\delta_2^\sharp) = X(1+2X)(X - Y^2)\delta_1^\flat + Y(1+2Y)(Y - X^2)\delta_2^\flat + XY(Y - X)(XY - 1)\delta_3^\flat \\ \frac{QR}{\sqrt{-3}P}(R\delta_1^\sharp - Q\delta_2^\sharp) = -X(X - Y^2)\delta_1^\flat + Y(Y - X^2)\delta_2^\flat + XY(X + Y)(XY - 1)\delta_3^\flat. \end{cases}$$

So the function $F_1(X^3, Y^3)$ in the right hand side satisfies the same hypergeometric differential equation for $\tilde{Z}(X, Y)$. Because $F_1(1 - x^3, 1 - y^3)_{|x=y=1} = F_1(0, 0) = 1$, we obtain the required equality.

q.e.d.

4 Main theorem

Theorem 4.1 *We have*

$$\frac{1}{M3(1, x, y)} = H(x, y) = F_1(1/3, 1/3, 1/3, 1; 1 - x^3, 1 - y^3).$$

in a neiborhood of $(x, y) = (1, 1)$.

proof].

Set

$$\varphi(x,y) = H(x,y)/F_1(1-x^3, 1-y^3), \Psi(1,x,y) = (\alpha, \beta, \gamma).$$

According to the functional equation, we have

$$\varphi(x,y) = \varphi(\beta/\alpha, \gamma/\alpha).$$

By the iteration of this argument we obtain

$$\varphi(x,y) = \varphi(1,1) = 1.$$

q.e.d.

5 $M3(1,x,y)$ as a period integral

According to Appell we have integral representations for $F_1(a,b,b',c;x,y)$ as follows:

Theorem 5.1 *(Appell [App])*

(1) If we have $\Re(a) > 0, \Re(c-a) > 0$, it holds

$$F_1(a,b,b',c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}(1-yu)^{-b'}du$$
$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^\infty u'^{b+b'-c}(u'-1)^{c-a-1}(u'-x)^{-b}(u'-y)^{-b'}du', \quad (|x|<1, |y|<1).$$

(2) If we have $\Re(b) > 0, \Re(b') > 0, \Re(c-b-b') > 0, |x| < 1, |y| < 1$, it holds

$$F_1(a,b,b',c;x,y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \iint_{u,v,1-u-v\geq 0} u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}(1-xu-yv)^{-a}dudv.$$

So our $M3(1,x,y)$ has expressions as period integrals for a family of Picard curves and at the same time for a family of certain elliptic $K3$ surfaces (see [S]).

Theorem 5.2 *We have*

$$\frac{1}{M3(1,x,y)} = \frac{1}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \int_1^\infty \frac{du}{v} = \frac{1}{(\Gamma(\frac{1}{3}))^3} \iint_{u,v,1-u-v\geq 0} \frac{dudv}{w},$$
$$v^3 = u(u-1)(u-1+x^3)(u-1+y^3),$$
$$w^3 = uv(1-u-v)(1-u(1-x^3)-v(1-y^3))$$
for $|x^3-1| < 1, |y^3-1| < 1.$

Where, we choose the real positive branches of v and w for real x, y.

Remark 5.1 *If we put $x = y$, our M3 councides with the "cubic AGM" discovered by Borweins* [B-B].

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