Isogeny formulas for the Picard modular form and a three terms AGM

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Introduction

In this paper we study the theta contants appeared in [S] those induced the modular function for the family of Picard curves $C(\xi)$ given by (1). Our theta constants $\vartheta_k(u,v) \quad (k = 0,1,2)$, given by (3), are "Neben type" modular forms of weight 1 defined on the complex 2-dimensional hyperball $\mathbb{B}$, given by (2), with respect to a index finite subgroup Γ_ϑ of the Picard modular group $\Gamma = PGL(M, \boldsymbol{Z}[\exp(2\pi i/3)])$. We define a simultaneous isogeny for the family of Jacobian varieties of $C(\xi)$. Our main result is stated in Theorem (3.1). There we show the explicit relations between theta constants $\vartheta_k(u,v)$ and $\vartheta_k(\sqrt{-3}u, 3v)$ which are corresponding to isogenous Jacobian varieties. In the theory of elliptic theta functions we have the relation

$$\begin{cases} \vartheta_{00}^2(2\tau) = \frac{1}{2}(\vartheta_{00}^2(\tau) + \vartheta_{01}^2(\tau)) \\ \vartheta_{01}^2(2\tau) = \vartheta_{00}(\tau)\vartheta_{01}(\tau). \end{cases}$$

This classical formula corresponds to the Gauss AGM process. Our formula plays an analogous role in the generalized AGM argument. We made an announcement of the result concerning our new AGM in Theorem 4.2, and the proof will be published elsewhere.

1 Picard modular revisted

The Picard modular functions are the counterparts of the elliptic λ-function for the family of the Picard curves of genus 3 ([P]):

$$C(\lambda_1, \lambda_2) : y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2).$$

Same as the case of $\lambda(\tau)$, we get an expression of λ_1 and λ_2 in terms of the Reiemann theta constants ([S]) that is the inverse of the period map and defined on the period domain. There are several detailed studies on the Picard modular forms ([F], [H], [S]).

Here we review the known results which we use in our arguments. We express the Picard curve with the projective parameters:

$$C(\xi) : y^3 = x(x-\xi_0)(x-\xi_1)(x-\xi_2), \tag{1}$$

where

$$\xi \in \Xi = \{[\xi_0 : \xi_1 : \xi_2] \in \mathbb{P}^2 : \xi_0\xi_1\xi_2(\xi_0-\xi_1)(\xi_1-\xi_2)(\xi_2-\xi_0) \neq 0\}.$$

The Jacobian variety $Jac(C(\xi))$ of $C(\xi)$ has a generalized complex multiplication by $\sqrt{-3}$ of type $(2,1)$. The moduli space of Picard curves is given as a quotient of a complex ball

$$\mathbb{B} = \{\eta = [\eta_0 : \eta_1 : \eta_2] \in \mathbb{P}^2(\mathbb{C}) : {}^t\eta M \overline{\eta} < 0\}, \quad M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

by the Picard modular group

$$\Gamma = \{g \in \mathrm{GL}_3(\mathbb{Z}[\omega]) : {}^t\overline{g} M g = M\}, \quad \omega = \exp(2\pi\sqrt{-1}/3).$$

Remark 1.1. *The period map $\Xi \longrightarrow \mathbb{B}$ is defined by*

$$\xi \mapsto [\int_{A_1} \frac{dx}{y} : -\omega^2 \int_{B_1} \frac{dx}{y} : \int_{A_2} \frac{dx}{y}]$$

for certain 1-cycles A_i and B_i on $C(\xi)$ ([P], [S]).

Henceforth we denote $g = (g_{ij})_{1\le i,j\le 3}$ whenever we need an elementwise argument of $g \in \Gamma$. Let $\mathbb{H}_3$ be the Siegel upper half space of degree 3. Setting

$$\Omega(u,v) = \begin{pmatrix} \frac{u^2+2\omega^2 v}{1-\omega} & \omega^2 u & \frac{\omega u^2-\omega^2 v}{1-\omega} \\ \omega^2 u & -\omega^2 & u \\ \frac{\omega u^2-\omega^2 v}{1-\omega} & u & \frac{\omega^2 u^2+2\omega^2 v}{1-\omega} \end{pmatrix}, \quad \text{with } u = \frac{\eta_2}{\eta_0},\ v = \frac{\eta_1}{\eta_0},$$

we have a modular embeding

$$\Omega : \mathbb{B} \longrightarrow \mathbb{H}_3, \quad (u,v) \mapsto \Omega(u,v).$$

It is compatible with the homomorphism $\rho : \Gamma \to \mathrm{Sp}_6(\mathbb{Z})$ given by

$$\rho(g) = \left(\begin{array}{ccc|ccc} a_{22}-b_{22} & a_{23}-b_{23} & -b_{22} & b_{21} & b_{23} & a_{21}-b_{21} \\ a_{32}-b_{32} & a_{33}-b_{33} & -b_{32} & b_{31} & b_{33} & a_{31}-b_{31} \\ b_{22} & b_{23} & a_{22} & -a_{21} & -a_{23} & b_{21} \\ \hline -b_{12} & -b_{13} & -a_{12} & a_{11} & a_{13} & -b_{11} \\ -b_{32} & -b_{33} & -a_{32} & a_{31} & a_{33} & -b_{31} \\ a_{12}-b_{12} & a_{13}-b_{13} & -b_{12} & b_{11} & b_{13} & a_{11}-b_{11} \end{array}\right)$$

with $g = (g_{ij}) \in \Gamma,\ g_{ij} = a_{ij} + \omega b_{ij}$. Namely we have $\rho(g)\cdot\Omega(u,v) = \Omega(g\cdot(u,v))$.

Remark 1.2. *The homomorphism ρ does not coincide with σ in [F], because there the affine coordinates are given by $u = \eta_2/\eta_1,\ v = \eta_0/\eta_1$.*

Let us consider the following Riemann theta constants and their Fourier expansions (see [S]):

$$\vartheta_k(u,v) = \vartheta\begin{bmatrix} 0 & 1/6 & 0 \\ k/3 & 1/6 & k/3 \end{bmatrix}(0,\Omega(u,v)) = \sum_{\mu\in\mathbb{Z}[\omega]} \omega^{2k\mathrm{tr}\mu} H(\mu u) q^{N(\mu)} \quad (3)$$

with an index $k \in \mathbb{Z}$, where $\mathrm{tr}\mu = \mu + \bar{\mu}$, $\mathrm{N}(\mu) = \mu\bar{\mu}$ and

$$H(u) = \exp[\frac{\pi}{\sqrt{3}}u^2]\vartheta\begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix}(u,-\omega^2), \quad q = \exp[\frac{2\pi}{\sqrt{3}}v].$$

Apparently it holds $\vartheta_k(u,v) = \vartheta_{k+3}(u,v)$, so k runs over $\{0,1,2\} = \mathbb{Z}/3\mathbb{Z}$.

According to [F],[P] and [S] , we have the following:

Fact (i) The map

$$\Lambda : \mathbb{B} \longrightarrow \mathbb{P}^2 \quad (u,v) \mapsto \xi = [\vartheta_0(u,v)^3 : \vartheta_1(u,v)^3 : \vartheta_2(u,v)^3]$$

gives the invese of the period map $\xi \mapsto (u,v)$ and is invariant under the action of the congruence group

$$\Gamma(\sqrt{-3}) = \{g \in \Gamma : g \equiv I_3 \mod \sqrt{-3}\}$$

of level $\sqrt{-3}$. It gives a biholomorphic isomorphism $\overline{\mathbb{B}/\Gamma(\sqrt{-3})} \cong \mathbb{P}^2$, where $\overline{X}$ indicates the Satake compactification of X.

Fact (ii) (Finis' equalities [F])

$$H(\omega u) = H(u), \quad H(0)^2 H(\sqrt{-3}u) = -wH(u)^3 - w^2 H(-u)^3. \quad (4)$$

Fact (iii) The projective group $\Gamma(\sqrt{-3})/\{1,\omega,\omega^2\}$ is generated by

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ \omega-\omega^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & 0 \\ \omega-1 & 1 & \omega-1 \\ 1-\omega^2 & 0 & 1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} 1 & \omega-\omega^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 1 & \omega-1 & \omega-1 \\ 0 & 1 & 0 \\ 0 & 1-\omega & 1 \end{pmatrix}.$$

Fact (iv) The detailed automorphic behavior of $\vartheta_k(u,v)$ is described in Part II section 4 of [S], and we use those exact automorphic factors.

2 Automorphic properties of $\vartheta_k(u,v)$

Acccording to Fact (i) we have

$$\frac{\vartheta_k(g\cdot(u,v))^3}{\vartheta_0(g\cdot(u,v))^3} = \frac{\vartheta_k(u,v)^3}{\vartheta_0(u,v)^3} \qquad (g \in \Gamma(\sqrt{-3})),$$

so there exist characters $\chi_k : \Gamma(\sqrt{-3}) \to \{1,\omega,\omega^2\}$ such that

$$\frac{\vartheta_k(g\cdot(u,v))}{\vartheta_0(g\cdot(u,v))} = \chi_k(g)\frac{\vartheta_k(u,v)}{\vartheta_0(u,v)} \qquad (g \in \Gamma(\sqrt{-3})).$$

Set

$$\Gamma_\vartheta = \{g \in \Gamma(\sqrt{-3}) : \chi_1(g) = \chi_2(g) = 1\}.$$

Then we have

Proposition 2.1. (i) $\Gamma_\vartheta = \{(g_{ij}) \in \Gamma(\sqrt{-3}) : g_{12} \equiv g_{13} \equiv g_{32} \equiv 0 \mod 3\}$,

$$\Gamma(\sqrt{-3})/\Gamma_\vartheta \cong (\mathbb{Z}/3\mathbb{Z})^3.$$

(ii) *The map*

$$\widehat{\Lambda} : \mathbb{B} \longrightarrow \mathbb{P}^2 \quad (u,v) \mapsto [\vartheta_0(u,v) : \vartheta_1(u,v) : \vartheta_2(u,v)]$$

gives a biholomorphic isomorphism $\overline{\mathbb{B}/\Gamma_\vartheta} \cong \mathbb{P}^2$.

Proof. All the automorphic factors of $\vartheta_k(u,v)$ for $g_1, \cdots, g_5$ are already calculated in Part II Lemma 4.2 in [S], then we obtain

$$(\chi_1(g_i), \chi_2(g_i)) = \begin{cases} (1,1) & (i = 1,2,3) \\ (\omega, 1) & (i = 4) \\ (\omega, \omega) & (i = 5) \end{cases}.$$

So we see that the map $(\chi_1, \chi_2) : \Gamma(\sqrt{-3}) \to \{1, \omega, \omega^2\}^2$ is surjective, and that $\Gamma(\sqrt{-3})/\Gamma_\vartheta \cong (\mathbb{Z}/3\mathbb{Z})^3$. Let us define a homomorphism $\psi_i : \Gamma(\sqrt{-3}) \to \mathbb{Z}/3\mathbb{Z} \quad (i = 1,2,3)$ by

$$\psi_1(g) = \frac{g_{12}}{\sqrt{-3}}, \quad \psi_2(g) = \frac{g_{13}}{\sqrt{-3}}, \quad \psi_3(g) = \frac{g_{32}}{\sqrt{-3}} \mod \sqrt{-3}.$$

Refering the explicit form of g_j $(j = 1, \ldots, 5)$ in Fact (iii) we have

$$\psi_1(g_i) = \begin{cases} 0 \ (i = 1,2,3) \\ 1 \ (i = 4) \\ 2 \ (i = 5) \end{cases} \quad \psi_2(g_i) = \begin{cases} 0 \ (i = 1,2,3) \\ 0 \ (i = 4) \\ 2 \ (i = 5) \end{cases} \quad \psi_3(g_i) = \begin{cases} 0 & (i = 1,2,3) \\ 0 & (i = 4) \\ 2 & (i = 5) \end{cases}.$$

Therefore we get

$$\chi_1(g) = \exp[\frac{2\pi}{3}(\psi_1(g) + \psi_2(g))], \quad \chi_2(g) = \exp[\frac{4\pi}{3}\psi_2(g)] = \exp[\frac{4\pi}{3}\psi_3(g)],$$

and

$$\begin{aligned} \chi_1(g) = \chi_2(g) = 1 &\Leftrightarrow \psi_1(g) = \psi_2(g) = \psi_3(g) = 0 \\ &\Leftrightarrow g_{12} \equiv g_{13} \equiv g_{32} \equiv 0 \mod 3. \end{aligned}$$

From the commutative diagram

$$\begin{array}{ccc} \overline{\mathbb{B}/\Gamma_\vartheta} & \xrightarrow{\widehat{\Lambda}} & \mathbb{P}^2 \\ \downarrow & & \downarrow {\scriptstyle (cube)} \\ \overline{\mathbb{B}/\Gamma(\sqrt{-3})} & \xrightarrow{\Lambda} & \mathbb{P}^2 \end{array} \qquad (cube) : [X : Y : Z] \mapsto [X^3 : Y^3 : Z^3]$$

we obtain the required isomorphism $\overline{\mathbb{B}/\Gamma_\vartheta} \cong \mathbb{P}^2$. $\square$

Now let us consider the elements of Γ

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 0 & 0 \\ \omega^2 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Note that $h_1 \cdot (u, v) = (-u, v), h_2 \cdot (u, v) = (u + 1, v - u + \omega^2)$.

Lemma 2.1. *We have*
(i) $\vartheta_k(h_1 \cdot (u, v)) = \vartheta_k(-u, v) = \vartheta_{-k}(u, v)$,
(ii) $\vartheta_k(h_2 \cdot (u, v)) = \vartheta_k(u + 1, v - u + \omega^2) = \vartheta_{k+2}(u, v)$.
Hence h_1 *and* h_2 *generate the permutations of* $\{\vartheta_0,\ \vartheta_1,\ \vartheta_2\}$.

Proof. (i)

$$\vartheta_k(-u, v) = \sum_{\mu \in \mathbb{Z}[\omega]} \omega^{2k\mathrm{tr}\mu} H(\mu(-u)) q^{\mathrm{N}(\mu)} = \sum_{\mu \in \mathbb{Z}[\omega]} \omega^{-2k\mathrm{tr}(-\mu)} H(-\mu u) q^{\mathrm{N}(-\mu)}$$
$$= \vartheta_{-k}(u, v).$$

(ii) The equality is shown by tedious calculation, and we omit details. We have

$$\vartheta_k(u + 1, v - u + \omega^2) = \sum_{\mu \in \mathbb{Z}[\omega]} \omega^{2k\mathrm{tr}\mu} H(\mu(u + 1)) \exp[\frac{2\pi}{\sqrt{3}}(v - u + \omega^2)]^{\mathrm{N}(\mu)}$$
$$= \sum_{\mu \in \mathbb{Z}[\omega]} \omega^{2k\mathrm{tr}\mu} q^{\mathrm{N}(\mu)} H(\mu(u + 1)) \exp[\frac{2\pi}{\sqrt{3}}(-u + \omega^2)]^{\mathrm{N}(\mu)}$$

and

$$H(\mu(u + 1)) \exp[\frac{2\pi}{\sqrt{3}}(-u + \omega^2)]^{\mathrm{N}(\mu)}$$
$$= \exp[\frac{\pi}{\sqrt{3}}\mu^2 u^2] \exp[\frac{\pi}{\sqrt{3}}\mu\{2(\mu - \bar{\mu})u + (\mu + 2\omega^2\bar{\mu})\}]\vartheta \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix} (\mu u + \mu, -\omega^2).$$

Putting $\mu = n - m\omega^2$,

$$\vartheta \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix} (\mu u + \mu, -\omega^2) = (-\omega^2)^{n-m} \exp[\pi\sqrt{-1}(m^2\omega^2 - 2m\mu u)]\vartheta \begin{bmatrix} 1/6 \\ 1/6 \end{bmatrix} (\mu u, -\omega^2).$$

We can show the desired equation by use of above equalities and $\omega^{2(n-m)} = \omega^{\mathrm{tr}\mu}$. □

3 Isogeny of Jacobians and the shift of modular forms

Let $L(u, v)$ denote the lattice $\Omega(u, v)\mathbb{Z}^3 + \mathbb{Z}^3$ in $\mathbb{C}^3$. And let $A(u, v)$ be the Jacobian variey $\mathbb{C}^3/L(u, v) = \mathrm{Jac}(C(\xi)),\ \xi = \Lambda(u, v)$.

Proposition 3.1. *We have an isogeny* $\phi : A(u, v) \to A(\sqrt{-3}u, 3v)$ *with the kernel being isomorphic to* $(\mathbb{Z}/3\mathbb{Z})^3$.

Proof. Let us define a linear map $\phi : \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$\phi : z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} 3 & 2(1-\omega)u & 0 \\ 0 & \sqrt{-3} & 0 \\ 0 & 2\sqrt{-3}u & 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

Then we have $\phi(L(u,v)) \subset L(\sqrt{-3}u, v)$ and ϕ gives an isogeny. The kernel is generated by

$$\frac{1}{3}e_1,\ \frac{1}{3}(\Omega e_2 + e_2),\ \frac{1}{3}e_3$$

where e_i is the i-th unit vector. □

Let $\mathfrak{a}$ denote the ideal $(\sqrt{-3})$ of $\mathbb{Z}[\omega]$. By an easy calculation we have

Lemma 3.1.

$$\mu \in \mathfrak{a} \quad \Longleftrightarrow \quad \mathrm{N}(\mu) \in 3\mathbb{Z} \quad \Longleftrightarrow \quad \mathrm{tr}\mu \in 3\mathbb{Z}.$$

Now we can show our first isogeny formula

Proposition 3.2 (Arithmetic Mean Formula).

$$\vartheta_0(\sqrt{-3}u, 3v) = \frac{1}{3}(\vartheta_0(u,v) + \vartheta_1(u,v) + \vartheta_2(u,v)).$$

Proof.

$$\sum_{k=0}^{2} \vartheta_k(u,v) = \sum_{k=0}^{2} \sum_{\mu \in \mathbb{Z}[\omega]} \omega^{2k\mathrm{tr}\mu} H(\mu u) q^{\mathrm{N}(\mu)} = \sum_{\mu \in \mathbb{Z}[\omega]} \Big(\sum_{k=0}^{2} \omega^{2k\mathrm{tr}\mu} \Big) H(\mu u) q^{\mathrm{N}(\mu)}.$$

Here we note that

$$\sum_{k=0}^{2} \omega^{2k\mathrm{tr}\mu} = 1 + (\omega^{\mathrm{tr}\mu})^2 + (\omega^{\mathrm{tr}\mu})^4 = \begin{cases} 0 & (\mathrm{tr}\mu \notin 3\mathbb{Z}) \\ 3 & (\mathrm{tr}\mu \in 3\mathbb{Z}). \end{cases}$$

According to the above Lemma we have

$$\begin{aligned} \sum_{k=0}^{2} \vartheta_k(u,v) = \sum_{\mu \in \mathfrak{a}} 3H(\mu u) q^{\mathrm{N}(\mu)} &= \sum_{\mu \in \mathbb{Z}[\omega]} 3H(\sqrt{-3}\mu u) q^{\mathrm{N}(\sqrt{-3}\mu)} \\ &= \sum_{\mu \in \mathbb{Z}[\omega]} 3H(\mu\sqrt{-3}u) q^{3\mathrm{N}(\mu)} = 3\vartheta_0(\sqrt{-3}u, 3v). \end{aligned}$$

□

This Proposition suggests us to define the isogenous modular forms

$$\vartheta_k^{\sharp}(u,v) = \vartheta_k(\sqrt{-3}u, 3v) \qquad (k = 0, 1, 2)$$

and to express them in terms of the original $\vartheta_k(u,v)$'s. For this purpose, we are requested to know the exact automorphic behaviors of $\vartheta_k^{\sharp}(u,v)$'s and $\vartheta_k(u,v)$'s. Let us define an automorphism of the group

$$G = \{ g \in \mathrm{GL}_3(\mathbb{Q}[\omega]) : {}^t\overline{g} M g = M \}$$

by

$$g \mapsto \widehat{g} = AgA^{-1}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \sqrt{-3} \end{pmatrix}.$$

Then we have

$$\vartheta_k^\sharp(g \cdot (u,v)) = \vartheta_k(\widehat{g} \cdot (\sqrt{-3}u, 3v)). \tag{5}$$

Lemma 3.2. *For any $g \in \Gamma_\vartheta$, we have $\widehat{g} \in \Gamma(\sqrt{-3})$.*

Proof. Let $g = (g_{ij})$ be an element of Γ_ϑ. By the condition ${}^t\overline{g}Mg = M$, we have

$$g_{12}\overline{g_{22}} + \overline{g_{12}}g_{22} + |g_{32}|^2 = 0$$

Because $g_{12} \equiv g_{32} \equiv 0 \mod 3$ and $g_{22} \equiv 1 \mod \sqrt{-3}$, we may put

$$g_{12} = 3x, \quad g_{22} = 1 + \sqrt{-3}y, \quad g_{32} = 3z \quad (x, y, z \in \mathbb{Z}[\omega]).$$

Then the above equation becomes

$$x(1 - \sqrt{-3}\overline{y}) + \overline{x}(1 + \sqrt{-3}y) + 3|z|^2 = 0.$$

We see that $\mathrm{tr}x = x + \overline{x} \equiv 0 \mod \sqrt{-3}$, and this implies $\mathrm{tr}x \in 3\mathbb{Z}$. By Lemma 3.1, we see $x \in \mathfrak{a}$, and therefore that g_{12} is a multiple of $3\sqrt{-3}$. We can perform the same procedure for g_{13}. So we see that

$$\widehat{g} = AgA^{-1} = \begin{pmatrix} g_{11} & g_{12}/3 & g_{13}/\sqrt{-3} \\ 3g_{21} & g_{22} & -\sqrt{-3}g_{23} \\ \sqrt{-3}g_{31} & -g_{32}/\sqrt{-3} & g_{33} \end{pmatrix}$$

is an element of $\Gamma(\sqrt{-3})$. □

Lemma 3.3. *Theta functcoins $\vartheta_0^\sharp(u,v)$, $\vartheta_1^\sharp(u,v)^3$ and $\vartheta_2^\sharp(u,v)^3$ belong to the polynimial ring $R = \mathbb{C}[\vartheta_0(u,v), \vartheta_1(u,v), \vartheta_2(u,v)]$.*

Proof. By Proposition 3.2, $\vartheta_0^\sharp(u,v)$ belongs to R. By the above lemma (and Fact (i)), we see

$$\frac{\vartheta_k^\sharp(g\cdot(u,v))^3}{\vartheta_0^\sharp(g\cdot(u,v))^3} = \frac{\vartheta_k(\widehat{g}\cdot(\sqrt{-3}u,3v))^3}{\vartheta_0(\widehat{g}\cdot(\sqrt{-3}u,3v))^3} = \frac{\vartheta_k(\sqrt{-3}u,3v)^3}{\vartheta_0(\sqrt{-3}u,3v)^3} = \frac{\vartheta_k^\sharp(u,v)^3}{\vartheta_0^\sharp(u,v)^3}$$

for any $g \in \Gamma_\vartheta$. Proposition 2.1 says $\overline{\mathbb{B}/\Gamma_\vartheta} = \mathrm{Proj}R = \mathbb{P}^2$. So we get the assertion. □

Lemma 3.4. *We have*
(i) $\vartheta_k^\sharp(h_1 \cdot (u,v)) = \vartheta_k^\sharp(-u,v) = \vartheta_{-k}^\sharp(u,v)$
(ii) $\vartheta_k^\sharp(h_2 \cdot (u,v)) = \vartheta_k^\sharp(u+1, v-u+\omega^2) = \vartheta_k^\sharp(u,v)$.

Proof. (i) is nothing but Lemma 2.1 (i).
(ii) By Lemma 2.1 (ii), we have

$$\vartheta_0^\sharp(h_2\cdot(u,v)) = \sum_{k=0}^{2}\vartheta_k(h_2\cdot(u,v)) = \sum_{k=0}^{2}\vartheta_{k+2}(u,v) = \vartheta_0^\sharp(u,v).$$

So the equality holds for $k = 0$. By (5) we have

$$\vartheta_k^\sharp(h_2 \cdot (u, v)) = \vartheta_k(\widehat{h_2} \cdot (\sqrt{-3}u, 3v)), \quad (k = 0, 1, 2)$$

and

$$\widehat{h_2} = \begin{pmatrix} 1 & 0 & 0 \\ -3-3\omega & 1 & 1+2\omega \\ 1+2\omega & 0 & 1 \end{pmatrix}, \quad \rho(\widehat{h_2}) = \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 2 & 1 & 3 & -1 & -3 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

We can apply the transformation formula of theta functions (see [I]), and we can check that the automorphic factor of $\vartheta_k(u, v)$ with repect to $\rho(\widehat{h_2})$ does not depend on k. Hence the equation for $k = 1, 2$ follows from the result for $k = 0$. □

Theorem 3.1. *We have the following identities*

$$\begin{aligned} \vartheta_0^\sharp(u, v) &= \frac{1}{3}(\vartheta_0(u, v) + \vartheta_1(u, v) + \vartheta_2(u, v)), \\ \vartheta_1^\sharp(u, v)^3 + \vartheta_2^\sharp(u, v)^3 &= \frac{1}{3}A(\vartheta_0(u, v), \vartheta_1(u, v), \vartheta_2(u, v)), \\ \vartheta_2^\sharp(u, v)^3 - \vartheta_1^\sharp(u, v)^3 &= \frac{1}{3\sqrt{-3}}\Delta(\vartheta_0(u, v), \vartheta_1(u, v), \vartheta_2(u, v)) \end{aligned}$$

where

$$\begin{aligned} A(x, y, z) &= x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2, \\ \Delta(x, y, z) &= (x - y)(y - z)(z - x). \end{aligned}$$

Proof. The first identity is obatined in Proposition 3.2.

By Lemma 2.1, h_1 and h_2 give a system of generaters of the symmetric group S_3 acting on $R = \mathbb{C}[\vartheta_0(u, v), \vartheta_1(u, v), \vartheta_2 u, v)]$. Lemma 3.4 means that $\vartheta_1^\sharp(u, v)^3 + \vartheta_2^\sharp(u, v)^3$ is an S_3-invariant and $\vartheta_2^\sharp(u, v)^3 - \vartheta_1^\sharp(u, v)^3$ is a proper A_3-invariant, where A_3 is the alternating group.

Put

$$\begin{aligned} \delta_1(x, y, z) &= x^3 + y^3 + z^3, \\ \delta_2(x, y, z) &= x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2, \\ \delta_3(x, y, z) &= xyz. \end{aligned}$$

Because we have the equality of graded rings $\mathbb{C}[\delta_1, \delta_2, \delta_3] = \mathbb{C}[x, y, z]^{S_3}$, it must hold

$$\vartheta_1^\sharp(u, v)^3 + \vartheta_2^\sharp(u, v)^3 = \alpha\delta_1(\vartheta_0, \vartheta_1, \vartheta_2) + \beta\delta_2(\vartheta_0, \vartheta_1, \vartheta_2) + \gamma\delta_3(\vartheta_0, \vartheta_1, \vartheta_2)$$

for some constants $\alpha, \beta, \gamma \in \mathbb{C}$. Putting $u = 0$ in (3) , we have the Fourier expansions

$$\vartheta_0(0, v) = c(1 + 6q + 6q^3 + \cdots), \quad \vartheta_1(0, v) = \vartheta_2(0, v) = c(1 - 3q + 6q^3 + \cdots)$$

with $c = H(0)$, and

$$\delta_1(\vartheta_0, \vartheta_1, \vartheta_2) = c^3(3 + 162q^2 + 216q^3 + \cdots),$$
$$\delta_2(\vartheta_0, \vartheta_1, \vartheta_2) = c^3(6 - 54q^3 + \cdots),$$
$$\delta_3(\vartheta_0, \vartheta_1, \vartheta_2) = c^3(1 - 27q^2 + 72q^3 + \cdots).$$

By the same procedure we have

$$\vartheta_1^\sharp(0, v)^3 + \vartheta_2^\sharp(0, v)^3 = c^3(2 - 18q^3 + \cdots).$$

By equating the coefficients we obtain $\alpha = \gamma = 0$ and $\beta = 1/3$.

The proper A_3-invariant of degree 3 is determined uniquely up to a constant factor, so we may put

$$\vartheta_2^\sharp(u, v)^3 - \vartheta_1^\sharp(u, v)^3 = \alpha(\vartheta_0 - \vartheta_1)(\vartheta_1 - \vartheta_2)(\vartheta_2 - \vartheta_0). \tag{6}$$

We must show $\alpha = 1/(3\sqrt{-3})$. According to Finis' equality (4), we have Fourier expansions

$$\vartheta_0(u, v) = H(0) + 3(H(u) + H(-u))q + \cdots,$$
$$\vartheta_1(u, v) = H(0) + 3(\omega H(u) + \omega^2 H(-u))q + \cdots,$$
$$\vartheta_2(u, v) = H(0) + 3(\omega^2 H(u) + \omega H(-u))q + \cdots.$$

Substituting these expansins to the right hand side of (6) it becomes to be

$$-81\sqrt{-3}\alpha(H(u)^3 - H(-u)^3)q^3 + \cdots,$$

By the same way the left hand side of (6) becomes to be

$$-9\sqrt{-3}H(0)^2(H(\sqrt{-3})u) - H(-\sqrt{-3}u))q^3 + \cdots.$$

By using again Finis' equality (4) we have

$$H(0)^2(H(\sqrt{-3}u) - H(-\sqrt{-3}u)) = -\sqrt{-3}(H(u)^3 - H(-u)^3).$$

This equality means $\alpha = 1/(3\sqrt{-3})$. □

4 a three terms AGM derived from the isogeny formula

Our isogeny formula works as that of Jacobii theta constants stated in the introduction. So it leads to define a three terms AGM process:

Definition 4.1. *Let a, b, c be real positive numbers with the condition $a \geq b \geq c$. Set*

$$\Psi(a, b, c) = (\alpha, \beta, \gamma) = (\frac{a + b + c}{3}, \sqrt[3]{A}, \sqrt[3]{B}), \tag{7}$$

where we put

$$\begin{cases} A = \frac{1}{6}(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2) - \frac{\sqrt{-1}}{6\sqrt{3}}(a - c)(a - b)(b - c) \\ \\ B = \frac{1}{6}(a^2b + b^2c + c^2a + ab^2 + bc^2 + ca^2) + \frac{\sqrt{-1}}{6\sqrt{3}}(a - c)(a - b)(b - c). \end{cases}$$

Here we choose the arguments of $\beta = \sqrt[3]{A}$ *and* $\gamma = \sqrt[3]{B}$ *such that*

$$0 \le \arg \sqrt[3]{A} < \frac{\pi}{6}, \; 0 < \beta + \gamma = \sqrt[3]{A} + \sqrt[3]{B}.$$

And we have the following results.

Theorem 4.1. *Let* a, b, c *be real positive numbers with the condition* $a \ge b \ge c$. *Set* $\Psi^n(a, b, c) = (a_n, b_n, c_n)$. *Then there is a common limit*

$$M3(a, b, c) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

Theorem 4.2. *We have*

$$\frac{1}{M3(1, x, y)} = H(x, y) = F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}1; 1 - x^3, 1 - y^3).$$

in a neiborhood of $(x, y) = (1, 1)$. *Where* $F_1(\alpha, \beta, \beta', \gamma; x, y)$ *stands for the Appell hypergeometric function* F_1.

The proofs will be published elsewhere.

References

[F] Tobias Finis, Some computational results on Hecke eigenvalues of modular forms on a unitary group, Manuscripta Math., 96 (1998), 149-180.

[H] R. P. Holzapfel, Ball and surface arithmetics. Aspects of Mathematics, E29. Friedr. Vieweg & Sohn, Braunschweig(1998).

[I] J. Igusa, Theta functions, Springer, Heidelberg, New-York(1972).

[KS] K. Koike and H. Shiga A three terms Arithmetic-Geometric mean, Tech. Rep. Chiba univ., 2005.

[LB] H. Lange and Ch. Birkenhake, Complex Abelian Varieties, *Springer-Verlag* (1992).

[M] D. Mumford, Tata Lectures on Theta I, Birkhäuser, Boston-Basel-Stuttgard(1983).

[P] E. Picard, Sur les fonctions de deux variables indépendantes analogues aux fonctions modulaires, Acta math., 2(1883), 114-135.

[S] H. Shiga, On the representation of the Picard modular function by θ constants I-II, Pub. R.I.M.S. Kyoto Univ. 24(1988), 311-360.

[Y] M. Yoshida, Fuchsian differential equations, Aspects of Mathematics, E11. Friedr. Vieweg & Sohn, Braunschweig(1987).