

# Extended Gauss AGM and corresponding Picard modular forms

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## Introduction

For two positive numbers  $a, b$ , set  $\psi_g(a, b) = (\frac{a+b}{2}, \sqrt{ab})$ , and set inductively  $\psi_g^n(a, b) = \psi_g(\psi_g^{n-1}(a, b)) = (a_n, b_n)$  ( $n = 1, 2, \dots$ ). Then we have a common limit

$$M_g(a, b) = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

This is called the Gauss Arithmetic Geometric Mean (we use the abbreviation Gauss AGM). Concerning this Gauss AGM we have two classical identities both we can find in the literature of Gauss himself.

**Theorem 1** (Gauss [GaT] 1799) *For  $x \in (0, 1)$  we have*

$$\frac{1}{AGM(1, x)} = \frac{1}{\pi} \int_1^\infty \frac{dz}{\sqrt{z(z-1)(z-\lambda)}} = F\left(\frac{1}{2}, \frac{1}{2}, 1; \lambda\right), \quad (\lambda = 1 - x^2). \quad (0.1)$$

*The right hand side  $F$  means the Gauss hypergeometric function.*

Setting the Jacobi theta function and theta constant for  $p, q \in \{0, 1\}^2$

$$\begin{aligned} \vartheta \begin{bmatrix} p \\ q \end{bmatrix} (z, \tau) &= \sum_{n \in \mathbf{Z}} \exp[\pi i(n + \frac{p}{2})^2 \tau + 2\pi i(n + \frac{p}{2})(z + \frac{q}{2})], \\ \vartheta \begin{bmatrix} p \\ q \end{bmatrix} (\tau) &= \sum_{n \in \mathbf{Z}} \exp[\pi i(n + \frac{p}{2})^2 \tau + \pi i(n + \frac{p}{2})q], \end{aligned}$$

we have

**Theorem 2** (Gauss [GaH] 1818, Nachlass, the equality 13 and the equality 21)

$$\begin{cases} \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^2 (2\tau) = \frac{1}{2} \left( \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}^2 (\tau) + \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 (\tau) \right) \\ \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 (2\tau) = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau) \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau). \end{cases} \quad (0.2)$$

The latter theorem shows the relation of the (coefficients of the realized) elliptic curves corresponding to two isogenous torus  $\mathbf{C}/\mathbf{Z} + \tau\mathbf{Z}$  and  $\mathbf{C}/\mathbf{Z} + 2\tau\mathbf{Z}$ . So in general this theorem is referred as the isogeny formula for the Jacobi theta constants.

Any way these two theorems are telling us a very interesting story concerned with AGM, periods of algebraic varieties, hypergeometric functions and modular forms at a same time. But till now there has been no sufficiently nice generalization.

In this article we show an extended story of Gauss AGM to two variables case by using Picard modular forms for  $\mathbf{Q}(\sqrt{-1})$  studied by Matsumoto [Mat1] which is coming from one of the Terada and Deligne-Mostow list. Our results are staed as two Main Theorems in Section 1 and Section 2.

Observing this result together with our previous work ([Ko-Shi]) for the case  $\mathbf{Q}(\frac{-1+\sqrt{-3}}{2})$  the authors imagine that there will be more general structure of AGM connected with hypergeometric systems and CM-isogenies.

# 1 Extended Gauss AGM and the Appell $F_1$

## 1.1 Extended AGM system

At first let us consider a trivially extended Gauss AGM system

$$(a, b, \sqrt{ab}) \mapsto \left( \frac{a+b}{2}, \sqrt{ab}, \sqrt{\frac{a+b}{2}\sqrt{ab}} \right).$$

We generalize it to the system of three terms. Let  $a, b$  and  $c$  be positive real numbers. Set

$$\psi(a, b, c) = \left( \frac{a+b}{2}, c, \sqrt[4]{\frac{1}{4}(c^2+ab)(c^2+\frac{a^2+b^2}{2})} \right), \quad (1.1)$$

and set inductively

$$(a_n, b_n, c_n) = \psi(a_{n-1}, b_{n-1}, c_{n-1}).$$

**Theorem 3** *Suppose  $0 < a, b, c$ , then the sequences  $\{a_n\}, \{b_n\}$  and  $\{c_n\}$  have a common limit.*

**Definition 1.1** *Let us denote the above limit by  $M(a, b, c)$ , and call it Extended Gauss AGM.*

**Lemma 1.1** *We have*

$$\min\{a, b, c\} \leq \min\{a_1, b_1, c_1\} \leq \text{Max}\{a_1, b_1, c_1\} \leq \text{Max}\{a, b, c\}.$$

[proof].

Set  $\min = \min\{a, b, c\}, \text{Max} = \text{Max}\{a, b, c\}$ .

Obviously we have  $a_1, b_1 \in [\min, \text{Max}]$ .

Set

$$c_1^\flat = \sqrt{\frac{c^2+ab}{2}}, \quad c_1^\sharp = \sqrt{\frac{1}{2}\left(c^2 + \frac{a^2+b^2}{2}\right)}.$$

We see easily  $c_1^\flat, c_1^\sharp \in [\min, \text{Max}]$ . Because  $c^2 + ab \leq c^2 + \frac{a^2+b^2}{2}$ , we have

$$c_1^\flat \leq c_1 \leq c_1^\sharp.$$

So we have  $c_1 \in [\min, \text{Max}]$

q.e.d.

**Remark 1.1** *It happens that*

$$\min\{a, b, c\} = \min\{a_1, b_1, c_1\} < \text{Max}\{a_1, b_1, c_1\} = \text{Max}\{a, b, c\}.$$

For example  $(a, b, c) = (1, 1, 2)$ .

The following Proposition assures Theorem 3.

**Proposition 1.1** *Suppose  $0 < \mu \leq a, b, c \leq \nu$  for fixed positive numbers  $\mu, \nu$ . Then*

$$\text{Max}\{a_2, b_2, c_2\} - \min\{a_2, b_2, c_2\} \leq \rho(\text{Max} - \min)$$

for some positive number  $\rho(\mu, \nu) < 1$ .

[proof].

Assume  $a \leq b \leq c$ . Then

$$a_2 = \frac{a_1 + b_1}{2} = \frac{(a+b)/2 + c}{2} \geq \frac{a+c}{2},$$

and

$$c_2 \geq c_2^\flat = \sqrt{\frac{c_1^2 + a_1 b_1}{2}} \geq \sqrt{\frac{(c_1^\flat)^2 + a_1 b_1}{2}} = \frac{1}{2} \sqrt{(c+a)(c+b)} \geq \frac{a+c}{2}.$$

We have

$$b_2 = c_1 \geq c_1^\flat = \sqrt{\frac{c^2 + ab}{2}} \geq \sqrt{\frac{c^2 + a^2}{2}} \geq \sqrt{ac}.$$

By considering obvious inequality  $a_2, b_2, c_2 \leq c$ , we have

$$\frac{\text{Max}\{a_2, b_2, c_2\} - \min\{a_2, b_2, c_2\}}{c-a} \leq \frac{c - \sqrt{ac}}{c-a} = \frac{\sqrt{c}}{\sqrt{c} + \sqrt{a}} \leq \frac{\sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} < 1.$$

Assume  $a \leq c \leq b$ , then we have

$$a_2 = \frac{a_1 + b_1}{2} = \frac{(a+b)/2 + c}{2} \geq \sqrt{\frac{a+b}{2}} a,$$

and

$$c_2 \geq c_2^\flat = \sqrt{\frac{c_1^2 + a_1 b_1}{2}} \geq \sqrt{\frac{(c_1^\flat)^2 + a_1 b_1}{2}} = \frac{1}{2} \sqrt{(c+a)(c+b)} \geq \sqrt{\frac{a+b}{2}} a.$$

We have

$$b_2 = c_1 \geq c_1^\flat = \sqrt{\frac{c^2 + ab}{2}} \geq \sqrt{\frac{a+b}{2}} a.$$

So we have

$$\frac{\text{Max}\{a_2, b_2, c_2\} - \min\{a_2, b_2, c_2\}}{b-a} \leq \frac{b - \sqrt{\frac{a+b}{2}} a}{b-a}.$$

By putting  $b/a = t$ , we have

$$\frac{\text{Max}\{a_2, b_2, c_2\} - \min\{a_2, b_2, c_2\}}{b-a} \leq \frac{t - \sqrt{(1+t)/2}}{t-1}.$$

We have

$$\frac{d}{dt} \left( \frac{t - \sqrt{(1+t)/2}}{t-1} \right) = \frac{2(t-1)^2}{3\sqrt{2} + \sqrt{2}t + 4\sqrt{1+t}} > 0$$

for  $t > 1$ . So

$$\text{Max}_{t \in (1, \nu/\mu]} \frac{t - \sqrt{(1+t)/2}}{t-1} = \frac{\nu - \sqrt{\mu(\mu+\nu)/2}}{\nu-\mu} < 1.$$

This argument works for the case  $c \leq a \leq b$  also.

So we obtain the required inequality.

q.e.d.

## 1.2 Functional equations

Set

$$x^\# y^\# = \frac{2(x+y)}{(1+\sqrt{xy})^2} \quad \frac{x^\# + y^\#}{2} = \frac{(\sqrt{x} + \sqrt{y})\sqrt{(1+x)(1+y)}}{(1+\sqrt{xy})^2}$$

for real numbers  $0 < x, y < 1$ , namely

$$x^\sharp = \frac{(\sqrt{x} + \sqrt{y})\sqrt{(1+x)(1+y)} + i(\sqrt{x} - \sqrt{y})\sqrt{(1-x)(1-y)}}{(1 + \sqrt{xy})^2}$$

$$y^\sharp = \frac{(\sqrt{x} + \sqrt{y})\sqrt{(1+x)(1+y)} - i(\sqrt{x} - \sqrt{y})\sqrt{(1-x)(1-y)}}{(1 + \sqrt{xy})^2},$$

here we choose real positive square roots.

We have

**Proposition 1.2**

$$M\left(1, \sqrt{xy}, \sqrt{\frac{x+y}{2}}\right) = \frac{1 + \sqrt{xy}}{2} M\left(1, \sqrt{x^\sharp y^\sharp}, \sqrt{\frac{x^\sharp + y^\sharp}{2}}\right).$$

[proof]. It holds

$$\begin{aligned} M\left(1, \sqrt{xy}, \sqrt{\frac{x+y}{2}}\right) &= M\left(\frac{1 + \sqrt{xy}}{2}, \sqrt{\frac{x+y}{2}}, \sqrt[4]{\frac{1}{4}\left(\frac{x+y}{2} + \sqrt{xy}\right)\left(\frac{x+y}{2} + \frac{1+xy}{2}\right)}\right) \\ &= M\left(\frac{1 + \sqrt{xy}}{2}, \sqrt{\frac{x+y}{2}}, \frac{1}{2}\sqrt[4]{(\sqrt{x} + \sqrt{y})^2(1+x)(1+y)}\right) \\ &= \frac{1 + \sqrt{xy}}{2} M\left(1, \sqrt{\frac{2(x+y)}{(1 + \sqrt{xy})^2}}, \sqrt[4]{\frac{(\sqrt{x} + \sqrt{y})^2(1+x)(1+y)}{(1 + \sqrt{xy})^4}}\right) \\ &= \frac{1 + \sqrt{xy}}{2} M\left(1, \sqrt{\frac{2(x+y)}{(1 + \sqrt{xy})^2}}, \sqrt{\frac{(\sqrt{x} + \sqrt{y})\sqrt{(1+x)(1+y)}}{(1 + \sqrt{xy})^2}}\right) \\ &= \frac{1 + \sqrt{xy}}{2} M\left(1, \sqrt{x^\sharp y^\sharp}, \sqrt{\frac{x^\sharp + y^\sharp}{2}}\right). \end{aligned}$$

q.e.d.

Let  $F_1(a, b, b', c; x, y)$  be the Appell hypergeometric function

$$F_1(a, b, b', c; x, y) = \sum_{m, n \geq 0} \frac{(a, m+n)(b, m)(b', n)}{(c, m+n)m!n!} x^m y^n, \quad (|x|, |y| < 1)$$

of complex variables  $x, y$  with the Pochhammer notation

$$(a, k) = \begin{cases} a(a+1)\cdots(a+k-1), & (k \geq 1) \\ 0, & (k = 0). \end{cases}$$

This is a solution of the differential equation

$$E_1(a, b, b', c) : \begin{cases} \lambda(1-x)f_{xx} + (1-x)yf_{xy} + (c - (a+b+1)x)f_x - byf_y - abf = 0, \\ y(1-y)f_{yy} + (1-y)xf_{xy} + (c - (a+b'+1)y)f_y - b'xf_x - ab'f = 0, \\ (x-y)f_{xy} - b'f_x + bf_y = 0, \end{cases}$$

and it is characterized as the solution  $f(x, y)$  which is holomorphic at  $(x, y) = (0, 0)$  and  $f(0, 0) = 1$ .

**Theorem 4** *We have*

$$F_1\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1, 1-x^2, 1-y^2\right) = \frac{2}{1 + \sqrt{xy}} F_1\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1, 1-(x^\sharp)^2, 1-(y^\sharp)^2\right)$$

for real variables  $0 < x, y < 1$ .

[proof]. First let us note the system  $E_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1)$  :

$$\begin{cases} P_1 = P_1(x, y, \partial_x, \partial_y) = x(1-x)\partial_x^2 + y(1-x)\partial_x\partial_y + (1-\frac{7}{4}x)\partial_x - \frac{1}{4}y\partial_y - \frac{1}{8} \\ P_2 = P_2(x, y, \partial_x, \partial_y) = y(1-y)\partial_y^2 + x(1-y)\partial_x\partial_y + (1-\frac{7}{4}y)\partial_y - \frac{1}{4}x\partial_x - \frac{1}{8} \\ P_3 = (1-x)\partial_x P_2 - (1-y)\partial_y P_1 + \frac{1}{4}(P_1 - P_2) = \frac{1}{2}(x-y)\partial_x\partial_y - \frac{1}{8}(\partial_x - \partial_y) \end{cases} \quad (1.2)$$

gives the differential equation for  $F_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1, x, y)$ .

From the definition of  $x^\sharp$  and  $y^\sharp$  we have

$$\begin{aligned} 1 - (x^\sharp)^2 &= \frac{(\sqrt{(1-x^2)(1-y^2)} - i(x-y))^2}{(1+\sqrt{xy})^4} \\ 1 - (y^\sharp)^2 &= \frac{(\sqrt{(1-x^2)(1-y^2)} + i(x-y))^2}{(1+\sqrt{xy})^4}. \end{aligned}$$

Putting  $x = X^2, y = Y^2$  we are requested to show

$$\frac{1+XY}{2} F_1(1-X^4, 1-Y^4) = F_1(\Phi_1(X, Y), \Phi_2(X, Y)) \quad (1.3)$$

with

$$\Phi_1(X, Y) = \left( \frac{\sqrt{(1-X^4)(1-Y^4)} - i(X^2 - Y^2)}{(1+XY)^2} \right)^2, \quad \Phi_2(X, Y) = \left( \frac{\sqrt{(1-X^4)(1-Y^4)} + i(X^2 - Y^2)}{(1+XY)^2} \right)^2.$$

Now we are going to describe the differential operators which annihilate  $F_1(1-X^4, 1-Y^4)$  and  $F_1(\Phi_1(X, Y), \Phi_2(X, Y))$ , respectively.

**Lemma 1.2** *The system*

$$\begin{cases} Q_1 = \frac{1-X^4}{16X^2} \partial_X^2 + \frac{X(1-Y^4)}{16Y^3} \partial_X \partial_Y - \frac{1}{4} X \partial_X + \frac{1-Y^4}{8Y^3} \partial_Y - \frac{1}{8} \\ Q_2 = \frac{1-Y^4}{16Y^2} \partial_Y^2 + \frac{Y(1-X^4)}{16X^3} \partial_X \partial_Y - \frac{1}{4} Y \partial_Y + \frac{1-X^4}{16X^3} \partial_X - \frac{1}{8} \\ Q_3 = -\frac{X^4-Y^4}{32X^3Y^3} \partial_X \partial_Y + \frac{1}{32X^3} \partial_X - \frac{1}{32Y^3} \partial_Y \end{cases} \quad (1.4)$$

annihilates  $F_1(1-X^4, 1-Y^4)$ .

[proof]. We get the above system by just putting

$$x = 1 - X^4, \quad y = 1 - Y^4, \quad \partial_x = -\frac{1}{4X^3} \partial_X, \quad \partial_y = -\frac{1}{4Y^3} \partial_Y$$

in the system (1.2).

**Lemma 1.3** *Putting  $R_i = (1+XY) \cdot Q_i \cdot (1+XY)^{-1}$  the system*

$$\begin{cases} R_1 = \frac{1-X^4}{16X^2} \partial_X^2 + \frac{X(1-Y^4)}{16Y^3} \partial_X \partial_Y - \frac{X^4+2Y^4+4X^3Y^3+X^4Y^4}{16X^2Y^3(1+XY)} \partial_X + \frac{(1-Y^4)}{16Y^3(1+XY)} \partial_Y - \frac{Y^3(X^2-Y^2)-X^3(1-Y^4)}{8X^2Y^3(1+XY)^2} \\ R_2 = \frac{1-Y^4}{16Y^2} \partial_Y^2 + \frac{Y(1-X^4)}{16X^3} \partial_X \partial_Y - \frac{Y^4+2X^4+4Y^3X^3+Y^4X^4}{16Y^2X^3(1+YX)} \partial_Y + \frac{(1-X^4)}{16X^3(1+YX)} \partial_X - \frac{X^3(Y^2-X^2)-Y^3(1-X^4)}{8Y^2X^3(1+YX)^2} \\ R_3 = -\frac{1}{32X^3Y^3} \left( (X^4 - Y^4) \partial_X \partial_Y - \frac{X^5+Y^3}{1+XY} \partial_X + \frac{X^2+Y^5}{1+XY} \partial_Y - \frac{2(X^4-Y^4)}{(1+XY)^2} \right) \end{cases}$$

annihilates  $\frac{1+XY}{2} F_1(1-X^4, 1-Y^4)$ .

Let  $S_1(S_2)$  be the operator obtained from  $P_1(P_2)$ , respectively, by substituting

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \Phi_1(X, Y) \\ \Phi_2(X, Y) \end{bmatrix}, \quad \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} = \begin{bmatrix} \partial\Phi_1/\partial X & \partial\Phi_2/\partial X \\ \partial\Phi_1/\partial Y & \partial\Phi_2/\partial Y \end{bmatrix}^{-1} \begin{bmatrix} \partial_X \\ \partial_Y \end{bmatrix}.$$

They take some complicated shapes, but

$$T_1 = 8(X^2 + Y^2)(1 - XY)^3(S_1 + S_2), \quad T_2 = \frac{8(X^2 - Y^2)(1 - XY)^3}{i\sqrt{(1 - X^4)(1 - Y^4)}}(S_1 - S_2)$$

can be represented in an explicit way:

**Lemma 1.4**

$$\begin{aligned} T_1 = & Y(1 - X^4)^2(X - Y^3)\partial_X^2 + (1 - X^4)(1 - Y^4)(X^2 + Y^2)(1 - XY)\partial_X\partial_Y \\ & + X(1 - Y^4)^2(Y - X^3)\partial_Y^2 - (1 - X^4)(X^3 - Y + 2X^4Y + 4XY^2 - 4X^2Y^3 - X^3Y^4 - Y^5)\partial_X \\ & - (1 - Y^4)(Y^3 - X + 2Y^4X + 4YX^2 - 4Y^2X^3 - Y^3X^4 - X^5)\partial_Y - 2(X^2 + Y^2)(1 - XY)^3 \\ T_2 = & Y(1 - X^4)(Y - X^3)\partial_X^2 - (X^4 - Y^4)(1 - XY)\partial_X\partial_Y - X(1 - Y^4)(X - Y^3)\partial_Y^2 \\ & + (X^5 - X^2Y + 2X^6Y - X^3Y^2 - Y^3)\partial_X - (Y^5 - Y^2X + 2Y^6X - Y^3X^2 - X^3)\partial_Y. \end{aligned}$$

By comparing the results in Lemma 1.3 and Lemma 1.4 we obtain:

**Lemma 1.5**

$$\begin{aligned} \frac{1}{16XY}T_1 = & X(1 - X^4)(X - Y^3)R_1 \\ & + Y(1 - Y^4)(Y - X^3)R_2 + 2(1 - X^4)(1 - Y^4)(X^2 - Y^2)R_3 \\ \frac{1}{16XY}T_2 = & X(Y - X^3)R_1 - Y(X - Y^3)R_2 \\ & + 2(XY + X^2Y^2 - X^3Y^3 + X^4Y^4 - X^4 - Y^4)R_3. \end{aligned}$$

So  $T_1$  and  $T_2$  annihilate  $\frac{1+XY}{2}F_1(1 - X^4, 1 - Y^4)$ . Hence  $S_1$  and  $S_2$  annihilate  $\frac{1+XY}{2}F_1(1 - X^4, 1 - Y^4)$  also. Namely  $\frac{1+XY}{2}F_1(1 - X^4, 1 - Y^4)$  satisfies the same differential equation  $S_1 + S_2$  with three dimensional solution space as  $F_1(\Phi_1(X, Y), \Phi_2(X, Y))$ .

This system has unique holomorphic solution at  $(X, Y) = (1, 1)$  up to a constant factor. So we obtain the required equality.

q.e.d.

### 1.3 First Main theorem

Now we have an extension of Theorem 1:

**First Main Theorem.** Assume  $0 < x, y < 1$  and set  $\lambda = 1 - x^2$ ,  $\mu = 1 - y^2$ . Then we have the following trinity theorem for Extended Gauss AGM:

$$\frac{1}{M(1, \sqrt{xy}, \sqrt{(x+y)/2})} = \frac{1}{\pi} \int_1^\infty \frac{du}{\sqrt[4]{u^2(u-1)^2(u-\lambda)(u-\mu)}} = F_1\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1; \lambda, \mu\right). \quad (1.5)$$

[proof]. According to Proposition 1.2 and Theorem 4, we know that  $\frac{1}{M(1, \sqrt{xy}, \sqrt{(x+y)/2})}$  and  $F_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1; \lambda, \mu)$  satisfy the same functional equation. Set

$$g(x, y) = F_1\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 1; \lambda, \mu\right)M(1, \sqrt{xy}, \sqrt{(x+y)/2}).$$

Then we have

$$g(x, y) = g(x^\sharp, y^\sharp).$$

Note that we have

$$\lim_{n \rightarrow \infty} \frac{b_n}{c_n} = 1$$

for the Extended AGM system  $(a_n, b_n, c_n)$ . By the iteration of the procedure  $(x, y) \mapsto (x^\sharp, y^\sharp)$ , we have  $g(x, y) = 1$ . It means that the left hand side and the right hand side of (1.5) coincide. The second equality in the statement of the Theorem is nothing but the integral representation of the Appell hypergeometric function. Namely we obtained the required equality.

q.e.d.

**Remark 1.2** By putting  $x = y$  the above theorem reduces to Theorem 1 of Gauss.

## 2 Modular interpretation and isogeny formula

### 2.1 Matsumoto hyperelliptic modular function for $C(x, y)$ (sum up with corrections)

Here we review the result of K. Matsumoto [Mat1] with some modifications and small corrections. Let us start from the family of algebraic curves

$$C(x, y) : w^4 = z^2(z-1)^2(z-x)(z-y) \quad ((x, y) \in \mathcal{C}^2 - \{xy(x-1)(y-1)(x-y) = 0\}). \quad (2.1)$$

$C(x, y)$  has a model of compact nonsingular hyperelliptic curve of genus 3. We denote it by the same notation. The differentials

$$dz/w, z(z-1)dz/w^3, z^2(z-1)dz/w^3$$

form a basis of holomorphic 1-forms on  $C(x, y)$ .

For the moment suppose  $x$  and  $y$  are real and satisfy  $1 < x < y$ . Let  $\alpha$  be the automorphism  $(z, w) \mapsto (z, \sqrt{-1}w)$  of  $C(x, y)$ .

Let us make three 1-cycles. Choose branches  $w^{(0)}, w^{(1)}, w^{(2)}, w^{(3)}$  of  $w$  over  $\{\text{Im } z \geq 0\} - \{0, 1, x, y\}$  so as to be  $\alpha(z, w^{(k)}) = (z, w^{(k+1)})$  with  $w^{(4)} = w^{(0)}$ . For real numbers  $z_1, z_2$  let  $[z_1, z_2 >$  be an oriented line segment connecting  $z_1$  and  $z_2$  in this order. Let  $[z_1, z_2 >^{(k)}$  be an arc on  $C(x, y)$  over  $[z_1, z_2 >^{(k)}$  with the branch  $w^{(k)}$ . Set

$$A_1 = [0, -\infty >^{(0)} + [-\infty, 0 >^{(2)}, \quad B_1 = [1, 0 >^{(0)} + [0, 1 >^{(2)}, \quad A_3 = [y, x >^{(0)} + [x, y >^{(1)}.$$

They are considered as 1-cycles on  $C(x, y)$ . By making their analytic continuations we can define multi-valued analytic functions

$$a_1(x, y) = \int_{A_1} \frac{dz}{w}, \quad a_2(x, y) = \int_{B_1} \frac{dz}{w}, \quad a_5(x, y) = \int_{A_3} \frac{dz}{w}$$

on  $\mathcal{C}^2 - \{xy(x-1)(y-1)(x-y) = 0\}$ . These setting and notation are the same in [Mat1].

[Fact 1]  $a_1(x, y), a_2(x, y), a_5(x, y)$  satisfy the differential equation  $E_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$  of the Appell hypergeometric function  $F_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}; x, y)$  and they form a basis of the space of solutions.

We define the Schwarz map for  $E_1(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$  by

$$\Phi : (x, y) \mapsto (u, v) = (a_5/a_1, a_2/a_1) \in \mathcal{C}^2.$$

[Fact 2] The image of  $\Phi$  is contained in the hyperball

$$\mathcal{B}^2 = \{(u, v) \in \mathcal{C}^2 : 2 \text{Im } v - |u|^2 > 0\}$$

and it is open dense in  $\mathcal{B}^2$ . The map  $\Phi$  has a continuation on  $\mathbf{P}^1 \times \mathbf{P}^1 - \{(0, 0), (1, 1), (\infty, \infty)\}$  and the image is equal to  $\mathcal{B}^2$ .

[Fact 3] The fundamental group  $\pi_1(\mathcal{C}^2 - \{xy(x-1)(y-1)(x-y) = 0\}, *)$  induces the monodromy group  $G$  of  $\Phi$  acting on  $\mathcal{B}^2$ . The following five transformations give a generator system of  $G$ :

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1+i & 1 & 1-i \\ -1-i & 0 & i \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 2+i & -1-i & -1-i \\ 1+i & -i & -1-i \\ 1-i & -1+i & i \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$g_4 = \begin{pmatrix} i & 1-i & 1-i \\ 0 & i & 0 \\ 0 & -1-i & -1 \end{pmatrix},$$

$$g_5 = \begin{pmatrix} 2+i & -1-i & 1-i \\ 1+i & -i & 1-i \\ -1-i & 1+i & i \end{pmatrix},$$

where  $g_i$  ( $i = 1, 2, 3, 4, 5$ ) acts on  ${}^t(1, v, u)$  from left. Their orders are 4, 4, 2, 4, 4 and eigen values are

$$\{1, 1, i\}, \{1, 1, i\}, \{1, 1, -1\}, \{-1, i, i\}, \{1, 1, i\},$$

respectively (the original article contains a printing error for  $g_5$ ).

The Schwarz map  $\Phi$  induces a biholomorphic correspondence

$$\mathbf{P}^1 \times \mathbf{P}^1 \cong \overline{\mathbf{B}^2/G},$$

and three points  $(0, 0), (1, 1), (\infty, \infty)$  correspond to the boundaries  $\overline{\mathbf{B}^2/G} - \mathbf{B}^2/G$ .

[Fact 5] We can realize  $\mathbf{B}^2$  in the form

$$D = \{\xi = [\xi_0, \xi_1, \xi_2] \in \mathbf{P}^2 : \xi H^t \bar{\xi} < 0\},$$

$$H = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The map

$$\Omega(u, v) = \begin{pmatrix} v + iu^2/2 & -u^2/2 & -iu \\ -u^2/2 & v - iu^2/2 & u \\ -iu & u & i \end{pmatrix}$$

gives an modular embedding of  $\mathbf{B}^2$  into the Siegel upper half space of degree 3. Set

$$G_0 = U(2, 1, \mathbf{Z}[\sqrt{-1}]) = \{g \in GL(3, \mathbf{Z}[\sqrt{-1}]) : gH^t \bar{g} = H\}.$$

$G_0$  becomes a restriction of  $Sp(6, \mathbf{Z})$  on  $\Omega(\mathbf{B}^2)$ . Note that the projective group  $\overline{G_0} = G_0/\langle i \rangle$  is isomorphic to

$$SU(2, 1, \mathbf{Z}[\sqrt{-1}]) = \{g \in SL(3, \mathbf{Z}[\sqrt{-1}]) : gH^t \bar{g} = H\}.$$

[Fact 6] Set

$$g_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

Then the group generated by  $G$  and  $g_6$  together with  $\sqrt{-1}$  becomes the congruent subgroup

$$G(1 + \sqrt{-1}) = \{g \in G_0 : g \equiv \text{id} \pmod{(1 + \sqrt{-1})}\}.$$

Fact 7] For  $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in \mathbf{Z}^3$ , let

$$\begin{aligned} \Theta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (u, v) &:= \vartheta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (\Omega(u, v)) \\ &= \sum_{n=(n_1, n_2, n_3) \in \mathbf{Z}^3} \exp[\pi i(n + \frac{p}{2})\Omega(u, v)^t(n + \frac{p}{2}) + \pi i(n + \frac{p}{2})^t q] \end{aligned}$$

be the Riemann theta constant with half integral characteristics defined on  $\mathbf{B}^2$  via the embedding  $\Omega$ .



**Theorem 5** (Matsumoto hyperelliptic theta map theorem)

$$(x, y) = \left( \frac{\Theta^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u, v)}{\Theta^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} (u, v)}, \frac{\Theta^4 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (u, v)}{\Theta^4 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} (u, v)} \right).$$

**Remark 2.1** The denominators and the numerators on the right hand side have common zeros on some divisors of  $\mathbf{B}^2$ . But according to the argument on the period map, we have extensions  $x = x(u, v)$ ,  $y = y(u, v)$  as holomorphic maps on the whole domain  $\mathbf{B}^2$ .

[Fact 8]

**Lemma 2.1** (Fourier expansion)

$$\begin{aligned} & \Theta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_1 & q_3 \end{bmatrix} (u, v) \\ &= \sum_{n_1, n_2} \exp[-\frac{\pi}{2} \{(n_1 + \frac{p_1}{2}) + i(n_2 + \frac{p_2}{2})\}^2 u^2] \exp[\pi i \{(n_1 + \frac{p_1}{2})q_1 + (n_2 + \frac{p_2}{2})q_2\}] \\ & \times \vartheta \begin{bmatrix} p_3 \\ q_3 \end{bmatrix} \left( \left( (n_2 + \frac{p_2}{2}) - i(n_1 + \frac{p_1}{2}) \right) u, i \right) \exp[\pi i \{(n_1 + \frac{p_1}{2})^2 + (n_2 + \frac{p_2}{2})^2\} v] \end{aligned}$$

**Lemma 2.2** (Matsumoto exchange formula (The multiplication factor is misplaced in the original paper)) We have

$$\exp[\pi i p_1 q_1 + \frac{\pi i}{2} p_3 q_3] \cdot \Theta \begin{bmatrix} p_2 & p_1 & q_3 \\ q_2 & q_1 & p_3 \end{bmatrix} (u, v) = \Theta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (u, v).$$

## 2.2 9 theta constants with monodromy invariant characteristics

Recall the transformation formula for theta constants:

**Proposition 2.1** (Transformation formula (see for example [I])) Let  $g \in G_0$  and  $N_g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$  be its symplectic representation. Then we have

$$\Theta \left[ N_g \circ \begin{bmatrix} a \\ b \end{bmatrix} \right] (g \circ (u, v)) = \varepsilon(g, a, b) (\det (C_g \Omega(u, v) + D_g))^{1/2} \Theta \begin{bmatrix} a \\ b \end{bmatrix} (u, v).$$

Where

$$N_g \circ \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} D_g a - C_g b + (1/2) \text{dv}(C_g^t D_g) \\ -B_g a + A_g b + (1/2) \text{dv}(A_g^t B_g) \end{bmatrix},$$

("dv" indicates the diagonal vector) and  $\varepsilon(g, a, b)$  is a certain 8-th root of unity.

By use of the transformation formula we obtain the following two propositions. Because the calculation is straight forward we omit the detail.

**Proposition 2.2** (1) There are 36 even theta constants  $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\Omega)$  with  $a^t b \equiv 0 \pmod{2}$ . Among them we have 20 different  $\Theta \begin{bmatrix} a \\ b \end{bmatrix} (u, v)$ :

number	name	characteristic $\{a, b\}$
<i>n.1</i>	$\Theta_{h1}$	$\{\{0, 0, 0\}, \{0, 0, 0\}\}$
<i>n.2</i>		$\{\{1, 0, 0\}, \{0, 0, 0\}\}$
<i>n.3</i>		$\{\{0, 0, 0\}, \{1, 0, 0\}\}$
<i>n.4</i>		$\{\{0, 0, 1\}, \{0, 0, 0\}\}$
<i>n.5</i>	$\Theta_{h2}$	$\{\{1, 1, 0\}, \{0, 0, 0\}\}$
<i>n.6</i>	$\Theta_{yN}$	$\{\{1, 0, 1\}, \{0, 0, 0\}\}$
<i>n.7</i>		$\{\{1, 0, 0\}, \{0, 1, 0\}\}$
<i>n.8</i>	$\Theta_{xN}$	$\{\{1, 0, 0\}, \{0, 0, 1\}\}$
<i>n.9</i>	$\Theta_{h3}$	$\{\{0, 0, 0\}, \{1, 1, 0\}\}$
<i>n.10</i>	$\Theta_{xZ}$	$\{\{0, 0, 1\}, \{1, 0, 0\}\}$
<i>n.11</i>	$\Theta_{yZ}$	$\{\{0, 0, 1\}, \{0, 1, 0\}\}$
<i>n.12</i>		$\{\{1, 1, 1\}, \{0, 0, 0\}\}$
<i>n.13</i>	$\Theta_{yD}$	$\{\{1, 0, 1\}, \{0, 1, 0\}\}$
<i>n.14</i>	$\Theta_{xD}$	$\{\{1, 0, 0\}, \{0, 1, 1\}\}$
<i>n.15</i>		$\{\{0, 0, 1\}, \{1, 1, 0\}\}$
<i>n.16</i>		$\{\{1, 0, 1\}, \{1, 0, 1\}\}$
<i>n.17</i>		$\{\{1, 1, 1\}, \{1, 1, 0\}\}$
<i>n.18</i>		$\{\{1, 1, 1\}, \{1, 0, 1\}\}$
<i>n.19</i>		$\{\{1, 0, 1\}, \{1, 1, 1\}\}$
<i>n.20</i>		$\{\{1, 1, 0\}, \{1, 1, 0\}\}$

Table 3.1

(2) We have 9 even theta's with  $N_g \circ \begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{bmatrix} a \\ b \end{bmatrix} \pmod{2}$  for  $g \in G$ , those are indicated in Table 3.1 with proper names.

Let us call these nine thetas  $G$ -invariant theta constants.

We note here that the rational representation of the transformation

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in G_0$$

is given by

$$\begin{pmatrix} \operatorname{Re} b_2 & -\operatorname{Im} b_2 & -\operatorname{Im} b_3 & \operatorname{Re} b_1 & -\operatorname{Im} b_1 & \operatorname{Re} b_3 \\ \operatorname{Im} b_2 & \operatorname{Re} b_2 & \operatorname{Re} b_3 & \operatorname{Im} b_1 & \operatorname{Re} b_1 & \operatorname{Im} b_3 \\ \operatorname{Im} c_2 & \operatorname{Re} c_2 & \operatorname{Re} c_3 & \operatorname{Im} c_1 & \operatorname{Re} c_1 & \operatorname{Im} c_3 \\ \operatorname{Re} a_2 & -\operatorname{Im} a_2 & -\operatorname{Im} a_3 & \operatorname{Re} a_1 & -\operatorname{Im} a_1 & \operatorname{Re} a_3 \\ \operatorname{Im} a_2 & \operatorname{Re} a_2 & \operatorname{Re} a_3 & \operatorname{Im} a_1 & \operatorname{Re} a_1 & \operatorname{Im} a_3 \\ \operatorname{Re} c_2 & -\operatorname{Im} c_2 & -\operatorname{Im} c_3 & \operatorname{Re} c_1 & -\operatorname{Im} c_1 & \operatorname{Re} c_3 \end{pmatrix}.$$

and that we have

$$\det g \cdot \det (C_g \Omega(u, v) + D_g) = (a_1 + a_2 v + a_3 u)^2,$$

$$\frac{\partial(g(u, v))}{\partial(u, v)} = \frac{\det g}{(a_1 + a_2 v + a_3 u)^3} \quad \text{for } g = \begin{pmatrix} a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \in G_0.$$

### 2.3 Generator systems of the structure ring

A holomorphic function  $f(u, v)$  on  $\mathbf{B}^2$  is said to be a modular form of weight  $d$  with respect to  $G_0$  provided

$$f(g(u, v)) = ((a_1 + a_2 v + a_3 u))^d f(u, v), \quad g = \begin{pmatrix} a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \in G.$$

We define the modular form for other discrete groups in the same way. We use the following notation:

$M_d(G)$  : the vector space of modular forms of weight  $d$  with respect to  $G$ ,

$M_d(G(1+i))$  : the vector space of modular forms of weight  $d$  with respect to  $G(1+i)$ ,

$M(G)(M(G(1+i)))$  : the graded ring of modular forms with respect to  $G(G(1+i))$ , respectively.

**Remark 2.2** If we define the "projective modular form" by the condition

$$f(g(u, v)) = ((\det g)(a_1 + a_2v + a_3u))^d f(u, v), \quad g = \begin{pmatrix} a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \in G,$$

it is equivalent to consider the modular form with respect to  $SU(2, 1; \mathbf{Z}[\sqrt{-1}])$ . The structure of the ring of these modular forms are studied by Resnikoff-Tai ([Re-Ta]). Our modular form does make sense only for  $d$  divisable by 4.

**Proposition 2.3** Every fourth power of the  $G$ -invariant theta constant is a modular form of weight 4 with respect to  $G$ , namely it satisfies

$$\Theta^4 \begin{bmatrix} a \\ b \end{bmatrix} (g \circ (u, v)) = (a_1 + a_2v + a_3u)^4 \Theta^4 \begin{bmatrix} a \\ b \end{bmatrix} (u, v), \quad g = \begin{pmatrix} a_1 & a_2 & a_3 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \in G.$$

Note that we have  $G(1+i) = \langle g_1, \dots, g_5, g_6, \sqrt{-1} \rangle$  and

$$[\overline{G(1+i)}, \overline{G}] = 2,$$

where  $\overline{G(1+i)}$  and  $\overline{G}$  indicate the corresponding projective group. And that  $g_6$  induces the involution  $\iota : (x, y) \mapsto (y, x)$ . So we obtain the diagram with the Segre embedding map  $(u, v) \mapsto [X_0, X_1, X_2, X_3] \in \mathbf{P}^3$ :

$$\begin{array}{ccc} \mathbf{P}^1 \times \mathbf{P}^1 & \xrightarrow{\Phi} & \overline{\mathbf{B}^2/G} \cong \{X_0X_3 = X_1X_2\} \subset \mathbf{P}^3 \\ (x, y) & & (u, v) \\ \downarrow \pi & & \downarrow \pi \\ \mathbf{P}^2 = (\mathbf{P}^1 \times \mathbf{P}^1)/\iota & \xrightarrow{\Psi} & \overline{\mathbf{B}^2/G(1+i)} \cong \mathbf{P}^2 \end{array}$$

Diagram 1: Period diagram

**Proposition 2.4** 1) We have

$$\dim M_4(G) = 4 \quad \dim M_8(G) = 9.$$

2)  $M_4(G)$  is generated by  $\Theta_{xN}^4, \Theta_{xD}^4, \Theta_{yD}^4, \Theta_{h2}^4$ .

[proof]. Note that  $\mathbf{B}^2$  is the uniformization of the orbifold  $\mathbf{P}^1 \times \mathbf{P}^1$  with the arrangement of weighted divisors:

name	divisor	weight
$D_{0x}$	$\{(x, y) : x = 0\}$	4
$D_{0y}$	$\{(x, y) : y = 0\}$	4
$D_{1x}$	$\{(x, y) : x = 1\}$	4
$D_{1y}$	$\{(x, y) : y = 1\}$	4
$D_{\infty x}$	$\{(x, y) : x = \infty\}$	4
$D_{\infty y}$	$\{(x, y) : y = \infty\}$	4
$D_{xy}$	$\{(x, y) : x = y\}$	2

So the divisor of  $du \wedge dv$  corresponds to

$$-\frac{3}{4}D_{0x} - \frac{3}{4}D_{0y} - \frac{3}{4}D_{1x} - \frac{3}{4}D_{1y} - \frac{1}{2}D_{xy} - \frac{3}{4}D_{\infty x} - \frac{3}{4}D_{\infty y}$$

via the period map  $\Phi$ . Suppose  $f(u, v)$  belongs to  $M_4(G)$ . Then the divisor  $(f(u, v)(du \wedge dv)^{4/3})$  should satisfy

$$(f(u, v)(du \wedge dv)^{4/3}) \geq -D_{0x} - D_{0y} - D_{1x} - D_{1y} - \frac{2}{3}D_{xy} - D_{\infty x} - D_{\infty y}.$$

That is equivalent to give an element of  $H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(\frac{11}{3}D_{0x} + \frac{11}{3}D_{0y} + \frac{4}{3}K))$ . Where  $K$  denotes the canonical divisor. Because we have  $K = -2D_{0x} - 2D_{0y}$ , we know the isomorphism

$$M_4(G) \cong H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(D_{0x} + D_{0y})).$$

The vector space  $H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(D_{0x} + D_{0y}))$  is generated by the system  $\{1, x, y, xy\}$ , and it is four dimensional.

By the same way we have

$$M_8(G) \cong H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(2D_{0x} + 2D_{0y})).$$

So we obtain  $\dim M_8(G) = 9$ .

Observing the definition of  $\Omega(u, v)$  we know that

$$\Theta \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} (0, v) = \vartheta \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} (v) \cdot \vartheta \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} (v) \cdot \vartheta \begin{bmatrix} p_3 \\ q_3 \end{bmatrix} (\sqrt{-1}).$$

So our modular forms  $\Theta_{xN}^4, \Theta_{xD}^4, \Theta_{yD}^4, \Theta_{h2}^4$  are linearly independent, and they give a basis of  $M_4(G)$ .  
q.e.d.

**Proposition 2.5** *The space  $M_4(G)$  is generated by fourth powers of  $G$ -invariant thetas, and all algebraic relations are induced from*

$$\begin{aligned} \Theta_{xN}^4 - \Theta_{xD}^4 &= \Theta_{yN}^4 - \Theta_{yD}^4 \\ \Theta_{h2}^4 &= 2(\Theta_{xN}^4 - \Theta_{xD}^4) \\ \Theta_{h1}^4 - \Theta_{h3}^4 &= 2(\Theta_{xD}^4 + \Theta_{yN}^4) \\ \Theta_{xZ}^4 - \Theta_{yZ}^4 &= \Theta_{xN}^4 - \Theta_{yN}^4 \\ \Theta_{h1}^4 &= 2(\Theta_{xZ}^4 + \Theta_{yZ}^4) \end{aligned}$$

$$\begin{aligned} \Theta_{xD}^4 \Theta_{yN}^4 &= \Theta_{xZ}^4 (\Theta_{xN}^4 - \Theta_{xD}^4) \\ \Theta_{xN}^4 \Theta_{yD}^4 &= \Theta_{yZ}^4 (\Theta_{yN}^4 - \Theta_{yD}^4). \end{aligned}$$

[proof]. Let us consider the truncation of the Fourier expansions of the fourth powers of our  $G$  invariant Theta constants up to some fixed hight  $\ell$ . Set  $V_\ell$  be the vector space generated by these polynomials. There is the canonical surjective homomorphism  $\bar{\omega} : M_4(G) \rightarrow V_\ell$ . We can easily find that  $V_\ell$  is four dimensional. So  $\bar{\omega}$  should be an isomorphism. Then we can obtain the above relations by explicit calculations of their coefficients. The situation is the same for the quadratic relations.

q.e.d.

**Remark 2.3** *By observing the last identity in the above Proposition we have a system of generators  $X_0 = \Theta_{xN}^4, X_1 = \Theta_{yZ}^4, X_2 = \Theta_{yN}^4 - \Theta_{yD}^4, X_3 = \Theta_{yD}^4$  of the structure ring  $\mathbf{C}[X_0, X_1, X_2, X_3]/(X_0X_3 - X_1X_2)$  of  $\overline{B^2/G} \cong \mathbf{P}^1 \times \mathbf{P}^1$ .*

## 2.4 CM-isogeny on $Jac(C(x, y))$

Let us consider an isogeny map of  $\mathbf{B}^2$ :

$$\sigma(u, v) = ((1+i)u, 2v).$$

It is induced from the  $\mathbf{Q}$ -symplectic transformation:

$$N_\sigma = \begin{pmatrix} A & O \\ O & {}^t A^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we have

$$N_\sigma \circ \tau = A\tau {}^t A, \quad \tau \in \mathbf{H}_3.$$

Set

$$A_0 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\Lambda = \{(n_1 + n_2, n_1 - n_2) \in \mathbf{Z}^2 : n_1, n_2 \in \mathbf{Z}\}$$

$$\vartheta_{a', b'}^\Lambda(\tau') := \sum_{m \in \Lambda} \exp[\pi i(m + \frac{a'}{2})\tau'^t(m + \frac{a'}{2}) + 2\pi i(m + \frac{a'}{2})\frac{{}^t b'}{2}], \quad \tau' \in \mathbf{H}_2, a', b' \in \mathbf{Z}^2$$

$$\vartheta_{a, b}^\sharp(\tau) := \vartheta_{a, b}(A\tau {}^t A), \quad \tau \in \mathbf{H}_3, a, b \in \mathbf{Z}^3$$

$$\vartheta_{a', b'}^\sharp(\tau') := \vartheta_{a', b'}(A_0 \tau' {}^t A_0), \quad \tau' \in \mathbf{H}_2, a', b' \in \mathbf{Z}^2,$$

where  $\mathbf{H}_g$  denotes the Siegel upper half space of degree  $g$ . We emphasize here that  $\vartheta_{a, b}^\sharp(\tau)$  will play an important role in the following argument.

By elementary calculations we have

### Lemma 2.3

$$\vartheta_{a', b'}^\sharp(\tau') = \vartheta_{a' A_0, b' {}^t A_0^{-1}}^\Lambda(\tau').$$

According to Matsumoto, Minowa and Nishimura [Mat-Min-Nish] we have

### Theorem 6 ( $\Lambda$ -theta formula)

$$\begin{pmatrix} \vartheta_{a, b}^\Lambda(\tau) \\ \vartheta_{a+\{2,0\}, b}^\Lambda(\tau) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \exp[-\frac{\pi i}{2}(a_1 + a_2)] \\ 1 & -\exp[-\frac{\pi i}{2}(a_1 + a_2)] \end{pmatrix} \begin{pmatrix} \vartheta_{a, b}(\tau) \\ \vartheta_{a, b+\{1,1\}}(\tau) \end{pmatrix}$$

By virtue of Theorem 6 together with Lemma 2.3 we obtain:

### Proposition 2.6

$$\begin{aligned} \vartheta^\sharp \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}(\tau) &= \frac{1}{2} \vartheta \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}(\tau) - \frac{1}{2} \vartheta \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}(\tau) \\ \vartheta^\sharp \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}(\tau) &= \frac{1}{2} \vartheta \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}(\tau) + \frac{1}{2} \vartheta \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}(\tau) \\ \vartheta^\sharp \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}(\tau) &= \vartheta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}(\tau) \\ \vartheta^\sharp \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}(\tau) &= \frac{1+i}{2} \vartheta \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}(\tau) \\ \vartheta^\sharp \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\tau) &= \frac{1}{2} \vartheta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\tau) + \frac{1}{2} \vartheta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}(\tau) \\ \vartheta^\sharp \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\tau) &= \frac{1}{2} \vartheta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(\tau) - \frac{1}{2} \vartheta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}(\tau). \end{aligned}$$

An algebraically closed system under the isogeny action

## 2.5 Second Main Theorem, modular interpretation of the extended Gauss AGM

Set

$$\begin{aligned} H_1(u, v) &= \Theta_{h1}(u, v) = \vartheta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\Omega(u, v)), \\ H_3(u, v) &= \Theta_{h3}(u, v) = \vartheta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} (\Omega(u, v)), \\ Z(u, v) &= \Theta_Z(u, v) = \vartheta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} (\Omega(u, v)). \end{aligned}$$

Now we have

Second Main Theorem (CM-isogeny formula) . It holds

$$\begin{aligned} H_1((1+i)u, 2v) &= \frac{1}{2}(H_1(u, v) + H_3(u, v)) \\ H_3((1+i)u, 2v) &= Z(u, v) \\ Z((1+i)u, 2v) &= \sqrt[4]{\frac{1}{4} \left( Z^2 + H_1 H_3 \right) \left( Z^2 + \frac{H_1^2 + H_3^2}{2} \right)}. \end{aligned}$$

[proof]. The first two equalities are direct consequence of Proposition 2.6. So we are only concerned with the last equality. We can show that  $\Theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$  and  $\Theta \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  make at most their exchange under the action of the monodromy group  $G$ . So by using Proposition 2.1 we can calculate the automorphic factors of  $\Theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^4 + \Theta \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^4$  and  $\Theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^2 \Theta \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^2$ . And we know that they belong to  $M_4(G)$ . According to Proposition 2.5 by looking for the linear relations of truncated forms of thetas we obtain

$$\begin{aligned} \Theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^4 + \Theta \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^4 &= \frac{1}{2}\Theta_{h1}^4 - \frac{1}{2}\Theta_{h2}^4 + \frac{1}{2}\Theta_{h3}^4 \\ \Theta \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^2 \Theta \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^2 &= \frac{1}{2}\Theta_{xN}^4 - \frac{1}{2}\Theta_{yN}^4. \end{aligned}$$

By Proposition 2.6 we can shift these equalities to the ones for

$\Theta^\# \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^4 + \Theta^\# \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^4$  and  $\Theta^\# \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^2 \Theta^\# \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}^2$ . By cancellation of  $\Theta^\# \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  we obtain

$$Z((1+i)u, 2v)^4 = \frac{1}{4} \{ Z(u, v)^2 + H_1(u, v)H_3(u, v) \} \{ (Z(u, v))^2 + \frac{1}{2}(H_1(u, v)^2 + H_3(u, v)^2) \}.$$

q.e.d.

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