A SUBGROUP OF SELF-HOMOTOPY EQUIVALENCES WHICH SATISFIES THE M-L CONDITION

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INTRODUCTION

Let $\mathcal{E}(X)$ denote the set of (based) homotopy classes of self-homotopy equivalences of a (based) space X. $\mathcal{E}(X)$ is a group with group operation given by composition of homotopy classes.

For a finite CW-complex X there are two natural subgroups $\mathcal{E}_*(X)$, the subgroup of homotopy classes which induce the identity on the homology groups of X and $\mathcal{E}_\#^{dim}(X)$, or simply denoted by $\mathcal{E}_\#(X)$, the subgroup of homotopy classes which induce the identity on homotopy groups of X in dimensions $\leq$ dim X. These subgroups are known to be nilpotent [DZ]. In this paper, mainly we will study another subgroup $\mathcal{E}^n_{\#/\tau}(X)$ which consists of elements inducing the identity on $\pi_i(X)/\tau$, for $i \leq n$ where τ is the subgroup of torsion elements of $\pi_i(X)$. If all $\pi_i(X)$ are finite groups, each $\mathcal{E}^n_{\#/\tau}(X)$ coincides with $\mathcal{E}(X)$. Thus $\mathcal{E}^n_{\#/\tau}(X)$ fails to be nilpotent in general. For example, $\mathcal{E}^n_{\#/\tau}(\bigvee^5 M(Z_2, 3)) \cong \mathcal{E}(\bigvee^5 M(Z_2, 3))$ are not nilpotent groups, where $\bigvee^5 M(Z_2, 3)$ is the 5-fold wedge sum of Z_2 -Moore spaces. On the other hand, $\mathcal{E}^n_{\#/\tau}(X)$ are nilpotent groups in many cases (Theorem 2.5). A remarkable point is that $\mathcal{E}^n_{\#/\tau}(X)$ satisfies a certain stable property (the Mittag-Leffler condition) (Theorem 1.2).

Let us denote by X_P the localization of X at P.

§1 $\mathcal{E}^n_{\#/\tau}(X)$ AND $\mathcal{E}^n_\#(X_0)$

Theorem 1.1 *Let X be a finite nilpotent complex, then a descending normal series $\mathcal{E}^n_\#(X_0)$, $n = 1,...$ is Mittag-Leffler, namely $\mathcal{E}^\infty_\#(X_0) = \mathcal{E}^N_\#(X_0)$ for some N.*

Proof. There exists the following action induced by composition of maps.

$$\mathcal{E}(X_0) \times \pi_i(X_0) \to \pi_i(X_0)$$

Let $\mathbf{x}_n = \{x_1, ..., x_k\}$ be generators for $\pi_i(X)/\tau \subset \pi_i(X_0)$, $i \leq n$. Note that $\pi_i(X_0)$ are Q-vector spaces for $i \geq 2$ and for $i = 1$ we can choose generators so that elements of $\pi_i(X_0)$ are expressed in the form of $x_{i_1}^{r_1} \cdots x_{i_1}^{r_k}$, where r_{i_j} are rational numbers [Ha, §6] . Let $\mathcal{E}(X_0)_{\mathbf{x}_n}$ be the intersection of the isotropy subgroups at the elements $x_k \in \mathbf{x}_n$.

$$\mathcal{E}^n_\#(X_0) = \mathcal{E}(\mathrm{X}_0)_{\mathbf{x}_n}$$

and

$$\mathcal{E}^\infty_\#(X_0) = \bigcap_n^\infty \mathcal{E}(X_0)_{\mathbf{x}_n}$$

$\mathcal{E}(X_0)$ is isomorphic to an algedraic group [Sul], and each $\mathcal{E}(X_0)_{\mathbf{x}_n}$ is isomorphic to its algebraic subgroup [Ma 3]. Therefore, by a fundamental property of algebraic sets we obtain that

$$\bigcap_n^{\infty} \mathcal{E}(X_0)_{\mathbf{x}_n} = \mathcal{E}(X_0)_{\mathbf{x}_N}$$

for some N. Hence the result follows.

Although it is not known that the subgroup $\mathcal{E}_\#^\infty(X)$ satisfies the M-L condition as above except for special cases such as X is a product of spheres (see [AM]), we obtain

Theorem 1.2. *Let X be a finite nilpotent complex, then a descending normal series $\mathcal{E}_{\#/\tau}^n(X)$, $n = 1, \ldots$ is Mittag-Leffler, namely $\mathcal{E}_{\#/\tau}^\infty(X) = \mathcal{E}_{\#/\tau}^N(X)$ for some N.*

Proof. Now we recall some results from [Ma 3]. Let

$$\ell : \mathcal{E}(X) \to \mathcal{E}(X_0)$$

be the homomorphism induced by rationalization. Let

$$\mathcal{E}(X) \times \pi_i(X) \to \pi_i(X)$$

be the natural action, $\mathbf{x} = \{x_1 \ldots, x_k \ldots\}$ be generators for $\pi_i(X), i \geq 1$. Then $Q\mathcal{E}(X)_{\mathbf{x}}$ is defined to be the subgroup of $\mathcal{E}(X)$ which consists of elements satisfying $(fx_k)_0 = x_{k0}$. Then obviously,

$$Q\mathcal{E}(X)_{\mathbf{x}} = \mathcal{E}_{\#/\tau}^\infty(X)$$

since X is a finite nilpotent complex. By [Proposition 2.4 (and its proof), [Ma 3]] , ℓ $(Q\mathcal{E}(X)_{\mathbf{x}})$ is an arithmetic subgroup of $\mathcal{E}(X_0)_{\mathbf{x}_0}$, where $\mathbf{x}_0 = \{(x_1)_0, \ldots, (x_k)_0, \ldots\}$. As in the proof of Theorem 1.1, the group $\mathcal{E}(X_0)_{\mathbf{x}_0}$ is isomorphic to $\mathcal{E}_\#^\infty(X_0)$. Therefore, ℓ $(\mathcal{E}_{\#/\tau}^\infty(X))$ is commensurable with $\mathcal{E}_\#^\infty(X_0)_{\mathbf{z}}$. But the latter is isomorphic to $\mathcal{E}_\#^M(X_0)_{\mathbf{z}}$, for some M by Theorem 1.1. Now ℓ $(\mathcal{E}_{\#/\tau}^M(X))$ is commensurable to $\mathcal{E}_\#^M(X_0)_{Z}$, by the same reason, and hence the two groups $\mathcal{E}_{\#/\tau}^\infty(X)$ and $\mathcal{E}_{\#/\tau}^M(X)$ are commensurable, namely the first group has finite index in the second group (ker ℓ is a finite subgroup). It follws that $\mathcal{E}_{\#/\tau}^N(X)$ and $\mathcal{E}_{\#/\tau}^\infty(X)$ are isomorphic for some N.

Corollary 1.3 *Let X be a finite nilpotent complex. $\mathcal{E}_{\#/\tau}^\infty(X)$ is finite if and only if $\mathcal{E}_\#^\infty(X_0) = \{1\}$. In this case $\mathcal{E}_\#^N(X)$ is finite for some N.*
Proof. $\mathcal{E}_{\#/\tau}^\infty(X)$ is finite $\Leftrightarrow \mathcal{E}_{\#/\tau}^N(X)$ is finite for some $N \Leftrightarrow \mathcal{E}_\#^N(X)$ is finite $\Leftrightarrow \mathcal{E}_\#^N(X_0)$ is trivial [Mal], [Mo] $\Leftrightarrow \mathcal{E}_\#^\infty(X_0)$ is trivial. The second equivalence follows from the fact that $\mathcal{E}_\#^N(X)$ has the finite index in $\mathcal{E}_{\#/\tau}^N(X)$.

Remark. *$\mathcal{E}_{\#/\tau}^n(X)$ or $\mathcal{E}_\#^n(X)$ are not finite groups in general* ([AM], Corollary 6.2]).

§2 More on $\mathcal{E}_{\#/\tau}^n(X)$

We define $K(X)$ to be the kernel of the homomorphism $\ell : \mathcal{E}(X) \to \mathcal{E}(X_0)$.

Theorem 2.1. *Let X be a finite nilpotent complex. If $\mathcal{E}_{\#/\tau}^n(X)$ is finite and $n \geq dimX$ or $n = \infty$, then $\mathcal{E}_{\#/\tau}^n(X) \cong K(X)$.*

Proof. Since $K(X)$ is a subgroup of $\mathcal{E}^n_{\#/\tau}(X)$ there exists the following exact seauence

$$0 \to K(X) \to \mathcal{E}^n_{\#/\tau}(X) \to \mathcal{E}^n_{\#}(X_0)$$

The group $\mathcal{E}^n_{\#}(X_0)$ is uniquely divisible and hence we obtain the result.

By the same argument we obtain

Proposition 2.2. *Let X be a finite nilpotent complex. Then $\mathcal{E}^n_{\#/\tau}(X)/K(X)$ is a nilpotent group for $n \geq$ dim X.*

A group G is said to be a finite-by-nilpotent group if it has a finite normal subgroup N such that G/N is nilpotent. By [HMR], $K(X)$ is a finite group and we obtain

Corollary 2.3. *Let X be a finite nilpotent complex. Then $\mathcal{E}^n_{\#/\tau}(X)$ is a finite-by-nilpotent group for $n \geq$ dim X or $n = \infty$.*

Next we consider the case where the homology groups of X have no torsion.

Lemma 2.4. *Let X be a finite nilpotent complex whose homology groups have no torsion. Then $K(X)$ is a finite subgroup of $\mathcal{E}_*(X)$.*

Proof. $K(X)$ is finite by [HMR]. By our assumptions, the result is clear.

Theorem 2.5. *Let X be a finite nilpotent complex whose homology groups have no torsion. Then $\mathcal{E}^n_{\#/\tau}(X)$ is a nilpotent group for $n \geq \dim X$.*

Proof. By Lemma 2.4, all the elements of $K(X)$ induces the identity on homology.

Let us consider three actions $\omega_1 : \mathcal{E}^n_{\#/\tau}(X) \times H_i(X) \to H_i(X)$, $\omega_2 : \mathcal{E}^n_{\#/\tau}(X)/K(X) \times H_i(X) \to H_i(X)$ and ω_3 : $\mathcal{E}^n_{\#}(X_0) \times H_i(X_0) \to H_i(X_0)$, where ω_2 is induced from ω_1. The following diagram is commutative.

$$\begin{array}{ccc}
\mathcal{E}^n_{\#/\tau}(X) \times H_i(X) & \longrightarrow & H_i(X) \\
\downarrow & & \| \downarrow \\
\mathcal{E}^n_{\#/\tau}(X)/K(X) \times H_i(X) & \longrightarrow & H_i(X) \\
\downarrow & & \downarrow \\
\mathcal{E}^n_{\#}(X_0) \times H_i(X_0) & \longrightarrow & H_i(X_0)
\end{array}$$

We obtain three lower central series defined in [HMR](section 4) correspondeing to these actions. Let us denote them by $\{\Gamma^i\omega_1(H_i)\}$, $\{\Gamma^j\omega_2(H_i)\}$ and $\{\Gamma^j\omega_3(H_{i0})\}$ respectively. By the above commutative diagram, $\Gamma^i_{\omega_1}(H_i) = \Gamma^j\omega_2(H_i)$ and they are subgroups of $\Gamma^i_{\omega_3}(H_{io})$ since homology has no torsion elements. If $n \geq$ dim X, the action ω_3 is a nilpotent action in the sense of [HMR], that is, $\Gamma^i_{\omega_3}(H_{i0}) = 0$ for some j (This can be achieved by induction using the postnikov system) and hence $\Gamma^i_{\omega_1}(H_i) = 0$ with this j. Now we obtain that the action ω_1 is a nilpotent action, and hence our result follows from Theorem D in [DZ].

Lemma 2.6. *Let X be a simply connected finite complex whose homology groups have no torsion. Then $K(X)$*

is a finite nilpotent group and $K(X_P) \cong K(X)_P$.

Proof. By Lemma 2.4, $K(X)$ is a subgroup of $\mathcal{E}_.(X)$. The latter group is nilpotent [DZ], and so is $K(X)$. On the other hand, we have an exact sequence

$$0 \to K(X) \to \mathcal{E}_.(X) \xrightarrow{\ell} \mathcal{E}_.(X_0)$$

By [Ma 2] $\mathcal{E}_.(X_P) \cong \mathcal{E}_.(X)_P$. Moreover, as localization commutes with pullbacks (Theorem 2.10 [HMR]), $(\ker \ell)_P \cong \ker \ell_P$. Hence the result follows.

By Theorem 2.1 and Lemma 2.6 wwe obtain

Theorem 2.7. *Let X be a simply connected finite complex whose homology groups have no torsion. Assume that* $\mathcal{E}^n_{\#/\tau}(X)$ *is finite and* $n \geq \dim X$ *or* $n = \infty$, *then* $\mathcal{E}^n_{\#/\tau}(X_P) \cong \mathcal{E}^n_{\#/\tau}(X)_P$.

Combining with Corollary 1.3 we obtain

Corollary 2.8. *Let X be a simply connected finite complex whose homology groups have no torsion. If* $\mathcal{E}^\infty_\#(X_0) = \{1\}$, *then* $\mathcal{E}^\infty_{\#/\tau}(X_P) \cong \mathcal{E}^\infty_{\#/\tau}(X)_P$.

§3 LOCALIZATION OF $\mathcal{E}^\infty_\#(X)$

In general we have

Proposition 3.1. $\mathcal{E}^\infty_\#(X_P)$ is P local for a finite nilpotent complex X and an arbitrary set of prime numbers P.

Lemma 3.2. *Let* $\{f_{i+1}: G_{i+1} \to G_i\}$ *be an inverse system such each* G_i *is P-local. Them* $\varprojlim G_i$ *is also P-local.*

Proof. Let q be an integer which is prime to the all the elemenets of P. Let $\cdot q$ be the q-th power map. The map $\cdot q : \varprojlim G_i \to \varprojlim G_i$ is induced from the maps $\cdot q : G_i \to G_i$. The following diagram is commutative.

$$\begin{array}{ccccccc}
\longrightarrow & G_{i+1} & \xrightarrow{f_{i+1}} & G_i & \xrightarrow{f_i} & G_{i-1} & \longrightarrow \\
 & \downarrow .q & & \downarrow .q & & \downarrow .q & \\
\longrightarrow & G_{i+1} & \xrightarrow{f_{i+1}} & G_i & \xrightarrow{f_i} & G_{i-1} & \longrightarrow
\end{array}$$

Each $\cdot q : G_i \to G_i$ is injective since G_i is p-local, and thus $\cdot q : \varprojlim G_i \to \varprojlim G_i$ is also injective. Let (g_i) be an element of $\varprojlim G_i$. For $g_i \in G_i$, there exists the unique element g_i' such that $\cdot q\,(g_i') = g_i$. $\cdot q\,(f_{i+1}(g_{i+1}')) = f_{i+1}(g_{i+1}) = g_i$ so, $f_{i+1}(g_{i+1}') = g_i'$. This shows that $(g_i') \in \varprojlim G_i$. Therefore $\cdot q : \varprojlim G_i. \to \varprojlim G_i$ is surjective.

Proof of proposition 3.1. By [Mal] [Mo] it holds that $\mathcal{E}_\#(X_p) \cong \mathcal{E}_\#(X)_P$, thus $\mathcal{E}_\#(X_P)$ is P- local. As the group $\mathcal{E}^\infty_\#(X_P)$ is isomorphic to $\varprojlim_n \mathcal{E}^n_\#(X_P)$. Our result is clerar from Lemma 3.2

Remark. Therefore $\mathcal{E}^\infty_\#(X_0)$ is finite if and only if $\mathcal{E}^\infty_\#(X_0) = \{1\}$ for finite nilpotent complexes.

For a more special case we obtain

Theorem 3.3. *Let X be a finite nilpotent complex. If $\mathcal{E}^{\infty}_{\#/\tau}(X)$ is finite or equivalently $\mathcal{E}^{\infty}_{\#}(X_0) = \{1\}$, then $\mathcal{E}^{\infty}_{\#}(X_P) = \mathcal{E}^{\infty}_{\#}(X)_P$.*

Proof. Under our condition, by Corollary 1.3, $\mathcal{E}^{\infty}_{\#}(X)$ is finite and isomorphic to $\mathcal{E}^{N}_{\#}(X)$ for some N. Now the result follows from [Ma1], [Mo].

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