Article

Fixed Point Structure of the "Principle of Effective Demand": An Exposition

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Abstract

This expository article will focus on the general (dis) equilibrium aspects of John Maynard Keynes' *General Theory* to provide a fixed-point characterization of macroeconomic equilibria. As an immediate consequence of our fixed-point characterization, the quantity adjustment process known as the "Principle of Effective Demand" will be identified as the counterpart to the Debreu mapping which represents the *tâtonnement process* in the traditional general equilibrium analysis.

1 Introduction

This expository article will focus on the general (dis) equilibrium aspects of John Maynard Keynes' *General Theory* to provide a fixed-point characterization of macroeconomic equilibria. As an immediate consequence of our fixed-point characterization, the quantity adjustment

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* Dedicated with many thanks to Professor Emeritus Masayuki Iwata on his retirement from Chiba University. This exposition might have constituted one of the subject matters of conversations in the Departmental Coffee Lounge I enjoyed so much with Professor Iwata.

process known as the "Principle of Effective Demand" will be identified as the counterpart to the Debreu mapping which represents the *tâtonnement process* in the traditional general equilibrium analysis.

Indeed, this finding is not surprising at all when one recalls the similarity between the 2-dimensional diagram of Brouwer's Fixed Point Theorem and the so-called 45 degree line analysis, which is a standard textbook treatment of the "Principle of Effective Demand." We have obtained our insight from the *self-fulfilled* nature of the effective demand, i.e., the effective demand is the *market-clearing* expectation among the entrepreneurs' expected aggregate demands out of the income computed as the monetary value of their aggregate supply.

As a general (dis)equilibrium model, one of the major virtues of Keynes' analysis is the emphasis on "spill-over" effects between such aggregate markets as the labor and commodity markets with a resort to the added simplicity made possible by his innovative aggregation of economic activities in many interrelated markets into three markets: Commodity, Money and Labor Markets.

> I have called my theory a *general* theory. I mean by this that I am chiefly concerned with the behaviour of the economic system as a whole, – with aggregate incomes, aggregate profits, aggregate output, aggregate employment, aggregate investment, aggregate saving rather than with the incomes, profits, output, employment, investment and saving of particular industries, firms or individuals. And I argue that important mistakes have been made through extending to the system as a whole conclusions which have been correctly arrived at in respect of a part of it taken in

isolation. [Keynes (1973, p. xxxi, Preface to the French Edition)]

The usual presumption of the economy-wide *ex post* "identity" of produced, distributed and expended monetary values is nothing but the Walras Law. Therefore, the commodity, money and labor markets are linearly dependent, and if any two of the three markets are in equilibrium, then automatically so is the third market. Throughout the present exposition, I shall follow the ordinary textbook treatment and drop the labor market from our formal analysis. The unemployment problem in the labor market will be taken care of indirectly as reflected in the commodity and money markets.

Of related interests, Uzawa (1962) noted the equivalence of Existence Theorem of general equilibria and Brouwer's Fixed Point Theorem, i.e., not only Brouwer's Fixed Point Theorem, in its generalized form by Kakutani (1941), implies existence of general equilibria as fixed points of the Debreu mapping which idealizes the tâtonnement process, but the converse is also true. Nikaido (1975) has successfully exploited a similar insight to ours, the self-fulfilled nature of effective demand, to construct an "objective demand" in the general equilibrium analysis of monopolistic competition.

2 Keynes' Theory of Employment or The Multiplier

Unlike Keynes who worked directly on the labor market to develop the general theory of employment, we shall first single out the commodity market, which is the *range* of the aggregate supply function or the aggregate demand function rather than their *domain*. Because of this difference, our aggregate supply and demand functions, to be introduced below, differ from their original characterization due to Keynes (1936, p. 25) who defines:

> Let Z be the aggregate supply price of the output from employing N men, the relationship between Z and N being written $Z = \phi(N)$, which can be called the *aggregate supply function*[1]. Similarly, let D be the proceeds which entrepreneurs expect to receive from the employment of N men, the relationship between N and D being written $D = f(N)$, which can be called the *aggregate demand function*.

In the preceding quotation, the "aggregate supply price of the output" may read as the "monetary value of the aggregate supply" in the current terminology, and the "proceeds which entrepreneurs expect to receive" as the "expected monetary value of aggregate demand." It also merits emphasis that, pertaining to the expected proceeds, Keynes might very well have been aware of the underlying fixed point structure when he idealized the *self-fulfilled* expected proceeds by his effective demand, and the interium adjustment by his multiplier process.

We may justify our choice of the commodity market by assuming away the crowding-out of investment and focusing on the case where investment I is independent of the interest rate r. In Section 3, we will

1) Its inverse $N = \phi^{-1}(Z)$ will be taken up again as the *employment function* to relate the effective demand with the employment demand [Keynes (1936, Chapter 20)].

introduce the crowding-out feature and take up the case with $I(r)$.

The "Principle of Effective Demand" may be summarized in terms of standard notation as[2]:

$C(Y)$ Consumption will depend on the level of aggregate income Y, the relation of which is governed by the psychological characteristic of the community, i.e., its *propensity to consume*. [Keynes (1936, p. 28, (2))]

$D(Y)$ The *effective demand* is "the sum of two quantities, namely $C(Y)$, the amount which the community is expected to spend on consumption, and I, the amount which it is expected to devote to new investment." [Keynes (1936, p. 29, (3))] In the context of a full-fledged open mixed economy, in addition to $C(Y)$ and I, the government expenditure G and the export demand X constitute $D(Y)$.

Y The aggregate supply Y is determined by "$C(Y)+I=D(Y)=Y$." [Keynes (1936, p. 29, (4))] In an open mixed economy, the aggregate supply includes the tax revenue T_{-1} (the supply from the economic activities of the private sector in the previous year -1) and the import M (the supply from the foreign countries) as well as the *GDP* Y (the domestic supply).

2) By focusing on the commodity market, we have managed to eliminate some of the proposed steps [Keynes (1936, pp. 28 and 29, Propositions (1), the second half of (3), (5), (6), (7) and (8))]. These eliminations have helped to reveal the bare bones of the Principle of Effective Demand.

It is worth emphasizing that the Principle of Effective Demand $Y = D(Y)$, "demand creates an equal amount of supply," in words, consists of two propositions:

1. "Demand creates supply," i.e., the aggregate supply is some function Φ of the aggregate demand $Y = \Phi(D(Y))$ in short,

and

2. "Creates an *equal* amount of," i.e., the function Φ takes a special form, the *identity map* $\mathrm{id}_{\mathbf{R}}$: $\mathbf{R} \to \mathbf{R}$ defined by $\mathrm{id}_{\mathbf{R}}(x) = x$ for all $x \in \mathbf{R}$.

Therefore, $Y = D(Y)$ for all $D(Y) \in \mathbf{R}$.

2. 1 A Quick Review of General Equilibrium Analysis

In order to be self-contained, we will summarize relevant equilibrium existence results. For more detail and precision, the readers are referred to the dictum by Gerard Debreu (1959).

A (*contingent*) *commodity* is a good or a service characterized with reference to commodity characteristics (commodity differentiation), date of delivery (intertemporal transactions), events or states (uncertainty), and location (urban economics).

A *commodity bundle* x is a specification of the quantity for each commodity. Every commodity bundle can be represented by an element in the *commodity space* X, i.e., $x \in X$. X could be (non-negative subset of) any linear space $\mathcal{L}_+$, e.g., $\mathbf{R}^l_+$, $\mathbf{R}_+ \times (\mathbf{N} \cup \{0\})^{l-1}$ ($(l-1)$ indivisible commodities with a requisite of the presence of (at least) one perfectly divisible commodity); $\mathbf{R}_\infty$, the space of real-valued sequences (e.g., for intertemporal consumption streams over a countable infinite horizon), or its bounded subsets $bd(\mathbf{R}_\infty)$ and l_∞; $\mathcal{L}_\infty$ the space of $\| \; \|$-norm bounded functions; or $\mathcal{L}_2$, the space of square integrable functions in finance.

A *consumption set* X is a subset of $\mathcal{X}$, i.e., $X \subset \mathcal{X}$. The distinction of the consumption set from the underlying commodity space is particularly important when each consumer has a different and/or nonconvex consumption set.

A *preference relation* (*preferences, preference* (*pre*)*ordering*) is defined by $(X, \succ)$ where $\succ$ is a binary relation on X, i.e., $\succ \subset X \times X$ or $\succ \in \mathcal{P}(X \times X)$, the *power set* of $X \times X$. Let $\mathcal{P}$ denote the *space of preferences* satisfying the following conditions: Let $x, y, z \in X$. (*i*) *Irreflexivity*: $x \not\succ x$; (*ii*) *Transitivity*: $x \succ y \wedge y \succ z \Rightarrow x \succ z$; and (*iii*) *Continuity*: $\{(x, y) \mid x \succ y\}$ is open relative to $(X, \mathcal{T})$.

Define two subsets of $\mathcal{P}$. $\succ \in \mathcal{P}_{mo}$ satisfies in addition (*iv*) (*Strict*) *Monotonicity*: $x \geq$ (or $>$) $y \Rightarrow x \succ y$; and $\succ \in \mathcal{P}_{con}$ satisfies (*v*) *Convexity*: $\{x \in X \mid x \succ z\}$ is convex for every $z \in X$.

Let A be the space of agents. A *finite exchange economy* is a mapping $\mathcal{E}: A \to \mathcal{P} \times X$, where $|A|$ is finite. Let $\succ_a$ be the projection of $\mathcal{E}(a)$ onto $\mathcal{P}$, and $e(a)$ the projection of $\mathcal{E}(a)$ onto X. $(X(a), \succ_a)$ is the preference of agent a, and $e(a)$ her initial endowment. In short, for $a \in A$, $\mathcal{E}(a)$ is the *consumer's characteristics* of agent a.

A *price* p is chosen from Δ, a compact and convex subset of the dual of $\mathcal{X}$.

For an agent $a \in A$, her *budget set* is $B(p, a) = \{x \in X(a) \mid p.x \leq p.e(a)\}$.

The (*individual*) *demand set* of agent a is the set of maximal elements in $B(p, a)$ with respect to $\succ_a$, or $\succ_a$-maximal in $B(p, a)$ in short, i.e.,

$$D(p, a) = \{x \in B(p, a) \mid (\forall y \in X(a)) y \succ_a X \Rightarrow p.y > p.e(a)\},$$

or in net terms, her *net* (or *excess*) *demand set* is

$$\begin{aligned} d(p, a) &= D(p, a) - e(a) \\ &= \{x - e(a) \mid x \in B(p, a), (\forall y \in X(a)) y \succ_a x \Rightarrow p.y > p.e(a)\}. \end{aligned}$$

Let $\mathcal{W}(\mathcal{E})$ be the set of *competitive equilibrium* (or *Walrasian*) allocations. $f \in \mathcal{W}(\mathcal{E})$ if there exists $p \in \Delta$ such that

(1) $(\forall a \in A) f(a) \in D(p, a)$, or $f(a) - e(a) \in d(p, a)$;

and

(2) $\sum_{a \in A} f(a) \leq \sum_{a \in A} e(a)$.

Given a finite exchange economy $\mathcal{E}: A \rightarrow \mathcal{P} \times X$ with $|A| = n$, let the *excess demand* D: $\Delta \rightarrow X \cup (-X)$ be defined by

$$\begin{aligned} D(p) &= \sum_{a \in A} d(p, a) \\ &= \sum_{a \in A} \Big\{ z \in X \cup (-X) \;\Big|\; p.z \leq 0, (\forall y \in X) y \succ_a z + e(a) \\ &\qquad\qquad \Rightarrow p.y > p.e(a) \Big\}. \end{aligned}$$

Define an *exchange process* μ: $\Delta \rightarrow \Delta$ by the Debreu-mapping

$$\begin{aligned} \mu(p) &= \{q \in \Delta \mid (\forall p \in \Delta) q.D(p) \geq p.D(p)\} \\ &= \{q \in \Delta \mid (\forall p \in \Delta)(q - p).D(p) \geq 0\} \end{aligned}$$

In words, "q maximizes $q.D(p)$ over Δ." This is another way of characterizing the *tâtonnement process*, alternative to the more familiar form: $q = p + \psi(D(p))$, where ψ is sign-preserving, and $D(p)$ is single-valued.

Theorem 2.1 (Brouwer's Fixed Point Theorem) [Knaster, Kuratowski and Mazurkiewicz (1929)]

Let $S \subset \mathbf{R}^l$ be a nonempty, compact and convex subset, and $f: S \to S$ a continuous into-itself function. Then, f has a fixed point x^* such that $x^* = f(x^*)$.

Theorem 2.2 (Kakutani's Fixed Point Theorem) [Kakutani (1941)]

Let $S \subset \mathrm{R}^l$ be a nonempty, compact and convex subset, and $f: S \to S$ an upper hemicontinuous into-itself correspondence, which is nonempty- and convex-valued for all $x \in S$. Then, f has a fixed point x^* such that $x^* \in f(x^*)$.

The existence of competitive equilibria $\mathcal{W}(\mathcal{E}) \neq \emptyset$ is translated as the existence of fixed points for the Debreu mapping, i.e., $p^* \in \mu(p^*)$.

Theorem 2.3 *Let $\mathcal{E}: A \to \mathcal{P}_{mo} \times \mathrm{R}^l_+$ be such that $\sum_{a \in A} e(a) \gg 0$ and $\mathcal{W}(\mathcal{E}) \neq \emptyset$.*

Let p be the nontrivial price associated with $f \in \mathcal{W}(\mathcal{E})$. Then, $p \gg 0$.

Proof: Suppose otherwise, i.e., $(\exists i \in \{1, \ldots, l\})[p^i = 0]$. Since p is nontrivial and $\sum_{a \in A} e(a) \gg 0$, $(\forall f \in \mathcal{W}(\mathcal{E}))(\exists j \in \{1, \ldots, l\})[p^j \neq 0]$. Define $g(a)$ by

$$g^k(a) = \begin{cases} f^i(a) + 1 & \\ f^j(a) - \varepsilon & \\ f^k(a) & \text{otherwise} \end{cases}.$$

Then, by monotonicity of $\succ_a$, $f(a) + u_i \succ_a f(a)$, where $u_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{R}^l_+$ with the 1 in the i-th coordinate; and by continuity of

$\succ_a$, for sufficiently small $\varepsilon > 0$, $g(a) \succ f(a)$. However, $p.g(a) =$

$\sum_{k=1}^{l} p^k.f^k(a) - p^j\varepsilon = p.e(a) - p^j\varepsilon < p.e(a)$, a contradiction to $f \in \mathcal{W}(\mathcal{E})$. ■

Theorem 2.4 *Let $\mathcal{E}_{con}$ be the convex exchange economy obtained by restricting $\mathcal{P}_{mo}$ to $\mathcal{P}_{mo} \cap \mathcal{P}_{con}$ in the definition of $\mathcal{E}$ in Theorem 2.3. Then, there exists $p \in \Delta = \left\{ p \in R^{l}_{+} \mid \|p\|_{\infty} \leq 1,\ p \gg 0 \right\}$ and an allocation $f(a) \in d(p, a)$ for all $a \in A$.*

Proof: Let

$$\Delta' = \left\{ p \in \Delta \mid p \geq \frac{1}{\sqrt{n}} e \right\}$$

and

$$X = \left\{ x \in R^{l} \mid \| x \|_1 \leq \left(n^{\frac{3}{2}} + n \right) \max \left\{ \| e(a) \|_1 \mid a \in A \right\} \right\}.$$

Define $\phi : \Delta' \times X \rightarrow \Delta' \times X$ by

$$\phi(p, x) = \mu(p) \times D(p) = \{ (q, y) \mid y \in D(p),\ q \text{ maximizes } q.x \text{ over } \Delta' \}.$$

Then, $D(p) \subset X$. ϕ is convex-valued and upper hemicontinuous. By Kakutani's Fixed Point Theorem (Theorem 2.2), ϕ has a fixed point $(p, x) \in \phi(p, x)$ such that $x \in D(p)$ and $q.x \leq p.x \leq 0$ for all $q \in \Delta'$. ■

2.2 Fixed Point Characterization of the *IS* Equilibrium

It is imperative to note the dual nature of *GDP* Y being "income" on which $C(Y)$ the consumption demand depends on the one hand, and "aggregate supply" on the other that is generated by the effective demand $D(Y) = C(Y) + I$ so that $Y = D(Y)$.

Therefore, on the diagram with the horozontal Y-axis and the vertical D-axis, to be precise, the aggregate supply $Y = D(Y)$ is depicted as the 45 *degree line against the vertical D-axis*.

The equilibrium *GDP* Y^* is determined at the intersection of the aggregate supply, as represented by the 45 degree line, and the aggregate demand i.e., such that $Y^*=D(Y^*)+I$ and $I=Y^*-C(Y^*)=S(Y^*)$ (The *IS* Equilibrium).

In this exposition, we will introduce yet another characterization: Y^* may be reckoned as the *fixed point* of the single-valued effective demand function $D(Y)$, i.e., $Y^*=D(Y^*)$. This characterization turns out to be particularly convenient when one attempts at an interpretation of the "Principle of Effective Demand" in terms of fulfilled expectations (See the Remark 2.2).

Theorem 2.5 *Let Y_F be the full employment GDP. Consider the effective demand function D: $[0, Y_F]\rightarrow[0, Y_F]$ defined by $D(Y)=C(Y)+I$. Then, there exists an IS equilibrium Y^* such that $I=S(Y^*)$.*

Proof: The closed interval $[0, Y_F]$ is a compact and convex subset of **R**, and the effective demand function D: $[0, Y_F]\rightarrow[0, Y_F]$ is a continuous into-itself function. By Brouwer's Fixed Point Theorem (Theorem 2.1), there exists a fixed point Y^* for D, i.e., $Y^*=D(Y^*)$.

Since the fixed point Y^* for D satisfies $Y^*=D(Y^*)=C(Y^*)+I$, $I=Y^*-C(Y^*)=S(Y^*)$, and consequently Y^* also qualifies as the *IS* equilibrium. ■

Remark 2.1 Our confinement of Y to $[0, Y_F]$ may be justified with reference to the following quotation from Keynes (1936, p. 28) who claims to the same effect in terms of levels of employment:

> This level [of employment] cannot be *greater* than full employment, i.e., the real wage cannot be less than the marginal disutility of labour.

Remark 2. 2 The adjustment process is conceived as follows: Suppose initially $Y_0 < D(Y_0)$, i.e., the aggregate supply Y_0 somehow falls short of the effective demand $D(Y_0)$. Then, **First Round:** Aggregate Supply $Y \uparrow (Y_0 \rightarrow Y_1)$ [By the "Principle of Effective Demand," the "realized" income is $Y_1 = D(Y_0)$] $\Rightarrow D(Y) \uparrow (D(Y_0) \rightarrow D(Y_1)) \Rightarrow$ **Second Round:** $Y \uparrow (Y_1 \rightarrow Y_2 (= D(Y_1))) \Rightarrow D(Y) \uparrow (D(Y_1) \rightarrow D(Y_2)) \Rightarrow \ldots \Rightarrow$ **n-th Round** $\Rightarrow \ldots$. This process will continue until the excess aggregate demand is eliminated, i.e., $Y^* = D(Y^*)$. At Y^*, the aggregate supply coincides with the effective demand that is the *self-fulfilled* or *market-clearing* one among the entrepreneurs' expectations of aggregate demand out of the income computed as the monetary value of their aggregate supply.

Alterrnative representations of this adjustment process are:

i. **Differential Equation:** $\dot{Y} = \alpha \{(C(Y) + I) - Y\} = \alpha \{I - S(Y)\}$ with $0 < \alpha < 1$. This is a special case of the characterization to be given in the subsequent Section 3. 1.

ii. **Multiplier:** Denote $Y^* - Y_0 = \Delta Y$ ("Total" Increment), $\Delta Y_1 = Y_1 - Y_0 = D(Y_0) - Y_0$ (Increment in the **First Round**), $\Delta Y_2 = Y_2 - Y_1 = c \Delta Y_1 = c(D(Y_0) - Y_0)$ (Increment in the **Second Round**), ... , $\Delta Y_n = Y_n - Y_{n-1} = c \Delta Y_{n-1} = c^{n-1}(D(Y_0) - Y_0)$ (Increment in the **n-th Round**), ... Then, $\Delta Y = \sum_{n=1}^{\infty} \Delta Y_n = \frac{1}{1-c}(D(Y_0) - Y_0)$, where c is the *marginal propensity to consume* and $\frac{1}{1-c}$ is the *multiplier* [Keynes (1936, p. 115)].

Remark 2. 3 The spill-over effect of the *deflationary gap* $Y_F - Y^*$ is reflected as the *involuntary unemployment* to the amount of $N_F - N^* = \phi^{-1}(Y_F) - \phi^{-1}(Y^*)$ in terms of the employment function ϕ^{-1} defined in the footnote 1.

3 The *IS–LM* Model

The contemporary characterization of the *IS–LM* equilibrium due to Hicks (1937) elucidates Keynes' analysis, perhaps too much simplified for some critics. Hicks' underlying methodology is expounded verbally in his subsequent book (1946, Chapter V, 4.), the mathematical content of which may be summarized as follows:

Let x and y be the state variable representing the interrelated X-market and Y-market, respetively. Suppose the adjustment process of each interrelated market is characterized as: $\frac{dx(t)}{dt} =_{Def} \dot{x} = \chi(x, y)$ and $\dot{y} = \psi(x, y)$. Then, the system-wide stability reduces to the stability of the simultaneous differential equations

$$\begin{cases} \dot{x} = \chi(x, y) \\ \dot{y} = \psi(x, y) \end{cases}.$$

In particular, the general equilibrium (x^*, y^*) satisfies

$$\begin{cases} \chi(x^*, y^*) = 0 \\ \psi(x^*, y^*) = 0 \end{cases}.$$

Let π denote the exchage rate in the denomination of home currency. Its inverse $\frac{1}{\pi}$ measures the strenghth of the home currency. Denote by Y_W the level of *GDP* in the rest of the world. We simplify the present analysis of *IS–LM* equilibrium by assuming both variables exogenously given[3].

3. 1 Dynamic Adjustment Process of the Commodity Market

$$\dot{Y} = \alpha \left\{ \left[\underbrace{I(r) - \overbrace{S(Y)}^{=Y-C(Y)}}_{\text{Private Balance}} \right] + \left[\underbrace{G - T_{-1}}_{\text{Fiscal Balance}} \right] + \left[\underbrace{X\left(\frac{1}{\pi}, Y_W\right) - M\left(\frac{1}{\pi}, Y\right)}_{\text{Current Balance}} \right] \right\}$$

$$= \alpha \left\{ \left[\underbrace{C(Y) + I(r) + G + X\left(\frac{1}{\pi}, Y_W\right)}_{\text{Aggergate Demand}} \right] - \left[\underbrace{Y + T_{-1} + M\left(\frac{1}{\pi}, Y\right)}_{\text{Aggergate Supply}} \right] \right\};$$

$0 < \alpha < 1$

- This process is a dynamic version of the Principle of Effective Demand in that the excess effective demand creates new sup-

3) Although we refrain from pursuing it here, it is certainly possible to incorporate the determination of π by introducing, in addition to those in Sections 3.1 and 3.2, the third differential equation which specifies the adjustment process of π in response to the balance of payment :

$$\dot{\pi} = \gamma \left\{ X\left[\frac{1}{\pi}, Y_w\right] - M\left[\frac{1}{\pi}, Y\right] + K\left[r - r_w\right] \right\};\ 0 < \gamma < 1,$$

where $K(r - r_w)$ denotes the (*Long-Term*) *Capital Balance*, the net flow of capital in response to the difference between domestic and foreign interest rates, the latter of which r_w is taken to be exogenously given under the *small country hypothesis*.

4) The rate of interest r (=marginal efficiency of investment) $\uparrow \Rightarrow$ investment demand $I(r) \downarrow \Rightarrow$ For $I = S$, saving $S(Y) \downarrow \Rightarrow$ *GDP* $Y \downarrow$

ply.

- $\dot{Y}=0$ yields the *IS*-curve, which is downward-sloped on the (Y, r) plane[4].

3. 2　Dynamic Adjustment Process of the Money Market

Given the exogenously determined money supply M_S, the adjustment of interest rate r reflects the excess demand for money, i.e.,

$$\dot{r} = \beta\,(L(Y, r) - M_S);\ 0 < \beta < 1$$

where $L(Y, r) = L_1(Y) + L_2(r)$ with $L_1(Y)$ being the transaction-cum-precautionary demand for money, and $L_2(r)$ its speculative demand.

- This is a regular price tâtonnement process.
- The *LM*-curve, corrsponding to $\dot{r}=0$, is upward-sloped on the (Y, r) plane[5].

3. 3　Fixed Point Characterization of the *IS*–*LM* Equilibrium

Theorem 3. 1 *Let Y_F be the full employment GDP. Denote by r_{lt} the "liquidity trap" interest rate, and by $\bar{r}$ the upper bound on the permissible values of interest rate r.*

Define the net effective demand function ("net" of T and M) D: $[0, Y_F] \times [r_{lt}, \bar{r}] \to [0, Y_F] \times [r_{lt}, \bar{r}]$ by $D(Y, r) = C(Y) + I(r) + G - T + X - M$. Define also the excess demand function for money l: $[0, Y_F] \times [r_{lt}, \bar{r}] \to [0, Y_F] \times [r_{lt}, \bar{r}]$ by $l(Y, r) = L(Y, r) - M_S$.

Then, there exists an IS–LM equilibrium (Y^*, r^*) such that

5) $r\uparrow$ ($\Leftrightarrow$ security price $\downarrow$) $\Rightarrow$ Traders anticipate no further drop of the security prices (*Bull Market* [Opposite: *Bear Market*]) $\Rightarrow$ Purchase of securities $\Leftrightarrow$ $L_2(r)\downarrow$ $\Rightarrow$ For $L=M$, $L_1(Y) = -L_2(r)\uparrow$ $\Rightarrow$ *GDP* $Y\uparrow$

(1) $Y^{*}+T+M=C(Y^{*})+I(r^{*})+G$

(2) $L(Y^{*}, r^{*})=M_S$.

Proof: Define the *interest rate tâtonnement process* $\lambda: [0, Y_F]\times[r_{lt}, \bar{r}]\to[0, Y_F]$ by

$$\lambda(Y, r)=\{r' \mid r' \text{ maximizes } r'. l(Y, r) \text{ over } [r_{lt}, \bar{r}]\}.$$

Define $\Lambda: [0, Y_F]\times[r_{lt}, \bar{r}]\to[0, Y_F]\times[r_{lt}, \bar{r}]$ by $\Lambda(Y, r)=D(Y, r)\cap\lambda(Y, r)$, i.e.,

$$\Lambda(Y, r)=\left\{(Y', r') \;\middle|\; \begin{array}{r} Y'=D(Y, r),\ r' \text{ maximizes } r'. l(Y, r) \\ \text{over } [r_{lt}, \bar{r}] \end{array}\right\}.$$

Then, Λ is convex-valued and upper hemicontinuous. By Kakutani's Fixed Point Theorem (Theorem 2.2), Λ has a fixed point $(Y^{*}, r^{*})\in\Lambda(Y^{*}, r^{*})$ such that $Y^{*}=D(Y^{*}, r^{*})$ and $r. l(Y^{*}, r^{*})\leq r^{*}. l(Y^{*}, r^{*})\leq 0$ for all $r\in[r_{lt}, \bar{r}]$. Therefore, (1) and (2) follow. ■

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