Projections onto Convex sets, Convex Functions and Their Subdifferentials

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Introduction

Let ϕ be a proper lower semicontinuous convex function on a (real) Hilbert space with the inner product $(\cdot\ ,\ \cdot)$ and norm $\|\cdot\|$, X be a closed convex subset of H and denote by T the projection from H onto X. Then we consider the property

(T) $\phi(u+T(v-u)) + \phi(v-T(v-u)) \leqq \phi(u) + \phi(v)$ for any $u, v \in H$,

and the following property of the subdifferential $\partial\phi$:

(i) $(u^*-v^*,\ u-v-T(u-v)) \geqq 0$ for any $[u, u^*], [v, v^*] \in G(\partial\phi)$,

where $G(\partial\phi)$ denotes the graph of $\partial\phi$.

The aim of the present paper is to show that (T) and (i) are equivalent. In fact, the assertion (T) → (i) follows directly from the definition of $\partial\phi$, but the converse will be proved via some properties which are equivalent to (i); for instance,

(ii) $\|J_\lambda u-J_\lambda v-T(J_\lambda u-J_\lambda v)\| \leqq \|u-v-T(u-v)\|$ for any $\lambda>0$ and $u, v \in H$;

(iii) $(\partial\phi_\lambda(u+Tv)-\partial\phi_\lambda(u),\ v-Tv) \geqq 0$ for any $\lambda>0$ and $u, v \in H$;

where $J_\lambda = (I+\lambda\partial\phi)^{-1}$ and ϕ_λ is the regularization of ϕ. Simultaneously we shall give further properties equivalent to (T) by means of ϕ_λ, $\partial\phi_\lambda$ and the contraction semigroup generated by $-\partial\phi$.

Especially in the space $L^2(\Omega)$, we are interested in the case

(a) $X=\{v\in L^2(\Omega);\ v \quad k \text{ a.e. on } \Omega\}$

or

(b) $X=\{v\in L^2(\Omega),\ 0\leqq v\leqq k \text{ a.e. on } \Omega\}$,

where k is a positive constant. The property (T) corresponding to the case (a) is written in the form

(c) $\phi(u\wedge(v+k)) + \phi(v\vee(u-k)) \leqq \phi(u)+\phi(v)$ for any $u, v \in L^2(\Omega)$.

This property was studied by Brézis [2; Chap. II] in which he showed that (c) implies

(d) $(u^*-v^*,\ (u-v-k)^+) \geqq 0$ for any $[u, u^*], [v, v^*] \in G(\partial\phi)$.

However the proof of (d) → (c) is not given there. The property (T) corresponding to the case (b) was investigated by Kenmochi-Mizuta [7, 8] in connection with (i), (ii) and (iii) in a special case of ϕ.

Finally we shall study convex functions satisfying (T) with the effective domains in the Sobolev space $W^{1,p}(\Omega)$ or $W^{1/p',p}(\Gamma)$ so as to apply our results to nonlinear elliptic boundary value problems and parabolic initial boundary value problems, where Ω is a domain in the Euclidean space with sufficiently smooth boundary Γ, $p \geqq 2$ and $1/p + 1/p' = 1$.

For other papers dealing with related topics, see Calvert [3, 4, 5] Konishi [9], Picard [10] and Sato [11].

§ 1. Preliminaries

In this paper we shall use some elementary facts in the nonlinear monotone operator theory without proof. For the definitions of maximal monotone operators, their resolvents and nonlinear contraction semigroups, etc., we refer to the textbook of Brézis [1].

In what follows, let H be a (real) Hilbert space with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let ϕ be a lower semicontinuous convex function on H and assume that ϕ is proper on H, i.e., $\phi(v) \in (-\infty, +\infty]$ for every $v \in H$ and $\phi \not\equiv +\infty$ on H. The set $D(\phi) = \{v \in H; \phi(v) < +\infty\}$ is called the effective domain of ϕ. The subdifferential $\partial\phi$ of ϕ is the (multivalued) operator in H defined by the following: $\partial\phi(v) = 0$ if $v \notin D(\phi)$ and

$$\partial\phi(v) = \{w \in H; (w, z-v) \leqq \phi(z) - \phi(v) \text{ for any } z \in H\}$$

if $v \in D(\phi)$. We define $D(\partial\phi) = \{v \in H; \partial\phi(v) \not\equiv 0\}$ and $G(\partial\phi) = \{[v, v^*]\gamma\ H^{\in} H; v \in D(\partial\phi), v^* \in \partial\phi(v)\}$. According to, for example, [1; Proposition 2.11], $D(\partial\phi) \subset D(\phi)$ and $\overline{D(\partial\phi)} = \overline{D(\phi)}$. Also, we define the operator $(\partial\phi)^\circ$ in H by

$$(\partial\phi)^\circ(v) = \{v^* \in \partial\phi(v); \|v^*\| = \inf \|w^*\|; w^* \in \partial\phi(v)\}$$

with $D((\partial\phi)^\circ) = \{v \in H; v \in D(\partial\phi), (\partial\phi)^\circ(v) \not\equiv \phi\}$. As is well-known (cf. [1; Proposition 2.7]), $(\partial\phi)^\circ$ is singlevalued and $D((\partial\phi)^\circ) = D(\partial\phi)$.

Now, consider the following type of regularization ϕ_λ of ϕ for each $\lambda > 0$:

$$\phi_\lambda(v) = \inf\{\|w-v\|^2/(2\lambda) + \phi(w); w \in H\}, v \in H.$$

Then we have

Lemma 1.1. (1) For each $\lambda > 0$, ϕ_λ is a convex function on H with $D(\phi_\lambda) = H$ which is everywhere differentiable on H in the sense of Gâteaux, and the Gâteaux-derivative of ϕ_λ coincides with the subdifferential $\partial\phi_\lambda$. Hence $D(\partial\phi_\lambda) = H$ and $\partial\phi_\lambda$ is singlevalued.

(2) $\partial\phi_\lambda = (I - J_\lambda)/\lambda$ for any $\lambda > 0$, where $J_\lambda = (I + \lambda\partial\phi)^{-1}$

with the identity I on H.

(3) $\partial\phi_\lambda(v) \to (\partial\phi)^\circ(v)$ in H as $\lambda \downarrow 0$ if $v \in D(\partial\phi)$.

(4) $\phi_\lambda(v) \uparrow \phi(v)$ as $\lambda \downarrow 0$ for any $v \in H$.

(5) For each $\lambda > 0$ and $u, v \in H$,

$$\phi_\lambda(v) - \phi_\lambda(u) = \int_0^1 (\partial\phi_\lambda(u+t(v-u)), v-u)\, dt.$$

For a proof of Lemma 1.1, for example, see [1; Propositions 2.6 and 2.11].

Next, let X be a closed convex subset of H and denote by T the projection from H onto X. Indeed, T is defined by

$$\|v-Tv\| = \inf\{\|v-w\|;\ w \in X\},\ v \in H.$$

We see that T is contractive on H, and have the following lemmas:

Lemma 1.2. The function ψ on H given by

$$\psi(v) = \frac{1}{2}\|v-Tv\|^2$$

is convex, everywhere differentiable on H in the sense of Gâteaux and $\partial\psi(v) = v-Tv$ for each $v \in H$.

Lemma 1.3. Let v be any element in H. Then

$$(Tv-w,\ v-Tv) \geqq 0 \quad \text{for any } w \in X.$$

Since the proofs of these lemmas are elementary, we omit them.

Lemma 1.4. Let z be an absolutely continuous function from $[0, r]$ into H. Then $t \to Tz(t)$ is also an absolutely continuous function from $[0, r]$ into H and

$$\left(\frac{dTz(t)}{dt},\ z(t)-Tz(t)\right) = 0 \quad \text{for a.e. } t \in [0, r]. \tag{1.1}$$

Proof. Since T is contractive on H, $t \to Tz(t)$ is absolutely continuous on $[0, r]$ and hence differentiable a.e. on $[0, r]$. Now, assume that it is differentiable at $t \in (0, r)$. Then we observe from Lemma 1.3 that

$$\lim_{h\downarrow 0} \frac{1}{h}(Tz(t)-Tz(t-h),\ z(t)-Tz(t)) \geqq 0$$

and

$$\lim_{h\downarrow 0} \frac{1}{h}(Tz(t+h)-Tz(t),\ z(t)-Tz(t)) \leqq 0.$$

Hence (1.1) holds.

§ 2. Property (T) and its characterization

Let ϕ be a proper lower semicontinuous convex function on H, X be a closed convex subset of H and denote by T the projection from H onto X. Then we say that T operates on H with respect to ϕ, if the following is satisfied:

(T) $\qquad \phi(u+T(v-u)) + \phi(v-T(v-u)) \leqq \phi(u) + \phi(v)$ for any $u, v \in H$.

We observe that if T operates on H with respect to ϕ, then $\overline{D(\phi)}$ has the property:

(D_T) $\qquad u+T(v-u) \in \overline{D(\phi)},\ v-T(v-u) \in \overline{D(\phi)}$ for any $u, v \in \overline{D(\phi)}$.

The following theorem gives a characterization of (T) by means of the subdifferential $\partial\phi$, the resolvent $J_\lambda = (I+\lambda\partial\phi)^{-1}$, the regularization ϕ_λ of ϕ, its subdifferential $\partial\phi_\lambda$ and the contraction semigroup $\{S(t);\, t \geqq 0\}$ on $\overline{D(\phi)}$ generated by $-\partial\phi$.

Theorem 2.1. The following statements are equivalent to each other:

(a1) T operates on H with respect to ϕ.

(a2) T operates on H with respect to ϕ_λ for every $\lambda > 0$.

(a3) $(u^*-v^*,\, u-v-T(u-v)) \geqq 0$ for any $[u, u^*], [v, v^*] \in G(\partial\phi)$.

(a4) $(\partial\phi_\lambda(u)-\partial\phi_\lambda(v),\, u-v-T(u-v)) \geqq 0$ for any $\lambda>0$ and $u, v \in H$.

(a5) $(\partial\phi_\lambda(u+Tv)-\partial\phi_\lambda(u),\, v-Tv) \geqq 0$ for any $\lambda>0$ and $u, v \in H$.

(a6) $\|J_\lambda u-J_\lambda v-T(J_\lambda u-J_\lambda v)\| \leqq \|u-v-T(u-v)\|$ for any $\lambda > 0$ and $u, v \in H$.

(a7) Condition (D_T) is satisfied and
$\|S(t)u-S(t)v-T(S(t)u-S(t)v)\| \leqq \|u-v-T(u-v)\|$ for any $t \geqq 0$ and $u, v \in \overline{D(\phi)}$.

Remark 2.1. The assertions (a3) ↔ (a4) ↔ (a6) are already known (cf. Brézis [1; Proposition 4.7] or Picard [10; Chap. II]). Also, as will be seen, the assertion (a6) ↔ (a7) is essentially included in [1; Proposition 4.7]. However, the assertions (a1) ↔ (a2) ↔ (a3) ↔ (a5) are new.

Remark 2.2. The inequality (a5) was investigated by Calvert [4] for some special kinds of T and for nonlinear monotone operators from Sobolev spaces into their dual spaces.

Proof of (a1) → (a3): Let $[u, u^*]$ and $[v, v^*]$ be in $G(\partial\phi)$. Then by the definition of $\partial\phi$ we have

$$(u^*, w-u) \leqq \phi(w) - \phi(u) \quad \text{for any } w \in H$$

and

$$(v^*, z-v) \leqq \phi(z) - \phi(v) \quad \text{for any } z \in H.$$

Taking $w = v + T(u-v)$ and $z = u-T(u-v)$, and adding the above two inequalities, we obtain

$$(u^*-v^*,\, u-v-T(u-v)) \geqq \phi(u)- \phi(v+T(u-v)) + \phi(v)- \phi(u-T(u-v)) \geqq 0.$$

Proof of (a3) → (a6): Let ψ be as in Lemma 1.2, $\lambda > 0$ and $u, v \in H$. Choose $[w, w^*]$ and $[z, z^*] \in G(\partial\phi)$ so that $u = w + \lambda w^*$ and $v = z + \lambda z^*$. Then, noting that $w = J_\lambda u$ and $z = J_\lambda v$, we have by (a3) and Lemma 1.2

$$\begin{aligned}\psi(u-v) - \psi(J_\lambda u-J_\lambda v) &= \psi(w-z+\lambda(w^*-z^*)) - \psi(w-z)\\ &\geqq \lambda(w^*-z^*,\, \partial\psi(w-z)) \geqq 0.\end{aligned}$$

Hence (a6) holds.

Proof of (a6) → (a5): Let $\lambda > 0$ and $u, v \in H$. By (a6) we see that $J_\lambda(u+Tv) - J_\lambda u \in X$, since $(u+Tv) - u = Tv \in X$. Hence it follows from Lemma 1.3 and (2) of Lemma 1.1 that

$$(\partial\phi_\lambda(u+Tv) - \partial\phi_\lambda(u),\, v-Tv) = \frac{1}{\lambda}\,(Tv-[J_\lambda(u+Tv) - J_\lambda u],\, v-Tv) \geqq 0.$$

Proof of (a5) → (a2): Let $\lambda>0$ and $u, v \in H$. Then, using (5) of Lemma 1.1 and (a5), we see that

$$\begin{aligned}&\phi_\lambda(v) - \phi_\lambda(u+T(v-u))\\ &= \int_0^1 (\partial\phi_\lambda(u+T(v-u)+t[v-u-T(v-u)]),\ v-u-T(v-u))dt\\ &\geqq \int_0^1 (\partial\phi_\lambda(u+t[v-u-T(v-u)]),\ v-u-T(v-u))dt\\ &= \phi_\lambda(v-T(v-u))-\phi_\lambda(u).\end{aligned}$$

The proof of (a2) $\to$ (a1) immediately follows from (4) of Lemma 1.1. Thus we have obtained the assertions (a1) $\leftrightarrow$ (a2) $\leftrightarrow$ (a3) $\leftrightarrow$ (a5) $\leftrightarrow$ (a6). The assertion (a2) $\leftrightarrow$ (a4) is nothing but the assertion (a1) $\leftrightarrow$ (a3) with ϕ replaced by ϕ_λ.

Next, in order to apply a result of Brézis [1; Proposition 4.7] we consider the indicator function K of $\overline{D(\phi)}$, i.e.,

$$K(v) = \begin{cases} 0 & \text{if } v \in \overline{D(\phi)},\\ +\infty & \text{otherwise.}\end{cases}$$

Clearly, K is a proper lower semicontinuous convex function on H with $D(K) = D(\partial K) = \overline{D(\phi)}$. For any $\lambda > 0$, the resolvent $(I+\lambda\partial K)^{-1}$ of ∂K coincides with the projection P from H onto $\overline{D(\phi)}$ (cf. [1; Example 2.8.2]).

Lemma 2.1. Condition (D_T) holds if and only if

$$\|Pu-Pv-T(Pu-Pv)\| \leqq \|u-v-T(u-v)\| \quad \text{for any } u, v \in H.$$

Proof. It is clear that (D_T) is equivalent to the following:

$$K(u+T(v-u)) + K(v-T(v-u)) \leqq K(u) + K(v) \quad \text{for any } u, v \in H,$$

so that we have the lemma by applying (a1) $\leftrightarrow$ (a6) to the function K.

On account of [1; Proposition 4.7] and Lemma 2.1, (a7) is equivalent to (a6) and hence the proof of Theorem 2.1 is complete.

Remark 2.3. In (a7) of Theorem 2.1, condition (D_T) can not be dropped; in fact we have the following example.

Example 2.1. Let us take $L^2(R^1)$ as H, $\{v \in L^2(R^1);\ 0 \leqq v \leqq 1$ a.e. on $R^1\}$ as X and the indicator function of the unit ball $U = \{v \in L^2(R^1);\ \|v\| \leqq 1\}$ as ϕ. Then it is easy to see that $S(t)v = v$ for every $t \geqq 0$ and $v \in U$, and hence for $t \geqq 0$ and $u, v \in U$

$$\|S(t)u - S(t)v - T(S(t)u-S(t)v)\| \leqq \|u-v-T(u-v)\|$$

holds with the projection T from $L^2(R^1)$ onto X. But, as is easily checked, (D_T) is not satisfied.

§3. Further properties equivalent to (T)

1. In addition to the properties in §2 equivalent to (T) we investigate some other properties which are apparently weaker than, but really equivalent to them.

Proposition 3.1. Each of the following is equivalent to any one of the statements listed in Theorem 2.1:

(a8) (D_T) is satisfied and

(3.1) $(\partial\phi_\lambda(u) - \partial\phi_\lambda(v),\ u-v-T(u-v)) \geqq 0$ for any $\lambda > 0$ and $u, v \in \overline{D(\phi)}$.

(a9) (D_T) is satisfied and

(3.2) $((\partial\phi)^\circ(u)-(\partial\phi)^\circ(v),\ u-v-T(u-v)) \geqq 0$ for any $u, v \in D(\partial\phi)$.

(a10) (D_T) is satisfied and
$\|J_\lambda u-J_\lambda v-T(J_\lambda u-J_\lambda v)\| \leqq \|u-v-T(u-v)\|$ for any $\lambda > 0$ and $u, v \in \overline{D(\phi)}$.

(a11) (D_T) is satisfied and
$(\partial\phi_\lambda(u+Tv) - \partial\phi_\lambda(u),\ v-Tv) \geqq 0$ for any $\lambda > 0$, $u \in \overline{D(\phi)}$ and $v \in H$ with $u+Tv \in \overline{D(\phi)}$.

(a12) $\phi_\lambda(u + T(v-u)) + \phi_\lambda(v-T(v-u)) \leqq \phi_\lambda(u) + \phi_\lambda(v)$ for any $\lambda > 0$ and $u, v \in \overline{D(\phi)}$.

Proof. Since (D_T) follows from (a4), the assertion (a4) → (a8) is true. Using (3) of Lemma 1.1 and letting $\lambda \downarrow 0$ in (3.1), we have (3.2), so that (a8) implies (a9). Again, according to [1; Proposition 4.7], (a9) is equivalent to (a6) and hence (a9) implies (a10). Also, we obtain (a10) → (a11) and (a12) → (a1) just as (a6) → (a5) and (a2) → (a1), respectively. Taking into account the fact that $u+t[v-u-T(v-u)] \in \overline{D(\phi)}$ and $u + T(v-u) + t[v-u-T(v-u)] \in \overline{D(\phi)}$ for every $t \in [0, 1]$ and $u, v \in \overline{D(\phi)}$ if (D_T) is satisfied, we have (a11) → (a12) in the same way as the proof of (a5) → (a2).

Proposition 3.2. Each of the statements (a1) ~ (a12) is equivalent to any one of the following:

(a13) $J_\lambda v-J_\lambda u \in X$ for any $\lambda > 0$ and $u, v \in H$ with $v-u \in X$.

(a14) (D_T) is satisfied and
$J_\lambda v-J_\lambda u \in X$ for any $\lambda > 0$ and $u, v \in \overline{D(\phi)}$ with $v-u \in X$.

(a15) (D_T) is satisfied and
$S(t)v-S(t)u \in X$ for any $t \geqq 0$ and $u, v \in \overline{D(\phi)}$ with $v-u \in X$.

Proof. In fact, we have (a13) ↔ (a6), (a14) ↔ (a10) and (a15) ↔ (a7). We give only a proof of (a15) ↔ (a7), since the others are similarly proved.

The direction (a7) → (a15) is trivial. Now, assume (a15) and let $u, v \in \overline{D(\phi)}$. Then, $S(t)(v+T(u-v))-S(t)v \in X$, since $[v+T(u-v)]-v = T(u-v) \in X$. Hence

$$\begin{aligned}
&\|S(t)u-S(t)v-T(S(t)u-S(t)v)\| \\
&\quad = \inf\{\|S(t)u - S(t)v-z\|;\ z \in X\} \\
&\quad \leqq \|S(t)u - S(t)(v + T(u-v))\| \\
&\quad \leqq \|u-v-T(u-v)\|.
\end{aligned}$$

2. Next, let us consider the evolution equation on $[0, r]$:

$$\text{(E)} \qquad \frac{du(t)}{dt} + \partial\phi^t(u(t)) \ni f(t),$$

where ϕ^t is a proper lower semicontinuous convex function on H for each $t \in [0, r]$ and $f \in L^1(0, r; H)$. By a strong solution u of (E) we mean an H-valued absolutely continuous function on $[0, r]$ such that $u(t) \in D(\partial\phi^t)$ for a.e. $t \in [0, r]$ and there is a function $u^* \in L^1(0, r; H)$ satisfying $u^*(t) \in \partial\phi^t(u(t))$ and $(d/dt)u(t) + u^*(t) = f(t)$ for a.e. $t \in [0, r]$. By a weak solution u of (E) we mean the limit of a sequence $\{u_n\}$ of strong solutions of (E) which converges to u in H uniformly on $[0, r]$ as $n \to +\infty$.

Proposition 3.3. Let X be a closed convex subset of H and T be the projection from H onto X. Assume that T operates on H with respect to ϕ^t for each $t \in [0, r]$. If u_1 and u_2 are weak solutions of (E), then

$$\|u_1(t)-u_2(t)-T(u_1(t)-u_2(t))\| \leqq \|u_1(s)-u_2(s)-T(u_1(s)-u_2(s))\| \tag{3.3}$$

holds for every $s, t \in [0, r]$ with $s \leqq t$.

Proof. It suffices to show (3.3) in case u_1 and u_2 are strong solutions of (E). In this case, there are $u_i^* \in L^1(0, r; H)$, $i = 1, 2$, such that $u_i^*(\tau) \in \partial\phi^\tau(u_i(\tau))$ and $(d/d\tau)u_i(\tau) + u_i^*(\tau) = f(\tau)$ for a.e. $\tau \in [0, r]$. By Theorem 2.1 and Lemma 1.4 we have

$$\frac{1}{2}\frac{d}{d\tau}\|u_1(\tau)-u_2(\tau)-T(u_1(\tau)-u_2(\tau))\|^2$$
$$= \left(\frac{du_1(\tau)}{d\tau} - \frac{du_2(\tau)}{d\tau} - \frac{d}{d\tau}T(u_1(\tau)-u_2(\tau)),\ u_1(\tau)-u_2(\tau)-T(u_1(\tau)-u_2(\tau))\right)$$
$$= -(u_1^*(\tau) - u_2^*(\tau),\ u_1(\tau)-u_2(\tau) - T(u_1(\tau) - u_2(\tau))) \leqq 0$$

for a.e. $\tau \in [0, r]$. Integrating the both sides over $[s, t] \subset [0, r]$, we have (3.3).

Remark 3.1. In case $H = L^2(\Omega)$, $\phi^t = \phi$ and $X = \{v \in L^2(\Omega);\ v \leqq k$ a.e. on $\Omega\}$ with a positive constant k, the above result is included in Brézis [2; Proposition II-7].

§ 4. Order theoretic properties in L^2-spaces

Let Ω be a locally compact Hausdorff space, ξ be a positive measure on Ω and denote by $L^2 = L^2(\Omega;\xi)$ the Hilbert space of all (real-valued) square ξ-integrable functions on Ω with the inner product

$$(v, w) = \int_\Omega vw\,d\xi$$

and the norm $\|v\| = \sqrt{(v, v)}$. For measurable functions v, w on Ω we define $v \vee w = max\{v, w\}$, $v \wedge w = min\{v, w\}$, $v^+ = v \vee 0$ and $v^- = -(v \wedge 0)$. Let ϕ be a proper lower semi-continuous convex function on L^2 and g be a non-negative measurable function on Ω.

1. Consider the set

$$X_g = \{v \in L^2;\ v \leqq g \text{ a.e. on } \Omega\}.$$

Then, clearly, X_g is closed and convex in L^2 and the projection T_g from L^2 onto X_g is given by $T_g v = v \wedge g$. In this case, a part of Theorem 2.1 and Proposition 3.2 can be rewritten in the following:

Theorem 4.1. The following statements are equivalent to each other:

(b1) T_g operates on L^2 with respect to ϕ.
(b2) $(u^*-v^*, (u-v-g)^+) \geqq 0$ for any $[u, u^*], [v, v^*] \in G(\partial\phi)$.

(b3) $(\partial\phi_\lambda(u+v\wedge g)-\partial\phi_\lambda(u),(v-g)^+)\geqq 0$ for any $\lambda>0$ and $u,v\in L^2$.

(b4) $\|(J_\lambda u-J_\lambda v-g)^+\|\leqq\|(u-v-g)^+\|$ for any $\lambda>0$ and $u,v\in L^2$.

(b5) $J_\lambda u\leqq J_\lambda v+g$ a.e. on Ω for any $\lambda>0$ and $u,v\in L^2$ with $u\leqq v+g$ a.e. on Ω.

(b6) Condition

$$(D_g)\quad u\wedge(v+g)\in\overline{D(\phi)},\ v\vee(u-g)\in\overline{D(\phi)}\ \text{for any}\ u,v\in\overline{D(\phi)}$$

is satisfied and

$$\|(S(t)u-S(t)v-g)^+\|\leqq\|(u-v-g)^+\|\ \text{for any}\ t\geqq 0\ \text{and}\ u,v\in\overline{D(\phi)}.$$

In fact, noting that $v-T_gv=(v-g)^+$, $u\wedge(v+g)=v+T_g(u-v)$ and $v\vee(u-g)=u-T_g(u-v)$, we see that (b1) ~ (b6) are (a1), (a3), (a5), (a6) (a13) and (a7) with T replaced by T_g, respectively.

2. Next, consider the set

$$X_g^+=\{v\in L^2;\ 0\leqq v\leqq g\ \text{a.e. on}\ \Omega\}.$$

Then this is closed and convex in L^2 and the projection T_g^+ from L^2 onto X_g^+ is given by $T_g^+v=v^+\wedge g$. In this case we have the following theorem as direct consequence of Theorem 2.1 and Proposition 3.2.

Theorem 4.2. The following are equivalent:

(c1) T_g^+ operates on L^2 with respect to ϕ.

(c2) $(u^*-v^*,u-v-(u-v)^+\wedge g)\geqq 0$ for any $[u,u^*],[v,v^*]\in G(\partial\phi)$.

(c3) $(\partial\phi_\lambda(u+v^+\wedge g)-\partial\phi_\lambda(u),v-v^+\wedge g)\geqq 0$ for any $\lambda>0$ and $u,v\in L^2$.

(c4) $\|J_\lambda u-J_\lambda v-(J_\lambda u-J_\lambda v)^+\wedge g\|\leqq\|u-v-(u-v)^+\wedge g\|$ for any $\lambda>0$ and $u,v\in L^2$.

(c5) $J_\lambda u\leqq J_\lambda v\leqq J_\lambda u+g$ a.e. on Ω for any $\lambda>0$ and $u,v\in L^2$ with $u\leqq v\leqq u+g$ a.e. on Ω.

(c6) Condition

$$(D_g^+)\quad u+(v-u)^+\wedge g\in\overline{D(\phi)},\ v-(v-u)^+\wedge g\in\overline{D(\phi)}\ \text{for any}\ u,v\in\overline{D(\phi)}$$

is satisfied and

$$\|S(t)u-S(t)v-(S(t)u-S(t)v)^+\wedge g\|\leqq\|u-v-(u-v)^+\wedge g\|\ \text{for any}\ t\geqq 0\ \text{and}\ u,v\in\overline{D(\phi)}.$$

3. Let $X^+=\{v\in L^2;\ v\geqq 0$ a.e. on $\Omega\}$ with the projection T^+ from L^2 onto X^+. Clearly, $T^+v=v^+$ for each $v\in L^2$.

Proposition 4.1. Assume that T^+ operates on L^2 with respect to ϕ. Then T_g^+ operates on L^2 with respect to ϕ if and only if T_g operates on L^2 with respect to ϕ.

Proof. Let $\lambda>0$. Then, by Proposition 3.2 with $T=T^+$,

(4.1) $J_\lambda w\leqq J_\lambda z$ a.e. on Ω for any $w,z\in L^2$ with $w\leqq z$ a.e. on Ω.

First assume that T_g^+ operates on L^2 with respect to ϕ. Let $u,v\in L^2$ with $u\leqq v+g$ a.e. on Ω. Then, noting that $v\leqq u\vee v\leqq v+g$ a.e. on Ω, we have by Theorem 4.2

$$J_\lambda v\leqq J_\lambda(u\vee v)\leqq J_\lambda v+g\ \text{a.e. on}\ \Omega.$$

Since $J_\lambda u \leqq J_\lambda (u \vee v)$ a.e. on Ω by (4.1), it follows that $J_\lambda u \leqq J_\lambda v + g$ a.e. on Ω. Thus (b5) holds and hence T_g operates on L^2 with respect to ϕ.

The converse is similarly proved by using Theorem 4.1.

§5. Examples

In this section, let Ω be a bounded domain in the m-dimensional Euclidean space R^m $(m \geqq 2)$ with sufficiently smooth boundary Γ, $2 \leqq p < +\infty$, $1/p + 1/p' = 1$ and denote by $\|\cdot\|_{1,p}$ and $\|\cdot\|_{1/p',p}$ the norms in $W^{1,p}(\Omega)$ and $W^{1/p',p}(\Gamma)$, respectively.

Let ϕ be a proper lower semicontinuous convex function on $L^2(\Omega)$ such that $D(\phi) \subset W^{1,p}(\Omega)$ and

(5.1) $\quad \phi(v) \geqq c_0 \|v\|_{1,p} - c_1$ for any $v \in D(\phi)$,

where c_0 and c_1 are positive constants. Now, define a function $\hat{\phi}$ on $L^2(\Gamma)$, associated with ϕ, by the following:

$$\hat{\phi}(h) = \begin{cases} inf\{\phi(v);\, v|_\Gamma = h,\, v \in D(\phi)\} & if\ h \in W^{1/p',p}(\Gamma), \\ +\infty & \text{otherwise,} \end{cases}$$

where $v|_\Gamma$ means the boundary values (the trace) of $W^{1,p}(\Omega)$. Then we have

Lemma 5.1. $\hat{\phi}$ is a proper lower semicontinuous convex function on $L^2(\Gamma)$ such that $D(\hat{\phi}) \subset W^{1/p',p}(\Gamma)$ and

(5.2) $\quad \hat{\phi}(h) \geqq c_2 \|h\|_{1/p',p} - c_3$ for any $h \in D(\hat{\phi})$,

where c_2 and c_3 are positive constants.

Proof. We note that $v \to v|_\Gamma$ is a linear continuous mapping from $W^{1,p}(\Omega)$ onto $W^{1/p',p}(\Gamma)$, so that there is a positive constant L such that $\|v|_\Gamma\|_{1/p',p} \leqq L\|v\|_{1,p}$ for all $v \in W^{1,p}(\Omega)$. By the definition of $\hat{\phi}$ and (5.1) we see that $\hat{\phi} \not\equiv +\infty$ on $L^2(\Gamma)$, $D(\hat{\phi}) \subset W^{1/p',p}(\Gamma)$ and

$$\hat{\phi}(h) \geqq c_0 \ inf\{\|v\|_{1,p};\, v \in W^{1,p}(\Omega),\, v|_\Gamma = h\} - c_1 \geqq (c_0/L)\|h\|_{1/p',p} - c_1$$

for any $h \in W^{1/p',p}(\Gamma)$. Hence $\hat{\phi}$ is proper on $L^2(\Gamma)$ and (5.2) holds.

We now show the convexity of $\hat{\phi}$. Let $h_1, h_2 \in D(\hat{\phi})$ and $0 \leqq t \leqq 1$. Then, for any $\epsilon > 0$ there are functions u_1 and u_2 in $D(\phi)$ such that $u_1|_\Gamma = h_1$, $u_2|_\Gamma = h_2$, $\hat{\phi}(h_1) \geqq \phi(u_1) - \epsilon$ and $\hat{\phi}(h_2) \geqq \phi(u_2) - \epsilon$.
Therefore

$$\begin{aligned} t\hat{\phi}(h_1) + (1-t)\hat{\phi}(h_2) &\geqq t\phi(u_1) + (1-t)\phi(u_2) - \epsilon \\ &\geqq \phi(tu_1 + (1-t)u_2) - \epsilon \\ &\geqq \hat{\phi}(th_1 + (1-t)h_2) - \epsilon \end{aligned}$$

since $[tu_1 + (1-t)u_2]|_\Gamma = th_1 + (1-t)h_2$. This implies that $\hat{\phi}(th_1 + (1-t)h_2) \leqq t\hat{\phi}(h_1) + (1-t)\hat{\phi}(h_2)$. Thus $\hat{\phi}$ is convex.

Finally we show the lower semicontinuity of $\hat{\phi}$. Let $\{h_n\} \subset D(\hat{\phi})$ be a sequence such that $h_n \to h$ in $L^2(\Gamma)$ as $n \to +\infty$ and $\liminf_{n\to+\infty} \hat{\phi}(h_n) < +\infty$. Then, given $\epsilon > 0$, we find a subsequence $\{h_{n_k}\}$ of $\{h_n\}$, a sequence $\{v_k\} \subset D(\phi)$ and $v \in W^{1,p}(\Omega)$ such that $v_k|_\Gamma = h_{n_k}$, $\hat{\phi}(h_{n_k})$

$\geq \phi(v_k) - \epsilon$, $v_k \to v$ weakly in $W^{1,p}(\Omega)$ and $\liminf_{n\to+\infty}\hat{\phi}(h_n) = \lim_{k\to+\infty}\hat{\phi}(h_{n_k})$. Since $v_{|\Gamma} = h$ and $\liminf_{k\to+\infty}\phi(v_k) \geqq \phi(v)$ by the lower semicontinuity of ϕ, we get

$$\liminf_{n\to+\infty} \hat{\phi}(h_n) \geqq \phi(v) - \epsilon \geqq \hat{\phi}(h) - \epsilon.$$

Thus $\hat{\phi}$ is lower semicontinuous on $L^2(\Gamma)$.

Theorem 5.1. Let g be a non-negative function in $W^{1,p}(\Omega)$ and denote $g_{|\Gamma}$ by $\hat{g}$. If T_g^+ (resp. T_g) operates on $L^2(\Omega)$ with respect to ϕ, then $T_{\hat{g}}^+$ (resp. $T_{\hat{g}}$) operates on $L^2(\Gamma)$ with respect to $\hat{\phi}$.

Proof. Let $h_1, h_2 \in D(\hat{\phi})$. Given $\epsilon > 0$, choose u_1 and u_2 in $D(\phi)$ so that $u_{1|\Gamma} = h_1$, $u_{2|\Gamma} = h_2$, $\hat{\phi}(h_1) \geqq \phi(u_1) - \epsilon$ and $\hat{\phi}(h_2) \geqq \phi(u_2) - \epsilon$. Then, noting that $[u_1 + T_g^+(u_2 - u_1)]_{|\Gamma} = h_1 + T_{\hat{g}}^+(h_2 - h_1)$ and $[u_2 - T_g^+(u_2 - u_1)]_{|\Gamma} = h_2 - T_{\hat{g}}^+(h_2 - h_1)$, we have

$$\begin{aligned}\hat{\phi}(h_1) + \hat{\phi}(h_2) &\geqq \phi(u_1) + \phi(u_2) - 2\epsilon \\ &\geqq \phi(u_1 + T_g^+(u_2 - u_1)) + \phi(u_2 - T_g^+(u_2 - u_1)) - 2\epsilon \\ &\geqq \hat{\phi}(h_1 + T_{\hat{g}}^+(h_2 - h_1)) + \hat{\phi}(h_2 - T_{\hat{g}}^+(h_2 - h_1)) - 2\epsilon.\end{aligned}$$

Hence $T_{\hat{g}}^+$ operates on $L^2(\Gamma)$ with respect to $\hat{\phi}$. The assertion for $T_{\hat{g}}$ can be quite similarly proved.

Example 5.1 (cf. [7; §6]). Let a_i $(i = 0, 1, \dots, m)$ be bounded measurable functions on Ω such that $a_i \geqq c$ a.e. on Ω for some positive constant c and consider the function ϕ on $L^2(\Omega)$ given by

$$(5.3)\qquad \phi(v) = \begin{cases} \dfrac{1}{p}\displaystyle\sum_{i=1}^{m}\int_\Omega a_i \left|\frac{\partial v}{\partial x_i}\right|^p dx + \frac{1}{p}\int_\Omega a_0 |v|^p dx & \text{if } v \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that ϕ is a proper lower semicontinuous convex function on $L^2(\Omega)$ satisfying (5.1), $D(\phi) = W^{1,p}(\Omega)$ and for every non-negative constant function g on Ω, T_g^+ and T_g operate on $L^2(\Omega)$ with respect to ϕ. In this case, the function $\hat{\phi}$ on $L^2(\Gamma)$ associated with ϕ is given by the following:

$$\hat{\phi}(h) = \begin{cases} \dfrac{1}{p}\displaystyle\sum_{i=1}^{m}\int_\Omega a_i \left|\frac{\partial u^h}{\partial x_i}\right|^p dx + \frac{1}{p}\int_\Omega a_0 |u^h|^p dx & \text{if } h \in W^{1/p',p}(\Gamma), \\ +\infty & \text{otherwise,} \end{cases}$$

where u^h is a unique function in $W^{1,p}(\Omega)$ such that $u^h{}_{|\Gamma} = h$ and

$$-\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(a_i \left|\frac{\partial u^h}{\partial x_i}\right|^{p-2} \frac{\partial u^h}{\partial x_i}\right) + a_0 |u^h|^{p-2} u^h = 0 \quad \text{on } \Omega.$$

Besides, $\partial\hat{\phi}$ is singlevalued and

$$(5.4)\qquad \partial\hat{\phi}(h) = \sum_{i=1}^{m} a_i \left|\frac{\partial u^h}{\partial x_i}\right|^{p-2} \frac{\partial u^h}{\partial x_i} \nu_i \quad \text{on } \Gamma$$

for every $h \in D(\partial\hat{\phi}) \subset W^{1/p',p}(\Gamma)$, where $(\nu_1(x), \nu_2(x),..., \nu_m(x))$ is the unit outward normal to Γ at $x \in \Gamma$ and the right hand side of (5.4) is taken in the sense of [6; §1]. Therefore, for $f \in L^2(\Gamma)$ and $h \in D(\partial\hat{\phi})$, there is $u \in W^{1,p}(\Omega)$ such that

$$-\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(a_i \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) + a_0 |u|^{p-2} u = 0 \quad \text{on } \Omega,$$

$$u_{|\Gamma} = h,$$

$$\sum_{i=1}^{m} a_i \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} \nu_i = f \quad \text{on } \Gamma$$

if and only if $f = \partial\hat{\phi}(h)$. By Theorem 5.1, for every non-negative constant function $\hat{g}$ on Γ, $T_{\hat{g}}^{+}$ and $T_{\hat{g}}$ operate on $L^2(\Gamma)$ with respect to $\hat{\phi}$.

Example 4.2 (cf. [10; Chap. 2]). Let $a_i(t, x)$ $(i = 0, 1,..., m)$ be bounded measurable functions on $[0, r] \times \Omega$ such that $a_i \geq c$ a.e. on $[0, r] \times \Omega$ for some positive constant c. Given $h_0 \in W^{1/p',p}(\Gamma)$, consider the problem to find a function $u = u(t, x)$ on $[0, r] \times \Omega$ such that $u \in L^p(0, r; W^{1,p}(\Omega))$, $u_{|\Gamma} \in C([0, r]; L^2(\Gamma))$, $u(0, \cdot)_{|\Gamma} = h_0$ and

$$-\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(a_i \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) + a_0 |u|^{p-2} u = 0 \quad \text{on } (0, r) \times \Omega$$

is satisfied with

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{m} a_i \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i} \nu_i = 0 \quad \text{on } (0, r) \times \Gamma$$

in a certain weak sense. Denoting by ϕ^t the function on $L^2(\Omega)$ given by (5.3) with $a_i = a_i(t, x)$, and by $\hat{\phi}^t$ the function on $L^2(\Gamma)$ associated with ϕ^t for each $t \in [0, r]$, the above problem can be transformed into the Cauchy problem to find $h \in L^p(0, T; W^{1/p',p}(\Gamma)) \cap C([0, r]; L^2(\Gamma))$ that is a strong or weak solution of

$$\frac{dh(t)}{dt} + \partial\hat{\phi}^t(h(t)) = 0$$

with $h(0) = h_0$. Proposition 3.3 can be applied to this problem.

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