Solvability of Nonlinear Evolution Equations with Time-Dependent Constraints and Applications

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Introduction

A class of nonlinear parabolic partial differential equations in physics and mechanics can be interpreted as nonlinear evolution equations in Hilbert spaces of the form

\[
(E) \quad \frac{du(t)}{dt} + \partial \phi^t(u(t)) \ni f(t), \quad 0 < t < T,
\]

involving the subdifferential operators $\partial \phi^t$ of convex functions $\phi^t$. The purpose of the present paper is to study equation $(E)$ from various view-points.

Throughout this paper we take an interest in the $t$-dependence of the mapping $t \mapsto \phi^t$, which allows the effective domain $D(\phi^t)$ of $\phi^t$ to vary smoothly (but $D(\phi^t) \cap D(\phi^s) = \phi$ may happen if $t \neq s$), and guarantees the solution of $(E)$ in a certain sense. The $t$-dependence of $t \mapsto \phi^t$ imposed in this paper is a modified version of that introduced by Attouch-Bénilan-Damlamian-Picard [1], which is given by means of the regularizations $\phi^t_\lambda$ of $\phi^t$ (see §1.1).

In Chapter 1 the Cauchy problem $CP(\phi^t; f, u_0)$ for $(E)$ with initial condition $u(0) = u_0$ is formulated and the question of existence and uniqueness is considered under the above-mentioned $t$-dependence of $t \mapsto \phi^t$. The main results of this chapter were already announced in Kenmochi [3] without detail proof; the complete proofs are here given. When one applies the abstract results obtained to a class of nonlinear partial differential equations, it is not easy to verify our condition of the $t$-dependence on $t \mapsto \phi^t$, because it is given by the regularizations $\phi^t_\lambda$. So at the end of this chapter we give a sufficient condition, which is given only by $\phi^t$, for the required $t$-dependence.

In Chapter 2, other aspects of equation $(E)$ are considered; such as regularity and stability of solutions, characterization of solutions in terms of variational inequalities, existence of periodic solutions, maximum and minimum periodic solutions, and convergence of solutions. In the final section of this chapter we consider

\[
(E') \quad \frac{du(t)}{dt} + \partial \phi^t(Bu(t)) \ni f(t), \quad 0 < t < T,
\]

where $B$ is a given nonlinear operator; $(E')$ is a modified equation of $(E)$ and is solved just as $(E)$.

In Chapter 3 we give some applications of abstract results obtained in Chapters 1 and 2 to a class of nonlinear parabolic partial differential equations with time-dependent constraints; for example, quasi-linear or semi-linear heat equations with obstacles, equations of the Stokes-type in non-cylindrical domains and free boundary problems of the Stefan-type for nonlinear parabolic equations.
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Chapter 0

Preliminaries

§0.1. Notations

Let $S$ be a subset of a topological vector space $X$ (over the reals $\mathbb{R}$). We then denote by $\overline{S}$ (resp. $\text{int} \ S$) the closure (resp. interior) of $S$ and by the symbol "" resp. "" the strong (resp. weak) convergence in $X$.

Given two topological vector spaces $X$ and $Y$, we mean by $X \subseteq Y$ that $X$ is a dense subspace of $Y$ with continuous natural injection from $X$ into $Y$. For a Banach space $X$ we denote by $\| \cdot \|_X$ the norm in $X$, by $X^*$ the dual space equipped with the dual norm and by $(\cdot, \cdot)_X : X^* \times X \to \mathbb{R}$ the natural duality pairing between $X^*$ and $X$. In particular, if $X$ is a Hilbert space and if $X$ is identified with $X^*$, then we mean by $(\cdot, \cdot)_X$ the inner product in $X$.

Let $X$ and $Y$ be two Banach spaces. Then, in general, by a (multivalued) operator $A$: 
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$X \to Y$ we mean that $A$ is a mapping from $X$ into $2^Y$; the domain, range and graph of $A$ are respectively the sets

$$D(A) = \{ z \in X; Az \neq \phi \},$$
$$R(A) = \bigcup_{z \in X} Az$$

and

$$G(A) = \{ [z, y] \in X \times Y; z \in D(A), y \in Az \}.$$

If the set $Az$ consists of at most one element for every $z \in X$, then $A$ is called single-valued. Associated with an operator $A: X \to Y$, the operator $A^o: X \to Y$ defined by

$$A^o z = \begin{cases} \{ y \in Az; |y|_Y = \min_{v \in Az} |v|_Y \} & \text{if } z \in D(A), \\ \phi & \text{otherwise} \end{cases}$$

is called the principal section of $A$; note that $G(A^o) \subset G(A)$. The inverse $A^{-1}: Y \to X$ of $A$ is the operator whose graph is given by

$$G(A^{-1}) = \{ [y, z] \in Y \times X; [z, y] \in G(A) \} ;$$

note that we can always define the inverse $A^{-1}$ as a multivalued operator. Also, if $\bigcup_{z \in S} Az$ is a bounded set in $Y$ for each bounded set $S$ in $X$, then $A$ is called bounded.

Let $X$ be a Banach space, $1 \leq p \leq \infty$ and $-\infty \leq T < T' \leq \infty$. Then $L^p(T, T'; X)$ stands for the Banach space of all (strongly) measurable $X$-valued functions $f$ on $(T, T')$ whose norm is given by

$$|f|_{L^p(T, T'; X)} = \begin{cases} \left\{ \int_T^{T'} |f(t)|_X^p \, dt \right\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \text{ess.sup}_{T \leq t \leq T'} |f(t)|_X & \text{if } p = \infty. \end{cases}$$

Furthermore by $W^{m, p}(T, T'; X)$, $m$ being a non-negative integer and $1 \leq p < \infty$, we denote the Banach space of Sobolev's type:

$$W^{m, p}(T, T'; X) = \{ f \in L^p(T, T'; X); f^{(j)} \in L^p(T, T'; X), j = 1, 2, \ldots, m \}$$

with norm

$$|f|_{W^{m, p}(T, T'; X)} = \left\{ |f|_{L^p(T, T'; X)}^p + \sum_{j=1}^m |f^{(j)}|_{L^p(T, T'; X)}^p \right\}^{1/p},$$

where $f^{(j)} = (d^j/dt^j)f$. Also, in case $T$ and $T'$ are finite, $C([T, T']; X)$ stands for the Banach space of all $X$-valued continuous functions on $[T, T']$ with the sup-norm. Usually, in case $X = R$, we write $L^p(T, T')$, $W^{m, p}(T, T')$ and $C(T, T')$ for $L^p(T, T'; X)$, $W^{m, p}(T, T'; X)$ and $C([T, T']; X)$, respectively.
§0.2. Monotone operators

In this section let $X$ be a reflexive Banach space. An operator $A: X \to X^*$ is called monotone if

$$\langle z_1^* - z_2^*, z_1 - z_2 \rangle_X \geq 0, \quad \forall \{z_i, z_i^*\} \in G(A), \ i = 1, 2.$$  

Especially it is called strictly monotone if

$$\langle z_1^* - z_2^*, z_1 - z_2 \rangle_X > 0, \quad \forall \{z_i, z_i^*\} \in G(A), \ i = 1, 2, z_1 \neq z_2.$$  

Also, a monotone operator $A: X \to X^*$ is called maximal if there is no monotone operator $A': X \to X^*$ such that $G(A)$ is a proper subset of $G(A')$.

**Proposition 0.2.1.** Let $A: X \to X^*$ be a monotone operator. Then we have:

(a) (cf. Rockafellar [2, Theorem 1]) $A$ is locally bounded in $\text{Int.} \ D(A)$, i.e. for each $z \in \text{Int.} \ D(A)$ there is a neighborhood $U$ of $z$ such that $U \cap A z$ is bounded in $X^*$.

(b) (cf. Browder [1, Chapter III]) If $A$ is maximal, then $A z$ is closed and convex in $X^*$ for every $z \in D(A)$ and $A$ has the following property: if $\{z_n, z_n^*\} \in G(A)$ $(n = 1, 2, \ldots)$, $z_n \to z$ in $X$, $z_n^* \to z^*$ in $X^*$ and $(z_n^*, z_n)_X \to (z^*, z)_X$, then $\{z, z^*\} \in G(A)$; hence $A$ is demiclosed (i.e. if $\{z_n, z_n^*\} \in G(A)$, $z_n \to z$ in $X$ and $z_n^* \to z^*$ in $X^*$, then $\{z, z^*\} \in G(A)$).

Let $\mu$ be a real-valued continuous strictly increasing function on $[0, \infty)$ such that $\mu(0) = 0$ and $\mu(r) \to \infty$ as $r \to \infty$. We then define an operator $F_\mu: X \to X^*$ by

$$F_\mu(z) = \{z^* \in X^*: |z^*|_{X^*} = \mu(|z|_X), (z^*, z)_X = |z|_X \mu(|z|_X)\}$$

for each $z \in X$, which is called the duality mapping from $X$ into $X^*$ associated with gauge function $\mu$. It is well-known (cf. Browder [1, Chapter I]) that $F_\mu$ is a maximal monotone operator from $X$ into $X^*$ with $D(F_\mu) = X$. Moreover we have

**Proposition 0.2.2.** (cf. Rockafellar [3, Proposition 1]) Let $A: X \to X^*$ be a monotone operator and $\mu$ be a gauge function. Then $R(F_\mu + A) = X^*$ if and only if $A$ is maximal.

**Proposition 0.2.3.** Let $A: X \to X^*$ be a demiclosed monotone operator with $D(A) = X$ such that $A z$ is closed and convex in $X^*$ for each $z \in X$. Then $A$ is maximal.

**Proof.** Let $A': X \to X^*$ be any maximal monotone operator satisfying $G(A) \subseteq G(A')$. Let $\{z, z^*\} \in G(A')$ be any element. Then by (a) of Proposition 0.2.1 there is a ball $B_r(z) = \{y \in X: |z - y|_X \leq r\} \ (r > 0)$ such that

$$\bigcup_{y \in B_r(z)} A y$$

is bounded in $X^*$, say for some $C \geq 0$

$$|y^*|_{X^*} \leq C, \quad \forall y^* \in Ay \text{ with } y \in B_r(z).$$

Now let $y$ be any point in $B_r(z)$, $z_t = ty + (1 - t)z$ and take $z_t^* \in A z_t$ for each $t \in (0, 1)$. Then we note that

$$|z_t^*|_X \leq C,$$

$$(z_t^* - z^*, z_t - z)_X \geq 0, \quad i.e. \ (z_t^* - z^*, y - z)_X \geq 0.$$
Hence there is a sequence \( \{ t_n \} \) with \( t_n \downarrow 0 \) such that \( z^*_n \rightharpoonup z^*(y) \) in \( X^* \) for some \( z^*(y) \in X^* \), and we get
\[
(z^*(y) - z^*, y - z)_X \geq 0.
\]
Moreover \( z^*(y) \in Az \) by the demiclosedness of \( A \). Thus we have shown that for each \( y \in B_r(z) \) there exists \( z^*(y) \in Az \) such that \( (z^*(y) - z^*, y - z)_X \geq 0 \). Therefore from a result of Browder [2; Lemma 1] it follows that \( z^* \in Az \), which implies \( A = A' \).

In addition to the above cited papers, for the systematic studies of nonlinear monotone operators we refer to Brézis [1], Browder-Hess [1], Kenmochi [1] and Lions [1].

\[ \text{§0.3. Subdifferentials of convex functions} \]

Let \( X \) be a reflexive Banach space and \( \phi \) be a proper (i.e. \( \phi \neq -\infty \) and \( -\infty < \phi \leq \infty \)) lower semicontinuous (1.s.c.) convex function on \( X \). The effective domain of \( \phi \) is the set
\[
D(\phi) = \{ z \in X; \phi(z) < \infty \}
\]
and the subdifferential \( \partial \phi \) of \( \phi \) is an operator from \( X \) into \( X^* \) defined by
\[
[z, z^*] \in G(\partial \phi) \iff z \in D(\phi) \text{ and } (z^*, y - z)_X \leq \phi(y) - \phi(z), \forall y \in X.
\]
As to subdifferential operators the following facts are well-known.

**Proposition 0.3.1.** (cf. Moreau [1], Rockafellar [1, 4]) Let \( \phi \) be a proper 1.s.c. convex function on \( X \). Then \( \partial \phi : X \to X^* \) is maximal monotone and
\[
\text{int} \ D(\phi) \subset D(\partial \phi).
\]

**Proposition 0.3.2.** Let \( \phi \) be as in Proposition 0.3.1. If \( \phi \) is strictly convex, i.e.
\[
\phi(ty + (1-t)z) < t\phi(y) + (1-t)\phi(z), \forall y, z \in D(\phi), y \neq z, \forall t \in (0, 1),
\]
then \( \partial \phi \) is strictly monotone.

**Proof.** Let \( y, z \) be any element in \( D(\partial \phi) \) with \( y \neq z \) and \( 0 < t < 1 \). Let \( y_t = ty + (1-t)z \), \( y^* \in \partial \phi(y) \) and \( z^* \in \partial \phi(z) \). Then
\[
t \{ z^*, y - z \}_X = (z^*, y - z)_X
\]
\[
\leq \phi(y_t) - \phi(z)
\]
\[
< t\phi(y) + (1-t)\phi(z) - \phi(z),
\]
\[
= t \{ \phi(y) - \phi(z) \}
\]
and hence \( (z^*, y - z)_X < \phi(y) - \phi(z) \). Similarly \( (y^*, z - y)_X < \phi(z) - \phi(y) \). Therefore \( (z^* - y^*, z - y)_X > 0 \).

Next let \( 1 < p < \infty, -\infty < T < T' < \infty \) and \( \phi^t \) be a proper 1.s.c. convex function on \( X \) for each \( t \in [T, T'] \) such that for some constant \( C \)
\[
(3.1) \quad \phi^t(z) + C \leq \phi(z)^p + C \geq 0, \forall t \in [T, T'], \forall z \in X
\]
and such that \( t \to \phi^t(v(t)) \) is measurable for each \( v \in L^p(T, T'; X) \). Then define a function \( \Phi \) on \( L^p(T, T'; X) \) as follows:
\[ \Phi(v) = \int_T^t \phi^f(v(t)) \, dt, \quad v \in L^p(T, T'; X); \]

note that this integral always makes sense by (3.1). Clearly $\Phi$ is proper, 1.s.c. and convex on $L^p(T, T'; X)$. Besides we have

**Proposition 0.3.3.** (cf. Rockafellar [5] or Kenmochi [2]) Assume that for each $t \in (T, T')$ and each $z \in D(\phi^f)$ there is a function $v \in L^p(T, T'; X)$ such that $v(t) = z$, $\phi^f(v(\cdot)) \in L^1(T, T')$, $v$ is right-continuous at $t$ and

\[ \limsup_{s \downarrow t} \phi^f(v(s)) \leq \phi^f(z). \]

Let $u$ be a function in $L^p(T, T'; X)$ such that $\phi^f(u(\cdot)) \in L^1(T, T')$ and $f$ be a function in $L^p(T, T'; X^*)/(1/p + 1/p' = 1)$. Then $f \in \partial \Phi(u)$ if and only if

\[ f(t) \in \partial \phi^f(u(t)) \quad \text{for a.e. } t \in [T, T'). \]

In the sequel we consider the subdifferential of a proper 1.s.c. convex function on a Hilbert space $H$.

**Proposition 0.3.4.** Let $\phi$ be a proper 1.s.c. convex function on a Hilbert space $H$. Then we have:

1. For each $\lambda > 0$, $R(I + \lambda \partial \phi) = H$ and $J_\lambda = (I + \lambda \partial \phi)^{-1}$ is singlevalued and contractive on $H$, where $I$ is the identity on $H$.
2. For any $\lambda, \mu > 0$ and $z \in H$,

\[ J_\lambda z = J_\mu \left( \frac{\mu}{\lambda} z + (I - \frac{\mu}{\lambda}) J_\lambda z \right). \]

3. $D((\partial \phi)^o) = D(\partial \phi)$ and $(\partial \phi)^o$ is singlevalued, where $(\partial \phi)^o$ is the principal section of $\partial \phi$.

Now, for each $\lambda > 0$ we define the regularization $\phi_\lambda$ of $\phi$ by

\[ \phi_\lambda(z) = \inf_{y \in H} \left\{ \frac{1}{2\lambda} \| z - y \|^2_H + \phi(y) \right\}, \quad \forall z \in H. \]

**Proposition 0.3.5.** Let $\phi$ be as in Proposition 0.3.4. Then we have:

1. $\phi_\lambda$ is finite continuous and convex on $H$ with $D(\partial \phi_\lambda) = H$ for each $\lambda > 0$.
2. $\phi_\lambda(z) = \frac{1}{2\lambda} \| z - J_\lambda z \|^2_H + \phi(J_\lambda z)$, $\forall z \in H$, $\forall \lambda > 0$.
3. $\partial \phi_\lambda$ is singlevalued and lipschitz continuous with $1/\lambda$ as a lipschitz constant on $H$ for each $\lambda > 0$.
4. $\partial \phi_\lambda(z) = (z - J_\lambda z)/\lambda \in \partial \phi(J_\lambda z)$, $\forall z \in H$, $\forall \lambda > 0$.
5. If $z \in D(\partial \phi)$, then $\| \partial \phi_\lambda(z) \|^2_H \leq \| (\partial \phi)^o(z) \|^2_H$ for each $\lambda > 0$ and $\partial \phi_\lambda(z) \to (\partial \phi)^o(z)$ in $H$ as $\lambda \downarrow 0$.
6. For each $z \in H$ and $\lambda > 0$, $\phi_\lambda(z) \leq \phi(z)$ and $\phi_\lambda(z) \uparrow \phi(z)$ as $\lambda \downarrow 0$.
7. If $\lambda_n > 0$ and $\{ z_n \}$ is a sequence in $H$ such that $\partial \phi_{\lambda_n}(z_n) \to z^*$ in $H$ and $z_n \to z$ in $H$, then $\{ z, z^* \} \in G(\partial \phi)$.
8. If $\lambda_n \downarrow 0$ and $\{ z_n \}$ is a sequence in $H$ such that $z_n \to z$ in $H$, then

\[ \liminf_{n \to \infty} \phi_{\lambda_n}(z_n) \geq \phi(z). \]
Moreover, if \( \{ \partial\phi_{\lambda_n}(z_n) \} \) is bounded in \( H \), then \( \phi_{\lambda_n}(z_n) \rightarrow \phi(z) \).

(9) \[ \phi_{\lambda}(x) \leq \phi_{\lambda}(y) + \| \partial\phi_{\lambda}(z) \|_H \| z - y \|_H, \forall z, y \in H, \forall \lambda > 0. \]

(10) \[ |\phi_{\lambda}(x) - \phi_{\lambda}(y)| \leq \{ 2|\partial\phi_{\lambda}(z)|_H + \frac{|x - z|_H}{\lambda} + \frac{|y - z|_H}{\lambda} \} |x - y|_H, \forall x, y, z \in H, \forall \lambda > 0. \]

For proofs of Proposition 0.3.4 and (1) \( \sim (8) \) of Proposition 0.3.5, for instance, see Brézis [4; Chapter II].

Proofs of (9) and (10) of Proposition 0.3.5: From the definition of subdifferential we obtain (9) directly. We observe that

\[ |\partial\phi_{\lambda}(x)|_H \leq |\partial\phi_{\lambda}(x) - \partial\phi_{\lambda}(z)|_H + |\partial\phi_{\lambda}(z)|_H \leq \frac{|x - z|_H}{\lambda} + |\partial\phi_{\lambda}(z)|_H, \]

so that it follows from (9) that

\[ \phi_{\lambda}(x) - \phi_{\lambda}(y) \leq \left\{ \frac{|x - z|_H}{\lambda} + |\partial\phi_{\lambda}(z)|_H \right\} |x - y|_H. \]

Similarly

\[ \phi_{\lambda}(y) - \phi_{\lambda}(x) \leq \left\{ \frac{|y - z|_H}{\lambda} + |\partial\phi_{\lambda}(z)|_H \right\} |x - y|_H. \]

Hence we obtain (10).

\section{Differential inequalities}

In this section we state results for two differential inequalities which will be used in Chapters 1 and 2.

**Proposition 0.4.1.** Let \( g_i \) \((i = 1, 2, 3)\) be real-valued functions on \([0, T]\) \((0 < T < \infty)\). Suppose that \( g_2, g_3 \in L^1[0, T], g_2 \) is non-negative and \( g_1 \) is differentiable a.e. on \([0, T]\) and its derivative \( g_1' \) is integrable on \([0, T]\) and satisfies

\[ g_1(t) - g_1(s) \leq \int_s^t g_1'(\tau) \, d\tau, \forall s, t \in [0, T], s \leq t. \]

Suppose further that

\[ g_1'(t) \leq g_2(t)g_1(t) + g_3(t) \]

for a.e. \( t \in [0, T] \). Then we have: for any \( s, t \in [0, T] \) with \( s \leq t \)

\[ g_1(t) \leq g_1(s) \exp \left( \int_s^t g_2(\tau) \, d\tau \right) + \int_s^t g_3(\tau) \exp \left( \int_s^\tau g_2(\tau) \, d\tau \right) \, d\tau. \]

**Proposition 0.4.2.** Let \( 2 < p < \infty, a_0 \) be a non-negative number, \( q \) be a non-negative integrable function on \([0, T]\) \((0 < T < \infty)\) and \( g \) be a non-negative absolutely continuous function on \([0, T]\) such that

\[ g'(t) + a_0g(t)^{p/2} \leq q(t) \]

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for a.e. $t \in [0, T]$. Then we have

$$g(t) \leq e^{\frac{1}{2} \int_0^t a_0(t-s)(p-2)^{-2/(p-2)} + \int_s^t q(t) \, dt}$$

for any $s, t \in [0, T]$ with $s \leq t$.

Proposition 0.4.1 is well-known and Proposition 0.4.2 is due to Simon [1; Lemma 1].

Chapter 1

Evolution Equation $u'(t) + \partial \phi^f(u(t)) \ni f(t)$:

Existence and Uniqueness of Solutions

Throughout this chapter let $H$ be a real Hilbert space and for simplicity denote respectively by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and the inner product in $H$.

§ 1.1. Existence and uniqueness theorems

Let $0 < T < \infty$ and $\phi^f$ be a proper l.s.c. convex function on $H$ for each $t \in [0, T]$. Let us consider the evolution equation involving the subdifferential $\partial \phi^f$:

(E) \quad $u'(t) + \partial \phi^f(u(t)) \ni f(t), \quad 0 < t < T,$

where $u'(t) = (d/dt)u(t)$ and $f$ is a given $H$-valued function on $[0, T]$.

Definition 1.1.1. (a) Given $f \in L^1(0, T; H)$, an $H$-valued absolutely continuous function $u$ on $[0, T]$ is called a strong solution to (E), if $u(t) \in D(\partial \phi^f)$ for a.e. $t \in [0, T]$ and there is $u^* \in L^1(0, T; H)$ such that

\[
\begin{align*}
& u'(t) + u^*(t) = f(t) \\
\text{and} \\
& u^*(t) \in \partial \phi^f(u(t))
\end{align*}
\]

for a.e. $t \in [0, T]$.

(b) Given $f \in L^1(0, T; H)$, a function $u \in C([0, T]; H)$ is called a weak solution to (E), if there are sequences $\{u_n\} \subset C([0, T]; H)$ and $\{f_n\} \subset L^1(0, T; H)$ such that $u_n$ is a strong solution to (E) with $f = f_n$ for each $n$, $u_n \rightarrow u$ in $C([0, T]; H)$ and $f_n \rightarrow f$ in $L^1(0, T; H)$ as $n \rightarrow \infty$.

Next we consider the Cauchy problem for (E).

Definition 1.1.2. Given $f \in L^1(0, T; H)$ and $u_0 \in H$, by CP($\phi^f, f, u_0$) we mean the Cauchy problem for (E) with initial condition $u(0) = u_0$; a strong (resp. weak) solution to (E) satisfying $u(0) = u_0$ is called a strong (resp. weak) solution to CP($\phi^f, f, u_0$).

As is easily seen from the above definitions, a strong solution to (E) is also a weak solution and as to weak solutions to (E) we have

Theorem 1.1.1. Let $u_i(t = 1, 2)$ be weak solutions to (E) with $f = f_i$ in $L^1(0, T; H)$. Then we have for any $s, t \in [0, T]$ with $s \leq t$

\begin{equation}
|u_1(t) - u_2(t)|^2 - |u_1(s) - u_2(s)|^2 \leq 2 \int_s^t (f_1(\tau) - f_2(\tau), u_1(\tau) - u_2(\tau)) \, d\tau.
\end{equation}
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In particular, if \( u_i (i = 1, 2) \) are strong solutions, then for any \( s, t \in [0, T] \) with \( s \leq t \) we have

\[
\begin{align*}
|u_1(t) - u_2(t)|^2 - |u_1(s) - u_2(s)|^2 + 2 \int_s^t (u_1^*(\tau) - u_2^*(\tau), u_1(\tau) - u_2(\tau)) d\tau \\
= 2 \int_s^t (f_1(\tau) - f_2(\tau), u_1(\tau) - u_2(\tau)) d\tau,
\end{align*}
\]

where \( u_i^* \) are functions in \( L^1(0, T; H) \) such that

\[
(1.3) \quad u_1^*(\tau) + u_2^*(\tau) = f(\tau), \quad u_i^*(\tau) \in \partial \phi^*(u_i(\tau)) \quad \text{for a.e.} \ \tau \in [0, T].
\]

Proof. First assume that \( u_i \) are strong solutions to (E) with \( f = f_i \). Then by definition there are \( u_i^* \in L^1(0, T; H) \) for which (1.3) holds, so that

\[
\begin{align*}
(u_1' (\tau) - u_2' (\tau), u_1(\tau) - u_2(\tau)) + (u_1^* (\tau) - u_2^* (\tau), u_1(\tau) - u_2(\tau)) \\
= (f_1 (\tau) - f_2 (\tau), u_1(\tau) - u_2(\tau)) \quad \text{for a.e.} \ \tau \in [0, T].
\end{align*}
\]

By integration by parts we obtain (1.2) for any \( s, t \in [0, T] \) with \( s \leq t \). Also, (1.1) for weak solutions follows from (1.2), the monotonicity of \( \partial \phi^* \) and the definition of weak solution.

Q.E.D.

We see easily from the above theorem that \( CP(\phi^*; f, u_0) \) has at most one weak (hence strong) solution. In order for \( CP(\phi^*; f, u_0) \) to admit a strong or weak solution, certain smoothness assumptions on \( t \rightarrow \phi^* \) are required; we impose here the following conditions on \( \{ \phi^*; 0 \leq t \leq T \} \):

(h1) There is a positive constant \( \alpha \) such that

\[
\phi^*(z) + \alpha |z| + \alpha \geq 0, \quad \forall \ t \in [0, T], \quad \forall \ z \in H;
\]

(h2) (i) For each \( z \in H \) and \( \lambda \in (0, 1] \) the function \( t \rightarrow \phi^\lambda (z) \) is differentiable at a.e. \( t \in [0, T] \) and its derivative is integrable on \( [0, T] \) and satisfies

\[
\phi^\lambda (z) - \phi^\lambda (z) \leq \int_s^t \frac{d}{dt} \phi^\lambda (z) d\tau, \quad \forall s, t \in [0, T], \ s \leq t,
\]

where \( \phi^\lambda \) stands for the regularization of \( \phi^* \);

(ii) For each \( r > 0 \) there are a number \( a_r \in (0, 1) \) and non-negative \( b_r, c_r \in L^1 (0, T) \) such that

\[
\frac{d}{dt} \phi^\lambda (z) \leq a_r |\phi^\lambda (z)|^2 + b_r(t) |\phi^\lambda (z)| + c_r(t) \quad \text{for a.e.} \ t \in [0, T]
\]

for any \( z \in H \) with \( |z| \leq r \) and \( \lambda \in (0, 1] \);

(h3) There are a function \( h: [0, T] \rightarrow H \) and a partition \( 0 = t_0 < t_1 < \ldots < t_N = T \) of \( [0, T] \) such that \( t \rightarrow \phi^* (h(t)) \) belongs to \( L^1 (0, T) \) and the restriction of \( h \) to \( (t_{k-1}, t_k) \) belongs to \( W^1,1( (t_{k-1}, t_k), H) \) for \( k = 1, 2, \ldots, N \).

We denote also by (h2)' the system of (i) of (h2) and the following (ii)'

(ii)' There are a number \( a \in (0, 1) \) and non-negative \( b, c \in L^1 (0, T) \) such that

\[
\frac{d}{dt} \phi^\lambda (z) \leq a |\phi^\lambda (z)|^2 + b(t) |\phi^\lambda (z)| + c(t) (1 + |z|^2) \quad \text{for a.e.} \ t \in [0, T]
\]

for any \( z \in H \) and \( \lambda \in (0, 1] \).
Our existence theorems are stated as follows.

**Theorem 1.1.2.** Suppose \( \{ (h1), (h2), (h3) \} \) (resp. \( \{ (h1), (h2)' \} \)). Then, for each \( u_0 \in D(\phi^t) \) and \( f \in L^2(0, T; H) \), \( CP(\phi^t; f, u_0) \) admits a weak solution \( u \) such that

\[
\sqrt{t} u^t \in L^2(0, T; H),
\]

\[
t \to \phi^t(u(t)) \text{ is integrable on } [0, T],
\]

\[
t \to t^{\phi^t}(u(t)) \text{ is bounded on } (0, T]
\]

and

\[
u^t(t) + \partial \phi^t(u(t)) \ni f(t) \quad \text{for a.e. } t \in (0, T].
\]

Moreover there is a number \( r_0 \leq \sup\{ |u(t)| : 0 \leq t \leq T \} \) such that for any \( r \geq r_0 \) and for any \( s, t \in (0, T] \) with \( s \leq t \)

\[
\phi^t(u(t)) - \phi^s(u(s)) + \int_s^t \left( u^t(\tau) - f(\tau) \right) d\tau
\]

\[
\leq \int_s^t \left\{ \|a_r|u^t(\tau) - f(\tau)\|^2 + b_r(\tau) \|\phi^t(u(\tau))\|^2 + c(\tau) \right\} d\tau
\]

\[
(\text{resp. } \leq \int_s^t \left\{ \|a_r|u^t(\tau) - f(\tau)\|^2 + b_r(\tau) \|\phi^t(u(\tau))\|^2 + c(\tau) \left(1 + \|u(\tau)\|^2 \right) \right\} d\tau,
\]

where \( a_r, b_r, c_r \) (resp. \( a, b, c \)) are as in \((ii)\) of \((h2)\) (resp. \((ii)'\) of \((h2)'\)). In particular, if \( u_0 \in D(\phi^0) \), then \( u^t \in L^2(0, T; H) \), \( t \to \phi^t(u(t)) \) is bounded on \([0, T] \) and (1.4) holds for any \( s, t \in [0, T] \) with \( s \leq t \).

**Theorem 1.1.3.** Suppose \( \{ (h1), (h2), (h3) \} \) or \( \{ (h1), (h2)' \} \). Then for each \( u_0 \in D(\phi^0) \) and \( f \in L^1(0, T; H) \), \( CP(\phi^t; f, u_0) \) admits a weak solution \( u \) such that \( t \to \phi^t(u(t)) \) is integrable on \([0, T] \).

The proofs of the theorems mentioned above will be given in sections 1.2, 1.3 and 1.4.

**Remark 1.1.1.** Hypotheses of the type \((h1), (h2)\) and \((h2)′\) were originally introduced by Attouch-Bénilan-Damlamian-Picard [1].

**Remark 1.1.2.** A part of results of Theorem 1.1.2 was announced in Kenmochi [3]. Many interesting results closely related to it are also found in Attouch-Bénilan-Damlamian-Picard [1] (or Picard [1]), Attouch-Damlamian [2], Biroli [1, 2], Brézis [2, 3], Kenmochi [4, 5], Moreau [2], Yamada [1] and Yotsutani [1].

§1.2. Lemmas

Before giving a proof of Theorem 1.1.2 we prepare some lemmas which are due to Attouch-Bénilan-Damlamian-Picard [1] (or in detail see Picard [1]).

Throughout this section, assume \( \{ (h1), (h2), (h3) \} \) or \( \{ (h1), (h2)' \} \). Denote by \( J^t_\lambda \) the operator \( (I + \lambda \phi^t)^{-1} \) for each \( t \in (0, T) \) and \( \lambda > 0 \).

**Lemma 1.2.1.** There is a positive constant \( M \) such that

\[(2.1) \quad \left| J^t_\lambda z \right| \leq M + |z| \]

and

\[(2.2) \quad \left| \partial \phi^t_\lambda (z) \right| \leq \frac{1}{\lambda} \left( M + 2 |z| \right) \]

---

---
for all $t \in [0, T], \lambda \in (0, 1)$ and $z \in H$.

**Proof.** Let $z_0 \in H$. By (h1),

$$
\phi_{t}^{\lambda}(z_0) = \phi_{t}(J_{t}^{\lambda}z_0) + \frac{1}{2} |z_0 - J_{t}^{\lambda}z_0|^2 \geq - \alpha |J_{t}^{\lambda}z_0| - \alpha + \frac{1}{2} |z_0 - J_{t}^{\lambda}z_0|^2,
$$

which implies $|J_{t}^{\lambda}z_0| \leq M'$ for all $t \in [0, T]$ with some positive constant $M'$. Using (2) of Proposition 0.3.4, we have

$$
|J_{t}^{\lambda}z_0| \leq |J_{t}^{\lambda}z_0| + (1 - \lambda) |J_{t}^{\lambda}z_0 - z_0| \leq 2M' + |z_0|, \quad \forall t \in [0, T], \quad \forall \lambda \in (0, 1).
$$

Since $J_{t}^{\lambda}$ is contractive on $H$, it follows that

$$
|J_{t}^{\lambda}z - J_{t}^{\lambda}z_0| \leq |z - z_0| + |J_{t}^{\lambda}z_0 - z_0| \leq 2M' + 2|z_0| + |z|
$$

for all $z \in H, t \in [0, T]$ and $\lambda \in (0, 1)$. Thus (2.1) holds for $M = 2M' + 2|z_0|$, and by the relation $\partial \phi_{t}^{\lambda} = (I - J_{t}^{\lambda})/\lambda$ we have (2.2).

**Lemma 1.2.2.** (1) The function $t \to \phi_{t}^{\lambda}(v(t))$ is measurable on $[0, T]$ for each $\lambda \in (0, 1)$ and $v \in L^{1}(0, T; H)$. Q.E.D.

(2) The function $t \to \phi_{t}^{\lambda}(v(t))$ is measurable on $[0, T]$ for each $v \in L^{1}(0, T; H)$.

(3) The function $\Phi$ given by

$$
\Phi(v) = \int_{0}^{T} \phi_{t}^{\lambda}(v(t)) dt, \quad \forall v \in L^{2}(0, T; H),
$$

is proper 1.s.c. and convex on $L^{2}(0, T; H)$ (note that the integration (2.3) always makes sense by (h1)).

(4) For each $\lambda > 0$, the function $\Psi^{\lambda}$ given by

$$
\Psi^{\lambda}(v) = \int_{0}^{T} \phi_{t}^{\lambda}(v(t)) dt, \quad \forall v \in L^{2}(0, T; H),
$$

is finite, continuous and convex on $L^{2}(0, T; H)$.

**Proof.** For each $z \in H$ and $\lambda \in (0, 1]$, by (i) of (h2) the function

$$
t \mapsto \phi_{t}^{\lambda}(z) - \int_{0}^{t} \frac{d}{dt} \phi_{t}^{\lambda}(z) dt
$$

is monotonically non-increasing in $t$ and hence $t \to \phi_{t}^{\lambda}(z)$ is measurable on $[0, T]$. Also, $z \to \phi_{t}^{\lambda}(z)$ is continuous on $H$ for fixed $t \in [0, T]$ and $\lambda \in (0, 1]$. Therefore, for each $\lambda \in (0, 1], t \to \phi_{t}^{\lambda}(v(t))$ is measurable on $[0, T]$ whenever $v$ is in $L^{1}(0, T; H)$. Noting (cf. (6) of Proposition 0.3.5) that $\phi_{t}^{\lambda}(z) \uparrow \phi_{t}^{\lambda}(z)$ as $\lambda \downarrow 0$ for each $t \in [0, T]$ and $z \in H$, we see that $t \to \phi_{t}^{\lambda}(v(t))$ is also measurable on $[0, T]$ if $v \in L^{1}(0, T; H)$. Thus we have (1) and (2). Moreover clearly $\Phi$ is convex on $L^{2}(0, T; H)$ as well as $\Psi^{\lambda}$, and by Fatou's lemma we see that $\Phi$ is proper and 1.s.c. on $L^{2}(0, T; H)$. Finally we show the finite continuity of $\Psi^{\lambda}$ on $L^{2}(0, T; H)$. In fact, let $z \in H$, $v, w \in L^{2}(0, T; H)$ and $\lambda \in (0, 1]$. Then it follows from (10) of Proposition 0.3.5 and Lemma 1.2.1 that
for a.e. $t \in [0, T]$, so that $\Psi^\lambda$ is finite and continuous on $L^2(0, T; H)$.

Q.E.D.

Lemma 1.2.3. For each $\lambda \in (0, 1)$, the $H$-valued function $t \mapsto \partial \phi^\lambda_t(\nu(t))$ is measurable and hence $t \mapsto J^\lambda_t \nu(t)$ is measurable on $[0, T]$, whenever $\nu \in L^1(0, T; H)$.

Proof. From Proposition 0.3.1 it follows that $\partial \Psi^\lambda(\nu) + \phi$ for every $\nu$ in $L^2(0, T; H)$. Now let $z \in H$ and $\nu^* \in \partial \Psi^\lambda(\nu)$ with $\nu(t) \equiv z$ and let $s \in (0, T)$ be any Lebesgue point of $\nu^*$ and $t \mapsto \phi^\lambda_t(z)$. Take any element $z' \in H$ and for $\delta > 0$ set

$$w(t) = \begin{cases} z' & \text{if } t \in [s, s + \delta], \\ z & \text{if } t \in [0, s) \cup (s + \delta, T]. \end{cases}$$

Then by the definition of $\partial \Psi^\lambda$,

$$\int_0^T \langle \nu^*(t), w(t) - z \rangle dt \leqslant \Psi^\lambda(w) - \Psi^\lambda(\nu),$$

that is,

$$(2.5) \quad \frac{1}{\delta} \int_s^{s+\delta} \langle \nu^*(t), z' - z \rangle dt \leqslant \frac{1}{\delta} \int_s^{s+\delta} \langle \phi^\lambda_t(z') - \phi^\lambda_t(z) \rangle dt.$$ 

Noting (cf. (i) of (h2)) that

$$\limsup_{\delta \downarrow 0} \frac{1}{\delta} \int_s^{s+\delta} \phi^\lambda_t(z') dt \leqslant \phi^\lambda(z'),$$

and letting $\delta \downarrow 0$ in (2.5), we get

$$(\nu^*(s), z' - z) \leqslant \phi^\lambda(z') - \phi^\lambda(z), \quad \forall z' \in H,$$

which implies $\nu^*(s) = \partial \phi^\lambda(z)$. Consequently, $t \mapsto \partial \phi^\lambda_t(z)$ is measurable in $t$ and hence $t \mapsto \partial \phi^\lambda_t(\nu(t))$ is measurable in $t$ whenever $\nu \in L^1(0, T; H)$.

Q.E.D.

Lemma 1.2.4. For each $\lambda \in (0, 1)$, $\Psi^\lambda$ coincides with the regularization $\Phi^\lambda$ of $\Phi$. Moreover,

$$(2.6) \quad [\partial \Phi^\lambda(\nu)](t) = \partial \phi^\lambda_t(\nu(t)) \quad \text{for a.e. } t \in [0, T],$$

whenever $\nu \in L^2(0, T; H)$.

Proof. Let $\nu \in L^2(0, T; H)$. By the definitions of $\Phi^\lambda$ and $\phi^\lambda_t$ we have

$$\Phi^\lambda(\nu) = \inf_{w \in L^2(0, T; H)} \left\{ \int_0^T \frac{1}{2\lambda} |\nu - w|^2_{L^2(0, T; H)} + \Phi(w) \right\}$$

$$= \inf_{w \in L^2(0, T; H)} \int_0^T \left\{ \frac{1}{2\lambda} |\nu - w|^2 + \phi^\lambda_t(\nu(t)) \right\} dt$$
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\[ \geq \int_0^T \Phi^T_{\lambda} (v(t)) \, dt \quad (= \Psi^\lambda (v)) \]
\[ = \int_0^T \frac{1}{2\lambda} |v(t) - J^T_{\lambda} v(t)|^2 + \phi^T (J^T_{\lambda} v(t)) \, dt \]
\[ \geq \Phi^\lambda (v), \]

because \( t \rightarrow J^T_{\lambda} v(t) \) belongs to \( L^2(0, T; H) \) by Lemmas 1.2.1 and 1.2.3. Therefore \( \Phi^\lambda = \Psi^\lambda \) and (2.6) is easily got.

The following lemma plays an essential role in our proof of Theorem 1.1.2.

Lemma 1.2.5. Let \( v \in W^{1,1} (T_0, T_1 \mid H) \) with \( 0 \leq T_0 < T_1 \leq T \) and \( 0 < \lambda \leq 1 \). Then we have:

(i) The function \( t \rightarrow \Phi^\lambda (v(t)) \) is differentiable a.e. on \( [T_0, T_1] \) and its derivative \( (d/dt) \Phi^\lambda (v(t)) \) is integrable on \( [T_0, T_1] \) and satisfies

\[ \Phi^\lambda (v(t)) - \Phi^\lambda (v(s)) \leq \int_s^t \frac{d}{d\tau} \Phi^\lambda (v(\tau)) \, d\tau, \quad \forall s, t \in [T_0, T_1], s \leq t. \]  

(ii) Let \( a_r \in [0, 1] \) (resp. \( a \in [0, 1] \)) and \( b_r, c_r \in L^1 (0, T) \) (resp. \( b, c \in L^1 (0, T) \)) be as in (ii) of (h2) with \( r \geq \sup \{ |v(t)| \mid T_0 \leq t \leq T_1 \} \) (resp. in (ii)' of (h2)'). Then

\[ \frac{d}{dt} \Phi^\lambda (v(t)) - (v'(t), \partial \Phi^\lambda (v(t))) \]
\[ \leq a_r |\partial \Phi^\lambda (v(t))|^2 + b_r(t) |\Phi^\lambda (v(t))| + c_r(t) \]  

(resp. \( \leq a |\partial \Phi^\lambda (v(t))|^2 + b(t) |\Phi^\lambda (v(t))| + c(t) (1 + |v(t)|^2) \))

and

\[ \frac{d}{dt} \{ t \Phi^\lambda (v(t)) \} - t (v'(t), \partial \Phi^\lambda (v(t))) \]
\[ \leq ta_r |\partial \Phi^\lambda (v(t))|^2 + (1 + tb_r(t)) |\Phi^\lambda (v(t))| + tc_r(t) \]  

(resp. \( \leq ta |\partial \Phi^\lambda (v(t))|^2 + (1 + tb(t)) |\Phi^\lambda (v(t))| + tc(t)(1 + |v(t)|^2) \))

for a.e. \( t \in [T_0, T_1] \)

Proof. We give proofs of (2.7) - (2.9) in case (h1), (h2) and (h3) are fulfilled, as they are similarly proved in another case. First, using (h2), for any \( s, t \in [T_0, T_1] \) with \( s \leq t \) and \( r \geq \sup \{ |v(t)| \mid T_0 \leq t \leq T_1 \} \) we observe that

\[ \Phi^\lambda (v(t)) - \Phi^\lambda (v(s)) - (v(t) - v(s), \partial \Phi^\lambda (v(t))) \]
\[ \leq \int_s^t \left\{ a_r |\partial \Phi^\lambda (v(s))|^2 + b_r(t) |\Phi^\lambda (v(s))| + c_r(t) \right\} \, d\tau. \]

Since \( \langle \tau, s \rangle \rightarrow \Phi^\lambda (v(s)) \) is bounded on \( [T_0, T_1] \times [T_0, T_1] \) by (10) of Proposition 0.3.5 as well as \( \langle \tau, s \rangle \rightarrow |\partial \Phi^\lambda (v(s))| \) by Lemma 1.2.1, it follows from (2.10) that for some non-negative \( \rho \in L^1 (T_0, T_1) \)

\[ \Phi^\lambda (v(t)) - \Phi^\lambda (v(s)) \leq \int_s^t \rho(\tau) \, d\tau, \quad \forall s, t \in [T_0, T_1], s \leq t. \]
Hence \( t \to \phi^t_\lambda(v(t)) \) is differentiable a.e. on \([T_0, T_1]\) and the derivative \((d/dt)\phi^t_\lambda(v(t))\) is integrable on \([T_0, T_1]\), so we get (2.7).

Next, we observe by (3) \((3.3.5)\) that

\[
\frac{1}{t-s} \int_s^t \frac{1}{t} \| \partial \phi^\tau_\lambda(v(s)) \| + \| \partial \phi^\tau_\lambda(v(\tau)) \|^2 \, d\tau
\]

\[
= \frac{a_r}{t-s} \int_s^t \left\{ \| \partial \phi^\tau_\lambda(v(s)) \| + \| \partial \phi^\tau_\lambda(v(\tau)) \| \right\} \, d\tau
\]

\[
\leq \frac{a_r}{(t-s)} \int_s^t \| \partial \phi^\tau_\lambda(v(s)) \| + \| \partial \phi^\tau_\lambda(v(\tau)) \| \, d\tau
\]

for any \( s < t \). Therefore

\[
(2.11) \quad \frac{1}{t-s} \int_s^t a_r \| \partial \phi^\tau_\lambda(v(s)) \|^2 + a_r \| \partial \phi^\tau_\lambda(v(\tau)) \|^2 \, d\tau \rightarrow a_r \| \partial \phi^t_\lambda(v(t)) \|^2 \quad \text{as} \quad s \uparrow t
\]

for a.e. \( t \in [T_0, T_1] \). Also, by (10) \((3.3.5)\)

\[
\frac{1}{t-s} \int_s^t b_r(\tau) \| \phi^\tau_\lambda(v(s)) - \phi^\tau_\lambda(v(\tau)) \| \, d\tau
\]

\[
\leq \frac{1}{t-s} \int_s^t b_r(\tau) \left\{ 2 \| \phi^\tau_\lambda(v(s)) \| + \frac{1}{\lambda} \| v(s) - v(\tau) \| \right\} \, d\tau
\]

for any \( s < t \), so that

\[
(2.12) \quad \frac{1}{t-s} \int_s^t b_r(\tau) \| \phi^\tau_\lambda(v(s)) \| \, d\tau \rightarrow b_r(t) \| \phi^t_\lambda(v(t)) \| \quad \text{as} \quad s \uparrow t
\]

for a.e. \( t \in [T_0, T_1] \). Consequently we derive (2.8) from (2.10) together with (2.11) and (2.12). Also, (2.8) multiplied by \( t \) yields (2.9). Q.E.D.

**Corollary.** In the above lemma, further suppose that \( v' \in L^2(T_0, T_1; H) \) and let \( v^* \in L^2(T_0, T_1; H) \) such that \( v^*(t) = \partial \phi^t_\lambda(v(t)) \) for a.e. \( t \in [T_0, T_1] \). Then we have:

(a) \( v(t) \in D(\phi^t) \) for all \( t \in (T_0, T_1) \).

(b) \( t \to (\partial \phi^t)^o(v(t)) \) belongs to \( L^2(T_0, T_1; H) \) with \( \| (\partial \phi^t)^o(v(t)) \| \leq \| v^*(t) \| \) for a.e. \( t \in [T_0, T_1] \).

(c) For any \( s, t \in (T_0, T_1) \) with \( s \leq t \) the following hold:

\[
\phi^t(\tau) - \phi^s(v(s)) - \int_s^t (v^*(\tau), (\partial \phi^\tau)^o(v(\tau))) \, d\tau
\]

\[
\leq \int_s^t \left\{ a_r (\partial \phi^\tau)^o(v(\tau)) \right\} + c_r(\tau) \right\} \, d\tau
\]

\[
(\text{resp.} \leq \int_s^t \left\{ a (\partial \phi^\tau)^o(v(\tau)) \right\} + c(\tau) \right\} (1 + \| v(\tau) \|^2) \right\} \, d\tau
\]

and

\[
\tau \phi^t(v(t)) - s \phi^s(v(s)) - \int_s^t (v^*(\tau), (\partial \phi^\tau)^o(v(\tau))) \, d\tau
\]

\[
\leq \int_s^t \left\{ a_r (\partial \phi^\tau)^o(v(\tau)) \right\} + c_r(\tau) \right\} \, d\tau
\]

\[
(\text{resp.} \leq \int_s^t \left\{ a (\partial \phi^\tau)^o(v(\tau)) \right\} + c(\tau) \right\} (1 + \| v(\tau) \|^2) \right\} \, d\tau.
\]
Furthermore if \( v(T_0) \in D(\Phi_{\lambda}^{T_0}) \), then (2.13) holds for every \( s, t \in [T_0, T_f] \) with \( s \leq t \).

**Proof.** We prove the Corollary in case (h1), (h2) and (h3) are satisfied. Let \( t_0 \) be a point in \([T_0, T_f]\) so that \( v(t_0) \in D(\Phi_{\lambda}^{T_0}) \). Then, using Proposition 0.4.1, we obtain from (2.8) of Lemma 1.2.5 that

\[
\phi_{\lambda}^s (v(t)) \leq \Phi_{\lambda}^{T_0}(v(t_0)) \exp \left( \int_{t_0}^t b_r(\tau) \, d\tau \right)
+ \int_{t_0}^t \left\{ a_{r}\left| \Phi_{\lambda}^s (v(s)) \right|^2 + c_{r}(s) \right\} \exp \left( \int_{s}^t b_r(\tau) \, d\tau \right) \, ds
\]

for all \( t \in [t_0, T_f] \). Since

\[
\partial \phi_{\lambda}^s (v(s)) \rightarrow (\partial \Phi_{\lambda}^{T_0}) (v(s)) \quad \text{in } H \text{ as } \lambda \downarrow 0 \text{ for a.e. } s \in [T_0, T_f],
\]

\[
|\partial \phi_{\lambda}^s (v(s))| \leq |\partial \Phi_{\lambda}^{T_0}(v(s))| \leq |v^*(s)| \quad \text{for a.e. } s \in [T_0, T_f]
\]

and

\[
\phi_{\lambda}^t (v(t)) \uparrow \Phi^t (v(t)) \quad \text{as } \lambda \downarrow 0 \text{ for all } t \in [T_0, T_f]
\]

cf. (5), (6) of Proposition 0.3.5, we infer from (2.15) that \( t \rightarrow \Phi^t [v(t)] \) is bounded on \([t_0, T_f]\). Hence, by integrating (2.8) over \([s, t] \subset [t_0, T_f]\) and letting \( \lambda \downarrow 0 \) we obtain

\[
\phi_{\lambda}^t (v(t)) - \Phi_{\lambda}^{T_0}(v(t_0)) - \int_{t_0}^t \left\{ a_{r}\left| \Phi_{\lambda}^s (v(s)) \right|^2 + c_{r}(s) \right\} d\tau
\leq \int_{t_0}^t \left\{ a_{r}\left| \Phi_{\lambda}^s (v(s)) \right|^2 + b_r(\tau) \right\} d\tau
\leq \int_{t_0}^t \left\{ a_{r}\left| \Phi_{\lambda}^s (v(s)) \right|^2 + b_r(\tau) \right\} d\tau
\]

As \( t_0 \) can be taken arbitrarily near \( T_0 \), (2.13) holds for any \( s, t \in [T_0, T_f] \) with \( s \leq t \). Similarly (2.14) is derived from (2.9).

### §1.3. Approximation for \( CP(\Phi; f, u_0) \)

In this section we fix \( u_0 \in D(\Phi_{\lambda}^{T_0}) \) and \( f \in L^2(0, T; H) \). According to, for example, Brézis [4; Theorem 1.41], under \( \{(h1), (h2), (h3)\} \) or \( \{(h1), (h2)'\} \) there exists a unique \( u_\lambda \) in \( W^1, 2(0, T; H) \) for each \( 0 < \lambda \leq 1 \) such that

\[
u_\lambda^s (t) + \partial \phi_{\lambda} (u_\lambda (t)) = f(t) \quad \text{for a.e. } t \in [0, T]
\]

with \( u_\lambda (0) = u_0 \).

(A) The case that \( (h1), (h2) \) and \( (h3) \) are satisfied.

Let \( h: [0, T] \rightarrow H \) be the same function as in (h3) with partition \( \{ 0 = t_0 < t_1 < \ldots < t_N = T \} \); for some conveniences assume that \( h \) is continuous on \([0, t_1]\) and on each \([t_{k-1}, t_k]\), \( k = 1, 2, \ldots, N \) (this assumption can be made without loss of generality). Then we have for a.e. \( s \in [t_{k-1}, t_k) \) \( (k = 1, 2, \ldots, N) \)

\[
\frac{d}{ds} |u_\lambda (s) - h(s)|^2 = 2 (u_\lambda^s (s) - h'(s), u_\lambda (s) - h(s))
\]

(3.1)
\[
\begin{align*}
&= -2 \langle \phi^*_\lambda(u_\lambda(s)), u_\lambda(s) - h(s) \rangle + 2 \langle f(s) - h'(s), u_\lambda(s) - h(s) \rangle \\
&\leq 2 \phi^*_\lambda(h(s)) - 2 \phi^*_\lambda(u_\lambda(s)) + 2 \langle f(s) \mid + h'(s) \rangle \| u_\lambda(s) - h(s) \|
\end{align*}
\]

and by (2.1) of Lemma 1.2.1
\[
\begin{align*}
\phi^*_\lambda(u_\lambda(s)) &\geq \phi^*(J^*_s u_\lambda(s)) \\
&\geq -\alpha |J^*_s u_\lambda(s)| - \alpha \\
&\geq -\alpha (M + |u_\lambda(s)|) - \alpha \\
&\geq -\alpha u_\lambda(s) - h(s) - \alpha |h(s)| - \alpha M - \alpha \\
&\geq -\frac{1}{2} u_\lambda(s) - h(s) + \alpha (\frac{\alpha}{2} + |h(s)| + M + 1).
\end{align*}
\]

From (3.1) and (3.2) we infer that
\[
\begin{align*}
\frac{d}{ds} |u_\lambda(s) - h(s)|^2 &\leq 2 |u_\lambda(s) - h(s)|^2 + g(s) \quad \text{for a.e. } s \in [0, T],
\end{align*}
\]

where
\[
g(s) = 2 \phi^*(h(s)) + (|f(s)| + |h'(s)|)^2 + 2 \alpha |h(s)| + 2 \alpha M + 2 \alpha + \alpha^2.
\]

**Lemma 1.3.1.** There is a non-negative constant \( N_1 = N_1 (|f|_{L^2(0, T; H)}, |u_0|_{V}) \) (depending only on \(|f|_{L^2(0, T; H)}\) and \(|u_0|_{V}\)) such that
\[
\begin{align*}
|u_\lambda(t)| &\leq N_1, \quad \forall t \in [0, T], \forall \lambda \in (0, 1), \\
\int_0^T \phi^*_\lambda(u_\lambda(s)) ds &\leq N_1, \quad \forall \lambda \in (0, 1).
\end{align*}
\]

**Proof.** Let \( 0 < \lambda \leq 1 \). Using Proposition 0.4.1 we obtain from (3.3) that
\[
\begin{align*}
|u_\lambda(t) - h(t)|^2 &\leq e^{2(t-t_{k-1})} |u_\lambda(t_{k-1}) - h(t_{k-1})|^2 + e^{2(t-t_{k-1})} \int_{t_{k-1}}^t g(\tau) d\tau \\
&\leq 2e^{2(t-t_{k-1})} \left( |u_\lambda(t_{k-1}) - h(t_{k-1})|^2 + |h(t_{k-1}) - h(t_{k-1})|^2 \right) \\
&\quad + e^{2(t-t_{k-1})} \int_{t_{k-1}}^t g(\tau) d\tau
\end{align*}
\]

for all \( t \in (t_{k-1}, t_k] \), \( k = 1, 2, \ldots, N \), where \( h(t_{k-1}+) \) stands for the limit of \( h(t) \) as \( t \downarrow t_{k-1} \).

Hence
\[
\begin{align*}
|u_\lambda(t) - h(t)|^2 &\leq 2Ne^{2T} \left( |u_0 - h(0)|^2 + \int_0^T g(\tau) d\tau + \sum_{k=1}^{N-1} |h(t_k) - h(t_{k+1})|^2 \right) \equiv R_1
\end{align*}
\]

for all \( t \in (t_{k-1}, t_k] \), \( k = 1, 2, \ldots, N \), and by the way
\[
\begin{align*}
|u_\lambda(t)| &\leq \sqrt{R_1} + \sup_{0 \leq s \leq T} |h(s)| \equiv R_2 \quad \text{for all } t \in [0, T].
\end{align*}
\]

Besides, by (3.1)
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\[
\int_{t_k-1}^{t_k} \phi_x^x(u_k(s)) \, ds + \frac{1}{2} \left| u_x(t_k) - h(t_k) \right|^2 - \frac{1}{2} \left| u_x(t_k-1) - h(t_k-1) \right|^2 
\leq \int_{t_k-1}^{t_k} \phi_x^s(h(s)) \, ds + \int_{t_k-1}^{t_k} (|f(s)| + |h'(s)|) |u_x(s) - h(s)| \, ds
\]
for all \( k = 1, 2, \ldots, N \), and hence by (3.6)

\[
\int_0^T \phi_x^s(u_x(s)) \, ds \leq \frac{1}{2} NR_1 + \sqrt{R_1} \int_0^T (|f(s)| + |h'(s)|) \, ds + \int_0^T |\phi_x^s(h(s))| \, ds \equiv R_3.
\]

Since by (2.1) of Lemma 1.2.1

(3.8) \quad |\phi_x^s(u_x(s))| \leq \phi_x^x(u_x(s)) + 2\alpha (M + R_2 + 1),

we get

\[
\int_0^T |\phi_x^s(u_x(s))| \, ds \leq R_3 + 2\alpha T (M + R_2 + 1) \equiv R_4.
\]

If we take \( N_1 = R_2 + R_4 \), then (3.4) and (3.5) hold. Q.E.D.

**Lemma 1.3.2.** There is a non-negative constant \( N_2 = N_2 (|f| \leq L^2(0, T; H), |u_0|) \) such that

(3.9) \quad |\sqrt{r} u_x^L(0, T; H) | \leq N_2, \quad \forall \lambda \in (0, 1),

(3.10) \quad |t \phi_x^s(u_x(t))| \leq N_2, \quad \forall t \in (0, T], \quad \forall \lambda \in (0, 1).

In particular, if \( u_0 \in D(\phi^0) \), then there is a non-negative constant \( N_3 = N_3 (|f|\leq L^2(0, T; H), |u_0|, \phi^0(u_0)) \) such that

(3.11) \quad |u_x^L(0, T; H) | \leq N_3, \quad \forall \lambda \in (0, 1),

(3.12) \quad |\phi_x^s(u_x(t))| \leq N_3, \quad \forall t \in (0, T], \quad \forall \lambda \in (0, 1).

**Proof.** Since

\[
|u_x^L(t)|^2 = -(u_x^L(t), \partial_x \phi_x^s(u_x(t)) - f(t))
\]

and

\[
|\phi_x^s(u_x(t))| \leq \phi_x^x(u_x(t)) + 2\alpha (M + R_2 + 1) \quad (cf. (3.8)),
\]

we have by (2.8) of Lemma 1.2.5

\[
\frac{d}{dt} \phi_x^s(u_x(t)) + |u_x^L(t)|^2
= \frac{d}{dt} \phi_x^s(u_x(t)) - (u_x^L(t), \partial_x \phi_x^s(u_x(t)) - f(t))
\leq a_p |\partial_x \phi_x^s(u_x(t))|^2 + b_p(t) |\phi_x^s(u_x(t))|^2 + c_p(t) + |u_x^L(t), f(t))
= a_p |u_x^L(t) - f(t)|^2 + b_p(t) |\phi_x^s(u_x(t))|^2 + c_p(t) + |u_x^L(t), f(t)),
\]

where \( a_p, b_p, c_p \) are as in (ii) of (h2) with \( r \geq N_1 \). Hence for any \( \delta > 0 \)

\[
\frac{d}{dt} \phi_x^s(u_x(t)) + |u_x^L(t)|^2
\]


\[ \leq \left\{ a_r + (a_r + 1) \frac{\delta}{\delta} \right\} |u_\lambda^\delta(t)|^2 + b_r(t) \phi_\lambda^\delta(u_\lambda(t)) + \left\{ \frac{1}{\delta} + a_r(1 + \frac{1}{\delta}) \right\} |f(t)|^2 + 2 \alpha (M + R_2 + 1) b_r(t) + c_r(t), \]

from which we obtain

\[ (3.13) \quad \left\{ 1 - a_r - (a_r + 1) \frac{\delta}{\delta} \right\} |u_\lambda^\delta(t)|^2 + \frac{d}{dt} \phi_\lambda^\delta(u_\lambda(t)) \leq b_r(t) \phi_\lambda^\delta(u_\lambda(t)) + g_1(t) \]

for a.e. \( t \in [0, T] \), where

\[ g_1(t) = \left\{ - \frac{1}{\delta} + a_r(1 + \frac{1}{\delta}) \right\} |f(t)|^2 + 2 \alpha (M + R_2 + 1) b_r(t) + c_r(t) \]

and \( \delta \) is chosen so that \( 1 - a_r - (a_r + 1) \frac{\delta}{\delta} > 0 \).

Now assume that \( u_0 \in D(\phi^0) \). Then, applying Proposition 0.4.1 to (3.13), we get

\[ \phi_\lambda^\delta(u_\lambda(t)) \leq \phi_\lambda^0(u_0) \exp \left( \int_0^t b_r(\tau) \, d\tau \right) + \int_0^t g_1(\tau) \exp \left( \int_0^\tau b_r(\sigma) \, d\sigma \right) \, d\tau \]

\[ \leq \left\{ |\phi^0(u_0)| + \int_0^T g_1(\tau) \, d\tau \right\} \exp \left( \int_0^T b_r(\tau) \, d\tau \right) \equiv R_5 \]

for all \( t \in [0, T] \). From (3.14) with (3.8) we see

\[ |\phi_\lambda^\delta(u_\lambda(t))| \leq R_5 + 2 \alpha (M + R_2 + 1) \equiv R_6, \quad \forall t \in [0, T]. \]

Moreover, by (3.13) again,

\[ \left\{ 1 - a_r - (a_r + 1) \frac{\delta}{\delta} \right\} \int_0^T |u_\lambda^\delta(\tau)|^2 \, d\tau \]

\[ \leq \phi_\lambda^0(u_0) - \phi_\lambda^\delta(u_\lambda(T)) + \int_0^T b_r(\tau) \phi_\lambda^\delta(u_\lambda(\tau)) \, d\tau + \int_0^T g_1(\tau) \, d\tau \]

\[ \leq 2R_6 + R_6 \int_0^T b_r(\tau) \, d\tau + \int_0^T g_1(\tau) \, d\tau \equiv R_7, \]

so that

\[ u_\lambda^\delta \in L^2(0, T; H) \leq \left\{ 1 - a_r - (a_r + 1) \frac{\delta}{\delta} \right\}^{-1/2} \sqrt{R_7} \equiv R_8. \]

If we take \( N_3 = R_6 + R_8 \), then (3.11) and (3.12) hold.

In the case of \( u_0 \in D(\phi^0) \), we observe by multiplying the both sides of (3.13) by \( t > 0 \) that

\[ \left\{ 1 - a_r - (a_r + 1) \frac{\delta}{\delta} \right\} \sqrt{t} |u_\lambda^\delta(t)|^2 + \frac{d}{dt} \left\{ t \phi_\lambda^\delta(u_\lambda(t)) \right\} \]

\[ \leq b_r(t) \left\{ t \phi_\lambda^\delta(u_\lambda(t)) \right\} + t g_1(t) + \phi_\lambda^\delta(u_\lambda(t)). \]

Applying Proposition 0.4.1 again to this inequality, we have for all \( t \in [0, T] \)

\[ t \phi_\lambda^\delta(u_\lambda(t)) \leq \int_0^t \exp \left( \int_0^\tau b_r(\sigma) \, d\sigma \right) \left( \phi_\lambda^\delta(\phi_\lambda^\delta(s)) \right) \, ds \]

\[ \leq \exp \left( \int_0^T b_r(\tau) \, d\tau \right) \left( \int_0^T \phi_\lambda^\delta(\phi_\lambda^\delta(s)) \, ds + N_7 \right) \equiv R_9 \]

and hence

\[ |r \phi_\lambda^\delta(u_\lambda(t))| \leq R_9 + 2 T \alpha (M + R_2 + 1) \equiv R_{10}. \]

By the way,
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\[
\left\{ 1 - a_r - (a_r + 1) \delta \right\} \int_0^T \sqrt{t} \, u_\lambda^\prime(t)^2 \, dt \\
\leq - T \phi^\prime_\lambda(u_\lambda(T)) + \int_0^T b_r(t) t \phi^\prime_\lambda(u_\lambda(t)) \, dt + \int_0^T g_f(t) \phi^\prime_\lambda(u_\lambda(t)) \, dt + \int_0^T \phi^\prime_\lambda(u_\lambda(t)) \, dt \\
\leq R_{10} + R_{10} \int_0^T b_r(t) \, dt + T \int_0^T g_1(t) \, dt + N_I \equiv R_{11}.
\]

Therefore, if we take

\[
N_2 = R_{10} + \left\{ 1 - a_r - (a_r + 1) \delta \right\}^{-1/2} R_{11},
\]

then (3.9) and (3.10) are valid. Q.E.D.

Remark 1.3.1. As is easily checked, \( N_I = N_I (|f|_L^2(0, T; H), \|u_0\|) \), \( i = 1, 2 \), are able to be chosen so as to be bounded when \( f \) and \( u_0 \) vary in bounded sets in \( L^2(0, T; H) \) and in \( H \), respectively. Also, \( N_2 = N_2 (|f|_L^2(0, T; H), \|u_0\|, \phi^0(u_0)) \) is able to be chosen so as to be bounded, when \( f, u_0 \) and \( \phi^0(u_0) \) vary in bounded sets in \( L^2(0, T; H) \), in \( H \) and in \( \mathbb{R} \), respectively.

(B) The case that (h1) and (h2)' are satisfied.

Just as in the case (A) we have

\[
\|u_\lambda^\prime(t)\|^2 + \frac{d}{dt} \phi^\prime_\lambda(u_\lambda(t)) \\
\leq |f(t)| |u_\lambda^\prime(t)| + a |\Delta \phi^\prime_\lambda(u_\lambda(t))|^2 + b(t) |\phi^\prime_\lambda(u_\lambda(t))| + c(t) (1 + |u_\lambda(t)|)^2
\]

for a.e. \( t \in (0, T) \). Now, note the following inequalities: for any \( \delta > 0 \),

\[
|f(t)| |u_\lambda^\prime(t)| \leq \delta |u_\lambda^\prime(t)|^2 + \frac{1}{4\delta} |f(t)|^2,
\]

\[
a |\Delta \phi^\prime_\lambda(u_\lambda(t))|^2 \leq a (1 + \delta) |u_\lambda^\prime(t)|^2 + a (1 + \frac{1}{\delta}) |f(t)|^2,
\]

\[
|u_\lambda(t)| \leq |u_0| + \int_0^t |u_\lambda^\prime(s)| \, ds \leq |u_0| + \sqrt{t} \left( \int_0^t |u_\lambda^\prime(s)|^2 \, ds \right)^{1/2}
\]

\[
|u_\lambda(t)| \leq 2 |u_0|^2 + 2 \int_0^t |u_\lambda^\prime(s)|^2 \, ds
\]

and

\[
|\phi^\prime_\lambda(u_\lambda(t))| \\
\leq \phi^\prime_\lambda(u_\lambda(t)) + 2a (M + |u_\lambda(t)| + 1)
\]

\[
\leq \phi^\prime_\lambda(u_\lambda(t)) + |u_\lambda(t)|^2 + \alpha^2 + 2a (M + 1)
\]

\[
\leq \phi^\prime_\lambda(u_\lambda(t)) + 2t \int_0^t |u_\lambda^\prime(s)|^2 \, ds + 2 |u_0|^2 + \alpha^2 + 2a (M + 1).
\]

From (3.15) together with (3.16) – (3.20) we infer that

\[
(1 - \delta - a(1 + \delta)) |u_\lambda^\prime(t)|^2 + \frac{d}{dt} \phi^\prime_\lambda(u_\lambda(t)) \\
\leq 2t (b(t) + c(t)) \int_0^t |u_\lambda^\prime(s)|^2 \, ds + b(t) \phi^\prime_\lambda(u_\lambda(t)) + \left\{ \frac{1}{4\delta} + a (1 + \frac{1}{\delta}) \right\} |f(t)|^2 \\
+ (2 |u_0|^2 + 1) c(t) + \left\{ 2 |u_0|^2 + \alpha^2 + 2a (M + 1) \right\} b(t)
\]

for a.e. \( t \in (0, T) \), where \( \delta \) is chosen so that \( 1 - \delta - a(1 + \delta) > 0 \). If \( u_0 \) is in \( D(\phi^0) \), then by applying Proposition 0.4.1 to this inequality we obtain
\[
\left\{1 - \delta - a(1 + \delta)\right\} \int_0^t |u'_\lambda(s)|^2 \, ds + \phi'_\lambda(u_\lambda(t)) \leq R_{12}, \quad \forall t \in [0, T],
\]
for a certain positive constant \(R_{12}\) depending only on \(\|f\|_{L^2(0, T; H)}\), \(\|u_0\|\) and \(\phi'(u_0)\), so that the same type of inequalities as (3.11) and (3.12) are obtained. In the case of \(u_0 \in D(\phi')\), the inequalities of the forms (3.4), (3.5), (3.9) and (3.10) are obtained just as in the case (A). Thus Lemmas 1.3.1 and 1.3.2 remain valid in case (h1) and (h2)' are satisfied.

\section{Convergence of approximate solutions}

In this section we assume that \(\{h1\}, \{h2\}, \{h3\}\) or \(\{h1\}, \{h2\}'\) are satisfied and shall show that \(u_\lambda\) converges to the solution of \(CP(\phi'; f, u_0)\) as \(\lambda \downarrow 0\).

Let \(0 < \lambda \leq 1\), \(0 < \mu \leq 1\) and \(u_\lambda, u_\mu\) be approximate solutions as constructed in the previous section. Then, using properties of subdifferentials of Proposition 0.3.4, we have

\[
\frac{1}{2} \frac{d}{dt} |u_\lambda(t) - u_\mu(t)|^2 = - \langle \partial \Phi'_\lambda(u_\lambda(t)) - \partial \Phi'_\mu(u_\mu(t)), u_\lambda(t) - u_\mu(t) \rangle
= - \langle \partial \Phi'_\lambda(u_\lambda(t)) - \partial \Phi'_\mu(u_\mu(t)), \lambda \partial \Phi'_\lambda(u_\lambda(t)) - \mu \partial \Phi'_\mu(u_\mu(t)) \rangle
- \langle \partial \Phi'_\lambda(u_\lambda(t)) - \partial \Phi'_\mu(u_\mu(t)), \lambda \partial \Phi'_\lambda(u_\lambda(t)) - \mu \partial \Phi'_\mu(u_\mu(t)) \rangle
\leq 0.
\]

for a.e. \(t \in [0, T]\). Hence

\[
|u_\lambda(t) - u_\mu(t)|^2 + 2 \int_0^t \langle \partial \Phi'_\lambda(u_\lambda(\tau)), \lambda \partial \Phi'_\lambda(u_\lambda(\tau)) - \mu \partial \Phi'_\mu(u_\mu(\tau)) \rangle \, d\tau \leq 0
\]

for any \(t \in [0, T]\); in particular,

\[
(\partial \Phi'_\lambda(u_\lambda) - \partial \Phi'_\mu(u_\mu), \lambda \partial \Phi'_\lambda(u_\lambda) - \mu \partial \Phi'_\mu(u_\mu))_{L^2(0, T; H)} \leq 0.
\]

Now, we recall a convergence lemma.

\textbf{Lemma 1.4.1.} (cf. Crandall-Pazy [1; Lemma 2.4]) Let \(V\) be a Hilbert space. Suppose that \(\{v_\lambda\} \ subseteq V\) is a bounded set in \(V\) such that

\[
\langle v_\lambda - v_\mu, \lambda v_\lambda - \mu v_\mu \rangle \leq 0, \quad \forall \lambda, \mu \in (0, 1].
\]

Then \(|v_\lambda| \ subseteq \subset V\) is non-decreasing in \(\lambda\) and \(v_\lambda\) converges in \(V\) as \(\lambda \downarrow 0\).

Making use of this lemma, we show the convergence of \(\{u_\lambda\}\).

\textbf{(A)} The case of \(u_0 \in D(\phi')\). In this case, since \(\{\partial \Phi'_\lambda(u_\lambda)\} \ subseteq L^2(0, T; H)\) (cf. Lemma 1.3.2), it follows from (4.1) that \(u_\lambda \ rightharpoonup u\) in \(C([0, T]; H)\) as \(\lambda \downarrow 0\) for some \(u \in C([0, T]; H)\) with \(u(0) = u_0\). Moreover, by applying Lemma 1.4.1 for \(V = L^2(0, T; H)\) and \(v_\lambda = \partial \Phi'_{\lambda}(u_\lambda)\) we derive from (4.2) that

\[
\partial \Phi'_{\lambda}(u_\lambda) \ rightharpoonup u^* \quad \text{in} \ L^2(0, T; H) \quad \text{as} \ \lambda \downarrow 0
\]

for some \(u^* \ subseteq L^2(0, T; H)\). Also, by (7) and (8) of Proposition 0.3.5 we see that

\[
\liminf_{\lambda \downarrow 0} \phi'_{\lambda}(u_\lambda(t)) \geq \phi'(u(t)) \quad \text{for all} \ t \in [0, T],
\]
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\[(4.4) \quad \lim_{\lambda \downarrow 0} \phi_\lambda^t(u_\lambda(t)) = \phi^t(u(t)) \quad \text{for a.e. } t \in [0, T],
\]
\[u^*(t) \in \partial \phi^t(u(t)) \quad \text{for a.e. } t \in [0, T]
\]

and
\[u_\lambda \to u^* \quad \text{in } L^2(0, T; H) \quad \text{as } \lambda \downarrow 0,
\]

so \(u^*(t) + u^*(t) = f(t)\) for a.e. \(t \in [0, T]\). Thus \(u\) is a strong solution to \(CP(\phi^t; f, u_0)\) such that \(t \to \phi^t(u(t))\) is bounded on \([0, T]\) and \(u^* \in L^2(0, T; H)\). Besides, to show (1.4) we observe from the Corollary to Lemma 1.2.5 that

\[(4.5) \quad \limsup_{t \uparrow s} \phi^t(u(t)) \leq \phi^t(u(s)) \quad \text{for any } s \in [0, T],
\]

and from (2.8) of Lemma 1.2.5 that

\[(4.6) \quad \phi_\lambda^t(u_\lambda(t)) - \phi_\lambda^t(u_\lambda(s)) + \int_s^t \left(\phi_\lambda^\tau(u_\lambda'(\tau), u_\lambda'(\tau) - f(\tau))\right) d\tau
\]

\[\leq \int_s^t \left(\phi_\lambda^\tau(u_\lambda'(\tau) - f(\tau)) + c_\lambda^t(\phi_\lambda^\tau(u_\lambda'(\tau)))\right) d\tau
\]

for any \(s, t \in [0, T]\) with \(s \leq t\). On account of (4.3), (4.4) and (4.5), letting \(\lambda \downarrow 0\) in (4.6) yields (1.4) for every \(s, t \in [0, T]\) with \(s \leq t\).

**Remark 1.4.1.** As is easily seen from our proof, the strong solution \(u\) obtained above satisfies the following estimates:

\[|u(t)| \leq N_1, \quad \forall t \in [0, T],
\]

\[\int_0^T |\phi^t(u(t))| dt \leq N_1,
\]

\[|u'|_{L^2(0, T; H)} \leq N_2,
\]

\[|\phi^t(u(t))| \leq N_2, \quad \forall t \in [0, T],
\]

where \(N_1\) and \(N_2\) are as in Lemmas 1.3.1 and 1.3.2.

**B.** The case of \(u_0 \in D(\phi^0)\). First choose a sequence \(\{u_{0,n}\} \subset D(\phi^0)\) such that \(u_{0,n} \to u_0\) in \(H\). Then, as was shown in the case (A), for each \(n\) there is a strong solution \(u_n\) to \(CP(\phi^t; f, u_{0,n})\). By Theorem 1.1.1 we have

\[|u_n(t) - u_m(t)| \leq |u_{0,n} - u_{0,m}|
\]

for all \(n, m\) and \(t \in [0, T]\), which implies that \(u_n \to u\) in \(C([0, T]; H)\) for some \(u \in C([0, T]; H)\) with \(u(0) = u_0\). By definition this limit \(u\) is a weak solution to \(CP(\phi^t; f, u_0)\). On account of estimations in Lemmas 1.3.1 and 1.3.2, \(\{\sqrt{T} u_n\} \) is bounded in \(L^2(0, T; H)\), \(\{\int_0^T |\phi_t^u(u_n(t))| dt\} \) is bounded and \(t \to \sqrt{T} u_n(t)\) is uniformly bounded on \([0, T]\), so that \(\sqrt{T} u' \in L^2(0, T; H)\), \(t \to \phi^t(u(t))\) is integrable on \([0, T]\) and \(t \to t \phi^t(u(t))\) is bounded on \([0, T]\). Also, the fact that (1.4) holds for every \(s, t \in [0, T]\) with \(s \leq t\) is similarly proved as in the case (A).

**Remark 1.4.2.** We note that the weak solution \(u\) obtained above satisfies the following estimates:

\[|u(t)| \leq N_1, \quad \forall t \in [0, T],
\]

\[\int_0^T |\phi^t(u(t))| dt \leq N_1,
\]

\[\int_0^T |\phi^t(u(t))| dt \leq N_1,
\]

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\[ |\sqrt{t} u'(t)|^2_{L^2(0, T; H)} \leq N_2, \]
\[ |t \phi^t(u(t))| \leq N_2, \quad t \in (0, T), \]

where \( N_1 \) and \( N_2 \) are as in Lemmas 1.3.1 and 1.3.2.

Thus the proof of Theorem 1.1.2 is complete.

Now that existence of strong solutions to \((E)\) was shown under the hypotheses \((h1)\) and \((h2)'\), we see that \((h3)\) follows from them, so from now on we have only to consider the case that \((h1)\), \((h2)\) and \((h3)\) are fulfilled.

**Proof** of Theorem 1.1.3: Given \( u_0 \in D(\phi^0) \) and \( f \in L^1(0, T; H) \), choose sequences \( \{u_{0,n}\} \) in \( D(\phi^0) \) and \( \{f_n\} \) in \( L^2(0, T; H) \) so that \( u_{0,n} \to u_0 \) in \( H \) and \( f_n \to f \) in \( L^1(0, T; H) \). By virtue of Theorem 1.1.2 there exists a strong solution \( u_n \) to \( CP(\phi^t; f_n, u_{0,n}) \) for each \( n \) such that \( u_n \to u \) in \( L^2(0, T; H) \) and \( t \to \phi^t(u_n(t)) \) is bounded on \( [0, T) \). Using Theorem 1.1.1, we easily see that \( \{u_n\} \) is a Cauchy sequence in \( C([0, T]; H) \), and that the limit \( u \) is a unique weak solution to \( CP(\phi^t; f, u_0) \). Since

\[ (f_n - u_n', v - u_n)_{L^2(0, T; H)} \leq \int_0^T \phi^t(v(t)) dt - \int_0^T \phi^t(u_n(t)) dt \]

for all \( v \in L^2(0, T; H) \) such that \( t \to \phi^t(v(t)) \) is integrable on \([0, T] \), we have by taking \( u_j \) as \( v \) and by integration by parts

\[ \int_0^T \phi^t(u_n(t)) dt \leq \int_0^T \phi^t(u_j(t)) dt + \int_0^T \frac{1}{2} |u_j - u_n|_{L^2(0, T; H)}^2 dt + \frac{1}{2} |u_{0,n} - u_{0,j}|_{L^2(0, T; H)}^2 \]

Hence the lower semicontinuity of \( \phi^t \) implies that \( t \to \phi^t(u(t)) \) is integrable on \([0, T) \).

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**§1.5. A sufficient condition for \((h1)\), \((h2)\) and \((h3)\)**

In this section we give a sufficient condition for \((h1)\), \((h2)\) and \((h3)\) so as to be convenient in applying Theorem 1.1.2 to various concrete problems.

Let \( \phi^t \) be a proper l.s.c. convex function on \( H \) for each \( t \in [0, T] \), \( 0 < T < \infty \), and consider the following type of condition for \( I_1 \leq \xi \leq \infty \):

\((H)_{\xi}\) For each \( r > 0 \) there are \( \alpha_r \in W^{1,1}(0, T) \) and \( \beta_r \in W^{1,1}(0, T) \) with the following property: for each \( s, t \in [0, T] \) with \( s \leq t \) and \( z \in D(\phi^s) \) with \( |z| \leq r \) there exists \( z_t \in D(\phi^t) \) such that

\[ |z_t - z| \leq |z_t - z_t| \leq \alpha_r(t - t) (1 + |\phi^s(z)|)^{1/2} \]

and

\[ \phi^t(z_t) - \phi^s(z) \leq |z_t - z_t| \leq |\beta_r(t - t) - \beta_r(s)| (1 + |\phi^s(z)|) \].

**Remark 1.5.1** If \( I_1 \leq \xi_2 \leq \infty \), then \((H)_{\xi_2}\) implies \((H)_{\xi_1}\).

Our theorem is stated as follows:

**Theorem 1.5.1** \((H)_{\xi_2}\) implies \((h1)\), \((h2)\) and \((h3)\).

Before proving this theorem we prepare some lemmas.

**Lemma 1.5.1.** (cf. Attouch-Damlamian [1; Lemma 1]) Under \((H)_{\xi_1}\) there is a constant \( \alpha > 0 \) satisfying

---
(5.1) \[ \phi^t(z) + a|z| + a \geq 0, \quad \forall t \in [0, T], \quad \forall z \in H; \]

and hence (h1) is valid.

**Proof.** By our assumption we can easily find a set \( \{ z_t \in H; 0 \leq t \leq T \} \) and \( r_0 > 0 \) such that \( z_t \in B_{r_0} \) (\( \{ w \in H; |w| \leq r_0 \} \)) and \( \phi^t(z_t) \leq r_0 \) for every \( t \in [0, T] \). Now, set \( r = r_0 + 1 \) and choose a partition \( \{ 0 = s_0 < s_1 < \ldots < s_n = T \} \) of \( [0, T] \) so that

\[
|\alpha_r(s_i) - \alpha_r(s_{i-1})| \leq \frac{1}{2}, \quad |\beta_r(s_i) - \beta_r(s_{i-1})| \leq \frac{1}{2}
\]

for \( i = 1, 2, \ldots, n \). Since \( \phi^t \) is proper, 1.s.c. and convex on \( H \) for any \( t \in [0, T] \), there are positive constants \( C_1 \) and \( C_2 \) having the property:

\[
\phi^t_z(z) \geq -C_1 |z| - C_2, \quad \forall z \in H, \quad \forall i = 1, 2, \ldots, n.
\]

Using our assumption again, for each \( i \in [s_{i-1}, s_i] \) and each \( z \in B_r \), we find \( \bar{z} \in D(\phi^t_z) \) such that

\[
|\bar{z} - z| \leq \frac{1}{2} \sqrt{1 + |\phi^t(z)|}
\]

and

\[
\phi^t(z) \geq \phi^t_z(\bar{z}) - \frac{1}{2}(1 + |\phi^t(z)|).
\]

Since

\[
|\bar{z} - z| \leq \frac{1}{2} + \frac{1}{2} |\phi^t(z)|^{1/2} \leq \frac{1}{2} + \frac{C_1}{4} + \frac{1}{4C_1} |\phi^t(z)|
\]

and

\[
\phi^t(z) + \frac{1}{2} |\phi^t(z)| \geq -C_1 |\bar{z}| - C_2 - \frac{1}{2} \geq -C_1 |z| - C_2 - C_1 |\bar{z} - z| - \frac{1}{2}, \quad \forall z \in B_r.
\]

we obtain that

\[
\phi^t(z) + \frac{3}{4} |\phi^t(z)| \geq -C_1 |z| - C_2 - \frac{C_1}{2} - \frac{C_2}{4} - \frac{1}{2}, \quad \forall z \in B_r.
\]

Thus there is a positive constant \( C_3 \) such that

(5.2) \[ \phi^t(z) + C_3 |z| + C_3 \geq 0, \quad \forall t \in [0, T], \quad \forall z \in B_r. \]

Next, let \( z \) be any element of \( H \) such that \( |z| > r \) and put

\[
\theta_t = \frac{1}{|z - z_t|}, \quad x_t = \theta_t z + (1 - \theta_t) z_t, \quad t \in [0, T].
\]

Then \( |x_t| \leq |x_t - z_t| + |z_t| \leq 1 + r_0 = r \), so that we have by (5.2)

\[
\theta_t \phi^t(z) + (1 - \theta_t) \phi^t(z_t) + C_3 |x_t| + C_3 \geq \phi^t(x_t) + C_3 |x_t| + C_3 \geq 0.
\]

Therefore

\[
\phi^t(z) + \theta_t^{-1} \left\{ (1 - \theta_t) \phi^t(z_t) + C_3 |x_t| + C_3 \right\} \geq 0,
\]

from which we get

\[
\phi^t(z) + |z| + r_0 \left\{ r_0 + C_3 r + C_3 \right\} \geq 0,
\]

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because \( \phi_t^{-1} = |z - z_t| \leq |z| + r_0 \). This inequality together with (5.2) implies that (5.1) holds for some \( \alpha > 0 \).

Q.E.D.

Lemma 1.5.2. Assume (H)\(_1\). If \( z_n \in D(\phi_t^n), z_n \rightarrow z \) in \( H \) and \( t_n \uparrow t \), then

\[
\phi_t^t(z) \leq \liminf_{n \to \infty} \phi_t^n(z_n).
\]

Proof. Take a number \( r > 0 \) so that \( |z_n| \leq r \) for all \( n \). By assumption, for each \( n \) there exists \( \tilde{z}_n \in D(\phi_{t_n}^t) \) such that

\[
|\tilde{z}_n - z_n| \leq |t_n - t| \alpha_r(t_n - t)/2
\]

and

\[
\phi_t^t(\tilde{z}_n) \leq \phi_t^n(z_n) + |t_n - t|/2
\]

Hence

\[
\phi_t^t(\tilde{z}_n) \leq \left\{ \begin{array}{ll}
(1 + |t_n - t|) \phi_t^n(z_n) + |t_n - t|/2 & \text{if } \phi_t^n(z_n) > 0, \\
(1 - |t_n - t|) \phi_t^n(z_n) + |t_n - t|/2 & \text{if } \phi_t^n(z_n) < 0.
\end{array} \right.
\]

It suffices to prove (5.3) in case \( \phi_t^n(z_n) \) is bounded above. In such a case \( \| \phi_t^n(z_n) \| \) is bounded by Lemma 1.5.1. Hence \( \tilde{z}_n \to z \) in \( H \), so that

\[
\phi_t^t(z) \leq \liminf_{n \to \infty} \phi_t^n(\tilde{z}_n) \leq \liminf_{n \to \infty} \phi_t^n(z_n)
\]

because of the lower semicontinuity of \( \phi_t^t \).

Q.E.D.

Lemma 1.5.3. Assume (H)\(_1\) with \( 1 < q < \infty \). Let \( 0 \leq T_0 < T \), \( z \in D(\phi_{T_0}^t) \), \( r \) and \( M \) be numbers satisfying \( r > |z| + 1 \) and \( M \geq |\phi_{T_0}^t(z)| + \alpha r + \alpha + 1 \) (\( \alpha \) is as in Lemma 1.5.1), and \( T_j \) be such that

\[
\{1 + M \exp \left( \int_0^T |\beta_t^r| \, dt \right) \int_0^T \alpha_r \, dt \leq 1.
\]

Then there exists \( h \in C([T_0, T_j]; H) \), with \( h^t \in L^q(T_0, T_j; H) \) such that

\[
\begin{align*}
h(T_0) &= z, \\
\limsup_{t \to T_0} \phi_t^t(h(t)) &\leq \phi_{T_0}^t(z), \\
|h(t)| &\leq r, \quad \forall t \in [T_0, T_j], \\
|\phi_t^t(h(t))| &\leq M + M \exp \left( \int_0^T |\beta_t^r| \, dt \right) \int_0^T \alpha_r \, dt, \quad \forall t \in [T_0, T_j], \\
|h'(t)| &\leq \left[ 1 + M \exp \left( \int_0^T |\beta_t^r| \, dt \right) \int_0^T \alpha_r \, dt \right] \text{ for } a.e. \ t \in [T_0, T_j].
\end{align*}
\]

Proof. For simplicity assume \( T_0 = 0 \). Let \( \{0 = t_0 < t_1 < \cdots < t_N = T_j \} \) be any partition of \( [0, T_j] \). By induction we are going to build a finite sequence \( \{z_k \in B_r; k = 0, 1, 2, \ldots, N \} \) with \( z_0 = z \) such that

\[
|z_k - z_{k-1}| \leq \left[ 1 + M \exp \left( \int_0^{T_k-1} |\beta_t^r| \, dt \right) \right] \int_{T_k-1}^{T_k} \alpha_r \, dt,
\]

\[
\phi_{T_k-1}^T(z_k) \leq \phi_{T_{k-1}}^T(z_{k-1}) + M \exp \left( \int_0^{T_{k-1}} |\beta_t^r| \, dt \right) \int_0^{T_{k-1}} \beta_t \, dt
\]

and

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(5.6) \[ 1 + |\phi^t (z_k)| \leq \text{Mexp} \left( \int_0^T \beta^r \, d\tau \right) \]

for \( k = 1, 2, \ldots, N \). Now suppose that \( z_k, k = 1, 2, \ldots, i - 1 (<N) \) were already defined so as to fulfill (5.4), (5.5) and (5.6). Then, using our assumption, we choose \( z_i \in D(\phi^t) \) so that

\[
|z_i - z_{i-1}| \leq \alpha_i (t_i) - \alpha_{i-1} (t_{i-1}) / (1 + |\phi^{t_i} (z_{i-1})|)^{1/2} \\
\leq \int_{t_{i-1}}^{t_i} \alpha^r \, d\tau + (1 + |\phi^{t_i} (z_{i-1})|) \int_{t_{i-1}}^{t_i} \beta^r \, d\tau
\]

and

\[
\phi^t (z_i) \leq \phi^{t_i} (z_{i-1}) / (1 + |\phi^{t_i} (z_{i-1})|) \int_{t_{i-1}}^{t_i} \beta^r \, d\tau.
\]

Since

\[
1 + |\phi^{t_i} (z_{i-1})| \leq \text{Mexp} \left( \int_0^{t_{i-1}} \beta^r \, d\tau \right)
\]

by the hypothesis of induction, we see that

\[
|z_i - z_{i-1}| \leq \left\{ 1 + \text{Mexp} \left( \int_0^{t_{i-1}} \beta^r \, d\tau \right) \right\}^{1/2} \int_{t_{i-1}}^{t_i} \alpha^r \, d\tau
\]

and

\[
\phi^t (z_i) \leq \phi^{t_i} (z_{i-1}) + \text{Mexp} \left( \int_0^{t_{i-1}} \beta^r \, d\tau \right) \int_{t_{i-1}}^{t_i} \beta^r \, d\tau.
\]

Moreover

\[
|z_i| \leq |z_0| + \sum_{k=1}^{i} |z_k - z_{k-1}|
\]

\[
\leq |z_0| + \left\{ 1 + \text{Mexp} \left( \int_0^T \beta^r \, d\tau \right) \right\} \int_0^{t_i} \alpha^r \, d\tau
\]

\[
\leq |z_0| + 1 (<r)
\]

and by Lemma 1.5.1

\[
1 + |\phi^{t_i} (z_i)|
\]

\[
\leq \left\{ \alpha r + \alpha + 1 \right\},
\]

if \( \phi^{t_i} (z_{i-1}) < 0 \),

\[
(1 + |\phi^{t_i} (z_{i-1})|) \left( 1 + \int_{t_{i-1}}^{t_i} \beta^r \, d\tau \right),
\]

if \( \phi^{t_i} (z_{i-1}) \geq 0 \).

Thus the required sequence \( \{ z_k \in B_r; k = 0, 1, \ldots, N \} \) can be built.

Next, for each positive integer \( n \), consider the partition \( \Delta_n = \{ 0 = t^n_0 < t^n_1 < \ldots < t^n_n = T_1 \} \) with \( \epsilon_n = T^n / n \) and \( t^n_k = \epsilon_n k (k = 1, 2, \ldots, n) \), and such a sequence \( \{ z^n_k \in B_r; k = 0, 1, \ldots, n \} \) as above corresponding to the partition \( \Delta_n \). Put \( v_n (0) = z_0 \) and

\[
v_n(t) = z^n_k - \frac{z^n_k - z^n_{k-1}}{\epsilon_n} \quad \text{if} \quad t \in (t^n_{k-1}, t^n_k).
\]

Then it follows from (5.4) that

\[
\int_s^t |\nabla v_n (\tau)|^q \, d\tau \leq \epsilon_n^{1-q} \sum_{k=i+1}^{j} |z^n_k - z^n_{k-1}|^q
\]

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\[ 1 + \text{Mexp} \left( \int_0^T |\beta_r| \, dt \right) \leq \int_0^T |\alpha_r| \, dt \]

for any \( j \in (t^n_j - \delta, t^n_j + \delta) \) and \( t \in (t^n_{j-1}, t^n_j) \) with \( s < t \), and

\[ |v_n(t) - v_n(s)| \leq M'(|t - s| + \epsilon_n)^{(\alpha - 1)}/\alpha, \quad \forall s, t \in [0, T_j]. \]

where \( M' \) is a positive constant. Hence there exists a subsequence \( \{v_{n'}\} \) of \( \{v_n\} \) and a function \( h \in C([0, T_1]; H) \) with \( h(0) = z \) and \( h' \in L^q(0, T_1; H) \) such that

\[ v_{n'}(t) \to h(t), \quad \forall t \in [0, T_1], \]

\[ \nabla v_{n'} \to h', \quad \text{in} \quad L^q(0, T_1; H) \]

as \( n' \to \infty \). Clearly, \( |h(t)| \leq r \) and

\[ \int_0^t |h(t)|^q \, dt \leq \frac{1 + \text{Mexp} \left( \int_0^T |\beta_{n'}| \, dt \right)}{\alpha} \int_0^t |\alpha_r| \, dt, \quad \forall s, t \in [0, T_j], s \leq t, \]

namely

\[ |h^*(t)| \leq \frac{1 + \text{Mexp} \left( \int_0^T |\beta_{n'}| \, dt \right)}{\alpha} \int_0^t |\alpha_r| \, dt \quad \text{for a.e. } t \in [0, T_j]. \]

Besides, noting that \( z_{n'}^* \to h(t) \) in \( H \) as \( n' \to \infty \) if \( \epsilon_{n'} k \uparrow t \) as \( n' \to \infty \), we have by Lemma 1.5.2.

\[ \phi^t(h(t)) \leq \liminf_{\epsilon_{n'} k \uparrow t} \phi^t(z_{n'}^*) \]

and hence by the property (5.5)

\[ \phi^t(h(t)) \leq \phi^t(z) + \text{Mexp} \left( \int_0^T |\beta_r| \, dt \right) \int_0^t |\beta_r| \, dt \]

This implies

\[ |\phi^t(h(t))| \leq \text{M} \left\{ 1 + \text{exp} \left( \int_0^T |\beta_r| \, dt \right) \int_0^t |\beta_r| \, dt \right\} \]

as well as

\[ \limsup_{t \to 0} \phi^t(h(t)) \leq \phi^0(z). \]

Thus this function \( h \) is a desired one.

**Lemma 1.5.4** Assume \((H)_1\). Then there is a positive constant \( K \) such that

\[ |J^t_\lambda z| \leq K + |z|, \quad |\partial J^t_\lambda z| \leq \frac{K + 2|z|}{\lambda} \]

for all \( t \in [0, T] \), \( \lambda \in (0, 1) \) and \( z \in H \).

**Proof.** By our assumption there is a set \( \{z_t \in H; 0 \leq t \leq T\} \) with a constant \( r_0 > 0 \) such that

\[ |z_t| \leq r_0, \quad |\phi^t(z_t)| \leq r_0, \quad \forall t \in [0, T]. \]

Therefore, on account of Lemma 1.5.1,

\[ r_0 \geq \phi^t(z_t) \geq \phi^t_0(z_t) = \frac{1}{2} |z_t - J^t_\lambda z_t|^2 + \phi^t(J^t_\lambda z_t) \geq \frac{1}{2} |z_t - J^t_\lambda z_t|^2 - \alpha |J^t_\lambda z_t| - \alpha \]

for all \( t \in [0, T] \). This shows that there is \( r_1 > 0 \) satisfying

\[ |J^t_\lambda z_t| \leq r_1, \quad \forall t \in [0, T], \]

so that for given \( z_0 \in H \) and \( t \in [0, T] \)

---
\[ |J_T^t z_0| \leq |J_T^t z_0 - J_T^t z_t| + |J_T^t z_t| \leq |z_0 - z_t| + r_t \leq |z_0| + r_0 + r_t. \]

Therefore we get the lemma just as Lemma 1.2.1. Q.E.D.

Proof of Theorem 1.5.1: Condition (h1) was already verified in Lemma 1.5.1. Now we show (h3). For this purpose we take a set \( \{ z_t \in H; 0 \leq t \leq T \} \) and a constant \( r_0 > 0 \) such that \( |z_t| < r_0 \) and \( |\phi^t(z_t)| < r_0 \) for all \( t \in [0, T] \). Putting \( r = r_0 + 1 \) and \( M = r_0 + \alpha r + \alpha + 1 \), we choose a partition \( \{ 0 = T_0 < T_1 < \ldots < T_N = T \} \) so that

\[ \left( 1 + M \exp \left( \int_0^T |\beta_t| \, dt \right) \right) \right) \right) \int_0^{T_k} |z_t| \, dt \leq 1 \]

for every \( k = 1, 2, \ldots, N \). Then, according to Lemma 1.5.3, there exists \( h_k \) in \( C([T_{k-1}, T_k]; H) \) for each \( k \) such that \( h_k(T_{k-1}) = z_{T_{k-1}} \), \( h_k \in L^2(T_{k-1}, T_k; H) \) and \( t \to \phi^t(h_k(t)) \) is bounded on \( [T_{k-1}, T_k] \). Making use of this family \( \{ h_k; k = 1, 2, \ldots, N \} \) we can easily build an \( H \)-valued function \( h \) on \( [0, T] \) satisfying the required properties in (h3).

Next, according to Lemma 1.5.4 for each \( r > 0 \) there is \( r_j > 0 \) such that

\[ |J_{r_j}^t z| \leq r_j, \quad \forall t \in [0, T], \quad \forall \lambda \in (0, 1), \quad \forall z \in H \text{ with } |z| \leq r; \]

in fact we can take \( r_j = K + r \) with \( K \) of Lemma 1.5.4. Therefore, by assumption for each \( z \in H \) with \( |z| \leq r \), \( \lambda \in (0, 1) \) and \( s, t \in [0, T] \) with \( s < t \) we find \( \tilde{z} \) in \( D(\phi^t) \) such that

\[ |\tilde{z} - J_{r_j}^t z| \leq |\alpha_{r_j}(t) - \alpha_{r_j}(s)| (1 + |\phi^s(J_{r_j}^t z)|)^{1/2} \]

and

\[ \phi^t(\tilde{z}) \leq |\beta_{r_j}(t) - \beta_{r_j}(s)| (1 + |\phi^s(J_{r_j}^t z)|)^{1/2}. \]

Hence, using (2) of Proposition 0.3.5, we see

\[ \phi^t(z) \leq \phi^t(z) + \frac{1}{2\lambda} \int \frac{1}{|z - J_{r_j}^t z|^2 - \phi^s(J_{r_j}^t z)} \]

\[ \leq |z - J_{r_j}^t z| + \frac{1}{2\lambda} \int \frac{1}{|z - J_{r_j}^t z|^2} \phi^s(J_{r_j}^t z)^2 \]

\[ \leq |z - J_{r_j}^t z| + \frac{1}{2\lambda} \int \frac{1}{|z - J_{r_j}^t z|^2} + \phi^s(J_{r_j}^t z)^2 \]

\[ \leq |\alpha_{r_j}(t) - \alpha_{r_j}(s)| (1 + |\phi^s(J_{r_j}^t z)|)^{1/2} \]

\[ + \frac{1}{2\lambda} |\alpha_{r_j}(t) - \alpha_{r_j}(s)|^2 (1 + |\phi^s(J_{r_j}^t z)|)^{1/2}. \]

Since \( s \to |\phi^s(J_{r_j}^t z)| \) is bounded on \( [0, T] \) by Lemma 1.5.4 as well as \( s \to |\phi^s(J_{r_j}^t z)| \), we see that \( \phi^t(z) \) is differentiable at a.e. \( s \in [0, T] \) and its derivative is integrable on \( [0, T] \) and satisfies (i) of (h2). Moreover we obtain

\[ \frac{d}{ds} \phi^s(z) \leq |\alpha_{r_j}(s)| (1 + |\phi^s(J_{r_j}^t z)|)^{1/2} + |\beta_{r_j}(s) (1 + |\phi^s(J_{r_j}^t z)|)^{1/2} \]

\[ \leq \delta |\phi^s(z)|^2 + \left( \frac{1}{2\delta} |\alpha_{r_j}(s)|^2 + |\beta_{r_j}(s)|^2 \right) \left( \frac{1}{2\delta} |\alpha_{r_j}(s)|^2 + |\beta_{r_j}(s)|^2 \right) \]

for a.e. \( s \in [0, T] \), where \( \delta \) is any positive number. Noting (cf. Lemma 1.5.1) that

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\begin{align*}
|\phi^L(x,z)| \leq \phi^L(J^L_x,z) + 2\alpha (|J^L_x| + 1) \leq |\phi^L(x,z)| + 2\alpha (r_1 + 1),
\end{align*}

we get from (5.7)
\begin{align*}
\frac{d}{ds} \phi^L(x) \leq \delta |\phi^L(x)|^2 + \left\{ \frac{1}{28} |\alpha^L_{\gamma_1}(s)|^2 + |\beta^L_{\eta_1}(s)| \right\} |\phi^L(x)| + \\
\left\{ \frac{1}{28} |\alpha^L_{\gamma_1}(s)|^2 + |\beta^L_{\eta_1}(s)| \right\} \{ 2\alpha (r_1 + 1) + 1 \}
\end{align*}

for a.e. \( s \in [0, T] \). Thus (ii) of (h2) holds.

Chapter 2

Evolution Equation \( u'(t) + \partial \phi^f(u(t)) \ni f(t) \):

Some Properties of Solutions

Throughout this chapter let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). For simplicity for the Hilbert space \( L^2(0, T; H) \) we denote by \( \langle \cdot, \cdot \rangle \) the inner product and by \( \| \cdot \| \) the norm.

§2.1. Regularity of solutions

Let \( 0 < T < \infty \) and \( \{ \phi^f; 0 \leq t \leq T \} \) be a family of proper l.s.c. convex functions on \( H \). In this section we investigate some regularity properties of solutions to the equation

\begin{align*}
(E) \quad u'(t) + \partial \phi^f(u(t)) \ni f(t), \quad 0 < t < T,
\end{align*}

under hypotheses (h1), (h3) (see §1.1) and the following (h*2):

(h*2) \begin{enumerate}
\item[(i)] For each \( x \in H \) and \( \lambda \in [0, 1] \) the function \( t \rightarrow \phi^L_\lambda(x) \) is absolutely continuous on \( [0, T] \).
\item[(ii)] For each \( r \geq 0 \) there are a number \( a^*_r \) in \( (0, 1) \) and non-negative \( b^*_r, c^*_r \) in \( L^1(0, T) \) such that
\begin{align*}
|\frac{d}{dt} \phi^L_\lambda(x)| \leq a^*_r |\partial \phi^L_\lambda(x)|^2 + b^*_r(t) |\phi^L_\lambda(x)| + c^*_r(t) \quad \text{for a.e.} \quad t \in [0, T],
\end{align*}
\end{enumerate}

whenever \( x \in H \) with \( |x| \leq r \) and \( \lambda \in (0, 1) \).

Clearly (h*2) implies (h2).

Theorem 2.1.1. Assume (h1), (h*2) and (h3), and let \( u \) be a unique weak solution to \( CP(\phi^f; f, u_0) \) with \( u_0 \in D(\phi^0) \) and \( f \in L^2(0, T; H) \). Then the function \( t \rightarrow t\phi^f(u(t)) \) is absolutely continuous on \( (0, T] \). If in particular \( u_0 \) is in \( D(\phi^0) \), then \( t \rightarrow \phi^f(u(t)) \) is absolutely continuous on \( [0, T] \).

Theorem 2.1.2. Assume (h1), (h*2) and (h3). Further suppose that as \( b^*_r \) in (ii) of (h*2) a function of \( L^1(0, T) \) can be taken for each \( r \geq 0 \). Let \( u \) be a unique weak solution to \( CP(\phi^f; f, u_0) \) with \( u_0 \in D(\phi^0) \) and \( f \in L^2(0, T; H) \). Then \( u' \in L^2(0, T; H) \) implies \( u_0 \in D(\phi^0) \).

Remark 2.1.1. In the time-independent case of \( \phi^f \) such a kind of investigation was made by Brézis [4].

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With the help of the following lemma we prove the above theorems.

Lemma 2.1. Assume (h1), (h*2) and (h3). Let \([T_0, T_1] \subset [0, T], 0 < \lambda \leq 1\) and \(u \in W^{1,1}(T_0, T_1; H)\). Then we have:

1. \(t \to \phi^f_\lambda(u(t))\) is absolutely continuous on \([T_0, T_1]\).
2. With \(a^*_\tau \in [0, 1]\) and \(b^*_\tau, c^*_\tau \in L^1(0, T)\) in (ii) of (h*2) for \(r \geq \sup \{ |u(t)|; T_0 \leq t \leq T_1 \}\) the following inequalities hold:

\[
\left| \frac{d}{dt} \phi^f_\lambda(u(t)) - (u'(t), \partial \phi^f_\lambda(u(t))) \right| \leq a^*_\tau |\partial \phi^f_\lambda(u(t))|^2 + b^*_\tau(t) |\phi^f_\lambda(u(t))| + c^*_\tau(t)
\]

and

\[
\left| \frac{d}{dt} \{ t \phi^f_\lambda(u(t)) \} - t (u'(t), \partial \phi^f_\lambda(u(t))) \right| \leq a^*_\tau t |\partial \phi^f_\lambda(u(t))|^2 + (1 + c^*_\tau(t)) |\phi^f_\lambda(u(t))| + t c^*_\tau(t)
\]

for a.e. \(t \in [T_0, T_1]\).

Proof. As was seen in the proof of Lemma 1.2.5, for every \(s, t \in [T_0, T_1]\) with \(s \leq t\) we have

\[
\phi^f_\lambda(u(t)) - \phi^f_\lambda(u(s)) = (u(t) - u(s), \partial \phi^f_\lambda(u(t)))
\]

\[
\leq \int_s^t \left\{ a^*_\tau |\partial \phi^f_\lambda(u(s))|^2 + b^*_\tau(\tau) |\phi^f_\lambda(u(s))| + c^*_\tau(\tau) \right\} d\tau
\]

and similarly

\[
\phi^f_\lambda(u(t)) - \phi^f_\lambda(u(s)) = (u(t) - u(s), \partial \phi^f_\lambda(u(s)))
\]

\[
\leq - \int_s^t \left\{ a^*_\tau |\partial \phi^f_\lambda(u(t))|^2 + b^*_\tau(\tau) |\phi^f_\lambda(u(t))| + c^*_\tau(\tau) \right\} d\tau.
\]

Therefore, taking into account the facts of Lemma 1.2.1, we see that \(t \to \phi^f_\lambda(u(t))\) is absolutely continuous on \([T_0, T_1]\). Moreover, just as (2.8) of Lemma 1.2.5, we get (1.1). By multiplying both sides of (1.1) by \(t > 0\) we obtain (1.2).

Q.E.D.

Corollary. Assume (h1), (h*2) and (h3), and let \(u \) be a unique weak solution to \(CP(\phi^f; f, u_0)\) with \(u_0 \in \overline{D(\phi^0)}\) and \(f \in L^2(0, T; H)\). Then we have

\[
|\phi^f(t) - \phi^f(s)| \leq \int_s^t \left\{ |u'(\tau)| |f(\tau) - u'(\tau)| + a^*_\tau |f(\tau) - u'(\tau)|^2 + b^*_\tau(\tau) |\phi^f(\tau)| + c^*_\tau(\tau) \right\} d\tau
\]

and

\[
|t \phi^f(t) - s \phi^f(s)| \leq \int_s^t \left\{ t |u'(\tau)| |f(\tau) - u'(\tau)| + t a^*_\tau |f(\tau) - u'(\tau)|^2 + (1 + t b^*_\tau(\tau)) |\phi^f(\tau)| + t c^*_\tau(\tau) \right\} d\tau
\]

for any \(0 \leq s \leq t \leq T\) and \(r \geq \sup \{ |u(t)|; 0 \leq t \leq T \}\). If in particular \(u_0 \) is in \(D(\phi^0)\), then (1.3) holds for any \(0 \leq s \leq t \leq T\).

In fact, using (5) and (6) of Proposition 0.3.5, we derive (1.3) and (1.4) easily from (1.1) and (1.2) respectively.
Proof of Theorem 2.1.1: It follows directly from (1.4) that \( t \to t \phi^f(u(t)) \) is absolutely continuous on \( [0, T] \), because \( \sqrt{t} u^* \in L^2(0, T; H) \) and \( t \to t \phi^f(u(t)) \) is bounded on \( [0, T] \) by Theorem 1.1.2. In case \( u_0 \in D(\phi^0) \), the absolute continuity of \( t \to \phi^f(u(t)) \) on \( [0, T] \) is similarly derived from (1.3).

Q.E.D.

Proof of Theorem 2.1.2: By (2) of Lemma 2.1.1,
\[
|\phi^0(u_0)| \leq \int_0^T |u(t)|^2 + \frac{1}{\lambda} \int_0^T |\phi^0(u(t))|^2 + c_1^* |\phi^0(u(t))| + c_2^* \int_0^T d\tau,
\]
for \( 0 < t < T \) and \( 0 < \lambda \leq 1 \). Since \( |\phi^0(u(t))| \leq |u(t)| \leq f(t) \) and \( |\phi^0(u(t))| \leq c_1 + c_2 f(t) \) for some constants \( c_1, c_2 \) (cf. (5), (6) of Proposition 0.3.5), the above inequality implies that \( \{ \phi^0(u_0), 0 < \lambda \leq 1 \} \) is bounded, so \( u_0 \in D(\phi^0) \) (cf. (6) of Proposition 0.3.5).

§2.2. Boundedness of solutions

Let \( \{ \phi^f; 0 \leq t \leq T \} \) be as in the previous section. In this section we investigate the set of all \( u(T) \), \( u \) being a weak solution to the equation

\[
\begin{align*}
(E) \quad u'(t) + \phi^f(u(t)) & \equiv f(t), \quad 0 < t < T (\leq \infty), \\
\text{under conditions (h2), (h3) and the following (h*1) with } p > 2:
\end{align*}
\]

(h*1) There are positive constants \( \alpha, \beta \) such that
\[
\phi^f(z) \geq \alpha |z|^p - \beta, \quad \forall t \in [0, T], \quad \forall z \in H.
\]

Theorem 2.2.1. Suppose (h*1) with \( p > 2 \), (h2) and (h3). Let \( f \in L^2(0, T; H) \). Then the set
\[
W_T = \{ u(T) \in H; u \text{ is a weak solution to (E) with initial value in } D(\phi^0) \}
\]
is bounded in \( H \).

The idea to our proof of this theorem is due to Simon [1], in which he showed the boundedness of \( W_T \) to evolution equations for operators with time-independent domains, and Proposition 0.4.2 is essentially to the proof.

Proof of Theorem 2.2.1: First take a strong solution \( w \to CP(\phi^f; f, u_0) \) with \( u_0 \in D(\phi^0) \). By Theorem 1.1.2, \( \phi^f \in L^2(0, T, H) \) and \( t \to \phi^f(u(t)) \) is bounded on \( [0, T] \). Now, let \( u \) be any strong solution to \( CP(\phi^f; f, u_0) \) with \( u \in D(\phi^0) \). Then we have by (1.2) of Theorem 1.1.1

\[
|u(t) - w(t)|^2 - |u(s) - w(s)|^2 + 2 \int_s^t (u^*(\tau) - w^*(\tau), u(\tau) - w(\tau)) d\tau \leq 0
\]
for any \( s, t \in [0, T] \) with \( s \leq t \), where \( u^* = f - u' \) and \( w^* = f - w' \). Since \( u^*(\tau) \in D(\phi^f(u(\tau))) \), it follows that
\[
(u^*(\tau), u(\tau) - w(\tau)) \geq \phi^f(u(\tau)) - \phi^f(w(\tau)).
\]
Choose suitable positive numbers \( \alpha', \beta' \) (depending on \( w \) but not on \( u \)) so that
\[
\phi^f(u(\tau)) \geq \alpha |u(\tau)|^p - \beta \geq \alpha' |u(\tau) - w(\tau)|^p - \beta'
\]
and
\[
(w^*(\tau), u(\tau) - w(\tau)) \leq \frac{\alpha'}{2} |u(\tau) - w(\tau)|^p + \beta' |w^*(\tau)|^{p'}
\]
where \( 1/p + 1/p' = 1 \). Then we obtain from (2.1) that
\[ |u(t) - w(t)|^2 - |u(s) - w(s)|^2 + \alpha' \int_s^t |u(\tau) - w(\tau)|^p d\tau \leq \int_s^t \left\{ 2 \phi^p(w(\tau)) \right\} + 2\beta' w^*(\tau)^p + 2\beta' \right\} d\tau \]

for any \( s, t \in [0, T] \), with \( s \leq t \). Now, putting \( g(t) = |u(t) - w(t)|^2 \), we see that \( g \) is absolutely continuous on \([0, T]\) and

\[ g'(t) + \alpha' g(t)^{p/2} \leq 2 \phi^p(w(t)) + 2\beta' w^*(t)^p + 2\beta' \equiv q(t) \]

for a.e. \( t \in [0, T] \). Therefore, applying Proposition 0.4.2 to this inequality, we get

\[ g(t) \leq \left\{ \frac{1}{2} \alpha' (p-2) \right\}^{-2/(p-2)} + \int_0^t q(\tau) d\tau, \quad \forall t \in (0, T). \]

This implies that

\[ W_T^S = \{ u(T) \in H; u \text{ is a strong solution to (E) with initial value in } D(\phi^0) \} \]

is bounded in \( H \). Taking into account the fact that every weak solution with initial value in \( D(\phi^0) \) is the uniform limit of strong solutions with initial values in \( D(\phi^0) \), we see that \( W_T^S \supset W_T \) and hence \( W_T \) is bounded in \( H \).

**Remark 2.2.1.** In the above theorem, the assumption \( p > 2 \) is essential. Indeed, in the case of \( p = 2 \) we have the following example: Consider the convex function

\[ \phi^f(z) = \frac{1}{2} |z|^2, \quad \forall t \in [0, T], \quad \forall z \in H. \]

Then for each \( u_0 \in D(\phi^0) = H \) the function \( u(t) = e^{-t}u_0 \) is a strong solution to \( CP(\phi^f; 0, u_0) \), so \( W_T = H \) and thus it is not bounded in \( H \).

**§ 2.3. Periodic solutions**

In this section, let \( \{ \phi^f; 0 \leq t \leq T \} \) be as in § 2.1 and consider equation (E) with periodic condition \( u(0) = u(T) \).

**Definition 2.3.1.** Given \( f \in L^1(0, T; H) \), we say that \( u \in C([0, T]; H) \) is a weak (resp. strong) periodic solution to (E), if \( u \) is a weak (resp. strong) solution to (E) satisfying \( u(0) = u(T) \).

For a fixed \( f \in L^2(0, T; H) \), we consider the mapping \( S_f \) from \( D(\phi^0) \) into \( D(\phi^T) \) which assigns to each \( u_0 \in D(\phi^0) \) the element \( u(T) \in D(\phi^T) \), \( u \) being a unique weak solution to \( CP(\phi^f; f, u_0) \). If we suppose that (h1), (h2) and (h3) are satisfied and \( D(\phi^0) \supset D(\phi^T) \), then by Theorem 1.1.1 the mapping \( S_f \) is a contraction on \( D(\phi^0) \). Also, \( S_f \) has a fixed point in \( D(\phi^0) \) if and only if there exists a weak periodic solution to (E); in view of the regularity property of Theorem 1.1.2 any weak periodic solution is necessarily a strong solution.

**Lemma 2.3.1.** (cf. Bénilan [1], Bénilan-Brézis [1] or Nagai [1]) Suppose that (h*1) with \( p = 2 \), (h2) and (h3) are satisfied and that \( D(\phi^0) \supset D(\phi^T) \) and let \( f \in L^2(0, T; H) \). Then for each \( u_0 \in D(\phi^0) \), \( \{ S^n_f u_0; n = 1, 2, \ldots \} \) is bounded in \( H \).

**Proof.** By Theorem 1.1.2 there exists a strong solution \( u_n \) of \( CP(\phi^f; f, S^n_f u_0) \), \( n = 1, 2, \ldots \) Since \( S_f \) is contractive on \( D(\phi^0) \), we see that

\[ ||u_n(0)| - |u_n(T)|| = ||S^n_f u_0| - |S^n_f u_0|| \leq |S_f u_0 - u_0| \]
and hence \( \{ |u_n(0)| - |u_n(T)| ; n = 1, 2, \ldots \} \) is bounded.

Now suppose for contradiction that \( \{ S_f u_0 \} = \{ u_n(T) \} \) is not bounded in \( H \) and let 
\( \{ u_{n'}(T) \} \) be a subsequence of \( \{ u_n(T) \} \) such that \( |u_{n'}(T)| \to \infty \) as \( n' \to \infty \); in this case we observe by Theorem 1.1.1 that
\[
\inf_{t \in [0, T]} |u_{n'}(t)| \to \infty \quad \text{as} \quad n' \to \infty.
\]
Taking some strong solution \( w \) to \( CP(\phi^t; f, w_0) \) with \( w_0 \in D(\phi^0) \), for any given \( \alpha > 0 \) we choose a positive integer \( n(\alpha') \) so that
\[
\phi^t(u_{n'}(t)) - \phi^t(w(t)) \geq \alpha' |u_{n'}(t) - w(t)| > 0
\]
for all \( t \in [0, T] \) and all \( n' \geq n(\alpha') \); in fact such an integer \( n(\alpha') \) exists because of (h*1) and the boundedness of \( t \to \phi^t(w(t)) \) on \([0, T]\). Therefore from (1.2) in Theorem 1.1.1 it follows that
\[
2 \int_s^t (w'(\tau), u_{n'}(\tau) - w(\tau)) \, d\tau \\
\geq |u_{n'}(t) - w(t)|^2 - |u_{n'}(s) - w(s)|^2 + 2 \int_s^t |\phi^t(u_{n'}(\tau)) - \phi^t(w(\tau))| \, d\tau \\
\geq |u_{n'}(t) - w(t)|^2 - |u_{n'}(s) - w(s)|^2 + 2 \alpha' \int_s^t |u_{n'}(\tau) - w(\tau)| \, d\tau
\]
and hence
\[
|u_{n'}(t) - w(t)| - |u_{n'}(s) - w(s)| \leq \int_s^t (|w'(|\tau)| - \alpha') \, d\tau
\]
for any \( s, t \in [0, T] \) with \( s \leq t \). In particular, if \( s = 0 \) and \( t = T \), then we get
\[
|u_{n'}(0) - w(0)| - |u_{n'}(T) - w(T)| \geq \alpha'T - \int_0^T |w'(\tau)| \, d\tau
\]
for all \( n' \geq n(\alpha') \). Since \( \alpha' \) is arbitrary, this inequality contradicts the fact that \( \{ |u_n(0)| - |u_n(T)| ; n = 1, 2, \ldots \} \) is bounded. Thus it must be true that \( \{ u_n(T) \} \) is bounded in \( H \). Q.E.D.

**Theorem 2.3.1.** Suppose that (h*1) with \( p = 2, (h2) \) and (h3) are satisfied and that \( D(\phi^0) \supset D(\phi^T) \), and let \( f \in L^2(0, T; H) \). Then there exists a strong periodic solution to (E).

**Proof.** It suffices to show that \( S_f \) has a fixed point in \( D(\phi^0) \). Since \( \{ S_f u_0 \} \) is bounded in \( H \) for \( u_0 \in D(\phi^0) \) by Lemma 2.3.1, we infer from a fixed point theorem of Browder-Petryshyn [11] that \( S_f \) has a fixed point in \( D(\phi^0) \). Q.E.D.

**Theorem 2.3.2.** Suppose that for each \( t \in [0, T] \), \( \phi^t \) is strictly convex on \( H \) and that \( f \in L^1(0, T; H) \). Then a strong periodic solution to (E) is unique.

**Proof.** Let \( u \) and \( v \) be any strong periodic solutions to (E). Then by (1.2) of Theorem 1.1.1,
\[
\int_0^T (u^*(\tau) - v^*(\tau), u(\tau) - v(\tau)) \, d\tau \leq 0,
\]
where \( u^*(\tau) \in \partial \phi^T(u(\tau)) \) and \( v^*(\tau) \in \partial \phi^T(v(\tau)) \) for a.e. \( \tau \in [0, T] \). Hence
\[
(u^*(\tau) - v^*(\tau), u(\tau) - v(\tau)) = 0 \quad \text{for a.e.} \quad \tau \in [0, T].
\]
Since the strict convexity of \( \phi^t \) implies the strict monotonicity of \( \partial \phi^t \) (cf. Proposition 0.3.2), we have \( u(\tau) = v(\tau) \) by (3.1). Therefore \( u = v \). Q.E.D.

In general, without the strict convexity of \( \phi^t \) a strong periodic solution is not unique, which is seen from the following example.

**Example 2.3.1.** Let \( K \) be a bounded closed convex subset of \( H \) and define
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\[ \phi^t(z) = \begin{cases} 
0 & \text{if } z \in K, \\
\infty & \text{otherwise}
\end{cases} \]

for each \( t \in [0, T] \). This family \( \{ \phi^t; 0 \leq t \leq T \} \) clearly fulfill \((h^*1)\) with \( p = 2, (h2), (h3)\) and \( D(\phi^0) = D(\phi^T) = K \). But for every \( u_0 \in K \) the constant function \( u(t) = u_0 \) is a strong periodic solution to \( u''(t) + \partial \phi^t(u(t)) \geq 0 \).

\section*{2.4. Characterization of weak solutions}

We now give a characterization of weak solutions to

\[(E) \quad u''(t) + \partial \phi^t(u(t)) \geq f(t), \quad 0 < t < T,\]

by means of variational inequality, where \( \phi^t \) is a proper 1.s.c. convex function on \( H \) for each \( t \in [0, T] \).

For a given \( u_0 \in D(\phi^0) \) we denote by \( M_{u_0} \) the operator whose graph is given as follows: \( [u, f] \in G(M_{u_0}) \subseteq L^2(0, T; H) \times L^2(0, T; H) \) if and only if \( f \in L^2(0, T; H) \) and \( u \) is the weak solution to \( \phi^t(u; f, u_0) \).

As was seen in Lemma 1.2.2, the function

\[ v \in L^2(0, T; H) \quad \Rightarrow \quad \Phi(v) = \int_0^T \phi^t(v(t)) \, dt \]

is proper 1.s.c. and convex on \( L^2(0, T; H) \) under \((h1), (h2)\) and \((h3)\).

\textbf{Theorem 2.4.1.} Suppose \((h1), (h2)\) and \((h3)\) and let \( u_0 \in D(\phi^0) \). Then we have:

(i) \( R(M_{u_0}) = L^2(0, T; H) \) and \( M_{u_0} : L^2(0, T; H) \rightarrow L^2(0, T; H) \) is maximal monotone.

(ii) \( [u, f] \in G(M_{u_0}) \) if and only if \( u \in D(\Phi), f \in L^2(0, T; H) \) and the following variational inequality holds:

\[ \forall v \in D(\Phi) \text{ with } v' \in L^2(0, T; H) \]

\[ <v' - f, u - v> - \frac{1}{2} |v(0) - u_0|^2 \leq \Phi(v) - \Phi(u), \]

\[(4.1) \quad u'(t) + \partial \phi(t(u(t))) \geq f(t), \quad 0 < t < T,\]

for all \( v \in L^2(0, T; H) \).

(iii) In particular, assume \( u_0 \in D(\phi^0) \). Then \( [u, f] \in G(M_{u_0}) \) if and only if \( f \in L^2(0, T; H) \), \( u \in D(\Phi), u' \in L^2(0, T; H) \), \( u(0) = u_0 \) and the following holds:

\[ <u' - f, u - v> \leq \Phi(v) - \Phi(u), \quad \forall v \in D(\Phi). \]

\[(4.2) \quad u'(t) + \partial \phi(t(u(t))) \geq f(t), \quad 0 < t < T,\]

\textbf{Proof.} By virtue of Theorems 1.1.1 and 1.1.2 we see that \( R(M_{u_0}) = L^2(0, T; H) \) and the inverse \( M^{-1}_{u_0} : L^2(0, T; H) \rightarrow L^2(0, T; H), D(M^{-1}_{u_0}) = L^2(0, T; H) \), is singlevalued, monotone and continuous. Therefore it follows from Proposition 0.2.3. that \( M^{-1}_{u_0} \) is maximal monotone, so is \( M_{u_0} \).

Now, denote by \( \bar{N}_{u_0} \) the operator from \( L^2(0, T; H) \) into itself given by the following:

\( [u, f] \in G(\bar{N}_{u_0}) \) if and only if \( f \in L^2(0, T; H), u \in D(\Phi) \) and (4.1) holds. Also, denote by \( \tilde{N}_{u_0} \) the operator from \( L^2(0, T; H) \) into itself given by the following:

\( [u, f] \in G(\tilde{N}_{u_0}) \) if and only if \( f \in L^2(0, T; H), u \in D(\Phi), u' \in L^2(0, T; H), u(0) = u_0 \) and (4.2) holds. Then our claim is to show that

\[ G(\bar{N}_{u_0}) = G(M_{u_0}) \]

\[ \bar{N}_{u_0} = M_{u_0} \]
as well as

$$G(\tilde{N}u_0) = G(\tilde{N}u_0) = G(\tilde{M}u_0)$$

in case \(u_0 \in D(\phi^0)\).

First let \(u_0 \in D(\phi^0)\) and \([u, f]\) be any element of \(G(Mu_0)\). Then by Theorem 1.1.2, \(u \in D(\Phi), u' \in L^2(0, T; H)\) and there is \(u^* \in L^2(0, T; H)\) satisfying \(u^*(t) \in \partial\phi^1(u(t))\) and \(u'(t) + u^*(t) = f(t)\) for a.e. \(t \in [0, T]\). Hence,

$$\langle u' - f, u - v \rangle = \langle u^*, v - u \rangle \leq \Phi(v) - \Phi(u), \quad \forall v \in D(\Phi)$$

and moreover by integration by parts

$$\langle v' - f, u - v \rangle - \frac{1}{2} |v(0) - u_0|^2 \leq \Phi(v) - \Phi(u),$$

$$\forall v \in D(\Phi) \text{ with } v' \in L^2(0, T; H).$$

Thus \(G(Mu_0) \subset G(\tilde{N}u_0) \subset G(\tilde{N}u_0)\).

Next, let \(u_0 \in D(\Phi^0)\) and \([u, f]\) be any element of \(G(Mu_0)\). Then, choosing a sequence \(\{u_{0,n}\} \subset D(\Phi^0)\) so that \(u_{0,n} \to u_0\) in \(H\) and denoting by \(u_n\) the strong solution to \(CP(\phi^1; f, u_{0,n})\), we see from (1.1) of Theorem 1.1.1 that \(u_n \to u\) in \(C([0, T]; H)\). Since \([u_n, f] \in G(\tilde{N}u_{0,n})\) as was seen above, it follows that

$$\langle v' - f, u_n - v \rangle - \frac{1}{2} |v(0) - u_{0,n}|^2 \leq \Phi(v) - \Phi(u_n),$$

$$\forall v \in D(\Phi) \text{ with } v' \in L^2(0, T; H).$$

Hence, letting \(n \to \infty\), we obtain that

$$\langle v' - f, u - v \rangle - \frac{1}{2} |v(0) - u_0|^2 \leq \Phi(v) - \Phi(u),$$

$$\forall v \in D(\Phi) \text{ with } v' \in L^2(0, T; H).$$

This shows that \(G(Mu_0) \subset G(\tilde{N}u_0)\).

Finally we show that \(G(\tilde{N}u_0) \subset G(Mu_0)\) for \(u_0 \in D(\phi^0)\). Let \(u_0 \in D(\phi^0), [w, g] \in G(\tilde{N}u_0)\) and \([u, f] \in G(Mu_0)\) and let \(\{u_{0,n}\} \in D(\phi^0)\) and \(\{u_n\}\) be as above. Then, noting that \([u_n, f] \in G(\tilde{N}u_{0,n})\), we see that

$$\langle u_n' - f, u_n - w \rangle \leq \Phi(w) - \Phi(u_n)$$

and by the definition of \(\tilde{N}u_0\)

$$\langle u_n' - g, w - u_n \rangle - \frac{1}{2} |u_{0,n} - u_0|^2 \leq \Phi(u_n) - \Phi(w).$$

Hence

$$\langle f - g, u_n - w \rangle \geq 0 - \frac{1}{2} |u_{0,n} - u_0|^2$$

and letting \(n \to \infty\) yields

$$\langle f - g, u - w \rangle \geq 0.$$
Remark 2.4.1. As is easily seen from our proof of Theorem 2.4.1 the following statements are valid under the same assumptions of Theorem 2.4.1:

(i) In case  \( u_0 \in \overline{D(\phi^0)} \), \([u, f] \in G(M u_0)\) if any only if \( u \in D(\Phi) \cap C([0, T]; H)\) with \( u(0) = u_0, f \in L^2(0, T; H)\) and

\[
\int_s^t (\nu'(\tau) - \nu(\tau), u(\tau) - \nu(\tau)) \, d\tau + \frac{1}{2} \left| u(t) - \nu(t) \right|^2 - \frac{1}{2} \left| u(s) - \nu(s) \right|^2 \\
\leq \int_s^t \left\{ \phi^\tau(\nu(\tau)) - \phi^\tau(\nu(t)) \right\} \, d\tau, \quad \forall \, \nu \in D(\Phi) \text{ with } \nu' \in L^2(0, T; H), \forall \, s, t \in [0, T], s \leq t.
\]

(ii) In case \( u_0 \in D(\phi^0) \), \([u, f] \in G(M u_0)\) if and only if \( u \in D(\Phi), u' \in L^2(0, T; H), u(0) = u_0, f \in L^2(0, T; H)\) and

\[
\int_s^t (u'(\tau) - f(\tau), u(\tau) - \nu(\tau)) \, d\tau \leq \int_s^t \left\{ \phi^\tau(\nu(\tau)) - \phi^\tau(u(\tau)) \right\} \, d\tau, \\
\forall \nu \in D(\Phi), \forall s, t \in [0, T] \text{ with } s \leq t.
\]

Associated with problem with periodic condition, we consider the following operator \( M_p \) from \( L^2(0, T; H) \) into itself: \([u, f] \in G(M_p)\) if and only if \( f \in L^2(0, T; H)\) and \( u \) is a weak periodic solution to \((E)\). We then have

Theorem 2.4.2. Suppose \((h1), (h2), (h3)\) and \( D(\phi^0) \supset D(\phi^T) \). Then we have:

(i) \( M_p : L^2(0, T; H) \rightarrow L^2(0, T; H) \) is maximal monotone.
(ii) \([u, f] \in G(M_p)\) if and only if \( u \in D(\Phi), f \in L^2(0, T; H)\) and

\[
<\nu' - f, u - \nu > \leq \Phi(\nu) - \Phi(u), \quad \forall \nu \in D(\Phi) \text{ with } \nu' \in L^2(0, T; H) \text{ and } \nu(0) = \nu(T).
\]

In order to prove Theorem 2.4.2 we observe

Lemma 2.4.1. Under the same assumptions of Theorem 2.4.2, \( R(M_p + I) = L^2(0, T; H)\).

Proof. Since \( M_{u_0} \) is maximal monotone for each \( u_0 \in \overline{D(\phi^0)} \) by (i) of Theorem 2.4.1, we see (cf. Proposition 0.2.2) that \( R(M_{u_0} + I) = L^2(0, T; H)\). Now, given \( f \in L^2(0, T; H)\), consider the mapping \( P_f \) from \( D(\phi^0) \) into itself which assigns to each \( u_0 \in D(\phi^0) \) the element \( u(T) \in D(\phi^T) \subset D(\phi^0) \), \( u \) being a unique function such that \( f \in M_{u_0} u + u \). Then we see that \( P_f \) is strictly contractive on \( D(\phi^0) \). Indeed, let \( u_0, v_0 \in D(\phi^0) \) and \( u, v \) be functions such that \( f \in M_{u_0} u + u \) and \( f \in M_{v_0} v + v \). From (1.1) of Theorem 1.1.1 it follows that for any \( s, t \in [0, T] \) with \( s \leq t \)

\[
|u(t) - v(t)|^2 - |u(s) - v(s)|^2 + 2 \int_s^t |u(\tau) - v(\tau)|^2 \, d\tau \leq 0
\]

and hence

\[
e^{-\tau} |u(t) - v(t)| \leq e^{-\tau} |u(s) - v(s)| ;
\]

in particular,

\[
|P_f u_0 - P_f v_0| \leq e^{-T} |u_0 - v_0| .
\]

Therefore \( P_f \) has a unique fixed point \( u_0 \in D(\phi^0) \), which implies that there exists a function \( u \) such that \( f \in M_{u_0} u + u \) and \( u(0) = u(T) \), i.e., \( f \in M_p u + u \).

Q.E.D.

Proof of Theorem 2.4.2: Let \([u, f]\) and \([v, g]\) be any elements of \( G(M_p)\). Then it follows from (1.1) of Theorem 1.1.1 that
Thus $M_p$ is monotone. Also, since $R(M_p + I) = L^2(0, T; H)$, by Lemma 2.4.1, Proposition 0.2.2 implies that $M_p$ is maximal monotone.

Next, to show (ii) we consider the operator $N_p: L^2(0, T; H) \rightarrow L^2(0, T; H)$ given by the following: $[u, f] \in G(N_p)$ if and only if $u \in D(\Phi)$, $f \in L^2(0, T; H)$; and (4.3) holds. Clearly $G(M_p) \subset G(N_p)$. Besides, let $[w, g] \in G(M_p)$ and $[u, f] \in G(N_p)$. Then, since $w$ is the strong solution to $CP(\Phi^t; g, w(0))$ such that $w(0) = w(T)$, we have

$$<w' - g, w - v> \leq \Phi(v) - \Phi(w), \quad \forall v \in D(\Phi).$$

Taking $u$ as $v$ in (4.4) and $w$ as $v$ in (4.3), we obtain

$$<w' - g, w - u> \leq \Phi(u) - \Phi(w)$$

and

$$<w' - f, u - w> \leq \Phi(w) - \Phi(u).$$

Adding these inequalities, we get

$$<f - g, u - w> \geq 0.$$

Since $M_p$ is maximal monotone, $G(M_p) \supset G(N_p)$. Consequently $M_p = N_p$. Q.E.D.

Remark 2.4.2 It should be noticed that in our characterization (such as (ii) of Theorem 2.4.1 and (ii) of Theorem 2.4.2) of weak solution $u$ to (E) by means of variational inequalities we do not require any regularity of $u$ except that $u \in D(\Phi)$.

Remark 2.4.3. Such observations on the operators $M_{u_0}$ and $M_p$ were also made in Brézis [2], Kenmochi-Nagai [1] and Nagai [1].

§2.5. Maximum and minimum periodic solutions

In this section we take the abstract $L^2$-space $L^2(\Omega; \mu)$ as the Hilbert space $H$, where $\Omega$ is a locally compact Hausdorff space and $\mu$ is a positive measure on $\Omega$; of course,

$$(v, w) = \int_\Omega\nu(x)w(x)\,d\mu(x), \quad |v| = (v, v)^{1/2}.$$ 

For simplicity we write “$v \geq w$" for “$\nu(x) \geq w(x)$ $\mu$-a.e. on $\Omega$”.

Let $\{\Phi^t: 0 \leq t \leq T\}$ be a family of proper l.s.c. convex functions on $H$, and assume that every $\Phi^t$ is $T$-monotone, i.e.

$$(5.1) \quad (z_1^* - z_2^*, z_1 - z_2) \leq 0, \quad \forall [z_1, z_2] \in G(\Phi^t) \quad (i = 1, 2),$$

where $z^*(x) = \max \{z(x), 0\}$. According to a well-known result (cf. Brézis [4; Chapter 4] or Kenmochi-Mizuuta-Nagai [1]), (5.1) is equivalent to the following:

$$\Phi^t(z_1 \wedge z_2) + \Phi^t(z_1 \vee z_2) \leq \Phi^t(z_1) + \Phi^t(z_2), \quad \forall z_1, z_2 \in H,$$

where

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(z₁ ∨ z₂)(x) = \max \{ z₁(x), z₂(x) \}, (z₁ ∧ z₂)(x) = \min \{ z₁(x), z₂(x) \}.

**Lemma 2.5.1.** (cf. Kenmochi-Mizuta-Nagai [1; Proposition 3.3]) Let \( u \) be weak solutions to CP(\( \phi^T; f, u_{0,1} \)) and \( f_{j} \in L^2(0, T; H) \) \((j = 1, 2)\). Then we have: for any \( s, t \in [0, T] \) with \( s \leq t \)
\[|u_1(t) - u_2(t)|^2 - |u_1(s) - u_2(s)|^2 \leq 2 \int_{s}^{t} (f_1(\tau) - f_2(\tau), (u_1(\tau) - u_2(\tau))^T) d\tau.\]
In particular, if \( f_{1} \leq f_{2} \) and \( u_{0,1} \leq u_{0,2} \), then \( u_1(t) \leq u_2(t) \) for all \( t \in [0, T] \).

**Lemma 2.5.2.** Suppose \((h*1)\) with \( p = 2 \), \((h_2), (h_3)\) and \( D(\phi^0) \supset D(\phi^T) \). Let \( f \in L^2(0, T; H) \) and \( S_f: D(\phi^0) \to D(\phi^T) \) be as in \( \S 2.3 \). If \( u_0 \in D(\phi^0) \) and \( S_f u_0 \geq u_0 \), then \( \{ S^n_f u_0 \} \) converges to some \( V_0 \) in \( H \) and \( V_0 \) is a fixed point of \( S_f \) with \( V_0 \geq u_0 \).

**Proof.** From Lemma 2.5.1 it follows that
\[v_0 \leq S^n_f u_0 \leq S^{n+1}_f u_0 \leq \ldots \leq S^n_f v_0 \leq \ldots\]
Also, by Lemma 2.3.1, \( \{ S^n_f u_0 \} \) is bounded in \( H \). Therefore \( S^n_f u_0 \to V_0 \) in \( H \) for some \( V_0 \in D(\phi^T) \) and clearly \( S_f V_0 = V_0 \).

Just as Lemmas 2.5.2 and 2.5.3 we prove

**Lemma 2.5.3.** Under the same assumptions of Lemma 2.5.2 as well as \( f \in L^2(0, T; H) \), if \( u_0 \in D(\phi^0) \) and \( S_f u_0 \leq u_0 \), then \( \{ S^n_f u_0 \} \) converges in \( H \) and the limit \( V_0 \) is a fixed point of \( S_f \) with \( V_0 \leq u_0 \).

By using Lemma 2.5.2 and 2.5.3 we prove

**Lemma 2.5.4.** Under the same assumptions of Lemma 2.5.2 as well as \( f \in L^2(0, T; H) \), we have the following: Let \( u_1 \) and \( u_2 \) be strong periodic solutions to \((E)\). Then there are strong periodic solutions \( U \) and \( u \) to \((E)\) such that
\[(5.2) \quad U(t) \geq u_1(t) \lor u_2(t), \quad \forall t \in [0, T],\]
\[(5.3) \quad u(t) \leq u_1(t) \land u_2(t), \quad \forall t \in [0, T].\]

**Proof.** Since \( u_1(0) \lor u_2(0) \geq u_1(0) \) and \( S^0_f u_1(0) = u_1(0) \) \((i = 1, 2)\), we see by Lemma 2.5.1
\[S_f(u_1(0) \lor u_2(0)) \geq (S^n_f u_1(0)) \lor (S^n_f u_2(0)) = u_1(0) \lor u_2(0).\]

Hence, by Lemma 2.5.2, \( S^n_f u_1(0) \lor u_2(0) \to U_0 \) in \( H \) and \( U_0 = S_f U_0 \geq u_1(0) \lor u_2(0) \), so that the strong solution \( U \) to \( CP(\phi^T; f, U_0) \) is a strong periodic solution satisfying (5.2). A strong periodic solution \( u \) satisfying (5.3) is similarly obtained by using Lemma 2.5.3. Q.E.D.

We already showed in \( \S 2.3 \) that a weak periodic solution to \((E)\) is not unique without the strict convexity of \( \phi^T \), so it is necessary to investigate the structure of the set \( M^{-1}_p f \) of all weak periodic solutions to \((E)\).

**Theorem 2.5.1.** Suppose \((h*1)\) with \( p = 2 \), \((h_2), (h_3)\) and \( D(\phi^0) \supset D(\phi^T) \). Let \( f \in L^2(0, T; H) \). Then there are strong periodic solutions \( U \) and \( V \) to \((E)\) such that
\[V(t) \geq v(t) \geq U(t), \quad \forall t \in [0, T],\]
for any strong periodic solution \( v \) to \((E)\).

**Proof.** First note (cf. Theorem 2.3.1, Theorem 2.4.2 and (b) of Proposition 0.2.1) that \( M^{-1}_p f \) is a non-empty bounded closed convex subset of \( L^2(0, T; H) \) and hence it is weakly compact in \( L^2(0, T; H) \). Let us introduce an order \( \leq \) in the set \( M^{-1}_p f \) as follows: \( v \leq w \) if and only if \( v(t) \leq w(t) \) for all \( t \in [0, T] \), and set for each \( u \in M^{-1}_p f \)

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\[ P(u) = \{ v \in \mathcal{M}_p^{-1}(f); v \leq u \}. \]

Then each \( P(u) \) is non-empty and weakly closed in \( L^2(0, T; H) \). Moreover, from Lemma 2.5.4 it follows that the family \( \{ P(u); u \in \mathcal{M}_p^{-1}(f) \} \) has the finite intersection property. Therefore,

\[ P_0 = \bigcap_{u \in \mathcal{M}_p^{-1}(f)} P(u) \neq \emptyset. \]

Now, taking an element \( U \) of \( P_0 \), we see that

\[ U \leq u, \quad \forall u \in \mathcal{M}_p^{-1}(f), \]

that is, \( U \) is the minimum element of \( \mathcal{M}_p^{-1}(f) \). The existence of the maximum element of \( \mathcal{M}_p^{-1}(f) \) is similarly proved.

§2.6. Stability of solutions

The aim of the present section is to investigate the stability of solutions to \( CP(\phi^t; f, u_0) \) for a class of \( \{ \phi^t; 0 \leq t \leq T \} \) \( (0 < T < \infty) \).

**Definition 2.6.1.** Let \( \alpha_0 \) be a non-negative number, \( \{ \delta_r; 0 \leq r < \infty \} \) be a subset of \( \{0, 1\} \) and \( \{ M_r; 0 \leq r < \infty \} \) be a subset of \( \{0, \infty\} \). Then we denote by \( G(\alpha_0, \{ \delta_r \}, \{ M_r \}) \) the class of all families \( \{ \phi^t; 0 \leq t \leq T \} \) of proper l.s.c. convex functions on \( H \) such that

(a) \( \phi^t(x) + \alpha_0 \lvert x \rvert + \alpha_0 \leq 0, \quad \forall t \in (0, T], \quad \forall x \in H, \)

(b) there is a function \( h: [0, T] \rightarrow H \) satisfying

\[ \lvert h \rvert_{C([0, T]; H)} \leq \alpha_0, \quad \lvert h \rvert_{L^1([0, T])} \leq \alpha_0, \quad \lvert \phi^t(h(t)) \rvert \leq \alpha_0, \quad \forall t \in (0, T), \]

(c) condition (h2) holds, where for each \( r \geq 0 \) there are \( a_r, b_r, c_r \) such that

\[ 0 \leq a_r \leq \delta_r, \quad \lvert b_r \rvert_{L^1((0, T])} \leq M_r, \quad \lvert c_r \rvert_{L^1((0, T])} \leq M_r \]

and for which (ii) of (h2) holds.

Our main theorem of this section is as follows.

**Theorem 2.6.1.** Let \( G = G(\alpha_0, \{ \delta_r \}, \{ M_r \}) \) be as above. Then we have:

(i) Given a number \( k_0 \geq 0 \), there are a constant \( K_0 = K_0(G, k_0) \) and \( r_0 = r_0(G, k_0) \) (depending only on the class \( G \) and \( k_0 \)) such that

\[ \left\{ \begin{array}{l}
\lvert u \rvert_{C([0, T]; H)} \leq K_0, \\
\lvert \sqrt{t} u \rvert_{L^2(0, T)} \leq K_0, \\
\int_0^T \lvert \phi^t(u(t)) \rvert dt \leq K_0, \\
\lvert t \phi^t(u(t)) \rvert \leq K_0, \\
\phi^t(u(t)) - \phi^s(u(s)) + \int_s^t (u'(r), u'(	au)) - f(r)) dr
\end{array} \right. \]

\[ \leq \int_s^t \left\{ a_r \lvert u'(	au) \rvert^2 + b_r \lvert u'(	au) \rvert + c_r \lvert u'(	au) \rvert \right\} d\tau, \]

\[ \forall s, t \in (0, T) \text{ with } s \leq t, \quad \forall r \geq r_0, \]

whenever \( \{ \phi^t; 0 \leq t \leq T \} \in G, f \in L^2(0, T; H) \) with \( \lvert f \rvert \leq k_0, u_0 \in D(\phi^0) \) with \( \lvert u_0 \rvert \leq k_0, u \) is a weak solution to \( CP(\phi^t; f, u_0) \) and \( a_r, b_r, c_r \) are as in (c) of Definition 2.6.1.

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(ii) Given a number \( k_1 \geq 0 \), there are constants \( K_1 = K_1(G, k_1) \) and \( r_1 = r_1(G, k_1) \) such that
\[ |u|_{W^{1,2}(0, T; H)} \leq K_1, \quad |\phi^t(u(t))| \leq K_1, \quad \forall t \in [0, T], \]
and (6.2) holds for every \( s \), \( t \in [0, T] \) with \( s \leq t \) and \( r \geq r_1 \), whenever \( \{ \phi^t; 0 \leq t \leq T \} \in G, f \in L^2(0, T; H) \) with \( \|f\| \leq k_1 \), \( u_0 \in D(\phi^0) \) with \( |u_0| + |\phi^0| \leq k_1 \), \( u \) is a strong solution to \( CP(\phi^t; f, u_0) \) and \( a_n, b_n, c_n \) as in (c) of Definition 2.6.1.

**Proof.** We note the following fact: there is a constant \( C \geq 0 \) depending only on the class \( G \) such that
\[ |\chi^t(z)| \leq C + |z|, \quad |\partial \phi^t_\lambda(z)| \leq \frac{\lambda}{C + 2|z|}, \quad \forall t \in [0, T], \quad \forall \lambda \in (0, 1], \quad \forall z \in H, \quad \forall \{ \phi^t; 0 \leq t \leq T \} \in G. \tag{6.3} \]
Indeed, this is easily understood from the proof of Lemma 1.2.1. Now, let \( \{ \phi^t; 0 \leq t \leq T \} \in G, f \in L^2(0, T; H) \) with \( \|f\| \leq k_0 \) and \( u_0 \in D(\phi^0) \) with \( |u_0| \leq k_0 \). Let \( u \) be a unique weak solution to \( CP(\phi^t; f, u_0) \) and \( \{ u_\lambda; 0 < \lambda \leq 1 \} \) be approximate solutions as constructed in §1.3. Then, taking account of (6.3) and reproducing carefully the proofs of Lemmas 1.3.1 and 1.3.2, we see that there is a certain positive constant \( K_0 \) depending only on \( G \) and \( k_0 \) and satisfying
\[ |u_\lambda|_{C((0, T]; H)} \leq K_0, \quad \|\sqrt{T}u_\lambda\| \leq K_0, \]
\[ \int_0^T |\phi^t_\lambda(u_\lambda(t))| |dt \leq K_0, \quad |\int_0^T \phi^t_\lambda(u_\lambda(t))| |dt \leq K_0, \quad \forall t \in [0, T], \]
for all \( \lambda \in (0, 1] \). From these estimations we obtain (6.1), and (6.2) is nothing but (1.4) of Theorem 1.1.2. Also (ii) of the theorem is similarly proved.

§2.7. Convergence of solutions

Let us recall a notion of convergence of convex functions. Given a sequence \( \{ \psi_n \} \) of proper 1.s.c. convex functions on a Hilbert space \( V \), we say that \( \psi_n \) converges to a proper 1.s.c. convex function \( \psi \) on \( V \) in the sense of Mosco [1] if the following two conditions (a) and (b) are satisfied:

(a) For each \( z \in D(\psi) \) there is a sequence \( \{ z_n \} \) in \( V \) such that \( z_n \to z \) in \( V \) and \( \psi_n(z_n) \to \psi(z) \).

(b) Let \( \{ \psi_{n_k} \} \) be any subsequence of \( \{ \psi_n \} \) and \( \{ z_k \} \) be a sequence in \( V \) such that \( z_k \to z \) in \( V \), then
\[ \liminf_{k \to \infty} \psi_{n_k}(z_k) \geq \psi(z). \]

Now, let \( G = G(a_0, \{ \delta_T \}, \{ M_T \}) \) be the class which was introduced in §2.6. We are given a family \( \{ \phi^t; 0 \leq t \leq T \} \in G \) and a sequence \( \{ \phi^t_n; 0 \leq t \leq T \} \in G, n = 1, 2, \ldots \) We then define proper 1.s.c. convex functions \( \Phi^T_n \) and \( \Phi^T_n \) on \( L^2(0, T'; H) \) for each \( T' \in (0, T) \) by
\[ \Phi^T_n(v) = \int_0^{T'} \phi^t_n(v(t)) dt, \quad \Phi^T_n(v) = \int_0^{T'} \phi^t_n(v(t)) dt, \quad v \in L^2(0, T'; H). \]

Before discussing the convergence of solutions we prove

**Proposition 2.7.1.** Let \( \{ \phi^t; 0 \leq t \leq T \} \in G = G(a_0, \{ \delta_T \}, \{ M_T \}) \) as well as \( \{ \phi^t_n; 0 \leq t \leq T \} \)
Suppose that \( \phi^t_n \to \phi^t \) on \( H \) in the sense of Mosco for every \( t \in [0, T] \). Then we have:

(i) Let \( v \in L^2(0, T; H) \) and \( \{ v_k \} \subset L^2(0, T; H) \), and suppose that at least one of the following conditions (7.1), (7.2), (7.3) is satisfied:

\[
\begin{align*}
(7.1) & \quad \{ v_k \} \text{ is bounded in } L^\infty(0, T; H) \text{ and } v_k(t) \rightharpoonup v(t) \text{ in } H \text{ for a.e. } t \in [0, T]; \\
(7.2) & \quad v_k \to v \text{ in } L^2(0, T; H); \\
(7.3) & \quad \{ v_k(t) \rightharpoonup v(t) \in H \text{ for a.e. } t \in [0, T] \text{ and for a positive constant } C_0 \\
& \quad \phi^t_n(z) + C_0 \geq 0, \quad \forall t \in [0, T], \quad \forall z \in H, \quad \forall n.
\end{align*}
\]

Then, for any subsequence \( \{ n_k \} \) of \( \{ n \} \) and any \( T' \in (0, T] \), we have

\[
\Phi^{T'}(v) \leq \liminf_{k \to \infty} \Phi^{T'}_{n_k}(v_k).
\]

(ii) For each \( T' \in (0, T] \) and \( u \in D(\phi^{T'}) \) there is a sequence \( \{ u_n \} \) in \( L^2(0, T'; H) \) such that

\[
u_n \rightharpoonup u \quad \text{in } L^2(0, T'; H)
\]

and

\[
\Phi^{T'}_{u_n}(u_n) \to \Phi^{T'}(u).
\]

**Proof** of (i) of Proposition 2.7.1: We give only a proof of (7.4) in the case of (7.1), since it can be similarly shown in other cases. Now, take a constant \( C > 0 \) so that

\[
|v_k(t)| \leq C \quad \text{for a.e. } t \in [0, T] \text{ and all } k.
\]

By assumption (cf. (a) of Definition 2.6.1),

\[
\phi^t_n(v_k(t)) + \alpha_0 C + \alpha_0 \geq \phi^t_{n_k}(v_k(t)) + \alpha_0 |v_k(t)| + \alpha_0 \geq 0,
\]

so that by Fatou's lemma

\[
\liminf_{k \to \infty} \Phi^{T'}_{n_k}(v_k) \geq \int_0^{T'} \liminf_{k \to \infty} \phi^t_{n_k}(v_k(t)) \, dt \geq \int_0^{T'} \phi^t(v(t)) \, dt = \Phi^{T'}(v).
\]

Q.E.D.

**Proof** of (ii) of Proposition 2.7.1: We prove (ii) in two steps.

(a) The case that \( u \) and \( t \to \phi^t(u(t)) \) are bounded on \( [0, T] \). We assume that

\[
|u(t)| \leq \rho, \quad |\phi^t(u(t))| \leq \rho, \quad \forall t \in [0, T],
\]

and for simplicity \( T' = T \). Let \( \varepsilon (\varepsilon < 1) \) be an arbitrary positive number. To this \( \varepsilon \), choose a closed set \( E^\varepsilon \subset [0, T] \) such that meas. \( f(1, T\setminus E^\varepsilon) \) (= the linear measure of \( [0, T\setminus E^\varepsilon] \) is not larger than \( \varepsilon \) and \( u, t \to \phi^t(u(t)) \) are continuous on \( E^\varepsilon \). Also, choose a partition \( \{ E_k^\varepsilon; k = 1, 2, \ldots, m \} \) of \( E^\varepsilon \) such that

\[
E^\varepsilon = \sum_{k=1}^m E_k^\varepsilon \text{ (direct sum)},
\]

\[
t_k^\varepsilon = \inf E_k^\varepsilon \subseteq E_k^\varepsilon, \quad k = 1, 2, \ldots, m,
\]

\[
\sup E_{k-1}^\varepsilon \leq t_k^\varepsilon, \quad k = 2, 3, \ldots, m,
\]

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diameter of $E^k_k \leq \epsilon$, \hspace{1em} k = 1, 2, ..., m.

\[ |u(t) - u(t^k_k)| \leq \epsilon, \quad \forall t \in E^k_k, \quad k = 1, 2, ..., m, \]

\[ |\phi^k(u(t)) - \phi^k(u(t^k_k))| \leq \epsilon, \quad \forall t \in E^k_k, \quad k = 1, 2, ..., m. \]

Next, using our assumption, take sequences \( \{x_{k,n}^e; n = 1, 2, ..., \} \) in $H$ for $k = 1, 2, ..., m$ such that

\[ x_{k,n}^e \to u(t_k^e) \text{ in } H, \quad \phi^k_n(x_{k,n}^e) \to \phi^k(u(t_k^e)) \quad \text{(as } n \to \infty), \]

and by the way a positive integer $N_e$ such that

\[ |x_{k,n}^e - u(t_k^e)| \leq \epsilon, \quad |\phi^k_n(x_{k,n}^e) - \phi^k(u(t_k^e))| \leq \epsilon, \quad \forall n \geq N_e, \quad k = 1, 2, ..., m. \]

Moreover, consider the following family \( \{ \psi_{k,n}^t; 0 \leq t \leq T \} \) of proper 1.s.c. convex functions on $H$:

\[ \psi_{k,n}^t(c) = \begin{cases} \phi^k_n(c) & \text{if } 0 \leq t \leq T - t_k^e, \\ \psi_{T}^t(c) & \text{if } T - t_k^e < t \leq T. \end{cases} \]

Then, clearly \( \{ \psi_{k,n}^t; 0 \leq t \leq T \} \in G \) for $k = 1, 2, ..., m$ and $n = 1, 2, ...,$, and by (7.5)

\[ |x_{k,n}^e| \leq \rho + 1, \quad |\psi_{k,n}^t(x_{k,n}^e)| = |\phi^k_n(x_{k,n}^e)| \leq \rho + 1, \quad \forall n \geq N_e, \quad k = 1, 2, ..., m. \]

Therefore, denoting by $v_{k,n}$ the strong solution to $CP(\psi_{k,n}^t; 0, x_{k,n}^e)$, we infer from Theorem 2.6.1 that for some positive constant $L$ independent of $\epsilon$, $k$ and $n$

\[ |v_{k,n}|_{L^1(0, T; H)} \leq L, \quad |v_{k,n}(v_{k,n}(t))| \leq L, \quad \forall t \in [0, T], \]

and for some positive constant $\rho$ independent of $\epsilon$, $k$ and $n$

\[ \psi_{k,n}(v_{k,n}(t)) - \psi_{k,n}^t(x_{k,n}^e) = \phi^k_n(t_k^e + s + c_{k,r}(t_k^e + s)) \]

\[ \leq \int_0^t \{ b_{n,r}(t_k^e + s) |\psi_{k,n}^s(t_k^e + s)| + c_{n,r}(t_k^e + s) \} ds \]

\[ \leq \int_0^t \{ L + c_{n,r}(t_k^e + s) \} ds, \quad \forall t \in [0, T], \]

where $b_{n,r}$ and $c_{n,r}$ are respectively as $b_r$ and $c_r$ in (c) of Definition 2.6.1 corresponding to the family \( \{ \phi^k_n; 0 \leq t \leq T \} \). Now, put

\[ u_n^e(t) = \begin{cases} v_{k,n}(t - t_k^e) & \text{if } t \in E^k_k, \quad k = 1, 2, ..., m, \\ h_n(t) & \text{otherwise} \end{cases} \]

with $h_n$ as in (b) of Definition 2.6.1 corresponding to \( \{ \phi^k_n; 0 \leq t \leq T \} \). Then, by (7.6), we see that for all $t \in E^k_k$, $k = 1, 2, ..., m$ and for all $n \geq N_e$,

\[ |u(t) - u_n^e(t)| \leq |u(t) - u(t_k^e)| + |u(t_k^e) - x_{k,n}^e| + |x_{k,n}^e - u_n^e(t)| \leq 2\epsilon + \sqrt{\epsilon} L, \]

so that for all $n \geq N_e$
\[ \| u_n^e - u \|^2 \leq \int_{[0, T]} \Lambda e \left| h_n(t) - u(t) \right|^2 dt + (2e + \sqrt{e} L)^2 T \]
\[ \leq e (\alpha_0 + \rho)^2 + (2e + \sqrt{e} L)^2 T \]

Also, by (7.7) we have for all \( k = 1, 2, \ldots, m \) and all \( n \geq N_e \)

\[
\int_{E_k} \phi^e_n (u_n^e(t)) \, dt - \int_{E_k} \phi^e (u(t)) \, dt
\]
\[
= \int_{E_k} \left\{ \phi^e_n (u_n^e(t)) - \phi^e_n (s_k, r(s)) \right\} \, dt + \text{meas.} (E_k^e) \left( \phi^e_n (s_k, r(s)) - \phi^e (u(t)) \right)
\]
\[
+ \int_{E_k} \left\{ \phi^e (u(t)) - \phi^e (u(t)) \right\} \, dt
\]
\[
\leq e \int_{t_k}^{t_{k+1}} \left\{ Lb_{n,r}(s) + c_{n,r}(s) \right\} \, ds + 2e \text{meas.} (E_k^e),
\]
so that

\[
\int_{E_k} \phi^e_n (u_n^e(t)) \, dt - \int_{E_k} \phi^e (u(t)) \, dt
\]
\[
\leq e \int_0^T \left\{ Lb_{n,r}(s) + c_{n,r}(s) \right\} \, ds + 2e \text{meas.} (E^e)
\]
\[
\leq e (LM_r + M_r + 2T), \quad \forall n \geq N_e.
\]

Therefore

\[
\Phi^T_n (u_n^e) - \Phi^T (u)
\]
\[
\leq \int_{[0, T]} \Lambda e \left\{ \phi^e_n (h_n(t)) - \phi^e (u(t)) \right\} \, dt + e (LM_r + M_r + 2T)
\]
\[
\leq e (\alpha_0 + \rho + LM_r + M_r + 2T), \quad \forall n \geq N_e.
\]

Thus we have shown above the following statement: Given any number \( \nu > 0 \), there are a positive integer \( N_0 \) and a sequence \( \{ u_n^\nu \} \) in \( L^2 (0, T; H) \) such that

\[ \| u_n^\nu - u \| \leq \nu, \quad \Phi^T_n (u_n^\nu) \leq \Phi^T (u) + \nu, \quad \forall n \geq N_0. \]

Making use of such a sequence \( \{ u_n^\nu \} \), we can easily construct a sequence \( \{ u_n \} \) having the properties that

\[ u_n \to u \quad \text{in} \quad L^2 (0, T; H) \]

and

\[ \limsup_{n \to \infty} \Phi^T_n (u_n) \leq \Phi^T (u). \]

Since (7.8) implies \( \Phi^T_n (u_n) \to \Phi^T (u) \) on account of (i) of Proposition 2.7.1, this sequence \( \{ u_n \} \) is a desired one.

(b) The general case. For simplicity we assume \( T' = T \). To each positive integer \( m \), choose
a closed subset $F_m$ of $[0, T]$ satisfying

$$\text{meas. } (0, T \setminus F_m) \leq \frac{1}{m}$$

and $u$ is bounded on $F_m$ as well as $t \to \phi^t(u(t))$ on $F_m$. Moreover, put

$$u^m(t) = \begin{cases} u(t) & \text{if } t \in F_m, \\ h(t) & \text{otherwise} \end{cases}$$

with $h$ of (b) of Definition 2.6.1. Then, clearly $u^m$ and $t \to \phi^t(u^m(t))$ are bounded on $[0, T]$ and

$$u^m \to u \quad \text{in } L^2(0, T; H), \quad \phi^T(u^m) \to \phi^T(u)$$

Here, by virtue of the result of the step (a), for each $m$ there is a sequence $\{u_n^m; n = 1, 2, \ldots \}$ in $L^2(0, T; H)$ such that

$$u_n^m \to u^m \quad \text{in } L^2(0, T; H), \quad \phi^T_n(u_n^m) \to \phi^T(u^m) \quad \text{as } n \to \infty.$$ 

Making use of these sequences $\{u_n^m\}$, we can easily obtain a sequence $\{u_n\}$ having the required properties.

Q.E.D.

Remark 2.7.1. Let $\{\phi^t; 0 \leq t \leq T\}$ and $\{\phi^t_n; 0 \leq t \leq T\}$ be families of proper l.s.c. convex functions on $H$, and suppose that $\phi^t_n \to \phi^t$ on $H$ in the sense of Mosco for every $t \in [0, T]$. Then it is interesting to find supplementary conditions in order that

$$\phi^T_n \to \phi^T \quad \text{on } L^2(0, T'; H) \quad \text{in the sense of Mosco}$$

for every $T' \in (0, T]$.

Our convergence theorem for solutions is stated as follows:

Theorem 2.7.1. Let $\{\phi^t; 0 \leq t \leq T\}$ and $\{\phi^t_n; 0 \leq t \leq T\}$, $n = 1, 2, \ldots$, be in the class $G = G(a_0, \{\delta_r\}, \{M_r\})$, and suppose that $\phi^t_n \to \phi^t$ on $H$ in the sense of Mosco for every $t \in [0, T]$. Further let $u_n$ be the weak solution to $CP(\phi^t_n; f_n, u_{0,n})$ with $f_n \in L^2(0, T; H)$ and $u_{0,n} \in D(\phi^0_n)$.

If $f_n \to f$ in $L^2(0, T; H)$ and if $u_{0,n} \to u_0$ in $H$ with $u_0 \in D(\phi^0)$, then $u_n$ converges in $C([0, T]; H)$ and the limit $u$ is the weak solution to $CP(\phi^t; f, u_0)$, and moreover $\phi^t_n(\{u_n\}) \to \phi^t(f, u_0)$ uniformly on $[0, T]$. In particular, if $\{\phi^t_n(\{u_{0,n}\})\}$ is bounded, then $u$ is the strong solution to $CP(\phi^t_n; f, u_0)$.

Proof. The first step. We consider the case that $\{\phi^0_n(\{u_{0,n}\})\}$ is bounded. In this case, by Theorem 2.6.1, $\{u_n\}$ is bounded in $W^{1,2}(0, T; H)$ as well as $t \to \phi^t_n(u_n(t))$ is uniformly bounded on $[0, T]$. Also, according to Theorem 2.4.1 and Remark 2.4.1, $u_n$ is characterized by the following

$$\int_0^t (u_n(s) - f_n(s), u_n(s) - w(s)) ds \leq \Phi^t_n(w) - \Phi^t_n(\{u_n\}),$$

$$\forall t \in [0, T], \quad \forall w \in L^2(0, t; H) \quad \text{with } \Phi^t_n(w) < \infty. \quad (7.9)$$

Now, choose a subsequence $\{u_{nk}\}$ of $\{u_n\}$ so that $u_{nk} \to u$ in $L^2(0, T; H)$ and $u_{nk} \to u'$ in $L^2(0, T; H)$ (hence $u_{nk}(t) \to u(t)$ in $H$ for all $t \in [0, T]$). Clearly $u \in W^{1,2}(0, T; H)$ with $u(0) = u_0$, and by (i) of Proposition 2.7.1

$$\phi^t(u) \leq \liminf_{k \to \infty} \Phi^t_{nk}(u_{nk}), \quad \forall t \in [0, T]. \quad (7.10)$$

Furthermore we take a sequence $\{v_k\}$ in $L^2(0, t; H)$ for each $t \in [0, T]$ such that
\( v_k \to u \) in \( L^2(0, t; H) \), \( \Phi^{t}_{n_k}(v_k) \to \Phi^t(u) \),

and substitute \( v_k \) as \( w \) of (7.9) in the case of \( n = n_k \). We then get

\[
\int_0^t (u'_{n_k} - f_{n_k}, u_{n_k} - u_k) \, ds \leq \Phi^{t}_{n_k}(v_k) - \Phi^{t}_{n_k}(u_{n_k}),
\]

so that

\[
limeqsup_{k \to \infty} \int_0^t (u'_{n_k}, u_{n_k} - u_k) \, ds \leq 0.
\]

Hence for every \( t \in [0, T] \)

\[
\frac{1}{2} \| u(t) \|^2 - \frac{1}{2} \| u_0 \|^2 = \int_0^t (u', u) \, ds \leq \limsup_{k \to \infty} \int_0^t (u'_{n_k}, u_{n_k}) \, ds
\]

\[
= \limsup_{k \to \infty} \left\{ \frac{1}{2} \| u_{n_k}(t) \|^2 - \frac{1}{2} \| u_{0, n_k} \|^2 \right\} = \limsup_{k \to \infty} \frac{1}{2} \| u_{n_k}(t) \|^2 - \frac{1}{2} \| u_0 \|^2,
\]

i.e.

\[
limeqsup_{k \to \infty} \| u_{n_k}(t) \| \leq \| u(t) \|
\]

This implies \( u_{n_k}(t) \to u(t) \) in \( H \) for every \( t \in [0, T] \), and hence

\( u_{n_k} \to u \) in \( C([0, T]; H) \)

and

(7.11)

\[
limeqsup_{k \to \infty} \Phi^{t}_{n_k}(u_{n_k}) \leq \Phi^t(u), \quad \forall t \in [0, T].
\]

By (7.10) and (7.11) we have

\[
\Phi^{t}_{n_k}(u_{n_k}) \to \Phi^t(u), \quad \forall t \in [0, T].
\]

Next, given \( w \in D(\Phi^T) \), we take a sequence \( \{w_n\} \) in \( L^2(0, T; H) \) such that

\( w_n \to w \) in \( L^2(0, T; H) \), \( \Phi^{T}_{n}(w_n) \to \Phi^{T}(w) \)

and substitute \( w_{n_k} \) as \( w \) of (7.9) with \( n = n_k \) and \( t = T \). Then, letting \( k \to \infty \), we obtain

\[
< u' - f, u - w > \leq \Phi^{T}(w) - \Phi^{T}(u), \quad \forall w \in D(\Phi^T),
\]

which implies by Theorem 2.4.1 that \( u \) is the strong solution to \( CP(\phi^T; f, u_0) \) and by the way \( u_n \to u \) in \( C([0, T]; H) \) and

\[
\Phi^{t}_{n}(u_n) \to \Phi^{t}(u), \quad \forall t \in [0, T].
\]

Moreover the last convergence is uniform with respect to \( t \) in \( [0, T] \), which is easily derived from the uniform boundedness of \( t \to \Phi^{t}_{n}(u_n(t)) \) on \( [0, T] \).

The second step. We prove the general case by using the result of the first step. Let \( \varepsilon \) be an arbitrary positive number. We then choose an element \( z^\varepsilon \) in \( D(\phi^0) \) and a positive integer \( n_\varepsilon(e) \) such that

\[
\|u_0 - z^\varepsilon\| \leq \varepsilon, \quad \|u_0 - u_{0,n}\| \leq \varepsilon, \quad \forall n \geq n_\varepsilon(e).
\]

Next, take a sequence \( \{z^\varepsilon_n\} \) with \( z^\varepsilon_n \in D(\phi^0_n) \) so that
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\[ z_n^e \to z^e \quad \text{in } H, \quad \phi_n^0(z_n^e) \to \phi_0(z^e), \]

and a positive integer \( n_2(\epsilon) \) so that

\[ |z_n^e - z^e| \leq \epsilon, \quad \forall n \geq n_2(\epsilon). \]

Moreover, denote by \( u, u^e, u_n \) and \( u_n^e \) the weak or strong solutions to \( CP(\phi^f; f, u_0) \), \( CP(\phi^f; f, z^e) \), \( CP(\phi_n^f; f_n, u_0, n) \) and \( CP(\phi_n^f; f_n, z_n^e) \), respectively. Then, by Theorem 1.1.1

\[ |u - u^e|_{C([0, T]; H)} \leq |u_0 - z^e| \leq \epsilon \]

and

\[ |u_n - u_n^e|_{C([0, T]; H)} \leq |u_0, n - z_n^e| \leq |u_0, n - u_0| + |u_0 - z^e| + |z^e - z_n^e| \leq 3 \epsilon, \quad \forall n \geq \max \{ n_1(\epsilon), n_2(\epsilon) \}. \]

Also, by the result of the first step we see that

\[ u_n^e \to u^e \quad \text{in } C([0, T]; H), \quad \Phi_n^f(u_n^e) \to \Phi^f(u^e) \quad \text{uniformly in } t \in [0, T]. \]

Therefore,

\[ \limsup_{n \to \infty} \{|u_n - u|_{C([0, T]; H)} \} \leq \limsup_{n \to \infty} \{ |u_n^e - u_n^e|_{C([0, T]; H)} + |u^e - u|_{C([0, T]; H)} \} \leq 4 \epsilon. \]

Since \( \epsilon \) is arbitrary, we conclude

\[ u_n \to u \quad \text{in } C([0, T]; H). \]

Finally we show that \( \Phi_n^f(u_n) \to \Phi^f(u) \) uniformly in \( t \in [0, T] \). To this purpose we observe from Theorem 2.4.1 and Remark 2.4.1 that

\[ (7.12) \quad \int_0^t \left( \frac{du_n^e}{ds} - f_n, u_n^e - u_n^e \right) ds - \frac{1}{2} \left| z_n^e - u_0, n \right|^2 \leq \Phi_n^f(u_n^e) - \Phi_n^f(u_n), \quad \forall t \in [0, T]. \]

We note here that there is a positive integer \( n_3(\epsilon) \) satisfying

\[ |\Phi_n^f(u_n^e) - \Phi^f(u^e)| \leq \epsilon, \quad \forall t \in [0, T], \]

\[ |z_n^e - z^e| \leq \epsilon, \quad |u_0, n - u_0| \leq \epsilon \]

for all \( n \geq n_3(\epsilon) \) and there is a positive constant \( K(\epsilon) \) satisfying

\[ \| \frac{du_n^e}{ds} - f_n \| \leq K(\epsilon), \quad \forall n \geq 1. \]

Therefore, from (7.12) it follows that

\[ \Phi_n^f(u_n) \leq \Phi^f(u^e) + \epsilon + 3 \sqrt{t K(\epsilon) \epsilon} + \frac{9}{2} \epsilon^2, \quad \forall t \in [0, T], \quad \forall n \geq n_3(\epsilon). \]

This shows the following statement: Given \( \nu > 0 \), there are \( \delta \in (0, T) \) and a positive integer
$N_v$ such that

$$\left| \Phi^n u_n \right| \leq v, \quad \forall n \geq N_v.$$  

Furthermore, since \( \{ \phi^n u_n(\delta) \} \) is bounded by Theorem 2.6.1, by applying the result of the first step to Cauchy problems \( CP(\phi^{t+\delta}; f(\tau+\delta), u(\delta)) \) and \( CP(\phi^n t+\delta; f_n(\tau+\delta), u_n(\delta)) \) we obtain that

$$\int_0^t \phi^n u_n(s) ds \to \int_0^t \phi^t u(s) ds$$  
uniformly in \( t \in [\delta, T] \).

From (7.13) and (7.14) we conclude that

$$\Phi^n u_n \to \Phi^t u$$  
uniformly on \( [0, T] \).

§ 2.8. Evolution equation $u^{+}(t) + \partial \phi^t(Bu(t)) \ni f(t)$

In this section we devote ourselves to the study of the Cauchy problem

$$CP(\phi^t; B; f, u_0) \quad \begin{cases} u^+(t) + \partial \phi^t(Bu(t)) \ni f(t), & 0 < t < T, \\ u(0) = u_0. \end{cases}$$

where $B: H \to H$ is a singlevalued operator with $D(B) = H$, \( \{ \phi^t; 0 \leq t \leq T \} \) is a family of proper 1.s.c. convex functions on $H$ and $u_0$ is given in $H$ as well as $f$ in $L^1(0, T; H)$.

**Definition 2.8.1** (i) Given $u_0 \in H$ and $f \in L^1(0, T; H)$, a function $u: [0, T] \to H$ is called a strong solution to $CP(\phi^t; B; f, u_0)$ if the following conditions (a) and (b) are fulfilled:

(a) $u$ is an $H$-valued absolutely continuous function on $[0, T]$ with $u(0) = u_0$.

(b) $Bu(t) \in D(\partial \phi^t)$ for a.e. $t \in [0, T]$ and there is $u^* \in L^1(0, T; H)$ such that

$$u^+(t) + u^*(t) \in \partial \phi^t(Bu(t)) \text{ for a.e. } t \in [0, T].$$

(ii) Given $u_0 \in H$ and $f \in L^1(0, T; H)$, a function $u: [0, T] \to H$ is called a weak solution, if there are sequences \( \{ u_n \} \subset C([0, T]; H) \) and \( \{ f_n \} \subset L^1(0, T; H) \) such that $u_n$ is a strong solution to $CP(\phi^t; B; f_n, u_n(0))$, $u_n \to u$ in $C([0, T]; H)$ and $f_n \to f$ in $L^1(0, T; H)$.

(A) Existence of strong solutions

We begin by stating an existence theorem.

**Theorem 2.8.1.** Suppose that condition $(H)_2$ (see § 1.5) is satisfied with the following:

$$\{ \text{for each } t \in [0, T] \text{ and } r \geq 0 \text{ the set } \{ z \in H; \ |z| \leq r, \ |\phi^t(z)| \leq r \} \text{ is relatively compact in } H. $$

Further suppose that there are positive constants $C_0$ and $C_0'$ such that

$$C_0 |Bz - Bz_1|^2 \leq (Bz - Bz_1, z - z_1), \quad C_0' |z - z_1| \leq |Bz - Bz_1|, \quad \forall z, z_1 \in H,$$

and that $B$ is the subdifferential of a finite continuous convex function $j$ on $H$ with $j(0) = 0$, i.e. $B = \partial j$. Then for each $u_0 \in H$ with $Bu_0 \in D(\phi^0)$ and $f \in L^2(0, T; H)$, $CP(\phi^t; B; f, u_0)$ admits at least one strong solution $u$ such that

(a) $u \in W^{1,2}(0, T; H)$;

(b) $t \to \phi^t(Bu(t))$ is bounded on $[0, T]$.
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(c) for any function \( h \in W^{1,2}(0, T; H) \) with \( \phi^1(r(t),.) \in L^2(0, T) \) we have

\[
J(u(t)) + \int_0^t \phi^1(Bu(\tau)) d\tau - (u(t), h(\tau)) + \int_0^t (u(\tau), h'(\tau)) d\tau - \int_0^t (f(\tau), Bu(\tau)) d\tau \\
\leq J(u_0) + \int_0^t \phi^1(h(\tau)) d\tau - (u_0, h(\tau)) - \int_0^t (f(\tau), h(\tau)) d\tau, \quad \forall \tau \in [0, T];
\]

(8.3)

(d) there is a positive constant \( r_0 = r_0 (\|u_0\|, \|u_0\|) \) such that

\[
\phi^1(Bu(t)) - \phi^2(Bu(s)) + \frac{C_0}{2} \int_s^t d\tau (Bu(\tau))^2 d\tau \leq \int_s^t \{k_{r, 1}(\tau) \phi^1(Bu(\tau)) + k_{r, 2}(\tau)\} d\tau
\]

and

\[
t\phi^1(Bu(t)) - s\phi^2(Bu(s)) + \frac{C_0}{2} \int_0^t \tau \frac{d}{d\tau} Bu(\tau)^2 d\tau \\
\leq \int_0^t \{\tau k_{r, 1}(\tau + 1) \phi^1(Bu(\tau)) + \tau k_{r, 2}(\tau)\} d\tau
\]

(8.5)

for all \( r \geq r_0 \) and all \( s, t \in [0, T] \) with \( s \leq t \), where

\[
k_{r, 1}(\tau) = \frac{4}{C_0 C_0'} \|\alpha(\tau)\|^2 + 4 \|\alpha(\tau)\| \|f(\tau)\| + \|\beta(\tau)\|
\]

with \( \alpha, \beta \) as in condition (H)2 and

\[
k_{r, 2}(\tau) = k_{r, 1}(\tau) \left(1 + 4\alpha(\tau + 1)\right) + \frac{1}{C_0} \|f(\tau)\|^2
\]

with the constant \( \alpha \) of Lemma 1.5.1.

To prove this theorem we prepare some lemmas. We make always all the assumptions of Theorem 2.8.1.

Lemma 2.8.1. (1) There are positive constants \( C_1, C_1' \) such that

\[
C_1 (|z|^2 - 1) \leq j(z) \leq C_1' (|z|^2 + 1), \quad \forall z \in H.
\]

(2) Let \( v \in W^{1,1}(0, T; H) \). Then we have

\[
(v'(t), Bu(t)) = \frac{d}{dt} j(v(t)) \text{ for a.e. } t \in [0, T]
\]

and

\[
(v'(t), \frac{d}{dt} Bu(t)) \geq C_0 \|\frac{d}{dt} Bu(t)\|^2 \text{ for a.e. } t \in [0, T],
\]

where \( C_0 \) is the constant of (8.2).

In fact this lemma follows immediately from assumptions imposed on \( B \). We also recall some results in §1.5: under (H)2 there are positive constants \( \alpha \) and \( K \) such that

\[
\phi^1(z) + \alpha |z| + \alpha \geq 0, \quad \forall t \in [0, T], \forall z \in H,
\]

and

\[
|\lambda|^2 \leq K + |z|, \quad \forall t \in [0, T], \forall z \in H, \forall \lambda \in [0, 1].
\]

Lemma 2.8.2. Let \( v \in W^{1,1}(0, T; H) \) and \( 0 < \lambda \leq 1 \). Then the function \( t \to \phi^1_{\lambda}(v(t)) \) is differentiable at a.e. \( t \in [0, T] \) and the derivative is integrable on \([0, T]\). Moreover we have
(8.8) \[ \phi^s_x(v(t)) - \phi^s_x(v(s)) \leq \int_s^t \frac{d}{dt} \phi^x_x(v(t)) \, dt \]
for any \( s, t \in [0, T] \) with \( s \leq t \), and

(8.9) \[
\frac{d}{dt} \phi^x_x(v(t)) - (v'(t), \partial \phi^x_x(v(t))) \\
\leq |\alpha'_r(t)| \|\partial \phi^x_x(v(t))\| (|\phi^x_x(v(t))|^{1/2} + 1 + \sqrt{2\alpha(K + |v(t)| + 1)}) \\
+ |\beta'_r(t)| (|\phi^x_x(v(t))| + 1 + 2\alpha(K + |v(t)| + 1))
\]
for a.e. \( t \in [0, T] \), where \( \alpha_r, \beta_r \) are as in \((H)_2\) corresponding to

\[ r \geq \sup \left\{ \|v(t)\|_z; 0 \leq t \leq T, 0 < \lambda \leq 1 \right\}. \]

**Proof.** Let \( \lambda \in (0, 1) \), \( z \in H \) and \( r \geq \sup \left\{ \|v(t)\|_z; 0 \leq t \leq T, 0 < \lambda \leq 1 \right\} \). Then taking account of (8.6) and (8.7), we have (cf. the proof of Theorem 1.5.1)

(8.10) \[
\phi^s_x(z) - \phi^s_x(z) \leq \int_s^t \frac{d}{dt} \phi^x_x(z) \, dt, \quad \forall s, t \in [0, T], s \leq t,
\]
and for a.e. \( t \in [0, T] \)

(8.11) \[
\frac{d}{dt} \phi^x_x(z) \leq |\alpha'_r(t)| \|\partial \phi^x_x(v(t))\| (|\phi^x_x(z)|^{1/2} + 1 + \sqrt{2\alpha(K + |z| + 1)}) \\
+ |\beta'_r(t)| (|\phi^x_x(z)| + 1 + 2\alpha(K + |z| + 1)).
\]

From (8.10) and (8.11) we obtain (8.8) and (8.9) just as Lemma 1.2.5. Q.E.D.

Let \( \lambda \in (0, 1) \). Then, the operator \( z \to \partial \phi^x_x(Bz) \) is lipschitz continuous on \( H \) with \((C_0\lambda)^{-1}\) as a lipschitz constant for each \( t \in [0, T] \), the function \( t \to \phi^x_x(Bz) \) is measurable on \([0, T]\) for each \( z \in H \) and \((t, z) \to \phi^x_x(Bz) \) is bounded on each bounded subset of \([0, T] \times H \) (cf. §1.2).

Therefore there exists a unique \( u_\lambda \in W^{1, 2}(0, T; H) \) satisfying

(8.12) \[
u^s_x(t) + \partial \phi^x_x(Bu_\lambda(t)) = f(t) \quad \text{for a.e. } t \in [0, T]
\]

\( \rightarrow \to \)

\( u_\lambda(0) = u_0, \)

where \( u_0 \in H \) and \( f \in L^2(0, T; H) \).

We are going to give some estimations for \( \{ u_\lambda; 0 < \lambda \leq 1 \} \) independent of \( \lambda \). For this purpose we take a function \( h \in W^{1, 2}(0, T; H) \) such that \( t \to \phi^f(h(t)) \) is integrable on \([0, T] \); for example, we can take as \( h \) a strong solution to \( CP(\phi^f; 0, h_0) \) with \( h_0 \in D(\phi^0) \). We multiply (8.12) by \( Bu_\lambda(t) - h(t) \) to get

(8.13) \[
(u^s_x(t), Bu_\lambda(t) - h(t)) + (\partial \phi^x_x(Bu_\lambda(t)), Bu_\lambda(t) - h(t)) \\
= (f(t), Bu_\lambda(t) - h(t)) \quad \text{for a.e. } t \in [0, T].
\]

Since

\[
(u^s_x(t), Bu_\lambda(t)) = \frac{d}{dt} \int (u^s_x(t)) \quad (\text{cf. (2) of Lemma 2.8.1})
\]

and

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\[(\partial \phi^f_B(u_H(t)), B_H(t) - h(t)) \geq \phi^f_B(u_H(t)) - \phi^f_B(h(t)) \geq \delta \phi^f_B(u_H(t)) - \phi^f_B(h(t))\]

for a.e. \(t \in [0, T]\), we obtain by integrating (8.13) on \([0, t]\)

\[
\begin{aligned}
&\int (u_H(t)) + \int_0^t \phi^f_B(B_H(\tau)) d\tau - (u_H(t), h(\tau)) + \int_0^t (u_H(\tau), h'(\tau)) d\tau \\
&- \int_0^t (f(\tau), B_H(\tau)) d\tau \leq \int (u_0) + \int_0^t \phi^f_B(h(\tau)) d\tau - (u_0, h(0)) - \int_0^t (f(\tau), h(\tau)) d\tau
\end{aligned}
\]  

(8.14)

for all \(t \in [0, T]\). From (8.14) we derive easily the following lemma.

**Lemma 2.8.3.** There is a positive constant \(N = N_d (\| f \|, |u_0|)\) such that

\[|B_H(t)| \leq N_d, \quad |u_H(t)| \leq N_d, \quad \forall t \in [0, T], \quad \forall \lambda \in (0, 1)\]

and

\[\int_0^T \| \phi_B^f(u_H(t)) \| dt \leq N_d, \quad \forall \lambda \in (0, 1).\]

Next, multiply (8.12) by \((d/dt)B_H(t)\) to get

\[\left(\alpha(t), \frac{d}{dt}B_H(t)\right) + (\partial \phi_B^f(B_H(t)), \frac{d}{dt}B_H(t)) = (f(t), \frac{d}{dt}B_H(t))\]

for a.e. \(t \in [0, T]\). We note here that for a.e. \(t \in [0, T]\)

\[\left(\alpha(t), \frac{d}{dt}B_H(t)\right) \geq C_0 |\frac{d}{dt}B_H(t)|^2 \quad \text{(cf. (2) of Lemma 2.8.1)},\]

\[\left(f(t), \frac{d}{dt}B_H(t)\right) \leq -C_0 |\frac{d}{dt}B_H(t)|^2 + \frac{1}{C_0} |f(t)|^2\]

and

\[|\phi_B^f(B_H(t))| \leq \phi_B^f(B_H(t)) + 2\alpha(K + N_d + 1).\]

Furthermore by Lemma 2.8.2 we see that for a.e. \(t \in [0, T]\) and any \(r \geq K + N_d\)

\[
\begin{aligned}
\frac{d}{dt} \phi_B^f(B_H(t)) - (\partial \phi_B^f(B_H(t)), \frac{d}{dt}B_H(t)) \\
\leq |\alpha_r(t)||\partial \phi_B^f(B_H(t))| \left\{|\phi_B^f(B_H(t))|^{1/2} + 1 + \sqrt{2\alpha(K + N_d + 1)} \right\} \\
+ |\phi_B^f(t)| \left\{|\phi_B^f(B_H(t))| + 1 + 2\alpha(K + N_d + 1) \right\}.
\end{aligned}
\]

Therefore we deduce from these inequalities that

\[
\frac{C_0}{2} |\frac{d}{dt}B_H(t)|^2 + \frac{d}{dt} \phi_B^f(B_H(t)) \leq k_r, 1(t) \phi_B^f(B_H(t)) + k_r, 2(t)
\]

for a.e. \(t \in [0, T]\) and all \(r \geq K + N_d\), where

\[k_r, 1(t) = \frac{4}{C_0 C_r^2} \alpha_r^2 + 4 |\alpha_r(t)||f(t)| + |\phi_B^f(t)|\]

and

\[k_r, 2(t) = k_r, 1(t) \left(1 + 4\alpha(r + 1) \right) + \frac{1}{C_0} |f(t)|^2.\]

By the way we have
(8.16) \[ \frac{C_0 t}{2} \left\{ \frac{d}{dt} B_{\lambda}(t) \right\}^2 + \frac{d}{dt} \left\{ t\phi_{\lambda}^{t}(B_{\lambda}(t)) \right\} \leq (tk_{r,1}(t) + 1) \phi_{\lambda}^{t}(B_{\lambda}(t)) + tk_{r,2}(t) \]

for a.e. \( t \in [0, T] \) and all \( r \geq K + N_{a} \).

**Lemma 2.8.4.** There is a positive constant \( N_{5} = N_{5}(\|f\|, \|u_{0}\|) \) such that

(8.17) \[ |t\phi_{\lambda}^{t}(B_{\lambda}(t))| \leq N_{5}, \quad \forall t \in [0, T], \quad \forall \lambda \in (0, 1) \]

and

(8.18) \[ \|t \frac{d}{dt} B_{\lambda}\| \leq N_{5}, \quad \forall \lambda \in (0, 1). \]

Moreover, if \( B_{u_{0}} \in D(\varphi^{0}) \), then there is a positive constant \( N_{6} = N_{6}(\|f\|, \|u_{0}\|, \varphi^{0}(B_{u_{0}})) \) such that

(8.19) \[ |\phi_{\lambda}^{t}(B_{\lambda}(t))| \leq N_{6}, \quad \forall t \in [0, T], \quad \forall \lambda \in (0, 1) \]

and

(8.20) \[ \|t \frac{d}{dt} B_{\lambda}\| \leq N_{6}, \quad \forall \lambda \in (0, 1). \]

In fact, applying Proposition 0.4.1 to the inequality (8.16) (resp. (8.15)), we get (8.17) and (8.18) (resp. (8.19) and (8.20)).

Now, assume that \( B_{u_{0}} \in D(\varphi^{0}) \). Then, taking account of (8.1) and the estimations of Lemmas 2.8.3 and 2.8.4, we can select a sequence \( \{\lambda_{n}\} \) with \( \lambda_{n} \downarrow 0 \) so that

\[ B_{\lambda_{n}} \rightharpoonup w \quad \text{weakly* in } L^\infty(0, T; H), \]

\[ \frac{d}{dt} B_{\lambda_{n}} \rightharpoonup w' \quad \text{in } L^{2}(0, T; H), \]

\[ B_{\lambda_{n}}(t) \rightharpoonup w(t) \quad \text{in } H, \quad \forall t \in [0, T] \]

and

\[ \phi_{\lambda_{n}}^{t}(B_{\lambda_{n}}(t)) \rightharpoonup \chi \quad \text{weakly* in } L^\infty(0, T). \]

Here, putting \( u(t) = B^{-1}w(t) \), i.e. \( w(t) = Bu(t) \), we see that \( u \in W^{1,2}(0, T; H) \) and \( u(0) = u_{0} \). Furthermore we have

**Lemma 2.8.5.** \( J_{\lambda_{n}}^{t}Bu_{\lambda_{n}}(t) \rightarrow Bu(t) \) and \( Bu_{\lambda_{n}}(t) \rightarrow Bu(t) \) in \( H \) uniformly on \( [0, T] \), and \( u_{\lambda_{n}} \rightharpoonup u' \) in \( L^{2}(0, T; H) \).

**Proof.** From (8.19) it follows that

\[ \frac{1}{2\lambda_{n}} |Bu_{\lambda_{n}}(t) - J_{\lambda_{n}}^{t}Bu_{\lambda_{n}}(t)|^{2} \leq N_{6} - \phi^{t}(J_{\lambda_{n}}^{t}Bu_{\lambda_{n}}(t)) \]

\[ \leq N_{6} + \alpha |J_{\lambda_{n}}^{t}Bu_{\lambda_{n}}(t)| + \alpha \leq N_{6} + \alpha (K + N_{a} + 1) \quad \text{for all } t \in [0, T], \]

so that

\[ |J_{\lambda_{n}}^{t}Bu_{\lambda_{n}}(t) - Bu_{\lambda_{n}}(t)| \rightarrow 0 \quad \text{uniformly on } [0, T] \]

and

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\[ |\phi(t) / J_{\lambda_n} Bu_{\lambda_n}(t)| \leq N_0 + \alpha (K + N_4 + 1), \quad \forall t \in [0, T]. \]

Therefore, by condition (8.1),

\[ J_{\lambda_n} Bu_{\lambda_n}(t) \rightarrow w(t) = Bu(t) \quad \text{in } H, \quad \forall t \in [0, T] \]

and hence

\[ Bu_{\lambda_n}(t) \rightarrow Bu(t) \quad \text{in } H, \quad \forall t \in [0, T]. \]

Since \( \{Bu_{\lambda_n}\} \) is equi-continuous on \([0, T]\), we get consequently that

\[ Bu_{\lambda_n} \rightarrow Bu \text{ (hence } u_{\lambda_n} \rightarrow u \text{)} \quad \text{in } C([0, T]; H) \]

and subsequently

\[ u_{\lambda_n} \rightarrow u' \quad \text{in } L^2(0, T; H), \]

\[ J_{\lambda_n} Bu_{\lambda_n}(t) \rightarrow Bu(t) \quad \text{in } H \text{ uniformly on } [0, T]. \]

Q.E.D.

Lemma 2.8.6. \( \chi = \phi^t(Bu(.)) \), \( f(t) - u'(t) \in \partial\phi^t(Bu(t)) \) for a.e. \( t \in [0, T] \) and \( u \) is a strong solution to \( CP(\phi^t, B; f, u_0) \).

Proof. We note that

\[ (f(t) - u_{\lambda_n}'(t), w(t) - Bu_{\lambda_n}(t)) \leq \phi^t_n(Bu_{\lambda_n}(t)) \]

(8.21) \( \quad \text{for any } w \in L^2(0, T; H) \text{ and a.e. } t \in [0, T] \)

and by the lower semicontinuity of \( \phi^t \)

\[ N_0 \leq \liminf_{n \rightarrow \infty} \phi^t_n(Bu_{\lambda_n}(t)) \leq \liminf_{n \rightarrow \infty} \phi^t(J_{\lambda_n} Bu_{\lambda_n}(t)) \leq \phi^t(Bu(t)) \]

(8.22) \( \quad \text{for every } t \in [0, T]. \) Let \( E \) be an arbitrary measurable subset of \([0, T]\). Then by integrating (8.21) on \( E \) and letting \( n \rightarrow \infty \) we obtain

\[ \int_E (f(t) - u'(t), w(t) - Bu(t)) dt \leq \int_E \phi^t(w(t)) dt - \limsup_{n \rightarrow \infty} \int_E \phi^t_n(Bu_{\lambda_n}(t)) dt \]

\[ \leq \int_E \phi^t(w(t)) dt - \int_E \phi^t(Bu(t)) dt, \quad \forall w \in L^2(0, T; H) \text{ with } \phi^t(w(.)) \in L^1(0, T); \]

in particular, if we take \( w = Bu \), then

\[ \limsup_{n \rightarrow \infty} \int_E \phi^t_n(Bu_{\lambda_n}(t)) dt \leq \int_E \phi^t(Bu(t)) dt. \]

Combining this with (8.22), we get

\[ \lim_{n \rightarrow \infty} \int_E \phi^t_n(Bu_{\lambda_n}(t)) dt = \int_E \phi^t(Bu(t)) dt, \]

which implies \( \chi = \phi^t(Bu(.)) \), because \( E \) is any measurable set in \([0, T]\). Also, (8.23) with \( E = [0, T] \) shows (cf. Proposition 0.3.3) that

\[ f(t) - u'(t) \in \partial\phi^t(Bu(t)) \quad \text{for a.e. } t \in [0, T]. \]

Hence \( u \) is a strong solution to \( CP(\phi^t, B; f, u_0) \).

Q.E.D.

Lemma 2.8.7. There are a countable dense subset \( E_0 \) of \([0, T]\) and a subsequence \( \{\lambda_n\} \).

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of \{\lambda_n\} such that

$$\lim_{n \to \infty} \phi_{n'}^t (Bu_{n'}(t)) = \phi^t(Bu(t)), \quad \forall t \in E_0.$$  

\textbf{Proof.} We denote by \(E\) the set of all \(t \in [0, T]\) such that the sequence \(\{f(t) - u_{n'}(t)\}\) contains a bounded subsequence in \(H\). Since \(\{f - u_{n'}\}\) is bounded in \(L^2(0, T; H)\), we see that \(\{f(t) - u_{n'}(t)\} \in L^2(0, T; H)\). Therefore we can find a countable dense subset \(E_0\) of \(E\) and a subsequence \(\{\lambda_{n'}\}\) of \(\{\lambda_n\}\) such that \(\{f(t) - u_{n'}(t) = \phi_{n'}^t(Bu_{n'}(t))\}\) is bounded in \(H\) for every \(t \in E_0\). According to (8) of Proposition 3.3.5, we have (8.24). Q.E.D.

For simplicity we denote again by \(\lambda_n\) the subsequence \(\lambda_{n'}\) obtained by Lemma 2.8.7.

\textbf{Proof of Theorem 2.8.1.} We showed above that the limit \(u\) of \(\{u_{n}\}\) is a strong solution to \(CP(\phi^t, B; f, u_0)\) having properties (a) and (b), so it remains to show (c) and (d). The inequality (8.3) of (c) is easily inferred from (8.14) with \(\lambda_n = \lambda_{n'}\) by passing to the limit in \(n\). In order to show (8.4) of (d) we observe from (8.15) that

$$\phi_{n'}^t(Bu_{n'}(t)) - \phi_{n'}^t(Bu_{n'}(s)) + \frac{C_0}{2} \int_s^t \left| \frac{d}{dt} Bu_{n'}(\tau) \right|^2 d\tau \leq \int_s^t \left[ k_{r,1}(\tau) \phi_{n'}^t(Bu_{n'}(\tau)) + k_{r,2}(\tau) \right] d\tau, \quad \forall s, \ t \in [0, T], s \leq t. \tag{8.25}$$

Letting \(n \to \infty\) in (8.25), we have with the aid of Lemma 2.8.6 with (8.22) and Lemma 2.8.7 that

$$\phi^t(Bu(t)) - \phi^t(Bu(s)) + \frac{C_0}{2} \int_s^t \left| \frac{d}{dt} Bu(\tau) \right|^2 d\tau \leq \int_s^t \left[ k_{r,1}(\tau) \phi^t(Bu(\tau)) + k_{r,2}(\tau) \right] d\tau, \quad \forall s \in [0, T], \ \forall t \in [0, T], \ s \leq t. \tag{8.26}$$

Here we note (cf. Corollary to Lemma 1.2.5) that

$$\limsup_{t \to \infty} \phi^t(Bu(\tau)) \leq \phi^t(Bu(s)), \quad \forall s \in [0, T].$$

Combining this with (8.26), we conclude that (8.4) holds for any \(s, t \in [0, T]\) with \(s \leq t\). The inequality (8.5) is similarly obtained. Q.E.D.

\textbf{(B) Uniqueness of weak solutions}

We discuss uniqueness of the weak solution under supplementary assumptions.

Let \(W\) be another Banach space such that \(H \subset W\), and let \(\gamma_0\) be a non-negative continuous convex functions on \(W\) such that

$$C_2 |z|_W \leq \gamma_0(z) + \gamma_0(-z) \leq C_2' |z|_W, \quad \forall z \in W \tag{8.27}$$

with positive constants \(C_2, C_2'\). Then it is easy to see that \(\gamma_0\) is also continuous on \(H\) and its subdifferential \(\partial \gamma_0\) on \(H\) satisfies that \(D(\partial \gamma_0) = H\) and

$$|z - z|_H \leq C_3 |z + l|, \quad \forall z \in H, \ \forall z \in \partial \gamma_0(z),$$

for a positive constant \(C_3\).

\textbf{Theorem 2.8.2.} Let \(\{\dot{\phi}^t; 0 \leq t \leq T\}\) be as in Theorem 2.8.1, and suppose that for each \(t \in [0, T]\), \(\dot{\phi}^t \circ B\) is \(\gamma_0\)-accretive on \(H\) (i.e. if \(z^* \in \partial \gamma_0(z)\) and \(z^* \in \partial \gamma_0(Bz)\), then \(z^* -\))

$$\{z, w\} \geq 0 \text{ for some } w \in \partial \gamma_0(z - z_1)\). Let \(u_1\) and \(u_2\) be weak solutions to \(CP(\phi^t, B; f_1, u_{0,1})\) and \(CP(\phi^t, B; f_2, u_{0,2})\), respectively, where \(f_1 \in L^1(0, T; H)\) and \(u_{0,i} \in H, i = 1, 2\). Then for any \(0 \leq s \leq t \leq T\)
\(\gamma_0(u_1(t) - u_2(t)) - \gamma_0(u_1(s) - u_2(s))\)
\[
\leq C_3 \int_s^t |f_1(\tau) - f_2(\tau)|/(|u_1(\tau) - u_2(\tau)| + 1) \, d\tau.
\]

**Proof.** It suffices to prove (8.29) in case \(u_1\) and \(u_2\) are strong solutions. In such a case, since \(f_i(\tau) - u_i(\tau) \in \partial \phi^T(Bu_i(\tau))\) \((i = 1, 2)\), we have for some \(w(\tau) \in \partial \gamma_0(u_1(\tau) - u_2(\tau))\)
\[
(f_1(\tau) - f_2(\tau) - u_1(\tau) + u_2(\tau), w(\tau)) \leq 0,
\]
so that by (8.28)
\[
\frac{d}{d\tau} \gamma_0(u_1(\tau) - u_2(\tau)) = (u_1'(\tau) - u_2'(\tau), w(\tau))
\]
\[
\leq (f_1(\tau) - f_2(\tau), w(\tau))
\]
\[
\leq C_3 |f_1(\tau) - f_2(\tau)|/(|u_1(\tau) - u_2(\tau)| + 1)
\]
for a.e. \(\tau \in [0, T]\). Integrating (8.30) over \([s, t]\), we have (8.29). Q.E.D.

**Corollary.** Under the same assumptions of Theorem 2.8.2, for given \(f\) in \(L^1(0, T; H)\) and \(u_0\) in \(H\), \(CP(\phi^t, B; f, u_0)\) admits at most one weak solution.

**Remark 2.8.1.** In the proof of Theorem 2.8.2, if as \(w(\tau)\) a measurable function in \(\tau\) can be taken, then for every \(0 \leq s \leq t \leq T\)
\[
\gamma_0(u_1(t) - u_2(t)) - \gamma_0(u_1(s) - u_2(s)) \leq \int_s^t (f_1(\tau) - f_2(\tau), w(\tau)) \, d\tau.
\]

**C. Existence of weak solutions**

Let \(W\) and \(\gamma_0\) be as in paragraph (B), and assume (8.27). Then we have

**Theorem 2.8.3.** Let \(\{\phi^t; 0 \leq t \leq T\}\) and \(B\) be as in Theorem 2.8.1, and suppose that \(\partial \phi^t \circ B\) is \(\gamma_0\)-accretive on \(H\) for every \(t \in [0, T]\). Then for each \(u_0 \in H\) with \(Bu_0 \in D(\phi^0)\) and \(f \in L^2(0, T; H)\) there exists a weak solution \(u\) to \(CP(\phi^t, B; f, u_0)\) such that \(\sqrt{t}u(\cdot) \in L^2(0, T; H)\), \(\phi^t(u(.) \in L^1(0, T)\), \(t \rightarrow t\phi^t(Bu(t))\) is bounded on \([0, T]\) and \(f(t) - u'(t) \in \partial \phi^t(Bu(t))\) for a.e. \(t \in [0, T]\).

**Proof.** First, take a sequence \(\{z_n\} \subset H\) so that \(Bz_n \in D(\phi^0)\) and \(z_n \rightarrow u_0\) in \(H\). By virtue of Theorem 2.8.1, for each \(n\) there is a strong solution \(u_n\) to \(CP(\phi^t, B; f, z_n)\) such that \(u_n \in W^{1,2}(0, T; H)\) and \(t \rightarrow \phi^t(Bu_n(t))\) is bounded on \([0, T]\) with the following inequality:
\[
\frac{1}{2} |u_n(t) + \int_0^t \phi^t(Bu_n(\tau)) \, d\tau - (u_n(t), h(\tau)) + \int_0^t (u_n(\tau), h'(\tau)) \, d\tau
\]
\[
- \int_0^t (f(\tau), Bu_n(\tau)) \, d\tau \leq \frac{1}{2} |z_n - z_m| + \int_0^t \phi^t(\tau) \, d\tau - (z_n, h(0)) - \int_0^t (f(\tau), h(\tau)) \, d\tau
\]
for all \(t \in [0, T]\), where \(h \in W^{1,2}(0, T; H)\) with \(\phi^t(h(.) \in L^1(0, T)\). Also, by Theorem 2.8.2,
\[
\gamma_0(u_n(t) - u_m(t)) + \gamma_0(u_m(t) - u_n(t)) \leq \gamma_0(z_n - z_m) + \gamma_0(z_m - z_n),
\]
\(\forall t \in [0, T], \forall n, m \geq 1\).

so that \(u_n \rightarrow u\) in \(C([0, T] ; W)\) for some \(u \in C([0, T]; W)\) with \(u(0) = u_0\). Moreover by estimations of Lemmas 2.8.3 and 2.8.4 there is a positive constant \(J\) independent of \(n\) satisfying
\[
|u_n(t)| \leq J, \quad t \phi^t(Bu_n(t)) \leq J, \quad \forall t \in [0, T], \forall n \geq 1,
\]
\[
|\sqrt{t} u_n(t)| \leq J, \quad \int_0^T \phi^t(Bu_n(t)) \, dt \leq J, \quad \forall n \geq 1.
\]
Therefore \( u \) is an \( H \)-valued weakly continuous function on \([0, T]\), and \( u_n(t) \to u(t) \) in \( H \) uniformly on \([0, T]\). Also, just as Lemmas 2.8.5 and 2.8.6, we can show with the aid of (8.1) that

\[
\begin{align*}
&u_n \to u \text{ in } C([\delta, T]; H), \\
&\frac{d}{dt}Bu_n \to \frac{d}{dt}Bu \text{ in } L^2(\delta, T; H), \\
&u_n' \to u' \text{ in } L^2(\delta, T; H) \quad \text{for every } 0 < \delta < T, \\
&\frac{d}{dt} {\phi}'(Bu_n(t)) \to {\phi}'(Bu(t)) \quad \forall t \in (0, T),
\end{align*}
\]

and that

(8.32) \( \phi(u_n(t)) \to \phi(u(t)) \) weakly * in \( L^\infty(\delta, T) \) for every \( 0 < \delta < T \),

\( \phi(u_n(t)) \to \phi(u(t)) \) weakly * in \( L^\infty(0, T) \)

as well as

\[ f(t) - u^*(t) \in \partial \phi'[Bu(t)] \quad \text{for a.e. } t \in [0, T]. \]

Furthermore,

(8.33) \[ \lim_{n \to \infty} \int_0^t \phi'(Bu_n(\tau)) \, d\tau = \int_0^t \phi'(Bu(\tau)) \, d\tau \quad \text{uniformly in } t \in [0, T]. \]

Indeed, noting that \( |\phi'(Bu_n(\tau))| \leq \phi'(K + |Bu_n(\tau)| + 1) \), \( j(u_n(t)) - j(z_n) \leq (Bz_n, u_n(t) - z_n) \to (Bu_0, u(t) - u_0) \) uniformly on \([0, T]\), we derive from (8.31) that

\[ \int_0^\delta \phi'(Bu_n(\tau)) \, d\tau \to 0 \quad \text{as } \delta \downarrow 0 \quad \text{uniformly in } n. \]

By this together with (8.32) we have (8.33). In order that \( u \) is a weak solution to \( CP(\phi^t, B; f, u_0) \) it remains to show that \( u \in C([0, T]; H) \) and \( u_n \to u \) in \( C(0, T); H \). For this purpose it suffices to show \( u_n(t) \to u_0 \) in \( H \) as \( t \downarrow 0 \) and \( n \to \infty \). This can be done as follows. By using (8.33) we see from (8.31) that

\[ \limsup_{n \to \infty} j(u_n(t)) \leq j(u_0), \]

so that \( j(u_n(t)) \to j(u_0) \) as \( n \to \infty \) and \( t \downarrow 0 \), because \( u_n(t) \to u_0 \) in \( H \) as \( n \to \infty \) and \( t \downarrow 0 \) and \( j \) is l.s.c. on \( H \) with respect to the weak topology of \( H \). Hence

\[ 0 = \lim_{n \to \infty} \left\{ j(u_n(t)) - j(u_0) - (Bu_0, u_n(t) - u_0) \right\} \]

\[ = \lim_{n \to \infty} \int_0^1 \left[ B(u_0 + r(u_n(t) - u_0)) - Bu_0, u_n(t) - u_0 \right] \, dr \]

\[ \leq \lim_{n \to \infty} C_0 |Bu_n(t) - Bu_0|^2. \]

Thus \( Bu_n(t) \to Bu_0 \) in \( H \) or equivalently \( u_n(t) \to u_0 \) in \( H \) as \( n \to \infty \) and \( t \downarrow 0 \). Q.E.D.

(D) Stability of solutions

We introduce a class of families of convex functions satisfying \((H)_2\).
Definition 2.8.2. Given a constant $\alpha \geq 0$ and a set $\{M_r\} = \{M_r; 0 \leq r < \infty\} \subset [0, \infty)$, we denote by $G_1(\alpha, \{M_r\})$ the class of all families $\{\phi^t; 0 \leq t \leq T\}$ of proper l.s.c. convex functions on $H$ such that

(a) $\phi^t(z) + \alpha|z| + \alpha \geq 0, \quad \forall t \in [0, T], \quad \forall z \in H$;

(b) for each $r \geq 0$ there are non-negative functions $\alpha_r \in W^{1,2}(0, T)$ with $|\alpha_r|_{W^{1,2}(0, T)} \leq M_r$ and $\beta_r \in W^{1,1}(0, T)$ with $|\beta_r|_{W^{1,1}(0, T)} \leq M_r$ for which $(H)_2$ holds;

(c) (8.1) is satisfied.

Also, let $W$ and $\gamma_0$ be as in paragraph (B). Then we denote by $G_1, \gamma_0(\alpha, \{M_r\})$ the subclass of all $\{\phi^t; 0 \leq t \leq T\} \in G_1(\alpha, \{M_r\})$ such that $\partial \phi^t B$ is $\gamma_0$-accretive on $H$ for every $t \in [0, T]$.

We establish a stability theorem.

Theorem 2.8.4. Let $G_1 = G_1(\alpha, \{M_r\})$ and $G_1, \gamma_0 = G_1, \gamma_0(\alpha, \{M_r\})$ be as above, and $B$ be as in Theorem 2.8.1. Then we have:

(i) Given a number $k_0 \geq 0$, there is a positive constant $K_0 = K_0(G_1, k_0)$ having the following property: for every $\{\phi^t; 0 \leq t \leq T\} \in G_1$ with $f \in L^2([0, T); H)$ with $\|f\| \leq k_0$ and $u_0 \in H$ with $\|u_0\| + \|\phi^0(Bu_0)\| \leq k_0$, $CP(\phi^t, B; f, u_0)$ admits at least one strong solution $u$ such that

\[ \|u\|_{W^{1,2}(0, T; H)} \leq K_0, \quad \|\phi^t(Bu(t))\| \leq K_0, \quad \forall t \in [0, T]. \]

(ii) Given a number $k_1 \geq 0$, there is a constant $K_1 = K_1(G_1, \gamma_0, k_1)$ having the following property: for every $\{\phi^t; 0 \leq t \leq T\} \in G_1, \gamma_0$ with $f \in L^2([0, T); H)$ with $\|f\| \leq k_1$ and $u_0 \in H$ with $Bu_0 \in D(\phi^0)$ and $\|u_0\| \leq k_1$, $CP(\phi^t, B; f, u_0)$ admits a unique weak solution $u$ such that

\[ \|u\|_{C([0, T]; H)} \leq K_1, \quad \|\phi^t(Bu(t))\| \leq K_1, \quad \forall t \in [0, T]. \]

Proof. As is easily seen from the proof of Theorem 1.5.1, $G_1 \subset G(\alpha_0, \{\delta_r\}, \{M_r\})$ for some $\alpha_0, \{\delta_r\}$ and $\{M_r\}$. Taking account of this fact with Theorems 2.8.1, 2.8.2 and 2.8.3, we obtain easily the assertions of the theorem.

Q.E.D.

(E) Convergence of solutions

Let $W$ and $\gamma_0$ be as in paragraph (B), and assume (8.27). For a number $\alpha \geq 0$, a set $\{M_r\}$ and a set $\{S^r\} = \{S^r; 0 \leq r < \infty\}$ of compact sets in $H$, denote by $G_1, \gamma_0 = G_1, \gamma_0(\alpha, \{M_r\}, \{S^r\})$ the class of all families $\{\phi^t; 0 \leq t \leq T\} \in G_1, \gamma_0(\alpha, \{M_r\})$ satisfying

$\{z \in H; |z| \leq r, |\phi^t(z)| \leq r\} \subset S^r, \quad \forall t \in [0, T], \quad \forall r \in [0, \infty).$

Theorem 2.8.5. Let $G_1, \gamma_0 = G_1, \gamma_0(\alpha, \{M_r\}, \{S^r\})$ be as above, and let $B$ as in Theorem 2.8.1. Let $\{\phi^t; 0 \leq t \leq T\}$ and $\{\phi_n^t; 0 \leq t \leq T\}$, $n = 1, 2, \ldots$, be in the class $G_1, \gamma_0$, and suppose that $\phi_n^t \to \phi^t$ in $H$ in the sense of Mosco for every $t \in [0, T]$. Further, let $u_n$ be the weak solutions to $CP(\phi_n^t, B; f_n, u_0, n)$ with $f_n \in L^2([0, T]; H)$ and $u_0, n \in H$ satisfying $Bu_0, n \in D(\phi^0)$. If $f_n \to f$ in $L^2([0, T]; H)$ and if $u_0, n \to u_0$ in $H$ with $Bu_0 \in D(\phi^0)$, then $u_n$ converges in $C([0, T]; H)$ and the limit $u$ is the weak solution to $CP(\phi^t, B; f, u_0)$, and moreover $\Phi^t_n(Bu_n) \to \Phi^t(Bu)$ uniformly in $t \in [0, T]$, where $\Phi^t_n$ and $\Phi^t$ are as in §2.7.

We prove this theorem by a sequence of lemmas.

Lemma 2.8.8. In addition to all the assumptions of Theorem 2.8.5, suppose that $\{\phi_n^0(Bu^0_n)\}$ is bounded. Then the conclusion of Theorem 2.8.5 is true and in this case the limit $u$ is the strong solution to $CP(\phi^t, B; f, u_0)$.

Proof. By (i) of Theorem 2.8.4, $\{u_n\}$ is bounded in $W^{1,2}(0, T; H)$ and $t \to \phi^t_n(Bu_n(t))$ is
uniformly bounded on \([0, T]\), so it holds for every \(t\) in \([0, T]\) that \(\{Bu_n(t)\} \subset S_r^f\) for some \(r \geq 0\). Therefore we can select a subsequence \(\{u_{n'}\}\) of \(\{u_n\}\) such that \(u_{n'}\) converges in \(C([0, T]; H)\) and evidently the limit \(u\) satisfies that \(u(0) = u_0, u \in W^{1, 2}(0, T; H)\) and

\[
\phi^f(Bu(t)) \leq \liminf_{n \to \infty} \phi^f_n(Bu_n(t)), \quad \forall t \in [0, T];
\]

hence \(t \to \phi^f(Bu(t))\) is bounded on \([0, T]\). We note here that for each \(n\) and \(t \in [0, T]\)

\[
\int_0^t (u_n(s) - f_n(s), Bu_n(s) - w(s)) \, ds \leq \Phi_n^f(w) - \Phi_n^f(Bu_n),
\]

\(\forall w \in L^2(0, t; H)\) with \(\Phi_n^f(w) < \infty\),

because \(f_n(s) - u_n(s) \in \partial \phi_n^f(Bu_n(s))\) for a.e. \(s \in [0, T]\). Now, we take a sequence \(\{v_n\} \subset L^2(0, t; H)\) for each \(t \in [0, T]\) such that

\[
v_n' \to Bu \text{ in } L^2(0, t; H), \quad \Phi_n^f(v_n') \to \Phi^f(Bu);
\]

indeed such a sequence \(\{v_n\}\) exists by (ii) of Proposition 2.7.1. Substituting \(v_n\) as \(w\) in (8.35) with \(n = n'\) and letting \(n' \to \infty\), we get by (8.35)

\[
\limsup_{n \to \infty} \Phi_n^f(Bu_n') \leq \Phi^f(Bu).
\]

Hence from (8.34) and (8.36) it follows that

\[
\lim_{n \to \infty} \Phi_n^f(Bu_n') = \Phi^f(Bu), \quad \forall t \in [0, T],
\]

and the fact that this convergence is uniform in \(t \in [0, T]\) is easily seen from the uniform boundedness of \(t \to \phi_n^f(Bu_n(t))\) on \([0, T]\). Next, given \(\tilde{w} \in D(\Phi^T)\), we take a sequence \(\{w_n\}\) in \(L^2(0, T; H)\) such that

\[
w_n \to \tilde{w} \quad \text{in } L^2(0, T; H), \quad \Phi_n^T(w_n) \to \Phi^T(\tilde{w})
\]

and substitute \(w_n\) as \(w\) of (8.35) with \(n = n'\) and \(t = T\). Then, on account of (8.37), letting \(n' \to \infty\) yields

\[
< u' - f, Bu - \tilde{w} > \leq \Phi^T(\tilde{w}) - \Phi^T(Bu), \quad \forall w \in D(\Phi^T),
\]

which shows by Proposition 0.3.3 that

\[
f(t) - u'(t) \in \partial \phi^f(Bu(t)) \quad \text{for a.e. } t \in [0, T].
\]

Thus \(u\) is a (unique) strong solution to \(CP(\phi^f; B; f; u_0)\) and by the way for the whole sequence \(\{u_n\}\) we see that \(u_n \to u\) in \(C([0, T]; H)\) and \(\phi_n^f(Bu_n) \to \phi^f(Bu)\) uniformly on \([0, T]\). Q.E.D.

**Lemma 2.8.9.** Under the same assumptions as in Theorem 2.8.5, \(\{u_n\}\) converges in \(C([0, T]; W)\) and the limit \(u\) is the weak solution to \(CP(\phi^f; B; f; u_0)\). Moreover

\[
u_n(t) \to u(t) \quad \text{in } H \text{ uniformly on } [0, T].
\]

**Proof.** For simplicity, let \(C_d\) be a positive constant satisfying

\[
|z|_W \leq C_d |z|, \quad \forall z \in H;
\]

such a constant \(C_d\) always exists by the assumption \(H \subset W\). Let \(\epsilon\) be an arbitrary positive number. We then choose \(\varepsilon \in D(\phi^0)\) and an integer \(n_\varepsilon(\varepsilon) > 0\) such that

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\[ |u_0 - z^e| \leq \varepsilon, \quad |u_0 - u_{0, n}| \leq \varepsilon, \quad \forall n \geq n_f(\varepsilon). \]

Next, we take a sequence \( \{ z^e_n \} \) with \( z^e_n \in D(\phi^{0}_n) \) such that

\[ z^e_n \rightarrow z^e \quad \text{in} \; H, \quad \phi^{0}_n(z^e_n) \rightarrow \phi^0(z^e), \]

and a positive integer \( n_2(\varepsilon) \) such that

\[ |z^e_n - z^e| \leq \varepsilon, \quad \forall n \geq n_2(\varepsilon). \]

Denoting by \( u, u^e, u_n \) and \( u^e_n \) the weak solutions to \( CP(\phi^f, B; f, u_0), \; CP(\phi^f, B; f, z^e), \; CP(\phi^{f}_n, B; f_n, u_{0, n}) \) and \( CP(\phi^{f}_n, B; f_n, z^e_n) \) respectively, we see from Lemma 2.8.8 that

\[ u^e_n \rightarrow u^e \quad \text{in} \; C([0, T]; H) \; \text{(hence in} \; C([0, T]; W)). \]

On account of Theorem 2.8.2 and (8.27) we have for all \( t \in [0, T] \)

\[
C_2 |u_n(t) - u^e_n(t)|_W \leq \gamma_0 (u_n(t) - u^e_n(t)) + \gamma_0 (u^e_n(t) - u_n(t)) \\
\leq \gamma_0 (u_{0, n} - z^e_n) + \gamma_0 (z^e_n - u_{0, n}) \\
\leq C_2 |u_{0, n} - z^e_n|_W \\
\leq C_2 C_4 \{ |u_{0, n} - u_0| + |u_0 - z^e| + |z^e - z^e_n| \} \\
\leq 3C_2 C_4 \varepsilon, \quad \forall n \geq n_3(\varepsilon) = \max \left\{ n_1(\varepsilon), n_2(\varepsilon) \right\},
\]

and hence

\[ |u_n - u^e_n|_{C([0, T]; W)} \leq 3C_2^{-1} C_2' C_4 \varepsilon, \quad \forall n \geq n_3(\varepsilon) \]

Similarly,

\[ |u - u^e|_{C([0, T]; W)} \leq 3C_2^{-1} C_2' C_4 \varepsilon, \quad \forall n \geq n_3(\varepsilon). \]

Therefore

\[ |u - u_n|_{C([0, T]; W)} \leq 4C_2^{-1} C_2' C_4 \varepsilon + |u^e - u^e_n|_{C([0, T]; W)}, \quad \forall n \geq n_3(\varepsilon), \]

so

\[ \limsup_{n \to \infty} |u - u_n|_{C([0, T]; W)} \leq 4C_2^{-1} C_2' C_4 \varepsilon. \]

Since \( \varepsilon \) is arbitrary, we get \( u_n \rightarrow u \) in \( C([0, T]; W) \). Also, from this fact and the boundeness of \( \{ u_n \} \) in \( C([0, T]; H) \) (cf. (ii) of Theorem 2.8.4) we derive (8.38).

Q.E.D.

**Lemma 2.8.10.** Under the same assumptions as in Theorem 2.8.5, the following hold:

\[ u_n(t) \rightarrow u_0 \quad \text{in} \; H \; \text{as} \; n \rightarrow \infty \; \text{and} \; t \downarrow 0. \]

\[ \Phi^{f}_n(Bu_n) \rightarrow 0 \quad \text{as} \; n \rightarrow \infty \; \text{and} \; t \downarrow 0. \]

**Proof.** First we take an element \( h_0 \in D(\phi^0) \) and a sequence \( \{ h_{0, n} \} \) with \( h_{0, n} \in D(\phi^{0}_n) \) such that

\[ h_{0, n} \rightarrow h_0 \quad \text{in} \; H, \quad \phi^{0}_n(h_{0, n}) \rightarrow \phi^0(h_0); \]

and denote by \( h_n \) and \( h \) the strong solutions to \( CP(\phi^{f}_n, 0; h_{0, n}) \) and \( CP(\phi^{f}; 0, h_0) \), respectively.
We see from (ii) of Theorem 2.6.1 that

\[(8.42) \quad |h_n|_{W^1,2(0, T; H)} \leq A, \quad |\phi_n^t(h_n(t))| \leq A, \quad \forall t \in [0, T],\]

for all \(n\), where \(A\) is a positive constant. Since the inequality of type (8.3) is also valid for weak solution to \(CP\) \((\phi_n^t, B; f_n, u_{0, n})\) as is easily verified, we have

\[(8.43) \quad j(u_n(t)) + \int_0^t \phi_n^s(Bu_n(s)) \, ds \leq j(u_{0, n}) + J_n(t),\]

where

\[J_n(t) = (u_n(t), h_n(t)) - (u_{0, n}, h_{0, n}) - \int_0^t (u_n, h_n) \, ds + \int_0^t (f_n, Bu_n) \, ds - \int_0^t (f_n, h_n) \, ds.\]

Here we note

\[\limsup_{n \to \infty, t \to 0} j(u_n(t)) \geq \liminf_{n \to \infty, t \to 0} j(u_n(t)) \geq j(u_0),\]

since \(j\) is l.s.c. on \(H\) in the weak topology of \(H\) and \(u_n(t) \to u_0\) as \(n \to \infty, t \downarrow 0\) (cf. Lemma 2.8.9). Besides, by Lemma 2.8.9, (8.41), (8.42) and the boundedness of \(\{u_n\}\) in \(C([0, T]; H)\), we see

\[|J_n(t)| \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad t \downarrow 0.\]

Accordingly, from (8.43) it follows that

\[\limsup_{n \to \infty, t \to 0} \int_0^t \phi_n^s(Bu_n(s)) \, ds \leq 0,\]

Combining this with the inequality \(|\phi_n^s(Bu_n(s))| \leq \phi_n^s(Bu_n(s)) + 2\alpha (|Bu_n(s)| + 1)\), we obtain

\[(8.40) \quad \text{and subsequently}\]

\[\limsup_{n \to \infty, t \to 0} j(u_n(t)) \leq j(u_0),\]

from which (8.39) follows (cf. the proof of Theorem 2.8.3). Q.E.D.

In order to accomplish the proof of Theorem 2.8.5 we use (ii) of Theorem 2.8.4. In fact, using this result and (8.1), we see that

\[u_n \to u \quad \text{in} \quad C([\delta, T]; H) \quad \text{for every} \quad 0 < \delta < T.\]

Also, applying Lemma 2.8.8 to the Cauchy problem with initial time \(\delta\), we see that for every \(0 < \delta < T\)

\[\int_\delta^t \phi_n^s(Bu_n(s)) \, ds \to \int_\delta^t \phi^s(Bu(s)) \, ds \quad \text{uniformly in} \quad t \in [\delta, T].\]

From these facts and Lemma 2.8.10 we infer that

\[u_n \to u \quad \text{in} \quad C([0, T]; H)\]

and

\[\Phi_n^t(Bu_n) \to \Phi^t(Bu) \quad \text{uniformly on} \quad [0, T].\]

Remark 2.8.2. A part of the results proved in this section was already discussed in Kenmochi [6].
Chapter 3

Applications

In this chapter we give some applications of abstract results obtained in Chapters 1 and 2 to nonlinear parabolic partial differential equations.

§3.1 Notations

We denote by $x = (x_1, x_2, \ldots, x_N)$ a generic point of $\mathbb{R}^N$, by $\alpha = \{ \alpha_1, \alpha_2, \ldots, \alpha_N \}$ a multi-index and by $D^\alpha = D_x^\alpha$ the differentiation

$$
\frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}, \quad |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N.
$$

For domains $\Omega, \mathcal{O}'$ in $\mathbb{R}^N$ and a diffeomorphism $\Theta$ from $\Omega$ onto $\Omega', J(\Theta)/(\text{resp.} \det J(\Theta))$ stands for the Jacobian matrix (resp. determinant).

Given a domain $\Omega$ in $\mathbb{R}^N$, a non-negative integer $m$ and a number $p$ with $1 \leq p \leq \infty$, we denote by $W^{m,p}(\Omega)$ the Sobolev space, i.e.,

$$
W^{m,p}(\Omega) = z \in L^p(\Omega); D^\alpha z \in L^p(\Omega), \forall \alpha \text{ with } |\alpha| \leq m
$$

with usual norm. We mean by $W^{m,p}_0(\Omega)$ the closure of $D(\Omega)$ (the space of all smooth functions on $\mathbb{R}^N$ with compact supports in $\Omega$) in $W^{m,p}(\Omega)$. Following Littman-Stampacchia-Weinberger [1], for a compact subset $E$ of $\Omega$ we mean by

"$z = 0$ (resp. $z \geq 0$) on $E$ in the sense of $W^{m,p}(\Omega)$"

that there is a sequence $\{ z_n \}$ of smooth functions on $\Omega$ such that $z_n = 0$ (resp. $z_n \geq 0$) on a neighborhood of $E$ and $z_n \to z$ in $W^{m,p}(\Omega)$.

Let $X \subseteq L^1(\Omega)$, where $\Omega$ is a domain in $\mathbb{R}^N$. Then any (strongly) measurable function $v$ from $(0, T)$ into $X$ is naturally regarded as a (Lebesgue) measurable function on $\Omega \times (0, T)$ and we often write $v(\cdot, t)$ for $\{v(t)\}(\cdot)$, namely

$$
[v(t)](x) = v(x, t) \quad \text{for a.e. } (x, t) \in \Omega \times (0, T).
$$

§3.2 Nonlinear parabolic equations with time-dependent constraints

We begin with some examples of $\{ \phi^t; 0 \leq t \leq T \}$ of proper l.s.c. convex functions on a Hilbert space $H$ satisfying condition $(H)_t^t$ (resp. $(H)_t^0)$, $l \leq t \leq \infty$, of the following type:

$(H)_t^t$ (resp. $(H)_t^0$) There is a non-negative function $a \in W^{1,1}(0, T)$ such that for each $s, t \in [0, T]$ (resp. $s, t \in [0, T]$ with $s \leq t$) and each $z \in D(\phi^t)$ there is $z_1 \in D(\phi^t)$ satisfying

$$
|z_1 - z|_H \leq |a(t) - a(s)|/(1 + |\phi^t(z)|^{1/2})
$$

and

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\[ | \Phi^t(z) - \Phi^s(z) | \leq | a(t) - a(s) | (1 + | \Phi^s(z) | ). \]

Clearly, \( (H')_1 \) and \( (H')_2 \) are stronger than \( (H)_1 \) which was considered in §1.5. By a slight modification of the proof of Theorem 1.5.1 we see that \( (H')_1 \) implies \( (\Phi^2) \) in §2.1.

(A) Application 1

Let \( H \) be a Hilbert space and \( X \) be a reflexive Banach space such that \( X \subseteq H \). Let \( 0 < T < \infty \), \( a^* \) be an \( X^* \)-valued function on \( [0, T) \) such that \( a^*(t) \neq 0 \) for all \( t \in [0, T] \). Given a real-valued function \( g \) on \( [0, T] \), we define

\[
K^1(t) = \{ z \in X; (a^*(t), z)_X = g(t) \}, \quad 0 \leq t \leq T
\]

and consider the family \( \{ \Phi^t; 0 \leq t \leq T \} \) given by

\[
\Phi^t(z) = \begin{cases} \frac{1}{p} | z |_X^p & \text{if } z \in K^1(t), \\ \infty & \text{otherwise,} \end{cases}
\]

where \( 2 \leq p < \infty \). Evidently \( \Phi^t \) is proper, l.s.c. and convex on \( H \) with \( D(\Phi^t) = K^1(t) \).

**Proposition 3.2.1.** Suppose that \( a^* \in W^{1,1}(0, T; X^*) \) and \( g \in W^{1,1}(0, T), \quad l \leq l \leq \infty \). Then \( (H')_1 \) holds for \( \{ \Phi^t; 0 \leq t \leq T \} \) given by (2.1) and (2.2).

**Proof.** First, take \( v(t) \in X \) for each \( t \in [0, T] \) so that

\[
(a^*(t), v(t))_X = 1, \quad \| v(t) \|_X \leq c
\]

with a positive constant \( c \) being independent of \( t \). Then we have for any \( z \in X \)

\[
(a^*(t), z - (a^*(t), z)_X - g(t))_X = g(t),
\]

which shows

\[
z_{\cdot t} = z - (a^*(t), z)_X - g(t) \big|_v(t) \in K^1(t).
\]

Now, if \( z \in K^1(s) \), then with \( J = [s, t] \) or \([t, s] \) we see

\[
| z - z |_X \leq | (a^*(t), z)_X - g(t) | \big|_v(t) \big|_X \leq c | (a^*(t) - a^*(s), z)_X + g(s) - g(t) | \leq c \int_J ( | a^*(r) |_X^* + | g'(r) | ) \, dr ( | z |_X + 1)
\]

and similarly

\[
\frac{1}{p} | z - z |_X^p \leq c | (a^*(t), z)_X - g(t) | (1 + | z |_X^{p-1}) \leq c' \int_J ( | a^*(r) |_X^* + | g'(r) | ) \, dr (1 + | z |_X^p)
\]

with positive constants \( c', c'' \) being independent of \( s, t \) and \( z \). Thus \( (H')_1 \) holds.

Q.E.D.

**Example 3.2.1.** Consider the case that

\[
H = L^2(\Omega), \quad X = W^{m,p}(\Omega); \quad 1 \leq m < \infty, \quad 2 \leq p < \infty \quad \text{and} \quad \Omega \text{ is a bounded domain in } \mathbb{R}^N (N \geq 2) \text{ with smooth boundary } \Gamma. \text{ Let}
\]

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\( g \in W^{1,2}(0, T), \; 0 < T < \infty, \) and \( a^*: [0, T] \to X^* \) be such that \( a^*(t) \neq 0 \) for all \( t \in [0, T] \) and

\[
(a^*(t), z)_X = \sum_{|\alpha| = m} \int_{\Omega} a^*_\alpha(x, t) D^\alpha x(x) dx + \sum_{|\beta| = m - 1} \int_{\Gamma} b^*_\beta(x, t) D^\beta x(x) d\Gamma \]

for \( z \in X \) and \( 0 \leq t \leq T \). Where \( a^*_\alpha \in W^{1,2}(0, T; L^p(\Omega)) \) for every \( \alpha \) with \( |\alpha| = m \) and \( b^*_\beta \in W^{1,2}(0, T; L^p(\Gamma)) \) for every \( \beta \) with \( |\beta| = m - 1 \). In this case, \( a^* \in W^{1,2}(0, T; X^*) \) and the family \( \{ \phi^f : 0 \leq t \leq T \} \) given by (2.1) and (2.2) satisfies \( (H^f) \) by the above proposition. Applying Theorems 1.1.1, 1.1.2 and 2.1.1 to \( CP(\phi^f; f, u_o) \) for \( f \) given in \( L^2(0, T; H) \) and \( u_o \) in the closure of \( K(0) \) in \( H \), we can find a unique function \( u \in C([0, T]; H) \) such that

(a) \( u(0) = u_o \); 
(b) \( u(t) \in K^I(t) \) for all \( t \in [0, T] \), \( u \in L^p(0, T; X), \sqrt{t}u' \in L^2(0, T; H) \)

and \( t \to t |u(t)|^p_X \) is absolutely continuous on \( [0, T] \);
(c) for a.e. \( t \in [0, T] \) the following inequality holds:

\[
(u'(t) - f(t), z)_H + \sum_{|\alpha| = m} \int_{\Omega} |D^\alpha u(x, t)|^p - 2D^\alpha u(x, t)D^\alpha z(x) dx = 0,
\]

\( \forall z \in X \) with \( (a^*(t), z)_X = 0 \).

In particular, if \( u_o \in K^I(0) \), then the following \( (b)' \) holds instead of (b):

(b)' \( u(t) \in K^I(t) \) for all \( t \in [0, T] \), \( u' \in L^2(0, T; H) \) and \( t \to |u(t)|^p_X \) is absolutely continuous on \( [0, T] \) (hence \( u \in C([0, T]; X) \)).

Moreover, in view of Theorems 2.3.1 and 2.3.2, if \( g(0) = g(T) \), then we can find a unique function \( u \in C([0, T]; H) \) satisfying \( u(0) = u(T) \), \( (b)' \) and (c).

(B) Application 2

Let \( \Omega \) be a bounded domain in \( R^N \) with smooth boundary \( \Gamma \), \( 1 \leq m < \infty, \; 2 \leq p < \infty, \; 0 < T < \infty \), and set

\[
H = L^2(\Omega), \quad X = W^{m, p}(n),
\]

Further, let \( \{ \omega(t); 0 \leq t \leq T \} \) be a family of non-empty compact subsets of \( \Omega \) and \( g \) be an \( X \)-valued function on \( [0, T] \). We then define for each \( t \in [0, T] \)

\[
K^2(t) = \{ z \in X; z - g(., t) = 0 \text{ on } \omega(t) \text{ in the sense of } X \},
\]

\[
K^3(t) = \{ z \in X; z - g(., t) \geq 0 \text{ on } \omega(t) \text{ in the sense of } X \}
\]

and

\[
K^4(t) = K^2(t) \cap \{ z \in X; z \geq g(., t) \text{ a.e. on } \Omega \}
\]

Clearly these sets are non-empty, closed and convex in \( X \). Besides we have

**Proposition 3.2.2** Let \( \{ K(t); 0 \leq t \leq T \} \) be anyone of the families \( \{ K^i(t); 0 \leq t \leq T \} \), \( i = 2, 3, 4 \), and \( \{ \phi^i; 0 \leq t \leq T \} \) be a family of proper l.s.c. convex functions on \( H \) given by

\[
(2.4) \quad \phi^i(z) = \begin{cases} 
\frac{1}{p} |z|^p_X & \text{if } z \in K(t), \\
\infty & \text{otherwise}.
\end{cases}
\]

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Then we have the following statements:

(i) If $\omega(t)$ is non-increasing in $t$, and $g \in W^{1,1}(0, T; X)$, $1 \leq s \leq \infty$, then $(H)''$ holds.

(ii) Suppose $g \in W^{1,1}(0, T; X) \cap L^{\infty}(0, T; W^{m+1,p} (\Omega))$ with $1 \leq s \leq \infty$ and the following geometric condition (**):

(**) There exists a diffeomorphism $\Theta_t = (\Theta_t^i, \Theta_t^j, \ldots, \Theta_t^N)$ of class $C^m$ from $\Omega$ onto itself with $\Theta_t(\omega(t)) = \omega(t)$ for every $t \in [0, T]$ such that $\Theta_t$ is the identity on $\Omega$ and $D_x^\alpha \Theta_t^i$ is continuously differentiable in $t$ on $\Omega \times [0, T]$ for every $\alpha$ with $|\alpha| \leq m$ and $i = 1, 2, \ldots, N$.

Then $(H)'''$ holds.

Proof of (i) of Proposition 3.2.2: Let $z$ be any element of $K(t)$ and $t \geq s$.

Then $z_t = z - g(s) + g(t)$ is clearly in $K(t)$ and

$$|z_t - z|_H = |g(t) - g(s)|_H \leq \int_s^t |g'(r)|_X \, dr$$

as well as

$$\frac{1}{p} |z_t|_X^p - |z|_X^p \leq c |g(t) - g(s)|_X \left( \int_0^t |g'(r)|_X \, dr \right)^{p-1} + |g(t)|_X^{p-1} + |z|_X^{p-1}$$

$$\leq c \int_s^t |g'(r)|_X \, dr \left( \frac{1}{p} |z|_X^p + 1 \right),$$

where $c, c'$ are positive constants independent of $t, s$ and $z$. Therefore if we take

$$a(t) = (c + c') \int_0^t |g'(r)|_X \, dr,$$

then $(H)''$ holds. Q.E.D.

In order to prove (ii) of Proposition 3.2.2 we prepare three lemmas. Under condition (**), consider the mapping $\Theta_{t,s} = \Theta_s \circ \Theta^{-1}_t$ for every pair $s, t \in [0, T]$. Clearly $\Theta_{t,s}$; $\Omega \to \Omega$ is a diffeomorphism of class $C^m$ such that $\Theta_{t,s}(\omega(t)) = \omega(s)$ and $\Theta^{-1}_{t,s} = \Theta_{s,t}$. It is also easy to see that there is a constant $R > 0$ such that

$$\frac{\partial}{\partial x} \Theta_{t,s}(x) \mid_{R^N} \leq R,$$

$$|\frac{\partial}{\partial x} \Theta_{t,s}(x) - \delta^i_j| \leq R |t - s|, \quad i, j = 1, 2, \ldots, N,$$

$$|\det J(\Theta_{t,s}(x)) - 1| \leq R |t - s|,$$

$$|D_x^\alpha \Theta_{t,s}(x)| \leq R |t - s|, \quad i = 1, 2, \ldots, N, \quad 2 \leq |\alpha| \leq m,$$

for all $s, t \in [0, T]$ and $x \in \Omega$, where $\theta^i_{t,s}$ is the $i$-th component of $\Theta_{t,s}$. We often write $|.|$ for the euclidian norm $|.|_{R^N}$.

Lemma 3.2.1 Suppose condition (**). Let $z \in X, |\alpha| \leq m$ and $s, t \in [0, T]$. Then we have

$$D^\alpha [z(\Theta_{t,s}(x))] = D^\alpha z (\Theta_{t,s}(x)) + \sum_{|\beta| \leq |\alpha|} \rho^\beta(x) D^\beta z (\Theta_{t,s}(x))$$

for a.e. $x \in \Omega$, with functions $\rho^\beta$, determined only by $\Theta_{t,s}$, of class $C^{-m-|\alpha|}$ satisfying

$$|D^\gamma \rho^\beta(x)| \leq C_{\alpha} |t - s|, \quad \forall x \in \Omega, \forall \gamma \text{ with } |\gamma| \leq m - |\alpha|,$$

$$\sum_{|\beta| \leq |\alpha|} |\rho^\beta(x)| \leq C_m |t - s|$$

where $C_{\alpha}$ and $C_m$ are positive constants independent of $x, \theta$ and $\alpha$.
where \( C_a \) is a positive constant independent of \( s, t \) and \( z \).

**Proof.** We show (2.9) by induction. In fact, in the case of \(|\alpha| = 0\) we have the trivial expression of the form (2.9) with \( f^\beta \equiv 0\). Now we suppose that for every \( m' \leq m - 1\) the expression of the form (2.9) with (2.10) was shown. Let \( \alpha = \{ \alpha_1, \alpha_2, ..., \alpha_N \} \) be any multi-index with \(|\alpha| = m'\) and

\[
\alpha(i) = \{ \alpha_1(i), \alpha_2(i), ..., \alpha_N(i) \}, \quad \alpha_j(i) = \alpha_j + \delta_j^i.
\]

Then using (2.5) – (2.8), we obtain by the hypothesis of induction

\[
D^{\alpha(i)}[z(\Theta_{t,s}(x))] = \frac{\partial}{\partial x_i} D^{\alpha} z(\Theta_{t,s}(x)) + \sum_{|\beta| \leq |\alpha|} f^\beta(x) D^\beta (\Theta_{t,s}(x))
\]

\[
= D^{\alpha(i)} z(\Theta_{t,s}(x)) + \sum_{j=1}^N \left( \frac{\partial}{\partial x_i} \theta^j_{t,s}(x) - \delta_j^i \right) D^{\alpha(i)} z(\Theta_{t,s}(x))
\]

\[
+ \sum_{|\beta| \leq |\alpha|} \frac{\partial\theta^j_{t,s}(x)}{\partial x_i} D^\beta z(\Theta_{t,s}(x)) + \sum_{|\beta| \leq |\alpha|} \sum_{j=1}^N f^\beta(x) \frac{\partial\theta^j_{t,s}(x)}{\partial x_i} \frac{\partial D^\beta z(\Theta_{t,s}(x))}{\partial x_j}
\]

and for all multi-index \( \gamma \) with \(|\gamma| \leq m - m' - 1\)

\[
|D^\gamma(\frac{\partial}{\partial x_i} \theta^j_{t,s}(x) - \delta_j^i)| \leq R |t - s|,
\]

\[
|D^\gamma(\frac{\partial\theta^j_{t,s}(x)}{\partial x_i})| \leq C_\alpha |t - s|,
\]

\[
|D^\gamma(f^\beta(x) \frac{\partial\theta^j_{t,s}(x)}{\partial x_i})| \leq C_\alpha |t - s|,
\]

where \( C_\alpha \) is a positive constant. Hence we have an expression of \( D^{\alpha(i)}[z(\Theta_{t,s}(x))] \) of the form (2.9) with (2.10). Thus we get the conclusion. Q.E.D.

We note by (2.5) that for a suitable positive number \( \delta_o \)

\[
\frac{1}{2} \leq \det J(\Theta_{t,s}(x)) \leq \frac{\bar{M}}{2}, \quad \forall s, t \in [0, T] \text{ with } |s - t| \leq \delta_o.
\]

**Lemma 3.2.2** Suppose condition (*), and let \( \delta_o \) be as above. Let \( z \in X \) and \( s, t \in [0, T] \) with \(|s - t| \leq \delta_o\). Then we have for any \( 2 \leq q \leq p \).

\[
|z \circ \Theta_{t,s} - z|_{L^q(\Omega)} \leq \frac{3}{2} R |t - s| \left( \int_{\Omega} |\nabla z|^q dx \right)^{1/q}
\]

**Proof.** For simplicity, assume \( t \geq s \). With \( 1/q + 1/q' = 1 \), we see

\[
|z \circ \Theta_{t,s} - z|_{L^q(\Omega)} \leq \left( \int_{\Omega} \int_s^t \frac{\partial}{\partial r} z(\Theta_{r,s}(x)) dr \right)^{1/q} \]

\[
\leq \left( \int_{\Omega} \int_s^t |\nabla z(\Theta_{r,s}(x))| \frac{\partial}{\partial r} \Theta_{r,s}(x) dr dx \right)^{1/q}
\]

\[
\leq R |t - s|^{1/q} \left( \int_{\Omega} \int_s^t |\nabla z(\Theta_{r,s}(x))| dr dx \right)^{1/q}
\]

\[
\leq \frac{3}{2} R |t - s| \left( \int_{\Omega} |\nabla z|^q dy \right)^{1/q}
\]

Q.E.D.
Lemma 3.2.3. Let \( s, t \in [0, T], z \in X \) and \( \alpha \) be a multi-index with \( |\alpha| \leq m \). Then we have

\[
\int_\Omega |D^\alpha z(\Theta_{t,s}(x))|^p \, dx \leq \int_\Omega |D^\alpha z|^p \, dx + R_\alpha' |t-s| \sum_{|\beta| = |\alpha|} \int_\Omega |D^\beta z|^p \, dx
\]

where \( R_\alpha \) is a positive constant which depends only on \( \alpha \) but not on \( s, t \) and \( z \).

Proof. By Lemma 3.2.1 we have

\[
\int_\Omega |D^\alpha z(\Theta_{t,s}(x))|^p \, dx
\]

\[
= \int_\Omega |D^\alpha (\Theta_{t,s}(x)) + \sum_{|\beta| \leq |\alpha|} f^\beta(x) D^\beta z(\Theta_{t,s}(x))|^p \, dx
\]

\[
\leq \int_\Omega |D^\alpha z(\Theta_{t,s}(x))|^p \, dx + R_\alpha' |t-s| \sum_{|\beta| = |\alpha|} \int_\Omega |D^\beta z(\Theta_{t,s}(x))|^p \, dx
\]

with a positive constant \( R_\alpha' \) independent of \( s, t \) and \( z \). Besides, for \( \beta \) with \( |\beta| \leq |\alpha| \),

\[
\int_\Omega |D^\beta z(\Theta_{t,s}(x))|^p \, dx
\]

\[
= \int_\Omega |D^\beta z(y)|^p \, dy + \int_\Omega |D^\beta z(y)|^p \, dy (\det J(\Theta_{t,s}(y)) - 1) \, dy
\]

\[
\leq \int_\Omega |D^\beta z(y)|^p \, dy + C |t-s| \int_\Omega |D^\beta z(y)|^p \, dy.
\]

Hence we get (2.12) for some constant \( R_\alpha' \). Q.E.D.

Proof of (ii) of Proposition 3.2.2: Given \( z \in K(t) \) and \( t \in [0, T] \) with \( |t-s| \leq \delta_0 \), we consider the function

\[ z_1 = z \circ \Theta_{t,s} - g(s) \circ \Theta_{t,s} + g(t). \]

Evidently \( z_1 \in K(t) \). Using Lemmas 3.2.2 and 3.2.3, we set that

\[
|z_1 - z|_H \leq |z \circ \Theta_{t,s} - z|_H + |g(s) \circ \Theta_{t,s} - g(s)|_H + |g(s) - g(t)|_H
\]

\[
\leq C_1 |t-s| \left\{ \left( \int_\Omega |\mathcal{W}|^2 \, dx \right)^{1/2} + \left( \int_\Omega |\mathcal{V}|^2 \, dx \right)^{1/2} + C \sum_{s} |g'(r)|_X \, dr \right\}
\]

and

\[
|z_1|^p_X \leq |z \circ \Theta_{t,s}|^p_X + C' \left( |g(s) \circ \Theta_{t,s} - g(s)|_X + |g(s) - g(t)|_X \right)
\]

\[
\leq |z|^p_X + C'' |t-s| \left( \left| z \right|^p_X + |g(s)|_{L^p(X)} + |g(t)|_{L^p(X)} \right) + C'' \int_s^t |g'(r)|_X \, dr,
\]

where \( C, C', C'' \) are positive constants independent of \( s, t \) and \( z \). Accordingly, if we take

\[ a(t) = \text{const.} \left\{ t + \int_0^t |g'(r)|_X \, dr \right\}, \quad t \geq 0, \]

then \( (H)_1' \) holds. Q.E.D.

Example 3.2.2. For instance, let \( \{ \phi^t; 0 \leq t \leq T \} \) be the family given by (2.4) corresponding to \( K(t) = K^4(t) \) under the assumptions of (ii) (with \( l = 2 \)) of Proposition 3.2.2. Then by virtue of Theorems 1.1.1, 1.1.2 and 2.1.1, for given \( f \in L^2(0, T; H) \) and \( u_0 \) in the closure of \( K^4(0) \) in \( H \) (resp. \( u_0 \) in \( K^4(0) \)) there exists a unique function \( u \in C ([0, T]; H) \) satisfying
Solvability of Nonlinear Evolution Equations

(a) $u(0) = u_0$;

(b) $(\text{resp. } (b')) u \in K^4(t)$ for all $t \in (0, T]$, $u \in L^2(0, T; X)$,
$\sqrt{t}u' \in L^2(0, T; H)$ (resp. $u' \in L^2(0, T; H)$) and $t \rightarrow |u(t)|_X^p$ (resp.
$t \rightarrow |u(t)|_X^p$) is absolutely continuous on $(0, T]$;

(c) for a.e. $t \in (0, T]$ the following inequality holds:

$$
(u'(t) - f(t), u(t) - z)_H + \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha u(x, t)|^p - 2 D^\alpha u(x, t)(D^\alpha u(x, t) - D^\alpha z(x)) \, dx \leq 0, \\
V z \in K^4(t).
$$

Moreover if $g(0) = g(T)$ and $\omega(0) = \omega(T)$, then by Theorems 2.3.1 and 2.3.2 there exists a
unique function $u \in C([0, T]; H)$ which satisfies $u(0) = u(T)$, (b') and (c). We note here that
(2.13) is equivalent to the following system (2.14) and (2.15):

$$
(u'(t) - f(t), z)_H + \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha u(x, t)|^p - 2 D^\alpha u(x, t)(D^\alpha u(x, t) - D^\alpha g(x, t)) \, dx \leq 0,
$$

$$
V z \in X \text{ with } z \equiv 0 \text{ on } \omega(t) \text{ in the sense of } X,
$$

$$
+ \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha u(x, t)|^p - 2 D^\alpha u(x, t)(D^\alpha u(x, t) - D^\alpha g(x, t)) \, dx = 0.
$$

Remark 3.2.1. Associated with initial-boundary value problems for parabolic partial differential equations formulated in non-cylindrical domains, such a kind of observations on variable function spaces as Proposition 3.2.2, was earlier studied by Yamada [1], in which only the case of $m = 1$ was dealt with.

Remark 3.2.2 In case $\Omega \subset R^N$ with $N \geq 2$, we consider a more general class of $\{ \phi^t; 0 \leq t \leq T \}$ than that given by (2.4). We define for each $t \in (0, T]$

$$
\phi^t (z) = \left\{ \begin{array}{ll}
\sum_{|\alpha| \leq m} \int_{\Omega} A_\alpha(x, t) |D^\alpha z(x)|^p \, dx + \sum_{|\beta| \leq m-1} \int_{\Gamma} B_\beta(x, t) |D^\beta z(x)|^p \, d\Gamma_x \\
\infty & \text{if } z \in K(t), \\
& \text{otherwise.}
\end{array} \right.
$$

where functions $A_\alpha$ on $\Omega \times (0, T]$ and $B_\beta$ on $\Gamma \times (0, T]$ satisfy respectively

- for every $\alpha$ with $|\alpha| \leq m$, $A_\alpha(x, t)$ is Lipschitz continuous in $(x, t)$ and $a_1 \leq A_\alpha(x, t) \leq a_2$
- for all $(x, t) \in \Omega \times (0, T]$ with some positive constants $a_1, a_2$

and

- for every $\beta$ with $|\beta| \leq m-1$, $B_\beta(x, t)$ is Lipschitz continuous in $(x, t)$ and $0 \leq B_\beta(x, t) \leq b_1$
- for all $(x, t) \in \Gamma \times (0, T]$ with some positive constant $b_1$.

We can similarly prove that $\mathcal{H}^t$ holds for the family $\{ \phi^t; 0 \leq t \leq T \}$ given by (2.14) under the same assumptions of (ii) of Proposition 3.2.2.

§3.3. Nonlinear equations of the Stokes-type with time-dependent constraints

Let $\Omega$ be a bounded domain in $R^N (N \geq 2)$ with smooth boundary $\Gamma$, $2 \leq p < \infty$ and $0 < T$.

Let $\{ \omega(t); 0 \leq t \leq T \}$ be a family of non-empty compact subsets of $\bar{\Omega}$. Also, let

$$
D_0(\bar{\Omega}) = \{ z \in (D(\bar{\Omega}))^N; \text{div } z = 0 \text{ on } \Omega \}.
$$
where $D(\tilde{\Omega})$ stands for the space of all smooth functions on $\tilde{\Omega}$. We set

$$H = \text{the closure of } D_0(\tilde{\Omega}) \text{ in } [L^2(\tilde{\Omega})]^N$$

and

$$X = \text{the closure of } D_0(\tilde{\Omega}) \text{ in } [W^{1,p}(\Omega)]^N$$

with

$$|z|_H = \left\{ \sum_{k=1}^N |z_p|_{L^2(\Omega)}^2 \right\}^{1/2}, \quad |z|_X = \left\{ \sum_{k=1}^N |z_k|^p_{W^{1,p}(\Omega)} \right\}^{1/p}$$

for $z = (z_1, z_2, \ldots, z_N)$. Clearly $H$ is a Hilbert space and $X$ is a reflexive Banach space with $X \subset H$. We define for each $t \in [0, T]$

$$X(t) = \{ z = (z_1, \ldots, z_N) \in X; \quad z_i = 0 \text{ on } \omega(t) \text{ in the sense of } W^{1,p}(\Omega), i = 1, 2, \ldots, N \},$$

which is a closed subspace of $X$.

**Proposition 3.3.1.** Let $\{ \phi^t; 0 \leq t \leq T \}$ be a family of proper l.s.c. convex functions on $H$ given by

$$\phi^t(z) = \begin{cases} \frac{1}{p} |z|^p_X, & \text{if } z \in X(t), \\ \infty, & \text{otherwise.} \end{cases}$$

(3.1)

Then with the same notations as in the previous section, we have:

(i) if $\omega(t)$ is non-increasing in $t$, then $(H)_t^n$ holds;

(ii) under condition (*) with $m = 2$, $(H)_t^\infty$ holds.

**Proof.** Since (i) is clear, we give below only a proof of (ii). Let $\Theta_{t,s}$ be as in Application 2 of §3.2 and denote by $a_{i,j}^{l,k}(x)$ the $(i,j)$-element of $J(\Theta_{t,s}(x))$, i.e. $a_{i,j}^{l,k}(x) = \partial / \partial x_l \delta_{i,j}^{l,k}(x)$, and by $\tilde{a}_{i,j}^{l,k}(x)$ its cofactor. As is well-known,

$$\sum_{l=1}^N a_{i,j}^{l,k}(x) \tilde{a}_{i,j}^{l,k}(x) = \delta_{i,j}^{l,k} \det J(\Theta_{t,s}(x)), \quad \forall x \in \Omega$$

and by a simple calculation

$$\sum_{l=1}^N \frac{\partial}{\partial x_l} \tilde{a}_{i,j}^{l,k}(x) = 0, \quad k = 1, 2, \ldots, N, \quad \forall x \in \Omega.$$ 

Now, let $z = (z_1, z_2, \ldots, z_N) \in X(s)$ and put

$$\tilde{z}_i(x) = z_i(\Theta_{t,s}(x)), \quad x \in \Omega, \quad i = 1, 2, \ldots, N$$

and

$$\begin{bmatrix} Z_1(x) \\ Z_2(x) \\ \vdots \\ Z_N(x) \end{bmatrix} = (\tilde{a}_{i,j}^{l,k}(x)) \begin{bmatrix} \tilde{z}_1(x) \\ \tilde{z}_2(x) \\ \vdots \\ \tilde{z}_N(x) \end{bmatrix}, \quad x \in \Omega.$$
Then clearly \( Z = (Z_1, Z_2, \ldots, Z_N, 0) \in \{W^{1,p}(\Omega)\}^N \) and \( Z_1 = 0 \) on \( \omega(t) \) in the sense of \( W^{1,p}(\Omega) \), \( i = 1, 2, \ldots, N \). Besides \( \text{div} \, Z = 0 \). In fact, \( Z_1 = \sum_{k=1}^{N} a_{t,s}^{i,k} \frac{\partial z_k}{\partial y_k} \), and

\[
\frac{\partial z_k}{\partial x_i} = \sum_{h=1}^{N} \frac{\partial z_k}{\partial y_h} \frac{\partial \theta^h}{\partial x_i} = \sum_{h=1}^{N} a_{t,s}^{i,h} \frac{\partial z_k}{\partial y_h},
\]

where \( y = (y_1, y_2, \ldots, y_N) \) with \( y_h = \theta^h_{t,s}(x) \), it follows that

\[
\text{div} \, Z = \sum_{t,s=1}^{N} \left( \frac{\partial z_k}{\partial x_i} \frac{\partial z_k}{\partial x_i} + \frac{\partial z_k}{\partial y_h} \frac{\partial z_k}{\partial y_h} \right)
\]

\[
= \sum_{k=1}^{N} \left( \frac{\partial z_k}{\partial x_i} \frac{\partial z_k}{\partial x_i} \right) + \sum_{t,s=1}^{N} a_{t,s}^{i,h} a_{t,s}^{i,k} \frac{\partial z_k}{\partial x_i} \frac{\partial z_k}{\partial y_h}
\]

\[
= \det \left( \Theta_{t,s} \right) \text{div} \, Z
\]

\[
= 0;
\]

we used here (3.2) and (3.3). Thus \( Z \in \mathcal{X}(t) \). Forthmore we see

\[
|Z_i - z_i|_{L^2(\Omega)} \leq R_1 |t - s| \sum_{k=1}^{N} |z_k|_{W^{1,2}(\Omega)}, \quad i = 1, 2, \ldots, N,
\]

and

\[
|\frac{\partial Z_i}{\partial x_j}|_{L^p(\Omega)} - |\frac{\partial z_i}{\partial x_j}|_{L^p(\Omega)}| \leq R_1 |t - s| \sum_{k=1}^{N} |z_k|_{W^{1,p}(\Omega)}, \quad i, j = 1, 2, \ldots, N,
\]

where \( R_1 \) is a positive constant independent of \( s, t \) and \( z \). Indeed,

\[
|Z_i - z_i|_{L^2(\Omega)} \leq |a_{t,s}^{i,l} (z_i - z_i)|_{L^2(\Omega)}
\]

\[
+ |(\hat{a}_{t,s}^{i,l} - 1) z_i|_{L^2(\Omega)} + \sum_{k \neq i} |a_{t,s}^{i,k} z_k|_{L^2(\Omega)}
\]

for all \( i = 1, 2, \ldots, N \). Applying Lemma 3.2.2, we have

\[
|\hat{z}_i - z_i|_{L^2(\Omega)} \leq R_2 |t - s| |z_i|_{W^{1,2}(\Omega)}, \quad i = 1, 2, \ldots, N,
\]

with a positive constant \( R_2 \) independent of \( s, t \) and \( z \). Since \( \Theta_{t,s}^{-1} = \Theta_{s,t} \), we see that

\[
\hat{a}_{t,s}^{i,j} = \det \left( \Theta_{t,s} \right) a_{s,t}^{i,j},
\]

so that by our assumption

\[
|\hat{a}_{t,s}^{i,l}(x) - 1| \leq R_3 |t - s|, \quad \forall x \in \Omega, \quad i = 1, 2, \ldots, N
\]

and

\[
|\hat{a}_{t,s}^{i,k}(x)| \leq R_3 |t - s|, \quad \forall x \in \Omega, \quad i \neq k,
\]

for any \( s, t \in [0, T] \), where \( R_3 \) is a positive constant independent of \( s, t \) and \( z \). With the aid of these inequalities we can easily obtain (3.4) from (3.6). Similarly, using Lemma 3.2.3, we get (3.5).

Q.E.D.
Now, consider the Cauchy problem $CP(\phi^t ; f, u_0)$ for the family $\{ \phi^t ; 0 \leq t \leq T \}$ given by (3.1), $f \in L^2(0, T; H)$ and $u_0$ in the closure of $X(0)$ in $H$ (resp. $u_0$ in $X(0)$) under the assumption of (ii) of Proposition 3.3.1. According to abstract results of Chapters 1 and 2, there exists a unique function $u$ in $C([0, T]; H)$ such that

(a) $u(0) = u_0$;
(b) $(\text{resp. } (b')) u(t) \in X(t)$ for all $t \in (0, T]$, $u \in L^p(0, T; X)$, $\sqrt{t}u' \in L^2(0, T; H)$ (resp. $u' \in L^2(0, T; H)$), $t \to t | u(t) |^p_X$ (resp. $t \to | u(t) |^p_X$) is absolutely continuous on $(0, T]$;
(c) for a.e. $t \in (0, T]$ the following inequality holds:

$$
(u'(t) - f(t), z)_H + \sum_{i,j=1}^{N} \int_{\Omega} \left| \frac{\partial u_i}{\partial x_j}(x,t) \right|^2 \left( \frac{\partial^2 u_i}{\partial x_j \partial x_i} \right)_x + \sum_{i,j=1}^{N} \int_{\Omega} | u_i(x,t) |^{p-2} u_i(x,t) \frac{\partial u_i}{\partial x_j}(x,t) \frac{\partial z_j}{\partial x_i}(x) dx = 0,
$$

$$
\forall z = (z_1, z_2, \ldots, z_N) \in X(t).
$$

Moreover, if $\omega(0) = \omega(T)$, then there exists a unique function $u$ in $C([0, T]; H)$ satisfying $u(0) = u(T)$, (b') and (c).

In particular, in case $\omega(t) = \Omega - \Omega(t)$ with a relatively compact subdomain $\Omega(t)$ of $\Omega$, then it is well-known that (3.7) is written in the following form:

$$
\frac{\partial u_i}{\partial t} - \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left( | u_i |^{p-2} \frac{\partial u_i}{\partial x_j} \right) + | u_i |^{p-2} u_i = f_i + \frac{\partial P}{\partial x_i}, \quad i = 1, 2, \ldots, N,
$$

on $U_{0 < t < T} \{ t \in \Omega(t) \}$, where $f = (f_1, f_2, \ldots, f_N)$.

**Remark 3.3.1.** The results of Proposition 3.3.1 were due to Yamada [2], in which he treated the case of $\omega(t) = \Omega - \Omega(t)$ and showed the existence of a periodic solution to a modified Stokes equation. Also, for related results, see Fujita [1], Fugita-Sauer [1], Morimoto [1], Otani-Yamada [1] and Kenmochi [5].

### §3.4 Quasi-linear heat equations with time-dependent constraints

In this section, we give an application of abstract results on the evolution equation $u'(t) + \partial \phi^t(Bu(t)) \ni f(t)$, which was studied in §3.2.8.

Throughout this section, let $\Omega$ be a bounded domain in $R^N (N \geq 2)$ with smooth boundary $\Gamma$, and let

$$
H = L^2(\Omega), \quad X = W^{1,2}(\Omega),
$$

and

$$
a(z, z_j) = \sum_{i=1}^{N} \int_{\Omega} \frac{\partial z_i}{\partial x_i} \frac{\partial z_j}{\partial x_i} dx \quad \text{for } z, z_j \in X.
$$

Given a non-negative function $\rho = \rho(x, t)$ on $\Gamma \times [0, T]$, $0 < T < \infty$, which is Lipschitz continuous, i.e., for a positive constant $C_\rho$

$$
| \rho(x, t) - \rho(x', s) | \leq C_\rho | x - x' | + | t - s |, \quad \forall (x, t), (x', s) \in \Gamma \times [0, T],
$$
we define
\[ a(t; z, z_1) = a(z, z_1) + \int_{\Gamma} \rho(x, t) z(x) z_1(x) \, d\Gamma_x \]
for \( t \in [0, T] \) and \( z, z_1 \in X \). Also, let \( \beta \) be a real-valued increasing function on \( R \).

We are given a family \( \{ \omega(t); 0 \leq t \leq T \} \) of non-empty compact subsets of \( \Omega \) and set for \( 0 \leq t \leq T \)
\[ K_\omega(t) = \{ z \in X; z = 0 \text{ on } \omega(t) \text{ in the sense of } X, z \geq 0 \text{ a.e. on } \Omega \} \]
and for a function \( \psi: [0, T] \rightarrow X \)
\[ K_{\psi}(t) = \{ z + \psi(\cdot, t); z \in K_\omega(t) \} \]
Evidently these sets are non-empty, closed and convex in \( X \).

Now, assuming that \( \beta \) is bi-Lipschitz (resp. locally bi-Lipschitz) continuous on \( R \), for \( f \in L^2(0, T; H), u_0 \in H, g \in L^2(0, T; X) \) and \( \psi \in L^2(0, T; X) \) we propose a variational problem
\[ V_{\beta}(f, u_0, g, \psi) \text{ (resp. } V_{\beta}'(f, u_0, g, \psi) \text{)} \]
to find a function \( u \in C([0, T]; H) \) (resp. \( u \in C([0, T]) \)) \( H \cap L^\infty(\Omega \times (0, T)) \) such that
\[
(V1) \quad u(0) = u_0; \\
(V2) \quad u(t) \in K_\psi(t) \text{ for a.e. } t \in [0, T], \quad u \in L^2(0, T; X) \text{ and } u \in W^{1,2}(\delta, T; H) \text{ for every } 0 < \delta < T; \\
(V3) \quad \text{for a.e. } t \in [0, T] \text{ the following inequality holds:}
\]
\[
(4.1) \quad (u'(t) - f(t), u(t) - z)_H + a(t; \beta(u(t)) - g(t), u(t) - z) \leq 0, \quad \forall z \in K_\psi(t).
\]
Such a function \( u \) is called a solution to \( V_{\beta}(f, u_0, g, \psi) \) (resp. \( V_{\beta}'(f, u_0, g, \psi) \)).

**Lemma 3.4.1.** Suppose \( \beta \) is bi-Lipschitz (resp. locally bi-Lipschitz) continuous on \( R \), \( f \in L^2(0, T; H), u_0 \in H, g \in L^2(0, T; X) \) and \( \psi \in L^2(0, T; X) \) (resp. \( \psi \in L^2(0, T; X) \cap L^\infty(\Omega \times (0, T)) \)). Let \( u \in C([0, T]; H) \) (resp. \( u \in C([0, T]) \)) \( H \cap L^\infty(\Omega \times (0, T)) \) for which (V2) holds. Then (V3) is equivalent to each of the following (a), (b) and (c).

(a) For a.e. \( t \in [0, T] \),
\[
(u'(t) - f(t), u(t) - \psi(t))_H + a(t; \beta(u(t)) - g(t), u(t) - \psi(t)) = 0
\]
and
\[
(4.2) \quad (u'(t) - f(t), z)_H + a(t; \beta(u(t)) - g(t), z) \leq 0, \quad \forall z \in K_\omega(t).
\]

(b) For a.e. \( t \in [0, T] \),
\[
(u'(t) - f(t), \beta(u(t)) - z)_H + a(t; \beta(u(t)) - g(t), \beta(u(t)) - z) \leq 0
\]
for all \( z \in K_{\beta(\psi)}(t) \), where
\[ K_{\beta(\psi)}(t) = \{ z + \beta(\psi(\cdot, t)); z \in K_\omega(t) \} \]

(c) For a.e. \( t \in [0, T] \), (4.2) holds together with
\[
(u'(t) - f(t), \beta(u(t)) - \beta(\psi(t)))_H + a(t; \beta(u(t)) - g(t), \beta(u(t)) - \beta(\psi(t))) = 0.
\]
Proof. First suppose (V3). Then, since $u(t) + z \in K_{\psi}(t)$ for any $z \in K_{\phi}(t)$ and a.e. $t \in [0, T]$, we derive (4.2) from (4.1). For simplicity, put

$$I(t; z) = (u'(t) - f(t), z)_{H} + a(t; \beta(u(t))) - g(t, z)$$

for $z \in X$ and $t \in [0, T]$. With this notation, taking $z = \psi(t)$ in (4.1), we have

$$I(t; u(t) - \psi(t)) \leq 0 \quad \text{for a.e. } t \in [0, T].$$

On the other hand, since $u(t) - \psi(t) \in K_{\phi}(t)$, we obtain from (4.2) that

$$I(t; u(t) - \psi(t)) > 0 \quad \text{for a.e. } t \in [0, T].$$

Hence (a) holds. Conversely, (a) → (V3) is easily shown, too. Similarly (b) ↔ (c).

Next we show (a) → (c). In fact, we see

$$\beta(u(t)) - \beta(\psi(t)) \in K_{\phi}(t)$$

and

$$M(u(t) - \psi(t)) - (\beta(u(t)) - \beta(\psi(t))) \in K_{\phi}(t) \quad \text{for a.e. } t \in [0, T]$$

with a positive constant $M$, because $\beta$ is Lipschitz or locally Lipschitz continuous on $R$. Therefore, if (a) holds, then

$$0 \leq I(t; \beta(u(t)) - \beta(\psi(t))) \leq MI(t; u(t) - \psi(t)) = 0$$

for a.e. $t \in [0, T]$, from which (c) follows. Similarly (c) → (a).

With the help of Lemma 3.4.1 we can prove the following comparison result for solutions to our variational problems.

**Theorem 3.4.1.** Suppose $\beta$ is bi-Lipschitz (resp. locally bi-Lipschitz) continuous on $R$. Let

$$f, \tilde{f} \in L^{2}(0, T; H), u_{o}, \tilde{u}_{o} \in H, g \in L^{2}(0, T; X)$$

and $\psi, \tilde{\psi} \in L^{2}(0, T; X)$ (resp. $\psi, \tilde{\psi} \in L^{2}(0, T; X)$) with $\tilde{\psi} \geq \psi$ a.e. on $\Omega \times (0, T)$. Let $u$ and $\tilde{u}$ be solutions to $V_{\beta}(f, u_{o}, g, \psi)$ (resp. $V_{\beta}^{\tilde{f}}(\tilde{f}, \tilde{u}_{o}, g, \tilde{\psi})$) and $V_{\beta}(\tilde{f}, \tilde{u}_{o}, g, \tilde{\psi})$ (resp. $V_{\beta}^{\tilde{f}}(\tilde{f}, \tilde{u}_{o}, g, \tilde{\psi})$), respectively. Then we have for any $s, t \in [0, T]$ with $s \leq t$.

$$\frac{(u(t) - \tilde{u}(t))^{+}}{L_{1}(\Omega)} \leq \frac{(u(s) - \tilde{u}(s))^{+}}{L_{1}(\Omega)} + \int_{s}^{t} \frac{(f(r) - \tilde{f}(r))^{+}}{L_{1}(\Omega)} \, dr.$$

**Proof.** Take a sequence $\{\sigma_{n}\}$ of smooth functions on $R$ with non-negative bounded derivatives $\sigma_{n}'$ such that $-1 \leq \sigma_{n} \leq 1, \sigma_{n}(0) = 0$ and

$$\sigma_{n}(r) \to \sigma_{o}(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0, \end{cases} \quad \text{(as } n \to \infty\text{)}$$

for each $r \in R$. Since

$$\sigma_{n}(\beta(\tilde{u}(t)) - \beta(u(t)))^{+} \in K_{\phi}(t)$$

and

$$M_{n}(u(t) - \tilde{u}(t)) - \sigma_{n}(\beta(\tilde{u}(t)) - \beta(u(t)))^{+} \in K_{\phi}(t)$$

$$-70-$$
for a.e. \( t \in [0, T] \) with some constant \( M_n > 0 \), by using Lemma 3.4.1 we have
\[
(u'(t) - f(t), \sigma_n(\beta(\bar{u}(t)) - \beta(\bar{u}(t)))^+) \leq 0
\]
and
\[
(\bar{u}'(t) - \bar{f}(t), \sigma_n(\beta(\bar{u}(t)) - \beta(\bar{u}(t)))^+) \leq 0
\]
for a.e. \( t \in [0, T] \). Noting that
\[
a(t; \beta(\bar{u}(t)) - \beta(\bar{u}(t)), \sigma_n(\beta(\bar{u}(t)) - \beta(\bar{u}(t)))^+)
= \int_\Omega \sigma_n'(|(\beta(\bar{u}(x, t)) - \beta(\bar{u}(x, t)))^+| \cdot \nabla(\beta(\bar{u}(x, t)) - \beta(\bar{u}(x, t)))^+) \, dx
+ \int_{\Gamma} \alpha(x, t) (\beta(\bar{u}(x, t)) - \beta(\bar{u}(x, t))) \sigma_n(\beta(\bar{u}(x, t)) - \beta(\bar{u}(x, t)))^+) \, d\Gamma_x
\geq 0,
\]
we infer from the above equality and inequality that
\[
(u'(t) - \bar{u}'(t), \sigma_n(\beta(\bar{u}(t)) - \beta(\bar{u}(t)))^+) \leq \int_\Omega (f(x, t) - \bar{f}(x, t)) \sigma_n(\beta(\bar{u}(x, t)) - \beta(\bar{u}(x, t)))^+) \, dx
\]
for a.e. \( t \in [0, T] \), so that letting \( n \to \infty \) gives
\[
(u'(t) - \bar{u}'(t), \sigma_n(\beta(\bar{u}(t)) - \beta(\bar{u}(t)))^+) \leq \int_\Omega (f(x, t) - \bar{f}(x, t))^+ \, dx
\]
for a.e. \( t \in [0, T] \), which implies
\[
\sigma_n(\beta(\bar{u}(t)) - \beta(\bar{u}(t)))^+ \sigma_n((u(t)) - \bar{u}(t))^+ \leq 0.
\]
Because \( \sigma_n((u(t)) - \bar{u}(t))^+ \sigma_n((u(t)) - \bar{u}(t))^+ \), we get (4.3). \( \Box \)

**Corollary.** Suppose \( \beta \) is bi-Lipschitz (resp. locally bi-Lipschitz) continuous on \( \mathbb{R} \), and let \( f \in L^2(0, T; H) \), \( u_0 \in H \), \( \sigma \in L^2(0, T; X) \) and \( \psi \in L^2(0, T; X) \) (resp. \( \psi \in L^2(0, T; X) \cap L^\infty(0, T; H) \cap L^\infty(0, T; H) \)). Then \( V_{\beta}^f(f, u_0, \sigma, \psi) \) (resp. \( V_{\beta}^f(f, u_0, \sigma, \psi) \)) has at most one solution.

**Remark.** The technic adopted above is found in Bénilang [2] and Demlaman [1]. As to problem \( V_{\beta}^f(f, u_0, \sigma, \psi, \psi') \) we establish

**Theorem 3.4.2.** Suppose condition (*) with \( m = 1 \) (see Proposition 3.2.2) and that \( \beta \) is bi-Lipschitz continuous on \( \mathbb{R} \). Let \( f \in L^2(0, T; H) \), \( g \in W^{1,2}(0, T; X) \cap L^\infty(0, T; H) \cap L^\infty(0, T; H) \) and \( \psi \in L^2(0, T; X) \) with \( \beta(\psi) \in W^{1,2}(0, T; X) \cap L^\infty(0, T; H) \cap L^\infty(0, T; H) \). Then we have:

(i) If \( u_0 \in K_{\psi}(0) \), then \( V_{\beta}^f(f, u_0, \sigma, \psi) \) has a solution in \( W^{1,2}(0, T; H) \) \cap L^\infty(0, T; H) \) and \( \sqrt{\mu} u \in L^\infty(0, T; H) \).

(ii) If \( u_0 \) is in the closure of \( K_{\psi}(0) \) in \( H \), then \( V_{\beta}^f(f, u_0, \sigma, \psi) \) has a solution \( u \) satisfying

\[
\sqrt{\mu} u \in L^2(0, T; H) \text{ and } \sqrt{\mu} u \in L^\infty(0, T; H).
\]

In order to apply the results obtained in §2.8 to \( V_{\beta}^f(f, u_0, \sigma, \psi) \), consider the family \( \{ \phi^t, 0 \leq t \leq T \} \) of proper l.s.c. convex functions on \( H \) given by
Lemma 3.4.2. Under the same assumptions of Theorem 3.4.2, the family \( \{ \phi^t ; 0 \leq t \leq T \} \) given by (4.4) satisfies \((H)_2'\).

Proof. We use the same notations as in Application 2 of § 3.2. Simply we set \( \psi_1 = \beta(\psi) \).
Given \( z \in K_{\psi_j}(s) \) and \( t \in [0, T] \), we see that the function
\[
z_1 = z \circ \Theta_{t,s} - \psi_1(s) \circ \Theta_{t,s} + \psi_1(t)
\]

belongs to \( K_{\psi_j}(t) \) and for a positive constant \( C \) independent of \( i, t \) and \( z, z_1 \), satisfies
\[
|z_1 - z|_H \leq C |t - s| \left( |z|_X + |\psi_1|_{L^\infty(0, T; L^\infty)} \right) + C \int_s^t |\psi_1'(r)|_X \, dr
\]
and
\[
\int_\Omega |\nabla z_1|^2 \, dx - \int_\Omega |\nabla z|^2 \, dx \leq C |t - s| \left( |z|^2_X + |\psi_1|_{L^\infty(0, T; W^{2,2}(\Omega))} \right)
\]
\[
+ C \int_s^t |\psi_1'(r)|_X \, dr,
\]
which were already shown in the proof of (ii) of Proposition 3.2.2. Similarly, using the Lipschitz continuity of \( \rho \), we have
\[
\int_\Gamma \rho(., t) |z_1|^2 \, d\Gamma - \int_\Gamma \rho(., s) |z|^2 \, d\Gamma \leq C' \left| t - s \right| \left( |z|^2_X + |\psi_1|_{L^\infty(0, T; W^{2,2}(\Omega))} \right) + C' |z|_X \left( \int_s^t |\psi_1'(r)|_X \, dr \right)
\]
with a positive constant \( C' \) independent of \( s, t \) and \( z \). From these inequalities it follows that \((H)'_2\) holds. Q.E.D.

By \( \gamma_o \) we denote the following function on \( L^1(\Omega) \):
\[
z \in L^1(\Omega) \rightarrow \int_\Omega z^+(x) \, dx,
\]
which is continuous, non-negative and convex on \( L^1(\Omega) \). Also by \( B \) we denote the operator from \( D(B) = H \) into itself given by
\[
(Bz)(x) = \beta(z(x)), \quad \forall \ z \in H, \quad \forall \ x \in \Omega,
\]
which is bi-Lipschitz continuous on \( H \) and is the subdifferential of \( f \) on \( H \) given by
\[
f(z) = \int_\Omega \tilde{\beta}(z(x)) \, dx, \quad \tilde{\beta}(r) = \int_0^r \beta(s) \, ds.
\]

Lemma 3.4.3. For every \( t \in [0, T] \) we have:
(i) \( z^* \in \partial \phi^t(z) \) implies
\[
(z^*, \eta, z)_H \leq a(t; z - g(t), \eta - z), \quad \forall \ \eta \in K_{\beta(\psi)}(t),
\]
and this is equivalent to the following system of (4.6) and (4.7):
\[
(z^*, \eta, z) \leq a(t; z - g(t), \eta - z), \quad \forall \ \eta \in K_{\beta(\psi)}(t),
\]
and
\[
(z^*, z - B\psi(t))_H = a(t; z - g(t), z - B\psi(t)).
\]
(ii) \( \partial \phi^t \circ B \) is \( \gamma_o \)-accretive on \( H \).

**Proof.** By a computation similar to that in the proof of Lemma 3.4.1 we get (i). In order to prove (ii), let \( z_i^* \in \partial \phi^t(Bz_i) \), \( i = 1, 2 \), and \( \sigma_n \) be as in the proof of Theorem 3.4.1. Then, since \( \sigma_n([Bz_1 - Bz_2]^+) \in K_o(t) \) and \( N_n(Bz_1 - B\psi(t)) - \sigma_n([Bz_1 - Bz_2]^+) \in K_o(t) \) for a certain positive number \( N_n \), we have by using (i) of this lemma

\[
(z_i^*, \sigma_n([Bz_1 - Bz_2]^+))_H = a(t; Bz_1 - g(t), \sigma_n([Bz_1 - Bz_2]^+)
\]

and

\[
(z_i^*, \sigma_n([Bz_1 - Bz_2]^+))_H \leq a(t; Bz_2 - g(t), \sigma_n([Bz_1 - Bz_2]^+)).
\]

Hence

\[
(z_1^* - z_2^*, \sigma_n([Bz_1 - Bz_2]^+))_H \geq a(t; Bz_1 - Bz_2, \sigma_n([Bz_1 - Bz_2]^+))
\]

\[
= \int_\Omega \sigma_n'(\beta(z_1(x)) - \beta(z_2(x))) \cdot [\nabla(\beta(z_1(x)) - \beta(z_2(x))] \, dx
\]

\[
+ \int_\Gamma \rho_0(x, t) (\beta(z_1(x)) - \beta(z_2(x))) \sigma_n((\beta(z_1(x)) - \beta(z_2(x))] \, d\Gamma_x \geq 0,
\]

so that letting \( n \to \infty \) yields

\[
(z_1^* - z_2^*, \sigma_n([z_1 - z_2]^+))_H \geq 0.
\]

Since \( \sigma_o([z_1 - z_2]^+) \in \partial \gamma_o(z_1 - z_2) \), this shows the \( \gamma_o \)-accretiveness of \( \partial \phi^t \circ B \) on \( H \).

Q.E.D.

Now, we are in a position to apply abstract results in §2.8 to our variational problem.

**Proof of Theorem 3.4.2:** As is easily seen, for each \( t \in [0, T] \) and \( r \geq 0 \), the set \( \{ z \in H; \| z \|_H \leq r, \| \phi^t(z) \| \leq r \} \) is relatively compact in \( H \). Therefore, all the assumptions of Theorems 2.8.1 – 2.8.3 are fulfilled, so for a given \( u_o \in K_\psi(0) \), \( CP(\phi^t, B; f, u_o) \) has a unique strong solution \( u \in W^{1,2}(0, T; H) \) such that \( t \to \phi^t(Bu(t)) \) is bounded on \( [0, T] \) (hence \( u \in L^\infty(0, T; X) \)). By Lemmas 3.4.1 and 3.4.3, this function \( u \) is a unique solution to \( V_\beta(f, u_o, g, \psi) \). Thus (i) is proved. Similarly (ii) can be proved.

Q.E.D.

In the remainder of this section we consider the problem \( V_\beta(f, u_o, g, \psi) \) in case \( \beta \) is locally bi-Lipschitz continuous on \( \mathcal{R} \).

**Theorem 3.4.3.** Suppose condition (*) with \( m = 1 \) and \( \beta \) is locally bi-Lipschitz continuous on \( \mathcal{R} \). Let \( f \in L^2(0, T; H) \cap L^\infty(\Omega \times (0, T)) \) and \( \psi \in L^2(0, T; X) \cap L^\infty(\Omega \times (0, T)) \) with \( \beta(\psi) \in W^{1,2}(0, T; X) \cap L^\infty(0, T; W^{1,2}(\Omega)) \). Then we have:

(i) If \( u_o \in K_\psi(0) \cap L^\infty(\Omega) \), then \( V_\beta'(f, u_o, 0, \psi) \) has a solution in \( W^{1,2}(0, T; H) \)

\( \cap L^\infty(0, T; X) \).

(ii) If \( u_o \) is in the closure of \( K_\psi(0) \) in \( H \) and \( u_o \in L^\infty(\Omega) \), then \( V_\beta'(f, u_o, 0, \psi) \) has a solution \( u \) satisfying

\[
\sqrt{u} \in L^2(0, T; H), \quad \sqrt{u} \in L^\infty(0, T; X).
\]

**Proof.** Let \( M \) be a positive number so that

\[
|f| \leq M, \quad |\psi| \leq M \quad \text{a.e. on } \Omega \times [0, T]
\]

and

\[
|u_o| \leq M \quad \text{a.e. on } \Omega.
\]

Q.E.D.
We then consider
\[ \tilde{u}(x, t) = Mt + M \quad \text{on } \tilde{\Omega} \times [0, T] \]
and for a number \( R > MT + M + 1 \)
\[ \beta_R(r) = \begin{cases} 
  r + \beta(R) - R, & \text{if } r > R, \\
  \beta(r), & \text{if } |r| \leq R, \\
  r + \beta(-R) + R, & \text{if } r < -R,
\end{cases} \]
which is clearly bi-Lipschitz continuous on \( R \). It is easy to see that \( \tilde{u} \) is a solution to \( V_{\beta}^f(\tilde{f}, \tilde{u}_0, 0, \tilde{\psi}) \) for \( \tilde{f} = M, \tilde{u}_0 = M \) and \( \tilde{\psi} = Mt + M \) as well as to \( V_{\beta_R}^f(f, \tilde{u}_0, 0, \tilde{\psi}) \). Now, denoting by \( u_R \) the solution to \( V_{\beta_R}^f(f, u_0, 0, \psi) \), we see from the comparison theorem (cf. Theorem 3.4.1) that
\[ \int_0^t |(u_R(t) - Mt - M)|^r_{L^1(\Omega)} \leq |(u_0 - M)|^r_{L^1(\Omega)} + \int_0^t |(f(r) - M)|^r_{L^1(\Omega)} \, dr = 0 \]
for all \( t \in [0, T] \), that is, \( u_R \leq Mt + M \) a.e. on \( \Omega \times [0, T] \). Similarly, comparing \( u_R \) with \( \tilde{v}(x, t) = -Mt - M \), we see that \( u_R \leq -Mt - M \) a.e. on \( \Omega \times [0, T] \). Accordingly, \( \beta_R(u_R) = \beta(u_R) \) a.e. on \( \Omega \times [0, T] \) and \( u_R \) is consequently a solution to \( V_{\beta}^f(f, u_0, 0, \psi) \).

**Remark 3.4.2.** The above type of problems was earlier studied by Kenmochi [6]. Also, see Brézis [5], Bénilan [2], Crandall [1] and Konishi [1] for related works, in which the case without constraints was dealt with.

§3.5. **Free boundary problems of the Stefan-type for nonlinear parabolic equations**

In this section we consider free boundary problems of the Stefan-type in one space dimension. Our problem is formally described as follows: Find a curve \( x = l(t) \) on \([0, T]\) and a function \( u = u(x, t) \) satisfying
\[
\begin{align*}
(5.1) & \quad u_t - A(u_x)_x = 0 \quad \text{for } 0 < x < l(t), \; 0 < t < T, \\
(5.2) & \quad u(x, 0) = u_0(x) \quad \text{for } 0 < x < l_0, \\
(5.3a) & \quad A(u_x)(0+, t) = -g(t) \quad \text{for } 0 < t < T \\
or \\
(5.3b) & \quad u(0, t) = g(t) \quad \text{for } 0 < t < T, \\
(5.4) & \quad u(l(t), t) = 0 \quad \text{for } 0 < t < T, \\
(5.5) & \quad \begin{cases} 
  l'(t) = \frac{d l(t)}{d t} = -A(u_x)(l(t) - , t) \quad & \text{for } 0 < t < T, \\
  l(0) = l_0, 
\end{cases}
\end{align*}
\]

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where $A(\cdot): R \to R$ is a given function and $u_o, g, l_o$ are given data.

Throughout this section we suppose that

(5.6) $A$ is strictly increasing and continuous on $R$;

(5.7) $c_o |r|^p \leq A(r) r \leq c_1 |r|^p, \quad \forall r \in R$;

(5.8) $c_o (r - r')^{p-1} \leq A(r) - A(r'), \quad \forall r, r' \in R$ with $r \geq r'$,

where $0 < c_o < 1, 0 < c_1, 2 \leq p < \infty$ are positive constants.

For simplicity we set

$$H = L^2(0, \infty), \quad X = W^{1, p}(0, \infty) \cap L^2(0, \infty).$$

By $\tilde{A}$ we denote the primitive of $A$ with $\tilde{A}(0) = 0$, i.e.

$$\tilde{A}(r) = \int_0^r A(s) ds, \quad r \in R.$$

Lemma 3.5.1 (i) $(c_o/p) |r|^p \leq \tilde{A}(r) \leq (c_1/p) |r|^p, \quad \forall r \in R.$

(ii) For any $r, r' \in R$ we have

$$\frac{1}{2} \tilde{A}(r) + \frac{1}{2} \tilde{A}(r') - \tilde{A}\left(\frac{r + r'}{2}\right) \leq \frac{c_o}{p} |r - r'|^p.$$

Proof. (i) is obvious. We have

$$\frac{1}{2} \tilde{A}(r) + \frac{1}{2} \tilde{A}(r') - \tilde{A}\left(\frac{r + r'}{2}\right)$$

$$= \frac{1}{2} \int_0^r A(s) ds + \frac{1}{2} \int_0^{r'} A(s) ds - \int_0^{(r + r')/2} A(s) ds$$

$$= \frac{1}{2} \int_0^r A(s) ds - \int_{r'}^{(r + r')/2} A(s) ds$$

$$= \frac{1}{2} \int_{r'}^r \{ A(s) - A\left(\frac{s + r'}{2}\right) \} ds$$

$$\leq \frac{c_o}{2p} \int_{r'}^r (s - r')^{p-1} ds$$

$$= \frac{c_o}{p} |r - r'|^p;$$

we used here (5.8). Thus (5.9) holds. Q.E.D.

Let $0 < T < \infty$ and for a given $0 < l_o < \infty$ denote by $\Lambda(l_o)$ the class

$$\{ x = l(t); l(0) = l_o, l \text{ is non-decreasing on } [0, T] \},$$

and by $\Lambda_c(l_o)$ the subclass $\Lambda(l_o) \cap C([0, T])$. Let $g \in C([0, T])$ and define for each $l \in \Lambda(l_o)$ and $t \in [0, T]$. 

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\[ K_{1,1}(t) = \begin{cases} z \in X; z = 0 \text{ on } [l(t), \infty) \end{cases}, \]
\[ K_{2,1}(t) = \begin{cases} z \in X; z = 0 \text{ on } [l(t), \infty), z(0) = g(t) \end{cases}, \]
\[ \phi_{1,1}^s(z) = \begin{cases} \int_0^\infty A(z_x(t)) \, dx - g(t) \, z(0), & \text{if } z \in K_{1,1}(t), \\ \infty, & \text{otherwise}, \end{cases} \]
\[ \phi_{2,1}^s(z) = \begin{cases} \int_0^\infty A(z_x(t)) \, dx, & \text{if } z \in K_{2,1}(t), \\ \infty, & \text{otherwise}. \end{cases} \]

Clearly \( K_{i,1}(t), i = 1, 2, \) are non-empty, closed and convex in \( X \) and \( \phi_{i,1}^s \) are proper lsc convex on \( H \) with \( D(\phi_{i,1}^s) = K_{i,1}(t). \)

We are going to give quasi-variational formulations corresponding to systems \( \{ (5.1), (5.2), (5.3a), (5.4), (5.5) \} \) and \( \{ (5.1), (5.2), (5.3b), (5.4), (5.5) \} \).

**Definition 3.5.1.** Let \( 0 < T < \infty, 0 < \xi_0 < \infty, u_0 \in H, g \in C([0, T]) \). Then a pair \( (l, u) \in \Lambda(1,0) \times C([0, T] ; H) \) is called a strong (resp. weak) solution to \( QV_1(l_0, u_0, g), i = 1, 2, \) if the following \( (QV_1) \) and \( (QV_2) \) hold:

\( (QV_1) \) \( u \) is the strong (resp. weak) solution to \( CP(\phi_{i,1}^s ; u_0, g) \) on \([0, T] \);

\( (QV_2) \) \( l \in W^{1,2}(0, T) \) (resp. \( l \in W^{1,2}(\delta, T) \) for every \( 0 < \delta < T \)) and

\[ l'(t) = -A(u_x)(l(t) - t) \quad \text{for a.e. } t \in [0, T]. \]  

Before solving these problems \( QV_1(l_0, u_0, f) \) we prepare some lemmas and propositions on \( CP(\phi_{i,1}^s ; 0, u_0) \).

**Lemma 3.5.2.** Let \( l \in \Lambda(1,0) \) and set

\[ C_{1,1} = \frac{l(T)}{p} \left( \frac{2}{c_0} P^p \right)^{1/p} \left( 1 + |g|_{C([0, T])} \right)^{p' + 1} \left( \frac{1}{2} \right)^{1/p + 1/p'} = 1. \]

Then the following statements hold:

(i) \( \phi_{1,1}^s(z) = \frac{1}{2} \int_0^\infty A(z_x(t)) \, dx - C_{1,1}, \quad \forall t \in [0, T], \quad \forall z \in K_{1,1}(t); \)

(ii) for each \( s, t \in [0, T] \) with \( s \leq t \) and \( z \in K_{1,1}(s) \) we have

\[ \phi_{1,1}^s(z) - \phi_{1,1}^t(z) \leq C_{1,1} \left| g(t) - g(s) \right| \left( 1 + |\phi_{1,1}^s(z)|^{1/p'} \right). \]

**Proof.** Let \( \delta \) be any positive number. We observe that for \( z \in K_{1,1}(s) \)

\[ |z(0)| \leq \int_0^{l(s)} |z_x(t)| \, dx \leq \int_0^{l(s)} \left( \frac{\delta |z_x(t)|^p}{p} + \frac{\delta^{1-p}}{p'} \right) \, dx \leq \frac{\delta}{c_0} \int_0^\infty A(z_x(t)) \, dx + \frac{\delta^{1-p}}{p'}, \]

from which we obtain

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\[ \phi_{1,1}^z (z) = (1 - \frac{\delta}{c_o} |g(s)|) \int_0^\infty \hat{A}(z_x) dx - \frac{\delta^{1-p'} |l(s)|}{p'} |g(s)| \]

and

\[ (1 - \frac{\delta}{c_o} |g(s)|) |z(0)| \leq \frac{\delta}{c_o} \phi_{1,1}^z (z) + \frac{\delta^{1-p'} |l(s)|}{p'} . \]

Now, if we take \( \delta = (c_o/2) (1 + |g|_{C(\{0, T\})}^{-1}) \), then we have the inequality of (i) and

\[ \phi_{1,1}^t (z) - \phi_{1,1}^s (z) \leq |g(t) - g(s)| |z(0)| \]

\[ \leq C_{1,1} [g(t) - g(s)] \left( \int |\phi_{1,1}^s (z)| + 1 \right) . \]

Q.E.D.

We choose a smooth function \( \varsigma^0 \in D(R) \) such that

\[ 0 \leq \varsigma^0 \leq 1 \quad \text{on} \quad R, \quad \varsigma^0 (0) = 1, \quad \varsigma^0 = 0 \quad \text{on} \quad (-\infty, -l_o/2] \quad \text{and} \quad [l_o/2, \infty), \]

and set

\[ C(\varsigma^0) = \sup_{x \in R} |\varsigma^0_x (x)| . \]

Using this function \( \varsigma^0 \) we prove

**Lemma 3.5.3.** Let \( g \in W^{1,1} (0, T), 1 \in \Lambda (l_o) \) and set

\[ C_{2,1} = \sqrt{1 + 2^{p-1} c_1 C(\varsigma^0) / L^1_{C(0, T)} C(\varsigma^0)^{p-1} + \frac{p}{c_o} + 1(T) + 1} . \]

Then for each \( s, t \in [0, T] \) with \( s \leq t \) and \( z \in K_{2,1} (s) \) there is \( z_1 \in K_{2,1} (t) \) such that

\[ |z_1 - z|_H \leq C_{2,1} |g(t) - g(s)| \]

and

\[ \phi_{2,1}^t (z_1) - \phi_{2,1}^s (z) \leq C_{2,1} |g(t) - g(s)| \left( \int |\phi_{2,1}^s (z)| + 1 \right) . \]

**Proof.** Given \( z \in K_{2,1} (s) \) and \( t \geq s \), we consider

\[ z_1 (x) = z(x) + (g(t) - g(s)) \varsigma^0 (x) . \]

Clearly \( z_1 \in K_{2,1} (t) \) and

\[ |z_1 - z|_H = |g(t) - g(s)| |\varsigma^0|_H \leq \sqrt{1} |g(t) - g(s)| . \]

By (5.7) and (i) of Lemma 3.5.1,

\[ \hat{A}(z_x (x)) + (g(t) - g(s)) \varsigma^0_x (x) - \hat{A}(z_x (x)) \]

\[ \leq C(\varsigma^0) |g(t) - g(s)| \left| A(z_x (x)) + (g(t) - g(s)) \varsigma^0_x (x) \right| \]

\[ \leq c_1 C(\varsigma^0) |g(t) - g(s)| \left| |z_x (x)| + |g(t) - g(s)| C(\varsigma^0) \right|^{p-1} \]

\[ \leq 2^{p-1} c_1 C(\varsigma^0) |g(t) - g(s)| \left( |z_x (x)| \right)^{p-1} + |g'|_{L^1 (0, T)} C(\varsigma^0)^{p-1} + 1 \]

\[ \leq C_{2,1} |g(t) - g(s)| \left( \hat{A}(z_x (x)) + \frac{1}{l(T)} \right) . \]
Hence we get (5.11) and (5.12). Q.E.D.

Lemma 3.5.4. Let \( l \in \Lambda (l_o) \) and \( l_n \in \Lambda (l_o) \), \( n = 1, 2, \ldots \). Suppose \( l_n (t) \to l(t) \) point-wise on \([0, T]\). Then \( \phi_{l, l_n}^t \to \phi_{l, l}^t \) on \( H \) in the sense of Mosco, \( i = 1, 2 \), for every \( t \in [0, T] \).

Proof. It is easy to see
\[
\phi_{l, l_n}^t (z) \leq \liminf_{k \to \infty} \phi_{l_n, l_{n_k}}^t (z_{n_k})
\]
for every \( t \in [0, T] \) and every subsequence \( \{ n_k \} \) of \( \{ n \} \) with \( z_{n_k} \to z \) in \( H \). Put \( \epsilon_n = \max \{ l(t) - l_n(t), 0 \} \). For a given \( z \in K_{1, l_n} \), consider \( z_n (x) = z (x + \epsilon_n) \). Then, clearly \( z_n \in K_{1, l_n} \) and \( z_n \to z \) in \( X \) with \( z_n (0) \to z (0) \), and hence \( \phi_{l, l_n}^t (z_n) \to \phi_{l, l}^t (z) \). Thus \( \phi_{l, l_n}^t \to \phi_{l, l}^t \) on \( H \) in the sense of Mosco. Next, for a given \( z \in K_{2, l_n} \), consider
\[
z_n (x) = z (x + \epsilon_n) + (g(t) - z (\epsilon_n)) \zeta^0 (x).
\]
This function \( z_n \) belongs to \( K_{2, l_n} \) and \( z_n \to z \) in \( X \). Hence \( \phi_{l, l_n}^t (z_n) \to \phi_{l, l}^t (z) \), and thus \( \phi_{l, l_n}^t \to \phi_{l, l}^t \) on \( H \) in the sense of Mosco. Q.E.D.

Lemma 3.5.5. Let \( l \in \Lambda (l_o) \) and \( t \in [0, T] \). Then the statements (i) and (ii) below hold:
(i) \( z^* \in \partial \phi_{l, l}^t (z) \) is equivalent to the following system of (5.13) and (5.14):

\[
(5.13) \quad z^* \in H \text{ and } z \in K_{1, l} (t).
\]

\[
(5.14) \quad (z^*, \eta)_{L^2} = \int_0^\infty A(z_x) \eta_x \, dx - g(t) \eta (0), \quad \forall \eta \in K_{1, l} (t).
\]

Moreover, under (5.13), (5.14) is equivalent to \( (5.15), (5.16) \):

\[
(5.15) \quad z^* = -A(z_x) \text{ on } (0, l(t)).
\]

\[
(5.16) \quad A(z_x (0 +)) = -g(t).
\]

(ii) \( z^* \in \partial \phi_{l, l}^t (z) \) is equivalent to the following system \( (5.17), (5.18) \):

\[
(5.17) \quad z^* \in H \text{ and } z \in K_{2, l} (t).
\]

\[
(5.18) \quad (z^*, \eta - z)_{L^2} = \int_0^\infty A(z_x) \eta_x - z_x \, dx, \quad \forall \eta \in K_{2, l} (t), \text{ or equivalently,}
\]
\[
z^* = -A(z_x) \text{ on } (0, l(t)).
\]

This lemma is quite standard, so we omit the proof. But we note here that (5.15) or (5.18) implies the absolute continuity of \( A(z_x (x)) \) on \((0, l(t))\) and hence the right hand limit \( A(z_x (0 +)) \) of \( A(z_x (x)) \) at \( x = 0 \) exists as well as the left hand limit \( A(z_x (l(t) -)) \).

Proposition 3.5.1. Let \( i = 1 \) or 2, and assume that \( g \in W^{1,1} (0, T) \). Then we have:
(i) For each \( l \in \Lambda (l_o) \) and \( u_o \) in the closure of \( K_{l, l} (0) \) in \( H \) (resp. \( u_o \in K_{l, l} (0) \)), \( CP (\phi_{l, l}^t, 0, u_o) \) has a unique weak (resp. strong) solution \( u \) such that \( \sqrt{u} \in L^2 (0, T; H) \) (resp. \( u \in L^2 (0, T; H) \)), \( t \to \phi_{l, l}^t (u(t)) \) is integrable on \([0, T] \), \( t \to t \phi_{l, l}^t (u(t)) \) (resp. \( t \to \phi_{l, l}^t (u(t)) \)) is bounded on \([0, T] \) and \( -u' (t) \in \partial \phi_{l, l}^t (u(t)) \) for a.e. \( t \in [0, T] \).

(ii) Let \( l \in \Lambda (l_o) \) and \( u_o \) be in the closure of \( K_{l, l} (0) \) in \( H \). Suppose that \( u_o \geq 0 \) on \([0, T] \),
and \( g \geq 0 \) on \([0, T]\). Then the weak solution to \( CP(\phi_{t,1}^t; 0, u_0) \) is non-negative on \([0, \infty) \times (0, T]\).

**Proof.** By virtue of Lemmas 3.5.2 and 3.5.3, we can apply Theorems 1.1.1 and 1.1.2 to \( CP(\phi_{t,1}^t; 0, u_0) \) and (i) is a direct consequence of them. Next, under assumptions of (ii), let \( u \) be the weak solution to \( CP(\phi_{t,1}^t; 0, u_0) \). Then by Lemma 3.5.5 we have

\[
(u'(t), u^-(t))_H + \int_0^\infty A(u_x) (x, t) u_x^-(x, t) dx - g(t)u^-(0, t) = 0,
\]

so that

\[
\frac{d}{dt} |u^-(t)|_H^2 \leq 0 \quad \text{for a.e. } t \in [0, T].
\]

This shows \( |u^-(t)|_H \leq |u_0^-|_H = 0 \), i.e. \( u(x, t) \geq 0 \) a.e. on \([0, \infty) \times (0, T]\). Since \( u \) is continuous on \([0, \infty) \times (0, T]\) (see Remark 3.5.1 mentioned below), we have \( u \geq 0 \) on \([0, \infty) \times (0, T]\). Q.E.D.

**Remark 3.5.1.** Since \( W^{1,2}(0, T; H) \cap L^\infty(0, T; X) \subset C([0, \infty) \times [0, T]) \), the strong (resp. weak) solution to \( CP(\phi_{t,1}^t; 0, u_0) \) obtained by (i) of Proposition 3.5.1 is continuous on \([0, \infty) \times [0, T] \) (resp. on \([0, \infty) \times (0, T]\)).

**Corollary to Proposition 3.5.1** Under the same assumptions of (ii) of Proposition 3.5.1,

\[
A(u_x)(l(t) -, t) \leq 0 \quad \text{for a.e. } t \in [0, T],
\]

where \( u \) is the weak solution to \( CP(\phi_{t,1}^t; 0, u_0) \).

**Proof.** According to Lemma 3.5.5 and (i) of Proposition 3.5.1, \( A(u_x)(x, t) \) is absolutely continuous in \( x \) on \([0, l(t)]\) for a.e. \( t \in [0, T] \). Hence \( A(u_x)(l(t) -, t) \) exists for a.e. \( t \in [0, T] \) as well as \( u_x(l(t) -, t) \). In this case

\[
u_x(l(t) -, t) = \lim_{\delta \downarrow 0} \frac{u(l(t), t) - u(l(t) - \delta, t)}{\delta} \leq 0,
\]

because \( u(l(t), t) = 0 \) and \( u(l(t) - \delta, t) \geq 0 \) by (ii) of Proposition 3.5.1. Thus \( A(u_x)(l(t) -, t) \leq 0 \). Q.E.D.

For problem \( CP(\phi_{t,1}^t; 0, u_0) \) we have

**Lemma 3.5.6.** Assume \( g \in W^{1,2}(0, T) \). Let \( r_o > 0 \). Then there is a positive constant \( R_o = R_o(1_{1_o}, r_o) \) such that

\[
|u|_{C([0, T]; H)} \leq R_o, \quad |u|^p_{L^p(0, T; L^p(0, \infty))} \leq R_o,
\]

whenever \( u \) is a weak solution to \( CP(\phi_{t,1}^t; 0, u_0) \) with \( l \) in \( (1_{1_o}) \) and \( u_o \) in the closure of \( K_{2,1}(0) \) in \( H \) satisfying \( |u_o|_H \leq r_o \).

**Proof.** Since \( g(t) \in C_{2,1}(t) \), we have by (ii) of Lemma 3.5.5

\[
(u'(t), u(t) - g(t)c^o) = \int_0^\infty A(u_x)(x, t) (u_x(x, t) - g(t)c^o(x)) dx = 0
\]

for a.e. \( t \in [0, T] \). Therefore, by making use of assumptions on \( A \) we obtain

\[
0 \geq \frac{d}{dt} |u(t) - g(t)c^o|_H^2 - 2 \sqrt{1_{1_o}} |g'(t)| \cdot |u(t) - g(t)c^o|_H
\]

\[
+ 2c_o \int_0^\infty |u_x(x, t)|^p dx - 2c_1 |g(t)| C(c^o) \int_0^1 |u_x(x, t)|^{p-1} dx
\]
\[ \frac{d}{dt} |u(t) - g(t) \phi^o|_{H^2}^2 - 2 \sqrt{l_o} |g'(t)| |u(t) - g(t) \phi^o|_H \\
+ c_o \int_0^\infty |u_x(x,t)|^p dx - (2c_1/p) |g(t) - C(\phi^o) I_1 \delta^{-p/p'} \\
, \]

where

\[ \delta = \frac{c_o p'}{2c_1} \left( 1 + |g|_{C((0, T))} C(\phi^o) \right)^{-1}. \]

Applying Proposition 0.4.1 to this inequality, we can find \( R_0 \) which has the required property.

Q.E.D.

Now, corresponding to \( 0 < l_o < \infty \), \( 0 < T < \infty \), \( g \in W^{1, i}(0, T) \), \( i = 1, 2 \), and a given \( r_o > 0 \) we choose positive numbers \( L_i \) so that

\[ L_i > l_o + \sqrt{l_o r_o} + \int_0^T g(t) \, dt \]

and

\[ L_2 > l_o^2 + 2l_o^{3/2} r_o + c_1 r_o^{1/p'} T^{1/p} L_2^{1/p}, \]

where \( R_0 = R_0(\Lambda(l_o^0, r_o) \) is as in Lemma 3.5.6. For such numbers \( L_i \), \( i = 1, 2 \), we set

\[ \Lambda(l_o^0; L_i) = \{ l \in \Lambda(l_o^0); l(T) \leq L_i \} \]

and

\[ \Lambda_c(l_o^0; L_i) = \Lambda(l_o^0; L_i) \cap C((0, T)). \]

We then note that the constants \( C_i \), appearing in Lemmas 3.5.2 and 3.5.3 are uniformly bounded with respect to \( l \) in \( \Lambda(l_o^0; L_i) \), and hence

\[ \{ \phi_{l, i}^t; \, 0 \leq t \leq T \} \in G, \quad \forall l \in \Lambda(l_o^0; L_i), \quad i = 1, 2, \]

where \( G \) is the class \( G(\alpha, \{ \delta \}, \{ M \}) \) (see Definition 2.6.1) corresponding to some \( \alpha, \{ \delta \}, \{ M \} \). Therefore we obtain the following proposition as a direct consequence of Theorem 2.6.1.

Proposition 3.5.2. Let \( r_o > 0, g \in W^{1, i}(0, T) \) and \( L_i \) be as given by (5.19) and (5.20), \( i = 1, 2 \). Then we have:

(i) There is a constant \( K_0 = K_0(\Lambda(l_o^0; L_i)) \) such that

\[ |u|_{C([0, T]; H)} \leq K_0, \quad |u|_{L^p(0, T; X)} \leq K_0, \]

\[ |\sqrt{t}u'|_{L^2(0, T; H)} \leq K_0, \quad t |u(t)|_X \leq K_0, \quad \forall t \in [0, T], \]

whenever \( u \) is the weak solution to \( CP(\phi_{l, i}^t; 0, u_o) \) for \( l \in \Lambda(l_o^0; L_i) \) and \( u_o \) in the closure of \( K_{l, i}(0) \) in \( H \) with \( |u_o|_{H} \leq r_o \).

(ii) Given a number \( r_1 > 0 \) there is a constant \( K_1 = K_1(\Lambda(l_o^0; L_i), r_1) \) such that

\[ |u|_{W^{1, 2}(0, T; H)} \leq K_1, \quad |u(t)|_X \leq K_1, \quad \forall t \in [0, T], \]

whenever \( u \) is the strong solution to \( CP(\phi_{l, i}^t; 0, u_o) \) for \( l \in \Lambda(l_o^0; L_i) \) and \( u_o \) in \( K_{l, i}(0) \) with \( |u_o|_{H} \leq r_0 \) and \( |u_o|_X \leq r_1 \).
Furthermore we have

**Proposition 3.5.3.** Let \( r_o > 0, \ g \in W^{1,1}(0, T) \) and \( L_i \) be as above, \( i = 1, 2 \). Let \( l \in \Lambda (l_o, \ L_i), l_n \in \Lambda (l_o; \ L_i), n = 1, 2, \ldots \) such that \( l_n(t) \to l(t) \) pointwise on \( [0, T] \). Suppose that \( u_{o,n} \) is in the closure of \( K_{l,1}(0) (= K_{l,1}(0) \) in \( H, |u_{o,n}|_H \leq r_o \) and \( u_{o,n} \to u_o \) in \( H \). Then, denoting by \( u_n \) and \( u \) the weak solutions to \( CP(\phi_{l,1}; 0, u_{o,n}) \) and \( CP(\phi_{l,1}; 0, u_o) \), respectively, we have

\[ u_n \to u \quad \text{in} \ C([0, T]; H) \quad \text{and in} \ L^p(0, T; X). \]

**Proof.** Applying Theorem 2.7.1 with Lemma 3.5.4, we get

\[ u_n \to u \quad \text{in} \ C([0, T]; H), \quad u_n \to u \quad \text{in} \ L^p(0, T; X) \]

and

\[ \int_0^T \phi_{l,1}^t(u_n(t)) \ dt \to \int_0^T \phi_{l,1}^t(u(t)) \ dt. \]  

It is easy to see that \( u_n(0, \cdot) \to u(0, \cdot) \) in \( L^1(0, T) \), so from (5.21) we infer

\[ \int_0^T \int_0^\infty A(u_{n,x}) \ dx\ dt \to \int_0^T \int_0^\infty A(u_x) \ dx\ dt. \]  

Putting \( v_n = (u_n + u)/2 \), we see \( v_n \to u \) in \( C([0, T]; H) \) and \( v_n \to u \) in \( L^p(0, T; X) \) and hence

\[ \liminf_{n \to \infty} \int_0^T \int_0^\infty A(v_{n,x}) \ dx\ dt \geq \int_0^T \int_0^\infty A(u_x) \ dx\ dt. \]  

On the other hand, using the inequality of (i) of Lemma 3.5.1, we have

\[ \frac{1}{2} \int_0^T \int_0^\infty A(u_{n,x}) \ dx\ dt + \frac{1}{2} \int_0^T \int_0^\infty A(u_x) \ dx\ dt \geq \int_0^T \int_0^\infty A(v_{n,x}) \ dx\ dt \]

\[ \geq \frac{c_o}{p^2} \int_0^T \int_0^\infty |u_{n,x} - u_x|^p \ dx\ dt. \]

The left hand side of this inequality tends to 0 as \( n \to \infty \) on account of (5.22) and (5.23), so that

\[ u_{n,x} \to u_x \quad \text{in} \ L^p(0, T; L^p(0, \infty)) \]

and hence

\[ u_n \to u \quad \text{in} \ L^p(0, T; X). \]

Q.E.D.

(A) **Problem \( QV_1(l_o, u_o, g) \)**

Our first existence theorem is as follows.

**Theorem 3.5.1.** Let \( 0 < l_o < \infty, \ 0 < T < \infty, \ g \in W^{1,1}(0, T) \) and \( u_o \in H. \) Suppose \( g \) is non-negative on \( [0, T] \), \( u_o \) is non-negative on \( [0, \infty) \) and \( u_o = 0 \) on \( [l_o, \infty) \). Then \( QV_1(l_o, u_o, g) \) admits a weak solution \( \{ l, u \} \in \Lambda (l_o, \ x \ C([0, T]; H) \) such that \( u \geq 0 \) on \( [0, \infty) \times (0, T] \), \( \sqrt{u} \in L^2(0, T; H), \ u \in L^p(0, T; X), t \to \sup \{ |u(t)|_X \} \) is bounded on \( (0, T] \) and

\[ -u'(t) \in \partial \phi_{l,1}^t(u(t)) \quad \text{for a.e.} \ t \in [0, T]. \]

In addition if \( u_o \in X, \) then \( QV_1(l_o, u_o, g) \) admits a strong solution \( \{ l, u \} \) such that \( u \geq 0 \) on
\[ [0, \infty) \times [0, T], u^i \in L^2(0, T; H) \text{ and } u \in L^\infty(0, T; X). \]

Given \( l_o, u_o \) and \( g \), we take \( r = |u_o|_H + 1 \) and a number \( L_I \) satisfying (5.19) and consider the classes \( \Lambda (l_o; L_I) \) and \( \Lambda_c (l_o; L_I) \). Using the weak solution \( u_i \) to \( CP \phi_{l,1}^t, 0, u_o \) for each \( l \in \Lambda (l_o; L_I) \), we define a mapping \( Q : \Lambda (l_o; L_I) \to C ([0, T]) \) by

\[ [Ql](t) = l_o + \int_0^t u_o(x)dx + \int_0^t g(t)dr - \int_0^\infty u_i(x, t)dx, \quad \forall t \in [0, T]. \]

**Lemma 3.5.7.** (i) \( Q \) maps \( \Lambda (l_o; L_I) \) into \( \Lambda_c (l_o; L_I) \cap W^{1,2}(\delta, T) \) for every \( 0 < \delta < T \).

(ii) For every \( l \in \Lambda (l_o; L_I) \),

\[ \frac{d}{dt} [Ql](t) = -A (u_{1,t}^l, x^l)(l(t) - t) \quad \text{for a.e. } t \in [0, T]. \]

(iii) If \( l_n \in \Lambda (l_o; L_I) \), \( n = 1, 2, \ldots \), and if \( l_n (t) \to l(t) \) pointwise on \([0, T] \), then

\[ Ql_n \to Ql \quad \text{in } C ([0, T]). \]

**Proof.** For every \( l \in \Lambda (l_o; L_I) \), since \( u_i \in C ([0, T]; H) \cap W^{1,2}(\delta, T; H) \) for every \( 0 < \delta < T \), we see that \( Ql \in C ([0, T]) \cap W^{1,2}(\delta, T) \) for every \( 0 < \delta < T \) and

\[ 0 \leq [Ql](t) \leq l_o + \int_0^1 u_o(x)dx + \int_0^T gdt \]

\[ \leq l_o + \sqrt{r_o} + \int_0^T gdt < L_I, \quad \forall t \in [0, T]. \]

Besides

\[ \frac{d}{dt} [Ql](t) = g(t) - \int_0^t u_{1,t}^l(x, t)dx \]

\[ = g(t) - \int_0^t A(u_{1,t}^l, x^l)(x, t)dx \]

\[ = -A (u_{1,t}^l, x^l)(l(t) - t) \quad \text{for a.e. } t \in [0, T], \]

which is non-negative on account of the Corollary to Proposition 3.5.1. Thus we get (i) and (ii). Next, assume \( l_n (t) \to l(t) \) pointwise on \([0, T] \) for \( l_n \in \Lambda (l_o; L_I) \), \( n = 1, 2, \ldots \). Then, from Proposition 3.5.3 it follows that \( u_i \to u_i \) in \( C ([0, T]; H) \) and hence \( Ql_n \to Ql \) in \( C ([0, T]) \).

Q.E.D.

**Proof of Theorem 3.5.1:** Suppose for a moment that \( u_o \) is in \( X \), non-negative and \( u_o = 0 \) on \([l_o, \infty) \), i.e. \( u_o \in K_{1,1} (0) \) for every \( l \in \Lambda (l_o) \). In this case, \( Q (\Lambda (l_o; L_I)) \) is contained in the class

\[ \Lambda_o = \{ l \in \Lambda_c (l_o; L_I) \cap W^{1,2}(0, T); |l'| \leq m, l' \in L^2(0, T) + \sqrt{L_I} K_1 \} \]

with the constant \( K_1 \) (with \( r_I = |u_o|_X \)) of (ii) of Proposition 3.5.2; in fact for every \( l \in \Lambda (l_o; L_I) \)

\[ \frac{d}{dt} [Ql](t) = g(t) - \int_0^t u_{1,t}^l(x, t)dx \]

and hence

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\[ \frac{d}{dt} |Ql| (t) \leq |g(t)| + \sqrt{L_1} |u(t)|_{H^1} \]

from which we see

\[ \frac{d}{dt} |Ql|_{L^2(0, T)} \leq |g|_{L^2(0, T)} + \sqrt{L_1} K_1. \]

Therefore Q maps \( \Lambda_0 \) into itself and is continuous in the topology of \( C([0, T]) \) by (iii) of Lemma 3.5.7. Since \( \Lambda_0 \) is non-empty, compact and convex in \( C([0, T]) \), by a well-known fixed point theorem, there is \( l \in \Lambda_0 \) such that \( Ql = l \). Clearly the pair \( \{ l, u_l \} \) gives a strong solution to \( QV_1(l, u_o, g) \) which has the required properties.

In general case, choose a sequence \( z_n \in X, n = 1, 2, \ldots, \) such that \( z_n \to z \) on \([l, \infty)\) and \( z_n \to u_o \) on \([l, \infty)\). Denote by \( \{ l_n, u_n \} \) the strong solutions to \( QV_1(l, u_n, g) \) as obtained above. Using (i) of Proposition 3.5.2, we can find a subsequence \( \{ u_{n_k} \} \) which converges in \( H \) pointwise on \([0, T] \), and denote by \( u(t) \) the limit of \( \{ u_{n_k}(t) \} \) for each \( t \in [0, T] \).

Since

\[ l_{n_k}(t) = l_o + \int_0^t z_{n_k} dx + \int_0^t gdr - \int_0^\infty u_{n_k}(x, t) dx, \quad \forall t \in [0, T], \]

it follows that \( l_{n_k}(t) \to l(t) \) pointwise on \([0, T] \) to some \( l \in \Lambda(l, l) \). Therefore, by Proposition 3.5.3, \( u_{n_k} \to u \) in \( C([0, T]; H) \) and hence \( l_{n_k} \to l \) in \( C([0, T]) \). Also, \( u \) is the weak solution to \( CP(\phi^t_{l, l}, 0, u_o) \) satisfying

\[ l(t) = l_o + \int_0^t u_o dx + \int_0^t gdr - \int_0^\infty u(x, t) dx, \quad \forall t \in [0, T] \]

and thus the pair \( \{ l, u \} \) is a weak solution to \( QV_1(l, u_o, g) \) having the required properties.

Q.E.D.

(B) Problem \( QV_2(l, u_o, g) \)

Our second existence theorem is as follows.

Theorem 3.5.2. Let \( 0 < l_o < \infty, 0 < T < \infty, g \in W^{1,2}(0, T) \) and \( u_o \in H \). Suppose \( g \) is non-negative on \([0, T] \), \( u_o \) is non-negative on \([0, \infty) \) and \( u_o = 0 \) on \([l_o, \infty) \). Then \( QV_2(l_o, u_o, g) \) admits a weak solution \( \{ l, u \} \in \Lambda_c(l_o, x) \times C([0, T]; H) \) such that \( u \geq 0 \) on \([0, \infty) \times [0, T] \), \( \sqrt{u} \) is bounded on \([0, T] \) and

\[ -u(t) \in \partial \phi^t_{2, l}(u(t)) \quad \text{for a.e. } t \in [0, T] \]

In addition, if \( u_o \in X \) and \( u_o(0) = g(0) \), then \( QV_2(l_o, u_o, g) \) admits a strong solution \( \{ l, u \} \) such that \( u \geq 0 \) on \([0, \infty) \times [0, T] \), \( u \in L^2(0, T; H) \) and \( u \in L^\infty(0, T; X) \).

Given \( l_o, u_o \) and \( g \), we take \( r_o = \| u_o \|_H + 1 \) and a number \( L_o \) satisfying (5.20), and consider the classes \( \Lambda(l, L) \) and \( \Lambda_c(l_o, L_2) \). By using \( u_i \), the weak solution to \( CP(\phi^t_{2, l}, 0, u_o) \) for each \( l \in \Lambda(l_o, L_2) \), we define a mapping \( Q: \Lambda(l_o, L_2) \to C([0, T]) \) by

\[ [Ql](t) = \int_0^t 2 \int_0^x u_i(x) dx - 2 \int_0^\infty u_i(x, t) dx + 2 \int_0^t \int_0^\infty A(u_i, x, r) dx dr \}

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for all \( t \in [0, T) \).

Just as Lemma 3.5.7 we prove

**Lemma 3.5.8.** (i) \( Q \) maps \( \Lambda (l, n) : L_2 \) into \( \Lambda _c (l, n) : L_2 \) \( \cap W^{1,2} (\delta, T) \) for every \( 0 < \delta < T \).

(ii) For every \( l \in \Lambda (l, n) : L_2 \),

\[
\frac{d}{dt} [Ql](t) = -A(u_{t,x}) (l(t) - t) \quad \text{for a.e. } t \in [0, T].
\]

(iii) If \( l_n \in \Lambda (l, n) : L_2 \), \( n = 1, 2, \ldots \), and if \( l_n (t) \rightarrow l(t) \) pointwise on \( [0, T] \), then

\[
Ql_n \rightarrow Ql \quad \text{in } C ([0, T]).
\]

**Proof** of Theorem 3.5.2: First suppose that \( u \in X \), \( u(0) = g(0) \), \( u \) is non-negative on \([0, \infty) \) and \( u = 0 \) on \([1, \infty) \), i.e. \( u \in K_2 \). In this case, \( Q (\Lambda (l, n) : L_2) \) is contained in the class

\[
\Lambda _c = \{ l \in \Lambda _c (l, n) : L_2 \cap W^{1,2} (0, T) \mid |l'(t)|_{L^2 (0, T)} \leq \frac{1}{2l_0} \left( l_2^{3/2} K_1 + c_1 T^{1/p} l_2^{1/p} R_o^{1/p'} \right) \}
\]

with the constant \( K_1 \) (with \( r_1 = |u|_{L^1} \) of (ii) of Proposition 3.5.2 and \( R_o \) of Lemma 3.5.6; in fact, for every \( l \in \Lambda (l, n) : L_2 \),

\[
2 |Ql|(t) \frac{d}{dt} [Ql](t) = -\int_0^L x u_{t,x}(x, t) dx - \int_0^L u_{t,x}(x, t) A(u_{t,x}) (x, t) dx
\]

and hence

\[
|\frac{d}{dt} [Ql](t)| \leq \frac{1}{2l_0} \left( l_2^{3/2} |u'(t)|_H + c_1 \int_0^L |u_{t,x}(x, t)|^{p-1} dx \right),
\]

from which we infer

\[
|\frac{d}{dt} Ql|_{L^2 (0, T)} \leq \frac{1}{2l_0} \left( l_2^{3/2} K_1 + c_1 T^{1/p} l_2^{1/p} R_o^{1/p'} \right).
\]

Therefore \( Q \) maps \( \Lambda _c \) into itself and is continuous in the topology of \( C ([0, T]) \) by (iii) of Lemma 3.5.8. Since \( \Lambda _c \) is non-empty, compact and convex in \( C ([0, T]) \), there is a fixed point \( l \) of \( Q \) in \( \Lambda _c \) and it is easy to see that the pair \( \{ l, u \} \) gives a strong solution to \( QV_2 (l, u, g) \).

In the case of general initial datum \( u_0 \) we can prove the existence of a weak solution to \( QV_2 (l, u, g) \) just as to \( QV_1 (l, u_0, g) \). Q.E.D.

**Remark 3.5.2.** In the case \( p > 2 \) the question of uniqueness for strong or weak solutions to \( QV_1 (l, u, g) \) remains open, but in the case of \( p = 2 \) the uniqueness can be proved under some supplementary conditions by employing a method in the author's forthcoming paper Kenmochi [7].

**Remark 3.5.3.** Free boundary problems of the Stefan-type for nonlinear parabolic equations have been studied by many authors, for example, Douglas [1], Fasano-Primicerio [1], Kyner [1] and Rubinstein [1].
References

H. Attouch, Ph. Bénilan, A. Damlamian and C. Picard

H. Attouch and A. Damlamian

Ph. Bénilan

Ph. Bénilan and H. Brézis

M. Biroli

H. Brézis

F.E. Browder

F.E. Browder and P. Hess

F.E. Browder and W.V. Petryshyn

M.G. Crandall

M.G. Crandall and A. Pazy

A. Damlamian

J. Douglas, Jr.
A. Fasano and M. Primicerio

H. Fujita

H. Fujita and N. Sauer

N. Kenmochi
7. Subdifferential operator approach to a class of free boundary problems, preprint.

N. Kenmochi, Y. Mizuta and T. Nagai

N. Kenmochi and T. Nagai

Y. Konishi

W.T. Kyner

J.L. Lions

W. Littman, G. Stampacchia and H.F. Weinberger

J.J. Moreau

H. Morimoto

U. Mosco
1. Convergence of convex sets and of solutions variational inequalities, Advances Math. 3, 510-585,
Solvability of Nonlinear Evolution Equations

T. Nagai

M. Otani and Y. Yamada

C. Picard

R.T. Rockafellar

L.I. Rubinstein

J. Simon

Y. Yamada

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