On the ring structure of SK_*

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1. Introduction.

In[4], Kosniowski proved that unoriented cutting and pasting ring is the integral polynomial ring on a real projective space RP^2. In this paper we shall study the oriented case.

Let M^n be a closed oriented manifold and let $L^{n-1} \subset M^n$ be a closed oriented submanifold of codimension 1 with trivial normal bundle. If we cut M open along L, we obtain a manifold M′ whose boundary is the disjoint union $L+(-L)$ of two (oppositely oriented) copies of L. By pasting these two copies together again in a different way (respecting orientation) we obtain a new closed oriented manifold N^n. Then we say that N has been obtained by cutting and pasting or,briefly,by SK (Schneiden und Kleben) from M. If N has been obtained from M by a sequence of cuttings and pastings, then we say that M and N are SK equivalent. Disjoint union makes the set of these equivalence classes into an abelian semigroup. The Grothendieck group of SK equivalence classes of the n-dimensional closed oriented manifolds is denoted by SK_n, elements of which are denoted by$[M^n]$. (In this paper we denote the oriented cobordism classes by $[M^n]_\Omega$)

Using cartesian product of manifolds as multiplication, we get a graded ring $SK_* = \sum_{n\geqq 0} SK_n$. Then we have the following theorem :

THEOREM. For the oriented cutting and pasting ring SK_*,

$$SK_* \otimes Z[\frac{1}{2}] \cong Z[\frac{1}{2}][[S^2],[CP^2]]/\vartheta$$

where ϑ is an ideal $([S^2][CP^2]^k - \frac{3^k}{2^{2k}}[S^2]^{2k+1})$ for $k \geqq 1$.

We examined how the elements of oriented cobordism ring appear in the SK theory. We, thereby, have shown that the ring structure of SK_* is deduced from cobordism ring. Our method is the same as [4]. However,in this paper, we used some results of the cobordism structure with semi-free S^1-actions.

Throughout this paper, manifolds will be smooth, orientable, and compact unless otherwise stated.

2. Preliminaries.

First of all we recall the following facts about SK_* that will be used later ([2], [4]).

(1°) $[S^1]=0$.

(2°) If M is a total space of the fibre bundle over S^n (resp. CP^n) with typical fibre F then $[M]=[S^n][F]$ (resp.$[CP^n][F]$)

(3°) If M_1 and M_2 are oriented n-dimensional cobordant manifolds then $[M_1]=[M_2]+\frac{1}{2}(\chi(M_1)-\chi(M_2))[S^n]$ in SK_n, where $\chi(M)$ is Euler characteristic of M.

(4°) Let $\mathrm{Tor}\Omega_n$ be the torsion part of Ω_n, then $\mathrm{Tor}\Omega_n \subset \mathscr{F}_n$. Where $\mathscr{F}_n$ is the subgroups of all elements which admit a representative M^n such that M^n is a total space of the fibre bundle over the circle S^1.

Next we define a semi-free S^1-action on a complex $2k$ dimensional projective space CP^{2k} as follows. Let define $\phi : S^1 \times CP^{2k} \to CP^{2k}$ by $\phi(\lambda,[z_0:z_1:\cdots:z_{2k}])=[z_0:z_1:\lambda z_2:\cdots:\lambda z_{2k}]$, where $\lambda \in S^1$ and $[z_0:z_1:\cdots:z_{2k}]\in CP^{2k}$. Then ϕ is a semi-free S^1-action with the fixed point set CP^1+CP^{2k-2}. Let ν^{2k-1} and ν^2 be the complex normal bundles to CP^1 and CP^{2k-2} respectively, and let θ be the trivial complex line bundle over CP^1 or CP^{2k-2}. Let $CP(\nu^{2k-1}\oplus\theta)$ and $CP(\nu^2\oplus\theta)$ be the complex projective bundles associated to $\nu^{2k-1}\oplus\theta$ and $\nu^2\oplus\theta$ respectively. Then we have the following lemma.

LEMMA (cf.[3] Lemma 2.2)

$$[CP^{2k}]_\Omega=[CP(\nu^{2k-1}\oplus\theta)]_\Omega+[CP(\nu^2\oplus\theta)]_\Omega.$$

PROOF. We consider the S^1-actions $(\phi_1, D^2\times CP^{2k})$ and $(\phi_2, D^2\times CP^{2k})$, where D^2 is a two dimensional disk and

$$\phi_1(\lambda,(z,w))=(\lambda z, w)\ ,$$
$$\phi_2(\lambda,(z,w))=(\lambda z, \phi(\lambda,w)),$$

for $\lambda\in S^1$, $z\in D^2$, $w\in CP^{2k}$. The fixed point set of ϕ_1 is $0\times CP^{2k}$, and the normal bundle to the fixed point set is a trivial comlex line bundle. The fixed point set of ϕ_2 is $0\times(CP^1+CP^{2k-2})$, and the normal bundle to the fixed point set is $\nu^{2k-1}\oplus\theta$ and $\nu^2\oplus\theta$. Define an equivariant diffeomorphism $f: S^1\times CP^{2k}\to S^1\times CP^{2k}$ by $f(\lambda,w)=(\lambda,\phi(\lambda,w))$ for $\lambda\in S^1, w\in CP^{2k}$.

We adjoin $(\phi_1, D^2\times CP^{2k})$ to $(\phi_2, D^2\times CP^{2k})$ along their boundaries via f. Then we obtain an oriented closed manifold $\Gamma(CP^{2k})=S^2\times_{S^1}CP^{2k}$ with semi-free S^1-action. Let N denote the equivariant open tublar neighborhood of the fixed point set in $\Gamma(CP^{2k})$. Then $\Gamma(CP^{2k})-N$ has a fixed point free S^1-action, so the quotient $\Gamma(CP^{2k})-N/S^1$ is an oriented manifold with boundary CP^{2k} and $CP(\nu^{2k-1}\oplus\theta)+CP(\nu^2\oplus\theta)$. q.e.d.

3. Proof of the Theorem.

LEMMA 3.1. $[S^{2n}]=\frac{1}{2^{n-1}}[S^2]^n$ in SK_*.

PROOF. By[2],

$$[S^k\times S^{n-k}]=\begin{cases}2[S^n] & \text{if k is even,}\\ 0 & \text{if k is odd.}\end{cases}$$

So we have $[S^{2n+2}]=\frac{1}{2}[S^{2n}\times S^2]=\frac{1}{2}[S^{2n}][S^2]$. Then we have the lemma by induction. q.e.d.

PROPOSITION 3.2. If M^n is cobordant to zero or to a representative of the torsion element in Ω_n then

$$[M^n]=\begin{cases}\frac{1}{2^k}\chi(M)[S^2]^k & \text{if } n=2k,\\ 0 & \text{if } n=2k+1.\end{cases}$$

PROOF. (i) If M^n is cobordant to zero then $M^n \sim S^n$ in Ω_n.

(a) If $n=2k+1$ then $\chi(M)=\chi(S^n)=0$. By (1°) and (3°), we have $[M]=[S^{2k+1}]=[CP^k]$. $[S^1]=0$ in SK_*, because S^{2k+1} is a fibre bundle over CP^k with the fibre S^1 and $[S^1]=0$.

(b) If $n=2k$ then $\chi(S^{2k})=2$ and by (3°)

$$[M]=[S^{2k}]+\frac{1}{2}(\chi(M)-\chi(S^{2k}))[S^{2k}]$$

$$=\frac{1}{2}\chi(M)[S^{2k}]=\frac{1}{2^k}\chi(M)[S^2]^k.$$

(ii) If M^n is cobordant to M_1 with $[M_1]_\Omega \in \mathrm{Tor}\,\Omega_n$, then by (4°) M_1 is cobordant to a total space M_2 of the fibre bundle over S^1 with some fibre F. Then $[M_2]=[S^1]\cdot[F]=0$ and $\chi(M_2)=\chi(S^1)\chi(F)=0$. So by (3°) we have $[M]=\frac{1}{2}\chi(M)[S^n]$. Therefore we have the same conclusion as (i). q.e.d.

PROPOSITION 3.3. In SK_*, we have

$$[CP^{2k}]=[CP^2][CP^{2k-2}]-\frac{k-1}{2^{2(k-1)}}[S^2]^{2k} \text{ for } k\geqq 1.$$

PROOF. By (3°) and Lemma, we have

$$\begin{aligned}[CP^{2k}]&=[CP(\nu^{2k-1}\oplus\theta)+CP(\nu^2\oplus\theta)]\\&\quad+\frac{1}{2}\{\chi(CP^{2k})-\chi((CP(\nu^{2k-1}\oplus\theta)+CP(\nu^2\oplus\theta))\}[S^{4k}]\\&=[CP^1][CP^{2k-1}]+[CP^{2k-2}][CP^2]\\&\quad+\frac{1}{2}\{\chi(CP^{2k})-\chi(CP^1)\chi(CP^{2k-1})-\chi(CP^{2k-2})\chi(CP^2)\}[S^{4k}]\\&=[S^2][CP^{2k-1}]+[CP^{2k-2}][CP^2]+2(1-2k)[S^{4k}],\end{aligned}$$

where CP^{2k-1} is a cobordant to zero, because all Pontrjagin numbers and Stiefel-Whitney numbers of CP^{2k-1} are zero.

Therefore from Proposition 3.2, we have

$$[CP^{2k-1}]=\frac{k}{2^{2(k-1)}}[S^2]^{2k-1}.$$

Also by Lemma 3.1, we have

$$[S^{4k}]=\frac{1}{2^{2k-1}}[S^2]^{2k}.$$

These make the proof of Proposition 3.3 complete.

REMARK. Let $\overline{SK_n}$ be SK_n factored by the cobordism relations, then $[CP^{2k}]=[CP^2]\cdot[CP^{2k-2}]$ in $\overline{SK_{4k}}$. This equality has been originally obtained by Jänich ([1]). So our Proposition 3.3 is a generalization of it.

COROLLARY. 3.4. $[CP^{2k}]=[CP^2]^k-\frac{3^k-2k-1}{2^{2k}}[S^2]^{2k}$.

PROOF. We can obtain the following equality by Proposition 3.3,

$$[CP^{2k}]=[CP^2]^k-\sum_{i=1}^{k-1}\frac{i}{2^{2i}}[CP^2]^{k-1-i}[S^2]^{2(i+1)}.$$

Well, $(CP^2)^{k-1-i}\times(S^2)^{2(i+1)}$ is cobordant to zero. Thus

$$[CP^2]^{k-1-i}[S^2]^{2(i+1)}=\frac{3^{k-1-i}}{2^{2(k-1-i)}}[S^2]^{2k},$$

by Proposition 3.2. Therefore we have

$$[CP^{2k}]=[CP^2]^k-\frac{3^{k-1}}{2^{2(k-1)}}\left(\sum_{i=1}^{k-1}\frac{i}{3^i}\right)[S^2]^{2k}$$

$$=[CP^2]^k-\frac{3^{k-1}}{2^{2(k-1)}}\cdot\frac{3^k-2k-1}{4\cdot 3^{k-1}}[S^2]^{2k}$$

$$=[CP^2]^k-\frac{3^k-2k-1}{2^{2k}}[S^2]^{2k}.$$ q.e.d.

Let $H_{m,n}$ $(m\leqq n)$ denote a hypersurface in the product $CP^m\times CP^n$. This hypersurface is defined by the equation

$$z_0w_0+z_1w_1+\cdots\cdots+z_mw_m=0,$$

where $[z_0:z_1:\cdots\cdots:z_m]\in CP^m$ and $[w_0:w_1:\cdots\cdots:w_n]\in CP^n$. Then $H_{m,n}$ is an oriented $2(m+n-1)$ dimensional submanifold of $CP^m\times CP^n$ and also a CP^{n-1}-bundle over CP^m with the canonical projection $H_{m,n}\to CP^m$. Therefore we have $[H_{m,n}]=[CP^m]\cdot[CP^{n-1}]$ in SK_*, and so we have next equalities.

PROPOSITION. 3.5.

$$[H_{2s,2t+1}]=[CP^2]^{s+t}-\frac{3^{s+t}-(2s+1)(2t+1)}{2^{2(s+t)}}[S^2]^{2(s+t)},$$

$$[H_{2s,2t}]=\frac{(2s+1)t}{2^{2(s+t-1)}}[S^2]^{2s+2t-1},$$

$$[H_{2s+1,2t+1}]=\frac{(s+1)(2t+1)}{2^{2(s+t)}}[S^2]^{2s+2t+1},$$

$$[H_{2s+1,2t}]=\frac{(s+1)t}{2^{2(s+t-1)}}[S^2]^{2(s+t)}.$$

PROOF. We can immediately calculate these equalities by Proposition 3.2 and Corollary 3.4. q.e.d.

From the above results we have the following main theorem.

THEOREM. For the oriented cutting and pasting ring SK_*,

$$SK_*\otimes Z[\tfrac{1}{2}]\cong Z[\tfrac{1}{2}][[S^2],[CP^2]]/\vartheta,$$

where ϑ is an ideal $\left([S^2][CP^2]^k-\frac{3^k}{2^{2k}}[S^2]^{2k+1}\right)$ for $k\geqq 1$.

PROOF. For any element $[M^n]\in SK_n$, M^n is cobordant to a manifold which can be expressed as a homogeneous polynomial of degree n in the generators x_i of Ω_*. Now we recall that $\Omega_*/\mathrm{Tor}\Omega_*$ is generated by even dimensional complex projective spaces CP^{2m} and hypersurfaces $H_{m,n}$. Therefore each x_i is represented by some even dimensional projective space CP^{2m}, some hypersurface $H_{m,n}$ or some torsion element in Ω_*. Hence, from the results which has calculated above , we see that any homogeneous polynomial in the x_i is SK equivalent to a polynomial of $[S^2]$ and $[CP^2]$ over $Z[\frac{1}{2}]$. Well, $S^2\times(CP^2)^k$ is cobordant to S^{4k+2}, so $[S^2][CP^2]^k=\frac{3^k}{2^{2k}}[S^2]^{2k+1}$ by Proposition 3.2. We have no any other relations, because Euler characteristic and signature are SK-invariants. Therefore the proof of the Theorem is completed.

q.e.d.

REFERENCES

[1] K.JÄNICH, On Invariants with the Novikov Additive Property, Math.Ann. 184 (1969), 65-77.

[2] U.KARRAS, M.KRECK, W.D.NEUMANN AND E.OSSA, Cutting and Pasting of Manifolds; SK-Groups, Publish or Perish Inc., (1973).

[3] K.KAWAKUBO AND F.UCHIDA, On the index of a semi-free S^1-action, Proc. Japan of Acad. 46 (1970), 620-622.

[4] C.KOSNIOWSKI, Actions of finite abelian groups, Reserch Notes in Math. 18, Pitmann (1978).

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