Some categorical properties of complex spaces

By

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§ 1. Introduction

Let $\mathcal{L}$ be a category of local ringed spaces and $\hat{C}$ be a category of complex spaces. Although they are not abelian, we can prove that some fundamental propositions in abelian categories are also valid in $\mathcal{L}$ or $\hat{C}$. In § 2, we shall introduce the category $\mathcal{L}$ and its some full subcategories $\hat{C}$, $\hat{C}_{red}$, etc. In § 3, some important categorical objects are defined in an arbitrary category $\mathcal{A}$. Let $v: X \longrightarrow S$ and $w: Y \longrightarrow S$ be any morphisms in $\mathcal{A}$, then the following properties are equivalent (Proposition 3);

(a) Z is a fiber product of X and Y over S.

(b) Z is a pullback for v and w.

(c) Z is a pullback for Δ and $v \times w$.

In § 4, we show the existences of products and pullbacks in $\hat{C}$ (Theorem 1, 2). Difference cokernels and pushouts exist always in $\mathcal{L}$ for any given morphisms in $\hat{C}$. But, in general, they are not objects of $\hat{C}$. In § 5, some sufficient conditions are given for their existences in $\hat{C}$.

§ 2. Categories of complex spaces

In this section, we shall introduce some categories of complex spaces [1].

Definition 1. A *local ringed space* is a pair $X=(|X|, \mathcal{H}_X)$, where $|X|$ is a topological space and $\mathcal{H}_X$ is a sheaf of local **C**-algebras. For each point $x \in |X|$, $\mathcal{H}_{X,x}$ is a commutative local **C**-algebra with a unit 1_x. Furthermore $\mathcal{H}_{X,x}/\mathfrak{m}_X$ is assumed to be isomorphic to **C**, here $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{H}_{X,x}$, i. e. $\mathcal{H}_{X,x} = \mathbf{C} \oplus \mathfrak{m}_x$.

Let $q_x: \mathcal{H}_{X,x} \longrightarrow \mathbf{C}$ be the natural projection. For any section $f \in \Gamma(U, \mathcal{H}_X)$ over an open set $U \subset |X|$, we define the *value* $f(x)$ of f at a point $x \in U$ as $q_x(f_x)$, where f_x is the germ of f at x.

Definition 2. A *morphism* $\phi: (|X|, \mathcal{H}_X) \longrightarrow (|Y|, \mathcal{H}_Y)$ of one local ringed space into another is a pair $\phi = (\phi_0, \phi_1)$, where $\phi_0: |X| \longrightarrow |Y|$ is a continuous map, and ϕ_1 is a continuous map $|X| \oplus_{\phi_0} \mathcal{H}_Y := \{(x,\sigma) \mid x \in |X|, \sigma \in \mathcal{H}_{Y,y}, y = \phi_0(x)\} \longrightarrow \mathcal{H}_X$. Let ϕ_{1x} be the restriction of ϕ_1 at $(x, \mathcal{H}_{Y,\phi_0(x)})$. we assume that ϕ_{1x} is always a unitary homomorphism of $(x, \mathcal{H}_{Y,y})$ into $\mathcal{H}_{X,x}$, $y = \phi_0(x)$. ϕ is an *isomorphism* if ϕ_0 and ϕ_1 are both topological maps and ϕ_{1x} is an isomorphism for each $x \in |X|$.

In the following, we sometimes identify the pair $(x, \mathcal{H}_{Y,y})$ with a local $\mathbf{C}$-algebra $\mathcal{H}_{Y,y}$. Since ϕ_{1x} is a unitary homomorphism of local $\mathbf{C}$-algebra, ϕ_{1x} maps the maximal ideal $\mathfrak{m}_{\phi 0(x)}$ of $\mathcal{H}_{Y,\phi 0(x)}$ into the maximal ideal $\mathfrak{m}_x$ of $\mathcal{H}_{X,x}$. A consequence of this is that a morphism $(\phi_0, \phi_1): X \longrightarrow Y$ preserves the value of the sections, in symbols $\phi_1(f)(x) = f(\phi_0(x)) \cdots (*)$, if $x \in |X|$ and f is a section of $\mathcal{H}_Y$ over an open set containing $\phi_0(x)$. Thus ϕ_0 and ϕ_1 are related, but the next example shows that ϕ_1 is not in general determined by ϕ_0.

Example 1. Let X be the local ringed space $(\{0\}, \mathbf{C}\{x\}/_{(x^2)})$, and let $Y = \mathbf{C}^n$ regarded as a local ringed space with the sheaf $\mathcal{O}_{\mathbf{C}^n}$ of germs of holomorphic functions on $\mathbf{C}^n$. Let (ϕ_0, ϕ_1) be a morphism of X into Y with $\phi_0(0) = 0$. Then ϕ_1 is a homomorphism $\phi_1: \mathbf{C}\{y_1, \cdots\cdots, y_n\} \longrightarrow \mathbf{C}\{x\}/_{(x^2)}$, here $\mathbf{C}\{y_1, \cdots\cdots, y_n\}$ denotes the ring of all converging power series in the variables $y_1, \cdots\cdots, y_n$. $\mathbf{C}\{x\}/_{(x^2)}$ is the ring of dual numbers representable as $a + b\epsilon$, where $a, b \in \mathbf{C}$, and $\epsilon^2 = 0$, ϵ being the class of x. Let us express $\phi_1(f)$ as $a(f) + \epsilon b(f)$. Since the maximal ideal of $\mathbf{C}\{x\}/_{(x^2)}$ is (ϵ), the value of $\phi_1(f)$ is $a(f)$. From $(*)$ it follows that $a(f) = \phi_1(f)(0) = f(\phi_0(0))$. Thus ϕ_0 determines only the zero order term of $\phi_1(f)$ at $0 \in \mathbf{C}$. It follows from the multiplication rule $\epsilon^2 = 0$ that $b(fg) = f(0)b(g) + g(0)b(f)$, hence b is a tangent vector at $0 \in \mathbf{C}^n$. Therefor the general form of ϕ_1 is of the type $\phi_1(f) = f(0) + \left(\sum_{i=1}^{n} \lambda_i \left(\frac{\partial f}{\partial x_i}\right)_{x=0}\right)\epsilon$ for some $\lambda_i \in \mathbf{C}$.

For any local ringed spaces $X = (|X|, \mathcal{H}_X)$, $Y = (|Y|, \mathcal{H}_Y)$, $Z = (|Z|, \mathcal{H}_Z)$ and morphisms $\phi = (\phi_0, \phi_1): X \longrightarrow Y$, $\psi = (\psi_0, \psi_1): Y \longrightarrow Z$, a morphism $\chi = (\chi_0, \chi_1): X \longrightarrow Z$ is defined as follows; $\chi_0 := \psi_0 \circ \phi_0$, $\chi_{1x} := \phi_{1x} \circ \psi_{1\phi 0(x)}$. $\psi \circ \phi := \chi$ is the composition of ϕ and ψ. The identity map $id_X: X \longrightarrow X$ with $id_X = (id_{|X|}, i)$, $i_x: id_{\mathcal{H}X,x}$, $x \in |X|$ is a morphism. Hence we have a *category* $\mathcal{L}$ of local ringed spaces. We denote by $\mathrm{Ob}\mathcal{L}$ the set of objects of $\mathcal{L}$ and by $\mathrm{Hom}_{\mathcal{L}}(X, Y)$ or simply by $\mathrm{Hom}\,(X, Y)$ the set of all morphisms of X into Y. Let $\mathcal{R}$ be a subcategory of $\mathcal{L}$ with the following properties: for any $X = (|X|, \mathcal{H}_X) \in \mathrm{Ob}\mathcal{R}$, $\mathcal{H}_X$ is a subsheaf of germs of complex valued continuous functions; for any morphism $(\phi_0, \phi_1) \in \mathrm{Hom}_{\mathcal{R}}\,(X, Y)$, ϕ_1 is the set of algebra homomorphisms $\phi_{1x}: (x, \mathcal{H}_{Y,\phi 0(x)}) \longrightarrow \mathcal{H}_{X,x}$ defined by lifting $f \longrightarrow f \circ \phi_0$ for any $f \in \mathcal{H}_{Y,\phi 0(x)}$. In this case, ϕ_1 is determined by ϕ_0. Therefor, we shall sometimes write ϕ_0 simply instead of (ϕ_0, ϕ_1). $\mathcal{R}$ is a full subcategory of $\mathcal{L}$, namely, $\mathrm{Hom}_{\mathcal{R}}(X, Y) = \mathrm{Hom}_{\mathcal{L}}(X, Y)$ for any $X, Y \in \mathrm{Ob}\mathcal{R}$. There is a natural covariant functor $Red: \mathcal{L} \longrightarrow \mathcal{R}$ which is identity functor on $\mathcal{R}$. The functor Red is called a *reduction functor*, and is defined as follows. For any $X = (|X|, \mathcal{H}_X) \in \mathrm{Ob}\,\mathcal{L}$, any open set $U \subset |X|$ and any $f \in \Gamma(U, \mathcal{H}_X)$, $red_U^* f$ is a continuous function on U defined by $(red_U^* f)\,(x) = q_x\,(f_x)$, $x \in U$. As a consequence of this, $red_x^*(f_x)$ is defined for each $f_x \in \mathcal{H}_{X,x}$. Let $\mathcal{F}$ be the sheaf of germs of complex valued continuous functions on $|X|$. As $red_x^*: \mathcal{H}_{X,x} \longrightarrow \mathcal{F}_x$ is an algebra homomorphism, $red^* := \{red_x^* \mid x \in |X|\}$ induces a sheaf homomorphism

Red: $\mathcal{H}_X \longrightarrow \mathcal{F}$. We have thus $\boldsymbol{Red}\,(X) := \{|X|, Red(\mathcal{H})\} \in \mathrm{Ob}\ \mathcal{R}$. On the other-hand, $red := (id_{|X|}, red^*) : Red(X) \longrightarrow X$ is a $\mathcal{L}$-morphism.

Remark 1. For any $\mathcal{L}$-morphism $\phi = (\phi_0, \phi_1) \in \mathrm{Hom}(X, Y)$, the next diagram is commutative (i. e. *Red* is a covariant functor);

$$\begin{tikzcd} X \arrow[r, "\phi"] & Y \\ Red\ X \arrow[u, "red"] \arrow[r, "Red\,(\phi):=\phi_0"] & Red\ Y \arrow[u, "red"'] \end{tikzcd}$$

Definition 3. A *closed subspace* of a local ringed space $X = (|X|, \mathcal{H}_X)$ is a local ringed space $A = (|A|, \mathcal{H}_A)$, where $|A| = \mathrm{supp}\,(\mathcal{H}_X/\mathcal{J})$ and $\mathcal{H}_A = \mathcal{H}_X/\mathcal{J}\,|\,|A|$ for some coherent sheaf $\mathcal{J}$ of ideals of $\mathcal{H}$. An *open subspace* of X is just a restriction $(U, \mathcal{H}_X|U)$, here U is an open set in $|X|$. For any point $x \in |X|$, a closed subspace $x = (\{x\}, \mathcal{H}_{X,x}/\mathfrak{m}_x)$ is called a *single point.*

For any closed subspace A of $X \in \mathrm{Ob}\mathcal{L}$, there is a *natural imbedding* $i: A \longrightarrow X$, here $i_0 : |A| \longrightarrow |X|$ is an inclusion map and $i_{1a} : \mathcal{H}_{X,a} \longrightarrow \mathcal{H}_{A,a} := \mathcal{H}_{X,a}/\mathcal{J}_a, a \in |A|$, is a natural projection map.

Example 2. A full subcategory $\mathcal{D}$ of $\mathcal{R}$. Its object is a pair $D = (|D|, \mathcal{O})$ where $|D|$ is a domain in some complex number space $\mathbf{C}^n$, and $\mathcal{O}$ is the sheaf of germs of holomorphic functions on $|D|$.

Example 3. A full subcategory C of $\mathcal{L}$. Its object $A = (|A|, \mathcal{H}_A)$ is a closed subspace of some object $D \in \mathrm{Ob}\mathcal{D}$.

For any $A = (|A|, \mathcal{H}_A) \in \mathrm{Ob}C$, with $|A| = \mathrm{supp}(\mathcal{O}/\mathcal{J})$, $\mathcal{H}_A = \mathcal{O}/\mathcal{J}\,|\,|A|$, let $\mathcal{G}$ be the sheaf of ideals of all holomorphic functions vanishing on $|A|$. As is well known, $\mathcal{G}$ is a coherent sheaf.

Remark 2. *Red* A is a closed subspace of A. *Red* $A \simeq (|A|, (\mathcal{O}/\mathcal{G})\,|\,|A|)$.

Considering the property (*), we see that for any morphism $(\phi_0, \phi_1) \in \mathrm{Hom}_C\,(X, Y)$, $\phi_1(\mathcal{H}_{Y,\phi_0(x)}) \subset \mathcal{H}_{X,x}$ for each point $x \in |X|$. Given two objects A, $B \in \mathrm{Ob}C$ with $\mathcal{H}_A = \mathcal{O}(G_1)/\mathcal{J}_1\,|\,|A|$, $\mathcal{H}_B = \mathcal{O}(G_2)/\mathcal{J}_2\,|\,|B|$. Let ψ be a holomorphic map of G_1 into G_2 such that $\psi(|A|) \subset |B|$. Then ψ generates a sheaf homomorphism $\hat{\psi} : G_1 \oplus_\psi \mathcal{J}_2 \longrightarrow \mathcal{O}(G_1)$. If $\hat{\psi}$ maps the sheaf $G_1 \oplus_\psi \mathcal{J}_2$ into $\mathcal{J}_1$, then $\hat{\psi}$ induces a sheaf homomorphism $\psi_1 : A \oplus_\psi \mathcal{H}_B \longrightarrow \mathcal{H}_A$. Thus we have a morphism $(\psi|\,|A|, \psi_1) \in \mathrm{Hom}_C\,(A, B)$. We say that this morphism is induced by ψ. Grauert [2] proved the following

Proposition 1. Given any morphism $\phi = (\phi_0, \phi_1) \in \mathrm{Hom}_C\,(A, B)$, then for any point $x \in |A|$, there exist an open neighborhood $U(x) \subset G_1$ and a holomorphic map $\psi : U \longrightarrow G_2$ such that the morphism ϕ is induced by ψ.

Example 4. A full subcategory C_{red} of C. For any object $X = (|X|, \mathcal{H}_X)$ of C_{red}, its structure sheaf $\mathcal{H}_X$ is characterized by the reduced structure, namely, $\mathcal{H}_{X,x}$ has no nilpotents for each $x \in |X|$. It is clear that $C_{red} = C \cap \mathcal{R}$. For instance, the space $X = (\{0\}, C\{x\}/(x^2)) \in \mathrm{Ob}C$ but $X \notin \mathrm{Ob}C_{red}$.

Example 5. A full subcategory C_{normal} of C_{red}. For any $X = (|X|, \mathcal{H}_X)$ of $\mathrm{Ob}C_{red}$, X belongs to $\mathrm{Ob}C_{normal}$ if and only if the ring $\mathcal{H}_{X,x}$ is integrally closed in its complete ring of quotient for each point $x \in |X|$.

Let $\mathcal{K}$ be any full subcategory of $\mathcal{L}$. We can extend $\mathcal{K}$ to a full subcategory $\hat{\mathcal{K}}$ of $\mathcal{L}$ in the following way. $X=(|X|, \mathcal{H}_X) \in \mathrm{Ob}\mathcal{L}$ belongs to $\mathrm{Ob}\hat{\mathcal{K}}$ if and only if every point $x \in |X|$ has an open neighborhood U such that the restriction of X to U is $\mathcal{L}$-isomorphic to some object of $\mathcal{K}$. $\hat{\mathcal{K}}$ is called a completion of $\mathcal{K}$.

Example 6. $\hat{\mathcal{D}}$: category of complex manifolds.

Example 7. $\hat{C}_{red}$: category of complex spaces in the sense of Serre. An object of $\hat{C}_{red}$ is called a *reduced complex space.*

Example 8. $\hat{C}$: category of complex spaces in the sense of Grauert. We call this category, the category of complex spaces. An object of $\hat{C}$ is called simply a *complex space.*

Example 9. Let $(\mathbf{C}^2, \mathcal{O}) \in \mathrm{Ob}\mathcal{D}$ and $p: \mathbf{C}^2 \longrightarrow \mathbf{C}$ be the projection $(x_1, x_2) \longrightarrow x_1$. For any open set $U \subset \mathbf{C}^2$, let $\mathcal{H}_U := \Gamma(U, \mathcal{H}) := \{f \in \Gamma(U, \mathcal{O}), f(p^{-1}(p(x)) \cap U) = \text{const.}, x \in U\}$. Let $\mathcal{H}$ be the sheaf generated by the presheaf $\mathcal{H}_U$. Then the space $X := (\mathbf{C}^2, \mathcal{H})$ is an object of $\mathcal{L}$ but not of $\hat{C}$.

If $(|X|, \mathcal{H}_X) \in \mathrm{Ob}\hat{C}$ is such that, $|X| = \{x\}$, and $\mathcal{H} = \mathbf{C}$, then $x = (\{x\}, \mathbf{C})$ may be regarded as a single point. In $\hat{C}$, a closed subspace is called a *complex subspace.* More generally, a morphism $u: A \longrightarrow X$ (or by abuse of language, A) is called a *complex g-subspace,* if there are complex subspace $\tilde{A}$ of X and isomorphism $j: A \longrightarrow \tilde{A}$ such that $u = i \circ j$, where $i: \tilde{A} \longrightarrow X$ is a natural imbedding. Let $(|X|, \mathcal{H}_X) \in \mathrm{Ob}\hat{C}$ and $\mathcal{N}$ be the sheaf of nilpotent elements of $\mathcal{H}_X$, $\mathcal{H}_{red} := \mathcal{H}_X/\mathcal{N}$. Then we have $(|X|, \mathcal{H}_{red}) \in \mathrm{Ob}\,\hat{C}_{red}$. The functor *Red* in the Remark 1 is a covariant functor of $\hat{C}$ into $\hat{C}_{red}$. From its definition, it is clear that, for any $(|X|, \mathcal{H}_X) \in \mathrm{Ob}\hat{C}$ (or $\in \mathrm{Ob}\hat{C}_{red}$), $\mathcal{H}_{X,x}$ is a noetherian local ring for each point $x \in |X|$. In the category $\hat{C}$, a morphism is called a holomorphic map and a section is called a holomorphic function. It is well known that every $X \in \mathrm{Ob}\hat{C}_{red}$ has a normalization $Y \in \mathrm{Ob}\hat{C}_{normal}$ [7]; there is a surjective, discrete, proper holomorphic map $\pi: Y \longrightarrow X$ such that the following conditions are satisfied. If S is the set of singular points of $|X|$, and $A = \pi_0^{-1}(S)$, then $|Y| - A$ is dense in $|Y|$ and $\pi \mid |Y| - A$ is an isomorphism onto $|X| - S$.

§ 3. Categorical notations

Let us recall briefly some categorical notations [6]. Throughout this section, let $\mathcal{A}$ be any fixed category.

Definition 4. Given a pair of object X_1 and X_2, we say that an object Z (or more precisely a pair (Z, π_1, π_2)) is a *product* of X_1 and X_2 if there exist morphisms $\pi_1: Z \longrightarrow X_1$ and $\pi_2: Z \longrightarrow X_2$ such that for every pair of morphisms $f_1: X \longrightarrow X_1$ and $f_2: X \longrightarrow X_2$ there is a unique morphism $f: X \longrightarrow Z$ such that the left diagram commutes.

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ X_1 & Z \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2 \end{tikzcd}$$

We denote this Z by $X_1 \times X_2$ and the morphism f by (f_1, f_2). π_1 and π_2 are called the *projection* morphisms from the product. A *coproduct* of X_1 and X_2 is dually defined to the product. Thus a coproduct is a pair of morphisms $u_1: X_1 \longrightarrow Z$, $u_2: X_2 \longrightarrow Z$ called *injections*, such that for any morphisms $f_1: X_1 \longrightarrow X$, $f_2: X_2 \longrightarrow X$, we have a unique morphism $f: Z \longrightarrow X$ with $f \circ f_1 = u_1$, $f \circ f_2 = u_2$. We denote this Z by $X_1 \dot{\cup} X_2$ and f by (f_1, f_2). Especially, when $X_1 = X_2 = X$, we have the *diagonal* morphism $\Delta: X \longrightarrow X \times X$ defined by $\pi_1 \circ \Delta = \pi_2 \circ \Delta = id_X$, and dually the *codiagonal* morphism $\nabla: X \dot{\cup} X \longrightarrow X$ defined by $\nabla \circ u_1 = \nabla \circ u_2 = id_X$. It is clear that Δ is necessarily a monomorphism and ∇ is an epimorphism.

Definition 5. Given two morphisms $f_1: X_1 \longrightarrow Y$ and $f_2: X_2 \longrightarrow Y$ with a common codomain, a commutative diagram (1) is called a *pullback* for f_1 and

$$\begin{tikzcd} L \arrow[r, "g_2"] \arrow[d, "g_1"'] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} L' \arrow[r, "g'_2"] \arrow[d, "g'_1"'] & X_2 \arrow[d, "f_2"] \\ X_1 \arrow[r, "f_1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} & L' \arrow[dl, "g'_1"'] \arrow[d, "l"] \arrow[dr, "g'_2"] & \\ X_1 & L \arrow[l, "g_1"] \arrow[r, "g_2"'] & X_2 \end{tikzcd}$$

(1) (2) (3)

f_2 if for every pair of morphisms $g'_1: L' \longrightarrow X_1$ and $g'_2: L \longrightarrow X_2$ such that the diagram (2) commutes, there exists a unique morphism $l: L' \longrightarrow L$ such that the diagram (3) commutes. L is also called a pullback for f_1 and f_2. The dual of a pullback is called a *pushout*. Thus a pushout diagram (or a pushout for f_1 and f_2) is obtained by reversing the direction of all arrows in the diagram (1).

Definition 6. Given two morphisms $f_1: X \longrightarrow Y$ and $f_2: X \longrightarrow Y$, a morphism $f: Z \longrightarrow X$ is called a *difference kernel* (or simply *kernel*) for f_1 and f_2 if $f_1 \circ f = f_2 \circ f$ and if for every morphism $f': Z' \longrightarrow X$ such that $f_1 \circ f' = f_2 \circ f'$, there exists a unique morphism $h: Z' \longrightarrow Z$ such that $f = f' \circ h$ (cf. diagram (4)). The dual of a

$$\begin{tikzcd} Z \arrow[rr, "f"] & & X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & Y \\ & Z' \arrow[ul, "h"] \arrow[ur, "f'"'] & & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & Y \arrow[rr, "f"] \arrow[dr, "f'"'] & & Z \arrow[dl, "h"] \\ & & Z' & \end{tikzcd}$$

(4) (5)

difference kernel is called a *difference cokernel* (or simply *cokernel*) for f_1 and f_2 (diagram (5)). In these cases, the sequences

$$Z \xrightarrow{f} X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \quad \text{and} \quad X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \xrightarrow{f} Z$$

are called *exact*.

Proposition 2. A difference kernel for double arrows $X \times X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$ is a diagonal morphism $\Delta: X \longrightarrow X \times X$.

Proof. Let $\pi: Z \longrightarrow X \times X$ be a morphism with $\pi_1 \circ \pi = \pi_2 \circ \pi$. Put $h := \pi_1 \circ \pi$. Then we have $\pi_1 \circ \Delta \circ h = \pi_1 \circ \Delta \circ \pi_1 \circ \pi = id_X \circ \pi_1 \circ \pi = \pi_1 \circ \pi = \pi_2 \circ \pi = \pi_2 \circ \Delta \circ h$. By the universal property of $X \times X$, we have $\pi = \Delta \circ h$. On the otherhand, let $h': Z \longrightarrow X$ be a morphism such that $\pi = \Delta \circ h = \Delta \circ h'$. Since Δ is a monomorphism, $h = h'$. Therefor we have an exact sequence $X \xrightarrow{\Delta} X\times X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$.

Dually a difference cokernel for double arrows $X \underset{u_2}{\overset{u_1}{\rightrightarrows}} X\dot{\cup} X$ is a codiagonal morphism $X\dot{\cup} X \xrightarrow{\nabla} X$.

Definition 7. Given two morphisms $f_1: X_1 \longrightarrow Y$ and $f_2: X_2 \longrightarrow Y$ and a product of X_1 and X_2, a difference kernel for double arrows $X\times X \underset{f_2\circ\pi_2}{\overset{f_1\circ\pi_1}{\rightrightarrows}} Y$ is called a *fiber product* of X_1 and X_2 over Y, and is denoted by $X_1\times_Y X_2$, here π_1 and π_2 are projections. Dually a *fiber coproduct* of Y_1 and Y_2 under X (denoted by $Y_1\cup_X Y_2$) is defined for any two given morphisms $f_1: X \longrightarrow Y_2$ and $f_2: X \longrightarrow Y_2$. Thus $Y_1\cup_X Y_2$ is a difference cokernel for double arrows $X \underset{u_2\circ f_2}{\overset{u_1\circ f_1}{\rightrightarrows}} Y_1\dot{\cup} Y_2$, where u_1 and u_2 are injections.

Remark 3. All these objects (products, coproducts, etc.) are not always exist, but if they exist, they are determined uniquely up to isomorphism.

Given two morphisms $v: X \longrightarrow S$ and $w: Y \longrightarrow S$. By the definition of $S\times S$, there exists a morohism $(v\circ\pi_1, w\circ\pi_2): X\times X \longrightarrow S\times S$ such that $p_1\circ(v\circ\pi_1, w\circ\pi_2) = v\circ\pi_1$ and $p_2\circ(v\circ\pi_1, w\circ\pi_2) = w\circ\pi_2$, here π_1, π_2, p_1 and p_2 are projections. We denote this morphism $(v\circ\pi_1, w\circ\pi_2)$ by $v\times w$. Let $\mathcal{A}$ be any category, in which products exist.

Proposition 3. The following properties are equivalent in the category $\mathcal{A}$.
(a) Z is a fiber product of X and Y over S.
(b) Z is a pullback for v and w.
(c) Z is a pullback for Δ and $v\times w$,

Proof. (a)$\Leftrightarrow$(b). Let $Z \xrightarrow{\alpha} X\times Y \underset{w\circ\pi_2}{\overset{v\circ\pi_1}{\rightrightarrows}} S$ be an exact sequence. Let $f = \pi_1\circ\alpha$ and $g = \pi_2\circ\beta$, then $v\circ f = v\circ\pi_1\circ\alpha = w\circ\pi_2\circ\alpha = w\circ g.$ ……………(1)

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "w"] \\ X \arrow[r, "v"] & S \end{tikzcd}$$

The right diagram commutes. Let $f': Z' \longrightarrow X$ and $g': Z' \longrightarrow Y$ are morphisms such that $v\circ f' = w\circ g'$. There exists a morphism $(f', g'): Z' \longrightarrow X\times Y$ such that $f' = \pi_1\circ(f', g')$ and $g' = \pi_2\circ(f', g')$. But since $v\circ f' = w\circ g'$, so $v\circ\pi_1\circ(f', g') = w\circ\pi_2\circ(f', g')$. Since Z is a difference kernel, there exists a unique morphism $\beta: Z' \longrightarrow Z$ such that $(f', g') = \alpha\circ\beta$. Then, we have $f\circ\beta = \pi_1\circ\alpha\circ\beta = \pi_1\circ(f', g') = f'$, and $g\circ\beta = g'$. ………(2) Given another morphism $\beta': Z' \longrightarrow Z$ such that $f' = f\circ\beta'$ and $g' = g\circ\beta'$, then $f' = \pi_1\circ\alpha\circ\beta'$ and $g' = \pi_2\circ\alpha\circ\beta'$. By the universal property of $X\times Y$, we have $\alpha\circ\beta = (f', g')$. Therefor $\alpha\circ\beta' = (f', g') = \alpha\circ\beta$. But since α is a monomorphism, we have $\beta' = \beta'$. ……………………………………………………………………………(3).

From the properties (1), (2) and (3), Z is a pullback for v and w. By the Remark 3, (a) and (b) are equivalent.

(a)$\Leftrightarrow$(c). Since $p_1\circ(v\circ\pi_2, w\circ\pi_2)\circ\alpha = p_2\circ(v\times w)\circ\alpha$ and $S \xrightarrow{\Delta} S\times S \underset{p_2}{\overset{p_1}{\rightrightarrows}} S$ is an exact

$$\begin{tikzcd}
 & & X \arrow[rr, "v"] & & S \\
Z \arrow[r, "\alpha"] \arrow[dr, "f"'] & X\times Y \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] \arrow[rr, "v\times w"] & & S\times S \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & \\
 & S \arrow[urr, "\Delta"] & Y \arrow[rr, "w"'] & & S
\end{tikzcd}$$

sequence by Proposition 2, there exists a morphism $f: Z \longrightarrow S$ such that

$\Delta\circ f=(v\times w)\circ\alpha$ .... (4).

Given two morphisms $f': Z' \longrightarrow S$ and $\alpha': Z' \longrightarrow X\times Y$ sueh that $\Delta\circ f'=(v\times w)\circ\alpha'$. Then $v\circ\pi_1\circ\alpha'=p_1\circ(v\times w)\circ\alpha'=p_1\circ\Delta\circ f=id_S\circ f'=f'=w\circ\pi_2\circ\alpha'$. Since Z is a difference kernel, there exists a morphism $\beta: Z' \longrightarrow Z$ such that $\alpha'=\alpha\circ\beta$. .... (5).

Furthermore, $\Delta\circ f\circ\beta=(v\times w)\circ\alpha\circ\beta=(v\times w)\circ\alpha'=\Delta\circ f'$. But since α is a monomorphism, $f\circ\beta=f'$. .... (6).

Let $\beta': Z' \longrightarrow Z$ be another morphism such that $\alpha\circ\beta'=\alpha'$, $f\circ\beta'=f'$. Then $\alpha\circ\beta'=\alpha'=\alpha\circ\beta$. Since α is a monomorphism, $\beta=\beta'$. .... (7).

By (4)$\sim$(7), Z is a pullback for Δ and $v\times w$. By the Remark 3, (a) and (c) are equivalent.

The above morphism $f: Z \longrightarrow S$ is sometimes written as $v\times_S w$. From the fact that $\Delta\circ f=(v\times w)\circ\alpha$, $p_1\circ\Delta=id_S$, we see that $v\times_S w=v\circ\pi_1\circ\alpha=w\circ\pi_2\circ\alpha$. Dually we can prove the following property for any category $\mathcal{A}'$ in which coproducts exist.

Proposition 4. Given two morphisms $v: S \longrightarrow X$ and $w: S \longrightarrow Y$, the following properties are equivalent in $\mathcal{A}'$.

(a′) Z is a fiber coproduct of X and Y under S.

(b′) Z is a pushout for v and w.

(c′) Z is a pushout for ∇ and $(u_1\circ v,\ u_2\circ w)$.

In other words,

(a′) means that $S \underset{u_2}{\overset{u_1}{\rightrightarrows}} X\dot{\cup} Y \xrightarrow{\alpha} Z$ is an exact sequence,

(b′) means that the diagram (6) is a pushout,

(c′) means that the diagram (7) is a pushout.

In the same way we have

Corollary 1. For any two morphisms $v: X \longrightarrow Y$ and $w: X \longrightarrow Y$ in $\mathcal{A}'$. the sequence $Z \xrightarrow{\alpha} X \underset{w}{\overset{v}{\rightrightarrows}} Y$ is exact if and only if the diagonal (8) is a pullback.

Corollary 2. For any two morphisms $v: X \quad Y$ and $w: X \quad Y$ in $\mathcal{A}'$, the sequence $X \underset{w}{\overset{v}{\rightrightarrows}} Y \xrightarrow{\alpha} Z$ is exact if and only if the diagram (9) is a pushout.

$$\begin{tikzcd}
S \arrow[r, "w"] \arrow[d, "v"'] & Y \arrow[d, "\alpha\circ u_2"] \\
X \arrow[r, "\alpha\circ u_1"'] & Z
\end{tikzcd}$$

(6)

$$\begin{tikzcd}
S\dot{\cup} S \arrow[r, "{(u_1\circ v,\ u_2\circ w)}"] \arrow[d, "\nabla"'] & X\dot{\cup} Y \arrow[d, "\alpha"] \\
S \arrow[r, "\alpha\circ u_1\circ v"'] & Z
\end{tikzcd}$$

(7)

$$\begin{tikzcd}
Z \arrow[r, "\alpha"] \arrow[d, "v\circ\alpha"'] & X \arrow[d, "{(v,\ w)}"] \\
Y \arrow[r, "\Delta"'] & X\times Y
\end{tikzcd}$$

(8)

$$\begin{tikzcd}
X\dot{\cup} X \arrow[r, "{(v,\ w)}"] \arrow[d, "\nabla"'] & Y \arrow[d, "\alpha"] \\
X \arrow[r, "\alpha\circ v"'] & Z
\end{tikzcd}$$

(9)

We must remark here, that the above Proposition 3 (or 4) itself does not guarantee the existence of pullbacks (or pushouts).

§ 4. Examples in the category $\hat{\mathcal{C}}$

Theorem 1. For any X, $Y \in \mathrm{Ob}\hat{\mathcal{C}}$, there exists a product in $\hat{\mathcal{C}}$.

Proof. Given two complex spaces $X_1=(|X_1|, \mathcal{H}_1)$ and $X_2=(|X_2|, \mathcal{H}_2)$. The problem being local, we may assume that $X_\nu(\nu=1,2)$ is an analytic set in some domain G_ν. Let $\mathcal{H}_\nu=\mathcal{O}(G_\nu)/\mathcal{J}_\nu$. Let $\mathcal{J}_Z$ be an ideal generated by $g \in \mathcal{J}_{1Z}$ and $g \in \mathcal{J}_{2Z}$ for $z=(z_1, z_2) \in G:=G_1 \times G_2$. Then $\mathcal{J}=\{\mathcal{J}_z\}$ is a coherent ideal sheaf with a supp $\mathcal{H}=|X_1| \times |X_2|$, $\mathcal{H}:=\mathcal{O}(G)/\mathcal{J}$. Thus we have $X_1 \times X_2:=(|X_1| \times |X_2|, \mathcal{H}) \in \mathrm{Ob}\hat{\mathcal{C}}$. Put $\pi_0^{(\nu)}(z_1,z_2):=z_\nu$ and $\tilde{\pi}_{1z}^{(\nu)}(g_\nu):=g_\nu \circ \pi_0^{(\nu)}$ for $g_\nu \in \mathcal{J}_{\nu z_\nu}$. Since $\tilde{\pi}_1^{(\nu)}(\mathcal{J}_\nu) \subset \mathcal{J}$, $\tilde{\pi}_1^{(\tau)}$ defines a unique homomorphism $\pi_{1z}^{(\nu)}:(z, \mathcal{H}_{\nu z_\nu}) \longrightarrow \mathcal{H}_z$. Hence we have a product $(X_1 \times X_2, \pi_1, \pi_2)$ with $\pi_\nu=(\pi_0^{(\nu)}, \pi_1^{(\nu)})$. By virtue of Proposition 1, the universal property of the product of $X_1 \times X_2$ is obvious from its construction.

$\mathcal{H}=:\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ is so called an *analytic tensor probuct* of $\mathcal{H}_1$ and $\mathcal{H}_2$. A coproduct $X_1 \dot{\cup} X_2$ of X_1 and X_2 in $\hat{\mathcal{C}}$ is clearly equal to their disjoint summ. By the definition, any complex space X is a complex g-subspace of $X \times X$.

Example 10. Given two holomorphic maps $\phi: X \longrightarrow Y$ and $i: A \longrightarrow Y$ in $\hat{\mathcal{C}}$, where $A=(|A|, \mathcal{H}_A)$ is a complex subspace of $Y=(|Y|, \mathcal{H}_Y)$ with $\mathcal{H}_A=\mathcal{H}_Y/_\mathcal{J}||A|$, and i is a natural imbedding. A pullback B for i and ϕ is constructed as follows. For $|B|:=\phi_0^{-1}(|A|) \subset |X|$, introduce the relative topology. Then the inclusion map $j_0: |B| \longrightarrow |X|$ and restriction map $\psi_0:=\phi_0|B| \longrightarrow |A|$ are both continuous. Difine the sheaf $\mathcal{H}_B$ on $|B|$ by $\mathcal{H}_{B,x}:=\mathcal{H}_{X,x}/\mathcal{J}_x$, $x \in |B|$, where $\mathcal{J}_x:=\mathcal{H}_{X,x} \circ \phi_{1x}(\mathcal{J}_{\phi 0(x)})$. Since $\phi_{1x}(\mathcal{J}_{\phi 0(x)}) \subset \mathcal{J}_x$, the map $\phi_{1x}: \mathcal{H}_{Y,\phi 0(x)} \longrightarrow \mathcal{H}_{X,x}$ induces an algebra homomorphism $\psi_{1x}: \mathcal{H}_{A,\phi 0(x)} \longrightarrow \mathcal{H}_{B,x}$. Let $j_{1x}: \mathcal{H}_{X,x} \longrightarrow \mathcal{H}_{B,x}$ be a natural quotient map. Put $\psi_1:=\{\psi_{1x}, x \in |B|\}$ and $j_1=\{j_{1x}, x \in |B|\}$. Then we have holomorphic maps $\psi=(\psi_0, \psi_1): B \longrightarrow A$ and $j=(j_0, j_1): B \longrightarrow X$ such that $i \circ \psi=\phi \circ j$. Given any holomorphic maps $\psi': B' \longrightarrow A$ and $j': B' \longrightarrow X$ with $i \circ \psi'=\phi \circ j'$, a holomorphic map $\beta': B' \longrightarrow B$ is defined as follows. $\beta'_0(b'):=j'_0(b')$ for $b' \in |B'|$ and $\beta'_{1b}(s):=j'_{1b}(t)$ for any $t \in \mathcal{H}_{X,j'_0(b')}$ with $j_{1j_0'(b')}(t)=s$. Then $\psi \circ \beta'=\psi'$ and $j \circ \beta'=j'$. Such a holomorphic map β' exists uniquely. Thus B is certainly a pullback for i and ϕ in $\hat{\mathcal{C}}$.

B is a complex subspace of X, and j is a natural imbedding. We say that B is an *inverse image* of A by ϕ and denoted by $\phi^{-1}(A)$. By Proposition 3, $\phi^{-1}(A)=A \times_Y X$. Especially when A is a single point, $a=(\{a\}, C)$, $\phi^{-1}(a)$ is called the *ϕ-fiber* over a. For any complex space S, a holomorphic map $\Delta: S \longrightarrow S \times S$ is a complex g-subspace. Given two holomorphic maps $v: X \longrightarrow S$ and $w: Y \longrightarrow S$, the right diagram is a pullback by Proposition 3 and Example 10. Thus we have

$$\begin{tikzcd} X \times_S Y \arrow[r, "\delta"] \arrow[d, "v \times_S w"'] & X \times Y \arrow[d, "v \times w"] \\ S \arrow[r, "\Delta"] & S \times S \end{tikzcd}$$

(10)

Theorem 2. There exist a fiber product and a pullback in $\hat{\mathcal{C}}$ for any given holomorphic maps $v: X \longrightarrow S$ and $w: X \longrightarrow S$.

Similarly, considering the Example 10 and Corollary 1 we have

Corollary 3. A difference kernel exists for any given holomorphic maps $v: X \longrightarrow Y$ and $w: X \longrightarrow Y$.

In order to give the difference kernel concretely, it is sufficient to consider it in the category $\mathcal{C}$. Let $|X| = \operatorname{supp}\mathcal{O}(G_1)/\mathcal{J}_1$. and $|Y| = \operatorname{supp}\mathcal{O}(G_2)/\mathcal{J}_2$. Suppose that v and w are the holomorphic maps induced by holomorphic maps $\tilde{v}: G_1 \longrightarrow G_2$ and $\tilde{w}: G_1 \longrightarrow G_2$ respectively (cf. Proposition 1). Then the difference kernel is fefined by $(\operatorname{supp}\mathcal{H}, \mathcal{H})$ with $\mathcal{H} = \mathcal{O}(G_1)/\mathcal{J}$, here $\mathcal{J}$ is an ideal generated by $\mathcal{J}_1$ and the components of holomorphic maps $\tilde{v} - \tilde{w}$.

Note that for any $X, Y \in \operatorname{Ob}\hat{\mathcal{C}}$, $X \times_S X = X \times Y$ if S is a single point. As an application, we can prove the following Theorem [3].

Theorem 3. Let $X = (|X|, \mathcal{H}_X) \in \operatorname{Ob}\hat{\mathcal{C}}$, and R be a proper equivalence relation in $|X|$. Let $|X|$ be a Hausdorff space. Then, for any complex structure $\mathcal{H} \subset \mathcal{H}_X/_R$ on $|X|/_R$ with $Y := (|X|/_R, \mathcal{H}) \in \operatorname{Ob}\hat{\mathcal{C}}$, $\mathcal{H}_X/_R$ is a coherent $\mathcal{H}$-sheaf.

Here a proper equivalence relation means that R is an equivalence relation such that R-saturated subset of any compact set $K \subset |X|$ is also compact. Let $p_0: |X| \longrightarrow |X|/_R$ be a quotient map, then $|X|/_R$ is a locally compact Hausdorff space and p_0 is a proper map. For any open set $U \subset |X|/_R$, let $\Gamma(U, \mathcal{H}_X/_R) := \{f \in \Gamma(p_0^{-1}(U), \mathcal{H}_X) \mid f: p_0^{-1}(y) \longrightarrow \mathbf{C} \text{ const. } y \in U\}$. Let $\mathcal{H}_X/_R$ be the sheaf generated by $\Gamma(U, \mathcal{H}_X/_R)$. Then, we have $X/_R := (|X|/_R, \mathcal{H}_X/_R) \in \operatorname{Ob}\mathcal{L}$. Let $p_{1x} : (\mathcal{H}_X/_R)_{p_0(x)} \longrightarrow \mathcal{H}_{X,x}$ be a natural homomorphism, and $p_1 : \{p_{1x} \mid x \in |X|\}$, $p := (p_0, p_1)$. Then $p \in \operatorname{Hom}_{\mathcal{L}}(X, X/_R)$.

Proof of Theorem. Let $\tilde{p}_1$ be the restriction of p_1 to $\mathcal{H}$, and $\tilde{p}_0 := p_0$, $\tilde{p} := (\tilde{p}_0, \tilde{p}_1)$. Then, $\tilde{p} \in \operatorname{Hom}_{\hat{\mathcal{C}}}(X, Y)$. Consider the following diagram (cf. diagram 10),

$$\begin{tikzcd}
X \arrow[d, "\tilde{p}"'] & X \times_Y X \arrow[l, shift right, "pr_1"'] \arrow[l, shift left, "pr_2"] \arrow[r, "\delta"] \arrow[d, "\tilde{p} \times_Y \tilde{p}"] \arrow[dll, dashed, "g"'] & X \times X \arrow[d, "\tilde{p} \times \tilde{p}"] \\
Y \arrow[r, "\Delta"] & (D, \mathcal{H}_D) \arrow[r, "i"] & Y \times Y
\end{tikzcd}$$

Here, $(D, \mathcal{H}_D)$ is the image of the diagonal map $\Delta: Y \longrightarrow Y \times Y$ and i is a narural imbedding. We have $\mathcal{H}_D = \mathcal{H} \hat{\otimes} \mathcal{H}/_{\mathcal{J}} \mid D$, $\mathcal{J} = \operatorname{Ker}\Delta$ with $\Delta = (\Delta_0, \Delta_1)$. $(\tilde{p} \times \tilde{p})_0^{-1}(D)$ is equal to the graph $R \subset |X| \times |X|$ of R. Let $\mathcal{G}$ be the sheaf of ideal generated by $(\tilde{p} \times \tilde{p})_1(\mathcal{J})$. Let $\mathcal{H}_R := (\mathcal{H}_X \hat{\otimes} \mathcal{H}_X)/\mathcal{G} \mid R$, then the space $(R, \mathcal{H}_R)$ can be identified with $X \times_Y X$, which is a complex space by Theorem 2. Let pr_i $(i = 1, 2)$ be the restriction to $(R, \mathcal{H}_R)$ of the holomorphic projection from $X \times X$ to X. Since $g := \Delta^{-1} \circ (\tilde{p} \times \tilde{p}) = \tilde{p} \circ pr_i$ and $\tilde{p}$ are both proper holomorphic maps, the direct images $\tilde{p}_0\mathcal{H}_X$ and $g_0\mathcal{H}_R$ are both coherent sheaves of $\mathcal{H}$-modules. The preceeding diagrams define two homomorphisms $u_i : \tilde{p}_0\mathcal{H}_X \longrightarrow g_0\mathcal{H}_X$ of $\mathcal{H}$-sheaves. we have $\mathcal{H} \subset \operatorname{Ker}(u_1 - u_2) \subset \mathcal{H}/_R \subset \tilde{p}_0\mathcal{H}_X$. On the otherhand, $\mathcal{H}/_R$ is equal to $(u_1 - u_2)^{-1}(\mathcal{N})$, here $\mathcal{N}$ is a sheaf of nilpotent elements of $g_0\mathcal{H}_R$. Since $\mathcal{N}$ is a direct image of a sheaf of nilpotent elements of $\mathcal{H}_R$, $\mathcal{N}$ and $\mathcal{H}/_R = (u_1 - u_2)^{-1}(\mathcal{N})$ are both coherent sheaves of $\mathcal{H}$-modules on Y. This compltes the proof.

Remark 4. Let $X=(|X|,\mathscr{H}_X)\in\mathrm{Ob}\hat{\mathcal{C}}$, and $\mathcal{S}$ be a category of coherent sheaves of $\mathscr{H}_X$-modules on $|X|$. Since $\mathcal{S}$ is abelian, there exist all objects of Definition (4)~(7). For instances, a product $\mathscr{F}_1\times\mathscr{F}_2$ of $\mathscr{F}_1$ and $\mathscr{F}_2$ is epual to the direct product as $\mathscr{H}_X$-modules. A fiber product $\mathscr{F}_1\times_{\mathscr{F}}\mathscr{F}_2$ for any given sheaf homomorphisms v: $\mathscr{F}_1\longrightarrow\mathscr{F}$ and w: $\mathscr{F}_2\longrightarrow\mathscr{F}$ is defined by the following presheaf; $|X|\supset U\longrightarrow\{(f,g)\in\Gamma(U,\mathscr{F}_1\times\mathscr{F}_2)\,|\,v_U(f)=w_U(g)\}$, for any open set $U\subset|X|$.

§ 5. Pushouts

Let $\mathscr{T}$ be category of topological spaces and continuous maps between them. Given two maps $v,w\in\mathrm{Hom}_{\mathscr{T}}(X,Y)$, a difference cokernel $Y\xrightarrow{\alpha}Z$ is defined as follows. Let R be a relation on Y defined by $y_1Ry_2\Longleftrightarrow y_1=v(x)$, $y_2=w(x)$. Let $\tilde{R}$ be an equivalence relation generated by R. Let $Z=Y/_{\tilde{R}}$ and $\alpha: Y\longrightarrow Z$ be a quotient map. Then we have an exact sequence $X\underset{w}{\overset{v}{\rightrightarrows}}Y\xrightarrow{\alpha}Z$.

A consequence of this and Proposition 4 is that, there exists a pushout for any given maps $v: S\longrightarrow X$ and $w: S\longrightarrow Y$ in $\mathscr{T}$.

Given two holomorphic maps $v,w\in\mathrm{Hom}_{\hat{\mathcal{C}}}(X,Y)$, there exists a difference cokernel $Y\xrightarrow{\alpha}Z$, in the category $\mathscr{T}$ and is defined as follows. Let $|Y|\xrightarrow{\alpha_0}|Z|$ be a difference cokernel for $v_0,w_0\in\mathrm{Hom}_{\mathscr{T}}(|X|,|Y|)$. The structure sheaf $\mathscr{H}_Z$, on $|Z|$ is defined by the following presheaf; $\Gamma(U,\mathscr{H}_Z):=\{f\in\Gamma(\alpha_0^{-1}(U),\mathscr{H}_Y)\,|\, v_1f=w_1f\in\Gamma(v_0^{-1}\circ\alpha_0^{-}(U)\mathscr{H}_X)\}$, for any open set $U\subset|Z|$. Let $\alpha_1:\mathscr{H}_Z\longrightarrow\mathscr{H}_Y$ be a natural sheaf homomorphism, then $Y\xrightarrow{\alpha}Z$ is a difference cokernel with $\alpha=(\alpha_0,\alpha_1)$. We have $Z:=(|Z|,\mathscr{H}_Z)\in\mathrm{Ob}\mathcal{L}$, but in general $Z\not\in\hat{\mathcal{C}}$.

Given two holomorphic maps $v: S\longrightarrow X$ and w: $S\longrightarrow Y$ in $\hat{\mathcal{C}}$, there exists a pushout for v and w in $\mathcal{L}$ and is defined as follows. Let $|Z|$ be a pushout for $v_0: |S|\longrightarrow|X|$ and $w_0: |S|\longrightarrow|Y|$. The structure sheaf $\mathscr{H}_Z$ on $|Z|$ is defined by the following presheaf (cf. diagram (6));

$\Gamma(U,\mathscr{H}_Z):=\{(f,g)\,|\,f\in\Gamma(u_{10}^{-1}\circ\alpha_0^{-1}(U),\mathscr{H}_X),\ g\in\Gamma(u_{20}^{-1}\circ\alpha_0^{-1}(U),\mathscr{H}_Y),\ v_1f=w_2g\}$.

Then we have a pushout $Z:=(|Z|,\mathscr{H}_Z)$ for v and w. $Z\in\mathrm{Ob}\mathcal{L}$ but in general $Z\not\in\mathrm{Ob}\hat{\mathcal{C}}$.

Example 11. Let $X=(|X|,\mathscr{H}_X):=(\mathbf{C}^2,\mathcal{O})\in\mathrm{Ob}\mathscr{D}$, ϕ_0 (resp. ψ_0) be a holomorphic map: $\mathbf{C}^2\longrightarrow\mathbf{C}^3$ defined by $(x_1,x_2)\longrightarrow(x_1^2,x_1^3,x_2)$ (resp. x_1+x_2,x_2^2,x_2^3). Let $A_1=\phi_0(\mathbf{C}^2)$ and $A_2=\psi_0(\mathbf{C}^2)$. We may regard $A_1,A_2\in\mathrm{Ob}\mathcal{C}_{red}$. Put $A_i=(|A_i|,\mathscr{H}_{A_i})(i=1,2)$. Induce the complex structure $\mathscr{H}_1$ (resp. $\mathscr{H}_2$) on X by ϕ (resp. ψ). Then, we have $X_i:=(|X|,\mathscr{H}_i)\in\mathrm{Ob}\hat{\mathcal{C}}_{red}$ and a pushout diagram in $\mathcal{L}$.

$$\begin{array}{ccc}(|X|,\mathscr{H}_X) & \longrightarrow & (|X|,\mathscr{H}_2)\\ \downarrow & & \downarrow\\ (|X|,\mathscr{H}_1) & \longrightarrow & (|X|\mathscr{H}_1\cap\mathscr{H}_2).\end{array}$$

By a simple calculus, we see that there is no neighborhood U of the point $(0,0)\in|X|$, which is holomorphic separable (as in the case of Example 9) [4].

Kaup [5] gave the following sufficient condition.

Theorem. Let $(|X|), \mathscr{H}_X), (|X|, \mathscr{H}_1), (|X|, \mathscr{H}_2) \in \mathrm{Ob}\hat{C}$ with $\mathscr{H}_1, \mathscr{H}_2 \subset \mathscr{H}_X$. Put $N_i := \{x \in |X| \,;\, (\mathscr{H}_i)_x \supsetneq (\mathscr{H}_j)_x$, for $i \neq j \in \{1,2\}\}$. If $N_1 \cap N_2$ is a discrete set in $|X|$, then $(|X|, \mathscr{H}_1 \cap \mathscr{H}_2) \in \mathrm{Ob}\hat{C}$. He also proved the following Theorem [5].

Theorem. Let $i: X \longrightarrow Y$ be a complex g-subspace, $h: X \longrightarrow Z$ be a proper, finite holomorphic map, and $|X|, |Y|$ be Hausdorff spaces. Then there exists a pushout $Y \cup_X Z \in \mathrm{Ob}\hat{C}$. Z is a complex g-subspace of $Y \cup_X Z$.

As a consequence of this, we have

Corollary 4. Let $X \in \mathrm{Ob}\hat{C}$, R be a proper, finite equivalence relation on $|X|$, and $|X|$ be a Hausdorff space. If $(Red\ X)/R \in \mathrm{Ob}\hat{C}$. then $X/R \in \mathrm{Ob}\hat{C}$.

$$\begin{tikzcd} Red\ X \arrow[r, "red"] \arrow[d] & X \arrow[d, "p"] \\ (Red\ X)/R \arrow[r] & X/R \end{tikzcd}$$

This follows immediately from the following pushout diagram. Here, R is called finite if $p_0: |X| \longrightarrow |X|/R$ is a finite map.

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