局所環のフィルトレーションと附随する次数付き環の研究

（課題番号　11640011）

平成 11 年度 － 平成 14 年度　科学研究費補助金（基盤研究(C)(2)）

研究成果報告書

平成 15 年 3 月

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はじめに

この報告書は、文部科学省科学研究費補助金 (基盤研究 (C)(2)) の交付を受けて、平成11年度から平成14年度の4年間に実施された研究「局所環のフルとレーションと附随する次数付き環の研究」に関するものである。

本研究は、可換環論における主要なテーマのひとつであるblow-up代数の環構造の解析を目的とするものであり、研究代表者・西田康二が中心となって研究分担者と密接に連絡をとりつつ行われたが、必要に応じて、外部からの研究協力を得ることとなった。研究費の多くの部分は、研究連絡のための旅費と計算機等の設備備品費に使用し、研究環境の充実を図った。

この様な4年間の研究の結果、一般のフィルトレーションに解析的差異を導入し、blow-up代数の研究を進めるという当初の計画は概ね達成することができた。特に、解析的差異が1以下である様なフィルトレーションに対しては、附随する次数付き環のCohen-Macaulay性を判定する実用的な判定法を与え、具体的な応用も見出すことに成功した。又、解析的差異が高い場合にも、イデアルのべき乗が定めるフィルトレーションに対して満足のいく結果が得られた。こうした研究成果の詳細については、本文の「研究成果」の項を参照していただきたい。

本研究の推進に当たっては、分担者のみならず、数多くの研究協力者や大学院生のお世話になっている。ここに記して感謝の意を申し上げたい。

科学研究費補助金(基盤研究(C)(2))研究成果報告書

(1) 課題番号 11640011

(2) 研究課題 局所環のフィルトレーションと附随する次数付き環の研究

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(5) 研究経費 総計：3,600 千円

平成 11 年度：1,200 千円

平成 12 年度：800 千円

平成 13 年度：800 千円

平成 14 年度：800 千円

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(7) 研究成果

この研究では、局所環 $(A/\mathfrak{m})$ のフィルトレーション $\mathcal{F}: A = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$ に附随する次数付き環 $\mathrm{G}(\mathcal{F}) = \oplus_{n\geq 0} F_n/F_{n+1}$ のホモロジカルな性質を中心に調べた。その為に、従来イデアルに対して定義されていた解析的差異 (analytic deviation) という概念を一般のフィルトレーションに対して拡張し、その不変量を尺度として分析を進めるという方針を採った。各年度ごとの進行は以下のようなものであり、概ね、当初の計画に沿った研究が達成できたと言える。

平成11年度

reduction の概念を中心とした基礎的部分の整理とフィルトレーションが equimultiple な場合の理論の構成を目標とし、下記に述べてある様な結果を得ることができた。

1. 局所環 A のフィルトレーション $A = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$ に対して、次の 2 条件：
   - ある $k_i > 0$ に対して $a_i \in F_{k_i}$ $(i = 1, 2, \cdots, r)$
   - $n \gg 0$ に対して $F_n = \sum_{i=1}^{r} a_i F_{n-k_i}$

   をみたす A の要素のシステム $a_1, a_2, \cdots, a_r$ をその reduction として捉えると、従来イデアルに対して定義されていた reduction の概念と理論が自然に一般化されることが分かった。実はこの見方はある意味では既に存在していたのだが、analytic spread という観点を通して考察したのはこの研究が最初のである。
2. 上で述べた $a_1, a_2, \cdots, a_r$ を r が F_1 の高さに一致する様に取れるとき、フィルトレーション $A = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$ は equimultiple であると言うことにする。こ

のとき、附随する次数付き環 $\bigoplus_{i\geq 0} F_i/F_{i+1}$ の Cohen-Macaulay 性を特徴付けることができた。

平成 12 年度

局所環のフィルトレーション $\mathcal{F}$ で解析的差異 が 1 のものが与えられたとき、附随する次数付き環 $\mathrm{G}(\mathcal{F})$ の Cohen-Macaulay 性を判定する実用的方法を見出すことを目標とし、次の様な結果を得た：

$\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ は d 次元 Cohen-Macaulay 局所環 A のフィルトレーションとし、F_1 の高さを s としたとき $\mathcal{F}$ は $s+1$ 個の元 $a_1, a_2, \cdots, a_{s+1}$ からなる reduction をもつとせよ (これは $\mathcal{F}$ の解析的差異が 1 であるということを意味する)。K は $a_1, a_2, \cdots, a_{s+1}$ が生成するイデアルとし $b = a_{s+1}$ とおく。さらに F_1 の任意の極小素因子 $\mathfrak{p}$ (この様な $\mathfrak{p}$ は有限個しかない) に対して $A_\mathfrak{p} \bigotimes_A \mathrm{G}(\mathcal{F})$ は Cohen-Macaulay 環であると仮定する。このとき正整数 α と β が定まり、$1 \leq n \leq \alpha$ をみたす n に対して剰余環 $A/K + F_n$ が Cohen-Macaulay で $1 \leq m \leq \beta$ をみたす任意の m に対して $A/K + bF_\alpha + F_m$ の depth が $d-s-1$ 以上であれば $\mathrm{G}(\mathcal{F})$ は Cohen-Macaulay 環になる。

これで、前年度の equimultiple な $\mathcal{F}$ の研究と合わせて、解析的差異が 1 以下の場合の理論の大枠はできたと思われる。

平成 13 年度

局所環のフィルトレーション $\mathcal{F}$ で解析的差異が 1 以下のものが与えられたとき、付随する次数付環 $\mathrm{G}(\mathcal{F})$ の Cohen-Macaulay 性を判定する方法が、前年度までの研究成果として得られたので、その判定法を適用することにより、様々なフィルトレーションを実際に調べることを目標とした。具体的には次のようなものに適用してみた：

- 3 次元正則局所環のイデアル I で、ある条件を充たしている長さ 2 の正則列によって生成されるものをとったとき、I のベキ乗の整閉包がなすフィルトレーション。このとき $\mathrm{G}(\mathcal{F})$ は代数として 1 次の元で生成され、Gorenstein 環（従って Cohen-Macaulay 環）となる。
- 4 次元正則局所環のイデアル I で、ある行列の小行列式で生成されるものをとったとき、I の記号的ベキ乗がなすフィルトレーション。このとき $\mathrm{G}(\mathcal{F})$ は 1 次と 3 次の元で生成され、やはり Gorenstein 環となる。

この年度の研究で、理論の有効性についてある程度の手応えが得られたと言える。

平成 14 年度

本研究課題の最終年度にあたり、これまでに得られた結果を統合し、解析的差異が一般のフィルトレーションに関する理論の構築を目標としたが、以下に述べる様な定理が得られた。これはイデアルの随伴次数環に関する Laura Ghezzi の結果を一般化したもので、フィルトレーション版への拡張が可能なものになっている。

定理 d 次元 Cohen-Macaulay 局所環 A のイデアル I に対して次の 4 条件をみたすイデアル $J = (a_1, \ldots, a_\ell)$ と非負整数 r が存在するとせよ：(1) $I^{r+1} = JI^r$, (2) $\mathfrak{p}$ が I を含む素イデアルで $\operatorname{ht}\mathfrak{p} \le i < \ell$ ならば $I^{i-\ell+r+1}A_\mathfrak{p} = J_i I^{i-\ell+r}A_\mathfrak{p}$ (但し $J_i = (a_1, \ldots, a_i)$), (3) $\operatorname{ht} I \le i < \ell - r$ ならば $A/J_i : I$ は Cohen-Macaulay, (4) $\mathfrak{p}$ が I を含む素イデアルで $1 \le n \le r$ ならば $A_\mathfrak{p}/I^n A_\mathfrak{p}$ の depth は $\operatorname{ht}\mathfrak{p} - \ell + r - n$ 又は $r - n$ 以上である。このとき I の随伴次数環の depth は

$$\{d\} \cup \{\operatorname{depth} A/I^n + \ell - r + n\}_{1 \le n \le r}$$

の最小値以上になる。

この主張の注目すべき点は、随伴次数環の depth を評価する為の条件が局所化で保たれるということにあり、それ故に ℓ についての帰納法が可能になる。又、$\ell - \operatorname{ht} I$ は解析的差異に対応する量と見ることができる。

以上のような研究成果の中から、本報告書では下記の 4 編の論文を以下につづるものとする。

Chapter 1

Hilbert-Samuel Function and Grothendieck Group

1.1 Introduction

The purpose of this paper is to establish the theory of Hilbert-Samuel function taking values in a Grothendieck group and to introduce a generalized notion of multiplicity for arbitrary ideals in local rings. This attempt was originated by M. Fraser [4] following the treatment of M. Auslander and D. Buchsbaum [1] by the methods of homological algebra, which is an approach first suggested by J. P. Serre. However the modern theory of multiplicity was produced originally by P. Samuel and M. Nagata applying the theory of Hilbert functions to local rings, and so it should be required to look at the subject from their point of view. In this paper we try to follow Nagata's trail [11, CHAPTER III] making the theory applicable to arbitrary ideals in local rings.

Let A be a Noetherian local ring with the maximal ideal $\mathbf{m}$ such that $A/\mathbf{m}$ is infinite and let I be a proper ideal. We denote by A mod the category of finitely generated A-modules. Let $\mathrm{K}_0(A/I)$ the Grothendieck group of A/I mod . For $L \in A$ mod with $I \subseteq \sqrt{\mathrm{ann}_A L}$, we can consider the class $[L] \in \mathrm{K}_0(A/I)$ by setting $[L] = \sum_{i \geq 0}[I^i L/I^{i+1}L]$, where $[I^i L/I^{i+1}L]$ denotes the class of A/I-module $I^i L/I^{i+1}L$ in $\mathrm{K}_0(A/I)$. Thus we derive, for $M \in A$ mod , the Hilbert-Samuel function $\chi_I^M : \boldsymbol{Z} \to \mathrm{K}_0(A/I)$ with $\chi_I^M(n) = [M/I^{n+1}M]$ for $n \in \boldsymbol{Z}$. The main result Theorem 4.1 of this paper insists that there exist uniquely determined elements $\mathrm{e}_0(I, M), \mathrm{e}_1(I, M), \cdots, \mathrm{e}_\ell(I, M)$ in $\mathrm{K}_0(A/I)$, where ℓ is the analytic spread of I (cf. [12]), such that

$$(\sharp) \qquad \chi_I^M(n) = \sum_{i=0}^{\ell} \binom{n+i}{i} \mathrm{e}_i(I, M)$$

for $n \gg 0$. Let us verify that the equality above corresponds to the well-known result on the coefficients of Hilbert polynomial in the case where I is $\mathbf{m}$-primary. In fact, if I is $\mathbf{m}$-primary, there exists an isomorphism $\sigma : \mathrm{K}_0(A/I) \xrightarrow{\sim} \boldsymbol{Z}$ of groups sending $[L]$ to $\mathrm{length}_A L$ for any $L \in A$ mod with $I \subseteq \sqrt{\mathrm{ann}_A L}$. Let $e'_{d-i} = (-1)^{d-i}\sigma(\mathrm{e}_i(I, M))$ for $0 \leq i \leq d$, where $d = \dim A$ (notice that $\ell = d$ as I is $\mathbf{m}$-primary). Then, mapping the both sides of ($\sharp$) by σ, we get

$$\mathrm{length}_A M/I^{n+1}M = \binom{n+d}{d}e'_0 - \binom{n+d-1}{d-1}e'_{d-1} + \cdots + (-1)^d e'_d$$

for $n \gg 0$. Thus we may say that the elements $\mathrm{e}_i(I, M)$ for $0 \leq i \leq \ell$ given above suitably generalize the notion of the coefficients of Hilbert polynomial for $\mathbf{m}$-primary ideals. In particular we notice that the element $\mathrm{e}_\ell(I, M)$ in the "top term", which is denoted by $\mathrm{e}_I(M)$, is mapped to the ordinary multiplicity. Furthermore we shall show that in general $\mathrm{e}_\ell(I, M)$ enjoy the same properties as the ordinary multiplicity of M with respect to an $\mathbf{m}$-primary ideal. For example, if J is a reduction of I, then the group homomorphism $\mathrm{K}_0(A/I) \to \mathrm{K}_0(A/J)$ induced from the canonical surjection $A/J \to A/I$ is isomorphic, and through this isomorphism we have $\mathrm{e}_I(M) = \mathrm{e}_J(M)$. Moreover if $J = (a_1, a_2, \cdots, a_\ell)A$ is a minimal reduction of I, then $\mathrm{e}_I(M)$ is equal to the Euler-Poincaré characteristic $\chi_A(a_1, \cdots, a_\ell; M)$ of the Koszul complex $\mathrm{K.}(a_1, \cdots, a_\ell; M)$, which is essentially due to Fraser [4, 2.6]. This fact immediately implies that if a short exact sequence $0 \to L \to M \to N \to 0$ in A mod is given, then $\mathrm{e}_I(M) = \mathrm{e}_I(L) + \mathrm{e}_I(N)$. Consequently, we see that there exists a group homomorphism $\mathrm{K}_0(A) \to \mathrm{K}_0(A/I)$ sending $[M]$ to $\mathrm{e}_I(M)$ for $M \in A$ mod .

Let us here recall Fraser's notion of general multiplicity map $\mathrm{K}_0(A) \to \mathrm{K}_0(A/I)$, which is defined to be the homomorphism sending $[M]$ to $\chi_A(a_1, \cdots, a_s; M)$ for $M \in A$ mod , where $a_1, \cdots, a_s$ is a system of generators for I. Of course it is equal to the homomorphism we saw above when $s = \ell$. However, if $s > \ell$, we see by the equality ($\sharp$) that Fraser's multiplicity map is a zero map since $\chi_A(a_1, \cdots, a_s; M) = \Delta^s \chi_I^M(n)$ for $n \gg 0$ as is proved in [4, 2.6] (see also 1.4.7 and 1.4.9 of this paper), where Δ^s denotes the difference of s-th order (see Section 2). For this reason, for $M \in A$ mod , we would like to employ the element $\mathrm{e}_I(M)$ as the multiplicity of M with respect to I and then we can develop a satisfactory theory for any ideals in A with no assumptions on the number of generators.

Let us explain how to organize this paper. In Section 2 we shall collect some basic facts on Grothendieck group, Euler-Poincaré characteristic of Koszul complexes and functions from $\boldsymbol{Z}$ to an additive group. Section 3 is also devoted to a preparation. We recall the theories of superficial element and analytic spread, slightly generalizing them. Although the results in Section 2 and Section 3 may be well-known, we give the proofs for them for the completeness of this paper. In Section 4 we state the main theorem on Hilbert-Samuel functions. In Section 5 we introduce an extended notion of multiplicity. A lot of properties of ordinary multiplicity for $\mathbf{m}$-primary ideals shall be generalized here. As an easy application of the theory, we consider when the multiplicity $\mathrm{e}_I(A)$ coincide with the class $[A/I]$ in $\mathrm{K}_0(A/I)$. Finally we give an example of non-equimultiple ideal I such that $\mathrm{e}_I(A) \neq 0$, showing that, for a certain class of ideals I, the vanishing of $\mathrm{e}_I(A)$ characterize the Gorensteinness of A/I.

Throughout this paper A is a Noetherian local ring with the maximal ideal $\mathbf{m}$ such that $A/\mathbf{m}$ is infinite. The category of finitely generated A-modules is denoted by A mod . For $M \in A$ mod , $\mu_A(M)$ is the number of elements in a minimal system of generators for M and $\mathrm{Min}_A\, M$ is the set of minimal elements in $\mathrm{Supp}_A\, M$. We further set $\mathrm{Assh}_A\, M = \{Q \in \mathrm{Min}_A\, M \mid \dim A/Q = \dim_A M\}$. For an ideal I in A, we denote by $\mathrm{V}(I)$ the set of all prime ideals in A containing I.

1.2 Preliminaries

In this section we first recall some basic facts on Grothendieck groups and next develop the theory on functions mapping $\boldsymbol{Z}$ to an additive group. We further review the theory of Euler-Poincaré characteristic of Koszul complexes.

Let $\overline{M}$ be the isomorphism class of $M \in A$ mod and let $\mathrm{F}(A) = \bigoplus \boldsymbol{Z} \cdot \overline{M}$ be the free Abelian group determined by the isomorphism classes of A mod . The Grothendieck group $\mathrm{K}_0(A)$ is the factor group of $\mathrm{F}(A)$ by the subgroup generated by the elements of the form $\overline{M} - \overline{L} - \overline{N}$, where L, M and $N \in A$ mod for which there exists an exact sequence $0 \to L \to M \to N \to 0$. The class of $\overline{M}$ in $\mathrm{K}_0(A)$ for $M \in A$ mod is denoted by $[M]$. Because any $M \in A$ mod has a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r = (0)$ such that, for all $0 \leq i < r$, $M_i/M_{i+1} \cong A/Q_i$ for some $Q_i \in \mathrm{Spec}\, A$, we see that $\mathrm{K}_0(A)$ is generated by $\{[A/Q] \mid Q \in \mathrm{Spec}\, A\}$. If A is Artinian, the group homomorphism $\varphi : \boldsymbol{Z} \to \mathrm{K}_0(A)$ with $\varphi(1) = [A/\mathbf{m}]$ is isomorphic. In fact, when A is Artinian, there exists "the length function" $\mathrm{K}_0(A) \to \boldsymbol{Z}$ sending $[M]$ to $\mathrm{length}_A M$ for $M \in A$ mod , which is the inverse homomorphism of φ. Let $A \to B$ be a flat homomorphism of rings. Then there exists a group homomorphism $\mathrm{K}_0(A) \to \mathrm{K}_0(B)$ sending $[M]$ to $[M \otimes_A B]$ for $M \in A$ mod . Let $Q \in \operatorname{Spec} A$. For $\xi \in \mathrm{K}_0(A)$, we denote by ξ_Q the image of ξ by the surjective homomorphism $\mathrm{K}_0(A) \to \mathrm{K}_0(A_Q)$ induced from the canonical homomorphism $A \to A_Q$. Now we notice that the surjective group homomorphism

$$\begin{array}{ccc} \mathrm{K}_0(A) & \to & \bigoplus_{Q \in \mathrm{Min}\, A} \mathrm{K}_0(A_Q) \\ \xi & \longmapsto & (\xi_Q)_Q \end{array}$$

always splits since $\mathrm{K}_0(A_Q) \cong \boldsymbol{Z}$ for any $Q \in \mathrm{Min}\, A$. Thus we see, letting m be the number of minimal primes of A,

$$\mathrm{K}_0(A) \cong \underbrace{\boldsymbol{Z} \oplus \cdots \oplus \boldsymbol{Z}}_{m \text{ times}} \oplus \widetilde{\mathrm{K}_0(A)},$$

where $\widetilde{\mathrm{K}_0(A)}$ is the subgroup of $\mathrm{K}_0(A)$ generated by $\{[A/Q] \mid Q \in \operatorname{Spec} A \backslash \mathrm{Min}\, A\}$. When we write

$$[M] = \sum_{Q \in \operatorname{Spec} A} m_Q \cdot [A/Q] \quad (m_Q \in \boldsymbol{Z})$$

for $M \in A$ mod , we have $m_Q = \mathrm{length}_{A_Q} M_Q$ for $Q \in \mathrm{Min}\, A$. If A is a normal domain, we have a natural homomorphism $\mathrm{K}_0(A) \to \boldsymbol{Z} \bigoplus \mathrm{Cl}(A)$ sending $[M]$ to $(\mathrm{rank}_A M, \mathrm{cl}(M))$ for $M \in A$ mod ,where $\mathrm{Cl}(A)$ denotes the divisor class group of A and $\mathrm{cl}(M)$ is the divisor class attached to M (cf. [2, Chapter VII § 4.7]). Moreover this is an isomorphism if A is a 2-dimensional normal domain such that $[A/\mathbf{m}] = \mathbf{0}$ in $\mathrm{K}_0(A)$ (cf. [16, (13.3)]).

Now we look at $\mathrm{K}_0(A/I)$ for an ideal I in A, which is the main tool in our investigation. Let $L \in A$ mod such that $I \subseteq \sqrt{\mathrm{ann}_A L}$. Because $I^i L / I^{i+1} L$ is an A/I-module, we may consider it's class $[I^i L / I^{i+1} L] \in \mathrm{K}_0(A/I)$. We set

$$[L] = \sum_{i \geq 0} [I^i L / I^{i+1} L] \in \mathrm{K}_0(A/I).$$

Notice that, for $Q \in \mathrm{V}(I)$, $IA_Q \subseteq \sqrt{\mathrm{ann}_{A_Q} L_Q}$ and $[L]_Q = [L_Q]$ by definition.

Lemma 1.2.1 *Let $L \in A$ mod such that $I \subseteq \sqrt{\mathrm{ann}_A L}$. If $L = L_0 \supseteq L_1 \supseteq \cdots \supseteq L_s = (0)$ is a filtration such that $IL_j \subseteq L_{j+1}$ for all $0 \leq j < s$, then $[L] = \sum_{j=0}^{s-1} [L_j / L_{j+1}]$ in $\mathrm{K}_0(A/I)$.*

Proof. We put $N_i = I^i L$. We have $N_r = (0)$ for some $r > 0$ as $I \subseteq \sqrt{\mathrm{ann}_A L}$. For integers $0 \le i \le r$ and $0 \le j \le s$, we put $N_{ij} = (N_i \bigcap L_j) + N_{i+1}$ and $L_{ij} = (N_i \bigcap L_j) + L_{j+1}$. Then we have the filtrations

$$N_i = N_{i0} \supseteq N_{i1} \supseteq \cdots \supseteq N_{is} = N_{i+1}$$

and

$$L_j = L_{0j} \supseteq L_{1j} \supseteq \cdots \supseteq L_{rj} = L_{j+1},$$

which imply

$$[N_i/N_{i+1}] = \sum_{j=0}^{s-1} [N_{ij}/N_{i,j+1}] \quad \text{and} \quad [L_j/L_{j+1}] = \sum_{i=0}^{r-1} [L_{ij}/L_{i+1,j}]$$

in $\mathrm{K}_0(A/I)$. On the other hand, we have

$$\begin{aligned} N_{ij}/N_{i,j+1} &\cong \frac{(N_i \cap L_j) + N_{i+1}}{(N_i \cap L_{j+1}) + N_{i+1}} \\ &\cong \frac{N_i \cap L_j}{(N_i \cap L_j) \cap \{(N_i \cap L_{j+1}) + N_{i+1}\}} \end{aligned}$$

and

$$\begin{aligned} & (N_i \cap L_j) \cap \{(N_i \cap L_{j+1}) + N_{i+1}\} \\ = \; & (N_i \cap L_{j+1}) + \{(N_i \cap L_j) \cap N_{i+1}\} \\ = \; & (N_i \cap L_{j+1}) + (N_{i+1} \cap L_j) \\ = \; & \{(N_i \cap L_j) \cap L_{j+1}\} + (N_{i+1} \cap L_j) \\ = \; & (N_i \cap L_j) \cap \{L_{j+1} + (N_{i+1} \cap L_j)\}, \end{aligned}$$

so we get

$$\begin{aligned} N_{ij}/N_{i,j+1} &\cong \frac{N_i \cap L_j}{(N_i \cap L_j) \cap \{L_{j+1} + (N_{i+1} \cap L_j)\}} \\ &\cong \frac{(N_i \cap L_j) + L_{j+1}}{(N_{i+1} \cap L_j) + L_{j+1}} \\ &\cong L_{ij}/L_{i+1,j}\,. \end{aligned}$$

Therefore we get

$$\begin{aligned}[L] &= \sum_{i=0}^{r-1}[N_i/N_{i+1}] \\ &= \sum_{0\le i\le r-1, 0\le j\le s-1}[N_{ij}/N_{i,j+1}] \\ &= \sum_{0\le i\le r-1, 0\le j\le s-1}[L_{ij}/L_{i+1,j}] \\ &= \sum_{j=0}^{s-1}[L_j/L_{j+1}],\end{aligned}$$

and the proof is completed.

Let L be as in 1.2.1. If I is $\mathbf{m}$-primary, then the length function $\mathrm{K}_0(A/I) \xrightarrow{\sim} \boldsymbol{Z}$ sends $[L]$ to $\mathrm{length}_A L$. Thus we may regard the class $[\,\cdot\,]$ defined above for finitely generated A-modules annihilated by some power of I as a notion generalizing "length". Unfortunately, unless I is $\mathbf{m}$-primary, L is not necessarily (0) even if $[L] = 0$ in $\mathrm{K}_0(A/I)$. However we have the following.

Lemma 1.2.2 *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence in* A mod *such that* $I \subseteq \sqrt{\mathrm{ann}_A M}$. *Then* $[M] = [L] + [N]$ *in* $\mathrm{K}_0(A/I)$.

Proof. There exists $r > 0$ such that $I^r N = (0)$. Let $M_i = I^i N \bigcap M$ for $0 \le i \le r$. Then $L \cong M_r$ and $M_i/M_{i+1} \cong I^i N/I^{i+1}N$ for all i. Hence, in $\mathrm{K}_0(A/I)$, we have

$$\begin{aligned}[M] &= \sum_{i=0}^{r-1}[M_i/M_{i+1}] + \sum_{j\ge 0}[I^j M_r/I^{j+1}M_r] \\ &= \sum_{i=0}^{r-1}[I^i N/I^{i+1}N] + \sum_{j\ge 0}[I^j L/I^{j+1}L] \\ &= [N] + [L],\end{aligned}$$

which is the required equality.

Let $A \to B$ be a homomorphism of commutative rings such that B is module-finite over A. Regarding B-module as A-module via $A \to B$ we have a group homomorphism $\mathrm{K}_0(B) \to \mathrm{K}_0(A)$. The next result plays an important role in Section 5.

Lemma 1.2.3 *Let* J *be an ideal contained in* I *such that* $\sqrt{J} = \sqrt{I}$. *Then the homomorphism* $\mathrm{K}_0(A/I) \to \mathrm{K}_0(A/J)$ *induced from the canonical surjection* $A/J \to A/I$ *is an isomorphism.*

Proof. Let M be an A/J-module. Then as $\sqrt{I} = \sqrt{J} \subseteq \sqrt{\mathrm{ann}_A M}$, we may consider the class $[M] \in \mathrm{K}_0(A/I)$, and so we get a homomorphism $\mathrm{F}(A/J) \to \mathrm{K}_0(A/I)$. By 1.2.2 we see that it induces a homomorphism $\mathrm{K}_0(A/J) \to \mathrm{K}_0(A/I)$, which is the inverse homomorphism of $\mathrm{K}_0(A/I) \to \mathrm{K}_0(A/J)$ stated in the assertion.

We shall mainly use 1.2.3 in the case where J is a reduction of I.

Now we proceed to the next topic in this section. Let G be an additive group. For a function $f : \boldsymbol{Z} \to G$, we define it's difference $\Delta f : \boldsymbol{Z} \to G$, by setting $\Delta f(n) = f(n) - f(n-1)$ for $n \in \boldsymbol{Z}$. The i times iterated Δ-operator will be denoted by Δ^i and we further set $\Delta^0 f = f$. For functions $f, g : \boldsymbol{Z} \to G$, $f + g$ and $-f$ are functions defined by setting $(f+g)(n) = f(n) + g(n)$ and $(-f)(n) = -f(n)$ for $n \in \boldsymbol{Z}$. We write $f \equiv g$ if $f(n) = g(n)$ for all $n \gg 0$. Notice that $\Delta^k(f+g) = \Delta^k f + \Delta^k g$ and $\Delta^k(-f) = -\Delta^k f$ for all $k \geq 0$. Now we define the degree of f as follows:

$$\deg f = \begin{cases} \sup\{k \mid \Delta^k f \not\equiv 0\} & \text{if} \quad f \not\equiv 0 \\ -1 & \text{if} \quad f \equiv 0, \end{cases}$$

here we denote by 0 the function sending all $n \in \boldsymbol{Z}$ to $0 \in G$. Obviously we have $\deg \Delta f = \deg f - 1$ if $f \not\equiv 0$, $\deg(-f) = \deg f$ and $\deg(f_1 + \cdots + f_r) \leq \sup\{\deg f_1, \cdots, \deg f_r\}$.

Lemma 1.2.4 *The following conditions are equivalent for an integer $d \geq 0$ and a function $f : \boldsymbol{Z} \to G$ with $f \not\equiv 0$.*

(1) $\deg f = d$.

(2) *There are elements $\xi_0, \xi_1, \cdots, \xi_d \in G$ such that $\xi_d \neq 0$ and*

$$f(n) = \sum_{i=0}^{d} \binom{n+i}{i} \xi_i$$

for $n \gg 0$.

When this is the case, the elements $\xi_0, \xi_1, \cdots, \xi_d$ are uniquely determined by f.

Proof. (1) $\Rightarrow$ (2) We prove by induction on d. Suppose $d = 0$. Then $\Delta f \equiv 0$ and so there exists $m \in \boldsymbol{Z}$ such that $\Delta f(n) = 0$ for all $n > m$, which means $f(n) = f(m)$ for all $n > m$. Because $f \not\equiv 0$, we can choose m so that $f(m) \neq 0$. Hence, setting $\xi_0 = f(m)$, we see that the condition (2) is satisfied in this case. Let $d > 0$. Then, as $\deg \Delta f = d - 1$, by the hypothesis of induction there are elements $\xi_1, \cdots, \xi_d \in G$ and an integer $m \geq 0$ such that $\xi_d \neq 0$ and

$$\triangle f(n) = \sum_{i=0}^{d-1} \binom{n+i}{i} \xi_{i+1}$$

for $n > m$. Because

$$f(n) - f(m) = \sum_{k=m+1}^{n} \triangle f(k),$$

we have

$$f(n) = \sum_{k=0}^{n} \{ \sum_{i=0}^{d-1} \binom{k+i}{i} \xi_{i+1} \} + \xi_0$$

for $n > m$, where

$$\xi_0 = f(m) - \sum_{k=0}^{m} \{ \sum_{i=0}^{d-1} \binom{k+i}{i} \xi_{i+1} \} .$$

So, for $n > m$, we get

$$\begin{aligned} f(n) &= \sum_{i=0}^{d-1} \{ \sum_{k=0}^{n} \binom{k+i}{i} \} \xi_{i+1} + \xi_0 \\ &= \sum_{i=0}^{d-1} \binom{n+i+1}{i+1} \xi_{i+1} + \xi_0 \\ &= \sum_{i=0}^{d} \binom{n+i}{i} \xi_i . \end{aligned}$$

(2) $\Rightarrow$ (1) We prove by induction on d. If $d = 0$, then $\xi_0 \neq 0$ and $f(n) = \xi_0$ for $n \gg 0$. Hence $f \not\equiv 0$ and $\triangle f \equiv 0$, which means $\deg f = 0$. Let $d > 0$. Because we have

$$\begin{aligned} \triangle f(n) &= \sum_{i=0}^{d} \binom{n+i}{i} \xi_i - \sum_{i=0}^{d} \binom{n-1+i}{i} \xi_i \\ &= \sum_{i=0}^{d-1} \binom{n+i}{i} \xi_{i+1} \end{aligned}$$

for $n \gg 0$, $\deg \triangle f = d - 1$ by the hypothesis of induction. Hence $\deg f = d$.

The uniqueness of $\xi_0, \xi_1, \cdots, \xi_d$ is a direct consequence of the next lemma.

Lemma 1.2.5 ([4, 2.3]) *Let $\xi_0, \xi_1, \cdots, \xi_d \in G$. If*

$$\sum_{i=0}^{d} \binom{n+i}{i} \xi_i = 0$$

for all $n \gg 0$, then $\xi_0 = \xi_1 = \cdots = \xi_d = 0$.

Proof. We prove by induction on d. Because it is obvious when $d = 0$, we consider the case where $d > 0$. Then, setting

$$f(n) = \sum_{i=0}^{d} \binom{n+i}{i} \xi_i,$$

we have

$$0 = \Delta f(n) = \sum_{i=0}^{d-1} \binom{n+i}{i} \xi_{i+1}$$

for $n \gg 0$, so the hypothesis of induction implies $\xi_1 = \cdots = \xi_d = 0$. Further, substituting $\xi_1 = \cdots = \xi_d = 0$ into the equality

$$\sum_{i=0}^{d} \binom{n+i}{i} \xi_i = 0$$

for $n \gg 0$, we see $\xi_0 = 0$ too.

For a function $f : \boldsymbol{Z} \to G$ with $0 \leq \deg f = d < \infty$, we denote by $c_i(f)$ $(i = 0, 1, \cdots, d)$ the element ξ_i stated in 1.2.4. We further set $c_i(f) = 0$ for $i > d$. In the case where $f \equiv 0$, we set $c_i(f) = 0$ for all $0 \leq i \in \boldsymbol{Z}$. It is easily seen from the proof of 1.2.4 that $c_i(\Delta f) = c_{i+1}(f)$ for all $i \geq 0$. Therefore we have the following

Lemma 1.2.6 *For a function $f : \boldsymbol{Z} \to G$ with $\deg f = d$, we have $c_d(f) = \Delta^d f(n)$ for $n \gg 0$.*

Let $f : \boldsymbol{Z} \to G$ be a function and α an integer. We define a function $f[\alpha] : \boldsymbol{Z} \to G$ by setting $f[\alpha](n) = f(n + \alpha)$ for $n \in \boldsymbol{Z}$. We can easily show that $\Delta^i(f[\alpha]) = (\Delta^i f)[\alpha]$ for all $i \geq 0$. Hence we get $\deg f[\alpha] = \deg f$. Moreover we have $\deg(f - f[\alpha]) \leq \deg \Delta f$. In fact, if $\alpha < 0$, we have $g := f - f[\alpha] = \Delta f + \Delta f[-1] + \cdots + \Delta f[\alpha + 1]$ and so $\deg g \leq \sup\{\deg \Delta f[\beta] \mid \alpha < \beta \leq 0\}$, from which we get $\deg g \leq \deg \Delta f$ since $\deg \Delta f[\beta] = \deg \Delta f$ for all β. If $\alpha > 0$, then setting $h = f[\alpha]$, we have $\deg(f - f[\alpha]) = \deg(h - h[-\alpha]) \leq \deg \Delta h = \deg \Delta f$. If $\alpha = 0$, the required inequality is obvious.

Lemma 1.2.7 *Let $f : \boldsymbol{Z} \to G$ be a function with $0 \leq \deg f = d < \infty$. Let α be an integer. Then $\mathrm{c}_d(f[\alpha]) = \mathrm{c}_d(f)$.*

Proof. Let X be an indeterminate. We set

$$P_i(X) = \binom{X+\alpha+i}{i} := \frac{(X+\alpha+i)(X+\alpha+i-1)\cdots(X+\alpha+1)}{i!}$$

for $0 \leq i \leq d$. Then $P_i(X)$ is a numerical polynomial of degree i (cf. [11, Section 20]). Hence by [11, (20.8)] there are integers $a_{i0}, a_{i1}, \cdots, a_{ii}$ such that

$$P_i(X) = \sum_{j=0}^{i} a_{ij}\binom{X+j}{j}.$$

Notice that we may choose $a_{d0}, a_{d1}, \cdots, a_{dd}$ so that $a_{dd} = 1$ since

$$P_d(X) - \binom{X+d}{d}$$

is a numerical polynomial of degree $d-1$. Therefore, for $n \gg 0$, we have

$$\begin{aligned} f[\alpha](n) &= \sum_{i=0}^{d} P_i(n) \cdot \mathrm{c}_i(f) \\ &= \binom{n+d}{d}\mathrm{c}_d(f) + \sum_{j=0}^{d-1}\binom{n+j}{j}\xi_j, \end{aligned}$$

where $\xi_j = \sum_{i=j}^{d} a_{ij}\mathrm{c}_i(f)$. This implies $\mathrm{c}_d(f[\alpha]) = \mathrm{c}_d(f)$, which is the required equality.

The rest of this section is devoted to reviewing the theory of Euler-Poincaré characteristic of Koszul complexes due to Auslander-Buchsbaum [1] and Fraser [4]. Let $a_1, a_2, \cdots, a_\ell$ ($\ell \geq 1$) be elements in A. We set $I = (a_1, a_2, \cdots, a_\ell)A$. We denote by $\mathrm{H}_i(a_1, \cdots, a_\ell; M)$ the i-th homology module of the Koszul complex $\mathrm{K}.(a_1, \cdots, a_\ell; M)$. Because $I \cdot \mathrm{H}_i(a_1, \cdots, a_\ell; M) = (0)$, the class $[\mathrm{H}_i(a_1, \cdots, a_\ell; M)] \in \mathrm{K}_0(A/I)$ can be considered for any i. We set

$$\chi_A(a_1, \cdots, a_\ell; M) = \sum_{i \geq 0}(-1)^i[\mathrm{H}_i(a_1, \cdots, a_\ell; M)] \in \mathrm{K}_0(A/I)$$

and call it the Euler-Poincaré characteristic.

Proposition 1.2.8 ([1, 3.2]) *Let $0 \to L \to M \to N \to 0$ be an exact sequence in A* mod . *Then we have*

$$\chi_A(a_1, \cdots, a_\ell; M) = \chi_A(a_1, \cdots, a_\ell; L) + \chi_A(a_1, \cdots a_\ell; N).$$

Proof. The long exact sequence

$$\cdots \to \mathrm{H}_i(a_1, \cdots, a_\ell; L) \to \mathrm{H}_i(a_1, \cdots, a_\ell; M) \to$$
$$\mathrm{H}_i(a_1, \cdots, a_\ell; N) \to \mathrm{H}_{i-1}(a_1, \cdots, a_\ell; L) \to \cdots$$

derived from the short exact sequence $0 \to L \to M \to N \to 0$ implies the required equality.

By 1.2.8 we see that there exists a group homomorphism $\chi_A(a_1, \cdots, a_\ell) : \mathrm{K}_0(A) \to \mathrm{K}_0(A/I)$ sending $[M]$ to $\chi_A(a_1, \cdots, a_\ell; M)$ for $M \in A \mod$.

Proposition 1.2.9 ([1, 3.2], [4, 1.2]) *Let $M \in A \mod$. If ${a_1}^n M = (0)$ for some $n > 0$, then $\chi_A(a_1, \cdots, a_\ell; M) = 0$.*

Proof. If $\ell = 1$, we may consider the class $[M] \in \mathrm{K}_0(A/I)$, so the exact sequence $0 \to (0) :_M a_1 \to M \xrightarrow{a_1} M \to M/a_1M \to 0$ implies $\chi_A(a_1; M) = [M/a_1M] - [(0) :_M a_1] = [M] - [M] = 0$. Suppose $\ell \geq 2$. Because ${a_1}^n M = (0)$ and $I^n = {a_1}^n A + (a_2, \cdots, a_\ell)I^{n-1}$, we have, for all $i \geq 0$, $I^n \cdot \mathrm{H}_i(a_2, \cdots, a_\ell; M) = (0)$, and so we may consider the class $[\mathrm{H}_i(a_2, \cdots, a_\ell; M)] \in \mathrm{K}_0(A/I)$. We set $\xi = \sum_{i \geq 0}(-1)^i[\mathrm{H}_i(a_2, \cdots, a_\ell; M)] \in \mathrm{K}_0(A/I)$. Now considering the long exact sequence

$$\cdots \to \mathrm{H}_{i+1}(a_1, \cdots, a_\ell; M) \to \mathrm{H}_i(a_2, \cdots, a_\ell; M) \xrightarrow{a_1}$$
$$\mathrm{H}_i(a_2, \cdots, a_\ell; M) \to \mathrm{H}_i(a_1, \cdots, a_\ell; M) \to \cdots$$

we get $\chi_A(a_1, \cdots, a_\ell; M) = \xi - \xi = 0$ and the proof is completed.

Proposition 1.2.10 ([1, 3.3], [4, 1.7]) *Let $M \in A \mod$. If $\ell \geq 2$, we have*

$$\chi_A(a_1, \cdots, a_\ell; M) = \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_\ell})(\chi_A(a_1; M)),$$

where $\overline{A} = A/a_1A$ and $\overline{a_i}$ denotes the class of a_i in $\overline{A}$.

Proof. We put $L = (0) :_M a_1$ and $\overline{M} = M/a_1M$. Let us consider the exact sequence

$$0 \to L \to \mathrm{K}.(1; M) \to \mathrm{K}.(a_1; M) \to \overline{M} \to 0$$

of complexes. Applying $\cdot \bigotimes_A \mathrm{K}.(a_2, \cdots, a_\ell; A)$ to it, we get the sequence

$$0 \to \mathrm{K}.(\overline{a_2}, \cdots, \overline{a_\ell}; L) \to \mathrm{K}.(1, a_2, \cdots, a_\ell; M) \to$$
$$\mathrm{K}.(a_1, a_2, \cdots, a_\ell; M) \to \mathrm{K}.(\overline{a_2}, \cdots, \overline{a_\ell}; \overline{M}) \to 0$$

of complexes, which is also exact as $\mathrm{K}.(a_2, \cdots, a_\ell; A)$ is a free complex. We divide this sequence into two short exact sequences

$$(\sharp) \qquad 0 \to \mathrm{K}.(\overline{a_2}, \cdots, \overline{a_\ell}; L) \to \mathrm{K}.(1, a_2, \cdots, a_\ell; M) \to X. \to 0$$

and

$$(\sharp\sharp) \qquad 0 \to X. \to \mathrm{K}.(a_1, a_2, \cdots, a_\ell; M) \to \mathrm{K}.(\overline{a_2}, \cdots, \overline{a_\ell}; \overline{M}) \to 0\,.$$

Because $\mathrm{H}_i(1, a_2, \cdots, a_\ell; M) = (0)$ for all i, by $(\sharp)$ we get $\mathrm{H}_i(X.) \cong \mathrm{H}_{i-1}(\overline{a_2}, \cdots, \overline{a_\ell}; L)$ for all i, and so $\sum_i (-1)^i [H_i(X.)] = -\chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_\ell}; L)$ in $\mathrm{K}_0(A/I)$. Furthermore the long exact sequence derived from $(\sharp\sharp)$ implies $\chi_A(a_1, a_2, \cdots, a_\ell; M) = \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_\ell}; \overline{M}) + \sum_i (-1)^i [\mathrm{H}_i(X.)]$. Therefore

$$\begin{aligned} \chi_A(a_1, a_2, \cdots, a_\ell; M) &= \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_\ell}; \overline{M}) - \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_\ell}; L) \\ &= \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_\ell}; \chi_A(a_1, M)) \end{aligned}$$

as $\chi_A(a_1, M) = [\overline{M}] - [L]$ in $\mathrm{K}_0(A/a_1A)$. Thus we have completed the proof.

Proposition 1.2.11 **([4, 1.7])** *Let $0 < k < \ell$. Then the following diagram*

$$\begin{array}{ccc} \mathrm{K}_0(A) & \xrightarrow{\chi_A(a_1, \cdots, a_k)} & \mathrm{K}_0(\overline{A}) \\ \| & & \downarrow \chi_{\overline{A}}(\overline{a_{k+1}}, \cdots, \overline{a_\ell}) \\ \mathrm{K}_0(A) & \xrightarrow{\chi_A(a_1, \cdots, a_\ell)} & \mathrm{K}_0(A/I) \end{array}$$

is commutative, where $\overline{A} = A/(a_1, \cdots, a_k)A$ and $\overline{a_i}$ denotes the class of a_i in $\overline{A}$.

Proof. We prove by induction on k. If $k = 1$, we immediately get the assertion by 1.2.10. Let $2 \leq k < \ell$. We set $A' = A/(a_1, \cdots, a_{k-1})A$ and denote by $a_i{}'$ the image of a_i in A'. Then by the hypothesis of induction

$$\begin{aligned} \chi_A(a_1, \cdots, a_\ell) &= \chi_{A'}(a_k{}', a_{k+1}{}', \cdots, a_\ell{}') \circ \chi_A(a_1, \cdots, a_{k-1}) \\ &= \chi_{\overline{A}}(\overline{a_{k+1}}, \cdots, \overline{a_\ell}) \circ \chi_{A'}(a_k{}') \circ \chi_A(a_1, \cdots, a_{k-1}) \\ &= \chi_{\overline{A}}(\overline{a_{k+1}}, \cdots, \overline{a_\ell}) \circ \chi_A(a_1, \cdots, a_k)\,, \end{aligned}$$

which is the required assertion.

1.3 Superficial element and analytic spread

In this section we recall the notions of superficial element (cf. [11]) and analytic spread (cf. [12]), generalizing them slightly. Let $\mathbf{G}$ be the associated graded ring $\mathrm{G}(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$. Let $M \in A$ mod and let X be the associated graded $\mathbf{G}$-module $\mathrm{G}(I, M) = \bigoplus_{n\geq 0} I^nM/I^{n+1}M$. For an element $a \in I$, we set $a^* = a \bmod I^2 \in \mathbf{G}_1$.

Lemma 1.3.1 *Let $a \in I$. Then the following conditions are equivalent.*

(1) *There exists $c > 0$ such that $(I^{n+1}M :_M a) \bigcap I^cM = I^nM$ for all $n > c$.*

(2) *There exists $c > 0$ such that a^* is a non-zero-divisor on $X|_{\geq c} := \bigoplus_{n\geq c} I^nM/I^{n+1}M$.*

(3) *If $\mathbf{G}_+ \not\subseteq Q \in \mathrm{Ass}_{\mathbf{G}}\, X$, then $a^* \notin Q$.*

Proof. (1) $\Leftrightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Suppose $\mathbf{G}_+ \not\subseteq Q \in \mathrm{Ass}_{\mathbf{G}}\, X$. Then $(\mathbf{G}_+)_Q = \mathbf{G}_Q$ and $Q\mathbf{G}_Q \in \mathrm{Ass}_{\mathbf{G}_Q} X_Q$. On the other hand, we have $(X|_{\geq c})_Q = X_Q$ as $X|_{\geq c} = {\mathbf{G}_+}^c \cdot X$. Hence $Q\mathbf{G}_Q \in \mathrm{Ass}_{\mathbf{G}_Q} (X|_{\geq c})_Q$ and so $Q \in \mathrm{Ass}_{\mathbf{G}}\, X|_{\geq c}$, which means $a^* \notin Q$ since a^* is a non-zero-divisor on $X|_{\geq c}$ by the assumption.

(3) $\Rightarrow$ (2) Let $\mathrm{Ass}_{\mathbf{G}}\, X = \{Q_1, \cdots, Q_n\}$ and $\bigcap_{i=1}^n Z_i = (0)$ be a primary decomposition of (0) in X such that $\mathrm{Ass}_{\mathbf{G}}\, X/Z_i = \{Q_i\}$. We may assume that, for some integer m with $0 \leq m \leq n$, $\mathbf{G}_+ \subseteq Q_i$ if $1 \leq i \leq m$ and $\mathbf{G}_+ \not\subseteq Q_i$ if $m+1 \leq i \leq n$. Because $Q_i = \sqrt{Z_i :_A X}$, there exists $c > 0$ such that ${\mathbf{G}_+}^c \cdot X = X|_{\geq c} \subseteq Z_i$ for $1 \leq i \leq m$. Suppose $f \in X|_{\geq c}$ and $a^* f = 0$. Then, for $m+1 \leq i \leq n$, we have $f \in Z_i$ since $a^* \notin Q_i$ and $\mathrm{Ass}_{\mathbf{G}}\, X/Z_i = \{\, Q_i \,\}$. Consequently, we see $f \in (X|_{\geq c}) \bigcap Z_{m+1} \bigcap \cdots \bigcap Z_n \subseteq \bigcap_{i=1}^n Z_i = (0)$, so $f = 0$. Therefore a^* is a non-zero-divisor on $X|_{\geq c}$ and the proof is completed.

We say that $a \in I$ is a superficial element of I with respect to M if one of the conditions of 1.3.1 is satisfied. Because we assume that $A/\mathbf{m}$ is infinite, the existence of a superficial element is always guaranteed by the condition (3) of 1.3.1.

Lemma 1.3.2 *Let a be a superficial element of I with respect to M. Then, for $n \gg 0$, we have*

(1) $aM \bigcap I^nM = aI^{n-1}M$,

(2) $I^{n+1}M :_M a = ((0) :_M a) + I^nM$,

(3) $((0) :_M a) \bigcap I^n M = (0)$ *and*

(4) $I^n((0) :_M a) = (0)$.

Proof. Let $c > 0$ be an integer such that $(I^{n+1}M :_M a) \bigcap I^c M = I^n M$ for any $n > c$.

(1) By Artin-Rees lemma, there exists a positive integer r such that $I^n M \bigcap aM = I^{n-r}(I^r M \bigcap aM) \subseteq aI^{n-r}M$ for all $n > r$. If $n > r + c$, then $I^{n-r}M \subseteq I^c M$, and so $I^n M \bigcap aM \subseteq I^n M \bigcap aI^c M = a((I^n M :_M a) \bigcap I^c M) = aI^{n-1}M$.

(2) Let $x \in I^{n+1}M :_M a$ for $n \gg 0$. Then $ax \in I^{n+1}M \bigcap aM = aI^n M$ by (1). Hence $ax = ay$ for some $y \in I^n M$, which implies $x - y \in (0) :_M a$ and so $x \in ((0) :_M a) + I^n M$. Thus we get $I^{n+1}M :_M a = ((0) :_M a) + I^n M$ since the right hand side is obviously contained in the left hand side.

(3) For any integer $n > c$, we have $((0) :_M a) \bigcap I^c M \subseteq (I^{n+1}M :_M a) \bigcap I^c M = I^n M$. Hence $((0) :_M a) \bigcap I^c M \subseteq \bigcap_{n>c} I^n M = (0)$.

(4) We have $I^c((0) :_M a) \subseteq ((0) :_M a) \bigcap I^c M = (0)$ by (3).

Lemma 1.3.3 *Let a be a superficial element of I with respect to M. Then the sequence*

$$0 \to (0) :_M a \to M/I^n M \xrightarrow{a} M/I^{n+1} \to \overline{M}/I^{n+1}\overline{M} \to 0$$

is exact for $n \gg 0$, where $\overline{M} = M/aM$.

Proof. Let us consider the exact sequence

$$0 \to I^{n+1}M :_M a/I^n M \to M/I^n M \xrightarrow{a} M/I^{n+1}M \to \overline{M}/I^{n+1}\overline{M} \to 0\,.$$

By 1.3.2, for $n \gg 0$, we have

$$\begin{aligned} I^{n+1}M :_M a/I^n M &= \frac{((0) :_M a) + I^n M}{I^n M} \\ &\cong \frac{(0) :_M a}{((0) :_M a) \cap I^n M} \\ &\cong (0) :_M a\,. \end{aligned}$$

Hence we get the required exact sequence.

We denote by $\ell(I, M)$ the Krull dimension of the $\mathbf{G}$-module $X/\mathbf{m}X$. In particular we write $\ell(I) = \ell(I, A)$, which is called the analytic spread of I (cf. [12]). In general, we have $0 \leq \ell(I, M) \leq \ell(I)$. Because we are assuming that $A/\mathbf{m}$ is infinite, $\ell(I) = \mu_A(J)$

for any minimal reduction J of I. The inequalities $\mathrm{ht}_A\, I \leq \ell(I) \leq \min\{\dim A, \mu_A(I)\}$ are valid for any ideal I in A. Hence if I is $\mathbf{m}$-primary, $\ell(I) = \dim A$. We note for future use that if $I = (a_1, \cdots, a_\ell)A$ and $\ell(I) = \ell$, then $\ell((a_1, \cdots, a_k)A) = k$ for $0 \leq k \leq \ell$. In fact, setting $K = (a_1, \cdots, a_k)A$ and $L = (a_{k+1}, \cdots, a_\ell)A$, we have $\ell(I) \leq \ell(K) + \ell(L)$ (cf. [12, §8 LEMMA 1]), $\ell(K) \leq k$ and $\ell(L) \leq \ell - k$, which imply $\ell(K) = k$. In particular, if $a_1, \cdots, a_k$ is a subsystem of parameters (ssop) for A, then $\ell((a_1, \cdots, a_k)A) = k$. We further notice that if $I = (a_1, \cdots, a_\ell)A$ and $a_1, \cdots, a_\ell$ is a d-sequence on A (cf. [7]), then $\ell(I) = \ell$ because by [8, 3.1] $\mathbf{G}/\mathbf{mG}$ is isomorphic to a polynomial ring over $A/\mathbf{m}$ with ℓ variables.

Lemma 1.3.4 *If $\ell(I, M) = 0$, then $I \subseteq \sqrt{\mathrm{ann}_A M}$.*

Proof. Because $\ell(I, M) = 0$, we have $\sqrt{\mathbf{mX} :_{\mathbf{G}} \mathbf{X}} = \mathbf{mG} + \mathbf{G}_+$. Hence $\mathbf{G}_+{}^n \cdot X \subseteq \mathbf{mX}$ for some $n > 0$. Then, looking at the n-th homogeneous components, we get $I^n M = \mathbf{mI^nM}$ and so $I^n M = (0)$ by Nakayama's lemma.

Lemma 1.3.5 *Suppose $\ell(I, M) > 0$. Then there exists an element $a \in I$ satisfying the following conditions:*

(1) *a is a part of a minimal system of generators for I.*

(2) *a is a superficial element of I with respect to M.*

(3) *$\ell(I, \overline{M}) = \ell(I, M) - 1$, where $\overline{M} = M/aM$.*

Proof. Let $\varphi : I \to \mathbf{G}$ be the A-linear map such that $\varphi(a) = a^*$ for $a \in I$. Let $\mathcal{F} = \{Q \in \mathrm{Ass}_{\mathbf{G}}\, X \mid \mathbf{G}_+ \not\subseteq Q\}$. For $Q \in \mathcal{F}$, we set $\mathrm{V}(Q) = \varphi^{-1}(Q) + \mathbf{mI}/\mathbf{mI}$, which is an $A/\mathbf{m}$-subspace of $I/\mathbf{mI}$. We notice $\mathrm{V}(Q) \neq I/\mathbf{mI}$. In fact, if $\mathrm{V}(Q) = I/\mathbf{mI}$, we have $I = \varphi^{-1}(Q)$ and so $\mathbf{G}_+ \subseteq Q$ as the image of φ is $\mathbf{G}_1$. But this contradicts to $Q \in \mathcal{F}$. Next, for $P \in \mathrm{Assh}_{\mathbf{G}}\, X/\mathbf{mX}$, we set $\mathrm{W}(P) = \varphi^{-1}(P) + \mathbf{mI}/\mathbf{mI}$. Suppose $\mathrm{W}(P) = I/\mathbf{mI}$. Then $I = \varphi^{-1}(P)$ and so $P = \mathbf{mG} + \mathbf{G}_+$, which is the graded maximal ideal of $\mathbf{G}$. Hence $\dim \mathbf{G}/P = 0$. But this contradicts to the assumption that $\ell(I, M) > 0$. Consequently $\mathrm{W}(P) \neq I/\mathbf{mI}$. Because we are assuming that $A/\mathbf{m}$ is infinite, we can choose an element $a \in I$ so that

(i) $\overline{a} \notin \mathrm{V}(Q)$ for any $Q \in \mathcal{F}$ and

(ii) $\overline{a} \notin \mathrm{W}(P)$ for any $P \in \mathrm{Assh}_{\mathbf{G}}\, X/\mathbf{mX}$, where $\overline{a}$ is the class of a in $I/\mathbf{mI}$. Then, as $\overline{a} \neq 0$, a is a part of a minimal system of generators for I. Moreover we have $a^* \notin Q$ for any $Q \in \mathcal{F}$ by (i), so a is a superficial element of I with respect to M. By (ii) we see $a^* \notin P$ for any $P \in \mathrm{Assh}_{\mathbf{G}} X/\mathbf{mX}$, so a^* is a ssop for $X/\mathbf{mX}$. Hence, in order to see $\ell(I, \overline{M}) = \ell(I, M) - 1$, it is enough to show $\dim_{\mathbf{G}} X/\mathbf{mX} + \mathbf{a^*X} = \dim_{\mathbf{G}} \mathrm{G}(\mathbf{I}, \overline{\mathbf{M}})/\mathbf{m}\mathrm{G}(\mathbf{I}, \overline{\mathbf{M}})$. In fact, as $[X/\mathbf{mX} + \mathbf{a^*X}]_{\mathbf{n}} = \mathbf{I^nM}/\mathbf{mI^nM} + \mathbf{aI^{n-1}M}$ and $[\mathrm{G}(I, \overline{M})/\mathbf{m}\mathrm{G}(\mathbf{I}, \overline{\mathbf{M}})]_{\mathbf{n}} = \mathbf{I^nM} + \mathbf{aM}/\mathbf{mI^nM} + \mathbf{aM}$ for any $n \in \boldsymbol{Z}$, there exists the canonical epimorphism $\rho : X/\mathbf{mX} + \mathbf{a^*X} \to \mathrm{G}(\mathbf{I}, \overline{\mathbf{M}})/\mathbf{m}\mathrm{G}(\mathbf{I}, \overline{\mathbf{M}})$ of graded $\mathbf{G}$-modules. Then we have

$$[\operatorname{Ker} \rho]_n \cong \frac{I^nM \cap (\mathbf{mI^nM} + \mathbf{aM})}{\mathbf{mI^nM} + \mathbf{aI^{n-1}M}} = \frac{\mathbf{mI^nM} + \mathbf{I^nM} \cap \mathbf{aM}}{\mathbf{mI^nM} + \mathbf{aI^{n-1}M}}.$$

Hence, for $n \gg 0$, we have $[\operatorname{Ker} \rho]_n = (0)$ since $I^nM \bigcap aM = aI^{n-1}M$ by 1.3.2. Therefore $\mathrm{length}_{\mathbf{G}} \operatorname{Ker} \rho < \infty$, and so we get the required equality of Krull dimensions. This completes the proof of 1.3.5.

1.4 Hilbert-Samuel function

For $M \in A$ mod , we define the function $\chi_I^M : \boldsymbol{Z} \to \mathrm{K}_0(A/I)$ by setting $\chi_I^M(n) = [M/I^{n+1}M]$ and call it the Hilbert-Samuel function of M with respect to I. We simply denote χ_I^A by χ_I.

Theorem 1.4.1 *Let $M \in A$ mod . Then*

$$\max\{\dim_{A_Q} M_Q \mid Q \in \mathrm{Min}_A A/I\} \leq \deg \chi_I^M \leq \ell(I, M).$$

In particular we have

$$\mathrm{ht}_A I \leq \deg \chi_I \leq \ell(I).$$

Here we notice that $\dim_A M = -\infty$ if $M = (0)$.

Lemma 1.4.2 *If $\deg \chi_I^M \leq 0$, then $\dim_{A_Q} M_Q \leq \deg \chi_I^M$ for any $Q \in \mathrm{Min}_A A/I$.*

Proof. Let us first consider the case where $\deg \chi_I^M = -1$. This means $\chi_I^M \equiv 0$, so $[M/I^nM] = 0$ in $\mathrm{K}_0(A/I)$ for $n \gg 0$. Hence if $Q \in \mathrm{Min}_A A/I$, we have, for $n \gg 0$, $M_Q = I^nM_Q$ as $[M_Q/I^nM_Q] = 0$ in $\mathrm{K}_0(A_Q/IA_Q)$ and as A_Q/IA_Q is Artinian, so

$M_Q = (0)$ by Nakayama's lemma. Suppose next $\deg \chi_I^M = 0$. Then as $\Delta\chi_I^M \equiv 0$, $[M/I^{n+1}M] = [M/I^nM]$ in $\mathrm{K}_0(A/I)$ for $n \gg 0$. Hence if $Q \in \mathrm{Min}_A A/I$ and $n \gg 0$, we have $I^{n+1}M_Q = I^nM_Q$ and so $I^nM_Q = (0)$, which means $\dim_{A_Q} M_Q \leq 0$. Thus we have proved the required assertion.

Proof of 1.4.1. We prove by induction on $\ell(I, M)$. Let $\ell(I, M) = 0$. Then $I^nM = (0)$ for $n \gg 0$ by 1.3.4, so $\chi_I^M(n) = [M]$ for $n \gg 0$, which implies $\deg \chi_I^M \leq 0$. Hence we get the required inequalities by 1.4.2. Let now $\ell(I, M) > 0$. Again by 1.4.2 it is enough to consider the case where $\deg \chi_I^M \geq 1$. By 1.3.5 we can choose an element $a \in I$ so that a is a superficial element of I with respect to M and $\ell(I, \overline{M}) = \ell(I, M) - 1$, where $\overline{M} = M/aM$. Then, by 1.3.3, we have an exact sequence

$$0 \to (0) :_M a \to M/I^nM \xrightarrow{a} M/I^{n+1}M \to \overline{M}/I^{n+1}\overline{M} \to 0$$

for $n \gg 0$. Notice that the class $[(0) :_M a]$ can be defined in $\mathrm{K}_0(A/I)$ by (4) of 1.3.2. Thus $\chi_I^{\overline{M}}(n) = \Delta\chi_I^M(n) + [(0) :_M a]$ for $n \gg 0$, and so $\chi_I^{\overline{M}} \equiv \Delta\chi_I^M + f$, where $f : \boldsymbol{Z} \to \mathrm{K}_0(A/I)$ is the constant function such that $f(n) = [(0) :_M a]$ for any $n \in \boldsymbol{Z}$. Because $\deg \chi_I^M \geq 1$ and $\deg f \leq 0$, we see

$$\deg \chi_I^{\overline{M}} = \begin{cases} \deg \chi_I^M - 1 & \text{if } \deg \chi_I^M \geq 2 \\ -1 \text{ or } 0 & \text{if } \deg \chi_I^M = 1\,. \end{cases}$$

Let $Q \in \mathrm{Min}_A A/I$. Then, by the hypothesis of induction, we have $\dim_{A_Q} \overline{M}_Q \leq \deg \chi_I^{\overline{M}}$, and so

$$\begin{aligned} \dim_{A_Q} M_Q &\leq \dim_{A_Q} \overline{M}_Q + 1 \\ &\leq \deg \chi_I^{\overline{M}} + 1 \\ &\leq \deg \chi_I^M\,. \end{aligned}$$

Moreover, when $\deg \chi_I^M \geq 2$, we get

$$\begin{aligned} \deg \chi_I^M &= \deg \chi_I^{\overline{M}} + 1 \\ &\leq \ell(I, \overline{M}) + 1 \\ &= \ell(I, M)\,. \end{aligned}$$

Because we are assuming $\ell(I, M) > 0$, the inequality $\deg \chi_I^M \leq \ell(I, M)$ holds obviously if $\deg \chi_I^M = 1$. Thus we have completed the proof.

Definition 1.4.3 *Let $M \in A$ mod . We set* $\mathrm{e}_i(I, M) = \mathrm{c}_i(\chi_I^M) \in \mathrm{K}_0(A/I)$ *for* $i \geq 0$. *Then* $\mathrm{e}_i(I, M) = 0$ *for* $i > \ell(I, M)$ *and*

$$\chi_I^M(n) = \sum_{i \geq 0} \binom{n+i}{i} \mathrm{e}_i(I, M)$$

in $\mathrm{K}_0(A/I)$ *for* $n \gg 0$.

Proposition 1.4.4 *Let $M \in A$ mod . Then we have the following assertions.*

(1) *Let a be a superficial element of I with respect to M. We set $\overline{M} = M/aM$. Then* $\mathrm{e}_i(I, \overline{M}) = \mathrm{e}_{i+1}(I, M)$ *for any* $i \geq 1$ *and* $\mathrm{e}_0(I, \overline{M}) = \mathrm{e}_1(I, M) + [(0) :_M a]$.

(2) $\mathrm{e}_i(I, M)_Q = \mathrm{e}_i(IA_Q, M_Q)$ *for any* $Q \in \mathrm{V}(I)$.

Proof. (1) Let $n \gg 0$. Then, by 1.3.3, there exists an exact sequence

$$0 \to (0) :_M a \to M/I^n M \xrightarrow{a} M/I^{n+1}M \to \overline{M}/I^{n+1}\overline{M} \to 0\,,$$

from which we see

$$\begin{aligned}
\chi_I^{\overline{M}}(n) &= \chi_I^M(n) - \chi_I^M(n-1) + [(0) :_M a] \\
&= \sum_{j \geq 0} \{ \binom{n+j}{j} - \binom{n-1+j}{j} \} \mathrm{e}_j(I, M) + [(0) :_M a] \\
&= \sum_{j \geq 1} \binom{n+j-1}{j-1} \mathrm{e}_j(I, M) + [(0) :_M a] \\
&= \sum_{i \geq 1} \binom{n+i}{i} \mathrm{e}_{i+1}(I, M) + (\mathrm{e}_1(I, M) + [(0) :_M a])\,.
\end{aligned}$$

Thus we get the required equalities.

(2) Because

$$[M/I^{n+1}M] = \sum_{i \geq 0} \binom{n+i}{i} \mathrm{e}_i(I, M)$$

for $n \gg 0$ and the localization $\mathrm{K}_0(A/I) \to \mathrm{K}_0(A_Q/IA_Q)$ is a group homomorphism, we have

$$\chi_{IA_Q}^{M_Q}(n) = [M/I^{n+1}M]_Q = \sum_{i \geq 0} \binom{n+i}{i} \mathrm{e}_i(I, M)_Q$$

for $n \gg 0$. Hence $\mathrm{e}_i(IA_Q, M_Q) = \mathrm{e}_i(I, M)_Q$ for any $i \geq 0$.

Corollary 1.4.5 *Let $M \in A$ mod and $Q \in \mathrm{V}(I)$. If $\mathrm{e}_i(I, M)_Q \neq 0$, then $i \leq \mathrm{ht}_A Q$.*

Proof. Suppose $\mathrm{e}_i(I, M)_Q \neq 0$. Then, by 1.4.1 and (2) of 1.4.4, we have $i \leq \ell(IA_Q, M_Q) \leq \ell(IA_Q) \leq \dim A_Q = \mathrm{ht}_A Q$. Thus the required inequality follows.

Proposition 1.4.6 ([4, 3.1]) *Let I be generated by an M-regular sequence of length m. Then $\mathrm{e}_m(I, M) = [M/IM]$ and $\mathrm{e}_i(I, M) = 0$ for any $i \neq m$.*

Proof. Let $i \geq 0$. Because $I^iM/I^{i+1}M$ is isomorphic to the direct sum of $\binom{i+m-1}{m-1}$ copies of M/IM, we have

$$[I^iM/I^{i+1}M] = \binom{i+m-1}{m-1}[M/IM]$$

in $\mathrm{K}_0(A/I)$. Then

$$\begin{aligned} \chi_I^M(n) &= \sum_{i=0}^{n}[I^iM/I^{i+1}M] \\ &= \{\sum_{i=0}^{n}\binom{i+m-1}{m-1}\}[M/IM] \\ &= \binom{n+m}{m}[M/IM], \end{aligned}$$

and so we get the required assertion.

The following result is due to Fraser [4]. We will give another proof using superficial element.

Proposition 1.4.7 ([4, 2.6]) *Let I be minimally generated by $a_1, a_2, \cdots, a_m$. Then for any $M \in A$ mod we have $\Delta^m \chi_I^M(n) = \chi_A(a_1, \cdots, a_m; M)$.*

We need the following

Lemma 1.4.8 *Let $M \in A$ mod, $\ell \geq 2$ and a_1 a superficial element of I with respect to M. Then*

$$\chi_A(a_1, a_2, \cdots, a_\ell; M) = \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_\ell}; \overline{M}),$$

where $\overline{A} = A/a_1A, \overline{M} = M/a_1M$ and $\overline{a_i}$ is the class of a_i in $\overline{A}$.

Proof. By 1.2.10 we have

$$\chi_A(a_1, a_2, \cdots, a_\ell; M) = \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_\ell}; \overline{M}) - \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_\ell}; (0) :_M a_1)\,.$$

Because ${a_2}^n((0) :_M a_1) = (0)$ for some $n > 0$ by (4) of 1.3.2, the required equality holds by 1.2.9.

Proof of 1.4.7. Let us prove by induction on m. Suppose first $m = 1$. Then by (3) of 1.3.1 a_1 is a superficial element of $I = a_1 A$ with respect to M. Hence, for $n \gg 0$,

$$0 \to (0) :_M a_1 \to M/{a_1}^n M \xrightarrow{a_1} M/{a_1}^{n+1} M \to M/a_1 M \to 0$$

is exact by 1.3.3, and so $\Delta\chi_I^M(n) = [M/a_1 M] - [(0) :_M a_1] = \chi_A(a_1; M)$. Let next $m \geq 2$. By 1.3.5 there exists a superficial element a of I with respect to M such that a is a part of a minimal system of generators for I. Suppose that $b_2, \cdots, b_m$ are elements with $I = (a, b_2, \cdots, b_m)A$. Then, as $\mathrm{H}_i(a_1, a_2, \cdots, a_m; M) \cong \mathrm{H}_i(a, b_2, \cdots, b_m; M)$ for all i, $\chi_A(a_1, a_2, \cdots, a_m; M) = \chi_A(a, b_2, \cdots, b_m; M)$. Consequently, we may assume $a_1 = a$. We set $\overline{A} = A/a_1 A$, $\overline{M} = M/a_1 M$ and denote by $\overline{a_i}$ the class of a_i in $\overline{A}$. Now, again by 1.3.3, the sequence

$$0 \to (0) :_M a_1 \to M/I^n M \xrightarrow{a_1} M/I^{n+1} M \to \overline{M}/I^{n+1}\overline{M} \to 0$$

is exact for $n \gg 0$, which means $\Delta\chi_I^M \equiv \chi_{I\overline{A}}^{\overline{M}} - f$ where $f : \boldsymbol{Z} \to \mathrm{K}_0(A/I)$ is the constant function with $f(n) = [(0) :_M a_1]$ for any $n \in \boldsymbol{Z}$. Therefore, for $n \gg 0$, we get

$$\begin{aligned}
\Delta^m \chi_I^M(n) &= \Delta^{m-1}(\chi_{I\overline{A}}^{\overline{M}} - f) \\
&= \Delta^{m-1}\chi_{I\overline{A}}^{\overline{M}}(n) && \text{(as } m \geq 2 \text{ and } \deg f = 0) \\
&= \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_m}; \overline{M}) && \text{(by the hypothesis of induction)} \\
&= \chi_A(a_1, a_2, \cdots, a_m; M) && \text{(by 1.4.8)}
\end{aligned}$$

and the proof is completed.

Corollary 1.4.9 *Let $M \in A$ mod . If I is minimally generated by $a_1, a_2, \cdots, a_m$ and $\ell(I, M) < m$, Then $\chi_A(a_1, \cdots, a_m; M) = 0$.*

Proof. By 1.4.1 the assumption implies $\deg \chi_I^M < m$, so $\Delta^m \chi_I^M \equiv 0$. Hence we get the assertion by 1.4.7.

1.5 Multiplicity

In this section we concentrate our attention on the "top term" in the expression of a Hilbert-Samuel function using binomial coefficients. Throughout this section $d = \dim A, \ell = \ell(I)$ and $M \in A \mod$.

Definition 1.5.1 *We set* $\mathrm{e}_I(M) = \mathrm{e}_\ell(I, M)$ *and call it the multiplicity of* M *with respect to* I.

Proposition 1.5.2 $\mathrm{e}_I(M) = \Delta^\ell \chi_I^M(n)$ *for* $n \gg 0$. *Hence* $\mathrm{e}_I(M) = 0$ *if* $\ell(I, M) < \ell$.

Proof. This follows immediately from 1.2.6.

Proposition 1.5.3 *Let* $m \geq 1$. *Then, identifying* $\mathrm{K}_0(A/I)$ *with* $\mathrm{K}_0(A/I^m)$ *through the isomorphism* $\mathrm{K}_0(A/I) \xrightarrow{\sim} \mathrm{K}_0(A/I^m)$ *induced from the canonical surjection* $A/I^m \to A/I$, *we get* $\mathrm{e}_{I^m}(M) = m^\ell \cdot \mathrm{e}_I(M)$.

Proof. Let us denote by σ the isomorphism $\mathrm{K}_0(A/I) \xrightarrow{\sim} \mathrm{K}_0(A/I^m)$. We notice that $\sigma(\chi_I^M(mn + m - 1) = \chi_{I^m}^M(n)$ for any $n \geq 1$. Let X be an indeterminate. We set

$$\mathrm{F}_i(X) = \binom{mX + m - 1 + i}{i}$$

for $0 \leq i \leq \ell$. Then $\mathrm{F}_i(X)$ is a numerical polynomial of degree i. Hence by [11, (20.8)] there exist integers $a_{i0}, a_{i1}, \cdots, a_{ii}$ such that

$$\mathrm{F}_i(X) = \sum_{j=0}^{i} a_{ij} \binom{X + j}{j}.$$

In particular we can choose $a_{\ell 0}, a_{\ell 1}, \cdots, a_{\ell\ell}$ so that $a_{\ell\ell} = m^\ell$ since

$$\mathrm{G}(X) := \mathrm{F}_\ell(X) - m^\ell \binom{X + \ell}{\ell}$$

is a numerical polynomial of degree $\ell - 1$. Sending by σ the both sides of the equality

$$\chi_I^M(mn + m - 1) = \sum_{i=0}^{\ell} F_i(n) \cdot \mathrm{e}_i(I, M)$$

in $\mathrm{K}_0(A/I)$ for $n \gg 0$, we have

$$\begin{aligned} \chi_{I^m}^M(n) &= \sum_{i=0}^{\ell} \mathrm{F}_i(n) \cdot \sigma(\mathrm{e}_i(I, M)) \\ &= \sum_{0 \leq j \leq i \leq \ell} a_{ij} \binom{n + j}{j} \cdot \sigma(\mathrm{e}_i(I, M)) \\ &= \binom{n + \ell}{\ell} \cdot m^\ell \cdot \sigma(\mathrm{e}_I(M)) + \sum_{j=0}^{\ell-1} \binom{n + j}{j} \cdot \xi_j, \end{aligned}$$

where $\xi_j = \sum_{i=j}^{\ell} a_{ij}\sigma(\mathrm{e}_i(I,M))$. Therefore we get $\mathrm{e}_{I^m}(M) = m^\ell \cdot \sigma(\mathrm{e}_I(M))$ since $\ell(I^m) = \ell(I) = \ell$, which is the required assertion.

Proposition 1.5.4 *Let $I = (a_1, \cdots, a_\ell)A$ and $a_1, \cdots, a_\ell$ is an M-regular sequence. Then* $\mathrm{e}_I(M) = [M/IM]$.

Proof. This follows immediately from 1.4.6.

Proposition 1.5.5 *Let $Q \in \mathrm{V}(I)$. If $\ell(IA_Q) = m$, then $\mathrm{e}_{IA_Q}(M_Q) = \mathrm{e}_m(I,M)_Q$.*

Proof. By definition $\mathrm{e}_{IA_Q}(M_Q) = \mathrm{e}_m(IA_Q, M_Q)$. Hence the assertion follows from (2) of 1.4.4.

Let us denote by $\mathrm{e}'_I(M)$ the ordinary multiplicity of M with respect to an $\mathbf{m}$-primary ideal I. Then, as is noticed in the introduction, when I is $\mathbf{m}$-primary, $\mathrm{e}_I(M)$ is sent to $\mathrm{e}'_I(M)$ by the length function $\mathrm{K}_0(A/I) \xrightarrow{\sim} \boldsymbol{Z}$. More generally we have the following.

Lemma 1.5.6 *Let $Q \in \mathrm{Min}_A\ A/I$ with $\mathrm{ht}_A\ Q = s$. Let*

$$\mathrm{e}_s(I,M) = \sum_{P \in \mathrm{V}(I)} m_P \cdot [A/P] \quad (m_P \in \boldsymbol{Z})$$

in $\mathrm{K}_0(A/I)$. Then $m_Q = \mathrm{e}'_{IA_Q}(M_Q)$.

Proof. Because $\ell(IA_Q) = \dim A_Q = s$, $\mathrm{e}_{IA_Q}(M_Q) = \mathrm{e}_s(I,M)_Q$ by 1.5.5. On the other hand, $\mathrm{e}_s(I,M)_Q = m_Q \cdot [A_Q/QA_Q]$ by the assumption. Thus $\mathrm{e}_{IA_Q}(M_Q) = m_Q \cdot [A_Q/QA_Q]$. Sending the both sides of this equality by the length function $\mathrm{K}_0(A_Q/IA_Q) \xrightarrow{\sim} \boldsymbol{Z}$, we get the required assertion.

Lemma 1.5.7 *Let N be an A-submodule of M such that $I \subseteq \sqrt{\mathrm{ann}_A M/N}$. If $\ell > 0$, we have* $\mathrm{e}_I(M) = \mathrm{e}_I(N)$.

Proof. By the lemma of Artin-Rees, there exists an integer $r > 0$ such that $I^n M \bigcap N = I^{n-r}(I^r M \bigcap N)$ for any $n > r$. Choosing r as big as enough, we may assume $I^r M \subseteq N$. Then $I^n M \bigcap N = I^{n-r} N$ for any $n > r$. Now we consider, for $n > r$, the exact sequence

$$0 \to N/I^{n-r}N \to M/I^n M \to M/N \to 0,$$

which implies $\chi_I^M \equiv \chi_I^N[-r] + f$ where $f : \boldsymbol{Z} \to \mathrm{K}_0(A/I)$ is the constant function such that $f(n) = [M/N]$ for any $n \in \boldsymbol{Z}$. Therefore we have

$$\begin{aligned} \mathrm{e}_I(M) &= \mathrm{c}_\ell(\chi_I^M) \\ &= \mathrm{c}_\ell(\chi_I^N[-r] + f) \quad \text{(by 1.2.5)} \\ &= \mathrm{c}_\ell(\chi_I^N[-r]) \quad \text{(as } \ell > 0) \\ &= \mathrm{c}_\ell(\chi_I^N) \quad \text{(by 1.2.7)} \\ &= \mathrm{e}_I(N), \end{aligned}$$

and the proof is completed.

Proposition 1.5.8 *Let J be a reduction of I. Then via the isomorphism $\mathrm{K}_0(A/I) \xrightarrow{\sim} \mathrm{K}_0(A/J)$ induced from the canonical surjection $A/J \to A/I$, we have $\mathrm{e}_I(M) = \mathrm{e}_J(M)$.*

Proof. We denote by σ the isomorphism $\mathrm{K}_0(A/I) \xrightarrow{\sim} \mathrm{K}_0(A/J)$. If $\ell = 0$, we have $\sigma(\mathrm{e}_I(M)) = \mathrm{e}_J(M) = [M]$. Let $\ell > 0$ and let $r > 0$ be an integer with $I^{r+1} = JI^r$. Then, for any $n \gg 0$, we have $[M/I^{n+r}M] = [M/J^nI^rM] = [M/I^rM] + [I^rM/J^nI^rM]$ in $\mathrm{K}_0(A/J)$, which means $\sigma \circ \chi_I^M[r] \equiv \chi_J^{I^rM} + f$ where $f : \boldsymbol{Z} \to \mathrm{K}_0(A/J)$ is the constant function with $f(n) = [M/I^rM]$ for any $n \in \boldsymbol{Z}$. Therefore we have

$$\begin{aligned} \sigma(\mathrm{e}_I(M)) &= \sigma(\mathrm{c}_\ell(\chi_I^M)) \\ &= \sigma(\mathrm{c}_\ell(\chi_I^M[r])) \quad \text{(by 1.2.7)} \\ &= \mathrm{c}_\ell(\sigma \circ \chi_I^M[r]) \\ &= \mathrm{c}_\ell(\chi_J^{I^rM} + f) \quad \text{(by 1.2.5)} \\ &= \mathrm{c}_\ell(\chi_J^{I^rM}) \quad \text{(as } \ell > 0) \\ &= \mathrm{e}_J(I^rM) \\ &= \mathrm{e}_J(M) \quad \text{(by 1.5.5).} \end{aligned}$$

Thus we get the required equality.

By virtue of 1.4.7 and 1.5.8, we immediately get the following.

Theorem 1.5.9 *Let $\ell \geq 1$ and $J = (a_1, a_2, \cdots a_\ell)A$ be a minimal reduction of I. Then $\mathrm{e}_I(M) = \chi_A(a_1, \cdots, a_\ell; M)$ via the isomorphism $\mathrm{K}_0(A/I) \xrightarrow{\sim} \mathrm{K}_0(A/J)$.*

Corollary 1.5.10 ([4, 1.12]) *Let $a_1, a_2, \cdots, a_m$ be elements in* $\mathbf{m}$. *Then for any positive integers $n_1, n_2, \cdots, n_m$, we have*

$$\chi_A(a_1{}^{n_1}, a_2{}^{n_2}, \cdots, a_m{}^{n_m}; M) = n_1 n_2 \cdots n_m \cdot \chi_A(a_1, a_2, \cdots, a_m; M)$$

through the isomorphism $\mathrm{K}_0(A/(a_1, a_2, \cdots, a_m)A) \xrightarrow{\sim} \mathrm{K}_0(A/(a_1{}^{n_1}, a_2{}^{n_2}, \cdots, a_m{}^{n_m})A)$.

Proof. It is enough to show $\chi_A(a_1{}^n, a_2, \cdots, a_m; M) = n \cdot \chi_A(a_1, a_2, \cdots, a_m; M)$ for any $n > 0$. Suppose first $m = 1$. Notice that $\ell(a_1{}^n A) = \ell(a_1 A)$. If $\ell(a_1 A) = 1$, by 1.5.3 and 1.5.9 we have $\chi_A(a_1{}^n; M) = \mathrm{e}_{a_1{}^n A}(M) = n \cdot \mathrm{e}_{a_1 A}(M) = n \cdot \chi_A(a_1; M)$. If $\ell(a_1 A) = 0$, then $\chi_A(a_1{}^n; M) = \chi_A(a_1; M) = 0$ by 1.4.9, so the required assertion is obviously true. Let now $m \geq 2$. We put $\overline{A} = A/a_1 A$ (resp. $A' = A/a_1{}^n A$) and denote by $\overline{a_i}$ (resp. $a_i{}'$) the image of a_i in $\overline{A}$ (resp. A'). As we have already seen, $\chi_A(a_1{}^n; M) = n \cdot \chi_A(a_1; M)$ through the isomorphism $\mathrm{K}_0(\overline{A}) \xrightarrow{\sim} \mathrm{K}_0(A')$. Hence we get

$$\chi_{A'}(a_2{}', \cdots, a_m{}')(\chi_A(a_1{}^n; M)) = n \cdot \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_m})(\chi_A(a_1; M))$$

through the isomorphism $\mathrm{K}_0(A/(a_1, a_2, \cdots, a_m)A) \xrightarrow{\sim} \mathrm{K}_0(A/(a_1{}^n, a_2, \cdots, a_m)A)$ since the diagram

$$\begin{array}{ccc} \mathrm{K}_0(\overline{A}) & \xrightarrow{\chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_m})} & \mathrm{K}_0(A/(a_1, a_2, \cdots, a_m)A) \\ \downarrow & & \downarrow \\ \mathrm{K}_0(A') & \xrightarrow{\chi_{A'}(a_2{}', \cdots, a_m{}')} & \mathrm{K}_0(A/(a_1{}^n, a_2, \cdots, a_m)A) \end{array}$$

is commutative. On the other hand, by 1.2.10 we have

$$\chi_A(a_1{}^n, a_2, \cdots, a_m; M) = \chi_{A'}(a_2{}', \cdots, a_m{}')(\chi_A(a_1{}^n; M))$$

and

$$\chi_A(a_1, a_2, \cdots, a_m; M) = \chi_{\overline{A}}(\overline{a_2}, \cdots, \overline{a_m})(\chi_A(a_1; M)).$$

Therefore we get the required equality.

The next proposition is a direct consequence of 1.2.8, 1.5.6 and 1.5.7.

Proposition 1.5.11 *Let $0 \to L \to M \to N \to 0$ be an exact sequence in A* mod . *Then* $\mathrm{e}_I(M) = \mathrm{e}_I(L) + \mathrm{e}_I(N)$.

By virtue of 1.5.11 we get the group homomorphism $\mathrm{e}_I : \mathrm{K}_0(A) \to \mathrm{K}_0(A/I)$ sending $[M]$ to $\mathrm{e}_I(M)$ for any $M \in A$ mod . If $J = (a_1, \cdots, a_\ell)A$ is a minimal reduction of I, the following diagram

$$\begin{array}{ccc} \mathrm{K}_0(A) & \xrightarrow{\mathrm{e}_I} & \mathrm{K}_0(A/I) \\ \| & & \downarrow \\ \mathrm{K}_0(A) & \xrightarrow{\chi_A(a_1,\cdots,a_\ell)} & \mathrm{K}_0(A/J) \end{array}$$

is commutative, where the vertical arrow denotes the isomorphism induced from the canonical surjection $A/J \to A/I$.

Proposition 1.5.12 *Let*

$$[M] = \sum_{Q \in \mathrm{Spec}\, A} m_Q \cdot [A/Q] \quad (m_Q \in \boldsymbol{Z})$$

in $\mathrm{K}_0(A)$. *Then*

$$\mathrm{e}_I(M) = \sum_{\substack{Q \in \mathrm{Spec}\, A \\ \ell(I+Q/Q)=\ell}} m_Q \cdot \mathrm{e}_I(A/Q)\,.$$

Proof. Notice that $0 \leq \ell(I+Q/Q) = \ell(I, A/Q) \leq \ell$ for all prime ideals Q and $\mathrm{e}_I(A/Q) = 0$ if $\ell(I, A/Q) < \ell$ by 1.5.2. Therefore we get the required equality since

$$\begin{aligned} \mathrm{e}_I(M) &= \mathrm{e}_I(\sum_Q m_Q \cdot [A/Q]) \\ &= \sum_Q m_Q \cdot \mathrm{e}_I(A/Q)\,. \end{aligned}$$

When I is $\mathbf{m}$-primary, 1.5.10 implies the additive formula:

$$\mathrm{e'}_I(M) = \sum_{Q \in \mathrm{Assh}\, A} \mathrm{length}_{A_Q} M_Q \cdot \mathrm{e'}_I(A/Q)\,,$$

because $m_Q = \mathrm{length}_{A_Q} M_Q$ for $Q \in \mathrm{Min}\ A$, $\ell = d$ and $\ell(I+Q/Q) = \dim A/Q$.

Proposition 1.5.13 *Let $J = (a_1, \cdots, a_\ell)A$ be a minimal reduction of I and $0 \leq k \leq \ell$. We put $K = (a_1, \cdots, a_k)A$. If $\ell(I/K) = \ell - k$, then $\mathrm{e}_I(M) = \mathrm{e}_{I/K}(\mathrm{e}_K(M))$.*

Proof. As the assertion is obvious when $k = 0$ or $k = \ell$, we consider the case where $0 < k < \ell$. Let $\overline{A} = A/K$ and $\overline{a_i}$ be the image of a_i in $\overline{A}$. Notice that $\ell(K) = k$ and

$J\overline{A} = (a_{k+1}, \cdots, a_\ell)\overline{A}$ is a minimal reduction of $I\overline{A}$. Then by 1.5.9 and 1.2.11 we have

$$\begin{aligned} \mathrm{e}_I(M) &= \chi_A(a_1, \cdots, a_\ell; M) \\ &= \chi_{\overline{A}}(\overline{a_{k+1}}, \cdots, \overline{a_\ell})(\chi_A(a_1, \cdots, a_k; M)) \\ &= \chi_{\overline{A}}(\overline{a_{k+1}}, \cdots, \overline{a_\ell})(\mathrm{e}_K(M)) \\ &= \mathrm{e}_{I\overline{A}}(\mathrm{e}_K(M)). \end{aligned}$$

Thus we get the required equality.

Let us notice that even if $I = (a_1, \cdots, a_\ell)A$, $\ell(I/(a_1, \cdots a_k)) < \ell - k$ can happen for some $0 < k < \ell$. For example, let $A = F[[X, Y]]$ be the formal power series ring over a field F and $I = (X^2, XY)A$. Then $\ell(I) = 2$. However $\ell(I/X^2A) = 0$ as I/X^2A is nilpotent. On the other hand, if $a_1, \cdots, a_\ell$ is a ssop for A or a d-sequence, then the equality $\ell(I/(a_1, \cdots, a_k)A) = \ell - k$ holds for all $0 \leq k \leq \ell$.

Corollary 1.5.14 *Under the same notations and assumptions as 1.5.13, let*

$$\mathrm{e}_K(M) = \sum_{Q \in \mathrm{V}(K)} m_Q \cdot [A/Q] \quad (m_Q \in \boldsymbol{Z})$$

in $\mathrm{K}_0(A/K)$. *Then*

$$\mathrm{e}_I(M) = \sum_{\substack{Q \in \mathrm{V}(K) \\ \ell(I+Q/Q) = \ell - k}} m_Q \cdot \mathrm{e}_{I/K}(A/Q).$$

Proof. By 1.5.13 we have

$$\begin{aligned} \mathrm{e}_I(M) &= \mathrm{e}_{I/K}(\mathrm{e}_K(M)) \\ &= \sum_{Q \in \mathrm{V}(K)} m_Q \cdot \mathrm{e}_{I/K}(A/Q). \end{aligned}$$

However $\mathrm{e}_{I/K}(A/Q) = 0$ if $\ell(I+Q/Q) = \ell(I/K, A/Q) < \ell - k$. Hence we get the required equality since $\ell(I + Q/Q) \leq \ell - k$ for all $Q \in \mathrm{V}(K)$.

When I is $\mathbf{m}$-primary, 1.5.14 means the associativity formula (cf. [11, (24.7)]). In fact, in that case, $\ell = d$ and $a_1, \cdots, a_d$ is a sop for A. So $\ell(I/K) = \dim A/K = d - k$. Furthermore $\ell(I + Q/Q) = \dim A/Q$ for all $Q \in \operatorname{Spec} A$. Therefore, as $m_Q = \mathrm{e'}_{KA_Q}(M_Q)$ for all $Q \in \operatorname{Min}_A A/K$ by 1.5.6, we have

$$\mathrm{e}_I(M) = \sum_{Q \in \operatorname{Assh}_A A/K} \mathrm{e'}_{KA_Q}(M_Q) \cdot \mathrm{e}_{I/K}(A/Q).$$

Now sending the both sides of the equality above by the length function $\mathrm{K}_0(A/I) \xrightarrow{\sim} \boldsymbol{Z}$, we get the associativity formula.

As is well known, when I is $\mathbf{m}$-primary, we always have inequalities $\mathrm{e}'_I(M) \geq 0$ and $\mathrm{e}'_I(M) \leq \mathrm{length}_A M/JM$ for any minimal reduction J of A. Now we generalize these facts. Let $\mathrm{K}_0(A/I)_+$ denotes the subset of $\mathrm{K}_0(A/I)$ consisting of the classes of finitely generated A/I-module.

Proposition 1.5.15 *We always have the following assertions.*

(1) $\mathrm{e}_I(M) \in \mathrm{K}_0(A/I)_+$.

(2) $[M/JM] - \mathrm{e}_I(M) \in \mathrm{K}_0(A/I)_+$ *for any minimal reduction J of I.*

Proof. By 1.5.9 we may assume $\mu_A(I) = \ell$ (Hence $I = J$ in (2)). We will prove by induction on ℓ. If $\ell = 0$, then $I = (0)$, so $\mathrm{e}_I(M) = [M]$ and the assertions are obviously true. Suppose $\ell = 1$. We write $I = aA$. Then, by 1.5.9, $\mathrm{e}_I(M) = \chi_A(a; M) = [M/aM] - [(0) :_M a]$, from which we get $[M/aM] - \mathrm{e}_I(M) = [(0) :_M a] \in \mathrm{K}_0(A/I)_+$. Because $(0) :_M a^i \subseteq (0) :_M a^{i+1}$ for all i, there exists $r > 0$ such that $(0) :_M a^r = (0) :_M a^n$ for any $n \geq r$. This implies that $a^r M/a^{r+1}M \overset{a^{n-r}}{\to} a^n M/a^{n+1}M$ is an isomorphism for $n \geq r$. Then, setting $E = a^r M/a^{r+1}M$ and $L = M/a^r M$, we have

$$\begin{aligned}
\chi_I^M(n) &= [M/a^{n+1}M] \\
&= [M/a^r M] + [a^r M/a^{r+1}M] + \cdots + [a^n M/a^{n+1}M] \\
&= (n - r + 1)[E] + [L] \\
&= \binom{n+1}{1}[E] + ([L] - r[E])
\end{aligned}$$

for $n \geq r$. Hence $\mathrm{e}_I(M) = [E] \in \mathrm{K}_0(A/I)_+$. Let now $\ell \geq 2$ and assume that (1) and (2) are true for any ideal whose analytic spread is less than ℓ. If $\ell(I, M) < \ell$, then $\mathrm{e}_I(M) = 0$ by 1.5.2 and the assertions are obvious. So let us consider the case where $\ell(I, M) = \ell$. We choose an element $a \in I$ satisfying the conditions of 1.3.5. We set $\overline{A} = A/aA$, $\overline{I} = I\overline{A}$ and $\overline{M} = M/aM$. Of course $\ell(\overline{I}) \leq \mu_{\overline{A}}(\overline{I}) = \ell - 1$. On the other hand, $\ell(\overline{I}) = \ell(I, \overline{A}) \geq \ell(I, \overline{M}) = \ell - 1$. Hence $\ell(\overline{I}) = \ell - 1$, and so $\mathrm{e}_{\overline{I}}(\overline{M}) = \mathrm{e}_{\ell-1}(\overline{I}, \overline{M})$. Then we get $\mathrm{e}_{\overline{I}}(\overline{M}) = \mathrm{e}_I(M)$ since $\mathrm{e}_{\ell-1}(\overline{I}, \overline{M}) = \mathrm{e}_{\ell-1}(I, \overline{M}) = \mathrm{e}_\ell(I, M)$ by 1.4.4. Therefore by the hypothesis of induction we easily see that the assertions (1) and (2) are true.

Proposition 1.5.16 *Let $I = (a_1, \cdots, a_\ell)A$ and let $n_1, \cdots, n_\ell$ be positive integers. We assume $\ell((a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})A) = \ell$. Then, through the isomorphism*

$$\mathrm{K}_0(A/I) \to \mathrm{K}_0(A/(a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})A),$$

we have

$$\mathrm{e}_{(a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})A}(M) = n_1 n_2 \cdots n_\ell \cdot \mathrm{e}_I(M).$$

Proof, By 1.5.9 and 1.5.10 we have

$$\begin{aligned} \mathrm{e}_{(a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})A}(M) &= \chi_A(a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell}; M) \\ &= n_1 n_2 \cdots n_\ell \cdot \chi_A(a_1, \cdots, a_\ell; M) \\ &= n_1 n_2 \cdots n_\ell \cdot \mathrm{e}_I(M). \end{aligned}$$

The next result is a generalization of the lemma of Lech. But in order to state it, we have to fix one more notation. Let m be a positive integer and

$$f : \underbrace{\boldsymbol{Z} \times \cdots \times \boldsymbol{Z}}_{m \text{ times}} \to G$$

a function, where G is an additive group. For $1 \leq i \leq m$, we define

$$\triangle_i f : \underbrace{\boldsymbol{Z} \times \cdots \times \boldsymbol{Z}}_{m \text{ times}} \to G$$

by setting $\triangle_i f(n_1, \cdots, n_i, \cdots, n_m) = f(n_1, \cdots, n_i, \cdots, n_m) - f(n_1, \cdots, n_i - 1, \cdots, n_m)$.

Proposition 1.5.17 *Let $I = (a_1, \cdots, a_\ell)A$. We assume that*

$$\ell((a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})A/(a_1{}^{n_1}, \cdots, a_k{}^{n_k})A) = \ell - k$$

for all positive integers $n_1, \cdots, n_\ell$ and $0 \leq k \leq \ell$. Let

$$f : \underbrace{\boldsymbol{Z} \times \cdots \times \boldsymbol{Z}}_{\ell \text{ times}} \to \mathrm{K}_0(A/I)$$

be the function such that $f(n_1, \cdots, n_\ell) = [M/(a_1{}^{n_1}, \cdots, a_\ell{}^{n_\ell})M]$. Then we have

$$\triangle_1 \triangle_2 \cdots \triangle_\ell f(n_1, \cdots, n_\ell) = \mathrm{e}_I(M)$$

for $n_1, \cdots, n_\ell \gg 0$.

Proof. We will prove by induction on ℓ. If $\ell = 1$, the assertion is a special case of 1.5.2. Let $\ell \geq 2$. We fix $n_1 > 0$ for a moment. We set $\widetilde{A} = A/{a_1}^{n_1}A$ and $\widetilde{M} = M/{a_1}^{n_1}M$. It is easy to see that

$$\ell(({a_2}^{n_2}, \cdots, {a_\ell}^{n_\ell})\widetilde{A}/({a_2}^{n_2}, \cdots, {a_k}^{n_k})\widetilde{A}) = \ell - 1 - k$$

for all positive integers $n_2, \cdots, n_\ell$ and $1 \leq k \leq \ell$ (the denominator is (0) when $k = 1$). Let

$$g : \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{\ell - 1 \text{ times}} \to \mathrm{K}_0(A/({a_1}^{n_1}, a_2, \cdots, a_\ell)A)$$

be the function such that $g(n_2, \cdots, n_\ell) = [\widetilde{M}/({a_2}^{n_2}, \cdots, {a_\ell}^{n_\ell})\widetilde{M}]$. Then, by the hypothesis of induction, we have

$$\triangle_1 \cdots \triangle_{\ell-1} g(n_2, \cdots, n_\ell) = \mathrm{e}_{(a_2, \cdots, a_\ell)\widetilde{A}}(\widetilde{M})$$

in $\mathrm{K}_0(A/({a_1}^{n_1}, a_2, \cdots, a_\ell)A)$ for $n_2, \cdots, n_\ell \gg 0$. Now we further set $\overline{A} = A/a_1A$ and $\overline{M} = M/a_1M$. Then, considering the commutative diagram

$$\begin{array}{ccc} \mathrm{K}_0(\widetilde{A}) & \overset{\mathrm{e}_{(a_2,\cdots,a_\ell)\widetilde{A}}}{\longrightarrow} & \mathrm{K}_0(A/({a_1}^{a_1}, a_2, \cdots, a_\ell)A) \\ \uparrow & & \uparrow \\ \mathrm{K}_0(\overline{A}) & \overset{\mathrm{e}_{(a_2,\cdots,a_\ell)\overline{A}}}{\longrightarrow} & \mathrm{K}_0(A/I), \end{array}$$

where the vertical arrows denote the isomorphisms induced from the canonical surjections $\widetilde{A} \to \overline{A}$ and $A/({a_1}^{n_1}, a_2, \cdots, a_\ell)A) \to A/I$, we get

$$\triangle_2 \cdots \triangle_\ell f(n_1, n_2, \cdots, n_\ell) = \mathrm{e}_{(a_2, \cdots, a_\ell)\overline{A}}([M/{a_1}^{n_1}M])$$

for $n_2, \cdots, n_\ell \gg 0$ in $\mathrm{K}_0(A/I)$. On the other hand, as $\ell(a_1A) = 1$,

$$[M/{a_1}^{n_1}M] = n_1 \cdot \mathrm{e}_{a_1A}(M) + \mathrm{e}_0(a_1A, M)$$

in $\mathrm{K}_0(\overline{A})$ for $n_1 \gg 0$. Hence, setting $\xi = \mathrm{e}_{(a_2, \cdots, a_\ell)\overline{A}}(\mathrm{e}_0(a_1A, M))$, we have

$$\begin{aligned} \mathrm{e}_{(a_2, \cdots, a_\ell)\overline{A}}([M/{a_1}^{n_1}M]) &= n_1 \cdot \mathrm{e}_{(a_2, \cdots, a_\ell)\overline{A}}(\mathrm{e}_{a_1A}(M)) + \xi \\ &= n_1 \cdot \mathrm{e}_I(M) + \xi \end{aligned}$$

for $n_1 \gg 0$. In conclusion we get

$$\triangle_1 \triangle_2 \cdots \triangle_\ell f(n_1, n_2, \cdots, n_\ell) = \mathrm{e}_I(M)$$

for $n_1, n_2, \cdots, n_\ell \gg 0$. Thus we have completed the proof.

So far we have verified that our multiplicities actually enjoy the same properties as the ordinary ones. Now it should be required to consider the influence of the value $\mathrm{e}_I(M)$ on I and M themselves. As the first step of the study in this aspect, the following two results are concerned with when $\mathrm{e}_I(A) = [A/I]$.

Proposition 1.5.18 *Let A be a Cohen-Macaulay ring. Then $\mathrm{e}_I(A) = [A/I]$ if and only if I is generated by a regular sequence.*

Proof. Suppose $\mathrm{e}_I(A) = [A/I]$. Let $Q \in \mathrm{Min}_A A/I$. Then $\mathrm{e}_I(A)_Q = [A_Q/IA_Q] \neq 0$ since it is mapped by the length function $\mathrm{K}_0(A_Q/IA_Q) \to \boldsymbol{Z}$ to $\mathrm{length}_{A_Q} A_Q/IA_Q \neq 0$. Because $\mathrm{e}_I(A) = \mathrm{e}_\ell(I, A)$, we get $\ell \leq \mathrm{ht}_A Q$ by 1.4.5. Hence $\ell \leq \mathrm{ht}_A I$, and so $\ell = \mathrm{ht}_A I$ as $\ell \geq \mathrm{ht}_A I$ in general. Let J be a minimal reduction of I. Notice that, as A is Cohen-Macaulay, J is generated by a regular sequence, from which we get $\mathrm{e}_J(A) = [A/J]$ by 1.4.6. Consequently the equality $[A/I] = [A/J]$ follows from $\mathrm{e}_I(A) = \mathrm{e}_J(A)$. Then, for any $Q \in \mathrm{Min}_A A/J = \mathrm{Min}_A A/I$, we have $[A_Q/IA_Q] = [A_Q/JA_Q]$, which implies $\mathrm{length}_{A_Q} A_Q/IA_Q = \mathrm{length}_{A_Q} A_Q/JA_Q$ and so $IA_Q = JA_Q$. Therefore $I = J$. Thus we see that I is generated by a regular sequence. The converse is a direct consequence of 1.4.6.

Proposition 1.5.19 *Let A/Q be a regular local ring. Assume $\mathrm{Ass}\,\hat{A} = \mathrm{Assh}\,\hat{A}$, where $\hat{A}$ is the completion of A. Then $\mathrm{e}_Q(A) = [A/Q]$ if and only if A is regular.*

Proof. Suppose $\mathrm{e}_Q(A) = [A/Q]$. By the same reason as in the proof of 1.5.18, we have $\ell(Q) = \mathrm{ht}_A Q =: s$. Let $J = (a_1, \cdots, a_s)$ be a minimal reduction of Q. We set $\overline{A} = A/J$. Because A is quasi-unmixed by the assumption, it is equidimensional and catenary, so $\dim \overline{A} = d - s$. Now choose the elements $a_{s+1}, \cdots, a_d \in \mathbf{m}$ so that $(a_{s+1}, \cdots, a_d)\overline{A}$ is a minimal reduction of $\mathbf{m}\overline{\mathbf{A}}$. Then, as $a_1, \cdots, a_d$ is a sop for A, by 1.5.8 and 1.5.13 we see $\mathrm{e}_{\mathbf{m}\overline{\mathbf{A}}}(\mathrm{e}_J(A)) = \mathrm{e}_{(a_{s+1},\cdots,a_d)\overline{A}}(\mathrm{e}_J(A)) = \mathrm{e}_{(a_1,\cdots,a_d)A}(A)$. On the other hand, we have $\mathrm{e}_{\mathbf{m}\overline{\mathbf{A}}}(\mathrm{e}_J(A)) = \mathrm{e}_{\mathbf{m}\overline{\mathbf{A}}}(\mathrm{e}_Q(A)) = \mathrm{e}_{\mathbf{m}\overline{\mathbf{A}}}(A/Q) = \mathrm{e}_{\mathbf{m}/\mathbf{Q}}(A/Q)$. Thus the equality $\mathrm{e}_{(a_1,\cdots,a_d)A}(A) = \mathrm{e}_{\mathbf{m}/\mathbf{Q}}(A/Q)$ follows. This implies $\mathrm{e}'_{(a_1,\cdots,a_d)A}(A) = \mathrm{e}'_{\mathbf{m}/\mathbf{Q}}(A/Q)$, and so $\mathrm{e}'_{(a_1,\cdots,a_d)A}(A) = 1$ since A/Q is regular. Then $\mathrm{e}'_{\mathbf{m}}(A) = 1$ since $0 < \mathrm{e}'_{\mathbf{m}}(A) \leq \mathrm{e}'_{(a_1,\cdots,a_d)A}(A)$. In conclusion A is regular by [11, (40.6)]. Conversely, if A is regular, then Q must be generated by a regular sequence since A/Q is regular. Hence $\mathrm{e}_Q(A) = [A/Q]$ by 1.4.6 and the proof is completed.

If I is equimultiple, then $\mathrm{e}_I(A) \neq 0$ by 1.4.1. The next proposition provides with examples of non-equimultiple ideal whose multiplicities are not vanished.

Proposition 1.5.20 *Let A be a Gorenstein ring and $Q \in \operatorname{Spec} A$ such that A/Q is a Cohen-Macaulay normal domain. We assume that $\mu_A(Q) = \operatorname{ht}_A Q + 1$ and A_Q is regular (such a prime ideal is said to be an almost complete intersection (cf. [5, (2.1)])). Then we have the following assertions.*

(1) $\mathrm{e}_Q(A) = [A/Q] - [K_{A/Q}]$, *where $K_{A/Q}$ denotes the canonical module of A/Q.*

(2) *If A/Q is not Gorenstein, then $\mathrm{e}_Q(A) \neq 0$. The converse is true when $\dim A/Q = 2$ and $[A/\mathbf{m}] = \mathbf{0}$ in $\mathrm{K}_0(A/Q)$.*

Proof. (1) We put $s = \operatorname{ht}_A Q$. Because $\operatorname{ht}_A Q \leq \ell(Q) \leq \mu_A(Q)$, we have $\ell(Q) = s$ or $s+1$. However, since $\ell(Q) = s$ implies $\mu_A(Q) = s$ (cf. [3, Theorem]), the equality $\ell(Q) = s+1$ must hold. By [5, (2.5)] there exist elements $a_1, \cdots, a_s, b$ of A satisfying the conditions

(i) $Q = (a_1, \cdots, a_s, b)A$ and $QA_Q = (a_1, \cdots, a_s)A_Q$,

(ii) $a_1, \cdots, a_s$ is an A-regular sequence and

(iii) $K :_A b = K :_A b^2$, where $K = (a_1, \cdots, a_s)A$.

Then by 1.4.6 and the condition (ii) above we have $\mathrm{e}_K(A) = [A/K]$. Moreover (ii) and (iii) imply that $a_1, \cdots, a_s, b$ is a d-sequence, and so by 1.5.13, setting $\overline{A} = A/K$, we get

$$\begin{aligned} \mathrm{e}_Q(A) &= \mathrm{e}_{b\overline{A}}(\mathrm{e}_K(A)) \\ &= \mathrm{e}_{b\overline{A}}(\overline{A}). \end{aligned}$$

Let $n > 0$ and $\varphi : A \xrightarrow{b^n} b^n\overline{A}/b^{n+1}\overline{A} = K + b^nA/K + b^{n+1}A$. If $x \in \operatorname{Ker}\varphi$, there exists $y \in K$ such that $b^n x \equiv b^{n+1}y \mod K$. Then $x - by \in K :_A b^n$. The condition (ii) implies $K :_A b^n = K :_A b$ for all $n \geq 1$. Hence $x - by \in K :_A Q$ as $K :_A b = K :_A Q$, so $x \in Q + (K :_A Q)$. Conversely, $Q + (K :_A Q) \subseteq \operatorname{Ker}\varphi$ is obvious. Thus we get $\operatorname{Ker}\varphi = Q + (K :_A Q)$. This implies $b^n\overline{A}/b^{n+1}\overline{A} \cong E$, where $E = A/Q + (K :_A Q)$. As a consequence, for any $n \geq 0$, we get

$$\begin{aligned}\chi_{b\overline{A}}(n) &= [\overline{A}/b^{n+1}\overline{A}] \\ &= [A/Q] + \sum_{i=1}^{n} [b^i\overline{A}/b^{i+1}\overline{A}] \\ &= \binom{n+1}{1}[E] + ([A/Q] - [E]).\end{aligned}$$

Therefore $\mathrm{e}_{b\overline{A}}(\overline{A}) = [E]$. Now we look at the exact sequence

$$0 \to Q + (K :_A Q)/Q \to A/Q \to E \to 0.$$

In order to prove $Q+(K :_A Q)/Q \cong K_{A/Q}$, we first notice the equality $(K :_A Q) \bigcap Q = K$, which is verified as follows: Because obviously $(K :_A Q) \bigcap Q \supseteq K$, it is enough to show $(K_P :_{A_P} QA_P) \bigcap QA_P = KA_P$ for all $P \in \mathrm{Ass}_A A/K$. But this is trivial if $Q \not\subseteq P$. Even if $Q \subseteq P$, we have $Q = P$ as $\mathrm{ht}_A P = s$, and so the required equality holds by the condition (i). Now we get

$$\begin{aligned}Q + (K :_A Q)/Q &\cong K :_A Q/(K :_A Q) \cap Q \\ &= K :_A Q/K \\ &= \mathrm{Hom}_{A/K}(A/Q, A/K).\end{aligned}$$

Because A/K is Gorenstein, by [9, 5.9 and 5.14] $K_{A/Q} \cong \mathrm{Hom}_{A/K}(A/Q, A/K)$. Thus the exact sequence

$$0 \to K_{A/Q} \to A/Q \to E \to 0$$

is induced. Hence $[E] = [A/Q] - [K_{A/Q}]$, and so we get the assertion (1).

(2) Let us consider the group homomorphism $\mathrm{K}_0(A/Q) \to \boldsymbol{Z} \bigoplus \mathrm{Cl}(A/Q)$ stated in Section 1. This homomorphism maps $\mathrm{e}_Q(A)$ to $(0, -\mathrm{cl}(K_{A/Q}))$. Notice that A/Q is Gorenstein if and only if $\mathrm{cl}(K_{A/Q}) = 0$ in $\mathrm{Cl}(A/Q)$. Therefore if A/Q is not Gorenstein, then $\mathrm{e}_Q(A) \neq 0$. In the case where $\dim A/Q = 2$ and $[A/\mathbf{m}] = \mathbf{0}$ in $\mathrm{K}_0(A/Q)$, the homomorphism above is isomorphic, which implies A/Q is not Gorenstein if $\mathrm{e}_Q(A) \neq 0$. Thus we have completed the proof.

The prime ideal in the formal power series ring $F[[X, Y, Z, U, V, W]]$ over a field F generated by the maximal minors of the matrix

$$\begin{pmatrix} X & Y & Z \\ U & V & W \end{pmatrix}$$

is a typical example of Q in 1.5.20

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Chapter 2

On the Integral Closures of Certain Ideals Generated by Regular Sequences

2.1 Introduction

Let $A = k[[X, Y, Z]]$ be the formal power series ring over a field k. Let $\mathfrak{p}$ be a prime ideal in A generated by the maximal minors of the matrix

$$\begin{pmatrix} X^{\alpha} & Y^{\beta'} & Z^{\gamma'} \\ Y^{\beta} & Z^{\gamma} & X^{\alpha'} \end{pmatrix}$$

where $\alpha, \beta, \gamma, \alpha', \beta'$ and γ' are all positive integers. For example, the defining ideal of a space monomial curve: $X = t^{n_1}, Y = t^{n_2}$ and $Z = t^{n_3}$ with $\gcd\{n_1, n_2, n_3\} = 1$ can be expressed in that way (cf. [2]). We put $a = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta'}, b = X^{\alpha+\alpha'} - Y^{\beta}Z^{\gamma'}$ and $c = Y^{\beta+\beta'} - X^{\alpha}Z^{\gamma}$. Then $\mathfrak{p} = (a, b, c)A$. Let $J = (a, b)A$. The purpose of this paper is to prove the following:

Theorem 2.1.1 *The integral closure $\overline{J^n}$ is equal to*

$$J^{n-1} \cdot (a, b, \{X^i Z^j c \mid i, j \geq 0 \text{ and } i/\alpha' + j/\gamma' \geq 1\})A$$

for all $n \geq 1$ and the Rees algebra $\mathrm{R}(\overline{J})$ is a Cohen-Macaulay ring.

Throughout this paper, we denote by $\overline{\mathfrak{A}}$ the integral closure of an ideal $\mathfrak{A}$ in a ring R. For a module M over R, $\ell_R(M)$ is the length of M. The multiplicity of M with respect to $\mathfrak{A}$ is denoted by $\mathrm{e}_{\mathfrak{A}}(M)$. We set $\mathrm{Assh}_R M = \{P \in \mathrm{Ass}_R M \mid \dim R/P = \dim_R M\}$, where $\mathrm{Ass}_R M$ is the set of associated primes of M.

2.2 $\overline{J^n A_P}$ for P in $\operatorname{Ass}_A A/J$

We begin with the following

Lemma 2.2.1 $(X^{\alpha'}, Z^{\gamma'})c \subsetneq J$

Proof. Because $X^{\alpha}a + Y^{\beta'}b + Z^{\gamma'}c = 0$ and $Y^{\beta}a + Z^{\gamma}b + X^{\alpha'}c = 0$, so $Z^{\gamma'}c \in J$ and $X^{\alpha'}c \in J$. Thus we get the assertion.

Lemma 2.2.2 $\operatorname{Ass}_A A/J = \operatorname{Assh}_A A/J = \{\mathfrak{p}, \mathfrak{q}\}$, *where* $\mathfrak{q} = (X, Z)A$.

Proof. Because a, b is a regular sequence in A, so $\operatorname{Ass}_A A/J = \operatorname{Assh}_A A/J$. Obviously, $\{\mathfrak{p}, \mathfrak{q}\} \subseteq \operatorname{Assh}_A A/J$. Now we take any $P \in \operatorname{Assh}_A A/J$. If $X \in P$, then $Z \in P$ as $a = Z^{\gamma+\gamma'} - X^{\alpha'}Y^{\beta'} \in J \subseteq P$, and so $P = \mathfrak{q}$. If $X \notin P$, then $c \in J$ as $X^{\alpha'}c \in J \subseteq P$ by 2.2.1, and consequently we get $P = \mathfrak{p}$.

Lemma 2.2.3 *For all* $n \geq 1$, *we have*

$$\overline{J^n A_{\mathfrak{p}}} = J^n A_{\mathfrak{p}}$$

and

$$\overline{J^n A_{\mathfrak{q}}} = J^{n-1} \cdot (\{X^i Z^j \mid i, j \geq 0 \text{ and } i/\alpha' + j/\gamma' \geq 1\})A_{\mathfrak{q}}.$$

Proof. Because $X \notin \mathfrak{p}$, we have $c \in JA_{\mathfrak{p}}$ by 2.2.1. Hence $JA_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, which implies the first equality. On the other hand, as $c \notin \mathfrak{q}$, we have $(X^{\alpha'}, Z^{\gamma'})A_{\mathfrak{q}} \subseteq JA_{\mathfrak{q}}$ by 2.2.1. The converse inclusion is obvious, so that we see $JA_{\mathfrak{q}} = (X^{\alpha'}, Z^{\gamma'})A_{\mathfrak{q}}$. Therefore, as is well known, $\overline{JA_{\mathfrak{q}}} = (\{X^i Z^j \mid i, j \geq 0 \text{ and } i/\alpha' + j/\gamma' \geq 1\})A_{\mathfrak{q}}$. Because $A_{\mathfrak{q}}$ is a two dimensional regular local ring, we get the last assertion (cf. [6, Appendix 5] or [3, 3.7]).

2.3 Proof of Theorem 2.1.1

Lemma 2.3.1 *Let* i, j *be non-negative integers with* $i/\alpha' + j/\gamma' \geq 1$. *Then* $X^i Z^j c \in \overline{J}$.

Proof. As $(X^i Z^j c)^{\alpha'\gamma'} \subseteq (X^{\alpha'}, Z^{\gamma'})^{i\gamma' + j\alpha'} \cdot c^{\alpha'\gamma'}$ and as $i\gamma' + j\alpha' \geq \alpha'\gamma'$, we get $(X^i Z^j c)^{\alpha'\gamma'} \in J^{\alpha'\gamma'}$ by 2.2.1. Hence $X^i Z^j c \in \overline{J}$.

Let $I = (a, b, \{X^i Z^j c \mid i, j \geq 0 \text{ and } i/\alpha' + j/\gamma' \geq 1\})A$. Notice $J \subseteq I \subseteq \overline{J}$.

Lemma 2.3.2 *A/I is a Cohen-Macaulay ring.*

Proof. It is enough to show $e_{YA}(A/I) = \ell_A(A/YA+I)$. Because $J \subseteq I \subseteq \overline{J}$, we have $\mathrm{Assh}_A\, A/I = \{\mathfrak{p}, \mathfrak{q}\}$ by ??. Then, by the additive formula of multiplicity (cf. [5, (23.5)]),

$$e_{YA}(A/I) = \ell_{A_\mathfrak{p}}(A_\mathfrak{p}/IA_\mathfrak{p}) \cdot e_{YA}(A/\mathfrak{p}) + \ell_{A_\mathfrak{q}}(A_\mathfrak{q}/IA_\mathfrak{q}) \cdot e_{YA}(A/\mathfrak{q}).$$

Obviously, $\ell_{A_\mathfrak{p}}(A_\mathfrak{p}/IA_\mathfrak{p}) = e_{YA}(A/\mathfrak{q}) = 1$ and $e_{YA}(A/\mathfrak{p}) = \ell_A(A/YA+\mathfrak{p})$. Let $B = k[[X,Z]]$ and $K = (\{X^iZ^j \mid i,j \geq 0 \text{ and } i/\alpha' + j/\gamma' \geq 1\})B$. It is easy to see that $IA_\mathfrak{q} = KA_\mathfrak{q}$ and $KA_\mathfrak{q}$ is $\mathfrak{q}A_\mathfrak{q}$-primary, which implies $\ell_{A_\mathfrak{q}}(A_\mathfrak{q}/IA_\mathfrak{q}) = \ell_B(B/K)$. Therefore

$$e_{YA}(A/I) = \ell_A(A/YA+\mathfrak{p}) + \ell_B(B/K).$$

In order to compute $\ell_A(A/YA+I)$, we consider the exact sequence

$$0 \to YA+\mathfrak{p}/YA+I \to A/YA+I \to A/YA+\mathfrak{p} \to 0.$$

Because

$$\begin{aligned}\frac{YA+\mathfrak{p}}{YA+I} &\cong \frac{(Z^{\gamma+\gamma'}, X^{\alpha+\alpha'}, X^\alpha Z^\gamma)B}{(Z^{\gamma+\gamma'}, X^{\alpha+\alpha'})B + X^\alpha Z^\gamma K} \\ &\cong \frac{X^\alpha Z^\gamma B}{(Z^{\gamma+\gamma'}, X^{\alpha+\alpha'})B \cap X^\alpha Z^\gamma B + X^\alpha Z^\gamma K}\end{aligned}$$

and

$$\begin{aligned}(Z^{\gamma+\gamma'}, X^{\alpha+\alpha'})B \cap X^\alpha Z^\gamma B &= (X^\alpha Z^{\gamma+\gamma'}, X^{\alpha+\alpha'} Z^\gamma)B \\ &\subseteq X^\alpha Z^\gamma K,\end{aligned}$$

we get $YA+\mathfrak{p}/YA+I \cong X^\alpha Z^\gamma B/X^\alpha Z^\gamma K \cong B/K$. Therefore

$$\ell_A(A/YA+I) = \ell_A(A/YA+\mathfrak{p}) + \ell_B(B/K).$$

Thus we get the required equality.

Lemma 2.3.3 *A/J^nI is Cohen-Macaulay for all $n \geq 0$.*

Proof. Because J^n/J^{n+1} is A/J-free, so $J^n/J^nI \cong J^n/J^{n+1} \otimes_A A/I$ is A/I-free. Hence, considering the exact sequence $0 \to J^n/J^nI \to A/J^nI \to A/J^n \to 0$, we get the assertion by 2.3.2.

Proof of Theorem 2.1.1. Let $n \geq 1$. By 2.2.2 and 2.3.3 we have $\mathrm{Ass}_A\, A/J^{n-1}I = \{\mathfrak{p}, \mathfrak{q}\}$. On the other hand, by 2.2.3, both of $\mathfrak{p}$ and $\mathfrak{q}$ do not support $\overline{J^n}/J^{n-1}I$. Hence $\overline{J^n} = J^{n-1}I$ since $\mathrm{Ass}_A\, \overline{J^n}/J^{n-1}I \subseteq \mathrm{Ass}_A\, A/J^{n-1}I$. Obviously $J^{n-1}I \subseteq I^n \subseteq \overline{J^n}$, so $I^n = J^{n-1}I$. In particular, $I^2 = JI$. As A/I is Cohen-Macaulay by 2.3.2, we see that the Rees algebra $\mathrm{R}(I)$ is Cohen-Macaulay by [1, (26.12)], and the proof of Theorem 2.1.1 is completed.

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Chapter 3

On Filtrations Having Small Analytic Deviation

3.1 Introduction

The purpose of this paper is to develop a general theory on filtrations in Noetherian local rings, which enables us to look at several kinds of filtrations from the same point of view.

Let A be a d-dimensional Noetherian local ring with the maximal ideal $\mathfrak{m}$ and let $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ be a family of ideals in A such that (i) $F_n \supseteq F_{n+1}$ for any $n \in \mathbb{Z}$, (ii) $F_0 = A$ and $F_1 \neq A$, and (iii) $F_m F_n \subseteq F_{m+n}$ for any $m, n \in \mathbb{Z}$. In this paper we simply call such $\mathcal{F}$ a filtration. When a filtration $\mathcal{F}$ is given, we can consider the following algebras:

$$\begin{aligned}
\mathrm{R}(\mathcal{F}) &= \sum_{n\geq 0} F_n T^n \subseteq A[T] \quad (T \text{ is an indeterminate}),\\
\mathrm{R}'(\mathcal{F}) &= \sum_{n\in\mathbb{Z}} F_n T^n \subseteq A[T, T^{-1}]\,, \text{ and}\\
\mathrm{G}(\mathcal{F}) &= \mathrm{R}'(\mathcal{F})/T^{-1}\mathrm{R}'(\mathcal{F}) = \bigoplus_{n\geq 0} F_n/F_{n+1}\,.
\end{aligned}$$

Those algebras are respectively called the Rees algebra, the extended Rees algebra, and the form ring associated to $\mathcal{F}$. We always assume that $\mathrm{R}(\mathcal{F})$ is Noetherian and $\dim \mathrm{R}(\mathcal{F}) = d + 1$. Typical examples of filtration are constructed from an ideal I. For example, setting $F_n = I^n$, we get the I-adic filtration. Symbolic filtration of I is defined by setting $F_n = I^{(n)}$, where $I^{(n)} = \bigcap_{\mathfrak{p}\in \mathrm{Min}_A A/I} I^n A_{\mathfrak{p}} \cap A$. It is also important to set $F_n = \overline{I^n}$, where $\overline{I^n}$ denotes the integral closure of I^n.

In order to treat these filtrations, we will use the extended notion of analytic spread, which was originally introduced for ideals by Northcott and Rees in [12]. We denote by $\ell(\mathcal{F})$ the Krull dimension of the ring $A/\mathfrak{m} \otimes_A \mathrm{G}(\mathcal{F})$ and call it the analytic spread of $\mathcal{F}$

(cf. [13]). If $\mathcal{F}$ is the I-adic filtration, then $\ell(\mathcal{F})$ is just the analytic spread of I in the sense of Northcott and Rees. It is easy to see that, similarly as the case of ideals, the inequality $\ell(\mathcal{F}) \geq \mathrm{ht}_A F_1$ always hold (here it should be noticed that $\mathrm{ht}_A F_1 = \mathrm{ht}_A F_n$ for any $n \geq 1$). So, following Huckaba and Huneke [8], we set $ad(\mathcal{F}) = \ell(\mathcal{F}) - \mathrm{ht}_A F_1$ and call it the analytic deviation of $\mathcal{F}$. For example, if $\mathcal{F}$ is the symbolic filtration of an ideal I with $\mathrm{ht}_A I = s < d$, then $ad(\mathcal{F}) < d - s$. On the other hand, if $I^n \subseteq F_n \subseteq \overline{I^n}$ for any n, then $ad(\mathcal{F})$ coincides with the analytic deviation of I in the sense of Huckaba and Huneke.

The main result of this paper is a characterization of the Cohen-Macaulay property of the form ring associated to a filtration having small analytic deviation. If $\mathcal{F}$ is a filtration with $\ell(\mathcal{F}) = \ell$, we can choose elements $a_1, \cdots, a_\ell$ in A so that $a_1 \in F_{k_1}, \cdots, a_\ell \in F_{k_\ell}$ for some positive integers $k_1, \cdots, k_\ell$ and $F_n = \sum_{i=1}^{\ell} a_i F_{n-k_i}$ for any $n \gg 0$. We will show that, in the case where $ad(\mathcal{F}) = 0$, $\mathrm{G}(\mathcal{F})$ is Cohen-Macaulay if and only if $A/(a_1, \cdots, a_\ell) + F_n$ is Cohen-Macaulay for finite number of n and $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ is Cohen-Macaulay for any $\mathfrak{p} \in \mathrm{Assh}_A A/F_1$, where $\mathcal{F}_\mathfrak{p}$ is the filtration $\{F_n A_\mathfrak{p}\}_{n\in\mathbb{Z}}$ of $A_\mathfrak{p}$ and $\mathrm{Assh}_A A/F_1 = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid F_1 \subseteq \mathfrak{p}$ and $\dim A/\mathfrak{p} = \dim A/F_1\}$. This characterization was already proved by Goto in the case where $\mathcal{F}$ is the symbolic filtration of a prime ideal $\mathfrak{p}$ such that $\dim A/\mathfrak{p} = 1$ (cf. [3]). We will also discuss the case where $ad(\mathcal{F}) = 1$. Although the statement is rather complicated, a condition for $\mathrm{G}(\mathcal{F})$ to be Cohen-Macaulay will be given similarly as the case where $ad(\mathcal{F}) = 0$.

Throughout this paper A is a d-dimensional Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ is a filtration of A such that $\mathrm{R}(\mathcal{F})$ is a $d+1$-dimensional Noetherian ring. In the case where $\mathcal{F}$ is the I-adic filtration, we write $\mathrm{R}(I)$, $\mathrm{R}'(I)$, and $\mathrm{G}(I)$ instead of $\mathrm{R}(\mathcal{F})$, $\mathrm{R}'(\mathcal{F})$, and $\mathrm{G}(\mathcal{F})$. Similarly we use the notation $\mathrm{R}_s(I)$, $\mathrm{R}'_s(I)$, and $\mathrm{G}_s(I)$ when $\mathcal{F}$ is the symbolic filtration of I. For a graded ring $S = \bigoplus_{n\geq 0} S_n$ and an integer m, we set $S_{\geq m} = \bigoplus_{n\geq m} S_n$, which is an ideal of S. In particular, $S_+ = S_{\geq 1}$. The i-th local cohomology module of an S-module L with respect to an ideal $\mathfrak{A}$ of S is denoted by $\mathrm{H}^i_{\mathfrak{A}}(L)$. When $(S_0, \mathfrak{n})$ is local and $L = \bigoplus_{n\in\mathbb{Z}} L_n$ is a graded S-module, we set

$$\mathrm{a}(L) = \max\{n \mid [\mathrm{H}^t_{\mathfrak{M}}(L)]_n \neq 0\},$$

where $t = \dim_S L$ and $\mathfrak{M} = \mathfrak{n}S + S_+$ (cf. [6]).

3.2 Preliminaries

In this section we summarize basic notion and facts we need throughout this paper. Let $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ be a filtration of A.

Definition 3.2.1 *We set* $\ell(\mathcal{F}) = \dim \mathrm{G}(\mathcal{F})/\mathfrak{m}\mathrm{G}(\mathcal{F})$ *and call it the analytic spread of* $\mathcal{F}$.

Definition 3.2.2 *We say that a system* $a_1, \cdots, a_r$ *of elements of* A *is a reduction of* $\mathcal{F}$, *if* $a_1 \in F_{k_1}, \cdots, a_r \in F_{k_r}$ *for some positive integers* $k_1, \cdots, k_r$ *and* $F_n = \sum_{i=1}^{r} a_i F_{n-k_i}$ *for any* $n \gg 0$.

If $a_1, \cdots, a_r$ is a reduction of $\mathcal{F}$ stated in 3.2.2, then we have $\ell(\mathcal{F}) \leq r$. On the other hand, we can always find a reduction of $\mathcal{F}$ consisting of $\ell(\mathcal{F})$ elements. Hence we have the following.

Lemma 3.2.3 $\mathrm{ht}_A F_1 \leq \ell(\mathcal{F}) \leq d$.

Definition 3.2.4 *We denote by* $\mathrm{ad}(\mathcal{F})$ *the difference* $\ell(\mathcal{F}) - \mathrm{ht}_A F_1$ *and call it the analytic deviation of* $\mathcal{F}$. *In particular,* $\mathcal{F}$ *is said to be equimultiple, if* $\mathrm{ad}(\mathcal{F}) = 0$.

Example 3.2.5 *Let* I *be an ideal of* A *with* $\mathrm{ht}_A I = s < d$. *Let* $F_n = I^{(n)}$ *for any* n. *Then* $\mathrm{ad}(\mathcal{F}) < d - s$. *In particular,* $\mathcal{F}$ *is equimultiple, if* $s = d - 1$.

Proof. Since $\operatorname{depth} A/F_n > 0$ for any n, we can choose $x \in \mathfrak{m}$ so that it is a non-zero-divisor on $\mathrm{G}(\mathcal{F})$. Then

$$\ell(\mathcal{F}) = \dim \mathrm{G}(\mathcal{F})/\mathfrak{m}\mathrm{G}(\mathcal{F}) \leq \dim \mathrm{G}(\mathcal{F})/x\mathrm{G}(\mathcal{F}) = d - 1\,.$$

Hence we get the required inequality.

Example 3.2.6 *Let* I *be an ideal with a reduction* $J = (a_1, \cdots, a_r)A$. *Let* $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ *be a filtration such that* $I^n \subseteq F_n \subseteq \overline{I^n}$ *for any* n. *Then* $a_1, \cdots, a_r$ *is a reduction of* $\mathcal{F}$. *In particular, if* I *is equimultiple, so is* $\mathcal{F}$.

Proof. As J is a reduction of I, the extension $\mathrm{R}(J) \subseteq \mathrm{R}(I)$ is module-finite. On the other hand, as $I^n \subseteq F_n \subseteq \overline{I^n}$ for any n, we see that $\mathrm{R}(I) \subseteq \mathrm{R}(\mathcal{F})$ is an integral extension, and so it is module-finite. Hence $\mathrm{R}(\mathcal{F})$ is finitely generated as a module over $\mathrm{R}(J) = A[a_1T, \cdots, a_rT]$. This implies that $a_1, \cdots, a_r$ is a reduction of $\mathcal{F}$.

Let M be a finitely generated A-module and $\mathcal{M} = \{M_n\}_{n\in\mathbb{Z}}$ a family of A-submodules of M such that (i) $M_n \supseteq M_{n+1}$ for any n, (ii) $M = M_n$ for any $n \ll 0$, and (iii) $F_m M_n \subseteq M_{m+n}$ for any m, n. Such $\mathcal{M}$ is called an $\mathcal{F}$-filtration. Let

$$\mathrm{R}'(\mathcal{M}) = \sum_{n\in\mathbb{Z}} M_n T^n \subseteq M[T, T^{-1}] \text{ and}$$
$$\mathrm{G}(\mathcal{M}) = \mathrm{R}'(\mathcal{M})/T^{-1}\mathrm{R}'(\mathcal{M}) = \bigoplus_{n\in\mathbb{Z}} M_n/M_{n+1}$$

Needless to say, $\mathrm{R}'(\mathcal{M})$ (resp. $\mathrm{G}(\mathcal{M})$) is a module over $\mathrm{R}'(\mathcal{F})$ (resp. $\mathrm{G}(\mathcal{F})$). We assume that $\mathrm{R}'(\mathcal{M})$ is finitely generated over $\mathrm{R}'(\mathcal{F})$. The next assertion is a generalization of the theorem of Vallabrega-Valla [15, Theorem 2.3].

Proposition 3.2.7 *Suppose that $k_1, \cdots, k_r$ are non-negative integers and $a_1 \in F_{k_1}, \cdots, a_r \in F_{k_r}$. Let us consider the following two conditions.*

(1) *$a_1T^{k_1}, \cdots, a_rT^{k_r}$ is a $\mathrm{G}(\mathcal{M})$-regular sequence.*

(2) *$a_1, \cdots, a_r$ is an M-regular sequence and*

$$(a_1, \cdots, a_r)M \cap M_n = \sum_{i=1}^{r} a_i M_{n-k_i}$$

for any $n \in \mathbb{Z}$.

Then we always have (1) $\Rightarrow$ (2). The converse holds if $k_1, \cdots, k_r$ are all positive. When the condition (1) is satisfied, there is a natural isomorphism

$$\mathrm{G}(\mathcal{M})/(a_1T^{k_1}, \cdots, a_rT^{k_r})\mathrm{G}(\mathcal{M}) \xrightarrow{\sim} \mathrm{G}(\overline{\mathcal{M}}),$$

where $\overline{\mathcal{M}}$ is the $\mathcal{F}$-filtration

$$\{(a_1, \cdots, a_r)M + M_n/(a_1, \cdots, a_r)M\}_{n\in\mathbb{Z}}$$

of $M/(a_1, \cdots, a_r)M$.

This assertion is well known and the proof is almost same as that of [15, Theorem 2.3].

Finally we state about localization of filtration. For a prime ideal $\mathfrak{p}$, we set $\mathcal{M}_\mathfrak{p} = \{(M_n)_\mathfrak{p}\}_{n\in\mathbb{Z}}$. Notice that $\mathcal{F}_\mathfrak{p}$ is a filtration of $A_\mathfrak{p}$ and $\mathcal{M}_\mathfrak{p}$ is an $\mathcal{F}_\mathfrak{p}$-filtration of $M_\mathfrak{p}$. We always have $\ell(\mathcal{F}_\mathfrak{p}) \leq \ell(\mathcal{F})$. Because $A_\mathfrak{p} \otimes_A \mathrm{G}(\mathcal{F}) \cong \mathrm{G}(\mathcal{F}_\mathfrak{p})$, once $\mathrm{G}(\mathcal{F})$ is Cohen-Macaulay, then so is $\mathrm{G}(\mathcal{F}_\mathfrak{p})$.

3.3 Key lemma

Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ be a filtration of A. Let M ba a finitely generated A-module with $\operatorname{depth}_A M = t$ and $\mathcal{M} = \{M_n\}_{n\in\mathbb{Z}}$ an $\mathcal{F}$-filtration of M. We fix integers r and ℓ such that $0 < r \le \ell \le t$ and take elements $a_1 \in F_{k_1}, \cdots, a_r \in F_{k_r}$ for some positive integers $k_1, \cdots, k_r$ so that $a_1, \cdots, a_r$ form an M-regular sequence. Let $\mathcal{P} = \{\mathfrak{p} \in \operatorname{Spec} A \mid F_1 \subseteq \mathfrak{p}$ and $\operatorname{ht}_A \mathfrak{p} \le d - t + \ell\}$. In this section, we aim to prove the following.

Lemma 3.3.1 *Assume that $a_1T^{k_1}, \cdots, a_rT^{k_r}$ is a $\mathrm{G}(\mathcal{M}_\mathfrak{p})$-regular sequence for any $\mathfrak{p} \in \mathcal{P}$ and $\operatorname{depth}_A M/(a_1, \cdots, a_r)M + M_n \ge t - \ell$ for any $n \le N$, where N is a fixed integer. Then we have the following assertion.*

(1) $\operatorname{depth}_A M/\sum_{i=1}^{r} a_iM_{n-k_i} \ge t - \ell$ *for any* $n \le N + 1$.

(2) $(a_1, \cdots, a_r)M \cap M_n = \sum_{i=1}^{r} a_iM_{n-k_i}$ *for any* $n \le N + 1$.

(3) $\operatorname{depth}_A M/M_n \ge t - \ell$ *for any* $n \le N$.

Let $J_q = (a_1, \cdots, a_q)A$ for $0 \le q \le r$ ($J_0 = 0$). In order to prove 3.3.1, let us consider the following conditions for any integer m.

(A_m) $\operatorname{depth}_A M/\sum_{i=1}^{q} a_iM_{n-k_i} \ge t - \ell$ if $0 \le q \le r$ and $n \le m$.

(B_m) $J_qM \cap M_n = \sum_{i=1}^{q} a_iM_{n-k_i}$ if $0 \le q \le r$ and $n \le m$.

(C_m) $\operatorname{depth}_A M/J_qM + M_n \ge t - \ell$ if $0 \le q \le r$ and $n \le m$.

Lemma 3.3.2 *Let m be an integer. Assume that the conditions* (B_m) *and* (C_m) *are satisfied. Then the condition* (A_{m+1}) *is satisfied.*

Proof. We take any integer $n \le m + 1$ and fix it. Let $N_q = \sum_{i=1}^{q} a_iM_{n-k_i}$ for $0 \le q \le r$ ($N_0 = 0$). We prove that $\operatorname{depth}_A M/N_q \ge t - \ell$ by induction on q. It is obvious for $q = 0$. Let us assume that $q > 0$ and $\operatorname{depth}_A M/N_{q-1} \ge t - \ell$. It is enough to show that $\operatorname{depth}_A N_q/N_{q-1} \ge t - \ell + 1$, since there exists an exact sequence $0 \to N_q/N_{q-1} \to M/N_{q-1} \to M/N_q \to 0$. For that, we consider the following isomorphisms:

$$\begin{aligned} N_q/N_{q-1} &\cong a_qM_{n-k_q}/a_q([N_{q-1} :_M a_q] \cap M_{n-k_q}) \\ &\cong M_{n-k_q}/[N_{q-1} :_M a_q] \cap M_{n-k_q}. \end{aligned}$$

As a_q is a non-zero-divisor on $M/J_{q-1}M$, we have $[N_{q-1} :_M a_q] \subseteq J_{q-1}M$. Moreover, as $n - k_q \leq m$, by the condition (B_m) we get $J_{q-1}M \cap M_{n-k_q} = \sum_{i=1}^{q-1} a_i M_{n-k_q-k_i}$, whose right-hand side is contained in $[N_{q-1} :_M a_q]$. Hence $[N_{q-1} :_M a_q] \cap M_{n-k_q} = J_{q-1}M \cap M_{n-k_q}$, and so

$$N_q/N_{q-1} \cong J_{q-1}M + M_{n-k_q}/J_{q-1}M\,.$$

As the condition (C_m) implies that $\operatorname{depth}_A M/J_{q-1}M + M_{n-k_q} \geq t-\ell$ and as $\operatorname{depth}_A M/J_{q-1}M \geq t-q+1 \geq t-\ell+1$, we get $\operatorname{depth}_A J_{q-1}M + M_{n-k_q}/J_{q-1}M \geq t-\ell+1$, and the proof is completed.

Lemma 3.3.3 *Let m be an integer. Assume that $a_1T^{k_1}, \cdots, a_rT^{k_r}$ is a $\mathrm{G}(\mathcal{M}_\mathfrak{p})$-regular sequence for any $\mathfrak{p} \in \mathcal{P}$ and the condition (A_m) is satisfied. Then the condition (B_m) is satisfied.*

Proof. Let $n \leq m$ and $0 \leq q \leq r$. We take any $\mathfrak{p} \in \operatorname{Ass}_A M/\sum_{i=1}^{q} a_i M_{n-k_i}$. Then $\operatorname{ht}_A \mathfrak{p} \leq d-t+\ell$, since $\operatorname{depth}_A M/\sum_{i=1}^{q} a_i M_{n-k_i} \geq t-\ell$ by the condition (A_m). If $F_1 \subseteq \mathfrak{p}$, we have $\mathfrak{p} \in \mathcal{P}$, and so by the assumption and 3.2.7, $J_q M_\mathfrak{p} \cap (M_n)_\mathfrak{p} = \sum_{i=1}^{q} a_i (M_{n-k_i})_\mathfrak{p}$. When $F_1 \not\subseteq \mathfrak{p}$, we get the same equality as $(M_n)_\mathfrak{p} = (M_{n-k_i})_\mathfrak{p} = M_\mathfrak{p}$. Therefore $J_q M \cap M_n = \sum_{i=1}^{q} a_i M_{n-k_i}$.

Lemma 3.3.4 *Let m be an integer. Assume* $\operatorname{depth}_A M/J_rM + M_n \geq t-\ell$ *for any $n \leq m$. Suppose that the conditions (B_m) and (C_{m-1}) are satisfied. Then the condition (C_m) is satisfied.*

Proof. We take any $n \leq m$ and fix it. We prove $\operatorname{depth}_A M/J_qM + M_n \geq t-\ell$ for any $0 \leq q \leq r$ by descending induction on q. Because this inequality is just the assumption when $q = r$, let us assume $q < r$ and $\operatorname{depth}_A M/J_{q+1}M + M_n \geq t-\ell$. It is enough to show that $\operatorname{depth}_A J_{q+1}M + M_n/J_qM + M_n \geq t-\ell$. Because $J_{q+1}M + M_n/J_qM + M_n \cong J_{q+1}M/J_qM + (J_{q+1}M \cap M_n)$ and the condition (B_m) implies $J_{q+1}M \cap M_n = \sum_{i=1}^{q+1} a_i M_{n-k_i}$, we have

$$\begin{aligned}
J_{q+1}M + M_n/J_qM + M_n &\cong J_{q+1}M/J_qM + a_{q+1}M_{n-k_{q+1}} \\
&\cong a_{q+1}M/(a_{q+1}M \cap J_qM) + a_{q+1}M_{n-k_{q+1}} \\
&= a_{q+1}M/a_{q+1}J_qM + a_{q+1}M_{n-k_{q+1}} \\
&\cong M/J_qM + M_{n-k_{q+1}}.
\end{aligned}$$

Thus we get the required inequality since $\operatorname{depth}_A M/J_qM + M_{n-k_{q+1}} \geq t - \ell$ by the condition (C_{m-1}).

Proof of 3.3.1. Let m be sufficiently small. Then all of the conditions (A_m), (B_m), and (C_m) are obviously satisfied. Hence, applying 3.3.2, 3.3.3, and 3.3.4 successively, we see that the conditions (A_{N+1}), (B_{N+1}), and (C_N) are satisfied. Now we get the assertions (1), (2), and (3) of 3.3.1 as special cases.

3.4 Equimultiple filtration

Let $\mathcal{F}$ be a filtration of A. The following theorem is a characterization of the Cohen-Macaulay property of the form ring associated to an equimultiple filtration of a Cohen-Macaulay ring. It was already proved by Goto [3, Theorem(1.2)] in the case where $\mathcal{F}$ is the symbolic filtration of a prime ideal $\mathfrak{p}$ with $\dim A/\mathfrak{p} = 1$.

Theorem 3.4.1 *Let A be a Cohen-Macaulay ring and $\operatorname{ht}_A F_1 = s$. Let $a_1, \cdots, a_s$ be elements in A such that $a_1 \in F_{k_1}, \cdots, a_s \in F_{k_s}$ for some positive integers $k_1, \cdots, k_s$ and $F_n = \sum_{i=1}^{s} a_i F_{n-k_i}$ for $n \gg 0$. Set $N = \sum_{i=1}^{s} k_i + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \operatorname{Assh}_A A/F_1\}$. Then the following conditions are equivalent.*

(1) $\mathrm{G}(\mathcal{F})$ *is a Cohen-Macaulay ring.*

(2) $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ *is Cohen-Macaulay for any $\mathfrak{p} \in \operatorname{Assh}_A A/F_1$ and $A/(a_1, \cdots, a_s)+F_n$ is Cohen-Macaulay for any $1 \leq n \leq N$.*

When this is the case, A/F_n is a Cohen-Macaulay ring for any $n \geq 1$, $F_n = \sum_{i=1}^{s} a_i F_{n-k_i}$ for any $n > N$, and

$$\mathrm{a}(\mathrm{G}(\mathcal{F})) = \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \operatorname{Assh}_A A/F_1\}.$$

Proof. We prove in the case where $s > 0$. Similarly one can prove the theorem in the case where $s = 0$, omitting most of the argument. We put $K = (a_1, \cdots, a_s)A$ and $\overline{\mathcal{F}} = \{K + F_n/K\}_{n \in \mathbb{Z}}$, which is a filtration of A/K.

(1) $\Rightarrow$ (2) As is noticed at the end of section 2, $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ is a Cohen-Macaulay ring for any $\mathfrak{p} \in \operatorname{Spec} A$. Let $x_{s+1}, \cdots, x_d$ be elements in $\mathfrak{m}$ which form an sop for A/F_1. Notice that $a_1T^{k_1}, \cdots, a_sT^{k_s}, x_{s+1}, \cdots, x_d$ is an sop for $\mathrm{G}(\mathcal{F})$, and so it is a $\mathrm{G}(\mathcal{F})$-regular

sequence. Then by 3.2.7

$$(*) \qquad \mathrm{G}(\mathcal{F})/(a_1T^{k_1}, \cdots, a_sT^{k_s})\mathrm{G}(\mathcal{F}) \cong \mathrm{G}(\overline{\mathcal{F}}) = \bigoplus_{n\geq 0} K + F_n/K + F_{n+1}$$

and $x_{s+1}, \cdots, x_d$ is a regular sequence on $K + F_n/K + F_{n+1}$ for any $n \geq 0$. Hence $\operatorname{depth}_A K + F_n/K + F_{n+1} = d - s$ for any $n \geq 0$. This implies that $A/K + F_n$ is a Cohen-Macaulay ring for any $n \geq 1$. On the other hand, as $x_{s+1}, \cdots, x_d$ is a $\mathrm{G}(\mathcal{F})$-regular sequence, we get also that A/F_n is Cohen-Macaulay for any $n \geq 1$.

(2) $\Rightarrow$ (1) We apply 3.3.1, setting $M = A$, $\mathcal{M} = \mathcal{F}$, and $r = \ell = s$. Notice that in the present case, $\mathcal{P} = \operatorname{Assh}_A A/F_1$, and so by the assumption $a_1T^{k_1}, \cdots, a_sT^{k_s}$ is a $\mathrm{G}(\mathcal{F}_\mathfrak{p})$-regular sequence for any $\mathfrak{p} \in \mathcal{P}$. Moreover, we are assuming that $\operatorname{depth}_A A/K + F_n \geq d - s$ for any $n \leq N$. Thus we get $\operatorname{depth} A/\sum_{i=1}^s a_iF_{N+1-k_i} = d - s$. Suppose that $F_{N+1} \neq \sum_{i=1}^s a_iF_{N+1-k_i}$. Then there exists an associated prime ideal $\mathfrak{q}$ of $F_{N+1}/\sum_{i=1}^s a_iF_{N+1-k_i}$. Since $\mathfrak{q} \in \operatorname{Ass}_A A/\sum_{i=1}^s a_iF_{N+1-k_i}$, we have $\mathfrak{q} \in \operatorname{Assh}_A A/F_1$, and so $a_1T^{k_1}, \cdots, a_sT^{k_s}$ is a $\mathrm{G}(\mathcal{F}_\mathfrak{q})$-regular sequence. Then

$$\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{q})/(a_1T^{k_1}, \cdots, a_sT^{k_s})\mathrm{G}(\mathcal{F}_\mathfrak{q})) = \mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{q})) + \sum_{i=1}^{s} k_i \leq N$$

and the left-hand side of the equality above coincides with

$$\max\{n \mid F_nA_\mathfrak{q} \neq \sum_{i=1}^{s} a_iF_{n-k_i}A_\mathfrak{q} + F_{n+1}A_\mathfrak{q}\}.$$

Hence $F_{N+1}A_\mathfrak{q} = \sum_{i=1}^s a_iF_{N+1-k_i}A_\mathfrak{q}$. However this contradicts that $\mathfrak{q}$ is an associated prime ideal of $F_{N+1}/\sum_{i=1}^s a_iF_{N+1-k_i}$. Thus we see that $F_{N+1} = \sum_{i=1}^s a_iF_{N+1-k_i} \subseteq K$. Therefore $K + F_n = K$ for any $n > N$, and so $\operatorname{depth}_A A/K + F_n \geq d - s$ for any n. Then, applying 3.3.1 again, we get $K \cap F_n = \sum_{i=1}^s a_iF_{n-k_i}$ for any n. This implies that $a_1T^{k_1}, \cdots, a_sT^{k_s}$ is a $\mathrm{G}(\mathcal{F})$-regular sequence, and so again we get the isomorphism $(*)$. Notice that $\mathrm{G}(\overline{\mathcal{F}})$ is a Cohen-Macaulay ring since $\operatorname{depth}_A K + F_n/K + F_{n+1} = d - s$ for any $n \geq 0$. Thus we see that $\mathrm{G}(\mathcal{F})$ is a Cohen-Macaulay ring.

Now we prove the last assertion of the theorem. If $n > N$, then

$$\begin{aligned} F_n &= K \cap F_n \\ &= \sum_{i=1}^{s} a_iF_{n-k_i}. \end{aligned}$$

Notice $\mathrm{a}(\mathrm{G}(\overline{\mathcal{F}})) = \max\{n \mid K + F_n \neq K + F_{n+1}\}$. As $F_n \subseteq K$ for any $n > N$, it follows that $\mathrm{a}(\mathrm{G}(\overline{\mathcal{F}})) \leq N$. Hence we get $\mathrm{a}(\mathrm{G}(\mathcal{F})) \leq \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \operatorname{Assh}_A A/F_1\}$

as $\mathrm{a}(\mathrm{G}(\overline{\mathcal{F}})) = \mathrm{a}(\mathrm{G}(\mathcal{F})) + \sum_{i=1}^{s} k_i$. The converse inequality is obvious and the proof is completed.

3.5 The case where $\mathrm{a}d(\mathcal{F}) = 1$

Let $\mathcal{F} = \{F_n\}_{n\in\mathbb{Z}}$ be a filtration of A with $\mathrm{ht}_A F_1 = s < d$. Throughout this section we always assume that $a_1, \cdots, a_s, a_{s+1}$ are elements in A such that $a_1 \in F_{k_1}, \cdots, a_s \in F_{k_s}, a_{s+1} \in F_{k_{s+1}}$ for some positive integers $k_1, \cdots, k_s, k_{s+1}$ and $F_n = \sum_{i=1}^{s+1} a_i F_{n-k_i}$ for $n \gg 0$. Moreover, we assume that $a_1, \cdots, a_s$ is an A-regular sequence and if $\mathfrak{q} \in \mathrm{Assh}_A A/F_1$, then $F_n A_\mathfrak{q} = \sum_{i=1}^{s} a_i F_{n-k_i} A_\mathfrak{q}$ for $n \gg 0$. It should be noticed that we can always find such $a_1, \cdots, a_s, a_{s+1}$ if $\ell(\mathcal{F}) \leq s+1$. We put

$$\begin{aligned}
\mathcal{P} &= \{\mathfrak{p} \in \mathrm{Spec}\, A \mid F_1 \subseteq \mathfrak{p} \text{ and } \mathrm{ht}_A \mathfrak{p} \leq s+1\}\,, \\
\alpha &= \sum_{i=1}^{s} k_i + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{q})) \mid \mathfrak{q} \in \mathrm{Assh}_A A/F_1\} + 1\,, \\
\beta &= \sum_{i=1}^{s+1} k_i + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \mathcal{P}\}, \text{ and} \\
K &= (a_1, \cdots, a_s)A\,.
\end{aligned}$$

We will often denote a_{s+1} (resp. k_{s+1}) by b (resp. k).

Lemma 3.5.1 *Let $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ be Cohen-Macaulay for any $\mathfrak{p} \in \mathcal{P}$. Then $\alpha \leq \beta - k + 1$.*

Proof. For any $\mathfrak{q} \in \mathrm{Assh}_A A/F_1$, there exists $\mathfrak{p} \in \mathcal{P}$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Then $\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{q})) \leq \mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p}))$. Consequently we get $\alpha - \sum_{i=1}^{s} k_i - 1 \leq \beta - \sum_{i=1}^{s+1} k_i$, and so $\alpha \leq \beta - k + 1$.

Lemma 3.5.2 *Assume that there exists $\mathfrak{q} \in \mathrm{Assh}_A A/F_1$ such that $\mathrm{G}(\mathcal{F}_\mathfrak{q})$ is Cohen-Macaulay. Then $\alpha > 0$ and $F_n A_\mathfrak{q} = \sum_{i=1}^{s} a_i F_{n-k_i} A_\mathfrak{q}$ for any $n \geq \alpha$.*

Proof. Let $S = \mathrm{G}(\mathcal{F}_\mathfrak{q})/(a_1 T^{k_1}, \cdots, a_s T^{k_s})\mathrm{G}(\mathcal{F}_\mathfrak{q})$. Since $a_1 T^{k_1}, \cdots, a_s T^{k_s}$ is a $\mathrm{G}(\mathcal{F}_\mathfrak{q})$-regular sequence, we get $\mathrm{a}(S) = \mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{q})) + \sum_{i=1}^{s} k_i < \alpha$. On the other hand, $\mathrm{a}(S) = \max\{n \mid F_n A_\mathfrak{q} \neq \sum_{i=1}^{s} a_i F_{n-k_i} A_\mathfrak{q}\} \geq 0$. Hence we get the assertion.

Lemma 3.5.3 *Assume that $\mathrm{G}(\mathcal{F}_\mathfrak{q})$ is Cohen-Macaulay for any $\mathfrak{q} \in \mathrm{Assh}_A A/F_1$. Then there exists $x \in \bigcap_{n\geq\alpha}[(\sum_{i=1}^{s} a_i F_{n-k_i}) : F_n]$ such that $a_1, \cdots, a_s, x+b$ is an ssop for A. Moreover, if we take $x_{s+2}, \cdots, x_d \in \mathfrak{m}$ so that $a_1, \cdots, a_s, x+b, x_{s+2}, \cdots, x_d$ is an sop for A, then $a_1 T^{k_1}, \cdots, a_s T^{k_s}, x - bT^k, x_{s+2}, \cdots, x_d$ is an sop for $\mathrm{G}(\mathcal{F})$.*

Proof. Set $I_n = \sum_{i=1}^{s} a_i F_{n-k_i}$ and $J = \mathfrak{m} \cap (\bigcap_{n \geq \alpha} [I_n : F_n])$. Suppose that there exists $\mathfrak{q} \in \mathrm{Assh}_A A/K$ such that $J + bA \subseteq \mathfrak{q}$. Then, for $n \gg 0$, we have $F_1{}^n \subseteq F_n = I_n + bF_{n-k} \subseteq \mathfrak{q}$. This implies $\mathfrak{q} \in \mathrm{Assh}_A A/F_1$, and so by 3.5.2 $F_n A_\mathfrak{q} = I_n A_\mathfrak{q}$ for any $n \geq \alpha$. Hence $JA_\mathfrak{q} = \bigcap_{n \geq \alpha} [I_n A_\mathfrak{q} : F_n A_\mathfrak{q}] = A_\mathfrak{q}$. However it is impossible as $J \subseteq \mathfrak{q}$. Consequently, $J + bA \not\subseteq \mathfrak{q}$ for any $\mathfrak{q} \in \mathrm{Assh}_A A/K$. Then, by [10, Theorem 124], there exists $x \in J$ such that $x + b \notin \mathfrak{q}$ for any $\mathfrak{q} \in \mathrm{Assh}_A A/K$. Needless to say, $a_1, \cdots, a_s, x + b$ is an ssop for A. Now choose $x_{s+2}, \cdots, x_d \in \mathfrak{m}$ so that $a_1, \cdots, a_s, x + b, x_{s+2}, \cdots, x_d$ form an sop for A. Set $\mathfrak{A} = (a_1 T^{k_1}, \cdots, a_s T^{k_s}, x - bT^k, x_{s+2}, \cdots, x_d, T^{-1})\mathrm{R}'(\mathcal{F})$. We would like to show that $\sqrt{\mathfrak{A}}$ is the graded maximal ideal of $\mathrm{R}'(\mathcal{F})$. Take an integer n such that $kn \geq \alpha$. Then, as $(bT^k)^n \cdot x \in I_{kn} T^{kn} = \sum_{i=1}^{s} a_i T^{k_i} \cdot F_{kn-k_i} T^{kn-k_i} \subseteq \mathfrak{A}$ and as $(bT^k)^n (x - bT^k) \in \mathfrak{A}$, we get $(bT^k)^{n+1} \in \mathfrak{A}$. Hence $bT^k \in \sqrt{\mathfrak{A}}$ and $x \in \sqrt{\mathfrak{A}}$. Now, for any $n > 0$, taking m large enough, we get $(F_n T^n)^m \subseteq \sum_{i=1}^{s+1} a_i T^{k_i} \cdot F_{nm-k_i} T^{nm-k_i} \subseteq \sqrt{\mathfrak{A}}$, and so $F_n T^n \subseteq \sqrt{\mathfrak{A}}$. Thus $\mathrm{R}'(\mathcal{F})_+ \subseteq \sqrt{\mathfrak{A}}$. On the other hand, as $a_i = a_i T^{k_i} \cdot (T^{-1})^{k_i} \in \sqrt{\mathfrak{A}}$ for any $1 \leq i \leq s+1$, we have $(a_1, \cdots, a_s, x + b, x_{s+2}, \cdots, x_d)\mathrm{R}'(\mathcal{F}) \subseteq \sqrt{\mathfrak{A}}$. This implies $\mathfrak{m}\mathrm{R}'(\mathcal{F}) \subseteq \sqrt{\mathfrak{A}}$. Thus we get the required assertion.

Theorem 3.5.4 *Let $\mathrm{G}(\mathcal{F})$ be a Cohen-Macaulay ring. Then we have the following assertions.*

(1) $\operatorname{depth} A/K + F_n \geq d - s - 1$ *for any* $n > 0$.

(2) bT^k *is a non-zero-divisor on* $\mathrm{G}(\overline{\mathcal{F}})_{\geq \alpha}$, *where* $\overline{\mathcal{F}}$ *is the filtration* $\{K + F_n/K\}_{n \in \mathbb{Z}}$ *of* A/K.

(3) $\operatorname{depth} A/K + bF_\alpha + F_n \geq d - s - 1$ *for any* $n > 0$.

(4) $F_n = \sum_{i=1}^{s+1} a_i F_{n-k_i}$ *for any* $n > \beta$.

(5) $\mathrm{a}(\mathrm{G}(\mathcal{F})) = \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \mathcal{P}\}$.

Proof. Set $\mathfrak{M} = \mathfrak{m}\mathrm{G}(\mathcal{F}) + \mathrm{G}(\mathcal{F})_+$ and choose $x, x_{s+2}, \cdots, x_d \in \mathfrak{m}$ as in 3.5.3. Then, as $a_1 T^{k_1}, \cdots, a_s T^{k_s}, x_{s+2}, \cdots, x_d, x - bT^k$ is a regular sequence on $\mathrm{G}(\mathcal{F})_\mathfrak{M}$,

$$\mathrm{G}(\mathcal{F})/(a_1 T^{k_1}, \cdots, a_s T^{k_s})\mathrm{G}(\mathcal{F}) \cong \mathrm{G}(\overline{\mathcal{F}}) = \bigoplus_{n \geq 0} K + F_n/K + F_{n+1}$$

and $x_{s+2}, \cdots, x_d, x - bT^k$ is a regular sequence on $\mathrm{G}(\overline{\mathcal{F}})_\mathfrak{M}$. Since $x_{s+2}, \cdots, x_d$ is a $\mathrm{G}(\overline{\mathcal{F}})$-regular sequence, we have $\operatorname{depth}_A K + F_n/K + F_{n+1} \geq d - s - 1$ for any $n \geq 0$, so we get the assertion (1).

Moreover, setting $\tilde{\mathcal{F}} = \{K + (x_{s+2}, \cdots, x_d) + F_n/K + (x_{s+2}, \cdots, x_d)\}_{n\in\mathbb{Z}}$, we have that

$$\mathrm{G}(\overline{\mathcal{F}})/(x_{s+2}, \cdots, x_d)\mathrm{G}(\overline{\mathcal{F}}) \cong \mathrm{G}(\tilde{\mathcal{F}})$$

and $x - bT^k$ is a non-zero-divisor on $\mathrm{G}(\tilde{\mathcal{F}})_{\mathfrak{M}}$. Now, by the choice of x, the following diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ (\mathrm{G}(\tilde{\mathcal{F}})_{\geq\alpha})_{\mathfrak{M}} & \hookrightarrow & \mathrm{G}(\tilde{\mathcal{F}})_{\mathfrak{M}} \\ \downarrow{\scriptstyle -bT^k} & & \downarrow{\scriptstyle x-bT^k} \\ (\mathrm{G}(\tilde{\mathcal{F}})_{\geq\alpha})_{\mathfrak{M}} & \hookrightarrow & \mathrm{G}(\tilde{\mathcal{F}})_{\mathfrak{M}} \\ & & \text{(ex.)} \end{array}$$

is commutative. Hence bT^k is a non-zero-divisor on $\mathrm{G}(\tilde{\mathcal{F}})_{\geq\alpha}$. As $x_{s+2}, \cdots, x_d$ are homogeneous elements of degree 0, they form a regular sequence also on $\mathrm{G}(\overline{\mathcal{F}})_{\geq\alpha}$ and

$$\mathrm{G}(\overline{\mathcal{F}})_{\geq\alpha}/(x_{s+2}, \cdots, x_d)\mathrm{G}(\overline{\mathcal{F}})_{\geq\alpha} \cong (\mathrm{G}(\overline{\mathcal{F}})/(x_{s+2}, \cdots, x_d)\mathrm{G}(\overline{\mathcal{F}}))_{\geq\alpha}.$$

This means that $x_{s+2}, \cdots, x_d, bT^k$ is a regular sequence on $\mathrm{G}(\overline{\mathcal{F}})_{\geq\alpha}$, and so $bT^k, x_{s+2}, \cdots, x_d$ is also a regular sequence on $\mathrm{G}(\overline{\mathcal{F}})_{\geq\alpha}$. In particular, we get the assertion (2).

Now we set $M = K + F_\alpha/K$ and

$$M_n = \begin{cases} K + F_n/K & \text{if} \quad n \geq \alpha \\ M & \text{if} \quad n < \alpha\,. \end{cases}$$

Then $\mathcal{M} = \{M_n\}_{n\in\mathbb{Z}}$ is an $\mathcal{F}$-filtration of M. Because $\mathrm{G}(\overline{\mathcal{F}})_{\geq\alpha} = \mathrm{G}(\mathcal{M})$, it follows that $bT^k, x_{s+2}, \cdots, x_d$ is a regular sequence on $\mathrm{G}(\mathcal{M})$. Hence, setting $\overline{\mathcal{M}}$ to be the filtration $\{bM + M_n/bM\}_{n\in\mathbb{Z}}$ of M/bM, by 3.2.7 we see that

$$\mathrm{G}(\mathcal{M})/bT^k\mathrm{G}(\mathcal{M}) \cong \mathrm{G}(\overline{\mathcal{M}}) = \bigoplus_{n\in\mathbb{Z}} bM + M_n/bM + M_{n+1}$$

and $x_{s+2}, \cdots, x_d$ is a regular sequence on $\mathrm{G}(\overline{\mathcal{M}})$. Therefore $\operatorname{depth}_A bM + M_n/bM + M_{n+1} \geq d - s - 1$ for any n. This implies $\operatorname{depth}_A M/bM + M_n \geq d - s - 1$ for any n. Thus we get the assertion (3) as $M/bM + M_n \cong K + F_\alpha/K + bF_\alpha + F_n$ for $n \geq \alpha$.

Let V be the cokernel of the inclusion $\mathrm{G}(\mathcal{M}) \hookrightarrow \mathrm{G}(\overline{\mathcal{F}})$. Then $[V]_n = 0$ unless $0 \leq n < \alpha$. Take any $\mathfrak{p} \in \mathcal{P}$ with $\operatorname{ht}_A \mathfrak{p} = s + 1$. Let $\mathfrak{N} = \mathfrak{p}\mathrm{G}(\mathcal{F}_\mathfrak{p}) + \mathrm{G}(\mathcal{F}_\mathfrak{p})_+$. Applying the local cohomology functor $\mathrm{H}^0_{\mathfrak{N}}(\cdot)$ to the exact sequence

$$0 \longrightarrow \mathrm{G}(\mathcal{M}_\mathfrak{p}) \longrightarrow \mathrm{G}(\overline{\mathcal{F}}_\mathfrak{p}) \longrightarrow A_\mathfrak{p} \otimes_A V \longrightarrow 0\,,$$

we get an exact sequence

$$(\natural) \qquad \mathrm{H}^0_{\mathfrak{N}}(A_{\mathfrak{p}} \otimes_A V) \to \mathrm{H}^1_{\mathfrak{N}}(\mathrm{G}(\mathcal{M}_{\mathfrak{p}})) \to \mathrm{H}^1_{\mathfrak{N}}(\mathrm{G}(\overline{\mathcal{F}}_{\mathfrak{p}})) \to \mathrm{H}^1_{\mathfrak{N}}(A_{\mathfrak{p}} \otimes_A V)\,.$$

Notice that $\mathrm{a}(\mathrm{G}(\overline{\mathcal{F}}_{\mathfrak{p}})) = \mathrm{a}(\mathrm{G}(\mathcal{F}_{\mathfrak{p}})) + \sum_{i=1}^{s} k_i \leq \beta - k$, and so $[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{G}(\overline{\mathcal{F}}_{\mathfrak{p}}))]_n = 0$ for any $n \geq \beta - k + 1$. On the other hand, $[\mathrm{H}^0_{\mathfrak{N}}(A_{\mathfrak{p}} \otimes_A V)]_n = 0$ for any $n \geq \alpha$. As $\alpha \leq \beta - k + 1$ by 3.5.1, we see that $[\mathrm{H}^1_{\mathfrak{N}}(\mathrm{G}(\mathcal{M}_{\mathfrak{p}}))]_n = 0$ for any $n \geq \beta - k + 1$, and so $\mathrm{a}(\mathrm{G}(\mathcal{M}_{\mathfrak{p}})) \leq \beta - k$. Then, as bT^k is a non-zero-divisor on $\mathrm{G}(\mathcal{M}_{\mathfrak{p}})$, we have $\mathrm{a}(\mathrm{G}(\overline{\mathcal{M}}_{\mathfrak{p}})) \leq \beta$. Since the left-hand side of this inequality coincides with $\max\{n \mid bM_{\mathfrak{p}} + (M_n)_{\mathfrak{p}} \neq bM_{\mathfrak{p}} + (M_{n+1})_{\mathfrak{p}}\}$, it follows that $bM_{\mathfrak{p}} + (M_n)_{\mathfrak{p}} = bM_{\mathfrak{p}} + (M_{n+1})_{\mathfrak{p}}$ for any $n > \beta$, which holds also in the case where $\mathrm{ht}_A \mathfrak{p} = s$. Let $n > \beta$. As is stated above, $\mathrm{depth}_A\, bM + M_n/bM + M_{n+1} \geq d - s - 1$, and this implies $\mathrm{Ass}_A\, bM + M_n/bM + M_{n+1} \subseteq \mathcal{P}$. Therefore we get $bM + M_n = bM + M_{n+1}$, and so

$$\begin{aligned} M_n &= (bM + M_{n+1}) \cap M_n \\ &= bM \cap M_n + M_{n+1} \\ &= bM_{n-k} + M_{n+1}\,. \end{aligned}$$

Then, as $n - k \geq \alpha$, we have $K + F_n = K + bF_{n-k} + F_{n+1}$, and so

$$\begin{aligned} F_n &= (K + bF_{n-k} + F_{n+1}) \cap F_n \\ &= K \cap F_n + bF_{n-k} + F_{n+1} \\ &= \sum_{i=1}^{s+1} a_i F_{n-k_i} + F_{n+1}\,. \end{aligned}$$

This proves the assertion (4).

Since $\mathrm{a}(\mathrm{G}(\overline{\mathcal{M}})) = \max\{n \mid bM + M_n \neq bM + M_{n+1}\}$, we have $\mathrm{a}(\mathrm{G}(\overline{\mathcal{M}})) \leq \beta$ by (4), and so $\mathrm{a}(\mathrm{G}(\mathcal{M})) \leq \beta - k$. Now we consider the exact sequence

$$\mathrm{H}^{d-s}_{\mathfrak{M}}(\mathrm{G}(\mathcal{M})) \longrightarrow \mathrm{H}^{d-s}_{\mathfrak{M}}(\mathrm{G}(\overline{\mathcal{F}})) \longrightarrow \mathrm{H}^{d-s}_{\mathfrak{M}}(V) \longrightarrow 0\,,$$

which is derived from the exact sequence $0 \to \mathrm{G}(\mathcal{M}) \to \mathrm{G}(\overline{\mathcal{F}}) \to V \to 0$. Because $[\mathrm{H}^{d-s}_{\mathfrak{M}}(\mathrm{G}(\mathcal{M}))]_n = 0$ for any $n \geq \beta - k + 1$, $[\mathrm{H}^{d-s}_{\mathfrak{M}}(V)]_n = 0$ for any $n \geq \alpha$, and $\alpha \leq \beta - k + 1$, it follows that $\mathrm{a}(\mathrm{G}(\overline{\mathcal{F}})) \leq \beta - k$. This implies $\mathrm{a}(\mathrm{G}(\mathcal{F})) \leq \beta - \sum_{i=1}^{s+1} k_i = \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_{\mathfrak{p}})) \mid \mathfrak{p} \in \mathcal{P}\}$. On the other hand, it is obvious that $\mathrm{a}(\mathrm{G}(\mathcal{F}_{\mathfrak{p}})) \leq \mathrm{a}(\mathrm{G}(\mathcal{F}))$ for any $\mathfrak{p} \in \mathrm{Spec}\, A$. Thus we get the assertion (5) and the proof is completed.

Lemma 3.5.5 *Let* $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ *be Cohen-Macaulay for any* $\mathfrak{p} \in \mathcal{P}$. *Then* $\beta < \infty$.

Proof. Let $\mathfrak{p} \in \mathcal{P}$ and $\mathrm{ht}_A \mathfrak{p} = s+1$. Then, setting

$$\alpha' = \sum_{i=1}^{s} k_i + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{q})) \mid \mathfrak{q} \in \mathrm{Assh}_A A/F_1 \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\} + 1\,,$$

we get by 3.5.4 (2) that bT^k is a non-zero-divisor on $\mathrm{G}(\overline{\mathcal{F}}_\mathfrak{p})_{\geq \alpha'}$, where $\overline{\mathcal{F}}_\mathfrak{p}$ is the filtration $\{KA_\mathfrak{p} + F_nA_\mathfrak{p}/KA_\mathfrak{p}\}_{n\in\mathbb{Z}}$ of $A_\mathfrak{p}/KA_\mathfrak{p}$. Set M and $\mathcal{M} = \{M_n\}_{n\in\mathbb{Z}}$ as in the proof of 3.5.4. Since $\alpha' \leq \alpha$ and $\mathrm{G}(\overline{\mathcal{F}}_\mathfrak{p})_{\geq\alpha} = \mathrm{G}(\mathcal{M}_\mathfrak{p})$, bT^k is a non-zero-divisor on $\mathrm{G}(\mathcal{M}_\mathfrak{p})$. Now we take $N > 0$ so that $F_n = \sum_{i=1}^{s+1} a_iF_{n-k_i}$ for any $n > N$ and $\alpha + k \leq N$. Then $M_n = bM_{n-k}$ for any $n > N$, and so $\mathrm{a}(\mathrm{G}(\mathcal{M}_\mathfrak{p})/bT^k\mathrm{G}(\mathcal{M}_\mathfrak{p})) \leq N$. Thus it follows that $\mathrm{a}(\mathrm{G}(\mathcal{M}_\mathfrak{p})) \leq N-k$. Now, considering the exact sequence ($\natural$) in the proof of 3.5.4, we get that $\mathrm{a}(\mathrm{G}(\overline{\mathcal{F}}_\mathfrak{p})) \leq N-k$. Hence $\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \leq N - \sum_{i=1}^{s+1} k_i$. Therefore $\beta \leq N$.

Theorem 3.5.6 *Let* A *be a Cohen-Macaulay ring. Let* $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ *be a Cohen-Macaulay ring for any* $\mathfrak{p} \in \mathcal{P}$ *and* $\mathrm{depth}\, A/K + bF_\alpha + F_n \geq d-s-1$ *for any* $1 \leq n \leq \beta$. *Then we have the following assertions.*

(1) $\mathrm{depth}\, A/F_n \geq d-s-1$ *for any* $n > 0$.

(2) *If* $A/K+F_n$ *is Cohen-Macaulay for any* $1 \leq n \leq \alpha$, *then* $\mathrm{G}(\mathcal{F})$ *is a Cohen-Macaulay ring.*

Proof. We may assume that $d \geq s+2$. Set $M = K + F_\alpha/K$. Then $\mathrm{depth}_A M = d-s$. Take x as in 3.5.3. Since M is a maximal Cohen-Macaulay A/K-module, $x+b$ is a non-zero-divisor on M. Hence b is a non-zero-divisor on M as $xM = 0$ by the choice of x. Set $\mathcal{M} = \{M_n\}_{n\in\mathbb{Z}}$ as in the proof of 3.5.4. Let $\mathfrak{p} \in \mathcal{P}$. Then, as is stated in the proof of 3.5.5, bT^k is a non-zero-divisor on $\mathrm{G}(\mathcal{M}_\mathfrak{p})$. Moreover, as

$$M/bM + M_n = \begin{cases} K + F_\alpha/K + bF_\alpha + F_n & \text{if} \quad n \geq \alpha \\ 0 & \text{if} \quad n < \alpha\,, \end{cases}$$

we have $\mathrm{depth}_A M/bM + M_n \geq d-s-1$ for any $n \leq \beta$. Now we apply 3.3.1, setting $r = \ell = 1, N = \beta, a_1 = b$, and $k_1 = k$. It follows that $\mathrm{depth}_A M/bM_{\beta+1-k} \geq d-s-1$. Assume that $M_{\beta+1} \neq bM_{\beta+1-k}$. Then there exists $\mathfrak{p} \in \mathrm{Ass}_A M_{\beta+1}/bM_{\beta+1-k}$. We have $\mathfrak{p} \in \mathcal{P}$ as $\mathfrak{p} \in \mathrm{Ass}_A M/bM_{\beta+1-k}$. Let $\beta' = \mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) + \sum_{i=1}^{s+1} k_i$. Then, by 3.5.4 (4), $F_nA_\mathfrak{p} = \sum_{i=1}^{s+1} a_iF_{n-k_i}A_\mathfrak{p}$ for any $n > \beta'$. Hence $F_{\beta+1}A_\mathfrak{p} \subseteq KA_\mathfrak{p} + bF_{\beta+1-k}A_\mathfrak{p}$, and so $(M_{\beta+1})_\mathfrak{p} = b(M_{\beta+1-k})_\mathfrak{p}$ as $\alpha \leq \beta + 1 - k$. However this contradicts $\mathfrak{p} \in \mathrm{Ass}_A M_{\beta+1}/bM_{\beta+1-k}$. As a consequence, we get $M_{\beta+1} = bM_{\beta+1-k} \subseteq bM$. It follows that $\operatorname{depth}_A M/bM+M_n \geq d-s-1$ for any n. Then, applying 3.3.1 again, we get $bM \cap M_n = bM_{n-k}$ and $\operatorname{depth}_A M/M_n \geq d-s-1$ for any n. In particular, by 3.2.7, bT^k is a non-zero-divisor on $\mathrm{G}(\mathcal{M})$.

Now, considering the exact sequence $0 \to M/M_n \to A/K + F_n \to A/K + F_\alpha \to 0$, we get $\operatorname{depth}_A A/K + F_n \geq d-s-1$ for any n. Therefore, by 3.3.1, $a_1T^{k_1}, \cdots, a_sT^{k_s}$ is a $\mathrm{G}(\mathcal{F})$-regular sequence and $\operatorname{depth}_A A/F_n \geq d-s-1$ for any n.

In order to investigate the Cohen-Macaulayness of $\mathrm{G}(\mathcal{F})$, we notice that $\mathrm{G}(\mathcal{M})$ is a Cohen-Macaulay $\mathrm{G}(\mathcal{F})$-module. In fact, setting $\overline{\mathcal{M}}$ to be the $\mathcal{F}$-filtration $\{bM + M_n/bM\}_{n\in\mathbb{Z}}$ of M/bM, we have

$$\mathrm{G}(\mathcal{M})/bT^k\mathrm{G}(\mathcal{M}) \cong \mathrm{G}(\overline{\mathcal{M}}) = \bigoplus_{n\geq\alpha} bM + M_n/bM + M_{n+1}\,.$$

Since $\operatorname{depth}_A bM+M_n/bM+M_{n+1} \geq d-s-1$ for any n, it follows that $\operatorname{depth}_{\mathrm{G}(\mathcal{F})} \mathrm{G}(\overline{\mathcal{M}}) = d-s-1$, and so $\operatorname{depth}_{\mathrm{G}(\mathcal{F})} \mathrm{G}(\mathcal{M}) = d-s$. Hence we get the required assertion.

Let $V = \mathrm{G}(\overline{\mathcal{F}})/\mathrm{G}(\mathcal{M})$. Assume that $A/K+F_n$ is Cohen-Macaulay for any $1 \leq n \leq \alpha$. Then, as $[V]_n = K+F_n/K+F_{n+1}$ for $0 \leq n < \alpha$ and $[V]_n = 0$ unless $0 \leq n < \alpha$, we have $\operatorname{depth}_{\mathrm{G}(\mathcal{F})} V = d-s$. Therefore, considering the exact sequence $0 \to \mathrm{G}(\mathcal{M}) \to \mathrm{G}(\overline{\mathcal{F}}) \to V \to 0$, we see that $\operatorname{depth} \mathrm{G}(\overline{\mathcal{F}}) = d-s$. Thus it follows that $\mathrm{G}(\mathcal{F})$ is Cohen-Macaulay and the proof is completed.

3.6 Applications

Let A be the formal power series ring $K[[X,Y,Z,W]]$ over a field K. Let I be the ideal of A generated by the maximal minors of the matrix

$$M = \begin{pmatrix} X & Y & Z & W^m \\ Y & Z & W & X \\ Z & W & X & Y^m \end{pmatrix},$$

where m is a positive integer. Then A/I is a Cohen-Macaulay ring with $\dim A/I = 2$. In the following, applying the results in previous sections, we will compute the symbolic powers of I.

For $1 \leq i \leq 4$, let a_i be the minor corresponding to the matrix derived from M deleting the i-th column. Usually, we denote a_1, a_2, a_3, and a_4 by a, b, c, and d, respectively. Then we have the following relations:

$$(\#_1) \qquad Xa - Yb + Zc - W^m d = 0\,,$$

$$(\#_2) \qquad Ya - Zb + Wc - Xd = 0\,, \text{ and}$$

$$(\#_3) \qquad Za - Wb + Xc - Y^m d = 0\,.$$

Lemma 3.6.1 *Let $\mathfrak{p} \in \mathrm{Assh}_A A/I$. Then $IA_\mathfrak{p} = (a, d)A_\mathfrak{p}$. Hence $I^{(n)}A_\mathfrak{p} = I^n A_\mathfrak{p}$ for any n and $\mathrm{G}(IA_\mathfrak{p})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(IA_\mathfrak{p})) = -2$.*

Proof. Let $\mathfrak{q}$ be the ideal of A generated by the maximal minors of the mtrix

$$\begin{pmatrix} Y & Z \\ Z & W \\ W & X \end{pmatrix}.$$

Then $I \not\subseteq \mathfrak{q}$ as $\mathfrak{q} \subseteq (Y, Z, W)A$ and $b \equiv -X^3 \mod (Y, Z, W)A$. It follows that $\mathfrak{q} \not\subseteq \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Assh}_A A/I$ as $\mathfrak{q}$ is a prime ideal with $\mathrm{ht}_A \mathfrak{q} = 2$. Because $\mathfrak{q}I \subseteq (a, d)A$, $IA_\mathfrak{p} \subseteq (a, d)A_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Assh}_A A/I$. Thus we get the assertion.

Theorem 3.6.2 *Let $m = 1$. Then there exists $e \in I^{(2)} \setminus I^2$ such that $\mathrm{R}_s(I) = A[IT, eT^2]$. When this is the case, $\mathrm{R}_s(I)$ is a Gorenstein ring.*

Proof. Set

$$\begin{aligned} u &= XZ - Y^2, \quad v = X^2 - YW, \quad w = XW - YZ\,, \\ f &= XY - ZW, \quad g = YW - Z^2, \quad h = XZ - W^2\,. \end{aligned}$$

Then we have the following relations:

$$(\#_4) \qquad v(c^2 - bd) = u(b^2 - ac)\,,$$

$$(\#_5) \qquad w(c^2 - bd) = u(bc - ad)\,,$$

$$(\#_6) \qquad f(c^2 - bd) = u(ab - cd)\,,$$

$$(\#_7) \qquad g(c^2 - bd) = u(ac - d^2)\,,$$

$$(\#_8) \qquad h(c^2 - bd) = u(a^2 - bd)\,.$$

Because u, v is a regular sequence, by $(\#_4)$ there exists $e \in A$ such that $ue = c^2 - bd$ and $ve = b^2 - ac$. Moreover, by $(\#_5)$ we have

$$\begin{aligned} we(c^2 - bd) &= ue(bc - ad) \\ &= (c^2 - bd)(bc - ad)\,, \end{aligned}$$

and so $we = bc - ad$. Similarly, using ($\#_6$), ($\#_7$), and ($\#_8$), we get $fe = ab - cd$, $ge = ac - d^2$, and $he = a^2 - bd$. Hence $e \in I^2 : \mathfrak{A}$, where $\mathfrak{A} = (u, v, w, f, g, h)A$. This implies $e \in I^{(2)}$ as $\mathrm{ht}_A \mathfrak{A} \geq 3$. We have $e \notin I^2$ since $(Y, Z, W)A + I^2 = (X^6, Y, Z, W)A$ and $e \equiv X^4 \mod (Y, Z, W)A$.

We set

$$F_n = \begin{cases} \displaystyle\sum_{\substack{i,j \geq 0 \\ 2i+j=n}} e^i I^j & \text{if } n \geq 0 \\ A & \text{if } n < 0. \end{cases}$$

Then $F_1 = I$, $F_2 = I^2 + eA$, and $I^n \subseteq F_n \subseteq I^{(n)}$ for any n. Let $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$. It is easy to see that $\mathcal{F}$ is a filtration such that $F_n = aF_{n-1} + eF_{n-2}$ for any $n \geq 2$. Hence $\mathcal{F}$ is an equimultiple filtration and a, e is a reduction of $\mathcal{F}$.

Let $\mathfrak{p} \in \mathrm{Assh}_A A/F_1$. Then by 3.6.1 $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ ($= \mathrm{G}(I_\mathfrak{p})$) is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) = -2$, and so $1 + 2 + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \mathrm{Assh}_A A/F_1\} = 1$. Notice that A/F_1 is a Cohen-Macaulay ring. Therefore by 3.4.1 we see that $\mathrm{G}(\mathcal{F})$ is Cohen-Macaulay and A/F_n is Cohen-Macaulay for any $n \geq 1$. Now, by [4, Theorem 1.2] it follows that $\mathrm{G}(\mathcal{F})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(\mathcal{F})) = -2$. Then [5, Corollary 1.4] implies that $\mathrm{R}(\mathcal{F})$ is a Gorenstein ring. Let n be a positive integer. Since A/F_n is Cohen-Macaulay, $F_n \subseteq I^{(n)}$, and $F_n A_\mathfrak{p} = I^{(n)} A_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Ass}_A A/F_n = \mathrm{Assh}_A A/I$, we get $F_n = I^{(n)}$. Therefore $\mathrm{R}(\mathcal{F}) = \mathrm{R}_s(I)$ and the proof is completed.

Theorem 3.6.3 *Let $m \geq 2$. Then there exists $e \in I^{(3)} \setminus I^3$ such that $\mathrm{R}_s(I) = A[IT, eT^3]$. When this is the case, $\mathrm{R}_s(I)$ is a Gorenstein ring.*

Proof. We divide the proof into several steps. Let us begin with the following

Claim 1 *Let $\mathfrak{p}$ be a prime ideal such that $I \subseteq \mathfrak{p}$ and $\mathrm{ht}_A \mathfrak{p} \leq 3$. Then $IA_\mathfrak{p}$ is generated by a regular sequence of length 2. Hence $I^{(n)} A_\mathfrak{p} = I^n A_\mathfrak{p}$ for any n and $\mathrm{G}(I_\mathfrak{p})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(I_\mathfrak{p})) = -2$.*

Proof of Claim 1. Let $\mathfrak{A}$ be the ideal of A generated by the 2-minors of the matrix M. As $Y^2 - XZ \in \mathfrak{A}$ and $Y^{m+1} - XZ \in \mathfrak{A}$, we have $Y^2(Y^{m-1} - 1) = Y^{m+1} - Y^2 \in \mathfrak{A}$. Hence, if $\mathfrak{A} \subseteq \mathfrak{p}$, it follows that $Y \in \mathfrak{p}$, and so $\mathfrak{p} = \mathfrak{m}$. However this is impossible. Consequently, $\mathfrak{A} \not\subseteq \mathfrak{p}$. Then there exists $1 \leq \alpha < \beta \leq 3$ and $1 \leq i < j \leq 4$ such that the minor f corresponding to the submatrix of M with rows α, β and columns i, j is not contained in $\mathfrak{p}$.

Let $\{k,l\} = \{1,2,3,4\} \setminus \{i,j\}$. Then, as $fI \subseteq (a_k, a_l)A$, it follows that $IA_\mathfrak{p} \subseteq (a_k, a_l)A_\mathfrak{p}$. Thus we get the claim.

Claim 2 $\operatorname{depth} A/(a,d) + I^2 = 1$.

Proof of Claim 2. We set

$$\varphi_1 = \begin{pmatrix} a & b^2 & bc & c^2 & d \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} Xb & Xc & Yb & Yc & Zb & Zc & d \\ -Y & 0 & -Z & 0 & -W & 0 & 0 \\ Z & -Y & W & -Z & X & -W & 0 \\ 0 & Z & 0 & W & 0 & X & 0 \\ -W^m b & -W^m c & -Xb & -Xc & -Y^m b & -Y^m c & -a \end{pmatrix}, \text{ and}$$

$$\varphi_3 = \begin{pmatrix} Z & 0 & W \\ -W & 0 & -X \\ -Y & W & 0 \\ Z & -X & 0 \\ 0 & -Z & -Y \\ 0 & W & Z \\ X^2 - YW^m & Y^{m+1} - XZ & XY^m - ZW^m \end{pmatrix}.$$

Since we have

$$Xab - Yb^2 + Zbc - W^m bd = 0$$

by $(\#_1)$, the $(1,1)$-component of $\varphi_1\varphi_2$ is 0. Similarly, we see that the other entries of $\varphi_1\varphi_2$ are also 0. Moreover, we get $\varphi_2\varphi_3 = 0$ from the relations:

$$(X^2 - YW^m)d = Y^2 b - XZb + XWc - YZc,$$

$$(Y^{m+1} - XZ)d = Z^2 b - YWb + XYc - ZWc,$$

$$(XY^m - ZW^m)d = YZb - XWb + X^2 c - Z^2 c,$$

$$(X^2 - YW^m)a = XYb - ZW^m b + W^{m+1} c - XZc,$$

$$(Y^{m+1} - XZ)a = Y^m Zb - XWb + X^2 c - Y^m Wc, \text{ and}$$

$$(XY^m - ZW^m)a = Y^{m+1} b - W^{m+1} b + XW^m c - Y^m Zc.$$

We get these relations computing $(\#_1) \times Y - (\#_2) \times X$, $(\#_2) \times Z - (\#_3) \times Y$, $(\#_1) \times Z - (\#_3) \times X$, $(\#_1) \times X - (\#_2) \times W^m$, $(\#_2) \times Y^m - (\#_3) \times X$, and $(\#_1) \times Y^m - (\#_3) \times W^m$. Therefore

$$0 \longrightarrow A^3 \xrightarrow{\varphi_3} A^7 \xrightarrow{\varphi_2} A^5 \xrightarrow{\varphi_1} A \longrightarrow A/(a, b^2, bc, c^2, d)A \longrightarrow 0$$

is a complex. One can see that it is an exact sequence by [2]. Hence we get the claim as $(a, d)A + I^2 = (a, b^2, bc, c^2, d)A$.

Claim 3 *Let $\mathfrak{p}$ be a prime ideal of A such that $I \subseteq \mathfrak{p}$ and $\mathrm{ht}_A \mathfrak{p} \leq 3$. Then aT, dT is a $\mathrm{G}(I_\mathfrak{p})$-regular sequence.*

Proof of Claim 3. Since $\mathrm{G}(I_\mathfrak{p})$ is Cohen-Macaulay by Claim 1, it is enough to show that aT, dT is an ssop for $\mathrm{G}(I_\mathfrak{p})$. As $(Y, Z, W)A \not\subseteq \mathfrak{p}$, by ($\#_1$), ($\#_2$), and ($\#_3$) we have $b \in (a, c, d)A_\mathfrak{p}$, and so $IA_\mathfrak{p} = (a, c, d)A_\mathfrak{p}$. Moreover, by 3.6.1 we have $IA_\mathfrak{q} = (a, d)A_\mathfrak{q}$ for any $\mathfrak{q} \in \mathrm{Assh}_A A/I$. Therefore we get the claim by 3.5.3.

Claim 4 $I^{(2)} = I^2$.

Proof of Claim 4. By Claim 2, Claim 3, and 3.3.1, we get $\mathrm{depth}\, A/I^2 > 0$. This yields $I^{(2)} = I^2$ since, by Claim 1, $I^{(2)}A_\mathfrak{p} = I^2 A_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ such that $I \subseteq \mathfrak{p}$ and $\mathrm{ht}_A \mathfrak{p} \leq 3$.

Claim 5 *There exists $e \in I^{(3)} \setminus I^3$ such that*

($\#_9$) $\quad Xe = b^3 + W^{m-1}a^2d + Y^{m-1}c^2d - Y^{m-1}W^{m-1}bd^2 - 2abc,$

($\#_{10}$) $\quad Ye = c^3 + ab^2 - a^2c + W^{m-1}ad^2 - (1 + W^{m-1})bcd,$

($\#_{11}$) $\quad Ze = Y^{m-1}W^{m-1}d^3 - (Y^{m-1} + W^{m-1})acd - b^2d + a^2b + bc^2,$ *and*

($\#_{12}$) $\quad We = Y^{m-1}cd^2 - (1 + Y^{m-1})abd - ac^2 + a^3 + b^2c.$

Proof of Claim 5. By ($\#_1$) and ($\#_2$) we have

$$W^m d = Xa - Yb + Zc \quad \text{and} \quad Wc = Xd - Ya + Zb.$$

Substituting these equations to

$$c \cdot W^m d = W^{m-1}d \cdot Wc \quad \text{and} \quad b \cdot Wc = c \cdot Wb,$$

we get

($\#_{13}$) $\quad Z(c^2 - W^{m-1}bd) = Y(bc - W^{m-1}ad) - X(ac - W^{m-1}d^2)$ and

($\#_{14}$) $\quad Z(b^2 - ac) = Y(ab - Y^{m-1}cd) - X(bd - c^2).$

Moreover, we substitute ($\#_{13}$) and ($\#_{14}$) to

$$Z(c^2 - W^{m-1}bd) \cdot (b^2 - ac) = (c^2 - W^{m-1}bd) \cdot Z(b^2 - ac)$$

and get

$$X(c^3 + ab^2 - a^2c + W^{m-1}ad^2 - (1 + W^{m-1})bcd)$$
$$= Y(b^3 + W^{m-1}a^2d + Y^{m-1}c^2d - Y^{m-1}W^{m-1}bd^2 - 2abc)\,.$$

Hence there exists $e \in A$ satisfying ($\#_9$) and ($\#_{10}$). The equation ($\#_{14}$) yields

$$Ze(b^2 - ac) = Ye(ab - Y^{m-1}cd) - Xe(bd - c^2)\,.$$

Substituting ($\#_9$) and ($\#_{10}$) to the right-hand side, we get

$$Ze(b^2 - ac) =$$
$$(Y^{m-1}W^{m-1}d^3 - (Y^{m-1} + W^{m-1})acd - b^2d + a^2b + bc^2)(b^2 - ac)\,.$$

This yields ($\#_{11}$). Finally, we get ($\#_{12}$) substituting ($\#_9$), ($\#_{10}$), and ($\#_{11}$) to

$$Wec = Xe \cdot d - Ye \cdot a + Ze \cdot b\,,$$

which is induced from ($\#_2$). Then $e \in I^3 : \mathfrak{m}$, and so $e \in I^{(3)}$. Since $(Y, Z, W)A + I^3 = (X^9, Y, Z, W)A$ and $e \equiv -X^8 \mod (Y, Z, W)A$, we see that $e \notin I^3$ and the proof of the claim is completed.

Now we set

$$F_n = \begin{cases} \displaystyle\sum_{\substack{i,j \geq 0 \\ 3i+j=n}} e^i I^j & \text{if } n \geq 0 \\ A & \text{if } n < 0\,. \end{cases}$$

In particular, $F_1 = I$, $F_2 = I^2$, and $F_3 = I^3 + eA$. Notice that ($\#_9$), ($\#_{10}$), ($\#_{11}$), and ($\#_{12}$) imply that b^3, c^3, bc^2, and b^2c are all contained in $(a, d, e)A$. Hence $F_3 \subseteq (a, d, e)A$. It is easy to see that $\mathcal{F} = \{F_n\}_{n \in \mathbb{Z}}$ is a filtration of A such that $\mathrm{R}(\mathcal{F}) = A[IT, eT^3]$. Moreover, the equalities ($\#_9$), ($\#_{10}$), ($\#_{11}$), and ($\#_{12}$) imply that $F_n = aF_{n-1} + dF_{n-1} + eF_{n-3}$ for any $n \geq 3$. Hence a, d, e is a reduction of $\mathcal{F}$.

Claim 6 *$A/[(a, d) : e]$ is a Cohen-Macaulay ring.*

Proof of Claim 6. Since a, d is a regular sequence contained in I, by [14, Proposition 1.3] $A/[(a, d) : I]$ is a 2-dimensional Cohen-Macaulay ring. Hence it is enough to show that $(a,d) : e = (a,d) : I$. Obviously $(a,d) : e \supseteq (a,d) : I$ as $e \in I$. Suppose $(a,d) : e \neq (a,d) : I$. Then there exists $\mathfrak{p} \in \mathrm{Ass}_A[(a,d) : e]/[(a,d) : I] \subseteq \mathrm{Ass}_A A/[(a,d) : I]$. It follows that $\mathrm{ht}_A \mathfrak{p} = 2$ as $\mathrm{depth}\, A/[(a,d) : I] = 2$. If $e \in \mathfrak{p}$, then $I^3 \subseteq F_3 \subseteq (a,d,e)A \subseteq \mathfrak{p}$, and so $\mathfrak{p} \in \mathrm{Assh}_A A/I$. Hence, in this case, by 3.6.1 we have $(a,d)A_\mathfrak{p} : IA_\mathfrak{p} = A_\mathfrak{p}$. This contradicts that $\mathfrak{p} \in \mathrm{Ass}_A A/[(a,d) : I]$. Consequently $e \notin \mathfrak{p}$, and so $(a,d)A_\mathfrak{p} : e = (a,d)A_\mathfrak{p} : IA_\mathfrak{p} = (a,d)A_\mathfrak{p}$. This is also impossible. Thus we get the required equality.

Claim 7 $\mathrm{depth}\, A/(a,d) + F_3 > 0$.

Proof of Claim 7. Notice that $(a,d) + F_3 = (a,d,e)$. Therefore, considering the exact sequence $0 \to A/[(a,d) : e] \xrightarrow{e} A/(a,d) \to A/(a,d,e) \to 0$, we get the claim as $\mathrm{depth}\, A/[(a,d) : e] = 2$ by Claim 6.

Claim 8 $\mathrm{depth}\, A/F_n > 0$ *for any* $n > 0$ *and* $\mathrm{G}(\mathcal{F})$ *is a Cohen-Macaulay ring.*

Proof of Claim 8. Notice that by 3.6.1, if $\mathfrak{q} \in \mathrm{Assh}_A A/F_1$, we have

$$F_n A_\mathfrak{q} = aF_{n-1}A_\mathfrak{q} + dF_{n-1}A_\mathfrak{q}$$

for any $n > 0$. Let $\mathcal{P} = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid I \subseteq \mathfrak{p} \text{ and } \mathrm{ht}_A \mathfrak{p} \leq 3\}$. Set $\alpha = 1 + 1 + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{q})) \mid \mathfrak{q} \in \mathrm{Assh}_A A/F_1\} + 1$ and $\beta = 1 + 1 + 3 + \max\{\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) \mid \mathfrak{p} \in \mathcal{P}\}$. By Claim 1 we see that, for any $\mathfrak{p} \in \mathcal{P}$, $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) = -2$. As a consequence, we have $\alpha = 1$ and $\beta = 3$. Notice that $(a,d) + eF_1 + F_n = (a,d) + F_n$ for $n = 2,3$. Hence, by Claim 2 and Claim 7, $\mathrm{depth}\, A/(a,d) + eF_1 + F_n > 0$ for any $1 \leq n \leq 3$. Therefore, by 3.5.6 we get the assertion of the claim.

Now we are ready to prove 3.6.3. Since $\mathrm{G}(\mathcal{F})$ is a Cohen-Macaulay ring such that $\mathrm{G}(\mathcal{F}_\mathfrak{p})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(\mathcal{F}_\mathfrak{p})) = -2$ for any $\mathfrak{p} \in \mathcal{P}$, by [4, Theorem 1.2] it follows that $\mathrm{G}(\mathcal{F})$ is a Gorenstein ring with $\mathrm{a}(\mathrm{G}(\mathcal{F})) = -2$. Then [5, Corollary 1.4] implies that $\mathrm{R}(\mathcal{F})$ is a Gorenstein ring. Let n be any positive integer. Since $\mathrm{depth}\, A/F_n > 0$, $F_n \subseteq I^{(n)}$, and $F_nA_\mathfrak{p} = I^{(n)}A_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Ass}_A A/F_n$, we get $F_n = I^{(n)}$. Therefore $\mathrm{R}(\mathcal{F}) = \mathrm{R}_s(I)$ and the proof is completed.

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Chapter 4

Hilbert Coefficients and Buchsbaumness of Associated Graded Rings

4.1 Introduction

Let A be a d-dimensional Noetherian local ring with the maximal ideal $\mathfrak{m}$ and I an $\mathfrak{m}$-primary ideal of A. Then there exist integers $\mathrm{e}_0(I), \mathrm{e}_1(I), \cdots, \mathrm{e}_d(I)$ such that

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0(I)\binom{n+d}{d} - \mathrm{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d(I)$$

for $n \gg 0$. These integers are called the Hilbert coefficients of I and a lot of results are known on them in the case where A is a Cohen-Macaulay ring. For example, as was proved by Northcott [8], we always have $\mathrm{e}_0(I) - \ell_A(A/I) \leq \mathrm{e}_1(I)$. Moreover, provided $A/\mathfrak{m}$ is infinite, Huneke and Ooishi proved that $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I)$ if and only if $I^2 = QI$ for some (any) minimal reduction Q of I, and when this is the case, by [11], the associated graded ring $\mathrm{G}(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a Cohen-Macaulay ring. The purpose of this paper is to extend their results without assuming that A is a Cohen-Macaulay ring.

Suppose that I contains a parameter ideal Q as a reduction. Then, from Northcott's inequality, one can easily deduce that $\mathrm{e}_0(I) - \ell_A(A/I) \leq \mathrm{e}_1(I) - \mathrm{e}_1(Q)$ (See 4.3.1). Assuming that Q is a standard ideal in the sense of [10, Definition 19 of Appendix], we will investigate when the equality $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I) - \mathrm{e}_1(Q)$ holds. In order to state our result, let us fix some notation. For an ideal $\mathfrak{q}$ of A which is minimally generated by $a_1, \cdots, a_s$, we set

$$\Sigma(\mathfrak{q}) = \mathfrak{q} + \sum_{i=1}^{s}[(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_s) :_A a_i].$$

It is easy to see that $\Sigma(\mathfrak{q})$ does not depend on the choice of the minimal system of generators. For a module M over a ring R, we denote by $\mathrm{H}^i_{\mathfrak{a}}(M)$ the i-th local cohomology module of M with respect to $\mathfrak{a}$. In particular, we set $W = \mathrm{H}^0_{\mathfrak{m}}(A)$. Then we have the following.

Theorem 4.1.1 *Suppose that I contains a standard parameter ideal Q as a reduction. Then $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I) - \mathrm{e}_1(Q)$ if and only if $I^2 \subseteq QI + W$ and $\Sigma(Q) \subseteq I$.*

If the length of $\mathrm{H}^i_{\mathfrak{m}}(A)$, which is denoted by $\mathrm{h}^i(A)$, is finite for any $0 \leq i < d$, we have that

$$-\mathrm{e}_1(Q) \leq \sum_{i=0}^{d-1} \binom{d-2}{i-1} \mathrm{h}^i(A)$$

with equality when Q is a standard ideal (See 4.2.4). Therefore, as a consequence of 4.1.1 and [3], we get the next result.

Corollary 4.1.2 *If A is a quasi-Buchsbaum ring, then*

$$\sup_{\sqrt{I}=\mathfrak{m}} \{\mathrm{e}_0(I) - \ell_A(A/I) - \mathrm{e}_1(I)\} = \sum_{i=0}^{d-1} \binom{d-2}{i-1} \mathrm{h}^i(A).$$

Moreover, assuming that A is a Buchsbaum ring or a slightly different condition, for ideals I which enjoy the property stated in 4.1.1, we will study the Buchsbaumness of $\mathrm{G}(I)$ together with $\mathrm{I}(\mathrm{G}(I))$ and $\mathrm{a}(\mathrm{G}(I))$, where $\mathrm{I}(*)$ and $\mathrm{a}(*)$ denote the I-invariant (cf. [10, p. 254]) and a-invariant (cf. [4]) respectively.

Theorem 4.1.3 *Suppose that either (i) A is a Buchsbaum ring or (ii) A is a quasi-Buchsbaum ring and $I \subseteq \mathfrak{m}^2$. If I contains a parameter ideal Q such that $I^2 \subseteq QI + W$ and $\Sigma(Q) \subseteq I$, then $\mathrm{G}(I)$ is a Buchsbaum ring with $\mathrm{I}(\mathrm{G}(I)) = \mathrm{I}(A)$ and $\mathrm{a}(\mathrm{G}(I)) \leq 2 - d$.*

Throughout this paper $(A, \mathfrak{m})$ denotes a commutative Noetherian local ring with $d = \dim A > 0$ and I an $\mathfrak{m}$-primary ideal of A. The Rees algebra $\mathrm{R}(\mathfrak{a})$ of an ideal $\mathfrak{a}$ of a ring R is the subring $R[It]$ of $R[t]$, where t is an indeterminate. The associated graded ring $\mathrm{G}(\mathfrak{a})$ is the quotient ring $\mathrm{R}(\mathfrak{a})/\mathfrak{a}\mathrm{R}(\mathfrak{a})$. For $f \in \mathrm{R}(\mathfrak{a})$, we denote it's image in $\mathrm{G}(\mathfrak{a})$ by $\overline{f}$.

4.2 Preliminaries

We begin with the following result of one dimensional case.

Lemma 4.2.1 *Let $d = 1$. If I contains a parameter ideal Q as a reduction, then we have that $\mathrm{e}_0(I) - \ell_A(A/I) \le \mathrm{e}_1(I) + \ell_A(I \cap W)$ with equality if and only if $I^2 \subseteq QI + W$.*

Proof. Let $B = A/W$. Then B is a Cohen-Macaulay ring with $\dim B = 1$ and QB is a parameter ideal of B contained in IB as a reduction. Hence, by Northcott's inequality and the result of Huneke and Ooishi stated in Introduction, we have that $\mathrm{e}_0(IB) - \ell_B(B/IB) \le \mathrm{e}_1(IB)$ with equality if and only if $I^2B = QIB$. On the other hand, as $\ell_B(B/I^{n+1}B) = \ell_A(A/I^{n+1}) - \ell_A(W)$ for $n \gg 0$, we have $\mathrm{e}_0(IB) = \mathrm{e}_0(I)$ and $\mathrm{e}_1(IB) = \mathrm{e}_1(I) + \ell_A(W)$. Moreover, $\ell_B(B/IB) = \ell_A(A/I) - \ell_A(W) + \ell_A(I \cap W)$. Therefore we get the required assertion as $I^2B = QIB$ if and only if $I^2 \subseteq QI + W$.

When we investigate higher dimensional case, we reduce the dimension using a superficial element (cf. [7, Section 22]), and the next result, which may be well known, plays a key role.

Lemma 4.2.2 *Let $d \ge 2$ and a be a superficial element of I. We set $B = A/aA$. Then $\dim B = d - 1$ and*

$$\mathrm{e}_i(IB) = \begin{cases} \mathrm{e}_i(I) & \text{if } 0 \le i < d-1 \\ \mathrm{e}_{d-1}(I) + (-1)^{d-1}\ell_A(0 :_A a) & \text{if } i = d-1\,. \end{cases}$$

Proof. Let $n \gg 0$. Then $I^{n+1} \cap aA = aI^n$ and $I^n \cap (0 :_A a) = 0$. Hence we have an exact sequence

$$0 \longrightarrow 0 :_A a \longrightarrow A/I^n \xrightarrow{a} (aA + I^{n+1})/I^{n+1} \longrightarrow 0,$$

so that

$$\begin{aligned} & \ell_B(B/I^{n+1}B) \\ = \ & \ell_A(A/I^{n+1}) - \ell_A(A/I^n) + \ell_A(0 :_A a) \\ = \ & \sum_{i=0}^{d}(-1)^i\mathrm{e}_i(I)\binom{n+d-i}{d-i} - \sum_{i=0}^{d}(-1)^i\mathrm{e}_i(I)\binom{n-1+d-i}{d-i} + \ell_A(0 :_A a) \\ = \ & \sum_{i=0}^{d-2}(-1)^i\mathrm{e}_i(I)\binom{n+d-1-i}{d-1-i} + (-1)^{d-1}\{\mathrm{e}_{d-1}(I) + (-1)^{d-1}\ell_A(0 :_A a)\}\,. \end{aligned}$$

Thus we get the required assertion.

Lemma 4.2.3 *Suppose that $A/\mathfrak{m}$ is infinite and J is a reduction of I. Then there exists an element $a \in J$ which is superficial for both of I and J. Moreover, for such element $a \in J$, setting $B = A/aA$, we have $\mathrm{e}_1(I) - \mathrm{e}_1(J) = \mathrm{e}_1(IB) - \mathrm{e}_1(JB)$ provided $d \ge 2$.*

Proof. By taking a general linear form in $\mathrm{G}(J)/\mathfrak{m}\mathrm{G}(J)$, we see the existence of $a \in J$ satisfying the required condition. If $d \geq 3$, we get the equality since $\mathrm{e}_1(IB) = \mathrm{e}_1(I)$ and $\mathrm{e}_1(JB) = \mathrm{e}_1(J)$. Even if $d = 2$, we have

$$\begin{aligned} \mathrm{e}_1(IB) - \mathrm{e}_1(JB) &= \{\mathrm{e}_1(I) - \ell_A(0 :_A a)\} - \{\mathrm{e}_1(J) - \ell_A(0 :_A a)\} \\ &= \mathrm{e}_1(I) - \mathrm{e}_1(J)\,. \end{aligned}$$

Lemma 4.2.4 *Let Q be a parameter ideal of A. We have the following statements provided $\mathrm{h}^i(A)$ is finite for any $0 \leq i < d$.*

(1) *Let $d = 1$. Then $-\mathrm{e}_1(Q) = \mathrm{h}^0(A)$.*

(2) *Let $d \geq 2$. Then we have that*

$$-\mathrm{e}_1(Q) \leq \sum_{i=1}^{d-1} \binom{d-2}{i-1} \mathrm{h}^i(A)$$

with equality if Q is a standard ideal.

Proof. Let $d = 1$. Then, taking $n \gg 0$ such that $W = 0 :_A Q^n$ and $\ell_A(A/Q^n) = \mathrm{e}_0(Q) \cdot n - \mathrm{e}_1(Q)$, we see that $-\mathrm{e}_1(Q) = \ell_A(W)$ since $\mathrm{e}_0(Q) \cdot n = \mathrm{e}_0(Q^n) = \ell_A(A/Q^n) - \ell_A(0 :_A Q^n)$. Thus we get the assertion (1).

Next we assume that $d \geq 2$. Moreover, in order to prove the assertion (2), we may assume that $A/\mathfrak{m}$ is infinite. Then we can choose $a \in Q \setminus \mathfrak{m}Q$ which is a superficial element of Q. Let $B = A/aA$ and $0 \leq i < d-1$. Considering the exact sequence

$$0 \longrightarrow 0 :_A a \longrightarrow A \xrightarrow{a} A \longrightarrow B \longrightarrow 0,$$

we get the exact sequence

$$(\#) \quad \mathrm{H}^i_{\mathfrak{m}}(A) \xrightarrow{a} \mathrm{H}^i_{\mathfrak{m}}(A) \longrightarrow \mathrm{H}^i_{\mathfrak{m}}(B) \longrightarrow \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \xrightarrow{a} \mathrm{H}^{i+1}_{\mathfrak{m}}(A)\,.$$

Hence it follows that $\mathrm{h}^i(B) \leq \mathrm{h}^i(A) + \mathrm{h}^{i+1}(A)$ with equality when Q is a standard ideal.

Let $d = 2$. Then $-\mathrm{e}_1(Q) = -\{\mathrm{e}_1(QB) + \ell_A(0 :_A a)\} = \mathrm{h}^0(B) - \ell_A(0 :_A a)$. Because the exact sequence (#) implies $\mathrm{h}^0(B) \leq \ell_A(0 :_W a) + \mathrm{h}^1(A)$, we have $-\mathrm{e}_1(Q) \leq \mathrm{h}^1(A)$. Furthermore, if Q is standard, then $\mathrm{h}^0(B) = \mathrm{h}^0(A) + \mathrm{h}^1(A)$ and $0 :_A a = W$, so that $-\mathrm{e}_1(Q) = \mathrm{h}^1(A)$.

Let $d \geq 3$. Then $\mathrm{e}_1(QB) = \mathrm{e}_1(Q)$. Hence we can easily verify the assertion (2) by induction on d.

4.3 General case

As a result in general case, we give the following assertion, which is a generalization of Northcott's inequality.

Theorem 4.3.1 *If I contains a parameter ideal Q as a reduction, then* $\mathrm{e}_0(I)-\ell_A(A/I) \leq \mathrm{e}_1(I) - \mathrm{e}_1(Q)$.

Proof. We prove by induction on d. If $d = 1$, the assertion follows from 4.2.1 and 4.2.4. Suppose that $d \geq 2$. We may assume that $A/\mathfrak{m}$ is infinite, so that there exists $a \in Q \setminus \mathfrak{m}Q$ which is superficial for both of I and Q. Then, setting $B = A/aA$, we have

$$\begin{aligned} \mathrm{e}_0(I) - \ell_A(A/I) &= \mathrm{e}_0(IB) - \ell_B(B/IB) \quad \text{by 4.2.2} \\ &\leq \mathrm{e}_1(IB) - \mathrm{e}_1(QB) \quad \text{by the inductive hypothesis} \\ &= \mathrm{e}_1(I) - \mathrm{e}_1(Q) \quad \text{by 4.2.3}. \end{aligned}$$

Thus we get the required inequality.

The next result gives a sufficient condition under which the inequality of 4.3.1 turns into an equality in the case where $I = \mathfrak{m}$.

Proposition 4.3.2 *Let Q be a parameter ideal which is a reduction of $\mathfrak{m}$. If there exists an ideal V of A such that $\dim_A V < d$ and $\mathfrak{m}^2 \subseteq Q\mathfrak{m}+V$, then* $\mathrm{e}_0(\mathfrak{m})-1 = \mathrm{e}_1(\mathfrak{m})-\mathrm{e}_1(Q)$.

Proof. We prove by induction on d. If $d = 1$, then $V \subseteq W \subseteq \mathfrak{m}$, so that by 4.2.1 we have $\mathrm{e}_0(\mathfrak{m}) - 1 = \mathrm{e}_1(\mathfrak{m}) + \ell_A(W)$, which yields the required equality since $-\mathrm{e}_1(Q) = \ell_A(W)$ by 4.2.4. Suppose that $d \geq 2$. As we may assume that $A/\mathfrak{m}$ is infinite, it is possible to take an element $a \in Q \setminus \mathfrak{m}Q$ such that $\dim_A V/aV < d-1$ and a is a superficial element for both of $\mathfrak{m}$ and Q. Let $B = A/aA$. Then $\dim_B VB < \dim B$ as VB is a homomorphic image of V/aV, so that by the inductive hypothesis we have $\mathrm{e}_0(\mathfrak{m}B) - 1 = \mathrm{e}_1(\mathfrak{m}B) - \mathrm{e}_1(QB)$, from which the required equality follows since $\mathrm{e}_0(\mathfrak{m}B) = \mathrm{e}_0(\mathfrak{m})$ and $\mathrm{e}_1(\mathfrak{m}B) - \mathrm{e}_1(QB) = \mathrm{e}_1(\mathfrak{m}) - \mathrm{e}_1(Q)$.

Corollary 4.3.3 *Let Q be a parameter ideal which is a reduction of $\mathfrak{m}$. Then* $\mathrm{e}_0(\mathfrak{m}) = 1$ *if and only if* $\mathrm{e}_1(\mathfrak{m}) = \mathrm{e}_1(Q)$.

Proof. Because $0 \leq \mathrm{e}_0(\mathfrak{m}) - 1 = \mathrm{e}_0(\mathfrak{m}) - \ell_A(A/\mathfrak{m}) \leq \mathrm{e}_1(\mathfrak{m}) - \mathrm{e}_1(Q)$, we get $\mathrm{e}_0(\mathfrak{m}) = 1$ if $\mathrm{e}_1(\mathfrak{m}) = \mathrm{e}_1(Q)$. In order to prove the converse implication, we may assume that A

is complete. Now suppose that $\mathrm{e}_0(\mathfrak{m}) = 1$. Let $\mathfrak{a}(\mathfrak{p})$ be the $\mathfrak{p}$-primary component of a primary decomposition of 0. We set $V = \bigcap_{\mathfrak{p} \in \mathrm{Assh}\, A} \mathfrak{a}(\mathfrak{p})$, where $\mathrm{Assh}\, A$ denotes the set of associated primes of A whose coheight is d, and $B = A/V$. Then $\dim_A V < d$ and $\mathrm{e}_0(\mathfrak{m}B) = \mathrm{e}_0(\mathfrak{m}) = 1$, which implies that B is a regular local ring. Hence we have $\mathfrak{m} = Q + V$, so that $\mathfrak{m}^2 \subseteq Q\mathfrak{m} + V$. Therefore, by 4.3.2 it follows that $\mathrm{e}_1(\mathfrak{m}) = \mathrm{e}_1(Q)$.

4.4 The case where Q is a standard ideal

Lemma 4.4.1 *Let $d \geq 2$ and $Q = (a_1, a_2, \cdots, a_d)$ be a standard parameter ideal of A. We set $a = a_1, b = a_d, J = (a_1, a_2, \cdots, a_{d-1})$ and $K = (a_2, a_3, \cdots, a_d)$. Then we have the following.*

(1) $aJ :_A b^2 = aJ :_A b$.

(2) $aJ \cap bA \subseteq aJI$ *provided* $\Sigma(Q) \subseteq I$.

(3) $I^2 \subseteq QI + W$ *provided* $\Sigma(Q) \subseteq I$, $I^2 \subseteq JI + [bA :_A a]$ *and* $I^2 \subseteq KI + [aA :_A b]$.

Proof. (1) Let us take any $x \in aJ :_A b^2$ and write $b^2x = ay$, with $y \in J$. Then, as $y \in [b^2A :_A a] \cap (b^2, a_1, \cdots, a_{d-1})$, there exists $z \in A$ such that $y = b^2z$. Here we notice that $bz \in J$ since $z \in J :_A b^2 = J :_A b$. On the other hand, as $b^2x = ab^2z$, we have $bx - abz \in [0 :_A b] \cap bA = 0$, so that $bx = a \cdot bz \in aJ$. Thus we get $aJ :_A b^2 \subseteq aJ :_A b$ and the converse inclusion is obvious.

(2) Let us take any $\xi \in aJ \cap bA$ and write $\xi = ay = bz$, with $y \in J$ and $z \in A$. Moreover, we write $y = a_1y_1 + \cdots + a_{d-1}y_{d-1}$, with $y_1, \cdots, y_{d-1} \in A$. It is enough to show $y_i \in I$ for any $1 \leq i \leq d-1$. However, as $y_1 \in K :_A a^2 = K :_A a \subseteq \Sigma(Q) \subseteq I$, we may consider only the case that $d \geq 3$ and $2 \leq i \leq d-1$. Because $ay_1 \in K$, we can express $ay_1 = a_2z_2 + \cdots + a_dz_d$, with $z_2, \cdots, z_d \in A$. Then $z_i \in (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) :_A a_i \subseteq I$ for any $2 \leq i \leq d$. On the other hand, as

$$\begin{aligned} bz &= a(a_2z_2 + \cdots + a_dz_d) + aa_2y_2 + \cdots + aa_{d-1}y_{d-1} \\ &= aa_2(y_2 + z_2) + \cdots + aa_{d-1}(y_{d-1} + z_{d-1}) + aa_dz_d\,, \end{aligned}$$

it follows that

$$\begin{aligned} y_i + z_i &\in (a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) :_A aa_i \\ &\subseteq (a_2, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) :_A a_i \subseteq I \end{aligned}$$

for $2 \leq i \leq d-1$, and hence we get $y_i \in I$.

(3) It is enough to show $[aA :_A b] \cap I^2 \subseteq JI + W$. Let us take any $x \in [aA :_A b] \cap I^2$. Then, $bx = ay$ for some $y \in A$, and $ax = a\xi + bz$ for some $\xi \in JI$ and $z \in A$. From these equalities we get $a^2y = ab\xi + b^2z$. Hence $z \in aJ :_A b^2 = aJ :_A b$, so that $bz = a\eta$ for some $\eta \in JI$. Then it follows that $ax = a\xi + a\eta$, which implies $x - \xi - \eta \in 0 :_A a = W$. Thus we have $x \in JI + W$ and the proof is completed.

Proof of Theorem 4.1.1. We prove by induction on d. By 4.2.1 and 4.2.4 we get the assertion when $d = 1$. Suppose that $d \geq 2$. As we may assume that $A/\mathfrak{m}$ is infinite, it is possible to choose a minimal system of generators $a_1, \cdots, a_d$ of Q such that a_1 and a_d are superficial for both of I and Q. We set $a = a_1, b = a_d, B = A/aA, J = (a_1, \cdots, a_{d-1}), K = (a_2, \cdots, a_d)$ and $Q_i = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d)$ for $1 \leq i \leq d$. Because $\mathrm{e}_0(I) = \mathrm{e}_0(IB), \ell_A(A/I) = \ell_B(B/IB)$ and $\mathrm{e}_1(I) - \mathrm{e}_1(Q) = \mathrm{e}_1(IB) - \mathrm{e}_1(KB)$, by the inductive hypothesis we have $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I) - \mathrm{e}_1(Q)$ if and only if $I^2B \subseteq KIB + \mathrm{H}^0_{\mathfrak{m}}(B)$ and $\Sigma(KB) \subseteq IB$, which holds if $I^2 \subseteq QI + W$ and $\Sigma(Q) \subseteq I$ since $WB \subseteq \mathrm{H}^0_{\mathfrak{m}}(B)$ and $\Sigma(KB) \subseteq \Sigma(Q)B$. Now we assume that $\mathrm{e}_0(I) - \ell_A(A/I) = \mathrm{e}_1(I) - \mathrm{e}_1(Q)$. Then it follows that $I^2 \subseteq KI + [aA :_A b]$ and $Q_i :_A a_i \subseteq I$ for $2 \leq i \leq d$. Moreover, by passing A/bA we get $I^2 \subseteq JI + [bA :_A a]$ and $Q_i :_A a_i \subseteq I$ for $1 \leq i \leq d-1$. Therefore, as $\Sigma(Q) \subseteq I$, we have $I^2 \subseteq QI + W$ by 4.4.1 and the proof is completed.

Proof of Corollary 4.1.2. We may assume that $A/\mathfrak{m}$ is infinite. Then any ideal of A has a minimal reduction, so that by 4.2.4 and 4.3.1 we have

$$\mathrm{e}_0(I) - \ell_A(A/I) - \mathrm{e}_1(I) \leq \sum_{i=1}^{d-2} \binom{d-2}{i-1} \mathrm{h}^i(A)$$

for any $\mathfrak{m}$-primary ideal I. Hence it is enough to find an $\mathfrak{m}$-primary ideal for which the equality holds. Let $x_1, \cdots, x_d$ be an sop for A contained in $\mathfrak{m}^2$ and $n_1, \cdots, n_d$ be integers not less than 2. We set $Q = ({x_1}^{n_1}, \cdots, {x_d}^{n_d})$ and $I = Q :_A \mathfrak{m}$. Then Q is a standard parameter ideal by [10, Proposition 2.1] and $I^2 = QI$ by [3]. Because we obviously have $\Sigma(Q) \subseteq I$, by 4.1.1 and 4.2.4 it follows that

$$\mathrm{e}_0(I) - \ell_A(A/I) - \mathrm{e}_1(I) = \sum_{i=1}^{d-2} \binom{d-2}{i-1} \mathrm{h}^i(A),$$

and the proof is completed.

Example 4.4.2 *Let $R = k[[X, Y, Z, W]]$ be the formal power series ring with variables X, Y, Z and W over an infinite field k. Let $\mathfrak{a} = (X^2, Y)R, \mathfrak{b} = (Z, W)R$ and $A = R/\mathfrak{a} \cap \mathfrak{b}$. Let x, y, z and w respectively denote the images of X, Y, Z and W in A. We set $Q = (x - z, y - w)A$ and $\mathfrak{m} = (x, y, z, w)A$. Then we have the following assertion.*

(1) $\dim A = 2, \operatorname{depth} A = 1, \mathrm{h}^1(A) = 2$ *and A is not a quasi-Buchsbaum ring.*

(2) $\mathfrak{m}^3 = Q\mathfrak{m}^2$, *but* $\mathfrak{m}^2 \neq Q\mathfrak{m}$.

(3) *If V is an ideal of A with $\dim_A V < 2$, then $V = 0$, so that $\mathfrak{m}^2 \not\subseteq Q\mathfrak{m} + V$.*

(4) $\mathrm{e}_0(\mathfrak{m}) = 3, \mathrm{e}_1(\mathfrak{m}) = 1$ *and* $\mathrm{e}_1(Q) = -1$, *so that* $\mathrm{e}_0(\mathfrak{m}) - \ell_A(A/\mathfrak{m}) = \mathrm{e}_1(\mathfrak{m}) - \mathrm{e}_1(Q)$.

Proof. From the exact sequence $0 \longrightarrow A \longrightarrow R/\mathfrak{a} \oplus R/\mathfrak{b} \longrightarrow R/\mathfrak{a} + \mathfrak{b} \longrightarrow 0$, we get the assertion (1). One can directly check the assertion (2). Because $\dim A/\mathfrak{p} = 2$ for any $\mathfrak{p} \in \operatorname{Ass} A$, we have the assertion (3). The associated graded ring $\mathrm{G}(\mathfrak{m})$ of $\mathfrak{m}$ is isomorphic to

$$k[X, Y, Z, W]/(X^2, Y) \cap (Z, W),$$

so that we have the exact sequence

$$0 \longrightarrow \mathrm{G}(\mathfrak{m}) \longrightarrow k[X, Z, W]/(X^2) \oplus k[X, Y] \longrightarrow k[X]/(X^2) \longrightarrow 0.$$

This implies that the Poincaré series $\mathrm{P}(\mathrm{G}(\mathfrak{m}), \lambda)$ of $\mathrm{G}(\mathfrak{m})$ is

$$\frac{1+\lambda}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} - (1+\lambda),$$

from which it follows that

$$\ell_A(A/\mathfrak{m}^{n+1}) = \frac{3}{2}n^2 + \frac{7}{2}n$$

for $n \geq 2$. Hence $\mathrm{e}_0(\mathfrak{m}) = 3$ and $\mathrm{e}_1(\mathfrak{m}) = 1$. Because k is infinite, there exists $\mu \in k$ such that $c = (x - z) + \mu(y - w)$ is a superficial element of Q. Let $B = A/cA$. Then $\mathrm{e}_1(Q) = \mathrm{e}_1(QB) = -\mathrm{h}^0(B)$ and the exact sequence $0 \longrightarrow A \xrightarrow{c} A \longrightarrow B \longrightarrow 0$ yields the exact sequence

$$0 \longrightarrow \mathrm{H}^0_{\mathfrak{m}}(B) \longrightarrow \mathrm{H}^1_{\mathfrak{m}}(A) \xrightarrow{c} \mathrm{H}^1_{\mathfrak{m}}(A).$$

Because $\mathrm{H}^1_{\mathfrak{m}}(A) \cong R/\mathfrak{a} + \mathfrak{b} \cong k[[X]]/(X^2)$ and $(X - Z) + \mu(Y - W) \equiv X \bmod \mathfrak{a} + \mathfrak{b}$, we have $\mathrm{H}^0_{\mathfrak{m}}(B) \cong [(X^2) :_{k[[X]]} X]/(X^2) = (X)/(X^2)$. Thus we get $\mathrm{e}_1(Q) = -1$ and the proof is completed.

4.5 Buchsbaumness of G(I)

Throughout this section we assume that I contains a parameter ideal $Q = (a_1, \cdots, a_d)$ as a reduction. We set $R = \mathrm{R}(I)$ and $G = \mathrm{G}(I)$. The graded maximal ideal of G is denoted by M. Furthermore, we set $f_i = a_i t \in R$ for $1 \leq i \leq d$. For certain elements $x_1, \cdots, x_n$ of a ring S and an S-module L, we denote by $\mathrm{e}(x_1, \cdots, x_n; L)$ the multiplicity symbol of $x_1, \cdots, x_n$ with respect to L (cf. [10, p. 24]).

Lemma 4.5.1 $\mathrm{e}(f_1{}^{n_1}, \cdots, f_d{}^{n_d}; G_M) = \mathrm{e}(a_1{}^{n_1}, \cdots, a_d{}^{n_d}; A)$ *for any* $n_1, \cdots, n_d > 0$.

Proof. Let G_+ be the ideal of G generated by homogeneous elements of positive degree. As $(f_1, \cdots, f_d)G$ is a reduction of G_+, we have $\mathrm{e}(f_1, \cdots, f_d; G_M) = \mathrm{e}_0((G_+)_M)$. On the other hand, as $\ell_{G_M}(G/(G_+)^n) = \ell_A(A/I^n)$ for any $n > 0$, we have $\mathrm{e}_0((G_+)_M) = \mathrm{e}_0(I)$. Hence it follows that $\mathrm{e}(f_1, \cdots, f_d; G_M) = \mathrm{e}(a_1, \cdots, a_d; A)$. Therefore, for any $n_1, \cdots, n_d >$ 0

$$\begin{aligned}\mathrm{e}(f_1{}^{n_1}, \cdots, f_d{}^{n_d}; G_M) &= n_1 n_2 \cdots n_d \cdot \mathrm{e}(f_1, \cdots, f_d; G_M) \\ &= n_1 n_2 \cdots n_d \cdot \mathrm{e}(a_1, \cdots, a_d; A) = \mathrm{e}(a_1{}^{n_1}, \cdots, a_d{}^{n_d}; A)\,.\end{aligned}$$

Thus we get the required equality.

In the rest of this section, we furthermore assume that Q is a standard ideal such that $I^2 \subseteq QI + W$, $I^3 \subseteq Q$ and $\Sigma(Q) \subseteq I$.

Lemma 4.5.2 *Let* $n_1, \cdots, n_d$ *be positive integers. Then*

$$(a_1{}^{n_1}, \cdots, a_i{}^{n_i}) \cap I^n = \sum_{j=1}^{i} a_j{}^{n_j} I^{n-n_j}$$

for any $n \in \mathbb{Z}$ *and* $1 \leq i \leq d$. *Hence we have*

$$G/(f_1{}^{n_1}, \cdots, f_i{}^{n_i})G \cong \mathrm{G}(IB)\,,$$

where $B = A/(a_1{}^{n_1}, \cdots, a_i{}^{n_i})$.

Proof. We may assume that $n > n_j$ for any $1 \leq j \leq i$. Let $x \in (a_1{}^{n_1}, \cdots, a_i{}^{n_i}) \cap I^n$. Then, as $x \in Q \cap (Q^{n-1}I + W) = Q^{n-1}I$, we can express

$$x = \sum_{\lambda \in \Lambda} y_\lambda a^\lambda \quad (y_\lambda \in I)\,,$$

where Λ is the set of $\lambda = (\lambda_1, \cdots, \lambda_d) \in \mathbb{Z}^d$ such that $\lambda_1 + \cdots + \lambda_d = n-1$ and $a^\lambda = {a_1}^{\lambda_1}{a_2}^{\lambda_2}\cdots{a_d}^{\lambda_d}$. On the other hand, as

$$x \in ({a_1}^{n_1}, \cdots, {a_i}^{n_i}) \cap Q^{n-1} = \sum_{j=1}^{i} {a_j}^{n_j} Q^{n-1-n_j},$$

we can write

$$x = \sum_{\gamma \in \Gamma} z_\gamma a^\gamma \quad (z_\gamma \in A),$$

where $\Gamma = \{\gamma \in \Lambda \mid \gamma_j \geq n_j$ for some $1 \leq j \leq i\}$. It is enough to show that $z_\gamma \in I$ for any $\gamma \in \Gamma$.

Let $B = A[T_1, \cdots, T_d]$ be the polynomial ring with variables $T_1, \cdots, T_d$ over A and $\varphi : B \longrightarrow \mathrm{R}(Q)$ be the homomorphism of A-algebras such that $\varphi(T_j) = f_j$ for $1 \leq j \leq d$. Because $a_1, \cdots, a_d$ is a d-sequence, $\ker\varphi$ is generated by homogeneous elements of degree one (cf. [5]), so that $\ker\varphi \subseteq IB$ as $\Sigma(Q) \subseteq I$. Now we set

$$f = \sum_{\lambda \in \Lambda \setminus \Gamma} y_\lambda T^\lambda + \sum_{\gamma \in \Gamma} (y_\gamma - z_\gamma) T^\gamma.$$

Then $f \in \ker\varphi$. Hence we get $z_\gamma \in I$ for any $\gamma \in \Gamma$.

Lemma 4.5.3 *We have*

(1) $[0 :_G f_1]_n = \{\overline{wt^n} \mid w \in W \cap I^n\}$,

(2) $0 :_G f_1 = [0 :_G f_1]_1 \oplus [0 :_G f_1]_2$,

(3) $\ell_{G_M}(0 :_G f_1) = \ell_A(W)$, *and hence* $\operatorname{depth} G > 0$ *if* $\operatorname{depth} A > 0$.

Proof. (1) Let $x \in I^n$ and $\overline{xt^n} \in 0 :_G f_1$. Then $a_1 x \in I^{n+2}$, so that by 4.5.2 we have $a_1 x = a_1 y$ for some $y \in I^{n+1}$, which implies $x \in I^{n+1} + W$ since $x - y \in 0 :_A a_1 = W$. Hence $\overline{xt^n} = \overline{wt^n}$ for some $w \in W \cap I^n$. Thus we get $[0 :_G f_1]_n \subseteq \{\overline{wt^n} \mid w \in W \cap I^n\}$, and the converse inclusion is obvious.

(2) This follow from the assertion (1) as $W \cap I^n \subseteq W \cap Q = 0$ for $n \geq 3$.

(3) We get this assertion since $[0 :_G f_1]_1 \cong W/W \cap I^2$ and $[0 :_G f_1]_2 \cong W \cap I^2$.

Lemma 4.5.4 $f_1, \cdots, f_d$ *is a standard system of parameters for* G_M. *In particular, it follows that* $\mathrm{H}^0_M(G) = 0 :_G f_1$, *so that* $\mathrm{h}^0(G_M) = \mathrm{h}^0(A)$. *Moreover, we have* $\mathrm{I}(G_M) = \mathrm{I}(A)$.

Proof. By 4.5.2 we have $G/(f_1, \cdots, f_d)G \cong \mathrm{G}(I/Q)$, so that

$$\ell_{G_M}(G/(f_1, \cdots, f_d)G) = \ell_A(A/Q)\,.$$

Similarly, setting $\mathfrak{a} = ({a_1}^2, \cdots, {a_d}^2)$, we have

$$\ell_{G_M}(G/({f_1}^2, \cdots, {f_d}^2)G) = \ell_A(A/\mathfrak{a})\,.$$

Then, using 4.5.1 and that $a_1, \cdots, a_d$ is a standard system of parameters for A, we get

$$\begin{aligned} & \ell_{G_M}(G/(f_1, \cdots, f_d)G) - \mathrm{e}(f_1, \cdots, f_d; G_M) \\ = \;& \ell_A(A/Q) - \mathrm{e}(a_1, \cdots, a_d; A) \\ = \;& \ell_A(A/\mathfrak{a}) - \mathrm{e}({a_1}^2, \cdots, {a_d}^2; A) \\ = \;& \ell_{G_M}(G/({f_1}^2, \cdots, {f_d}^2)G) - \mathrm{e}({f_1}^2, \cdots, {f_d}^2; G_M)\,. \end{aligned}$$

Therefore by [10, Theorem and Definition 17 in Appendix], we have the required assertion.

Lemma 4.5.5 *We have the following.*

(1) *If $0 < i < d$, then $\mathrm{H}^i_M(G)$ is concentrated in degree $1 - i$.*

(2) $\mathrm{a}(G) \leq 2 - d$.

Proof. We prove by induction on d. Let $d = 1$. In this case, the assertion (1) insists nothing. In order to prove the assertion (2), let us consider the exact sequence

$$0 \longrightarrow \mathrm{H}^0_M(G)(-1) \longrightarrow G(-1) \xrightarrow{f_1} G \longrightarrow G/f_1G \longrightarrow 0\,.$$

This sequence yields the exact sequence

$$\mathrm{H}^0_M(G/f_1G) \longrightarrow \mathrm{H}^1_M(G)(-1) \xrightarrow{f_1} \mathrm{H}^1_M(G) \longrightarrow 0\,,$$

which implies $[\mathrm{H}^1_M(G)]_{n-1} \cong [\mathrm{H}^1_M(G)]_n$ for $n \geq 3$ since $[G/f_1G]_n \cong I^n/QI^{n-1} + I^{n+1} = 0$ for $n \geq 3$. Hence we get $[\mathrm{H}^1_M(G)]_n = 0$ for $n \geq 2$, so that $\mathrm{a}(G) \leq 1$.

Now we assume that $d \geq 2$. Let $B = A/W$. Then the kernel of the graded homomorphism $G \longrightarrow \mathrm{G}(IB)$ of A-algebras induced from the canonical surjection $A \longrightarrow B$ has finite length, so that we have $\mathrm{H}^i_M(G) \cong \mathrm{H}^i_M(\mathrm{G}(IB))$ for $i > 0$. On the other hand, QB is a standard parameter ideal of B such that $I^2B = QIB$ and $\Sigma(QB) \subseteq IB$. Hence by 4.5.3 and 4.5.4 we have that f_1 is $\mathrm{G}(IB)$-regular and $f_1 \cdot \mathrm{H}^i_M(\mathrm{G}(IB)) = 0$ for any $0 \leq i < d$.

Furthermore, setting $C = B/a_1B$, we have $\mathrm{G}(IB)/f_1\mathrm{G}(IB) \cong \mathrm{G}(IC)$ by 4.5.2. Therefore we get the exact sequence

$$0 \longrightarrow \mathrm{G}(IB)(-1) \xrightarrow{f_1} \mathrm{G}(IB) \longrightarrow \mathrm{G}(IC) \longrightarrow 0,$$

from which we see that $\mathrm{H}^i_M(\mathrm{G}(IB)) \hookrightarrow \mathrm{H}^i_M(\mathrm{G}(IC))$ for $0 \le i < d$ and $\mathrm{H}^{d-1}_M(\mathrm{G}(IB))$ is a homomorphic image of $\mathrm{H}^{d-2}_M(\mathrm{G}(IC))(1)$. Because $QC = (a_2, \cdots, a_d)C$ is a standard parameter ideal of C such that $I^2C = QIC$ and $\Sigma(QC) \subseteq IC$, the inductive hypothesis insists that $\mathrm{H}^i_M(\mathrm{G}(IC)) = [\mathrm{H}^i_M(\mathrm{G}(IC))]_{1-i}$ for any $0 \le i < d-1$ and $\mathrm{a}(\mathrm{G}(IC)) \le 3-d$. Now the assertion (1) can be verified easily. In order to see the assertion (2), let us consider the exact sequence

$$\mathrm{H}^{d-1}_M(\mathrm{G}(IC)) \longrightarrow \mathrm{H}^d_M(\mathrm{G}(IB))(-1) \xrightarrow{f_1} \mathrm{H}^d_M(\mathrm{G}(IB)) \longrightarrow 0.$$

If $n > 3-d$, then $[\mathrm{H}^{d-1}_M(\mathrm{G}(IC))]_n = 0$, so that $[\mathrm{H}^d_M(\mathrm{G}(IB))]_{n-1} \cong [\mathrm{H}^d_M(\mathrm{G}(IB))]_n$. Hence we have $[\mathrm{H}^d_M(G)]_n \cong [\mathrm{H}^d_M(\mathrm{G}(IB))]_n = 0$ for any $n \ge 3-d$. Therefore we get the assertion (2) and the proof is completed.

Lemma 4.5.6 *Suppose that $a_1, \cdots, a_d$ form a weak sequence (cf. [10, Definition 1.1]) in any order. We arbitrary take $x_i \in \mathfrak{m}$ for $1 \le i \le d$ and set $\xi_i = x_i - a_it$. Then*

$$(\xi_1, \cdots, \xi_d)G \cap \mathrm{H}^0_M(G) = 0.$$

Proof. Let us take any $\varphi \in (\xi_1, \cdots, \xi_d)G \cap \mathrm{H}^0_M(G)$. As $\mathrm{H}^0_M(G) = 0 :_G f_1$ by 4.5.4, we can express $\varphi = \overline{w_1t + w_2t^2}$, with $w_j \in W \cap I^j$ for $j = 1, 2$. We would like to show that $w_j \in I^{j+1}$ for $j = 1, 2$. For that, we write $\varphi = \sum_{i=1}^d \overline{\xi_i} \cdot \overline{\eta_i}$, with $\eta_i \in R$ for $1 \le i \le d$. Taking $N \gg 0$, we can express $\eta_i = \sum_{j=1}^N \eta_{ij}t^j$ ($\eta_{ij} \in I^j$) for $1 \le i \le d$. Our assumption implies $\mathfrak{m}W = 0$, so that $\mathfrak{m}I^2 \subseteq \mathfrak{m}QI$. Hence $I^j \subseteq Q$ for $j \ge 3$. Then, by 4.5.1 we have $\eta_{ij} \in QI^{j-1}$ for $j \ge 3$. Furthermore, we can choose η_{i2} in QI since $\xi_i \in \mathfrak{m}A[t]$, $I^2 \subseteq QI + W$ and $\mathfrak{m}W = 0$. Because

$$w_1t + w_2t^2 \equiv \sum_{i=1}^d \xi_i\eta_i \mod IR,$$

we get the following congruence equations:

$$\begin{aligned}
0 &\equiv \sum_{i=1}^{d} x_i \eta_{i0} \bmod I\,, \\
w_1 &\equiv \sum_{i=1}^{d} (x_i \eta_{i1} - a_i \eta_{i0}) \bmod I^2\,, \\
w_2 &\equiv \sum_{i=1}^{d} (x_i \eta_{i2} - a_i \eta_{i1}) \bmod I^3\,, \\
0 &\equiv \sum_{i=1}^{d} (x_i \eta_{ij} - a_i \eta_{i,j-1}) \bmod I^{j+1} \quad \text{for } 3 \le j \le N \text{ and} \\
0 &\equiv \sum_{i=1}^{d} a_i \eta_{iN} \bmod I^{N+2}\,.
\end{aligned}$$

The third equation implies $w_2 \in Q$, so that $w_2 = 0$. Hence it is enough to show $w_1 \in I^2$. We need the following.

Claim *There exist elements $y_{\alpha\beta}^{(j)} \in I^j$ for any $1 \le j \le N$ and $1 \le \alpha < \beta \le d$ such that*

$$\sum_{i=1}^{d} a_i(\eta_{ij} + \sum_{\alpha<i} x_\alpha y_{\alpha i}^{(j)} - \sum_{i<\beta} x_\beta y_{i\beta}^{(j)}) \in I^{j+2}\,.$$

If this is true, setting

$$v_i = \eta_{i1} + \sum_{\alpha<i} x_\alpha y_{\alpha i}^{(1)} - \sum_{i<\beta} x_\beta y_{i\beta}^{(1)}\,,$$

we have $\sum_{i=1}^{d} a_i v_i \in I^3 = QI^2$. Hence there exist $v_i' \in I^2$ for $1 \le i \le d$ such that $\sum_{i=1}^{d} a_i(v_i - v_i') = 0$. Then, for any $1 \le i \le d$ we get

$$\begin{aligned}
v_i - v_i' &\in (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) :_A a_i \\
&= (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_d) :_A \mathfrak{m}\,,
\end{aligned}$$

so that $x_i(v_i - v_i') \in Q$, which implies $x_i v_i \in Q$ as $x_i v_i' \in \mathfrak{m} I^2 \subseteq Q$. On the other hand, we have $\sum_{i=1}^{d} x_i v_i = \sum_{i=1}^{d} x_i \eta_{i1}$, so that $q \in Q$, where $q = \sum_{i=1}^{d}(x_i \eta_{i1} - a_i \eta_{i0})$. Because $w_1 - q \in I^2$, we have $w_1 - q = q' + w'$ for some $q' \in QI$ and $w' \in W$. Then, as $w_1 - w' = q + q' \in Q \cap W = 0$, we get $w_1 \in I^2$.

Proof of Claim. We prove by descending induction on j. First, we set $y_{\alpha\beta}^{(N)} = 0$ for any $1 \leq \alpha < \beta \leq d$. Next, we assume that $2 \leq j \leq N$ and we have the required elements $y_{\alpha\beta}^{(j)}$. Of course, $y_{\alpha\beta}^{(j)} \in QI^{j-1}$ if $j \geq 3$. However, even if $j = 2$ we can choose $y_{\alpha\beta}^{(j)}$ in QI^{j-1} since $I^2 \subseteq QI + W$ and $\mathfrak{m}W = 0$. Now we set

$$v_{ij} = \eta_{ij} + \sum_{\alpha<i} x_\alpha y_{\alpha i}^{(j)} - \sum_{i<\beta} x_\beta y_{i\beta}^{(j)} .$$

Let $K_\bullet = \mathrm{K}_\bullet(f_1, \cdots, f_d; G)$ be the Koszul complex with the differential maps $\partial_p : K_p \longrightarrow K_{p-1}$ and let $T_1, T_2, \cdots, T_d$ be the free bases of K_1. We set

$$\sigma = \sum_{i=1}^{d} \overline{v_{ij}t^j} \cdot T_i .$$

Then $\sigma \in (f_1, \cdots, f_d)K_1$ as $v_{ij} \in QI^{j-1}$ for any $1 \leq i \leq d$. On the other hand,

$$\partial_1(\sigma) = \sum_{i=1}^{d} f_i \cdot \overline{v_{ij}t^j} = \overline{(\sum_{i=1}^{d} a_i v_{ij})t^{j+1}} = 0$$

in G, so that $\sigma \in \mathrm{Z}_1(K_\bullet)$. Because $f_1, \cdots, f_d$ is a d-sequence on G, we have

$$(f_1, \cdots, f_d)K_1 \cap \mathrm{Z}_1(K_\bullet) = \mathrm{B}_1(K_\bullet) .$$

As a consequence, it follows that there exist elements $y_{\alpha\beta}^{(j-1)} \in I^{j-1}$ for any $1 \leq \alpha < \beta \leq d$ such that

$$\partial_2(\sum_{\alpha<\beta} \overline{y_{\alpha\beta}^{(j-1)} t^{j-1}} \cdot T_\alpha \wedge T_\beta) = \sigma .$$

The left hand side is equal to

$$\sum_{i=1}^{d} \overline{(\sum_{\alpha<i} a_\alpha y_{\alpha i}^{(j-1)} - \sum_{i<\beta} a_\beta y_{i\beta}^{(j-1)})t^j} \cdot T_i ,$$

so that we have

$$v_{ij} \equiv \sum_{\alpha<i} a_\alpha y_{\alpha i}^{(j-1)} - \sum_{i<\beta} a_\beta y_{i\beta}^{(j-1)} \mod \quad I^{j+1}$$

for any $1 \leq i \leq d$. This implies

$$\sum_{i=1}^{d} x_i v_{ij} \equiv \sum_{\alpha<\beta} a_\alpha x_\beta y_{\alpha\beta}^{(j-1)} - \sum_{\alpha<\beta} x_\alpha a_\beta y_{\alpha\beta}^{(j-1)} \mod \quad I^{j+1} .$$

On the other hand,

$$\sum_{i=1}^{d} x_i v_{ij} = \sum_{i=1}^{d} x_i \eta_{ij} \equiv \sum_{i=1}^{d} a_i \eta_{i,j-1} \mod \quad I^{j+1}.$$

Therefore we get

$$\sum_{i=1}^{d} a_i(\eta_{i,j-1} + \sum_{\alpha<i} x_\alpha y_{\alpha i}^{(j-1)} - \sum_{i<\beta} x_\beta y_{i\beta}^{(j-1)}) \in I^{j+1}$$

and the proof is completed.

Proof of Theorem 4.1.3. Only the Buchsbaumness of G is left to show. We prove by induction on d. Because $\mathrm{H}_M^0(G) = \{\overline{w_1 t + w_2 t^2} \mid w_1 \in W, w_2 \in W \cap I^2\}$ and $\mathfrak{m}W = 0$, we have $M \cdot \mathrm{H}_M^0(G) = 0$. Hence G is a Buchsbaum ring if $d = 1$.

Suppose that $d \geq 2$. Let $B = A/W$ and $C = B/a_1 B$. Then C and IC inherits the assumption on A and I in 4.1.3 (cf. Proof of 4.5.5). Therefore the inductive hypothesis implies that $\mathrm{G}(IC)$ is a Buchsbaum ring, so that $\mathrm{G}(IB)$ is also a Buchsbaum ring since $\mathrm{G}(IB)/f_1\mathrm{G}(IB) \cong \mathrm{G}(IC)$, f_1 is $\mathrm{G}(IB)$-regular and $f_1 \cdot \mathrm{H}_M^i(\mathrm{G}(IB)) = 0$ for any $i < d$ (cf. [10, Proposition 2.19]). Furthermore, it is easy to see that the kernel of the graded homomorphism $G \longrightarrow \mathrm{G}(IB)$ coincides with $\mathrm{H}_M^0(G)$. Thus we get that $G/\mathrm{H}_M^0(G)$ is a Buchsbaum ring.

Let $V = \mathfrak{m} + It \subseteq R$. Because we may assume that $A/\mathfrak{m}$ is infinite, we can choose a system of generators $\xi_1, \cdots, \xi_\ell$ of V such that $\{\xi_i\}_{i \in \Lambda}$ form an sop for G_M for any subset $\Lambda \subseteq \{1, 2, \cdots, \ell\}$ with d-elements. In order to prove the Buchsbaumness of G, it is enough to show that

$$(\{\xi_i\}_{i\in\Lambda})G \cap \mathrm{H}_M^0(G) = 0$$

for any Λ stated above (cf. [10, Proposition 2.22]). Let $\Lambda = \{i_1 < i_2 < \cdots < i_d\}$ and $\xi_{i_k} = x_k - b_k t$ ($x_k \in \mathfrak{m}, b_k \in I$) for $1 \leq k \leq d$. Because $(b_1 t, \cdots, b_d t)G + \mathfrak{m}G$ coincides with the M-primary ideal $(\xi_{i_1}, \cdots, \xi_{i_d})G + \mathfrak{m}G$, we have that $b_1 t, \cdots, b_d t$ is an sop for $G/\mathfrak{m}G$. Hence $Q' = (b_1, \cdots, b_d)$ is a reduction of I. Then, by our assumption that (i) A is a Buchsbaum ring or (ii) A is a quasi-Buchsbaum ring and $I \subseteq \mathfrak{m}^2$, we have that Q' is a standard parameter ideal of A, and hence by 4.1.1 we get $I^2 \subseteq Q'I + W$ and $\Sigma(Q') \subseteq I$. Therefore, by 4.5.6 we have $(\xi_{i_1}, \cdots, \xi_{i_d}) \cap \mathrm{H}_M^0(G) = 0$ and the proof is completed.

The next example insists that the assumption of 4.1.3 that $I \subseteq \mathfrak{m}^2$ is necessary when A is a quasi-Buchsbaum ring but not a Buchsbaum ring.

Example 4.5.7 *Let $F = k[[X, Y, Z, W]]$ be the formal power series ring with variables X, Y, Z and W over a field k. Let $\mathfrak{a} = (X, Y)F \cap (Z, W)F \cap (X^2, Y, Z^2, W)F$ and $A = F/\mathfrak{a}$. Let x, y, z and w respectively denote the images of X, Y, Z and W in A. We set $\mathfrak{m} = (x, y, z, w)A$, $a = x - z$, $b = y - w$ and $Q = (a, b)A$. Then we have the following.*

(1) *A is a 2-dimensional quasi-Buchsbaum ring but not a Buchsbaum ring.*

(2) *Q is a standard parameter ideal of A such that $\mathfrak{m}^2 = Q\mathfrak{m} + W$. We obviously have $\Sigma(Q) \subseteq \mathfrak{m}$.*

(3) *$\mathrm{G}(\mathfrak{m})$ is not a Buchsbaum ring.*

Proof. Let $\mathfrak{n} = (X, Y, Z, W)F$ and $\mathfrak{b} = (X, Y)F \cap (Z, W)F$. Then we have the exact sequence $0 \longrightarrow F/\mathfrak{b} \longrightarrow F/(X, Y)F \oplus F/(Z, W)F \longrightarrow F/\mathfrak{n} \longrightarrow 0$, which implies that $F/\mathfrak{b}$ is a 2-dimensional Buchsbaum ring such that $\operatorname{depth} F/\mathfrak{b} = 1, \mathrm{H}^1_{\mathfrak{n}}(F/\mathfrak{b}) \cong k$ and $\mathrm{e}_0(\mathfrak{n}/\mathfrak{b}) = 2$. Because $\mathfrak{b} = \mathfrak{a} + XZF$ and $XZ\mathfrak{n} \subseteq \mathfrak{a}$, considering the exact sequence $0 \longrightarrow \mathfrak{b}/\mathfrak{a} \longrightarrow A \longrightarrow F/\mathfrak{b} \longrightarrow 0$, we get

$$W = \mathrm{H}^0_{\mathfrak{m}}(A) = \mathfrak{b}/\mathfrak{a} = xzA \cong k\,,$$
$$\mathrm{H}^1_{\mathfrak{m}}(A) \cong \mathrm{H}^1_{\mathfrak{n}}(F/\mathfrak{b}) \cong k\,,$$
$$\mathrm{e}_0(\mathfrak{m}) = \mathrm{e}_0(\mathfrak{n}/\mathfrak{b}) = 2\,.$$

Hence A is a 2-dimensional quasi-Buchsbaum ring with $\mathrm{I}(A) = \mathrm{h}^0(A) + \mathrm{h}^1(A) = 2$.

On the other hand, It is easy to see that $A/Q \cong k[[X, Y]]/(X^3, XY, Y^2)$ and Q is a reduction of $\mathfrak{m}$. Then $\ell_A(A/Q) = 4$ and $\mathrm{e}(a, b; A) = \mathrm{e}_0(\mathfrak{m}) = 2$, so that $\ell_A(A/Q) - \mathrm{e}(a, b; A) = \mathrm{I}(A)$, which implies that Q is a standard ideal of A. Because $F/\mathfrak{b}$ is a Buchsbaum ring with $\mathrm{e}_0(\mathfrak{n}/\mathfrak{b}) = 2$ and $\operatorname{depth} F/\mathfrak{b} > 0$, by [1] and [2] it follows that $F/\mathfrak{b}$ has maximal embedding dimension, so that we have $\mathfrak{n}^2 = (X - Z, Y - W)\mathfrak{n} + \mathfrak{b}$. Hence we get $\mathfrak{m}^2 = Q\mathfrak{m} + W$.

Let $a' = x - w$ and $b' = y - z$. Then $A/(a', b')A \cong k[[X, Y]]/(X^2, XY, Y^2)$ and $(a', b')A$ is a reduction of $\mathfrak{m}$. Hence $\ell_A(A/(a', b')A) = 3$ and $\mathrm{e}(a', b'; A) = 2$, so that $\ell_A(A/(a', b')A) - \mathrm{e}(a', b'; A) \neq \mathrm{I}(A)$. Therefore a', b' is not a standard sop for A, which implies that A is not a Buchsbaum ring. Then $\mathrm{G}(\mathfrak{m})$ is also not a Buchsbaum ring since

$$\mathrm{G}(\mathfrak{m}) \cong S/\{(X, Y)S \cap (Z, W)S \cap (X^2, Y, Z^2, W)S\}\,,$$

where $S = k[X, Y, Z, W]$, and the proof is completed.

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