Order Structures of Hypergroup Extensions with respect to Subhypergroups and their Quotients

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Abstract

In this paper we make it clear that there are several results on subhypergroups of a finite commutative hypergroup and their quotients. By considering extensions of a finite commutative hypergroup of order two by another of order two, we can get the complete parametrization of finite commutative hypergroup extensions of this type, whose orders are less than or equal to four. In this parametrization, we can characterize all strong hypergroup extensions. Using N. Wildberger's results we determine all strong hypergroups of order four which have non-trivial subhypergroups.

In our argument, we can estimate orders of extensions of this type. Our result is that there exist many extensions of this type, whose orders are between five and their limit.

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Contents

1 Introduction

The theory of hypergroup has been started as one of extended group theory which has the property that the values of binary operation are non-empty subsets of the total set. Let $\mathbb{Z} = \{(n)\}$ be a total space and "$\oplus$" be its binary operation. For an element $(1) \in \mathbb{Z}$, roughly speaking, an operation is defined as $(1) \oplus (1) = \{(1), (2)\} \subset \mathbb{Z}$, so called a multivalued operation. Besides in the theory of group $(1) \oplus (1) = (2)$, i.e. the value of an operation is a single point as usually considered.

This means that the composition $(1) \oplus (1)$ generates two elements (1) or (2). Assumed that the results (1) and (2) occur in the same probabilities, we can write that $(1) \oplus (1) = 1/2 \cdot (1) + 1/2 \cdot (2)$. The right hand means that the composition of terms (1) and (2) with the probability 1/2 are linearly jointed. Thus hypergroup is developed as in the form of possessing binary operation which admits multi-values with probabilities. This binary operation is needed to be associative. In addition, a binary operation of an element and its "involution" includes a unit element but other binary operations do not. This condition plays an important role in hypergroup theory.

N. Wildberger described a physical image of finite commutative hypergroups and their applications to others fields [W1]. We can recognize very clearly the meaning of products, involutions and unit of a hypergroup when we read his description about quantum mechanics in physics.

> So what is a finite commutative hypergroup? We will give a precise mathematical definition in latter, here is a precise physical definition. A finite commutative hypergroup is a finite collection of particles, (strictly speaking, particle types) say $\{c_0, \ldots, c_n\}$, which are allowed to interact by colliding. When two particle collide, they coalesce to form a third particle. The results of collisions are however not definite; if we collide c_i with c_j the probability of emerging with the single particle c_k is n_{ij}^k and is fixed. The particle c_0 is absorbed in any collision we call it a photon. Each particle has an anti-particle which is uniquely specified by the following rule: collision of two particles has a non-zero probability of resulting in a photon if and only if the two particles are each others anti-particles. Particle interactions are independent of their order in time or their position in space. The structure of the entire system is thus determined completely by the probability n_{ij}^k which are invariant under interchange of all particles with their anti-particles.

N. Wildberger [W1]p.414

The images of theory of hypergroup tells us that multi-valued operations should be described a quantum phenomenon such that some interactions cause several pure states more than one.

We start with the sequence of the sections 2.1-2.5 to recognize hypergroups and their fundamental structures, and give their proofs and construct useful tools for our study.

In the section 2.1 we introduce two definitions of topological hypergroup, not necessarily commutative, and finite commutative hypergroups, in order to observe their differences. The direct products and the join hypergroups for given two finite commutative hypergroups allow us to construct new hypergroups. In the section 2.2, as standard constructions of hypergroups, we introduce examples of class hypergroups by automorphism subgroups, conjugacy class hypergroups and hypergroups obtained from graph theory.

In the section 2.3, we state the harmonic analysis for finite commutative signed hypergroups and their dual hypergroups, which are detailed in [W1]. Using the harmonic analysis of signed hypergroups, N. Wildberger[W2] has determined all strong hypergroups of order three. N. Wildberger's harmonic analysis includes many useful tools, i.e. metric, duality and so on, to construct hypergroup extensions. On sketches of proofs of these statements given here, we develop N. Wildberger's tools for analyzing finite commutative hypergroups. They are consisted of weights, characters and inner products. We can produce the condition of character tables which determines signed hypergroups or not.

In the section 2.4, we study an action of a hypergroup on a set defined as in the manner of [SW1]. Proposition 2.9 determines the equivalent class of a set with respect to an action of a finite commutative hypergroup, so that it can be seen the equivalent relation, in a similar way to group theory. Using Theorem 2.1, all finite sets acted by a finite commutative hypergroup are decomposed into the disjoint summands of irreducible subsets.

The typical examples are actions of subhypergroups on a total hypergroup by its multiplication. In the section 2.5, the quotient class of a finite commutative hypergroup is always a hypergroup. The description of the equivalence with respect to the category of hypergroups is more complicated than one of groups. By the way we can yield the homomorphism theorem such that the quotient hypergroup exists uniquely for any subhypergroup and their isomorphism from the quotient onto its image is induced by is given from a surjective homomorphism. These preparations lead several propositions, so called five lemma, as in an additive categories.

Let $\mathcal{K}$ be a finite commutative hypergroup. If $\mathcal{K}$ has a subhypergroup $\mathcal{H}$, then the quotient $\mathcal{K}/\mathcal{H} = \mathcal{L}$ is a hypergroup, which will be shown in the section 2.5. Under the above preparation, $\mathcal{K}$ is called a *hypergroup extension* of $\mathcal{L}$ by $\mathcal{H}$. Now we set up the problem of hypergroup extensions: Given commutative hypergroups $\mathcal{L}$ and $\mathcal{H}$, require hypergroup extensions in the category of commutative hypergroups, i.e. $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ and analyze them and dual signed hypergroups $\mathcal{K}^\wedge$.

S. Kawakami and his group have studied many examples of hypergroups and analyzed extension type hypergroups of famous non trivial hypergroups. The extension problem of hypergroups in [HKKK], [HK1] and [K] has discussed in general situation for splitting extensions as in a sense of very similar way to group theory. In the papers [HK2] and [KST], all extensions in the case that $\mathcal{H}$ is a group are determined by some parameters and are analyzed their good structures.

On the other hand, when $\mathcal{H}$ is not a group, namely general finite commutative hypergroups, useful tools have not built yet, but only element-wise calculations, in order to construct new hypergroups. Restricting to case of subhypergroups of order two and quotients are group, we have studied almost all the classes of hypergroup extensions when $\mathcal{L}$ are small group [IKS].

The section 3 is devoted to state the order conditions for hypergroup extension problem. Provided that subhypergroups and quotients are fixed, this condition is described in the form of their weights and orders. It is shown, however, that any hypergroup of order under this estimation does not exist when a subhypergroup is a group. In order to show that the evidence of our estimation are valid, we must give examples of hypergroups of orders satisfying these estimation. This is shown in the later sections.

In the section 4, we introduce the results in the paper [IK1]. We consider the extension problem in the case of order four that both of $\mathcal{H}$ and $\mathcal{L}$ are hypergroups of order two. This is the first approach to the model case of hypergroup extensions that $\mathcal{H}$ and $\mathcal{L}$ are not necessarily groups. We can analyze all hypergroup extensions of this type and list strong extensions among them. According to N. Wildberger's results [W2] for hypergroups of order three, we determine all the hypergroups of order four which has a non-trivial subhypergroup and characterized strong hypergroups among them.

The section 5 is devoted to giving explicit answers to the question whether there exists a hypergroup extension of order five or higher, which is described in paper [IK2]. There exist two series of signed hypergroup models are determined by character tables with arbitrary orders in Proposition 5.1 and Proposition 5.2. These models give examples of hypergroup which attain all orders in the range estimated in Proposition 3.3. Theorem 5.1 and Theorem 5.2 show the existence of hypergroups of order between five and the maximal number.

Finally we can show two facts immediately given from our examples which do not occur in group theory. There exists a subhypergroup system $\mathcal{H} \subset \mathcal{K}$ such that a number of elements in some $\mathcal{H}$-equivalence class is larger than order of $\mathcal{H}$. There exists a short exact sequence such that a cross section homomorphism exists but its extension is not isomorphic to the direct product of its subhypergroup and its quotient. From this fact we state the question what is the split extension and when the extension is the form of direct product in the category of finite commutative hypergroups.

2 Preliminaries

2.1 Definitions of Hypergroups

We first introduce the definition of a hypergroup on a locally compact Hausdorff space $\mathcal{K}$, which is a general position of the axiom corresponding to finite commutative hypergroups for our main object. Let $M_b(\mathcal{K})$ be the space of all bounded Radon measures on $\mathcal{K}$. The subsets $M_b^+(\mathcal{K}), M^1(\mathcal{K}) \subset M_b(\mathcal{K})$ are of all non-negative bounded measures and of all probability measures respectively. Let $C_c(\mathcal{K})$ be the space of all continuous complex-valued functions with compact supports on $\mathcal{K}$. For each $\mu \in M_b(\mathcal{K})$, the support of μ is denoted by $\mathrm{supp}(\mu)$ and the norm of μ is given by $||\mu||_1 := \sup\{\, |\mu(f)| : f \in C_c(\mathcal{K}),\, ||f||_\infty \leq 1\,\}$, where $||f||_\infty = \max\{|f(c)| : c \in \mathcal{K}\}$ is the uniform norm. The symbol ε_x means the Dirac measure at a point $x \in \mathcal{K}$, i.e., $\varepsilon_x(f) = f(x)$ for all $f \in C_c(\mathcal{K})$.

Definition 1. (Michael Topology) Let X be a locally compact (Hausdorff) space and $\mathfrak{C}(X)$ be the set of all compact subsets in X which is not void. Michael topology on $\mathfrak{C}(X)$ is given by the subbasis of all

$$\mathfrak{S}_U(V) := \{C \in \mathfrak{C}(X) : C \cap U \neq \emptyset,\, C \subset V\}$$

for open subsets $U, V \in X$. We note that

$$\bigcap_{i=1}^{n} \mathfrak{S}_{U_i}(V_i) = \big\{C \in \mathfrak{C}(X) : C \cap U_i \neq \emptyset \ \forall i,\, C \subset \bigcap_{i=1}^{n} V_i\big\}.$$

These subsets are open basis of the Michael topology. Then it is shown that the Michael topology has the following properties:

- If X is compact, then $\mathfrak{C}(X)$ is compact.
- $\mathfrak{C}(X)$ is a locally compact Hausdorff space.
- $X \ni x \mapsto \{x\} \in \mathfrak{C}(X)$ is a homeomorphism of X onto a closed subset of $\mathfrak{C}(X)$.
- The collection of nonvoid finite subsets of X is dense in $\mathfrak{C}(X)$.
- If Ω is a compact subset of $\mathfrak{C}(X)$ then $B := \bigcup\{A : A \in \Omega\}$ is compact.

Moreover if X is metrizable, then the metric for two compact sets is induced by the metric of X. It is shown that Michael topology is stronger than the topology given by the induced metric. Michael topology was developed by E. Michael [M].

We now define a topological hypergroup.

Definition 2. (Topological hypergroup) A *hypergroup* $\mathcal{K} := (\mathcal{K}, \star, ^\diamond)$ consists of a locally compact space $\mathcal{K}$ together with an associative product, so called convolution, "$\star$" and an involution "$^\diamond$" satisfying the following condition:

(1) A pair $(M_b(\mathcal{K}), \star)$ is a Banach algebra with respect to the norm $||\cdot||_1$.

(2) The mapping $(\mu, \nu) \mapsto \mu\star\nu$ from $M_b^+(\mathcal{K}) \times M_b^+(\mathcal{K})$ into $M_b^+(\mathcal{K})$ is continuous with respect to the weak topology in $M_b^+(\mathcal{K})$.

(3) For $x, y \in \mathcal{K}$ the product $\varepsilon_x \star \varepsilon_y$ belongs to $M^1(\mathcal{K})$ and has a compact support.

(4) The mapping $\mathcal{K} \times \mathcal{K} \ni (x, y) \mapsto \mathrm{supp}(\varepsilon_x \star \varepsilon_y)$ is continuous with respect to the Michael-Hausdorff topology.

(5) There exists a unit element $e \in \mathcal{K}$ such that $\varepsilon_e \star \varepsilon_x = \varepsilon_x \star \varepsilon_e = \varepsilon_x$ for all $x \in \mathcal{K}$.

(6) The involution $(\cdot)^\diamond$ is a homeomorphism on $\mathcal{K}$ inducing an isomorphism of $M_b(\mathcal{K})$ by $(\varepsilon_x)^\diamond = \varepsilon_{x^\diamond}$ and $(\varepsilon_x^\diamond)^\diamond = \varepsilon_x$ such that for $x, y \in \mathcal{K}$, $(\varepsilon_x \star \varepsilon_y)^\diamond = \varepsilon_y^\diamond \star \varepsilon_x^\diamond$ and $e \in \mathrm{supp}(\varepsilon_x \star \varepsilon_y) \Leftrightarrow y^\diamond = x$.

A hypergroup $\mathcal{K}$ is said to be commutative if the Banach algebra $M_b(\mathcal{K})$ is commutative, and *hermitian* if the involution $^\diamond$ is the identity mapping. If $\mathcal{K}$ is finite, then a hypergroup $\mathcal{K}$ has a finite generator.

When the index set is finite and its algebras is commutative, we can define a hypergroup in more simple form. Let $\mathcal{K}$ be finite. Dirac measures generate all of continuous functions in the case of finite hypergroups. Definition 2 is reconstructed to a simple form such that all topological expressions are omitted. We recall some notions and facts on finite commutative hypergroups from Wildberger's paper [W1] and Bloom-Heyer's book [BH].

Definition 3. (Finite commutative hypergroup) A pair $(\mathcal{K}, \mathfrak{A}(\mathcal{K}))$ is called a finite commutative *signed hypergroup* of order $n+1$ if the following conditions (a1)-(a6) are satisfied.

(a1) *Unit*: $\mathfrak{A}(\mathcal{K})$ is a $*$-algebra over $\mathbb{C}$ with unit c_0,

(a2) *Basis*: $\mathcal{K} = \{c_0, c_1, \ldots, c_n\}$ is a $\mathbb{C}$-linear basis of $\mathfrak{A}(\mathcal{K})$,

(a3) *Involution*: $\mathcal{K}^* = \mathcal{K}$,

(a4) *Structure constants*: $c_i c_j = \sum_{k=0}^n n_{ij}^k \, c_k$, where $n_{ij}^k \in \mathbb{R}$ such that

(i) $c_i^* = c_j \iff n_{ij}^0 > 0$, (ii) $c_i^* \neq c_j \iff n_{ij}^0 = 0$,

(a5) *Stochasticity*: $\sum_{k=0}^n n_{ij}^k = 1$ for any i, j,

(a6) *Commutativity*: $c_i c_j = c_j c_i$ for any i, j.

We simply write $\mathcal{K} = (\mathcal{K}, \mathfrak{A}(\mathcal{K}))$ and the order $|\mathcal{K}| = n+1$. In addition if the condition that $n_{ij}^k \geq 0$ for any i, j, k is satisfied, then $\mathcal{K}$ is called a finite commutative *hypergroup*. If $c_i^* = c_i$ for all $i = 1, 2, \ldots, n$, then $\mathcal{K}$ is called a *hermitian* hypergroup. We note that finite commutative groups are hypergroups.

Let $\mathbf{1} = \{c_0\}$ be a hypergroup with a single element of unit c_0. This hypergroup is called to be trivial. We use description in multiplicative manner $\mathbf{1}$ which is necessary to avoid confusion of $*$-algebraic summation.

Let $\mathscr{L}(q) = \{\ell_0, \ell_1 \text{ s.t. } \ell_1^2 = q\ell_0 + (1-q)\ell_1,\ \ell_0 \text{ unit}\}$ be the smallest non trivial hypergroup of order two for $0 < q \leq 1$. Specially $\mathscr{L}(1)$ equals $\mathbb{Z}_2$. $\mathfrak{A}(\mathscr{L}(q))$ is isomorphic as $*$-algebra to $\mathbb{C} \oplus \mathbb{C}$. For a real q, we have the following table on the structures of $\mathscr{L}(q)$:

q	$q < 0$	$q = 0$	$0 < q < 1$	$q = 1$	$1 < q$
$\mathscr{L}(q)$	nothing	ℓ_1 idempotent	hypergroups	group $\cong \mathbb{Z}_2$	signed HG's

The *weight* of an element $c_i \in \mathcal{K}$ is defined by $w(c_i) := (n_{ij}^0)^{-1}$ where $c_j = c_i^*$. Then it is shown that $w(c_i) \geq 1$ if $\mathcal{K}$ is a hypergroup. We define the *weight* of a subset $S \subset \mathcal{K}$:

$$w(S) = \sum_{s \in S} w(s)$$

and the *total weight* $w(\mathcal{K}) := \sum_{i=0}^n w(c_i)$. For example, we have $w(\mathscr{L}(q)) = 1+1/q$.

For a finite commutative signed hypergroup $\mathcal{K}$, a complex valued function χ on $\mathcal{K}$ is called a *character* if

$$(1)\ \chi(c_0) = 1, \quad (2)\ \chi(c_i^*) = \chi(c_i)^-, \quad (3)\ \chi(c_i)\chi(c_j) = \sum_{k=0}^n n_{ij}^k \chi(c_k).$$

A character of $\mathcal{K}$ is also a character of $*$-algebra $\mathfrak{A}(\mathcal{K})$. It is well known that the conjugate function χ^- of a character χ also is a character. Let $\mathcal{K}^\wedge$ be the set of characters on $\mathcal{K}$. In the section 2.3, we prove the following facts: $\mathcal{K}^\wedge = (\mathcal{K}^\wedge, \mathfrak{A}(\mathcal{K}^\wedge))$ becomes a finite commutative signed hypergroup of order $|\mathcal{K}^\wedge| = |\mathcal{K}|$, and $\mathcal{K}^{\wedge\wedge} = (\mathcal{K}^{\wedge\wedge}, \mathfrak{A}(\mathcal{K}^{\wedge\wedge}))$ also becomes a finite commutative signed hypergroups isomorphic to $\mathcal{K}$ as a signed hypergroup [W1][Z]. Moreover $\mathcal{K}$ is said to be self-dual if $\mathcal{K}^\wedge \cong \mathcal{K}$.

For example, $\mathscr{L}(q) = \{\ell_0, \ell_1\}$ has a following table of characters,

	ℓ_0	ℓ_1	$w(\chi_i)$
χ_0	1	1	1
χ_1	1	$-q$	q^{-1}
$w(c_j)$	1	q^{-1}	$1 + q^{-1}$

The values $1, -q$ are immediately obtained from the equation of structure constants $c_1^2 = qc_0 + (1-q)c_1$. Since the table is symmetric, $\mathscr{L}(q)^\wedge = \{\chi_0, \chi_1\}$ is isomorphic to $\mathscr{L}(q)$, hence $\mathscr{L}(q)$ is self dual and consequently strong.

Now we introduce the fundamental methods generating the new hypergroup from given hypergroups.

1. *Direct product hypergroup*

Let $\mathcal{K}, \mathcal{L}$ be two finite commutative hypergroups. The direct product $\mathcal{K} \times \mathcal{L}$ has structure of a hypergroup as in the following way. Let $\mathcal{K} = \{c_0, \ldots, c_m\}$ and $\mathcal{L} = \{c'_0, \ldots, c'_n\}$. The product is defined as component wise product. To see them more precisely, write elements in the form of double index like as $\mathcal{K} \times \mathcal{L} = \{c_{(i,j)} = (c_i, c'_j)$ s.t. $c_i \in \mathcal{K}, c'_j \in \mathcal{L}\}$. The unit of $\mathcal{K} \times \mathcal{L}$ is $c_{(0,0)}$ and the involution is given by $(c_{(i,j)})^* = (c_i^*, c_j^*)$. The structure constants are calculated as

$$c_{(i_1,j_1)} c_{(i_2,j_2)} = (c_{i_1} c_{1_2}, c'_{j_1} c'_{j_2}) = (\sum_{k_1} n_{i_1 i_2}^{k_1} c_{k_1}, \sum_{k_2} n_{j_1 j_2}^{k_2} c'_{k_2})$$

$$= \sum_{(k_1,k_2)} n_{i_1 i_2}^{k_1} n_{j_1 j_2}^{k_2} c_{(k_1,k_2)} = \sum_{(k_1,k_2)} n_{(i_1,j_1)(i_2,j_2)}^{(k_1,k_2)} c_{(k_1,k_2)},$$

where $n_{(i_1,j_1)(i_2,j_2)}^{(k_1,k_2)} = n_{i_1 i_2}^{k_1} \cdot n_{j_1 j_2}^{k_2}$. Thus $\mathcal{K} \times \mathcal{L}$ is checked to be a hypergroup since the axiom (a4) is satisfied in the above structure. It is obviously shown that $|\mathcal{K} \times \mathcal{L}| = |\mathcal{K}| \times |\mathcal{L}|$.

For $\chi_i \in \mathcal{K}^\wedge, \chi'_j \in \mathcal{L}^\wedge$, the double character (χ_i, χ'_j) is defined as

$$(\chi_i, \chi'_j)(c_{i_1}, c'_{j_1}) = \chi_i(c_{i_1}) \chi'_j(c'_{j_1}).$$

It is also checked that all of double characters are characters of $\mathcal{K} \times \mathcal{L}$, namely $(\mathcal{K} \times \mathcal{L})^\wedge = \mathcal{K}^\wedge \times \mathcal{L}^\wedge$. We remark that the last equation explains the dual of the direct product equals the direct product of the duals as hypergroups.

2. *Join hypergroup*

In the category of hypergroups, there exists another construction of new hypergroup from given two hypergroups. This construction is similar with the amalgam sum of sets in the set theory as below.

Let $\mathcal{K}, \mathcal{L}$ be two finite commutative hypergroups. Let $\mathcal{K} = \{c_0, \ldots, c_m\}$ and $\mathcal{L} = \{c'_0, \ldots, c'_n\}$ with the structure constants n_{ij}^k.

Consider the set $\mathcal{K} \vee \mathcal{L} = \{c_0, \ldots, c_n, c'_1, \ldots, c'_m\}$ which is made by smashing one point $\{c'_0\}$ of $\mathcal{K} \cup \mathcal{L}$ into void. The products on $\mathcal{K} \vee \mathcal{L}$ are defined by

(i) $c_i c_j$ is the same expression as in $\mathcal{K}$,

(ii) $c_i c'_j = c'_j \quad$ for $c'_j \in \mathcal{K} \vee \mathcal{L}$,

(iii) $c'_i c'_j = n_{ij}^0 e_0^{\mathcal{K}} + \sum_k n_{ij}^k c'_k \quad$ for $c'_i, c'_j \in \mathcal{K} \vee \mathcal{L}$,

where $e_0^{\mathcal{K}} := w(\mathcal{K})^{-1} \sum_l w(c_l) c_l$ is the normalized Haar measure appeared in the section 2.3. The most useful equation $c_i e_0^{\mathcal{K}} = e_0^{\mathcal{K}}$ leads the associative products are defined over $\mathfrak{A}(\mathcal{K} \vee \mathcal{L})$ from the above definition. By the way this equation of absorbing will be proved in the subsequent section.

It is obviously shown that $|\mathcal{K} \vee \mathcal{L}| = |\mathcal{K}| + |\mathcal{L}| - 1$. We can, in straight forward, check that c_0 is unit of $\mathcal{K} \vee \mathcal{L}$ and the involutions $c_i^*, (c'_j)^*$ are as same as in $\mathcal{K}$

and $\mathcal{L}$. Thus $\mathcal{K} \vee \mathcal{L}$ is a hypergroup. We note that $\mathcal{K} \vee \mathcal{L}$ is a signed hypergroup if $\mathcal{K}$ and $\mathcal{L}$ are signed hypergroups. Remarkable fact is that clearly the left hand $\mathcal{K} \subset \mathcal{K} \vee \mathcal{L}$ but the right hand $\mathcal{L}$ is not canonically included in $\mathcal{K} \vee \mathcal{L}$. A hypergroup $\mathcal{L}$ is realized as a set $\{e_0^{\mathcal{K}}, c_1', \ldots, c_m'\}$ in $\mathfrak{A}(\mathcal{K} \vee \mathcal{L})$. So that $\mathcal{K} \vee \mathcal{L}$ is called a *join* hypergroup bonded $\mathcal{L}$ to $\mathcal{K}$.

We remark that $\mathcal{K} \vee \mathcal{L}$ is not always isomorphic to $\mathcal{L} \vee \mathcal{K}$. We have immediately the following results for weight functions $w'(\cdot)$ on $\mathcal{K} \vee \mathcal{L}$ such that

1. $w'(c_i) = w(c_i)$, i.e. the same function as in $\mathcal{K}$,
2. $w'(c_j') = w(\mathcal{K}) w(c_j') \quad$ for $c_j' \in \mathcal{K} \vee \mathcal{L}$,
3. $w'(\mathcal{K} \vee \mathcal{L}) = w(\mathcal{K}) w(\mathcal{L})$.

These are implied from (i)(iii) and the definition of weights.

2.2 Examples and Constructions of Hypergroups

There are many famous examples of finite commutative hypergroups and constructions in this section. They are described in many papers around in 1980's. N. Wildberger explains the series of examples and several constructions of hypergroups in [W1]. The methods of creating hypergroups are obtained from group and representation theory, and from the graph theory. We now introduce class hypergroups which are driven from an automorphism group and its conjugacy class.

Example 1. Let G be a finite commutative group and $\Gamma \subset \mathrm{Aut}(G)$ be a subgroup. Then the probability Γ-invariant measures on the Γ-orbits in G form a hypergroup under convolution and involution in the same way of a group $*$-algebra $\mathbb{C}G$. We denote this hypergroup by $\mathcal{K}(G;\Gamma)$. Precisely we define this

$$C_g = \sum_{\alpha \in \Gamma} g^{\alpha} \in \mathbb{C}G \qquad \text{and} \qquad c_g = C_g / |C_g|,$$

where C_g is identified with a class of Γ-orbit of $g \in G$. We rewrite these classes $\{C_0, \ldots, C_n\}$ and $C_0 = e$ unit of G. The structure constants are calculated as

$$C_i C_j = \sum_k N_{ij}^k C_k \qquad \text{and} \qquad c_i c_j = \sum_k n_{ij}^k c_k,$$

where

$$n_{ij}^k = \frac{N_{ij}^k |C_k|}{|C_i| \cdot |C_j|}.$$

The involution of c_i is defined by $c_j = c_i^*$ where $C_j \ni g^{-1}$ for some $g \in C_i$. The stochasticity $\sum_k n_{ij}^k = 1$ is easily checked from the structure constants equations applied by trivial character χ_0 of G, where it is noted the fact that $\chi_0(c_i) = 1$ for all c_i. This leads that $\mathcal{K}(G;\Gamma) = \{c_0, \dots, c_n\}$ is a hypergroup. It is easily seen that

$$w(c_i) = |C_i| \text{ for all } c_i \in \mathcal{K}(G;\Gamma) \quad \text{and} \quad w(\mathcal{K}(G;\Gamma)) = |G|.$$

[**1**] In the case of Γ is trivial, i.e. $\Gamma = \{id\}$.

A hypergroup $\mathcal{K}(G;\Gamma)$ equals a group G. The category of all finite commutative hypergroups includes the category of all finite commutative groups. The dual hypergroup $\mathcal{K}(G;\{id\})^\wedge = \mathcal{K}(G^\wedge;\{id\})$, namely the dual hypergroup $G^\wedge$ is the dual group of G.

In particular a group $(\mathbb{Z}_2)^m$ for some integer m is hermitian (hypergroup). Because the values of characters are real, i.e. $\{1, -1\}$.

[**2**] The case of non-trivial $|\Gamma_2| = 2$, that is a very typical generating hermitian hypergroup, so called self involutive.

Since G is commutative, the map $\beta : G \ni g \mapsto g^{-1}$ is isomorphism on G. Then $\beta^2 = id$. Define $\Gamma_2 = \{id, \beta\}$. If G is not isomorphic to a group $(\mathbb{Z}_2)^m$ for some integer m, then $|\Gamma_2| = 2$. We note that $\mathcal{K}(G;\Gamma_2)$ is hermitian because a class includes an element and its inverse. There are many hypergroup examples as below.

(1) $G = \mathbb{Z}_{2m+1}$ where m is a integer.

Let $G = \langle\, g \,|\, g^{2m+1} = e \,\rangle$ and $\Gamma_2 = \{id, \beta\}$. As shown in the above Γ-orbit classes are calculated as, for $1 \leq i \leq m$,

$$c_0 = e \quad \text{and} \quad c_i = \frac{g^i + g^{-i}}{2}.$$

The structure constants are followed by

$$c_i c_j = \frac{1}{2} c_{i+j} + \frac{1}{2} c_{i-j} \quad \text{and} \quad c_i^2 = \frac{1}{2} c_{2i} + \frac{1}{2} c_0,$$

where the identification $c_k = c_{2m-k+1}$, $c_k = c_{-k}$ are admitted when the index is out of a range $1 \leq k \leq m$.

The class hypergroup $\mathcal{K}(G;\Gamma_2) = \{c_0, \dots, c_m\}$ is analyzed as follows. The character function of c_i is driven from the character of G, χ_j is in the form:

$$\chi_j(g) = \exp\Big(\frac{2\pi j\sqrt{-1}}{2m+1}\Big).$$

Therefore

$$\chi_j(c_i) = \cos\big(\frac{2\pi ij}{2m+1}\big).$$

We note that $\chi_{-j}(c_i) = \chi_j(c_i)$ and $\chi_{-j} = \beta(\chi_j)$. Hence Γ_2-orbit of χ_j take the same value for an element c_i. We obtains that $w(c_i) = 2$ for $i \neq 0$ and $w(\chi_j) = 2$ for $j \neq 0$.

Thus we can obviously recognize that $\mathcal{K}(\mathbb{Z}_{2m+1};\Gamma_2)^\wedge = \mathcal{K}((\mathbb{Z}_{2m+1})^\wedge;\Gamma_2)$ is isomorphic to $\mathcal{K}(\mathbb{Z}_{2m+1};\Gamma_2)$, i.e. to be hermitian and self dual because the character table has real components and is symmetric with respect to χ_j, c_i.

We illustrate with low order examples as follows: 1-a),1-b)

1-a) $\mathcal{K}(\mathbb{Z}_3;\Gamma_2) = \mathscr{L}(1/2)$.

Indeed when $m = 1$ in the above, it is easily seen that $\mathcal{K}(\mathbb{Z}_3;\Gamma_2) = \{c_0, c_1\}$ and the structure constants are

$$(c_1)^2 = \frac{1}{2}c_0 + \frac{1}{2}c_1.$$

1-b) $\mathcal{K}(\mathbb{Z}_5;\Gamma_2)$ is called to be *Golden Hypergroup.*

The class hypergroup $\mathcal{K}(\mathbb{Z}_5;\Gamma_2) = \{c_0, c_1, c_2\}$ has structure in the form:

$$(c_1)^2 = \frac{1}{2}c_0 + \frac{1}{2}c_2, \quad (c_2)^2 = \frac{1}{2}c_0 + \frac{1}{2}c_1, \quad c_1c_2 = \frac{1}{2}c_1 + \frac{1}{2}c_2.$$

Characters of c_1 except the trivial character take the values

$$\Re(\exp(\frac{2\pi}{5}\sqrt{-1})) = \cos\Big(\frac{2\pi}{5}\Big) = \frac{\sqrt{5}-1}{4}, \quad \cos\Big(\frac{4\pi}{5}\Big) = \frac{-\sqrt{5}-1}{4}.$$

Their weights are $w(c_1) = w(c_2) = 2$ and total weight $w(\mathcal{K}(\mathbb{Z}_5;\Gamma_2)) = 5$.

N. Wildberger [W1] gives us a sight of maps on strong hermitian hypergroups of order three, and shows that the Golden hypergroup takes a place of a cusp point in the set of strong hermitian hypergroups. The set of strong hypergroups has real dimension three as a topological set. We obtain that

$$\mathrm{Aut}(\mathcal{K}(\mathbb{Z}_5;\Gamma_2)) = \mathbb{Z}_2.$$

(2) $G = \mathbb{Z}_{2m}$ where m is a integer. In this case some classes contain Γ-stable points.

Let $G = \langle\, g \,|\, g^{2m} = e \,\rangle$ and $\Gamma_2 = \{id, \beta\}$. As shown in the above Γ-orbit classes are calculated as, for $1 \le i \le m-1$,

$$c_0 = e, \quad c_m = g^m, \quad \text{and} \quad c_i = \frac{g^i + g^{-i}}{2}.$$

The structure constants are followed by

$$c_ic_j = \frac{1}{2}c_{i+j} + \frac{1}{2}c_{i-j} \quad \text{and} \quad c_i^2 = \frac{1}{2}c_{2i} + \frac{1}{2}c_0,$$

where the identification $c_k = c_{2m-k+1}$, $c_k = c_{-k}$ are admitted when the index is out of a range $1 \le k \le m$.

We illustrate with low order examples as follows: 2-a),2-b).

2-a) $\mathcal{K}(\mathbb{Z}_4;\Gamma_2)$ There are two points which Γ_2-stable.

The class hypergroup $\mathcal{K}(\mathbb{Z}_4;\Gamma_2)=\{c_0,c_1,c_2\}$ has structure in the form:

$$(c_1)^2=\frac{1}{2}c_0+\frac{1}{2}c_2,\quad (c_2)^2=c_0,\quad c_1c_2=c_1.$$

We also have the value for characters χ_1,χ_2 except trivial:

$$\chi_1(c_1)=0\quad \chi_2(c_1)=-1,$$

$$\chi_1(c_2)=-1\quad \chi_2(c_2)=1.$$

2-b) $\mathcal{K}(\mathbb{Z}_6;\Gamma_2)$ There are two points which Γ_2-stable.

The class hypergroup $\mathcal{K}(\mathbb{Z}_6;\Gamma_2)=\{c_0,c_1,c_2,c_3\}$ has structure in the form:

$$(c_1)^2=\frac{1}{2}c_0+\frac{1}{2}c_2,\quad (c_2)^2=\frac{1}{2}c_0+\frac{1}{2}c_2,\quad (c_3)^2=c_0,$$

$$c_1c_2=\frac{1}{2}c_1+\frac{1}{2}c_3,\quad c_1c_3=c_2,\quad c_2c_3=c_1.$$

(3) Observe case that an automorphism group Γ has larger order than the above examples. Let p be a prime. The automorphism group of a cyclic group $\mathbb{Z}_p=\langle\, g\mid g^p=e\rangle$ is well known to be a cyclic group $\mathbb{Z}_{(p-1)}$. Consider a class hypergroup $\mathcal{K}(\mathbb{Z}_p;\mathbb{Z}_{(p-1)})$. We have that the set of their class is $\{c_0,c_1\}$, where $c_1=(p-1)^{-1}\sum_{i=1}^{p-1}g^i$. The structure constants are

$$(c_1)^2=\frac{1}{p}c_0+\frac{p-1}{p}c_1.$$

Thus $\mathcal{K}(\mathbb{Z}_p;\mathbb{Z}_{(p-1)})\cong\mathscr{L}(1/p)$. This class construction gives us a series of hypergroups of order two.

We next introduce the wide examples which are generated by finite group theory described in [W1], i.e. groups are not commutative.

Example 2. Let G be a finite group. We denote its conjugacy classes $C_0=\{e\},C_1,\ldots,C_n$, where $C_i=\sum_{g\in C_i}g\in\mathbb{C}G$. There exists the relation

$$C_iC_j=\sum_k N_{ij}^k\,C_k,$$

where N_{ij}^k are non-negative integers. Let $c_i=C_i/|C_i|$. We have

$$c_ic_j=\sum_k n_{ij}^kc_k,\qquad n_{ij}^k=\frac{N_{ij}^k|C_k|}{|C_i|\cdot|C_j|}.$$

The involution is given by $c_i^*=|C_i|^{-1}\sum_{g\in c_i}g^{-1}$. The class hypergroup of G is denoted by $\mathcal{K}(G)=\{c_0,\ldots,c_n\}$ and is noted that $w(c_i)=|C_i|$ and $w(\mathcal{K}(G))=|G|$.

Now let $\{\rho_0, \dots, \rho_n\}$ be a equivalent class of irreducible representations of G where ρ_0 is a trivial representation. There exist non-negative integers M_{ij}^k such that

$$\rho_i \otimes \rho_j = \sum_k M_{ij}^k \rho_k.$$

The normalized character of ρ_i is denoted by $\chi_i(g) = \operatorname{tr} \rho_i(g)/\dim \rho_i$. Then the set of these characters $\mathcal{K}(G^\wedge) = \{\chi_0, \dots, \chi_n\}$ yields to be a finite commutative hypergroup having the structure as:

$$\chi_i \chi_j = \sum_k m_{ij}^k \chi_k, \qquad m_{ij}^k = \frac{M_{ij}^k \dim \rho_k}{\dim \rho_i \cdot \dim \rho_j}$$

and

$$\chi_i^* = \chi_i^-, \qquad \chi_0 \text{ is unit (trivial character).}$$

$\mathcal{K}(G^\wedge)$ is called to be a *character hypergroup.*

We note obviously that the weights of a character hypergroup is given by $w(\chi_i) = (\dim \rho_i)^2$ and the total weight $w(\mathcal{K}(G^\wedge)) = |G|$.

$\diamondsuit$ S_3 : The symmetric group of order three. This example is a simple and famous hypergroup of order three.

A class hypergroup $\mathcal{K}(S_3)$ has the structure constants as follows. Conjugacy classes are

$$C_0 = \{e\},\ C_1 = \{(12), (23), (31)\},\ C_2 = \{(123), (132)\}.$$

Then $c_1 = C_1/3$, $c_2 = C_2/2$ leads that

$$c_1^2 = \frac{1}{3}c_0 + \frac{2}{3}c_2, \quad c_2^2 = \frac{1}{2}c_0 + \frac{1}{2}c_2, \quad c_1 c_2 = c_1.$$

From the middle equation we have that character has $\{1, -1/2\}$ as the value of c_2. The right equation implies that character of c_1 is 0 when the value of c_2 is $-1/2$. Thus we obtain the next table:

	c_0	c_1	c_2	$w(\chi_i)$
χ_0	1	1	1	1
χ_1	1	-1	1	1
χ_2	1	0	$-1/2$	4
$w(c_j)$	1	3	2	6

and

$$\chi_1^2 = \chi_0, \quad \chi_1 \chi_2 = \chi_2, \quad \chi_2 = \frac{1}{4}\chi_0 + \frac{1}{4}\chi_1 + \frac{1}{2}\chi_2.$$

We introduce visual examples which are generated by finite graph theory [W1].

Example 3. Let $\mathfrak{X}$ be a finite edge-vertex graph with the special point, which is called *initial vertex* x_0. An automorphism is a bijective map on vertexes such that any edge is mapped to an edge. Assume that the automorphism group on $\mathfrak{X}$ acts distance transitively, i.e. for two pairs of vertexes which have the same distance, there exits an automorphism which maps one of pairs to the other. In this case we call $\mathfrak{X}$ to be distance transitive.

Consider the random walk problem on distance transitive $\mathfrak{X}$. A vertex $x \in \mathfrak{X}$ is said to have the distance i from the initial vertex if there exists a minimal i-steps path of edges which connects x_0 and x. The classes C_i denotes the set of all points which have distance i.

Let $y \in C_i$. Choose a point z which has distance j from x_0 to y. Let n_{ij}^k be the probability that z is of distance k from x_0. It is well defined because This defines a hermitian hypergroup $\mathcal{K}(\mathfrak{X}) = \{c_0, \ldots, c_n\}$ by

$$c_i c_j = \sum_k n_{ij}^k c_k.$$

We illustrate with visual example, ring type, of lower number vertexes.

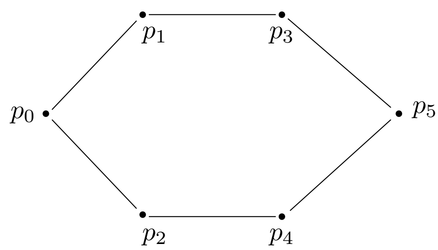

Figure 1: Ring graph hypergroup

See the below hexagon graph $\mathfrak{X}$ with vertexes $\{p_0, p_1, p_2, p_3, p_4, p_5\}$, where p_0 is an initial vertex, and edges appeared in it. Then it is distance transitively.

The classes are decomposed in

$$C_0 = \{p_0\},\ C_1 = \{p_1, p_2\},\ C_2 = \{p_3, p_4\},\ C_3 = \{p_5\}.$$

The structure constants are calculated in the form:

$$(c_1)^2 = (c_2)^2 = \frac{1}{2}c_0 + \frac{1}{2}c_2, \quad c_1 c_2 = \frac{1}{2}c_1 + \frac{1}{2}c_3$$

$$(c_3)^2 = c_0, \quad c_2 c_3 = c_1, \quad c_1 c_3 = c_2.$$

Therefore this $\mathcal{K}(\mathfrak{X})$ is equal to a class hypergroup $\mathcal{K}(\mathbb{Z}_6; \Gamma_2)$ of cyclic group appeared in 2-b).

We illustrate with another simple example. Let $\mathfrak{X}$ is all vertexes and edges in a simplex of order $n+1$. Write the vertexes $\{p_0, p_1, \ldots, p_n\}$ and p_0 is an initial vertex. There exists two distances

$$C_0 = \{p_0\}, \quad C_1 = \{p_1, \ldots, p_n\}.$$

The structure constants are in the following:

$$(c_1)^2 = \frac{1}{n}c_0 + \frac{n-1}{n}c_1.$$

Therefore $\mathcal{K}(\mathfrak{X})$ is equal to $\mathscr{L}(1/n)$. We establish larger family of hypergroups of order two than the class hypergroups in Example. 1.

2.3 Harmonic Analysis and its proof

There exists many theories about harmonic analysis on general topological hypergroups in the form of extended harmonic analysis of topological groups. N. Wildberger explains the harmonic analysis on the category of finite commutative signed hypergroups in [W1]. The basic facts in this theory bring us useful tools to analyze hypergroups. Now let us present the partial facts in [W1] which are needed for our analysis.

Let $\mathcal{K}$ be a finite commutative signed hypergroup and $\mathcal{K} = \{c_0, \ldots, c_n\}$. Then the following (w1)-(w9) are satisfied.

(w1) There exists a base $\{e_0^{\mathcal{K}}, \ldots, e_n^{\mathcal{K}}\}$ of $\mathfrak{A}(\mathcal{K})$ consisting of mutually orthogonal idempotents satisfying

$$c_j e_i^{\mathcal{K}} = \chi_i(c_j) e_i^{\mathcal{K}} \quad \text{for all} \quad 0 \le i, j \le n$$

for some character χ_j on $\mathcal{K}$.

(w2) $\mathcal{K}^\wedge = \{\chi_0, \ldots, \chi_n\}$, i.e. $|\mathcal{K}^\wedge| = n+1$ and each χ_j satisfies $\chi_j(c_i^*) = \chi_j(c_i)^-$.

(w3) The functions $\{\chi_j\}_j$ are orthogonal with respect to the inner product

$$\langle f, g \rangle_d = \frac{1}{w(\mathcal{K})} \sum_k f(c_k) g(c_k)^- w(c_k)$$

for f, g in $\mathfrak{A}(\mathcal{K})$.

(w4) $\mathcal{K}$ is itself a signed hypergroup under pointwise multiplication and complex conjugation. Thus $w(\chi_j)$ is well-defined and positive.

(w5) $w(\mathcal{K}^\wedge) = w(\mathcal{K})$.

(w6) $(\mathcal{K}^\wedge)^\wedge \cong \mathcal{K}$ under the canonical map.

(w7) The relationships between the two bases $\{c_0, \ldots, c_n\}$ and $\{e_0^{\mathcal{K}}, \ldots, e_n^{\mathcal{K}}\}$ of $\mathfrak{A}(\mathcal{K})$ are given by

$$\begin{aligned} c_i &= \sum_j \chi_j(c_i) e_j^{\mathcal{K}}, \\ e_j^{\mathcal{K}} &= \frac{w(\chi_j)}{w(\mathcal{K})} \sum_i w(c_i) \chi_j(c_i)^- c_i. \end{aligned}$$

In particular we have the following formulae for the Haar measure $e_0^{\mathcal{K}}$ and unit c_0:

$$\begin{aligned} c_0 &= \sum_j e_j^{\mathcal{K}}, \\ e_0^{\mathcal{K}} &= \frac{1}{w(\mathcal{K})} \sum_i w(c_i) c_i. \end{aligned}$$

(w8) The hypergroup character table is the matrix $\left(\chi_i(c_j)\right)_{ij}$. Its rows are orthogonal with respect to the inner product in (w3) and its columns are orthogonal with respect to the corresponding inner product on $\mathcal{K}^\wedge$. The character table allows an explicit realization of $\mathcal{K}$ as column vectors or of $\mathcal{K}^\wedge$ as row vectors.

(w9) If the structure constants of $\mathcal{K}^\wedge$ are given by m_{ij}^k, then the n_{ij}^k and m_{ij}^k can be obtained from the character table by

$$\begin{aligned} n_{ij}^k &= \frac{w(c_k)}{w(\mathcal{K})} \sum_l w(\chi_l)\chi_l(c_i)\chi_l(c_j)\chi_l(c_k)^-, \\ m_{ij}^k &= \frac{w(\chi_k)}{w(\mathcal{K})} \sum_l w(c_l)\chi_i(c_l)\chi_j(c_l)\chi_k(c_l)^-. \end{aligned}$$

The above is almost part of N. Wildberger [W1] to be useful for our assertion of this paper.

Let $\mathcal{K}$ be a finite commutative signed hypergroup. We give a sketch of the sequence of proofs which $\mathcal{K}^\wedge$ is a signed hypergroup and several fundamental results for finite commutative hypergroups described in [W1].

The start of Proof (w1)-(w9)

Since $*$-algebra $\mathfrak{A}(\mathcal{K})$ is semi-simple, we have $\mathfrak{A}(\mathcal{K}) \cong \mathbb{C}^{n+1}$ as $*$-algebras and there exist self adjoint projections $\{e_0^{\mathcal{K}}, \ldots, e_n^{\mathcal{K}}\}$ such that the spectral decomposition: $\mathfrak{A}(\mathcal{K}) = \mathbb{C}e_0^{\mathcal{K}} + \cdots + \mathbb{C}e_n^{\mathcal{K}}$. The mapping of $\xi \in \mathfrak{A}(\mathcal{K})$ to the restriction $\xi e_k^{\mathcal{K}} \in \mathbb{C}e_k^{\mathcal{K}}$ gives a $*$-homomorphism from $\mathfrak{A}(\mathcal{K})$ to $\mathbb{C}$, i.e. some character in $\mathcal{K}^\wedge$. Therefore there exist $n+1$ characters of $\mathcal{K}$. From (a4) and (a5), there exists a trivial character χ_0 such that $\chi_0(c_j) = 1$ for all j. We can give the numbering of $\{e_0^{\mathcal{K}}, \ldots, e_n^{\mathcal{K}}\}$ such that $e_k^{\mathcal{K}} \xi e_k^{\mathcal{K}} = \chi_k(\xi) e_k^{\mathcal{K}}$ for all $\chi_k \in \mathcal{K}^\wedge$. Thus it is obtained that

$$c_j e_i^{\mathcal{K}} = \chi_i(c_j) e_i^{\mathcal{K}} \quad \text{for all} \quad 0 \le i, j \le n.$$

Obviously we note that $(e_i^{\mathcal{K}})^* = e_i^{\mathcal{K}}$ and $e_i^{\mathcal{K}} e_j^{\mathcal{K}} = \delta_{ij} e_i^{\mathcal{K}}$, so that $\chi_i(e_j^{\mathcal{K}}) = \delta_{ij}$. where δ_{ij} is Kronecker's delta symbol. This shows (w1) and obviously (w2).

We consider a linear functional $f_0 : \mathfrak{A}(\mathcal{K}) \to \mathbb{C}$ such that $f_0(\xi) = \alpha_0$ for $\xi = \sum_k \alpha_k c_k \in \mathfrak{A}(\mathcal{K})$. Then $f_0(\xi\xi^*) = \sum_k |\alpha_k|^2$, so that f_0 is faithful and positive.

The standard inner product $\langle \cdot, \cdot \rangle$ is given by $\langle c_i, c_j \rangle = f_0(c_i c_j^*)$. From the condition (a3) in an axiom of hypergroups, there exists an integer $0 \le i^* \le n$ such that $c_{i^*} = (c_i)^*$. The condition (a4) leads

$$\langle c_i, c_j \rangle = n_{ii^*}^0 \delta_{ij} = w(c_i)^{-1} \delta_{ij}.$$

Thus the basis $c_0, \ldots, c_n$ is an orthogonal system in $\mathfrak{A}(\mathcal{K})$ with respect to $\langle \cdot, \cdot \rangle$.

Expand $e_0^{\mathcal{K}} = \sum_k a_k c_k$ for $a_k \in \mathbb{C}$. Since $c_i e_0^{\mathcal{K}} = e_0^{\mathcal{K}}$ and $e_0^{\mathcal{K}}$ is a projection, we note that $\langle e_0^{\mathcal{K}}, c_j \rangle = \langle c_j^* e_0^{\mathcal{K}}, c_0 \rangle = \langle e_0^{\mathcal{K}}, c_0 \rangle = \langle (e_0^{\mathcal{K}})^2, c_0 \rangle = \langle e_0^{\mathcal{K}}, e_0^{\mathcal{K}} \rangle$. We check that $\langle e_0^{\mathcal{K}}, c_j \rangle = f_0(\sum_k a_k c_k c_j^*) = a_j f_0(c_j c_j^*) = a_j w(c_j)^{-1}$. On calculating these inner products, it yields that $a_k w(c_k)^{-1} = a_0 w(c_0)^{-1} = \sum_k |a_k|^2 w(c_k)^{-1}$. The equation $w(c_0) = 1$ implies that $a_k = a_0 w(c_k)$. Hence $a_0 = |a_0|^2 \sum_k w(c_k) = |a_0|^2 w(\mathcal{K})$. It is shown that $a_0 = w(\mathcal{K})^{-1}$ is positive and $a_k = w(c_k) a_0$ for all k. Therefore

$$e_0^{\mathcal{K}} = w(\mathcal{K})^{-1} \sum_{k=0}^{n} w(c_k) c_k.$$

The projection $e_0^{\mathcal{K}}$ is called the *normalized Haar measure* on $\mathcal{K}$. Therefore this shows the last half of (w7).

The structure constant of $\mathcal{K}$ has expressed as $n_{ij}^k = w(c_k) \langle c_i c_j, c_k \rangle$ for all i, j, k, because $\langle c_i c_j, c_k \rangle = \sum_{k'} \langle n_{ij}^{k'} c_{k'}, c_k \rangle = n_{ij}^k \langle c_k, c_k \rangle$. According to a partition of unit $c_0 = \sum_i e_i^{\mathcal{K}}$ and $f_0(c_0) \ge f_0(e_i^{\mathcal{K}})$, there exists a positive number $w(\chi_i)$ for any projection $e_i^{\mathcal{K}}$ such that

$$\langle e_i^{\mathcal{K}}, e_i^{\mathcal{K}} \rangle = w(\chi_i) \cdot w(\mathcal{K})^{-1} \le 1.$$

We also have that

$$1 = \langle c_0, c_0 \rangle = \langle \sum_k e_k, c_0 \rangle = \sum_k \langle e_k, c_0 \rangle = \sum_k \langle e_k, e_k \rangle = \sum_k w(\chi_k)/w(\mathcal{K}).$$

This implies that $\sum_k w(\chi_k) = w(\mathcal{K})$, i.e. (w5).

The formula of inner products by characters is obtained that

$$\begin{aligned}
\langle c_i, c_j \rangle = \langle c_i \sum_k e_k^{\mathcal{K}}, c_j \sum_{k'} e_{k'}^{\mathcal{K}} \rangle &= \langle \sum_k \chi_k(c_i) e_k^{\mathcal{K}}, \sum_{k'} \chi_{k'}(c_j) e_{k'}^{\mathcal{K}} \rangle \\
&= \sum_k \chi_k(c_i) \chi_k(c_j)^- \langle e_k^{\mathcal{K}}, e_k^{\mathcal{K}} \rangle \\
&= w(\mathcal{K})^{-1} \sum_k \chi_k(c_i) \chi_k(c_j)^- w(\chi_k). \qquad \text{(Eq1)}
\end{aligned}$$

We note that this inner product is the corresponding inner product for the column vectors in (w8) and the orthogonality of c_j has been already shown as in the previous definition of $\langle \cdot, \cdot \rangle$.

Hence we have

$$\delta_{ij} = w(\mathcal{K})^{-1} \sum_k \chi_k(c_i) \chi_k(c_j)^- w(\chi_k) w(c_i). \qquad \text{(Eq2)}$$

It implies that the matrix $\left(\chi_i(c_j)w(\chi_i)^{1/2}w(c_j)^{1/2}w(\mathcal{K})^{-1/2}\right)_{ij}$ is unitary. Thus

$$\delta_{ij} = w(\mathcal{K})^{-1}\sum_k \chi_i(c_k)\chi_j(c_k)^- w(c_k)w(\chi_i). \tag{Eq3}$$

Defining an inner product in $\mathfrak{A}(\mathcal{K}^\wedge)$ as

$$\langle \chi_i, \chi_j\rangle_d = w(\mathcal{K})^{-1}\sum_k \chi_i(c_k)\chi_j(c_k)^- w(c_k), \tag{Eq4}$$

we have that $\langle \chi_i, \chi_j\rangle_d = w(\chi_i)^{-1}\delta_{ij}$ and $w(\chi_i) = w(\bar{\chi}_i)$. So that we can regard $\{\chi_0, \ldots, \chi_n\}$ as an orthogonal basis of $*$-algebra $\mathfrak{A}(\mathcal{K}^\wedge)$ with respect to the above inner product. This shows the statement (w3). Consequently it is shown that $m_{ij}^k := w(\chi_k)\langle \chi_i\chi_j, \chi_k\rangle_d$ turn to structure constants which make $\mathcal{K}^\wedge$ a signed hypergroup. This completes (w9). Indeed, it is easily to check that $m_{ij}^k \in \mathbb{R}$ since $\chi(c_i^*) = \chi(c_i)^-$ for $\chi \in \mathcal{K}^\wedge$ and $w(c_i) = w(c_i^*)$. Now we obtain that expansion $\chi_i\chi_j = \sum_k m_{ij}^k \chi_k$ satisfies the stochastic condition (4), applying the both terms to unit c_0. Therefore all conditions in the axiom of a signed hypergroup are shown. This shows (w8).

In addition we note that $w(\chi_i)$ is a weight of $\chi_i \in \mathcal{K}^\wedge$ and can recognize that a symbol $w(\chi_i)$ is written in the form of a weight of χ_i as an element of a signed hypergroup. This completes (w4).

The statement (w6) immediately is shown by considering c_j to be a function on χ_i as in the above character table. The canonical map between $(\mathcal{K}^\wedge)^\wedge$ and $\mathcal{K}$ is given in this way. The relation between two basis $\{c_0, \ldots, c_n\}$ and $\{e_0^{\mathcal{K}}, \ldots, e_n^{\mathcal{K}}\}$ is given by using matrix chasing. First we present forms in the spectral decomposition such that

$$c_i = c_i \sum_k e_k^{\mathcal{K}} = \sum_k c_i e_k^{\mathcal{K}} = \sum_k \chi_k(c_i)e_k^{\mathcal{K}}.$$

This is turned into the system form:

$$(c_0, \ldots, c_n) = (e_0^{\mathcal{K}}, \ldots, e_n^{\mathcal{K}})\Big(\chi_i(c_j)\Big)_{ij}.$$

Hence

$$\begin{aligned} & (w(c_0)^{1/2}c_0, \ldots, w(c_n)^{1/2}c_n) \\ = \ & (w(\chi_0)^{-1/2}e_0^{\mathcal{K}}, \ldots, w(\chi_n)^{-1/2}e_n^{\mathcal{K}})\Big(\chi_i(c_j)w(c_j)^{1/2}w(\chi_i)^{1/2}\Big)_{ij} \\ = \ & w(\mathcal{K})^{1/2}(w(\chi_0)^{-1/2}e_0^{\mathcal{K}}, \ldots, w(\chi_n)^{-1/2}e_n^{\mathcal{K}})\Big(\chi_i(c_j)w(c_j)^{1/2}w(\chi_i)^{1/2}w(\mathcal{K})^{-1/2}\Big)_{ij}. \end{aligned}$$

The last matrix is unitary from the equation (Eq2) and (Eq3), then it is shown that

$$\begin{aligned} & w(\mathcal{K})^{1/2}(w(\chi_0)^{-1/2}e_0^{\mathcal{K}}, \ldots, w(\chi_n)^{-1/2}e_n^{\mathcal{K}}) \\ = \ & (w(c_0)^{1/2}c_0, \ldots, w(c_n)^{1/2}c_n)\Big(\chi_i(c_j)w(c_j)^{1/2}w(\chi_i)^{1/2}w(\mathcal{K})^{-1/2}\Big)_{ij}^* \\ = \ & (w(c_0)^{1/2}c_0, \ldots, w(c_n)^{1/2}c_n)\Big(\chi_j(c_i)^- w(c_i)^{1/2}w(\chi_j)^{1/2}w(\mathcal{K})^{-1/2}\Big)_{ij}. \end{aligned}$$

Consequently

$$(e_0^{\mathcal{K}}, \ldots, e_n^{\mathcal{K}}) = (c_0, \ldots, c_n)\Big(\chi_j(c_i)^{-} w(c_i) w(\chi_j) w(\mathcal{K})^{-1}\Big)_{ij}.$$

Thus it yields the expanded formula of the spectral projections with respect to the bases $\{c_0, \ldots, c_n\}$:

$$e_j^{\mathcal{K}} = \frac{w(\chi_j)}{w(\mathcal{K})} \sum_k \chi_j(c_k)^{-} w(c_k) c_k.$$

These show the former part of (w7).

Therefore we complete our conclusion (w1)-(w9).

The end of proof (w1)-(w9)

From the above construction in the proof, we notice that the orthogonal projections $e_0^{\mathcal{K}} \in \mathfrak{A}(\mathcal{K})$ is obviously different from $e_0^{\mathcal{K}^\wedge} \in \mathfrak{A}(\mathcal{K}^\wedge)$ with respect to the dual signed hypergroup.

The next basic formulas for structure constants in Proposition 2.1 appear in [SW2], which is proved in more general condition without commutativity. Remark the symbol of the integer i^* such that $c_{i^*} = (c_i)^*$, so that this map is regarded as the involution of i on the index set.

Proposition 2.1. *Let $\mathcal{K}$ be a finite commutative signed hypergroup. Then the following statements for structure constants are satisfied* :

(1) $\quad n_{ij}^k = n_{ji}^k,$

(2) $\quad w(c_k)^{-1} n_{ij}^k = w(c_j)^{-1} n_{ik^*}^{j^*}$

for all i, j, k. Moreover, if $\mathcal{K}$ is hermitian, the expression of (2) *turns into*

$$w(c_k)^{-1} n_{ij}^k = w(c_j)^{-1} n_{ik}^j.$$

Proof. (1) It is obtained that $n_{ij}^k = w(c_k)\langle c_i c_j, c_k\rangle = w(c_k)\langle c_j c_i, c_k\rangle = n_{ji}^k$.

(2) It is obtained from $\langle c_i c_j, c_k\rangle = \langle c_{k^*} c_i c_j, c_0\rangle = \langle c_{k^*} c_i, c_{j^*}\rangle$.

The last part is straight forward. □

When a finite commutative signed hypergroup is given, the character table and weights are calculated as the above. Conversely let us require the condition of character tables which determines a signed hypergroup, in another word, prove that there exists a signed hypergroup such that their character table and weights are coincide with given one. Now we define the next condition of matrix.

Definition 4. (Character matrix) Let $U \in M_{n+1}(\mathbb{C})$ be a square matrix of order $n+1$. If the conjugate of any column vector of U is some column vector and the conjugate of any row vector of U is some row vector, then the matrix U is called *real constants type.*

Lemma 2.2. *Let a matrix $\big(\chi_i(c_j)\big)_{ij}$ with positive values $\{w(c_i)\}_i$, $\{w(\chi_i)\}_i$ be given in the form :*

	c_0	c_1	$\cdots$	c_n	
χ_0	1	1	$\cdots$	1	$w(\chi_0)=1$
χ_1	1	$\chi_1(c_1)$	$\cdots$	$\chi_1(c_n)$	$w(\chi_1)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
χ_n	1	$\chi_n(c_1)$	$\cdots$	$\chi_n(c_n)$	$w(\chi_n)$
	$w(c_0)=1$	$w(c_1)$	$\cdots$	$w(c_n)$	W

where $W=\sum_i w(c_i)=\sum_i w(\chi_i)$. If the matrix $\big(\chi_i(c_j)w(\chi_i)^{1/2}w(c_j)^{1/2}W^{-1/2}\big)_{ij}$ is a real constants type unitary, then there exists a signed hypergroup $\mathcal{K}=\{c_0,\ldots,c_n\}$ whose characters $\mathcal{K}^\wedge=\{\chi_0,\ldots,\chi_n\}$ and weights table coincide with the above form.

Proof. Consider a column vector of the above table $c_i={}^T\big(\chi_0(c_i),\ldots,\chi_n(c_i)\big)$ in $*$-algebra $\mathbb{C}^{n+1}$ and $\mathcal{K}=\{c_0,\ldots,c_n\}$, where ${}^TX$ is a transposed matrix of X. Then products and involution are given as:

$$c_ic_j={}^T\big(\chi_0(c_i)\chi_0(c_j),\ldots,\chi_n(c_i)\chi_n(c_j)\big),\ c_i^*={}^T\big(\chi_0(c_i)^-,\ldots,\chi_n(c_i)^-\big).$$

It is obvious that (a1)(a6) in the axiom of signed hypergroups hold. Now (a2) is shown as matrix $\big(\chi_i(c_j)\big)_{ij}$ is regular and (a3) $c_i^*\in\mathcal{K}$ is from the assumption of "real constant type". By the assumption of "unitary" we have

$$\delta_{ij}=w(\mathcal{K})^{-1}\sum_k\chi_k(c_i)\chi_k(c_j)^-w(\chi_k)w(c_i)$$

and

$$\delta_{ij}=w(\mathcal{K})^{-1}\sum_k\chi_i(c_k)\chi_j(c_k)^-w(c_k)w(\chi_i).$$

Hence we have that $\langle c_i,c_j\rangle=w(c_i)^{-1}\delta_{ij}$ and $w(c_i^*)=w(c_i)$ where the standard inner product $\langle c_i,c_j\rangle=w(\mathcal{K})^{-1}\sum_k\chi_k(c_i)\chi_k(c_j)^-w(\chi_k)$ [W2]. Similarly $w(\chi_i^-)=w(\chi_i)$. Noting that $\{c_0,\ldots,c_n\}$ is an orthogonal system, the structure constants are given by $n_{ij}^k=w(c_k)\langle c_ic_j,c_k\rangle$ for all i,j,k, where $c_ic_j=\sum_k n_{ij}^kc_k$ is expanded. Since the sum of two summands χ,χ^-:

$$\chi(c_ic_j)\chi(c_k^*)w(\chi)+\chi(c_ic_j)^-\chi(c_k^*)^-w(\chi^-)=2w(\chi)\Re\big(\chi(c_ic_j)\chi(c_k^*)\big)$$

is real, it is obtained that all n_{ij}^k are real. Thus (a4) is shown. Applying the trivial character χ_0 to $c_ic_j=\sum_k n_{ij}^kc_k$, we have (a5).

Therefore it follows that $\mathcal{K}$ is a signed hypergroup and its character table is a given one. □

Remark 2.1. The unitary appeared in Lemma 2.2 corresponding to a signed hypergroup $\mathcal{K}$ is written by $U(\mathcal{K})$. A signed hypergroup $\mathcal{K}$ with the hypothesis in Lemma 2.2 becomes a hypergroup if it is checked that $\langle c_i c_j, c_k \rangle \geq 0$ for all i, j, k. However it is not easy to see these values positive in case of general situation.

Example 4. For the character table of a hypergroup $\mathscr{L}(q)$, a unitary $U(\mathscr{L}(q))$ of the above Lemma 2.2 is calculated in the following form:

$$U(\mathscr{L}(q)) = \begin{pmatrix} \frac{\sqrt{q}}{\sqrt{1+q}} & \frac{1}{\sqrt{1+q}} \\ \frac{1}{\sqrt{1+q}} & -\frac{\sqrt{q}}{\sqrt{1+q}} \end{pmatrix}.$$

It is easily seen to be an orthogonal matrix of order two. It is difficult to establish the weights and the structure constants by seeing the above unitary at once.

We will illustrate with the structures of join hypergroups.

Proposition 2.3. *Let $\mathcal{K}, \mathcal{L}$ be finite commutative hypergroups. Then the join hypergroup $\mathcal{K} \vee \mathcal{L}$ has characters of $\psi \in \mathcal{L}^\wedge$ and $\chi \in \mathcal{K}^\wedge$ where χ is not a trivial character. If $\mathcal{K} = \{c_0, \dots, c_n\}, \mathcal{L} = \{\ell_0, \dots, \ell_m\}$ and $\mathcal{K}^\wedge = \{\chi_0, \dots, \chi_n\}, \mathcal{L}^\wedge = \{\psi_0, \dots, \psi_m\}$, then $(\mathcal{K} \vee \mathcal{L})^\wedge = \{\psi_0, \dots, \psi_m, \chi_1, \dots, \chi_n\}$.*
Furthermore we have the identification

$$(\mathcal{K} \vee \mathcal{L})^\wedge = \mathcal{L}^\wedge \vee \mathcal{K}^\wedge,$$

from the character table in the following:

	c_0	$\cdots$	c_n	ℓ_1	$\cdots$	ℓ_m	$w(\text{char.})$
ψ_0	1	$\cdots$	1	1	$\cdots$	1	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
ψ_m	1	$\cdots$	1	$\psi_m(\ell_1)$	$\cdots$	$\psi_m(\ell_m)$	$w(\psi_m)$
χ_1	1	$\cdots$	$\chi_1(c_n)$	0	$\cdots$	0	$w(\mathcal{L})w(\chi_1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
χ_n	1	$\cdots$	$\chi_n(c_n)$	0	$\cdots$	0	$w(\mathcal{L})w(\chi_n)$
$w(\text{elem.})$	1	$\cdots$	$w(c_n)$	$w(\mathcal{K})w(\ell_1)$	$\cdots$	$w(\mathcal{K})w(\ell_m)$	$w(\mathcal{K})w(\mathcal{L})$

Proof. For $\psi_i \in \mathcal{L}^\wedge$, a character ψ_i is extended to the join $\mathcal{K} \vee \mathcal{L}$ as $\psi_i|_{\mathcal{K}} = 1$. For non trivial character $\chi_i \in \mathcal{K}^\wedge$, χ_i is extended as $\chi_i(\ell_j) = 0$ for all $\ell_j (\neq \ell_0) \in \mathcal{L}$. From the definition (i)-(iii) of a join hypergroup, it is clear that extended functions ψ_i and χ_i are characters. Thus the order of $\mathcal{K} \vee \mathcal{L}$ leads that all character are in the form of these extended characters. The character table is shown to be the above. From this, the identification is also obtained. □

R.C. Vrem showed the above statement in [Vr] which is described in the category of infinite and non-commutative hypergroups. From Proposition 2.3 we can determine by seeing a character table whether a finite commutative hypergroup is a join or not.

Recall the character table of $\mathcal{K}(S_3)$ in N. Wildberger [W1] and view its dual $\mathcal{K}(S_3)^\wedge$ which appeared in the section 2.1. From the structure constants in definition of class hypergroups in the previous section, we easily obtain that

	c_0	c_2	c_1	$w(\chi_n)$
χ_0	1	1	1	1
χ_1	1	1	-1	1
χ_n	1	$-1/2$	0	4
$w(c_n)$	1	2	3	6

where the order of elements $\{c_0, c_1, c_2\}$ is changed. It is trivial that a set $\{c_0, c_2\} \cong \mathscr{L}(1/2)$ is a subhypergroup of $\mathcal{K}(S_3)$ and a set $\{\chi_0, \chi_1\} \cong \mathbb{Z}_2$ is a subhypergroup of $\mathcal{K}(S_3)^\wedge$. Exchanging the column vectors c_1, c_2 from the table in the section 2.1, we regard the above table as in the form of Proposition 2.3. Thus it is shown that

$$\mathcal{K}(S_3) \cong \mathscr{L}(1/2) \vee \mathbb{Z}_2$$

and

$$\mathcal{K}(S_3)^\wedge \cong \mathbb{Z}_2 \vee \mathscr{L}(1/2).$$

This form leads that they are hermitian because $\mathbb{Z}_2$ and $\mathscr{L}(1/2)$ are hermitian.

2.4 Actions of a hypergroup

Let us recall the notions about actions of a hypergroup over a set [SW1]. For a finite set $X = \{x_1, \ldots, x_m\}$, let $\sigma X = \{\alpha_1 x_1 + \cdots + \alpha_m x_m \text{ s.t. } \alpha_i \geq 0,\ \alpha_1 + \cdots + \alpha_k = 1\}$, which is called an *absolute simplex* of X. We denote $\mathrm{Aff}(X)$ the set of all affine maps from σX into σX. Moreover, $\mathrm{Aff}(X, z_0) = \{\psi \in \mathrm{Aff}(X) \text{ s.t. } \psi(z_0) = z_0\}$ means the set of all affine maps fixing a point z_0 which is called a *general centroid* $z_0 \in \sigma X$.

Definition 5. An *action* of a hypergroup $\mathcal{K} = \{c_0, \ldots, c_n\}$ on a finite set X is an affine homomorphism $\pi : \sigma\mathcal{K} \to \mathrm{Aff}(X)$, i.e. $\pi(c_i) \in \mathrm{Aff}(X)$ for every $c_i \in \mathcal{K}$, satisfy that

(1) $\pi(c_0)$ is the identity, i.e. non-degenerate,

(2) $\pi(c_i)\pi(c_j) = \pi(c_i c_j)$ for c_i, $c_j \in \mathcal{K}$.

Moreover, π is called an action with a general centroid z_0 if every image of the above action π is involved in $\mathrm{Aff}(X, z_0)$

An absolute simplex σX is considered as the set of all non-negative probability measures over X, i.e. $\sigma X = M^1(X)$.

For an element $\xi = \alpha_1 x_1 + \cdots + \alpha_m x_m \in \sigma X$, we define a *support* of ξ as

$$\mathrm{supp}(\xi) = \{x_k \text{ s.t. } \alpha_k > 0\} \subset X.$$

We denote $x_k \in \xi$ if $x_k \in \mathrm{supp}(\xi)$, and also $S \subset \xi$ if $S \subset \mathrm{supp}(\xi)$.

It is easily seen that $\mathrm{supp}(e_0^{\mathcal{K}}) = \mathcal{K}$ from the expression in the section 2.3. If $\mathrm{supp}(\xi) = \{c_j\}$, then $\xi = c_j$. If the cardinal number $\#\,\mathrm{supp}(\xi) \geq 2$, then ξ belongs to the interior of general simplex $\sigma(\mathrm{supp}(\xi))$. So that the normalized Haar measure $e_0^{\mathcal{K}}$ belongs to the interior of $\sigma\mathcal{K}$.

Remark 2.2. Any hypergroup $\mathcal{K}$ acts on itself by multiplication. This action admits a general centroid $e_0^{\mathcal{K}}$ since $c_i e_0^{\mathcal{K}} = e_0^{\mathcal{K}}$ for all $c_i \in \mathcal{K}$.

Proposition 2.4. *Let $\mathcal{K}$ be a finite commutative hypergroup and consider the standard inner product over $\mathfrak{A}(\mathcal{K})$. If $\mathcal{K}$ does not equal to trivial hypergroup* **1***, then the following statements hold*:

(1) *The normalized Haar measure $e_0^{\mathcal{K}}$ is not involved in $\mathcal{K}$,*

(2) *$e_0^{\mathcal{K}}$ is a point which has the minimal distance $w(\mathcal{K})^{-1/2}$ in $\sigma\mathcal{K}$.*

Proof. The fact of $\langle e_0^{\mathcal{K}}, c_j\rangle = \langle c_j^* e_0^{\mathcal{K}}, c_0\rangle = \langle e_0^{\mathcal{K}}, c_0\rangle$ for all $c_j \in \mathcal{K}$ implies that $e_0^{\mathcal{K}}$ is orthogonal to the plane σK, i.e. the distance to the origin is achieved by $e_0^{\mathcal{K}}$. The square of this distance is calculated to be $\langle e_0^{\mathcal{K}}, e_0^{\mathcal{K}}\rangle = w(\mathcal{K})^{-1}$. Thus $e_0^{\mathcal{K}}$ is characterized as a unique point which has the smallest norm in simplex $\sigma\mathcal{K}$. This completes our assertion.

□

Proposition 2.5. *Let $\mathcal{K}$ be a finite commutative hypergroup. The following statements hold* :

(1) $|\chi(c)| \leq 1$ *for* $\chi \in \mathcal{K}^\wedge, c \in \mathcal{K}$,

(2) *The maximal norm in $\sigma\mathcal{K}$ is achieved by unit c_0.*

Proof. From the axiom $w(c) \geq 1$ of hypergroups, unit c_0 achieves the maximal norm one in $\mathcal{K}$. A general simplex $\sigma\mathcal{K}$ has the maximal norm on their vertexes. This shows (2). Note that $\sigma\mathcal{K}$ is compact. The set $\{|\chi(\sigma\mathcal{K})|\}$ is bounded. For an arbitrary integer n, we can see that $c^n \in \sigma\mathcal{K}$. Therefore $\{|\chi(c)|^n\}_n$ is bounded, so that it is obtained $|\chi(c)| \leq 1$, that is (1). □

Remark 2.3. Let $\mathcal{K}$ be a signed hypergroup. Define an *observable* of $\mathcal{K}$ to be any element of the form $c_{i_1} c_{i_2} \ldots c_{i_n}$ [W1]. Proposition 2.5 explains that all observables of a hypergroup are always bounded.

Corollary 2.6. *Let $\mathcal{K}$ be a finite commutative hypergroup. Then the subset $S = \{\, c \in \mathcal{K} \text{ s.t. } \langle c, c\rangle = 1\}$ forms a group.*

Proof. In general, from the formula of inner products (Eq1),

$$\langle c, c\rangle \;=\; w(\mathcal{K})^{-1} \sum_{\chi\in\mathcal{K}^\wedge} |\chi(c)|^2 w(\chi) \;\leq\; w(\mathcal{K})^{-1} \sum_{\chi\in\mathcal{K}^\wedge} w(\chi) \;=\; 1.$$

It implies that $\langle c, c\rangle = 1$ if and only if $|\chi(c)| = 1$ for all $\chi \in \mathcal{K}^\wedge$. Thus if $c \in S$ then $c^* \in S$, and if $c, c' \in S$ then $cc' \in S$. □

Definition 6. Let π be a $\mathcal{K}$-action on a set X. We call $\xi \in \sigma X$ a $\mathcal{K}$-stable point if $\pi(c_i)\xi = \xi$ for all $c_i \in \mathcal{K}$. A point $\xi \in \sigma X$ is said to be minimal $\mathcal{K}$-stable if there exists no $\eta \in \sigma X$ such that $\mathrm{supp}(\eta) \subsetneq \mathrm{supp}(\xi)$.

Lemma 2.7. *Let $\mathcal{K}$ be a finite commutative hypergroup and π be a $\mathcal{K}$-action on a set X. The following statements hold :*

(1) *$\xi \in \sigma X$ is a $\mathcal{K}$-stable point if and only if $\pi(e_0^{\mathcal{K}})\xi = \xi$,*

(2) *If $\xi \in \sigma X$ is a minimal $\mathcal{K}$-stable point, then $\xi = \pi(e_0^{\mathcal{K}})x$ for any $x \in \mathrm{supp}(\xi)$.*

Proof. (1) $\pi(e_0^{\mathcal{K}})\xi = w(\mathcal{K})^{-1}\sum_k w(c_k)\pi(c_k)\xi = w(\mathcal{K})^{-1}\sum_k w(c_k)\xi = \xi$. Conversely $\pi(c_k)\xi = \pi(c_k)\pi(e_0^{\mathcal{K}})\xi = \pi(e_0^{\mathcal{K}})\xi = \xi$ for $c_k \in \mathcal{K}$.
(2) Let $\xi \ni x$. From the assumption we have $\mathrm{supp}(\xi) = \mathrm{supp}(\pi(e_0^{\mathcal{K}})x)$. Suppose that $\xi \neq \pi(e_0^{\mathcal{K}})x$. Consider a $\mathcal{K}$-stable point $\eta = (\xi - p\pi(e_0^{\mathcal{K}})x)/(1-p)$ for some $0 < p < 1$. We can choose a suitable number p such that $\eta \in \sigma X$ is $\mathcal{K}$-stable and $\mathrm{supp}(\eta) \subsetneq \mathrm{supp}(\xi)$. This is contradiction. Thus $\xi = \pi(e_0^{\mathcal{K}})x$. □

Lemma 2.8. *Let π be a $\mathcal{K}$-action on a set X. Then the following statements hold : for $x, y, z \in X$,*

(1) *$\pi(e_0^{\mathcal{K}})x \ni x$,*

(2) *If $\pi(e_0^{\mathcal{K}})x \ni y$, then $\pi(e_0^{\mathcal{K}})y \ni x$,*

(3) *If $\pi(e_0^{\mathcal{K}})x \ni y$ and $\pi(e_0^{\mathcal{K}})y \ni z$, then $\pi(e_0^{\mathcal{K}})x \in z$.*

Proof. (1) is obviously shown that $\pi(e_0^{\mathcal{K}})x \ni x$ since $\mathcal{K}$ includes a unit.

(3) Let $\pi(e_0^{\mathcal{K}})x \ni y$ and $\pi(e_0^{\mathcal{K}})y \ni z$. We have $\pi(e_0^{\mathcal{K}})x \ni y$ implies that $\pi(e_0^{\mathcal{K}})x \supset \mathrm{supp}(\pi(e_0^{\mathcal{K}})y)$. We also have $\pi(e_0^{\mathcal{K}})y \ni z$. Thus, $\pi(e_0^{\mathcal{K}})x \ni z$.

We will finally show the reflexivity (2). Let $\pi(e_0^{\mathcal{K}})x \ni y$, i.e. $\mathrm{supp}(\pi(e_0^{\mathcal{K}})x) \supset \mathrm{supp}(\pi(e_0^{\mathcal{K}})y)$. Suppose that $\mathrm{supp}(\pi(e_0^{\mathcal{K}})x) \supsetneq \mathrm{supp}(\pi(e_0^{\mathcal{K}})y)$. There exists a $\mathcal{K}$-stable point $\pi(e_0^{\mathcal{K}})z$ having a minimal support such that $\pi(e_0^{\mathcal{K}})x \supset \mathrm{supp}(\pi(e_0^{\mathcal{K}})y) \supset \mathrm{supp}(\pi(e_0^{\mathcal{K}})z)$. By the same argument in Lemma 2.7 (2), we can choose a point $\eta = \big(\pi(e_0^{\mathcal{K}})x - p\pi(e_0^{\mathcal{K}})z\big)/(1-p)$ such that $\mathrm{supp}(\eta) \subsetneq \mathrm{supp}(\pi(e_0^{\mathcal{K}})x)$. Then $\eta \ni x$ shows that $\mathrm{supp}(\pi(e_0^{\mathcal{K}})\eta) = \mathrm{supp}(\pi(e_0^{\mathcal{K}})x)$. This is contradiction. It is obtained that $\mathrm{supp}(\pi(e_0^{\mathcal{K}})x) = \mathrm{supp}(\pi(e_0^{\mathcal{K}})y)$ for all $y \in \mathrm{supp}(\pi(e_0^{\mathcal{K}})x)$. Suppose that $\pi(e_0^{\mathcal{K}})x \neq \pi(e_0^{\mathcal{K}})y$. By the same argument, there exists a $\mathcal{K}$-stable point which has a support smaller than $\mathrm{supp}(\pi(e_0^{\mathcal{K}})x)$. This is contradiction. Consequently $\pi(e_0^{\mathcal{K}})x = \pi(e_0^{\mathcal{K}})y$. This means that $\pi(e_0^{\mathcal{K}})y \in x$. □

Definition 7. (equivalence) Let π be a $\mathcal{K}$-action on a set X. We write $x \sim_{\mathcal{K}} y$ for $x, y \in X$ if $\pi(e_0^{\mathcal{K}})x \ni y$. From the above Lemma 2.8, it is shown that $\sim_{\mathcal{K}}$ is an equivalence relation. Furthermore x is said to be $\mathcal{K}$-equivalent to y if $x \sim_{\mathcal{K}} y$.

The sequence of these Lemma's shows the next Proposition.

Proposition 2.9. *Let π be a $\mathcal{K}$-action on a set $X$and $x, y \in \mathcal{K}$. The following statements are equivalent* :

(1) $\pi(c)x \ni y$ *for some* $c \in \mathcal{K}$,

(2) $x \sim_{\mathcal{K}} y$,

(3) $\pi(e_0^{\mathcal{K}})x = \pi(e_0^{\mathcal{K}})y$.

Remark 2.4. These expressions are equivalence relations, so that (1) in the above is the most useful relation to check that $\mathcal{K}$-orbit includes an element. There is a short proof [SW1] in the case that X is a hypergroup which includes a subhypergroup $\mathcal{K}$. In this proof it is needed that $x \in X$ has an involution x^*. However we have achieved the above proof of Lemma 2.8 without the assumption that an element in X does have so called an 'involution'.

Assume that $\pi(e_0^{\mathcal{K}})x \ni y$. We remark that y is not necessarily included in $\mathbb{C}$-linear space $\pi(\mathfrak{A}(\mathcal{K}))x$. However in the category of groups, $\pi(\mathfrak{A}(\mathcal{K}))x$ does always include y. In the category of hypergroups we can show that a subspace $\pi(\mathfrak{A}(\mathcal{K}))x$ is not always equals $\mathbb{C}\,\mathrm{supp}(\pi(e_0^{\mathcal{K}})x)$. In order to see an example such that $\pi(\mathfrak{A}(\mathcal{K}))x$ is a pure subspace of $\mathbb{C}\,\mathrm{supp}(\pi(e_0^{\mathcal{K}})x)$, we will give examples of hypergroups $\mathcal{K}(n,p)$ or more general one $\mathcal{K}(n,p,q)$ in the latter section 5.

We prepare a definition for the decomposition of a given action, which is stated in the paper [SW1].

Definition 8. Let π be an action of a finite commutative hypergroup $\mathcal{K}$ on a finite set X. π is *reducible* if there exists a nonempty set $Y \subset X$ and $Y \neq X$ such that $\pi(\mathcal{K})Y \subset \sigma Y$. Otherwise π is *irreducible.*

V. Sunder and N. Wildberger [SW1] present the next equivalence statements for *irreducibility.*

Theorem 2.1. *Suppose π is an action of a hypergroup $\mathcal{K}$ on a finite set X. Then the following are equivalent*:

(1) *π is irreducible,*

(2) *All components of a matrix $\pi(e_0^{\mathcal{K}})$ on X are strictly positive,*

(3) *A matrix $\pi(e_0^{\mathcal{K}})$ on X is a rank one projection.*

Proof. It is straight forward from Proposition 2.9. □

Remark 2.5. When $X = \{x_1, \dots, x_m\}$ and the unique $\mathcal{K}$-stable point obtained from Proposition 2.9 $\pi(e_0^{\mathcal{K}})x_1 = \sum_j \alpha_j x_j$, the above matrix in Theorem 2.1 can be written in the form of

$$\pi(e_0^{\mathcal{K}}) = {}^T(1, \dots, 1)(\alpha_1, \dots, \alpha_m).$$

We note that $\sum_j \alpha_j x_j \in \sigma X$ is a general centroid.

Example 5. Viewing the preparation as above, we can illustrate with more general examples of actions of hypergroups.

Let $\mathcal{K} \supset \mathcal{H}$ be a finite commutative hypergroup and a subhypergroup. Then it is obviously seen that $\mathcal{H}$ acts on $\mathcal{K}$ by canonical multiplication. According to Theorem 2.1, total hypergroup $\mathcal{K}$ is decomposed into disjoint summand of $\mathcal{H}$-equivalence sets. That is

$$\mathcal{K} = \mathcal{K}_0 \cup \mathcal{K}_1 \cup \cdots \cup \mathcal{K}_m$$

where $\mathcal{K}_0 = \mathcal{H}$ and $\mathcal{K}_i$ is a $\mathcal{H}$-equivalence class.

For a class $\mathcal{K}_i = \{s_1, \dots, s_{n_i}\}$, we define the *general centroid*

$$z_w = \Big(\sum_{j=1}^{n_i} w(s_j)\Big)^{-1} \sum_{j=1}^{n_i} w(s_j) s_j \in \sigma\mathcal{K}_i.$$

In the case that $w(s_i) = w_0 > 0$ for all $s_i \in \mathcal{K}_i$, i.e. independent to i, it is easy to see that z_w equals the *centroid* $z = (s_1 + \cdots + s_{n_i})/n_i$ [SW1].

The action π of $\mathcal{H}$ on $\mathcal{K}$ has a decomposition into the direct summands of the irreducible action restricted to a set $\mathcal{K}_i$ for i. This means that

$$\pi = \bigoplus_{\ell \in \mathcal{L}} \pi|_{\mathcal{K}_i} \qquad \text{(irreducible decomposition)},$$

where $\pi(h)|_{\mathcal{K}_i} \in \mathrm{Aff}(\mathcal{K}_i, z_w)$ for all $h \in \mathcal{H}$.

2.5 Subhypergroups, Quotients and Homomorphisms

Definition 9. Let $\mathcal{K}$ be a finite commutative hypergroup. A subset $\mathcal{H}$ of $\mathcal{K}$ is said to be a *subhypergroup* if $\mathrm{supp}(\mathcal{H}\mathcal{H}) = \mathcal{H}$ and $\mathcal{H}^* = \mathcal{H}$.

Remark 2.6. We remark that $\mathrm{supp}(\mathcal{H}\mathcal{H}) \supset \mathcal{H}$ for a subset $\mathcal{H} \ni c_0$. It is easily seen that unit c_0 belongs to $\mathcal{H}$ from the fact $\mathrm{supp}(hh^*) \ni c_0$.

Let $\mathcal{H}$ be a subhypergroup of a finite commutative hypergroup $\mathcal{K}$. A subhypergroup $\mathcal{H}$ acts on $\mathcal{K}$ by multiplication since the axiom leads $c_i c_j \in \sigma\mathcal{K}$ for $c_i \in \mathcal{H}$, $c_j \in \mathcal{K}$. The total hypergroup $\mathcal{K}$ is decomposed into the set of $\mathcal{H}$-equivalent class as in the previous section.

Definition 10. Let $\mathcal{H} \subset \mathcal{K}$ be a subhypergroup of a finite commutative hypergroup. A set $\{e_0^{\mathcal{H}} c_i \text{ s.t. } c_i \in \mathcal{K}\}$ is said to be a *quotient* of $\mathcal{K}$ by a subhypergroup $\mathcal{H}$ and is denoted by $\mathcal{K}/\mathcal{H}$ or $\mathcal{H}\backslash\mathcal{K}$.

Lemma 2.10. *Let $\mathcal{H} \subset \mathcal{K}$ be a subhypergroup of a finite commutative hypergroup. The quotient $\mathcal{K}/\mathcal{H}$ turns to be a commutative hypergroup.*

Proof. We first remark that $(e_0^{\mathcal{H}})^* = e_0^{\mathcal{H}}$ and is a projection. Then $e_0^{\mathcal{H}} \cdot e_0^{\mathcal{H}} c_i = e_0^{\mathcal{H}} c_i$ shows that $e_0^{\mathcal{H}}$ is unit. So the relations of $(e_0^{\mathcal{H}} c_i)^* = e_0^{\mathcal{H}}(c_i^*)$ and $e_0^{\mathcal{H}} c_i \cdot e_0^{\mathcal{H}} c_j = e_0^{\mathcal{H}}(c_i c_j)$ leads that a $\mathbb{C}$-linear space $\{\mathbb{C}e_0^{\mathcal{H}} + \mathbb{C}e_0^{\mathcal{H}} c_i + \cdots\}_{e_0^{\mathcal{H}} c_{k'} \in \mathcal{K}/\mathcal{H}}$ is considered as a $*$-algebra. Moreover,

$$e_0^{\mathcal{H}}(c_i c_j) = \sum_{c_k \in \mathcal{K}} n_{ij}^k e_0^{\mathcal{H}} c_k = \sum_{e_0^{\mathcal{H}} c_{k'} \in \mathcal{K}/\mathcal{H}} \Big(\sum_{c_k \in e_0^{\mathcal{H}} c_{k'}} n_{ij}^k \Big) e_0^{\mathcal{H}} c_{k'}$$

shows that the products have stochastic expansions. Since $e_0^{\mathcal{H}} c_i \cdot e_0^{\mathcal{H}} c_i^* = e_0^{\mathcal{H}}(c_i c_i^*)$ includes unit, the involution of $e_0^{\mathcal{H}} c_i$ equals to $e_0^{\mathcal{H}} c_i^*$. Suppose that $e_0^{\mathcal{H}} c_i \cdot e_0^{\mathcal{H}} c_j \ni e_0^{\mathcal{H}}$. We have $c_i c_j \ni h$ for some $h \in \mathcal{H}$. Then $c_i(h^* c_j) \ni c_0$, so that $h^* c_j \ni c_i^*$. Hence $e_0^{\mathcal{H}} c_j \ni c_i^*$ and $e_0^{\mathcal{H}} c_j = e_0^{\mathcal{H}} c_i^*$. We obtain that the involution is unique.

Thus it is already shown that the quotient is a hypergroup.

□

We now state the general examples of actions of hypergroup. Let $\mathcal{H} \subset \mathcal{K} \subset \mathcal{L}$ be finite commutative hypergroups. An action π of $\mathcal{K}$ on the quotient hypergroup $\mathcal{L}/\mathcal{H}$ is defined by usual multiplication.

$$\mathcal{K} \times \mathcal{L}/\mathcal{H} \xrightarrow{\pi} \mathcal{L}/\mathcal{H}$$

$$(c, x) \quad \mapsto \quad \varphi(c)x$$

where φ is a quotient map $\mathcal{L} \to \mathcal{L}/\mathcal{H}$.

Let $\mathcal{K}$ and $\mathcal{L}$ be finite commutative hypergroups. A map φ from $\mathcal{K}$ to $\mathcal{L}$ is called a morphism if uniquely extended $\mathbb{C}$-linear mapping of φ on $\mathfrak{A}(\mathcal{K})$ to $\mathfrak{A}(\mathcal{L})$ is recognized as a unital $*$-homomorphism over $*$-algebras, namely,

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\varphi} & \mathcal{L} \\ \iota\downarrow & & \downarrow\iota \\ \mathfrak{A}(\mathcal{K}) & \xrightarrow[\varphi]{} & \mathfrak{A}(\mathcal{L}). \end{array}$$

We denote simply a homomorphism in the category of finite commutative hypergroups in the forms: $\varphi : \mathcal{K} \to \mathcal{L}$ or $\mathcal{K} \xrightarrow{\varphi} \mathcal{L}$. A homomorphism φ is called a monomorphism if it is injective, and an epimorphism if it is surjective.

Let $\mathrm{Ker}(\varphi) := \varphi^{-1}(\ell_0) = \{c_i \text{ s.t. } \varphi(c_i) = \ell_0\}$, where ℓ_0 is unit of $\mathcal{L}$, and be said a kernel of φ. Let $\mathrm{Im}(\varphi) := \varphi(\mathcal{K})$ and be said an image of φ.

Lemma 2.11. *If a homomorphism $\varphi : \mathcal{K} \to \mathcal{L}$ over finite commutative hypergroups is given, then* $\mathrm{Ker}(\varphi)$ *is a subhypergroup of* $\mathcal{K}$ *and* $\mathrm{Im}(\varphi)$ *is a subhypergroup of* $\mathcal{L}$.

Proof. It is obviously shown that $\mathrm{Im}(\varphi)$ is a subhypergroup. It is easily seen from the fact (a4) in the axiom of hypergroups that $\mathrm{Ker}(\varphi)$ is a hypergroup. □

Proposition 2.12. *Let $\varphi : \mathcal{K} \to \mathcal{L}$ be a homomorphism and c_0 be unit of $\mathcal{K}$. If* $\mathrm{Ker}(\varphi) = \{c_0\}$ *then φ is injective.*

Proof. Assume that there exists $\ell \in \mathcal{L}$ such that $\varphi(c) = \varphi(c') = \ell$ for $c \neq c'$. We have $c'c^*$ does not include unit. However $\varphi(c'c^*) = \ell\ell^*$ includes unit. It is contradictory. Thus φ is injective.

□

We will build up the following theorem of homomorphisms in the category of finite commutative hypergroups as well as the category of commutative topological groups.

Theorem 2.2. (*Homomorphism*) *Let $\varphi : \mathcal{K} \to \mathcal{L}$ be a surjective homomorphism. If* $\mathrm{Ker}(\varphi) = \mathcal{H}$, *then the quotient hypergroup $\mathcal{K}/\mathcal{H}$ is isomorphic to $\mathcal{L}$ whose isomorphism is induced by φ. In another word, there exists a canonical isomorphism such that*

$$\mathcal{K}/\operatorname{Ker}(\varphi) \cong \operatorname{Im}(\varphi) = \mathcal{L}.$$

Proof. Consider the map $\dot{\varphi} : e_0^{\mathcal{H}} c_i \mapsto \varphi(c_i)$. Suppose that $e_0^{\mathcal{H}} c_i = e_0^{\mathcal{H}} c_j$. On writing that unit $\ell_0 \in \mathcal{L}$, since $\varphi(e_0^{\mathcal{H}}) = \ell_0$, we have that $\varphi(c_i) = \varphi(c_j)$. So this shows that $\dot{\varphi}$ is well defined.

It is obvious that $\dot{\varphi}$ is multiplicative and involutive and $\mathrm{Ker}(\dot{\varphi}) = \{e_0^{\mathcal{H}}\}$, i.e. injective. From Proposition 2.12, it is shown that $\dot{\varphi}$ is an isomorphism. □

Remark 2.7. When φ is not surjective, if we replace $\mathcal{L}$ with $\mathrm{Im}(\varphi)$, then the above theorem also holds. We select the simple form as above. In a similar argument as commutative groups the quotient hypergroups are always determined by their subhypergroups and structure constants.

We state the next proposition which let us know that if $\mathcal{H} \subset \mathcal{K} \subset \mathcal{L}$, then any $\mathcal{H}$-equivalent class in $\mathcal{K}$ equals a $\mathcal{H}$-equivalent class in $\mathcal{L}$.

Proposition 2.13. *Let $\varphi : \mathcal{K} \to \mathcal{L}$ be a monomorphism and $\mathcal{H}$ be a subhypergroup of $\mathcal{K}$. If $y \sim_{\varphi(\mathcal{H})} \varphi(x)$ in $\mathcal{L}$ for $x \in \mathcal{K}$, then there exists $z \in \mathcal{K}$ such that $\varphi(z) = y$.*

Moreover, the cardinal number of a set $\{x' \in \mathcal{K}$ s.t. $x' \sim_{\mathcal{H}} x\}$ equals one of a set $\{y' \in \mathcal{L}$ s.t. $y' \sim_{\varphi(\mathcal{H})} \varphi(x)\}$, namely, for $x \in \mathcal{K}$

$$\#\operatorname{supp}(e_0^{\mathcal{H}} x) = \#\operatorname{supp}(e_0^{\varphi(\mathcal{H})} \varphi(x)).$$

Proof. Consider $\varphi(e_0^{\mathcal{H}}x) = e_0^{\varphi(\mathcal{H})}\varphi(x)$. This is $\varphi(\mathcal{H})$-stable and $\varphi(\mathcal{H})$-equivalent class which encloses $\varphi(x)$. By uniqueness of a $\varphi(\mathcal{H})$-stable set, we have $\varphi(e_0^{\mathcal{H}}x) \ni y$. Hence there exists $z \in \mathcal{K}$ such that $\varphi(z) = y$. Therefore $e_0^{\varphi(\mathcal{H})}\varphi(x) = e_0^{\varphi(\mathcal{H})}y$. This shows the first conclusion. A monomorphism φ maps $\mathrm{supp}(e_0^{\mathcal{H}}x)$ to $\mathrm{supp}(e_0^{\varphi(\mathcal{H})}\varphi(x))$. The cardinal numbers are equal. □

Remark 2.8. Notice that subhypergroup $\mathcal{H}$ is arbitrary. If a finite commutative hypergroup $\mathcal{K}$ has many subhypergroups, then the equivalent classes with respect to a subhypergroup have the same cardinal numbers. The hypergroup $\mathcal{K}$ is limited as the manner as above.

Our main object in this paper is presented in the next definition in order to describe formally subhypergroups and quotients.

Definition 11. (Exact sequences) Let $\mathcal{K}_i$ be finite commutative hypergroups indexed by integers i. Let $\varphi_i : \mathcal{K}_i \to \mathcal{K}_{i+1}$ be homomorphisms for i. The sequence

$$\mathcal{K}_1 \xrightarrow{\varphi_1} \mathcal{K}_2 \xrightarrow{\varphi_2} \mathcal{K}_3$$

is said to be *exact* at $\mathcal{K}_2$ if $\mathrm{Im}(\varphi_1) = \mathrm{Ker}(\varphi_2)$. Moreover,

$$\cdots \to \mathcal{K}_i \xrightarrow{\varphi_i} \mathcal{K}_{i+1} \xrightarrow{\varphi_{i+1}} \mathcal{K}_{i+2} \to \cdots$$

is called to be an exact sequence if it is exact at $\mathcal{K}_i$ for all i.

We define a cokernel of φ_i as $\mathrm{Coker}(\varphi_i) = \mathcal{K}_{i+1}/\,\mathrm{Im}(\varphi_i)$.

We can define the zero sequence provided that $\mathrm{Im}(\varphi_1) \subset \mathrm{Ker}(\varphi_2)$, but we do not use it in this paper, so that we omit it. The fundamental results for the exact sequences also are satisfied like as in the category of commutative groups.

Lemma 2.14. *The following statements hold*:

(1) $\mathbf{1} \to \mathcal{H} \overset{\varphi}{\to} \mathcal{K}$ *is exact if and only if* φ *is injective,*

(2) $\mathcal{H} \overset{\varphi}{\to} \mathcal{K} \to \mathbf{1}$ *is exact if and only if* φ *is surjective,*

(3) $\mathbf{1} \to \mathcal{H} \overset{\varphi}{\to} \mathcal{K} \to \mathbf{1}$ *is exact if and only if* φ *is an isomorphism,*

(4) *If a homomorphism* $\mathcal{H} \overset{\varphi}{\to} \mathcal{K}$ *is given, then a sequence*

$$\mathbf{1} \to \mathrm{Ker}(\varphi) \to \mathcal{H} \overset{\varphi}{\to} \mathcal{K} \to \mathrm{Coker}(\varphi) \to \mathbf{1}$$

is exact.

Proof. It is very clear all statements are satisfied using Proposition 2.12. □

Definition 12. The following diagram of homomorphisms $\varphi_1, \dots, \varphi_4$ is said to be commutative if $\varphi_4 \circ \varphi_1 = \varphi_3 \circ \varphi_2$.

$$\begin{array}{ccc} \mathcal{K}_1 & \xrightarrow{\varphi_1} & \mathcal{K}_2 \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\varphi_4} \\ \mathcal{L}_1 & \xrightarrow[\varphi_3]{} & \mathcal{L}_2 \end{array}$$

We can state the next two famous propositions as in an abelian category.

Proposition 2.15. (Five Lemma) *Let a commutative diagram be given*

$$\begin{array}{ccccccccc} \mathcal{K}_1 & \xrightarrow{\varphi_1} & \mathcal{K}_2 & \xrightarrow{\varphi_2} & \mathcal{K}_3 & \xrightarrow{\varphi_3} & \mathcal{K}_4 & \xrightarrow{\varphi_4} & \mathcal{K}_5 \\ {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\alpha_3}\downarrow & & {\scriptstyle\alpha_4}\downarrow & & {\scriptstyle\alpha_5}\downarrow \\ \mathcal{L}_1 & \xrightarrow[\varphi_5]{} & \mathcal{L}_2 & \xrightarrow[\varphi_6]{} & \mathcal{L}_3 & \xrightarrow[\varphi_7]{} & \mathcal{L}_4 & \xrightarrow[\varphi_8]{} & \mathcal{L}_5 \end{array}$$

where rows are exact. Then we have the following statements :

(1) *If α_1 is surjective, and α_2, α_4 are injective, then α_3 is injective,*

(2) *If α_5 is injective, and α_2, α_4 are surjective, then α_3 is surjective,*

(3) *If α_1 is surjective, α_5 is injective, and α_2, α_4 are bijective, then α_3 is bijective.*

Proof. (1) is proved as the same as in the category of commutative groups. Indeed, suppose that $x \in \mathcal{K}_3$ with $\alpha_3(x) = e$ mapped to unit. By commutative diagrams, we have $\varphi_7 \circ \alpha_3(x) = \alpha_4 \circ \varphi_3(x) = e$. Since α_4 is injective, $\varphi_3(x) = e$. Then there exists $x' \in \mathcal{K}_2$ such that $\varphi_2(x') = x$. Viewing $\varphi_6 \circ \alpha_2(x') = \alpha_3 \circ \varphi_2(x') = e$, we have $\alpha_2(x') \in \mathrm{Ker}(\varphi_2)$ since α_2 is injective. Thus there exists $x'' \in \mathcal{K}_1$ such that $\varphi_5 \circ \alpha_1(x'') = \alpha_2(x')$ since α_1 is surjective and exactness. Therefore $\alpha_2 \circ \varphi_1(x'') = \alpha_2(x')$ leads $\varphi_1(x'') = x'$. It implies that $x = \varphi_2(x') = \varphi_2 \circ \varphi_1(x'') = e$, so that α_3 is injective.

(2) will be proved in similar way in the category of commutative groups, but it is needed to express elements in an equivalent class carefully. Indeed, suppose that $x \in \mathcal{L}_3$. There exists $x' \in \mathcal{K}_4$ such that $\alpha_4(x') = \varphi_7(x)$ since α_4 is surjective. Then $\alpha_5 \circ \varphi_4(x') = \varphi_8 \circ \alpha_4(x') = \varphi_8 \circ \varphi_7(x) = e$ implies $\varphi_4(x') = e$, since α_5 is injective. By exactness, there exists $x'' \in \mathcal{K}_3$ such that $\varphi_3(x'') = x'$. Viewing $\alpha_4 \circ \varphi_3(x'') = \varphi_7 \circ \alpha_3(x'') = \varphi_7(x)$, we have $\alpha_3(x'') \sim_{\mathrm{Im}(\varphi_6)} x$, namely $\varphi_6(y)\alpha_3(x'') \ni x$ for some $y \in \mathcal{L}_2$.

Since α_2 is surjective, there exists $z' \in \mathcal{K}_2$ such that $\alpha_2(z') = z$ for any $z \in \mathcal{L}_2$. Therefore $\alpha_3(\varphi_2(z')) = \alpha_3 \circ \varphi_2(z') = \varphi_6 \circ \alpha_2(z') = \varphi_6(z)$. This shows that $\mathrm{Im}(\alpha_3) \supset \mathrm{Im}(\varphi_6)$. Since $\alpha_3(x'')$ is in the image of α_3 and x is $\mathrm{Im}(\varphi_6)$-equivalent to x'', Proposition 2.13 shows that x is in the image of α_3.

(3) is obviously proved by combining (1) and (2). ☐

Proposition 2.16. (Snake lemma) *Let a commutative diagram with exact rows be given.*

$$
\begin{array}{ccccccccc}
\mathbf{1} & \longrightarrow & \mathcal{K}_2 & \xrightarrow{\iota} & \mathcal{K}_3 & \xrightarrow{\varphi_3} & \mathcal{K}_4 & \longrightarrow & \mathbf{1} \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \\
\mathbf{1} & \longrightarrow & \mathcal{L}_2 & \xrightarrow[\iota]{} & \mathcal{L}_3 & \xrightarrow[\varphi_3]{} & \mathcal{L}_4 & \longrightarrow & \mathbf{1}
\end{array}
$$

Then there exists the exact sequence as follow:

$$\mathbf{1} \to \mathrm{Ker}(\alpha_2) \to \mathrm{Ker}(\alpha_3) \to \mathrm{Ker}(\alpha_4) \to \mathrm{Coker}(\alpha_2) \to \mathrm{Coker}(\alpha_3) \to \mathrm{Coker}(\alpha_4) \to \mathbf{1}$$

Proof. In a similar way in the category of finite commutative group, we can obtain the conclusion. However we exchange classes for a subgroup into classes for a subhypergroup as in the manner of multiplying the Haar measure of a subhypergroup. □

3 Estimation of Orders

Let $\mathcal{K}$ be a finite commutative hypergroup and $\mathcal{H} \subset \mathcal{K}$ be a subhypergroup. It is well known that the quotient $\mathcal{K}/\mathcal{H}$ is also a hypergroup [SW1]. In order to describe this situation, we often use the form of short exact sequence $\mathbf{1} \to \mathcal{H} \to \mathcal{K} \overset{\varphi}{\to} \mathcal{L} \to \mathbf{1}$ where $\mathcal{L} = \mathcal{K}/\mathcal{H}$ and φ is the quotient map.

The extension problem is set up in the case of general finite commutative hypergroups.

Problem. (General style) For given finite commutative hypergroups $\mathcal{H}, \mathcal{L}$, require all finite commutative hypergroups $\mathcal{K}$ such that $\mathbf{1} \to \mathcal{H} \to \mathcal{K} \to \mathcal{L} \to \mathbf{1}$ is exact. Moreover analyze the total of $\mathcal{K}$'s and parametrize some useful families of them.

Now fix a finite commutative hypergroup $\mathcal{K}$ and a subhypergroup $\mathcal{H}$ of $\mathcal{K}$.

$$\mathcal{H}(s) = \{h \in \mathcal{H} \text{ s.t. } hs = s\}$$

is called a *stabilizer* of $s \in \mathcal{K}$ under $\mathcal{H}$-multiplication.

Lemma 3.1. *Let $1 \to \mathcal{H} \to \mathcal{K}$ be an exact sequence of finite commutative hypergroups. An element $h \in \mathcal{H}(s)$ is characterized by the following property $(*)$:*
$(*)$ $\quad \chi(h) = 1$ *for* $\chi \in \mathcal{K}^\wedge$ *such that* $\chi(s) \neq 0$.
Furthermore $\mathcal{H}(s)$ is a subhypergroup.

Proof. For $\chi \in \mathcal{K}^\wedge$ such that $\chi(s) \neq 0$, we have that $\chi(hs) = \chi(h)\chi(s) = \chi(s)$, so that $\chi(h) = 1$. Conversely it is shown that $\chi(hs) = \chi(s)$ for all $\chi \in \mathcal{K}$. Thus $hs = s$ and $h \in \mathcal{H}(s)$. To start with showing that $\mathcal{H}(s)$ is a subhypergroup, we notice that $\mathcal{H}(s) \ni h^*$ if $\mathcal{H}(s) \ni h$ from the property $(*)$. Therefore $\mathcal{H}(s)^* = \mathcal{H}(s)$. For $h, h' \in \mathcal{H}(s)$, we write $hh' = \sum_i n_i c_i$ where $\sum_i n_i = 1$, $c_i \in \mathcal{K}$. Then we get $1 = \chi(h)\chi(h') = \chi(hh') = \sum_i n_i \chi(c_i) = \sum_i n_i \cdot \Re(\chi(c_i)) \leq \sum_i n_i = 1$ for χ in $(*)$. Since $|\chi(c_i)| \leq 1$, we have $\Re(\chi(c_i)) \leq 1$, and note that $\Re(\chi(c_i)) = 1$ if and only if $\chi(c_i) = 1$. It is clear to see that $\chi(c_i) = 1$ for i such that $n_i \neq 0$. Thus hh' belongs to a real linear span of elements in $\mathcal{H}(s)$. $\square$

Provided that $\mathcal{K}$ is a group, it is well known that the above subhypergroup $\mathcal{H}(s) = \{c_0\}$, so that a trivial group. We can find an example which has the maximal trivial stabilizer, i.e. $\mathcal{H} \vee \mathcal{L}$ has $\mathcal{H}(s) = \mathcal{H}$ for an element $s \notin \mathcal{H}$.

We give some relations among weights of hypergroups in a short exact sequence.

Lemma 3.2. *Let $1 \to \mathcal{H} \to \mathcal{K} \overset{\varphi}{\to} \mathcal{L} \to 1$ be an exact sequence of finite commutative hypergroups. For any $\ell \in \mathcal{L}$ and not equal to unit ℓ_0, the following conditions* (1)-(3) *hold*:

(1) $w(\mathcal{K}) = w(\mathcal{H})w(\mathcal{L})$,

(2) $w(\varphi^{-1}(\ell)) = w(\mathcal{H})w(\ell)$,

(3) $w(\ell) \leq w(s) \leq w(\mathcal{H})w(\ell)$ *for* $s \in \varphi^{-1}(\ell)$.

Proof. Let $e_0^{\mathcal{K}} = w(\mathcal{K})^{-1} \sum_{c_i \in \mathcal{K}} w(c_i)c_i$ be the normalized Haar measure of $\mathcal{K}$. The image $\varphi(e_0^{\mathcal{K}})$ equals the normalized Haar measure $e_0^{\mathcal{L}}$ of $\mathcal{L}$. Let ℓ_0 be the unit of $\mathcal{L}$, and $\varphi^{-1}(\ell_0) = \mathcal{H}$. From the expression

$$\begin{aligned}\varphi(e_0^{\mathcal{K}}) &= w(\mathcal{K})^{-1}\Big(\sum_{c\in\mathcal{H}} w(c)\ell_0 + \cdots + \sum_{d\in\varphi^{-1}(\ell)} w(d)\ell + \cdots\Big) \\ &= w(\mathcal{K})^{-1}\Big(\ w(\mathcal{H})\ell_0\ + \cdots +\ w(\varphi^{-1}(\ell))\ell\ + \cdots\Big),\end{aligned}$$

it is obtained that $w(\mathcal{K})^{-1}w(\mathcal{H}) = w(\mathcal{L})^{-1}$ and $w(\mathcal{K})^{-1}w(\varphi^{-1}(\ell)) = w(\mathcal{L})^{-1}w(\ell)$. This shows (1),(2) and the second inequality of (3).

The first one of (3) is shown by the fact $w(s)^{-1} \leq w(\ell)^{-1}$ since

$$\varphi(ss^*) = \varphi(w(s)^{-1}c_0 + \cdots) = (w(s)^{-1} + \cdots)\ell_0 + \cdots = w(\ell)^{-1}\ell_0 + \cdots .$$

□

In the category of signed hypergroup extensions, namely if $\mathcal{K}$ is a signed hypergroup, the conditions (1),(2) and (3) $w(s) \leq w(\mathcal{H})w(\ell)$ in Lemma 3.2 also hold but (3) $w(\ell) \leq w(s)$ does not always hold. It turns that (3′) $0 < w(s) \leq w(\mathcal{H})w(\ell)$. Because a signed hypergroup admits a weight which is less than one.

Remark 3.1. From Lemma 3.2 (1), the weights of hypergroups play a part of orders in the category of finite commutative groups, but orders of hypergroups does not. The next result gives us to look for the existence of an counterexample, i.e. it will be seen in Theorem 5.1.

Using Lemma 3.2 on the relations about weights, we give a general condition about orders of hypergroups in a short exact sequence.

Proposition 3.3. (Order condition) *Let* $1 \to \mathcal{H} \to \mathcal{K} \overset{\varphi}{\to} \mathcal{L} \to 1$ *be an exact sequence of finite commutative hypergroups. Then the inequalities*:

$$|\mathcal{H}| + |\mathcal{L}| - 1 \ \leq\ |\mathcal{K}| \ \leq\ |\mathcal{H}| + \lfloor w(\mathcal{H}) \rfloor (|\mathcal{L}| - 1)$$

hold, where the floor function $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$ *for real* x.

Proof. The first inequality is obvious, equality occurs if and only if $\mathcal{K} = \mathcal{H} \vee \mathcal{L}$, i.e. a join hypergroup to $\mathcal{H}$ with $\mathcal{L}$. From the above Lemma 3.2 (2)(3), $w(\mathcal{H})w(\ell) = \sum_{s\in\varphi^{-1}(\ell)} w(s) \geq \sum_{s\in\varphi^{-1}(\ell)} w(\ell) = |\varphi^{-1}(\ell)|w(\ell)$. Hence $w(\mathcal{H}) \geq |\varphi^{-1}(\ell)|$.

□

This estimation is strict since we will show that there exist hypergroup extensions of several order in the inequalities. It is trivial that the direct product hypergroup $\mathcal{K} = \mathcal{H} \times \mathcal{L}$ is a hypergroup extension of order $|\mathcal{H}| \cdot |\mathcal{L}|$.

At first we will show the cases in which the maximal order of hypergroup extension $\mathcal{K}$ is $|\mathcal{H}| \cdot |\mathcal{L}|$.

When $\mathcal{H}$ is a group, we have $w(\mathcal{H}) = \lfloor w(\mathcal{H}) \rfloor = |\mathcal{H}|$. The condition of Proposition 3.3 is in the form $|\mathcal{H}| + |\mathcal{L}| - 1 \leq |\mathcal{K}| \leq |\mathcal{H}| \cdot |\mathcal{L}|$ which is established in [K]. In the case of $|\mathcal{H}| \leq w(\mathcal{H}) < |\mathcal{H}| + 1$, the above condition also holds.
We define for subsets M, N of $\mathcal{K}$,

$$\begin{aligned} M \diamond N &:= \bigcup_{M \ni c_i, N \ni c_j} \mathrm{supp}(c_i c_j) \\ &= \{c_{k_0} \in \mathcal{K} \text{ s.t. } n_{ij}^{k_0} > 0 \text{ for some } c_i c_j = \sum_k n_{ij}^k c_k,\ c_i \in M, c_j \in N\}. \end{aligned}$$

Proposition 3.4. *Let $1 \to \mathcal{H} \to \mathcal{K} \overset{\varphi}{\to} \mathcal{L} \to 1$ be an exact sequence of finite commutative hypergroups and $\mathcal{H}$ be a group. Fixed $\ell \in \mathcal{L}$, we have the following* (1)-(4) :

(1) $hs \in \varphi^{-1}(\ell)$ *for* $s \in \varphi^{-1}(\ell)$ *and* $h \in \mathcal{H}$,

(2) $\mathcal{H} \diamond \{s\} = \varphi^{-1}(\ell)$ *for* $s \in \varphi^{-1}(\ell)$,

(3) $\mathcal{H}(s) = \mathcal{H}(s')$ *for* $s, s' \in \varphi^{-1}(\ell)$, *and* $|\varphi^{-1}(\ell)| = |\mathcal{H}/\mathcal{H}(s)|$,

(4) $w(s) = w(\ell) \cdot |\mathcal{H}|/|\varphi^{-1}(\ell)|$ *for* $s \in \varphi^{-1}(\ell)$.

Proof. We first note that the maximal normed points in an absolute simplex $\sigma\varphi^{-1}(\ell)$ lie in their vertexes, where the norm is given from the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{K}$. Assume that $s \in \varphi^{-1}(\ell)$ has the maximal norm in $\varphi^{-1}(\ell)$. Then $\langle hs, hs \rangle = \langle s, h^* hs \rangle = \langle s, s \rangle$ implies that the multiplication by h is isometric. Hence hs is a vertex, so that $hs \in \varphi^{-1}(\ell)$ (1). The statement (2) follows from the fact that $e_0^{\mathcal{H}} s = e_0^{\mathcal{H}} s'$ for $s, s' \in \varphi^{-1}(\ell)$. When $s' = h_1 s$ for some $h_1 \in \mathcal{H}$, the inclusion $\mathcal{H}(s) \subset \mathcal{H}(s')$ is implied from $hs' = h(h_1 s) = h_1 hs = h_1 s = s'$ for any $h \in \mathcal{H}(s)$. Converse inclusion also holds, then $\mathcal{H}(s) = \mathcal{H}(s')$ and (3). Finally Lemma 3.2 (3) and the fact $w(s')^{-1} = \langle s', s' \rangle = \langle s, s \rangle = w(s)^{-1}$ show the last statement (4). □

In the paper [K], Proposition 3.4 (1)-(3) were showed using a different proof from the above. Proposition 3.4 gives many examples of hypergroups whose stabilizer have any subgroup of $\mathcal{H}$.

In the paper [IKS], it was showed that if $\mathcal{H} = \mathscr{L}(q)$ and $\mathcal{L}$ is a group, then $|\mathcal{H}| + |\mathcal{L}| - 1 \leq |\mathcal{K}| \leq |\mathcal{H}| \cdot |\mathcal{L}|$ holds, more precisely $|\mathcal{L}| + 1 \leq |\mathcal{K}| \leq 2 \cdot |\mathcal{L}|$.

It is obvious from Proposition 3.2 that the maximal order $\mathcal{K}$ is $|\mathcal{H}| \cdot |\mathcal{L}|$ provided that $|\mathcal{H}| \leq w(\mathcal{H}) < |\mathcal{H}| + 1$. For example, $\mathcal{H} = \mathscr{L}(q)$ for $1/2 < q \leq 1$.

What happens in the case of a general finite commutative hypergroup $\mathcal{H}$ and of any finite commutative group $\mathcal{L}$? The following is an answer to this question.

Proposition 3.5. *Let $1 \to \mathcal{H} \to \mathcal{K} \overset{\varphi}{\to} \mathcal{L} \to 1$ be an exact sequence of finite commutative hypergroups. If $\mathcal{L}$ is a group, then $|\varphi^{-1}(\ell)| \leq |\mathcal{H}|$ for $\ell \in \mathcal{L}$ and $w(\chi) = w(\chi')$ for $\chi, \chi' \in \mathcal{K}^\wedge$ while their restrictions $\chi|_{\mathcal{H}} = \chi'|_{\mathcal{H}} \in \mathcal{H}^\wedge$.*

Proof. We first obtain that the matrix $[\chi(s_i)]_{\chi\in\mathcal{K}^\wedge,\, s_i\in\varphi^{-1}(\ell)}$ of the characters matrix has rank $m+1$ since the column vectors are linearly independent from Axiom of hypergroups. Now we will calculate this rank in another way.

Fix a non-trivial character $\chi^{\mathcal{H}} \in \mathcal{H}^\wedge$. We define the subset $\Sigma(\chi^{\mathcal{H}}) = \{\chi \in \mathcal{K}^\wedge \text{ s.t. } \chi|_{\mathcal{H}} = \chi^{\mathcal{H}}\}$ of $\mathcal{K}^\wedge$. Write $\varphi^{-1}(\ell) = \{s_0, \ldots, s_m\} \subset \mathcal{K}$.

Assume that $\chi(s_0) \neq 0$ for some character $\chi \in \Sigma(\chi^{\mathcal{H}})$. Since $\mathcal{L}$ is a group, we have $(s_0)^* s_i \in A(\mathcal{H})$, so that the value $\chi((s_0)^* s_i)$ is independent of the choice of $\chi \in \Sigma(\chi^{\mathcal{H}})$. Then $V(\chi^{\mathcal{H}}, 0, i) := \chi((s_0)^* s_i)$. It is obvious that $|\chi(s_0)|^2 = V(\chi^{\mathcal{H}}, 0, 0) \neq 0$. This shows that the absolute value of $\chi(s_0)$ is independent of χ. It is also obtained that $\chi(s_i) = V(\chi^{\mathcal{H}}, 0, i)\chi(s_0^*)^{-1}$. Hence the vector $(\chi(s_0), \ldots, \chi(s_m))$ is parallel to $(V(\chi^{\mathcal{H}}, 0, 0), \ldots, V(\chi^{\mathcal{H}}, 0, m))$. The sub-matrix $[\chi_j(s_i)]$ for $\chi_j \in \Sigma(\chi^{\mathcal{H}})$ and $s_i \in \varphi^{-1}(\ell)$ has rank one.

If $\chi_j(s_i) = 0$ for all χ_j, $s_i \in \varphi^{-1}(\ell)$, then the sub-matrix $[\chi_j(s_i)]$ for $\chi_j \in \Sigma(\chi^{\mathcal{H}})$ turns to be zero, i.e. the rank is zero. Hence the sub-matrix $[\chi_j(s_i)]$ for $\chi_j \in \Sigma(\chi^{\mathcal{H}})$ and $s_i \in \varphi^{-1}(\ell)$ has at most rank one. Therefore the rank of the matrix $[\chi(s_i)]_{\chi\in\mathcal{K}^\wedge,\, s_i\in\varphi^{-1}(\ell)}$ is at most $|\mathcal{H}^\wedge| = |\mathcal{H}|$. This shows that $|\varphi^{-1}(\ell)| = m+1 \leq |\mathcal{H}|$.

It is already shown that the values of $\chi(s_i)$ have the same absolute value for all $\chi \in \Sigma(\chi^{\mathcal{H}})$ which depends only on $s_i \in \varphi^{-1}(\ell)$. If $c \in \mathcal{K}$ is fixed then $|\chi(c)|$ has the same value for $\chi \in \Sigma(\chi^{\mathcal{H}})$. From the formula for weights of characters, we have that for $\chi, \chi' \in \Sigma(\chi^{\mathcal{H}})$,

$$\begin{aligned} w(\chi)^{-1} &= w(\mathcal{K})^{-1} \sum_{c\in\mathcal{K}} |\chi(c)|^2 w(c) \\ &= w(\mathcal{K})^{-1} \sum_{c\in\mathcal{K}} |\chi'(c)|^2 w(c) = w(\chi')^{-1}. \end{aligned}$$

This implies that $w(\chi) = w(\chi')$. □

We have the next corollary on the order of hypergroup extensions.

Corollary 3.6. *Let $1 \to \mathcal{H} \to \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \to 1$ be an exact sequence of finite commutative hypergroups. If $\mathcal{L}$ is a group, then*

$$|\mathcal{H}| + |\mathcal{L}| - 1 \leq |\mathcal{K}| \leq |\mathcal{H}| \cdot |\mathcal{L}|.$$

Proof. It is a trivial consequence of $|\varphi^{-1}(\ell)| \leq |\mathcal{H}|$ together with the arguments in the proof of Proposition 3.3. □

We note that if a hypergroup $\mathcal{K}$ is strong, Corollary 3.6 is immediately consequence of the duality theory and S. Kawakami's Lemma [K]. The maximal order $|\mathcal{H}| \cdot |\mathcal{L}|$ is achieved when $\mathcal{K}$ is a direct product $\mathcal{H} \times \mathcal{L}$, but there does not always exist hypergroup extensions with an order of any integer between $|\mathcal{H}| + |\mathcal{L}| - 1$ and $|\mathcal{H}| \cdot |\mathcal{L}|$ in Corollary 3.6, which was proved in [IKS].

4 Extensions of Minimum Problems

By preparation of the general theory in the previous sections we set up the next extension Problem such that sub-hypergroup and its quotient are in the smallest non-trivial hypergroups. This is the first approach for the case that subhypergroup is not a group. In order to describe them more simply, we define very useful tilde symbol $\tilde{x} = 1 - x$ for a number x.

Problem. (Minimal Extensions) Fix finite commutative hypergroups $\mathscr{L}(\tilde{p})$ and $\mathscr{L}(\tilde{q})$ for two numbers $0 < \tilde{p} \leq 1$ and $0 < \tilde{q} \leq 1$. Require all finite commutative hypergroup $\mathcal{K}$ such that

$$\mathbf{1} \to \mathscr{L}(\tilde{p}) \to \mathcal{K} \to \mathscr{L}(\tilde{q}) \to \mathbf{1}$$

is exact. Moreover analyze the total of $\mathcal{K}$'s and parametrize some useful families of them like as strong hypergroup extensions.

At first, we shall approach to a general result in the case of $|\mathcal{K}| = 4$.

Assume that a subhypergroup $\mathcal{H} \cong \mathscr{L}(\tilde{p})$ with $0 < \tilde{p} \leq 1$ and its quotient $\mathcal{K}/\mathcal{H} = \mathcal{L} \cong \mathscr{L}(\tilde{q})$ with $0 < \tilde{q} \leq 1$. We write $\mathcal{H} = \{h_0, h_1\}$ and $\mathcal{L} = \{\ell_0, \ell_1\}$ with structure equations $h_1^2 = \tilde{p}h_0 + ph_1$ and $\ell_1^2 = \tilde{q}\ell_0 + q\ell_1$ since $\mathcal{K}$ is of order four.

We can write $\mathcal{K} = \{h_0, h_1, s_0, s_1\}$ where $\varphi(s_i) = \ell_1$ $(i = 0, 1)$.

Proposition 4.1. *An extension $\mathcal{K}$ of order four in the above Problem for given two positive number p, q satisfies the following conditions*:

(Ha) $h_1 s_0 = \tau s_0 + \tilde{\tau} s_1$,

(Hb) $h_1 s_1 = (p + \tau)s_0 + (\tilde{p} - \tau)s_1$.

where τ is a parameter such that $0 \leq \tau \leq p$.

Proof. Since $\varphi(h_1 s_i) = \ell_1$ for $i = 0, 1$, we can write $h_1 s_0 = \tilde{x}s_0 + xs_1$ and $h_1 s_1 = \tilde{y}s_0 + ys_1$ for $x, y \in [0, 1]$. Consider a triple product

$$(s_0 h_1)h_1 = (\tilde{x}s_0 + xs_1)h_1 = (\tilde{x}^2 + x\tilde{y})s_0 + (\tilde{x}x + xy)s_1$$

and

$$s_0(h_1)^2 = s_0(\tilde{p}h_0 + ph_1) = \tilde{p}s_0 + p(\tilde{x}s_0 + xs_1) = (\tilde{p} + p\tilde{x})s_0 + pxs_1.$$

Compare the coefficients of s_1. Then we have $\tilde{x}x + xy = px$.

Assume that $x = 0$, then $s_0 h_1 = s_0$. Hence $e_0^{\mathcal{H}} s_0 = s_0$, where $e_0^{\mathcal{H}}$ is the normalized Haar measure of $\mathcal{H}$. On the other hand $\mathrm{supp}(e_0^{\mathcal{H}} s_0) \ni s_1$. It is a contradiction, so that $x \neq 0$. Therefore, $\tilde{x} + y = p$.

Put $\tau = \tilde{x}$. We note that $y = p - \tau$ and $\tilde{y} = \tilde{p} + \tau$. By the axiom of hypergroups we have $0 \leq \tau \leq p$. □

4.1 Non Hermitian Extensions

Now we consider the case that $\mathcal{K}$ is not a hermitian hypergroup, i.e. $s_0^* = s_1$ and $s_1^* = s_0$. Then we have the following theorem about multiplicative structure of $\mathcal{K}$.

Theorem 4.1. *If $\mathcal{K}$ is not a hermitian hypergroup with order $|\mathcal{K}|$ equals four, then $\mathcal{K} = \mathcal{K}_{nh}^{p;q} = \{h_0, h_1, s_0, s_1\}$ is determined by the following multiplicative structure:*

(a) $h_1 s_0 = \frac{p}{2} s_0 + (1 - \frac{p}{2}) s_1, \quad h_1 s_1 = (1 - \frac{p}{2}) s_0 + \frac{p}{2} s_1,$

(b) $s_0^2 = s_1^2 = \tilde{q} h_1 + \frac{q}{2} s_0 + \frac{q}{2} s_1,$

(c) $s_0 s_1 = \frac{2\tilde{p}\tilde{q}}{1+\tilde{p}} h_0 + \frac{p\tilde{q}}{1+\tilde{p}} h_1 + \frac{q}{2} s_0 + \frac{q}{2} s_1.$

Proof. The results (Ha)(Hb) in Proposition 4.1 also hold in this case.

By the fact that $\varphi(s_0^2) = \varphi(s_1^2) = \varphi(s_0 s_1) = \ell_1^2$, we can set the expression of multiplications as:

$$
\begin{aligned}
s_0^2 &= \tilde{q} h_1 + q\tilde{z} s_0 + q z s_1, && \text{(Eq5)}\\
s_1^2 &= \tilde{q} h_1 + q\tilde{w} s_0 + q w s_1, && \text{(Eq6)}\\
s_1 s_0 &= \tilde{q}\tilde{x} h_0 + \tilde{q} x h_1 + q\tilde{y} s_0 + q y s_1, && \text{(Eq7)}
\end{aligned}
$$

where $0 \le x, y, z, w \le 1$ but $x \neq 1$. Operate the inverse $*$ on the above (1)–(3). Then we have $q\tilde{w} = qz$ and $q\tilde{y} = qy$. Hence, $q\tilde{y} = qy = q/2$. Moreover it is easy to see that $\tau = p/2$ by $(h_1 s_0)^* = h_1 s_1$, $(h_1 s_1)^* = h_1 s_0$. This shows (a). Next consider a triple product $(s_0 s_1) h_1, (s_0 h_1) s_1$ and $s_0(h_1 s_1)$. From the coefficients of h_0 of them, it shows that $\tilde{q} x \tilde{p} = \tau \tilde{q} \tilde{x} = (p - \tau)\tilde{q}\tilde{x}$. Since $\tilde{q}\tilde{x} \neq 0$ and $\tau = p/2$, we have $2x\tilde{p} = p\tilde{x} = p(1-x)$. Hence $x = p/(1+\tilde{p})$ and $\tilde{x} = 2\tilde{p}/(1+\tilde{p})$, and this means (b). Compare the coefficients of h_0 on a product $(s_0 s_1) s_0 = (s_0)^2 s_1$. Then $qy\tilde{q}\tilde{x} = q\tilde{z}\tilde{q}\tilde{x}$. Hence $q\tilde{z} = qy = q/2$. Consequently $qz = qw = q/2$. Therefore we have (c).

As the coefficients of s_0 in products $(s_0 s_1) h_1, (s_0 h_1) s_1$ and $s_0(h_1 s_1)$ equal to $q/2$, we have the associative law: $(s_0 s_1) h_1 = (s_0 h_1) s_1 = s_0(h_1 s_1)$.

Finally it is easy to see the associativity on $h_1 s_0 s_0, h_1 s_1 s_1, s_0 s_0 s_1, s_0 s_1 s_1$.

□

Remark 4.1. In the previous paper [KST] and [IKS], we studied the cases that $\mathcal{H}$ or $\mathcal{L}$ is a group. When $p = 0$, i.e. $\mathcal{H} \cong \mathbb{Z}_2$, $\mathcal{K}_{nh}^{0,q}$ is parameterized as same as Model 1 in [KST]. When $q = 0$, i.e. $\mathcal{L} \cong \mathbb{Z}_2$, $\mathcal{K}_{nh}^{p,0}$ is parameterized as in [IKS]. When $p = q = 0$, we have $\mathcal{K}_{nh}^{0,0} \cong \mathbb{Z}_4$. This is non-splitting exact sequence as in abelian group category.

We remark that all of extensions in Theorem 4.1 are not splitting as hypergroup, i.e. there does not exist cross section homomorphism for the canonical quotient homomorphism from an extension $\mathcal{K}$ into $\mathcal{L}$. If it is splitting case, we can find a subhypergroup of an extension $\mathcal{K}$ from the calculated structure constants.

Now we start to calculate characters of $\mathcal{K}_{nh}^{p,q}$. We denote the dual signed hypergroup $\widehat{\mathcal{K}}_{nh}^{p,q} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$. Then we can have achieved the structure of the dual $\widehat{\mathcal{K}}_{nh}^{p,q}$ from the result of the following Proposition in order to check that the exact sequence of dual hypergroups

$$\mathbf{1} \to \mathcal{L}^\wedge \to \widehat{\mathcal{K}}_{nh}^{p,q} \to \mathcal{H}^\wedge \to \mathbf{1}$$

holds. Since a character in the dual hypergroup $\mathcal{L}^\wedge$ is recognized to be a character of an extension $\widehat{\mathcal{K}}_{nh}^{p,q}$, we will choose a character χ_1 such that it is satisfied the duality of being $\mathcal{L}^\wedge = \{\chi_0, \chi_1\} \cong \mathscr{L}(q)$. Next we will establish the quotient hypergroup $\widehat{\mathcal{K}}_{nh}^{p,q}/\mathcal{L}^\wedge$ is isomorphic to $\mathcal{H}$.

Proposition 4.2. *A hypergroup $\mathcal{K}_{nh}^{p,q}$ has characters $\{\chi_0, \chi_1, \chi_2, \chi_3\}$ which is determined by the following table with values of weights:*

	h_0	h_1	s_0	s_1	$w(\chi_i)$
χ_0	1	1	1	1	1
χ_1	1	1	$-\tilde{q}$	$-\tilde{q}$	$1/\tilde{q}$
χ_2	1	$-\tilde{p}$	$\sqrt{-\tilde{p}\tilde{q}}$	$-\sqrt{-\tilde{p}\tilde{q}}$	$\dfrac{\tilde{q}+1}{2\tilde{p}\tilde{q}}$
χ_3	1	$-\tilde{p}$	$-\sqrt{-\tilde{p}\tilde{q}}$	$\sqrt{-\tilde{p}\tilde{q}}$	$\dfrac{\tilde{q}+1}{2\tilde{p}\tilde{q}}$
$w(h_i), w(s_i)$	1	$1/\tilde{p}$	$\dfrac{\tilde{p}+1}{2\tilde{p}\tilde{q}}$	$\dfrac{\tilde{p}+1}{2\tilde{p}\tilde{q}}$	$\dfrac{(\tilde{p}+1)(\tilde{q}+1)}{\tilde{p}\tilde{q}}$

Proof. From $\mathcal{L}^\wedge \subset \mathcal{K}^\wedge$ we have that $\chi_1(h_j) = 1,\ \chi_1(s_j) = -\tilde{q}$ for $j = 0, 1$. Assume a character $\chi \neq \chi_0, \chi_1$. Since $\chi \notin \mathcal{L}^\wedge$, it follows that $\chi(h_1) \neq 1$. This means $\chi(h_1) = -\tilde{p}$. We have $\chi(s_0) = -\chi(s_1)$ from (a) in Theorem 4.1. From the conditions (b) and (c), we have the same equation $\chi(s_0)^2 = -\tilde{p}\tilde{q}$. Therefore we write $\chi_2(s_0) = \sqrt{-\tilde{p}\tilde{q}}$ and $\chi_3(s_0) = -\sqrt{-\tilde{p}\tilde{q}}$.

Applying the method of [W2] which calculates the structure coefficients m_{ij}^k of the dual signed hypergroup, we have $m_{22}^0 = w(\chi_2)^{-1}$ and

$$\begin{aligned} m_{22}^0 &= \Big(1 + \chi_2(h_1)^2 w(h_1) + |\chi_2(s_0)|^2 w(s_0) + |\chi_2(s_1)|^2 w(s_1)\Big) w(\mathcal{K}_{nh}^{p,q})^{-1} \\ &= \Big(1 + \tilde{p} + \frac{1+\tilde{p}}{2} + \frac{1+\tilde{p}}{2}\Big) \cdot \frac{\tilde{p}\tilde{q}}{(\tilde{p}+1)(\tilde{q}+1)} \\ &= \frac{2\tilde{p}\tilde{q}}{\tilde{q}+1}. \end{aligned}$$

Hence $w(\chi_2) = (\tilde{q}+1)/(2\tilde{p}\tilde{q})$. It is shown that $w(\chi_3) = w(\chi_3^*) = w(\chi_2)$.

The value on the right and bottom corner in the table is total weight $w(\mathcal{K}_{nh}^{p,q}) = w(\mathcal{H})w(\mathcal{L})$. □

We determine the dual signed hypergroup of $\mathcal{K}_{nh}^{p,q}$.

Theorem 4.2. *The dual* $\widehat{\mathcal{K}}_{nh}^{p,q} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$ *is determined by the structure equations:*

(A) $\chi_1^2 = \tilde{q}\chi_0 + q\chi_1, \quad \chi_1\chi_2 = \frac{q}{2}\chi_2 + (1-\frac{q}{2})\chi_3, \quad \chi_1\chi_3 = (1-\frac{q}{2})\chi_2 + \frac{q}{2}\chi_3,$

(B) $\chi_2^2 = \chi_3^2 = \tilde{p}\chi_1 + \frac{p}{2}\chi_2 + \frac{p}{2}\chi_3,$

(C) $\chi_2\chi_3 = \tilde{p}\cdot\frac{2\tilde{q}}{\tilde{q}+1}\chi_0 + \tilde{p}\cdot\frac{q}{\tilde{q}+1}\chi_1 + \frac{p}{2}\chi_2 + \frac{p}{2}\chi_3.$

Furthermore the hypergroup $\mathcal{K}_{nh}^{p,q}$ *is strong.*

Proof. It is easily obtained from the symmetry of the table with respect to p, q in Proposition 4.2. It is shown that all coefficients of the above are non negative. Therefore the dual $\widehat{\mathcal{K}}_{nh}^{p,q}$ is a hypergroup. □

Remark 4.2. It is easy to see that $\widehat{\mathcal{K}}_{nh}^{p,q} \cong \mathcal{K}_{nh}^{q,p}$ from the structure in Theorem 4.1 and Theorem 4.2. In the special case of $p = q$, a hypergroup $\mathcal{K}_{nh}^{p,p}$ is self dual.

4.2 Hermitian type hypergroups with order four

In this section we consider the extension of hermitian type. We have the following theorem, which is more complicated than non hermitian case in the previous section. There is two parameters with real dimension which generates all hermitian extensions.

Theorem 4.3. *Let* $\mathcal{K}$ *be an extension of* $\mathcal{L} \cong \mathscr{L}(\tilde{q})$ *by* $\mathcal{H} \cong \mathscr{L}(\tilde{p})$ *whose order is four. If* $\mathcal{K}$ *is hermitian type, then* $\mathcal{K} = \mathcal{K}_h^{p,q}(\tau,\sigma) = \{h_0, h_1, s_0, s_1\}$ *is determined by the following structure and condition:*

(a) $h_1 s_0 = \tau s_0 + \tilde{\tau} s_1, \quad h_1 s_1 = (\tilde{p}+\tau)s_0 + (p-\tau)s_1,$

(b) $s_0 s_1 = \tilde{q}h_1 + \sigma s_0 + (q-\sigma)s_1,$

(c) $s_0^2 = \tilde{q}\cdot\frac{\tilde{p}}{\tilde{p}+\tau}h_0 + \tilde{q}\cdot\frac{\tau}{\tilde{p}+\tau}h_1 + \left(q - \sigma\cdot\frac{\tilde{\tau}}{\tilde{p}+\tau}\right)s_0 + \sigma\cdot\frac{\tilde{\tau}}{\tilde{p}+\tau}s_1,$

(d) $s_1^2 = \tilde{q}\cdot\frac{\tilde{p}}{\tilde{\tau}}h_0 + \tilde{q}\cdot\frac{p-\tau}{\tilde{\tau}}h_1 + (q-\sigma)\cdot\frac{\tilde{p}+\tau}{\tilde{\tau}}s_0 + \left(q - (q-\sigma)\cdot\frac{\tilde{p}+\tau}{\tilde{\tau}}\right)s_1,$

(e) $q\cdot\frac{\tilde{p}+\tau}{\tilde{\tau}} \geq \sigma \geq q\cdot(1-\frac{\tilde{\tau}}{\tilde{p}+\tau}),$

where τ, σ *are parameters such that* $0 \leq \tau \leq p$ *and* $0 \leq \sigma \leq q$.

Proof. The statement (a) is showed in Proposition 4.1 with a parameter $0 \le \tau \le p$. Since $\varphi(s_0^2) = \varphi(s_1^2) = \varphi(s_0 s_1) = \boldsymbol{\ell}_1^2$, we can write

$$s_1 s_0 = \tilde{q} h_1 + \sigma s_0 + (q - \sigma) s_1, \tag{Eq8}$$
$$s_0^2 = \tilde{q}\tilde{x} h_0 + \tilde{q} x h_1 + q\tilde{y} s_0 + q y s_1, \tag{Eq9}$$
$$s_1^2 = \tilde{q}\tilde{z} h_0 + \tilde{q} z h_1 + q\tilde{w} s_0 + q w s_1, \tag{Eq10}$$

where $0 \le \sigma \le q$ and $0 \le x, y, z, w \le 1$ but x, z are not 1. Then (b) is the first equation (4).

Now consider triple products $(s_0^2)h_1$ and $(s_0 h_1)s_0$. From the coefficients of h_0, it shows that $\tilde{q} x \tilde{p} = \tau \tilde{q}\tilde{x}$. Since $\tilde{q} \ne 0$, we have $x\tilde{p} = \tau\tilde{x}$. Hence $x = \tau/(\tilde{p} + \tau)$ and $\tilde{x} = \tilde{p}/(\tilde{p} + \tau)$. Next compare the coefficients of h_0 on products $(s_1)^2 h_1$ and $(s_1 h_1)s_1$. Then $\tilde{q} z \tilde{p} = (p - \tau)\tilde{q}\tilde{z}$. Hence $z = (p - \tau)/\tilde{\tau}$ and $\tilde{z} = \tilde{p}/\tilde{\tau}$.

Moreover, compare the coefficients of s_0 on the above triple product $(s_0^2)h_1$ and $(s_0 h_1)s_0$. Then we have $\tau q \tilde{y} + \tilde{\tau}\sigma = q\tilde{y}\tau + qy(\tilde{p} + \tau)$. Hence $\tilde{\tau}\sigma = qy(\tilde{p} + \tau)$, so that $qy = \sigma\tilde{\tau}/(\tilde{p} + \tau)$. Compare the coefficients of s_1 on the above triple product $(s_1^2)h_1$, then $q\tilde{w} = (q - \sigma)(\tilde{p} + \tau)/\tilde{\tau}$. Therefore we prove the statements (c) and (d), and get associativity for $(s_0^2)h_1 = (s_0 h_1)s_0$ and similarly $(s_1)^2 h_1 = (s_1 h_1)s_1$.

The condition (e) is immediate from the fact that the coefficients in products on (c) and (d) are non-negative.

It is easy to see that the associativity on $s_0 s_0 s_1, s_0 s_1 s_1, h_1 s_0 s_1$ holds. □

Remark 4.3. Let $\mathfrak{D}_h$ in Figure 2 be a region of two parameter (τ, σ) in which the condition (e) satisfies. A region $\mathfrak{D}_h$ includes the central point $(p/2, q/2)$ of symmetry. The curve (e_b^1) in Figure 2, is the equation of the first inequality of (e) and a curve (e_b^2) is also the one with respect to the second inequality of (e). When a point (τ, σ) is out of $\mathfrak{D}_h$, i.e. in two parts in left top and right bottom corners, $\mathcal{K}_h^{p,q}(\tau, \sigma)$ is a signed hypergroup.

When $p = 0$, i.e. $\mathcal{H} \cong \mathbb{Z}_2$, we have that $\mathcal{K}_h^{0,q}(0, \sigma)$ is parameterized as same as in Model 1 in [KST]. In this case, the condition (e) of the above becomes $0 \le \sigma \le q$.

When $q = 0$, i.e. $\mathcal{L} \cong \mathbb{Z}_2$, we have that $\mathcal{K}_h^{p,0}(\tau, 0)$ is parameterized in [IKS]. The condition (e) is always satisfied because all terms of (e) is 0.

Moreover when $p = q = 0$, we have a hypergroup $\mathcal{K}_h^{0,0}(0, 0) = \mathbb{Z}_2 \times \mathbb{Z}_2$, indeed it is a group.

Corollary 4.3. *$\mathcal{K}_h^{p,q}(\tau, \sigma) \cong \mathcal{K}_h^{p,q}(\tau', \sigma')$ if and only if the following* (1) *or* (2) *is satisfied*:

(1) $\tau' = p - \tau$ *and* $\sigma' = q - \sigma$,
(2) $\tau' = \tau$ *and* $\sigma' = \sigma$.

Proof. The isomorphism of $\mathcal{K}_h^{p,q}(\tau, \sigma) \to \mathcal{K}_h^{p,q}(q - \tau, p - \sigma)$ is given by the flip Φ, i.e. $\Phi(h_i) = h_i$ $(i = 0, 1)$ identical map on $\mathcal{H}$ and $\Phi(s_0) = s_1$, $\Phi(s_1) = s_0$ from (a),(b) in Theorem 4.3.

□

In the figure 2, two points (τ, σ) and (τ', σ') of the condition in Corollary 4.3 are reflection of each other with respect to the central point $(p/2, q/2)$.

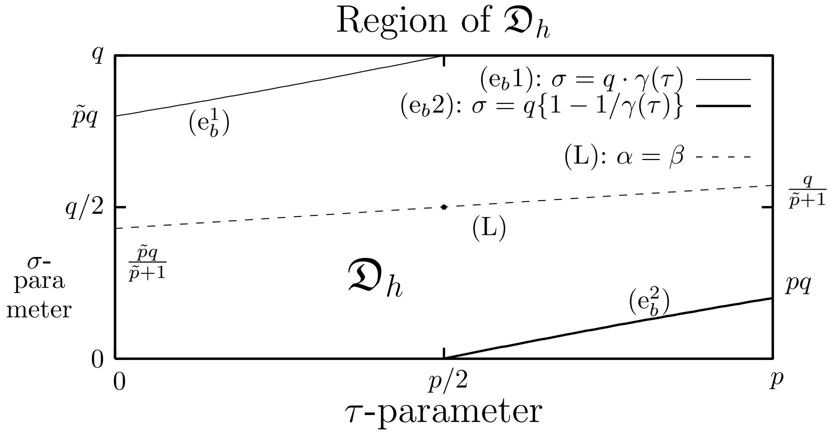

Figure 2: Hypergroup condition (e) of σ, τ

Now we will calculate the characters of hermitian hypergroups $\mathcal{K}_h^{p,q}(\tau, \sigma)$ in the same way of non-hermitian case in the previous section.

Proposition 4.4. *A hypergroup* $\mathcal{K}_h^{p,q}(\tau, \sigma)$ *has characters* $\{\chi_0, \chi_1, \chi_2, \chi_3\}$ *which is determined by the following table with values of weights*:

	h_0	h_1	s_0	s_1	$w(\chi_i)$
χ_0	1	1	1	1	1
χ_1	1	1	$-\tilde{q}$	$-\tilde{q}$	$1/\tilde{q}$
χ_2	1	$-\tilde{p}$	α	$-\gamma\alpha$	$\dfrac{(\tilde{q}+1)\beta}{\tilde{p}\tilde{q}(\alpha+\beta)}$
χ_3	1	$-\tilde{p}$	$-\beta$	$\gamma\beta$	$\dfrac{(\tilde{q}+1)\alpha}{\tilde{p}\tilde{q}(\alpha+\beta)}$
$w(h_i), w(s_i)$	1	$1/\tilde{p}$	$\dfrac{\tilde{p}+\tau}{\tilde{p}\tilde{q}}$	$\dfrac{\tilde{\tau}}{\tilde{p}\tilde{q}}$	$\dfrac{(\tilde{p}+1)(\tilde{q}+1)}{\tilde{p}\tilde{q}}$

where two real numbers α, β *have the relation*:

$$(*) \qquad \alpha - \beta = q - \sigma(1 + \gamma^{-1}), \;\; \alpha\beta = \tilde{p}\tilde{q}\gamma^{-1}.$$

Proof. Using the same argument in Proposition 4.2, we suppose that $\mathcal{L}^\wedge = \{\chi_0, \chi_1\}$. Let $\chi \in \widehat{\mathcal{K}}_h^{p,q}(\tau, \sigma)$ with $\chi \notin \mathcal{L}^\wedge$. From (a) in Theorem 4.3 under the case of $\chi(h_1) = -\tilde{p}$, it implies that

$$(**) \qquad \chi(s_1) = -\gamma \cdot \chi(s_0),$$

where the ratio of weights $\gamma := (\tilde{p} + \tau)/\tilde{\tau} = w(s_0)/w(s_1)$. Then the equations (b), (c) and (d) become single equation

$$-\gamma \cdot \chi(s_0)^2 = -\tilde{p}\tilde{q} + \{\sigma - (q - \sigma)\gamma\}\chi(s_0).$$

Therefore

$$(***)\qquad \gamma\cdot\chi(s_0)^2+\{\sigma-(q-\sigma)\gamma\}\chi(s_0)-\tilde{p}\tilde{q}=0.$$

Two real numbers $\alpha, -\beta$ with $-\beta < 0 < \alpha$ is a pair of solutions of $(***)$ with respect to $\chi(s_0)$, i.e. the relation $(*)$ in our conlusion holds.

We determine the last two character χ_2, χ_3 by $\chi_2(s_0) = \alpha$, $\chi_2(s_1) = -\gamma\alpha$ and $\chi_3(s_0) = -\beta$, $\chi_3(s_1) = \gamma\beta$.

Using the method of [W2], the weights of these characters are calculated as

$$\begin{aligned}
m^0_{22} = w(\chi_2)^{-1} &= \Big(1+\chi_2(h_1)^2 w(h_1)+\chi_2(s_0)^2 w(s_0)+\chi_2(s_1)^2 w(s_1)\Big)\Big/ w(\mathcal{K}^{p,q}_{nh})\\
&= \Big(1+\tilde{p}+\frac{\alpha^2(\tilde{p}+\tau)}{\tilde{p}\tilde{q}}+\frac{\gamma^2\alpha^2\tilde{\tau}}{\tilde{p}\tilde{q}}\Big)\cdot\frac{\tilde{p}\tilde{q}}{(\tilde{p}+1)(\tilde{q}+1)}\\
&= \Big(1+\tilde{p}+\frac{\alpha^2(\tilde{p}+\tau)(\tilde{\tau}+\tilde{p}+\tau)}{\tilde{\tau}\tilde{p}\tilde{q}}\Big)\cdot\frac{\tilde{p}\tilde{q}}{(\tilde{p}+1)(\tilde{q}+1)}\\
&= (1+\frac{\alpha^2\gamma}{\tilde{p}\tilde{q}})\cdot\frac{\tilde{p}\tilde{q}}{(\tilde{q}+1)}\\
&= \frac{\alpha+\beta}{\beta}\cdot\frac{\tilde{p}\tilde{q}}{\tilde{q}+1},
\end{aligned}$$

and this leads the value of a weight $w(\chi_2)$. In a similar way we get the value of $w(\chi_3)$.

□

The curve of $\alpha = \beta$, on which the ratio of dual weights $\beta/\alpha = w(\chi_2)/w(\chi_3) = 1$, is a dotted line (L) including the center point $(p/2, q/2)$ in Figure 2. This is obtained from the equation $\sigma = \frac{q}{\tilde{p}+1}\cdot(\tilde{p}+\tau)$ which comes from $q = \sigma(1+1/\gamma)$.

Now we calculate the structure equations of dual hypergroups $\widehat{\mathcal{K}}^{p,q}_h(\tau,\sigma)$ as well as non hermitian case.

Theorem 4.4. *The dual* $\widehat{\mathcal{K}}^{p,q}_h(\tau,\sigma) = \{\chi_0,\chi_1,\chi_2,\chi_3\}$ *has the structure equations as a signed hypergroup*:

$$\text{(A)}\ \ \chi_1^2=\tilde{q}\chi_0+q\chi_1,\ \ \chi_1\chi_2=\frac{\beta-\tilde{q}\alpha}{\alpha+\beta}\chi_2+\frac{\alpha+\tilde{q}\alpha}{\alpha+\beta}\chi_3,\ \ \chi_1\chi_3=\frac{\beta+\tilde{q}\beta}{\alpha+\beta}\chi_2+\frac{\alpha-\tilde{q}\beta}{\alpha+\beta}\chi_3,$$

$$\text{(B1)}\ \ \chi_2^2=\frac{\tilde{p}\tilde{q}(\alpha+\beta)}{(1+\tilde{q})\beta}\chi_0+\frac{\tilde{p}(\beta-\tilde{q}\alpha)}{(1+\tilde{q})\beta}\chi_1+\frac{p\beta+\alpha^2(1-\gamma)}{\alpha+\beta}\chi_2+\frac{p\alpha-\alpha^2(1-\gamma)}{\alpha+\beta}\chi_3,$$

$$\text{(B2)}\ \ \chi_3^2=\frac{\tilde{p}\tilde{q}(\alpha+\beta)}{(1+\tilde{q})\alpha}\chi_0+\frac{\tilde{p}(\alpha-\tilde{q}\beta)}{(1+\tilde{q})\alpha}\chi_1+\frac{p\beta+\beta^2(1-\gamma)}{\alpha+\beta}\chi_2+\frac{p\alpha-\beta^2(1-\gamma)}{\alpha+\beta}\chi_3,$$

$$\text{(C)}\ \ \chi_2\chi_3=\tilde{p}\chi_1+\frac{p\beta-\alpha\beta(1-\gamma)}{\alpha+\beta}\chi_2+\frac{p\alpha+\alpha\beta(1-\gamma)}{\alpha+\beta}\chi_3.$$

Proof. It is obvious that $\{\chi_0,\chi_1\} = \mathcal{L}^\wedge \cong \mathcal{L}(q)$ from Proposition 4.2. This shows the first equation of (A). With the relation of the structure and its characters[W2], for example, the coefficient m_{12}^3 of χ_3 in a product of $\chi_1\chi_2$ is

$$\begin{aligned}
m_{12}^3 &= \Big(1+\tilde{p}^2/\tilde{p}+\tilde{q}\alpha\beta\cdot\frac{\tilde{p}+\tau}{\tilde{p}\tilde{q}}+\tilde{q}\gamma^2\alpha\beta\cdot\frac{\tilde{\tau}}{\tilde{p}\tilde{q}}\Big)\cdot\frac{w(\chi_3)}{w(\mathcal{K})}\\
&= \Big(\tilde{p}+1+\frac{\alpha\beta}{\tilde{p}}(\tilde{p}+\tau+\gamma^2\tilde{\tau})\Big)\cdot\frac{w(\chi_3)}{w(\mathcal{K})}\\
&= \Big(\tilde{p}+1+\frac{\alpha\beta}{\tilde{p}}\gamma(\tilde{p}+1)\Big)\cdot\frac{\alpha}{(\tilde{p}+1)(\alpha+\beta)}\\
&= \frac{(\tilde{q}+1)\alpha}{\alpha+\beta}.
\end{aligned}$$

It is shown that $m_{13}^2=(\tilde{q}+1)\beta/(\alpha+\beta)$ in a similar way. Therefore, we have equations (A).

The coefficient m_{22}^2 of χ_2 in a product of χ_2^2 is

$$\begin{aligned}
m_{22}^2 &= \Big(1-\tilde{p}^3/\tilde{p}+\alpha^3\cdot\frac{\tilde{p}+\tau}{\tilde{p}\tilde{q}}-\gamma^3\alpha^3\cdot\frac{\tilde{\tau}}{\tilde{p}\tilde{q}}\Big)\cdot\frac{w(\chi_2)}{w(\mathcal{K})}\\
&= \Big(1-\tilde{p}^2+\frac{\alpha^3}{\tilde{p}\tilde{q}}(\tilde{p}+\tau-\gamma^3\tilde{\tau})\Big)\cdot\frac{w(\chi_2)}{w(\mathcal{K})}\\
&= \Big(1-\tilde{p}^2+\frac{\alpha^3}{\tilde{p}\tilde{q}}(\tilde{p}+\tau)(1-\gamma^2)\Big)\cdot\frac{\beta}{(\tilde{p}+1)(\alpha+\beta)}\\
&= \Big(1-\tilde{p}^2+\frac{\alpha^3\gamma}{\tilde{p}\tilde{q}}(\tilde{p}+1)(1-\gamma)\Big)\cdot\frac{\beta}{(\tilde{p}+1)(\alpha+\beta)}\\
&= \Big(p+\frac{\alpha^3\gamma}{\tilde{p}\tilde{q}}(1-\gamma)\Big)\cdot\frac{\beta}{(\alpha+\beta)}\\
&= \frac{p\beta+\alpha^2(1-\gamma)}{\alpha+\beta}.
\end{aligned}$$

The equation (B1) is implied from $m_{22}^0=w(\chi_2)^{-1}$ and $m_{12}^2/w(\chi_2)=m_{22}^1/w(\chi_1)$. We also have (B2). The equation (C) is obtained from $m_{23}^2/w(\chi_2)=m_{22}^3/w(\chi_3)$. □

Theorem 4.5. *The dual* $\widehat{\mathcal{K}}_h^{p,q}(\tau,\sigma)$ *is a hypergroup if the following conditions* (E) *and* (F) *are satisfied:*

- (E) $\beta-\tilde{q}\alpha\geq 0,\ \ \alpha-\tilde{q}\beta\geq 0,$
- (F) $p\beta+\alpha^2(1-\gamma)\geq 0,\ \ p\alpha-\beta^2(1-\gamma)\geq 0.$

Proof. We will check that all coefficients are non negative. First we will show that the inequalities

$$\text{(D):}\ \ p-\alpha(1-\gamma)\geq 0,\ \ p+\beta(1-\gamma)\geq 0$$

hold in the region $\mathfrak{D}_h$ of Figure 2. Notice that $d\gamma/d\tau=1/\tilde{\tau}^2>0$. We can consider that the positive numbers α,β determined by the relation $(*)$ in Proposition 4.4 are functions of two variables γ,σ, where $\tilde{p}\leq\gamma\leq 1/\tilde{p}$.

Since $\partial\alpha/\partial\sigma - \partial\beta/\partial\sigma = -(1+1/\gamma) > 0$ and $\beta \cdot \partial\alpha/\partial\sigma + \alpha \cdot \partial\beta/\partial\sigma = 0$, we have $\partial\alpha/\partial\sigma < 0$ and $\partial\beta/\partial\sigma > 0$. Moreover,

$$\partial\alpha/\partial\gamma(\gamma,0) = \partial/\partial\gamma(1/2 \cdot \sqrt{q^2+4\tilde{p}\tilde{q}\gamma^{-1}} + q/2) = -\tilde{p}\tilde{q}/(\gamma^2\sqrt{q^2+4\tilde{p}\tilde{q}\gamma^{-1}}) < 0.$$

At first we will prove that $p - \alpha(1-\gamma) \geq 0$. When $\gamma \geq 1$, then the first inequality in (D) holds. If $\gamma < 1$, then

$$p - \alpha(\gamma,\sigma)(1-\gamma) \geq p - \alpha(\gamma,0)(1-\gamma) \geq p - \alpha(\tilde{p},0)(1-\tilde{p}) = p - 1 \cdot p = 0.$$

Hence $p - \alpha(1-\gamma) \geq 0$.

The next it is obtained that $p + \beta(1-\gamma) \geq 0$ from the relations of

$$\partial\beta/\partial\gamma(\gamma,q) < 0$$

and

$$\beta(1/\tilde{p},q) = 1/2 \cdot \{\sqrt{(q-q(1+\tilde{p}))^2+4\tilde{p}^2\tilde{q}} - q + q(1+\tilde{p})\} = \tilde{p}.$$

Therefore inequalities (D) are always satisfied when $\mathcal{K}_h^{p,q}(\tau,\sigma)$ is a hypergroup.

Hence it is shown that the 4th coefficient of (B1) and the 3rd one of (B2) are non-negativity.

The conditions (E) and (F) assure that non-negativity of coefficients in (A), (B1) and (B2) respectively.

□

Remark 4.4. Let $\mathfrak{D}'_h$ be a region in which the conditions (E) and (F) in Theorem 4.5 are satisfied. We give an atlas of $\mathfrak{D}'_h$ and the region $\mathfrak{D}_h$ of Theorem 4.3. In order to view $\mathfrak{D}'_h$, the special values of $\alpha(\gamma,\sigma)$ and $\beta(\gamma,\sigma)$ as functions with respect to γ,σ are the followings:

$\alpha(\tilde{p},0) = 1$	$\alpha(1,q/2) = \sqrt{\tilde{p}\tilde{q}}$	$\alpha(1/\tilde{p},q) = \tilde{p}\tilde{q}$	$\alpha(\gamma,0) > \beta(\gamma,0)$
$\beta(\tilde{p},0) = \tilde{q}$	$\beta(1,q/2) = \sqrt{\tilde{p}\tilde{q}}$	$\beta(1/\tilde{p},q) = \tilde{p}$	$\alpha(\gamma,q) < \beta(\gamma,q)$

$\alpha\Big(\tilde{p}, \dfrac{2\tilde{p}q}{\tilde{p}+1}\Big) = \tilde{q}$	$\alpha\Big(\dfrac{1}{\tilde{p}}, \dfrac{pq}{\tilde{p}+1}\Big) = \tilde{p}$
$\beta\Big(\tilde{p}, \dfrac{2\tilde{p}q}{\tilde{p}+1}\Big) = 1$	$\beta\Big(\dfrac{1}{\tilde{p}}, \dfrac{pq}{\tilde{p}+1}\Big) = \tilde{p}\tilde{q}$

.

When $p\beta+\alpha^2(1-\gamma) = 0$ and $p+\beta(1-\gamma) = 0$, it is clear that $\alpha = \beta = p/(\gamma-1)$. The intersection of (F) and (D) are on the line (L) in Figure 2. It is obvious that

$$\frac{q}{\tilde{p}+1} > pq > \frac{pq}{\tilde{p}+1} \quad \text{and} \quad \frac{2\tilde{p}q}{\tilde{p}+1} > \tilde{p}q > \frac{\tilde{p}q}{\tilde{p}+1}.$$

There are many varieties of intersections of conditions:(e),(E) and (F), so that we give a typical figure.

Assume $p,q < 3/5$. The thick curves $(\mathrm{D}_b), (\mathrm{E}_b)$ and (F_b) are boundaries of conditions (D)–(F) in the Figure 3 and the boundaries (e_b^1),(e_b^2) of Figure 2 are drown by the thin curves.

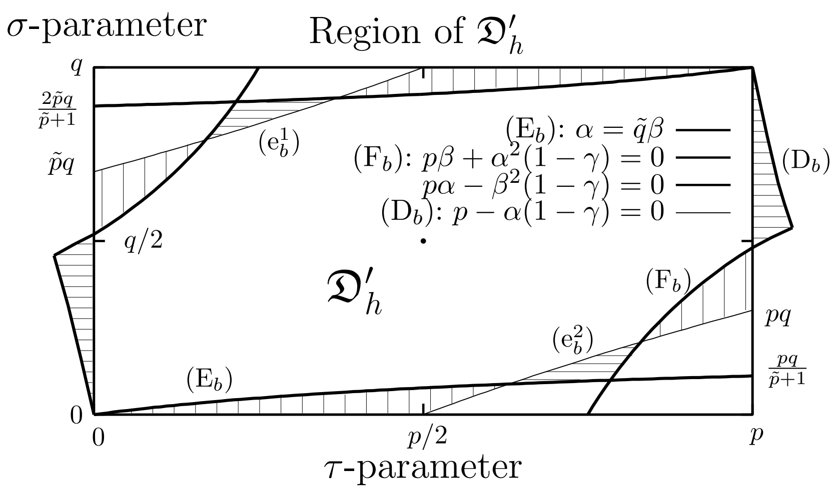

Figure 3: Strong hypergroup condition of σ, τ in the case $p, q < 3/5$

Let $\mathfrak{D}'_h$ be the region with the thick curves (D_b),(E_b),(F_b) and etc. including a central point $(p/2, q/2)$ in Figure 3. Using the result in Theorem 4.5, the dual $\widehat{\mathcal{K}}_h^{p,q}(\tau, \sigma)$ is a hypergroup in $\mathfrak{D}'_h$. Since $\mathcal{K}_h^{p,q}(\tau, \sigma)$ is a hypergroup in $\mathfrak{D}_h$, we have that hypergroups $\mathcal{K}_h^{p,q}(\tau, \sigma)$ are strong when (τ, σ) is in the intersection $\mathfrak{D}_h \cap \mathfrak{D}'_h$. In the area of $\mathfrak{D}'_h \setminus \mathfrak{D}_h$ with the horizontal strip, $\widehat{\mathcal{K}}_h^{p,q}(\tau, \sigma)$ is a hypergroup but $\mathcal{K}_h^{p,q}(\tau, \sigma)$ is not.

In the horizontal striped regions out of the rectangle: $0 \leq \tau \leq p$ and $0 \leq \sigma \leq q$, which is the extremely right area with the boundary (D_b) and the extremely left one, the dual $\mathfrak{D}'_h$ is a hypergroup but $\mathfrak{D}_h$ is not a hypergroup.

But if p or q is nearly to 1, then there are many varieties of the regions in which the conditions (E),(F) and (e) are satisfied.

We apply our theorems to determine the structure of strong hypergroup of order four which have non-trivial subhypergroups. Given a exact sequence of hypergroups: $\mathbf{1} \to \mathcal{H} \to \mathcal{K} \to \mathcal{L} \to \mathbf{1}$, where $|\mathcal{K}| = 4$. By the order condition: $|\mathcal{H}| + |\mathcal{L}| - 1 \leq |\mathcal{K}| \leq |\mathcal{H}| \cdot |\mathcal{L}|$, the possible orders of $\mathcal{H}$ and $\mathcal{L}$ are the following cases:

$$|\mathcal{H}| = |\mathcal{L}| = 2, \tag{Eq11}$$

$$|\mathcal{H}| = 3,\ |\mathcal{L}| = 2, \tag{Eq12}$$

$$|\mathcal{H}| = 2,\ |\mathcal{L}| = 3. \tag{Eq13}$$

In the case of (Eq11), $\mathcal{K}$ is already determined by Theorem 4.1 and Theorem 4.3. Using the results of Theorem 4.2 and Theorem 4.5, we can estimate the strong hypergroups.

In the case of (Eq12) or (Eq13), we have $|\mathcal{H}| + |\mathcal{L}| - 1 = |\mathcal{K}| = 4$. When $|\mathcal{H}|$ and $|\mathcal{L}|$ are fixed, $|\mathcal{K}|$ has a minimal order. Hence it is shown that $\mathcal{K}$ is a *join* hypergroup $\mathcal{H} \vee \mathcal{L}$ which is announced in Section 2.1 Preliminary.

Obvious proposition about the general property of being strong is stated as follows.

Proposition 4.5. *Let $\mathcal{K}$, $\mathcal{L}$ be two finite commutative hypergroups. Then the following statements hold*:

(1) *$\mathcal{K} \vee \mathcal{L}$ is strong if and only if the both $\mathcal{K}$ and $\mathcal{L}$ are strong,*

(2) *$\mathcal{K} \times \mathcal{L}$ is strong if and only if the both $\mathcal{K}$ and $\mathcal{L}$ are strong.*

Proof. It is clearly obtained from the fact that

$$(\mathcal{K} \vee \mathcal{L})^{\wedge} = \mathcal{L}^{\wedge} \vee \mathcal{K}^{\wedge}$$

because the character matrix of a join hypergroup is in the form in Proposition 2.3 and

$$(\mathcal{K} \times \mathcal{L})^{\wedge} = \mathcal{K}^{\wedge} \times \mathcal{L}^{\wedge}$$

because the character of the direct product are multiplicative characters, see 1. Direct product hypergroup in Preliminaries. □

N.J. Wildberger [W2] completely analyzed all hypergroups $\mathcal{W}_3$ of order three and described whether they are strong or not, which includes Jewett's example [J]. In the case of (Eq12), it is obvious that $\mathcal{K} = \mathcal{W}_3 \vee \mathcal{L}(\rho)$, and $\mathcal{K}^{\wedge} = \mathcal{L}^{\wedge}(\rho) \vee {\mathcal{W}_3}^{\wedge} \cong \mathcal{L}(\rho) \vee \mathcal{W}_3^{\wedge}$. In the case of (Eq13), it is obvious that $\mathcal{K} = \mathcal{L}(\rho) \vee \mathcal{W}_3$ and $\mathcal{K}^{\wedge} \cong {\mathcal{W}_3}^{\wedge} \vee \mathcal{L}(\rho)$.

We remark that the joins $\mathcal{W}_3 \vee \mathcal{L}(\rho)$ and $\mathcal{L}(\rho) \vee \mathcal{W}_3$, for a strong hypergroup $\mathcal{W}_3$, are also strong.

5 Higher Order Extensions

Problem. Fix finite commutative hypergroups $\mathscr{L}(p)$ and $\mathscr{L}(q)$ for two numbers $0 < p \leq 1$ and $0 < q \leq 1$, require all finite commutative hypergroup $\mathcal{K}$ such that

$$\mathbf{1} \to \mathscr{L}(p) \to \mathcal{K} \to \mathscr{L}(q) \to \mathbf{1}$$

is exact. Moreover analyze the total of $\mathcal{K}$'s and parametrize some useful families of them.

The previous section present us the existence of extensions of order four and three about the above problem. Now the next question is naturally appeared.

Does there exist an extension $\mathcal{K}$ in the above problem with higher order than four that equals $|\mathcal{H}| \cdot |\mathcal{L}|$?

From Proposition 3.3, the possibility for orders of extensions is announced as the maximal order is limited by a number $|\mathcal{H}| + \lfloor w(\mathcal{H}) \rfloor (|\mathcal{L}| - 1)$. Now we give the series of models of hermitian signed hypergroups for the answer to this question.

Proposition 5.1. *Let n be an integer and $0 < q \leq 1$. The following character table $\chi_i(c_j)$ with values of weights $w(\chi_i), w(c_j)$ defines a signed hypergroup $\mathcal{K}(n,q) = \{c_0, \ldots, c_{n+1}\}$ of order $n+2$ and the total weight $w(\mathcal{K}) = (1+q)^2 q^{-2}$.*

	c_0	c_1	c_2	c_3	$\cdots$	c_{n+1}	$w(\chi_i)$
χ_0	1	1	1	1	$\cdots$	1	1
χ_1	1	1	$-q$	$-q$	$\cdots$	$-q$	$1/q$
χ_2	1	$-q$	$(n-1)q$	$-q$	$\cdots$	$-q$	$\frac{1+q}{nq^2}$
χ_3	1	$-q$	$-q$	$(n-1)q$	$\cdots$	$-q$	$\frac{1+q}{nq^2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
χ_{n+1}	1	$-q$	$-q$	$-q$	$\cdots$	$(n-1)q$	$\frac{1+q}{nq^2}$
$w(c_i)$	1	$1/q$	$\frac{1+q}{nq^2}$	$\frac{1+q}{nq^2}$	$\cdots$	$\frac{1+q}{nq^2}$	$\frac{(1+q)^2}{q^2}$

Moreover, $\mathcal{K}(n,q)^\wedge = \{\chi_0, \ldots, \chi_{n+1}\} \cong \mathcal{K}(n,q)$.

Proof. Since the values of characters and weights of c_i are real, it is clear that the matrix $\left(\chi_i(c_j) w(\chi_i)^{1/2} w(c_j)^{1/2} w(\mathcal{K})^{-1/2}\right)_{ij}$ is a real constant type. In order to show that $\mathcal{K}(n,q)$ is a signed hypergroup from Lemma 2.2, it is sufficient to show that the above matrix is unitary.

It is clear that the values of weights of c_i $(i \geq 2)$ equal $(1+q)n^{-1}q^{-2}$, which is calculated by using Wildberger's harmonic analysis [W2]. In fact

$$\sum_{k=0}^{n+1} w(c_k) = 1 + \frac{1}{q} + \frac{1+q}{nq^2} \cdot n = \frac{(1+q)^2}{q^2} = w(\mathcal{K})$$

shows that the total weight is correct. The results

$$\sum_{k=0}^{n+1} |\chi_k(c_1)|^2 w(\chi_k) = 1 + \frac{1}{q} + q^2 \frac{1+q}{nq^2} \cdot n$$

$$= (1+q)^2 q^{-1} = (1+q)^2 q^{-2} \cdot q = \frac{w(\mathcal{K})}{w(c_1)}$$

and

$$\sum_{k=0}^{n+1} |\chi_k(c_i)|^2 w(\chi_k) = 1 + q^2 \cdot \frac{1}{q} + (n-1)^2 q^2 \cdot \frac{1+q}{nq^2} + q^2 \frac{1+q}{nq^2} \cdot (n-1)$$

$$= (1+q)n = \frac{(1+q)^2}{q^2} \cdot \frac{nq^2}{1+q} = \frac{w(\mathcal{K})}{w(c_i)}$$

shows that all weights of elements are correct. For $i \geq 2$,

$$\begin{aligned} \langle c_1, c_i \rangle &:= w(\mathcal{K})^{-1} \sum_{k=0}^{n+1} \chi_k(c_1)\chi_k(c_i)^- w(\chi_k) \\ &= w(\mathcal{K})^{-1}\Big(1 + \frac{(-q)}{q} + \big((-q)(n-1)q + (n-1)(-q)^2\big)\frac{(1+q)}{nq^2}\Big) \\ &= 0 \end{aligned}$$

and for $j \geq 2$, $j \neq i$

$$\begin{aligned} \langle c_i, c_j \rangle &= w(\mathcal{K})^{-1}\Big[1 + \frac{(-q)^2}{q} + \big(2(-q)(n-1)q + (n-2)(-q)^2\big)\frac{(1+q)}{nq^2}\Big] \\ &= 0. \end{aligned}$$

It is shown that the column vectors of c_i are mutually orthogonal with respect to the standard inner product $\langle\ ,\ \rangle$. Hence $\{c_0, \dots, c_{n+1}\}$ is linear independant. Thus it is trivial that $\mathcal{K}(n,q)$ satisfies (a1),(a2),(a3) and (a6) in Axiom of a hypergroup. Since $\chi_i(e_j)$ is real and (a2), it is expanded as $c_i c_j = \sum_0^{n+1} n_{ij}^k c_k$ and n_{ij}^k is real. Applying the trivial character χ_0, we have that $\chi_0(c_i c_j) = \sum_{k=0}^{n+1} n_{ij}^k \chi_0(c_k)$ means (a5). Moreover $\langle c_i c_j, c_k \rangle = n_{ij}^k \langle c_k, c_k \rangle = n_{ij}^k w(c_k)^{-1}$ and $n_{ij}^0 = \langle c_i c_j, c_0 \rangle = \langle c_i, c_j^* \rangle$. Hence (a4). Therefore $\mathcal{K}(n,q)$ is a signed hypergroup.

Symmetry of the table leads to $\mathcal{K}(n,q)^\wedge \cong \mathcal{K}(n,q)$.

□

Now we calculate structure constants of $\mathcal{K}(n,q)$ to determine which model $\mathcal{K}(n,q)$ is a hypergroup or not.

Theorem 5.1. *For the models $\mathcal{K}(n,q)=\{c_0,\dots,c_{n+1}\}$ for an integer n and $0<q\le 1$ in Proposition 5.1, the following statements hold:*

(T1) *$\mathcal{K}(n,q)$ is a hypergroup if $1\ge q(n-1)$.*

(T2) *$\{c_0,c_1\}\subset\mathcal{K}(n,q)$ is a subhypergroup isomorphic to $\mathscr{L}(q)$,*

(T3) *$\mathcal{K}(n,q)/\{c_0,c_1\}\cong\mathscr{L}(q)$.*

Proof. According to Wildberger's formula for structure constants and values of characters and weights [W2], we have for $i,j\ge 2$, $i\ne j$

$$\begin{aligned}
\langle c_i^2,c_1\rangle &= w(\mathcal{K})^{-1}\Big(1+\frac{(-q)^2}{q}+\big((-q)(n-1)^2q^2+(n-1)(-q)^3\big)\frac{(1+q)}{nq^2}\Big)\\
&= w(\mathcal{K})^{-1}\Big(1+q-(n-1)q(1+q)\Big)\\
&= \frac{q^2(1-(n-1)q)}{1+q},
\end{aligned}$$

and

$$\begin{aligned}
\langle c_ic_j,c_1\rangle &= w(\mathcal{K})^{-1}\Big(1+\frac{(-q)^2}{q}+\big(2(-q)^2(n-1)q+(n-2)(-q)^3\big)\frac{(1+q)}{nq^2}\Big)\\
&= w(\mathcal{K})^{-1}\Big(1+q+\big(2(n-1)-(n-2)\big)\frac{q(1+q)}{n}\Big)\\
&= w(\mathcal{K})^{-1}\Big(1+q+q(1+q)\Big)\\
&= q^2.
\end{aligned}$$

Hence

$$n_{ii}^1=\frac{q(1-(n-1)q)}{1+q}\quad\text{and}\quad n_{ij}^1=q.$$

Using Proposition 2.1, we have

$$n_{1i}^i=\frac{1-(n-1)q}{n}\quad\text{and}\quad n_{1i}^j=\frac{1+q}{n}.$$

In addition, we have

$$\begin{aligned}
\langle c_i^2,c_i\rangle &= w(\mathcal{K})^{-1}\Big(1+\frac{(-q)^3}{q}+\big((n-1)^3q^3+(n-1)(-q)^3\big)\frac{(1+q)}{nq^2}\Big)\\
&= w(\mathcal{K})^{-1}\Big(1-q^2-(n-1)(n-2)q(1+q)\Big)\\
&= \frac{q^2(1-q+(n-1)(n-2)q)}{1+q}\\
&= \frac{q^2(1-(n-1)q+n(n-2)q)}{1+q},
\end{aligned}$$

and

$$\begin{aligned}
\langle c_i c_i, c_j\rangle &= w(\mathcal{K})^{-1}\Big(1+\frac{(-q)^3}{q}-\big((n-1)^2q^3-(n-1)q^3+(n-2)q^3\big)\frac{(1+q)}{nq^2}\Big)\\
&= w(\mathcal{K})^{-1}\Big(1-q^2-\big((n-1)^2-1\big)\frac{q(1+q)}{n}\Big)\\
&= w(\mathcal{K})^{-1}\Big(1-(n-1)q\Big)(1+q)\\
&= \frac{q^2(1-(n-1)q)}{1+q}.
\end{aligned}$$

Hence

$$n_{ii}^i=\frac{1-(n-1)q}{n}+(n-2)q \quad \text{and} \quad n_{ii}^j=\frac{1-(n-1)q}{n}=n_{ij}^i.$$

The multiplication has the following expression:

(1) $c_1^2=qc_0+(1-q)c_1$,

(2) $c_1c_i=(-q)c_i+\dfrac{1+q}{n}(c_2+\cdots+c_{n+1})$,

(3) $c_i^2=\dfrac{nq^2}{1+q}c_0+\dfrac{1-q(n-1)}{1+q}qc_1+(n-2)qc_i+\dfrac{1-q(n-1)}{n}(c_2+\cdots+c_{n+1})$,

(4) $c_ic_j=qc_1+(-q)(c_i+c_j)+\dfrac{1+q}{n}(c_2+\cdots+c_{n+1})$

for $2\le i,\ 2\le j$ and $i\ne j$.

(T2) is obvious from (1). Hence non negativity of coefficients in the above (2)-(4) implies (T1). From (2), it leads that

$$\frac{1}{1+q}(qc_0+c_1)c_i=\frac{1}{n}(c_2+\cdots+c_{n+1}),$$

whence,

$$\frac{1}{1+q}(qc_0+c_1)\frac{1}{n}(c_2+\cdots+c_{n+1})=\frac{1}{n}(c_2+\cdots+c_{n+1})=:\boldsymbol{\ell}_1.$$

Summarizing (3) and (4), we have

$$\begin{aligned}
\boldsymbol{\ell}_1c_i &= \frac{1}{n}\Big[\frac{nq^2}{1+q}c_0+\frac{1-q(n-1)}{1+q}qc_1+(n-1)qc_1+(n-2)qc_i+(-q)(n-1)c_i\\
&\qquad +(-q)n\boldsymbol{\ell}_1+qc_i+(1-q(n-1)+(1+q)(n-1))\boldsymbol{\ell}_1\Big]\\
&= q(\frac{q}{1+q}c_0+\frac{1}{1+q}c_1)+(1-q)\boldsymbol{\ell}_1.
\end{aligned}$$

Therefore, write $\boldsymbol{\ell}_0 := (1+q)^{-1}(qc_0 + c_1)$,

$$\boldsymbol{\ell}_1^2 = \frac{1}{n}\sum_{i=2}^{n+1} \boldsymbol{\ell}_1 c_i \;=\; q(\frac{q}{1+q}c_0 + \frac{1}{1+q}c_1) + (1-q)\boldsymbol{\ell}_1$$
$$= \; q\boldsymbol{\ell}_0 + (1-q)\boldsymbol{\ell}_1.$$

Thus the quotient $\mathcal{K}(n,q)/\{c_0, c_1\} = \{\boldsymbol{\ell}_0, \boldsymbol{\ell}_1\} \cong \mathscr{L}(q)$, which shows (T4). □

The statements in Theorem 5.1 mean that

$$\mathbf{1} \to \mathscr{L}(q) \to \mathcal{K}(n,q) \to \mathscr{L}(q) \to \mathbf{1}$$

is exact. Since $\mathcal{K}(n,q)$ is a self-dual hypergroup, it is a strong hypergroup.

Remark 5.1. We comment here about the condition between n and q in which $\mathcal{K}(n,q)$ is a hypergroup. It is easy to see that

$$\mathcal{K}(1,q) = \mathscr{L}(q) \vee \mathscr{L}(q)$$

of order three, which is said to be a join hypergroup. The condition (t1) means $n \le w(\mathcal{H}) = 1 + 1/q$ which is proved in Proof of Proposition 3.3. If an integer $n > 1 + 1/q$, then $\mathcal{K}(n,q)$ is a signed hypergroup but not a hypergroup. There exists a signed hypergroup extension with any order higher than three. It is well known that hypergroup extensions of a group by another group are also groups. As $\mathscr{L}(1) = \mathbb{Z}_2$, there exist well known group extension $\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathcal{K}(2,1)$ and $\mathbb{Z}_4$, only two group extensions, but there exist infinite many signed hypergroup extensions $\mathcal{K}(n,1)$ for $n \ge 3$ which have subhypergroup $\mathbb{Z}_2$ and the quotient $\mathbb{Z}_2$. Although non splitting group extension $\mathbb{Z}_4$ does not appear in our models. When the maximal integer n such that $1 + 1/q \ge n$, namely $n = 1 + \lfloor 1/q \rfloor$, this model $\mathcal{K}(n,q)$ is a self dual hypergroup with the maximal order mentioned in Proposition 3.3. Moreover there exits a hypergroup extension $\mathcal{K}(n,q)$ whose order is any integer between 3 and $3 + \lfloor 1/q \rfloor$.

When $1/2 < q \le 1$, we have $w(\mathscr{L}(q)) < 3$, so that $\lfloor w(\mathscr{L}(q)) \rfloor = 2$. According to estimation in Proposition 3.3, the hypergroup extensions of $\mathscr{L}(q)$ by $\mathscr{L}(q)$ must have an order less than or equal to four. Thus there exists no extension of order higher than four.

When $0 < q \le 1/2$, there exist many extensions of orders higher than four.

At the beginning, if $q = 1/2$, then it is easily shown that hermitian extension of order five is isomorphic to $\mathcal{K}(3, 1/2)$ provided that the character table is symmetric, i.e. uniquely determined hermitian hypergroup extension of this type. The reason of this fact will be shown in the latter. Now see the character table as follows:

	c_0	c_1	c_2	c_3	c_4	$w(\chi_i)$
χ_0	1	1	1	1	1	1
χ_1	1	1	$-1/2$	$-1/2$	$-1/2$	2
χ_2	1	$-1/2$	1	$-1/2$	$-1/2$	2
χ_3	1	$-1/2$	$-1/2$	1	$-1/2$	2
χ_4	1	$-1/2$	$-1/2$	$-1/2$	1	2
$w(c_i)$	1	2	2	2	2	9

In this case it is easily seen from the table that $\mathcal{K}(3, 1/2)$ has four subhypergroups

$$\{c_0, c_1\} \cong \{c_0, c_2\} \cong \{c_0, c_3\} \cong \{c_0, c_4\} \cong \mathscr{L}(1/2)$$

and their quotients are isomorphic to $\mathscr{L}(1/2)$. The automorphism group of $\mathcal{K}(3, 1/2)$ is a permutation group $\mathfrak{S}_4$ of degree four on the set $\{c_1, c_2, c_3, c_4\}$, namely

$$\mathrm{Aut}(\mathcal{K}(3, 1/2)) = \mathfrak{S}_4$$

Thus there exist three cross sections ε_i for $i = 2, 3, 4$ in the following exact sequence:

$$\mathbf{1} \to \mathscr{L}(1/2) \to \mathcal{K}(3, 1/2) \leftrightarrows \mathscr{L}(1/2) \to \mathbf{1}$$

where hypergroup homomorphism ε_i maps $\ell_1 \in \mathscr{L}(1/2)$ to c_i. However $\mathcal{K}(3, 1/2)$ is not isomorphic to a direct product $\mathscr{L}(1/2) \times \mathscr{L}(1/2)$. Even if there exists a cross section for a exact sequence, hypergroup extension is not alway the direct product of a subhypergroup and a quotient.

Furthermore $\mathcal{K}(n + 1, 1/n)$ for an integer $n \geq 3$ has the cross sections in the short exact sequences in a similar way as the above. It is uniquely determined as a hermitian hypergroup extension of order $n + 2$ in the main problem provided that character table is symmetric. It is shown the property that

$$\mathrm{Aut}(\mathcal{K}(n + 1, 1/n)) = \mathfrak{S}_{n+2}.$$

When $1/3 < q < 1/2$, there exists many hermitian extensions which are not isomorphic to $\mathcal{K}(3, q)$. After suitable calculations, the set of all hypergroup extensions is parametrized by two real dimension with respect to two weights, but we omit them. Therefore the entire list of hypergroup extensions of $\mathscr{L}(q)$ by $\mathscr{L}(q)$ for a fixed positive number $0 < q < 1/2$ and $q \neq 1/n$ $(n = 2, 3, \dots)$ is intricate for us to express their all structure constants completely. When the number q is near to 0, it is very intricate for us to done it completely even if in the hermitian case.

Remark 5.2. The action π of $\mathcal{H}$ on the set $\{c_2, \dots, c_{n+1}\}$ is irreducible and has the matrix form T_{ij} appeared in [SW1]. For an example, $\pi(e_0^{\mathcal{H}}) = (1/n)J_n$ where J_n is $n \times n$-matrix all components 1. The action π is a irreducible $*$-action in Example 1 in [SW1].

Since the value of characters in $\mathcal{K}(n, q)$ does not vanish if $n \neq 1$, so that stabilizer $\mathscr{L}(q)(c_i) = \{c_0\} \subset \mathscr{L}(q)$ for any $c_i \in \mathcal{K}(n, q)$. Hypergroup $\mathcal{K}(n, q)$ for

$n > 2$ and $q \leq 1/2$ leads an example of which section $\varphi^{-1}(\ell)$ is not given from the index with respect to the quotient $\mathscr{L}(q)/\mathscr{L}(q)(c_i)$. Therefore the method of stabilizers can not go well in general.

Finally we illustrate with more general models for hermitian signed hypergroup extensions having three parameters.

Proposition 5.2. *The following character table with values of weights defines a signed hypergroup $\mathcal{K}(n,p,q)$ for an integer n and $0 < p \leq 1$, $0 < q \leq 1$. In addition $\mathcal{K}(n,p,q)$ has order $n+2$.*

	c_0	c_1	c_2	c_3	$\cdots$	c_{n+1}	$w(\chi_i)$
χ_0	1	1	1	1	$\cdots$	1	1
χ_1	1	1	$-q$	$-q$	$\cdots$	$-q$	$1/q$
χ_2	1	$-p$	$(n-1)\sqrt{pq}$	$-\sqrt{pq}$	$\cdots$	$-\sqrt{pq}$	$\frac{1+q}{npq}$
χ_3	1	$-p$	$-\sqrt{pq}$	$(n-1)\sqrt{pq}$	$\cdots$	$-\sqrt{pq}$	$\frac{1+q}{npq}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
χ_{n+1}	1	$-p$	$-\sqrt{pq}$	$-\sqrt{pq}$	$\cdots$	$(n-1)\sqrt{pq}$	$\frac{1+q}{npq}$
$w(c_i)$	1	$1/p$	$\frac{1+p}{npq}$	$\frac{1+p}{npq}$	$\cdots$	$\frac{1+p}{npq}$	$\frac{(1+p)(1+q)}{pq}$

Proof. Similarly as Proposition 5.1, it is obvious that the condition of Lemma 2.2 is satisfied. Therefore $\mathcal{K}(n,p,q)$ is a signed hypergroup. □

Here the structure of $\mathcal{K}(n,p,q)$ is calculated in a similar way and is expressed as follows:

(1) $c_1^2 = pc_0 + (1-p)c_1$,

(2) $c_1c_i = (-p)c_i + \dfrac{1+p}{n}(c_2 + \cdots + c_{n+1})$,

(3) $$c_i^2 = \frac{npq}{1+p}c_0 + \frac{1+p-np}{1+p}qc_1 + (n-2)\sqrt{pq}c_i + \frac{1-q-\sqrt{pq}(n-2)}{n}(c_2 + \cdots + c_{n+1}),$$

(4) $$c_ic_j = qc_1 + (-\sqrt{pq})(c_i + c_j) + \frac{1-q+2\sqrt{pq}}{n}(c_2 + \cdots + c_{n+1})$$

for $2 \leq i,\ 2 \leq j$ and $i \neq j$.

Checking a coefficient of c_i in (2) and those of c_1 in (3), we obtain the following condition (i). Checking a coefficient of c_j in (3) and those of c_i, c_j in (3), we also obtain the following condition (ii). Thus the above structure constants are non-negative if the following conditions (i)-(ii) hold:

(i) $1 + p - pn \geq 0$,

(ii) $1 - q - \sqrt{pq}(n-2) \geq 0$.

Hence $\mathcal{K}(n, p, q)$ is a hypergroup under the above condition. Thus the next theorem follows.

Theorem 5.2. *The model* $\mathcal{K}(n, p, q)$ *is a hypergroup if and only if the above conditions* (i)-(ii) *are satisfied. Moreover* $\mathcal{K}(n, p, q)$ *has subhypergroup* $\{c_0, c_1\} \cong \mathscr{L}(p)$ *such that a quotient* $\mathcal{K}(n, p, q)/\{c_0, c_1\} \cong \mathscr{L}(q)$*, namely* $\mathcal{K}(n, p, q)$ *is a hypergroup extension of* $\mathscr{L}(q)$ *by* $\mathscr{L}(p)$*.*

Proof. The first statement is already proved. The statement about the quotient hypergroup is clear in a similar way to the proof of Theorem 5.1. □

Obviously we see that $\mathcal{K}(n, p, p) = \mathcal{K}(n, p)$. From Theorem 5.2 we have similarly that

$$\mathbf{1} \to \mathscr{L}(p) \to \mathcal{K}(n, p, q) \to \mathscr{L}(q) \to \mathbf{1}$$

is exact. This sequence gives the models of hypergroup extensions to the problem: an exact sequence $\mathbf{1} \to \mathscr{L}(p) \to \mathcal{K} \to \mathscr{L}(q) \to \mathbf{1}$ in the hermitian case of orders higher than four. These problems of hypergroup extensions in the case of order four are described in [IK1], and are completely analyzed.

Remark 5.3. When $p = 1$, the condition (i) turns out $n \leq 2$, which is shown in [K]. When $q = 1$, the condition (ii) also turns out $n \leq 2$, which is shown in [IKS].

We notice that the condition (ii) is in the form

$$(n-2)p \leq \frac{(1-q)^2}{q}.$$

This curve in p, q-plane is decreasing for $0 < q \leq 1$ and has an asymptote $q = 0$.

Now we possess many hypergroup extensions of $\mathscr{L}(q)$ by $\mathscr{L}(p)$ with order higher than four in some case. Furthermore given $0 < q < 1$ and an integer $n \geq 3$, there exists a suitable $0 < p < 1$ such that $\mathcal{K}(n-2, p, q)$ is a hypergroup with the order n. This is shown by checking that (i),(ii) hold as $p \to +0$.

If $\mathcal{L}$ of order two is not a group, then we can choose a suitable hypergroup $\mathcal{H}$ of order two such that there exists a hypergroup extension of any order higher than four. Fixed n larger than two, it is shown that the region of all points (p, q) for $0 < p < 1$ and $0 < q < 1$ such that there exists a hypergroup extension of order $n + 2$ is not symmetric with respect to two parameters p, q. Thus $\mathcal{K}(n, p, q)$ is not always strong even if $\mathcal{K}(n, p, q)$ is a hypergroup.

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