On the connection problem and Stokes coefficients for differential equations with an irregular singular point on the complex plane

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CHIBA UNIVERSITY

（千葉大学審査学位論文）

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Preface

In 1864, Stokes studied the asymptotic behavior of the Airy integral [St]:

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{i\left(\frac{s^3}{3} + zs\right)\right\} ds.$$

Airy attempted to numerically compute $Ai(z)$ for z real, but his method using a convergent series only worked when $|z|$ was small. Stokes found asymptotic expansions that yielded good approximations for large $|z|$, but he noticed that the expansions for positive z and negative z were different. Further investigation showed that in the complex domain, the asymptotic expansions depend discontinuously on the angle of approach. This behavior is called the Stokes phenomenon.

In this paper, we study the Stokes phenomenon for solutions to ordinary differential equations and systems of differential equations. In particular, we wish to investigate the dependence of the asymptotic behavior of a solution on the sectorial neighborhood near an irregular singularity.

For any sector, any fundamental set of solutions near a regular singular point, and any fundamental set of solutions near the irregular singular point on the sector, there is a linear transformation relating the two fundamental sets of solutions. The problem of finding the coefficients of this linear transformation is called the connection problem. By composing these linear transformations, we can analyze the Stokes phenomenon.

In Chapter 1, we introduce a method for reducing a single differential equation with a finite number of regular singular point and one irregular singular point to a generalized Schlesinger system. Using this reduction, the multi-point connection problem may then be solved by a method of Kohno [K3].

Okubo [O1] treated the connection problem of a Birkhoff system with an irregular singular point of rank one at infinity and one regular singular point at the origin:

$$t\frac{dX}{dt} = (A + Ct)X,$$

where A and C are n by n constant matrices, X is a vector with n entries, and t is a complex variable. He also solved the reduction problem from hypergeometric equations to the hypergeometric system

$$(tI - B)\frac{dX}{dt} = AX,$$

also called the Okubo system. The special case where all eigenvalues of B are distinct had been briefly addressed in lectures by Hukuhara [H], and the existence of a solution in the general case was worked out in [O2]. In the preface of [O2], Okubo credits Kohno and Suzuki [KS] with proving "the constructive version, and reduction program in Reduce III" where they gave an explicit description of the matrix A. Moreover, Kohno studied the two point connection problem for a single differential equation which has one regular singular point at the origin and one irregular singular point of rank greater than one at infinity [K1], and in the last section of [K3], he sketched a method for finding the solutions of the multi-point connection problem for a generalized Schlesinger system.

Given the above results, it is worth investigating reduction problems in more general settings, e.g., where one has more than two regular singular points. For this reason, we wish to reduce to a generalized Schlesinger system.

In Chapter 2, we compute the Fuchsian relations for non-holomorphic solutions for a single ordinary differential equation, and we find the difference between this and the Fuchsian relation for the corresponding system. The Fuchsian relation plays an essential role in the global analysis of linear differential equations with regular or irregular singularities.

In Chapter 3, we consider single differential equations on the complex projective line which have one regular singular point and one irregular singular point. We construct a family of functions whose asymptotic expansions match those of a fundamental solution at a regular singular point. These functions are particular solutions of first order nonhomogeneous differential equations that can be derived from the fundamental solutions at the regular singular point and formal solutions at the irregular singular point of the original differential equation. We call these functions the fundamental functions associated with this two point connection problem. The series expansions of the associated fundamental functions are described by systems of difference equations, and the coefficients relating them to the fundamental solutions can be found by a recursive process. This yields a method for calculating the linear relation between the two fundamental sets of solutions, i.e., solving the connection problem.

The method of associated fundamental functions was first applied to the two-point connection problem for a differential system with an irregular singular point of rank one by K. Okubo in 1963 [O1]. In 1974, M. Kohno applied it to a single differential equation with a regular singular point and an irregular singular point of arbitrary rank [K1]. In 1999, he also sketched an argument that would allow one to apply the associated fundamental functions to the problem in the case where one has an arbitrary number of regular singular points and one irregular singular point [K2].

It seems that this last advance has gone largely unnoticed, and there have been no further developments. In future work, we intend to apply this method to solve the multi-point connection problem.

Finally, in the last chapter, we consider the Stokes coefficients. The Stokes phenomenon for a linear differential system with an irregular singularity at zero is the appearance of distinct sector-dependent analytic solutions that are asymptotic to a single formal solution. To each anti-Stokes direction there is a Stokes matrix which is a meromorphic invariant for the system. For analyzing the Stokes coefficients, Sibuya gave a characterization of meromorphic equivalence classes of differential equations in terms of Stokes data, using the Cauchy-Heine integral [S]. B. Malgrange reinterpreted this characterization as an isomorphism with a sheaf cohomology group [M], now called the Sibuya-Malgrange Isomorphism. M. Loday-Richaud proved a formula for the Stokes multipliers for the Birkhoff canonical system of size two, using the Cauchy-Heine integral [L]. The Stokes matrices are, in general, transcendental with respect to the coefficients of the differential system. As we do not have an algebraic method for finding the Stokes Multipliers, we turn to numerical manipulation of examples for insight.

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1 Reduction to a system of first degree differential equations

In this chapter, we introduce a method of a reduction from a single differential equation with a finite number of regular singular points and one irregular singular point to a generalized Schlesinger system. In Section 1.1, we shall introduce a method of our reduction. In Section 1.2, we shall describe an explicit form for our system. And in Section 1.3, we shall describe how our methods can be applied to a more general reduction problem, where we remove a constraint on the degree. In Section 1.4, we consider an example of the reduction of a fourth order linear differential equation. We believe that an explicit example is valuable for understanding our algorithm of the reduction. Furthermore, it has been helpful in our work on the multi-point connection problem.

In the beginning, we define a regular singular point, and an irregular singular point.

Definition 1.1. *(regular singular point of a function) Assume that $a \in \mathbb{C}$ and $f(t)$ is a holomorphic function on $0 < |t-a| < r, r \in \mathbb{R}$.*

We say that $t = a$ is a regular singular point, if $f(t)$ cannot be extended to a holomorphic function at $t = a$ and $\exists N > 0, \forall \alpha, \beta$ s.t.

$$|t-a|^N |f(t)| \to 0 \quad (|t-a| \to 0 \quad \alpha < \arg(t-a) < \beta).$$

1. *If $t = a$ is a regular singular point or a holomorphic point, we say that a is at least a regular singular point.*

2. *If $t = a$ is neither a regular singular point nor a holomorphic point, we call a an irregular singular point.*

Definition 1.2. *(regular singular point of a differential equation) Assume that $a \in \mathbb{C}$. Consider*

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \tag{1.1}$$

where $p_j(t)(j = 1, 2, \cdots, n)$ are holomorphic on $0 < |t-a| < r$ and they are either holomorphic at $t = a$, or have poles of finite order there. We say that $t = a$ is a regular singular point of (1.1), if for any solution $y(t)$ of (1.1) $t = a$ is at least a regular singular point of $y(t)$.

In the paper [AK], we considered the reduction of the linear differential equation

$$P_n(t)\, y^{(n)} = P_{n-1}(t)\, y^{(n-1)} + \cdots + P_1(t)\, y' + P_0(t)\, y, \tag{1.2}$$

where

$$P_n(t) = \prod_{j=1}^{n} (t - \lambda_j),$$

and the coefficients $P_j(t)$ $(j = 0, 1, \ldots, n-1)$ are polynomials of degree at most n, to the system of linear differential equations

$$(tI - B)\frac{dX}{dt} = (A + Ct)X,$$

with

$$B = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

and A, C being constant matrices. We then showed a method of reduction for the case where all the λ_j are mutually distinct. In this paper we shall consider the more general reduction problem, in which $P_n(t)$ may have multiple roots. Moreover, multiplying $(tI - B)^{-1}$ from the left side, we shall show that our system can be reduced to a generalized Schlesinger system:

$$\frac{dX}{dt} = \left(\sum_{i=1}^{q} \frac{\bar{A}_i}{t - \lambda_i} + C\right) X,$$

where q is a number of regular singular points and $\bar{A}_i (i = 1, 2, \ldots, q)$ are n by n constant matrices.

Given the above results, it is worth investigating reduction problems in more general settings, e.g., where one has at least two regular singular points. For this reason, we wish to reduce to a generalized Schlesinger system. In [AK], we treated the case where all of the regular singular points are mutually distinct. In this chapter, we consider such a reduction problem, in which the regular singular points are not necessarily distinct.

1.1 Method of the reduction

Consider the single linear differential equation with a finite number of regular singular points at $t = \lambda_\nu$ $(\nu = 1, 2, \cdots, q)$ and one irregular singularity of rank one at $t = \infty$ in the whole complex plane. A general form of such differential equations can be expressed as follows :

$$P_n(t)y^{(n)} = P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y, \tag{1.3}$$

where we assume that the coefficients $P_j(t)\,(j = 0,\,1,\,\cdots,n-1,\,n)$ are polynomials of degree at most n. This hypothesis is not essential, and we shall show how to eliminate it in a remark at the end of Section 1.3. The roots of $P_n(t)$ are the regular singular points.

We now assume that

$$P_n(t) = (t-\lambda_1)^{n_1}(t-\lambda_2)^{n_2}\cdots(t-\lambda_q)^{n_q}, \tag{1.4}$$

with

$$\begin{cases} n_1 + n_2 + \cdots + n_q = n \quad (1 \le q \le n), \\ \\ 1 \le n_q \le n_{q-1} \le \cdots \le n_1 \le n. \end{cases}$$

In this case, in order that $t = \lambda_\nu$ be a regular singularity of (1.3), the functions

$$\frac{(t-\lambda_\nu)^i P_{n-i}(t)}{P_n(t)} \qquad (i = 1,\,2,\,\cdots,\,n)$$

must be holomorphic at $t = \lambda_\nu$, and hence for each ν, the polynomials $P_{n-i}(t)$ $(1 \le i \le n_\nu)$ have the factor $(t-\lambda_\nu)^{n_\nu - i}$.

From the above fact, it is easy to see that the coefficients $P_{n-i}(t)$ are written as :

$$\begin{cases} P_{n-i}(t) = \left[\prod_{\nu=1}^{q}(t-\lambda_\nu)^{n_\nu - i}\right]\widehat{P}_{n-i}(t) \quad (0 < i \le n_q), \\ P_{n-i}(t) = \left[\prod_{\nu=1}^{k-1}(t-\lambda_\nu)^{n_\nu - i}\right]\widehat{P}_{n-i}(t) \quad (n_k < i \le n_{k-1}\,;\, k = q,\, q-1,\cdots,\,2), \\ P_{n-i}(t) = \widehat{P}_{n-i}(t) \quad (n_1 < i \le n), \end{cases} \tag{1.5}$$

with $\widehat{P}_{n-i}(t)$ being a polynomial for all $i = 0, 1, \ldots, n$.

Then, introducing the notation

$$\begin{cases} N_k = n_1 + n_2 + \cdots + n_k \quad (k = 1,\,2,\,\cdots,\,q), \\ \\ N_0 = 0, \quad N_q = n, \end{cases}$$

one can see that for $n_k < i \le n_{k-1}\,(k = 1,\,2,\,\cdots,\,q+1\,;\, n_0 = n,\, n_{q+1} = 0)$, the degree of $\widehat{P}_{n-i}(t)$ is at most

$$n - \{n_1 + n_2 + \cdots + n_{k-1} - i(k-1)\} \equiv n - N_{k-1} + i(k-1).$$

Hence, the single differential equation includes

$$\begin{aligned}\mathcal{N} &= \sum_{k=1}^{q+1} \sum_{i=n_k+1}^{n_{k-1}} \{n - N_{k-1} + i(k-1) + 1\} \\ &= \sum_{k=1}^{q+1} \left\{ (n+1-N_{k-1})(n_{k-1}-n_k) + (k-1)\,\frac{(n_{k-1}-n_k)(n_{k-1}+n_k+1)}{2} \right\}\end{aligned}$$

constants.

Since

$$\sum_{k=1}^{q+1} (k-1)(n_{k-1}-n_k) = \sum_{k=1}^{q} n_k \;=\; n,$$

$$\sum_{k=1}^{q+1} (k-1)(n_{k-1}^2-n_k^2) = \sum_{k=1}^{q} n_k^2,$$

$$\begin{aligned}\sum_{k=1}^{q+1} N_{k-1}(n_{k-1}-n_k) &= \sum_{k=1}^{q} N_k n_k - \sum_{k=2}^{q+1} N_{k-1} n_k \\ &= N_1 n_1 + \sum_{k=2}^{q} (N_k - N_{k-1}) n_k - N_q n_{q+1} \\ &= \sum_{k=1}^{q} n_k^2,\end{aligned}$$

we have

$$\begin{aligned}\mathcal{N} &= n(n+1) - \sum_{k=1}^{q} n_k^2 + \frac{1}{2}\left\{ \sum_{k=1}^{q} n_k^2 + n \right\} \\ &= \frac{n(2n+3)}{2} - \frac{1}{2}\sum_{k=1}^{q} n_k^2.\end{aligned}$$

We now investigate the characteristic exponents of convergent power series solutions near regular singularities and the characteristic constants of formal solutions at the irregular singularity.

Near each regular singular point at $t = \lambda_\nu$ ($\nu = 1, 2, \ldots, q$), there exist convergent power series solutions of the form

$$y(t) = (t-\lambda_\nu)^\rho \sum_{m=0}^{\infty} g(m)\,(t-\lambda_\nu)^m, g(0) \neq 0. \tag{1.6}$$

The characteristic exponent ρ is a root of the equation

$$[\rho]_n = \sum_{i=1}^{n_\nu} \gamma_i\,[\rho]_{n-i},$$

where $[\rho]_k$ is the Pochhammer symbol for $k = 0, 1, 2, \ldots$:

$$[\rho]_k \equiv \rho(\rho-1)\cdots(\rho-k+1), \qquad [\rho]_0 \equiv 0,$$

and the coefficients γ_i are given by

$$\gamma_i = \left[\frac{P_{n-i}(t)}{P_n(t)}(t-\lambda_\nu)^i\right]_{t=\lambda_\nu}$$

$$= \frac{\widehat{P}_{n-i}(\lambda_\nu)}{\prod\limits_{\substack{\ell=1\\ \ell\neq\nu}}^{k-1}(\lambda_\nu-\lambda_\ell)^i \prod\limits_{\ell=k}^{q}(\lambda_\nu-\lambda_\ell)^{n_\ell}} \qquad (n_k < i \leq n_{k-1} \leq n_\nu).$$

Then, ρ is a root of

$$[\rho]_{n-n_\nu} = 0, \quad \text{i.e.,} \quad 0,\ 1,\ \cdots,\ n-n_\nu-1,$$

or

$$[\rho-n+n_\nu]_{n_\nu} = \sum_{i=1}^{n_\nu} \gamma_i\,[\rho-n+n_\nu]_{n_\nu-i}. \tag{1.7}$$

This implies that there exist $n - n_\nu$ holomorphic solutions and possibly, n_ν nonholomorphic solutions near the regular singular point $t = \lambda_\nu$.

Near the irregular singularity at infinity, one can find formal solutions of the form

$$y(t) = e^{\mu\, t}\, t^{\eta} \sum_{s=0}^{\infty} h(s)\, t^{-s}, h(0) \neq 0, \tag{1.8}$$

where the characteristic constant μ is a root of the equation

$$\mu^{n} \ = \sum_{i=1}^{n} p_{n-i}\, \mu^{n-i}, \tag{1.9}$$

with p_{n-i} being the coefficient of highest degree in $\widehat{P}_{n-i}(t)$.

Now we shall consider the reduction of the single differential equation (1.3) to the system of differential equations

$$(t - B)\frac{dY}{dt} = (A \,+\, C\, t)\, Y, \tag{1.10}$$

where

$$B = \mathrm{diag}(\overbrace{\lambda_1, \cdots, \lambda_1}^{n_1}, \ \overbrace{\lambda_2, \cdots, \lambda_2}^{n_2}, \ \ \cdots \ , \ \overbrace{\lambda_q, \cdots, \lambda_q}^{n_q}),$$

where we recall the notation

$$n = N_q = n_1 + n_2 + \cdots + n_q,$$

and $A,\ C$ are n by n constant matrices with the following form :

$$A \,+\, C\, t \ \equiv \ D(t) \ = \left(\begin{array}{c|c|c|c} D_1(t) & 1 & & \mathbf{0} \\ \hline & D_2(t) & 1 & \\ \hline d_{j,i}(t) & & \ddots \quad \ddots & 1 \\ \hline & & & D_q(t) \end{array} \right).$$

All the elements $d_{j,\,i}(t)$ are polynomials of degree at most 1 and the $D_k(t)$ are $n_k \times n_k$ submatrices of the form

$$D_k(t) \;=\; \begin{pmatrix} d_{N_{k-1}+1,\,N_{k-1}+1}(t) & 1 & & & \\ & d_{N_{k-1}+2,\,N_{k-1}+2}(t) & 1 & & \mathbf{0} \\ d_{j,\,i}(t) & & \ddots & \ddots & \\ & & & \ddots & 1 \\ d_{N_k,\,N_{k-1}+1}(t) & d_{N_k,\,N_{k-1}+2}(t) & \cdots & \cdots & d_{N_k,\,N_k}(t) \end{pmatrix}.$$

In order to reduce (1.3) to (1.10), we apply the transformation introduced in [AK]:

$$\begin{cases} y_1 = y, \\ y_2 = \varphi_1\, y' \,+\, e_{2,0}(t)\, y, \\ \quad\vdots \\ y_j = \varphi_{j-1}\, y^{(j-1)} \,+\, e_{j,\,j-2}(t)\, y^{(j-2)} \,+ \cdots +\, e_{j,1}(t)\, y' + e_{j,0}(t)\, y, \\ \quad\vdots \\ y_n = \varphi_{n-1}\, y^{(n-1)} \,+\, e_{n,\,n-2}(t)\, y^{(n-2)} \,+ \cdots +\, e_{n,1}(t)\, y' \,+\, e_{n,0}(t)\, y. \end{cases} \tag{1.11}$$

Here we set

$$\varphi_j \;=\; \prod_{k=1}^{j}(t-\lambda_{a(k)}) \quad (j=1,2,\cdots,n), \qquad \varphi_0 \;\equiv\; 1,$$

where $a : \{1,2,\dots,n\} \to \{1,2,\dots,q\}$ is the unique weakly increasing function such that $a^{-1}(i)$ has cardinality n_i for all $1 \le i \le q$. Obviously, $\varphi_n \;=\; P_n(t)$.

We shall attempt to drive a system of linear differential equations for the column vector $Y = (y_1, y_2, \ldots, y_n)_*$:

$$\begin{pmatrix} t-\lambda_{a(1)} & & & \text{\Large 0} \\ & t-\lambda_{a(2)} & & \\ & & \ddots & \\ \text{\Large 0} & & \ddots & t-\lambda_{a(n)} \end{pmatrix} Y' = \begin{pmatrix} d_{1,1} & 1 & & & \text{\Large 0} \\ d_{2,1} & d_{2,2} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & 1 \\ d_{n,1} & d_{n,2} & \cdots & \cdots & d_{n,n} \end{pmatrix} Y,$$

where all $d_{j,i}(t)$ are polynomials of <u>the first degree</u>.

For this transformation, we have the following :

Theorem 1.3. *Assume that y is a solution of the single differential equation (1.3), and that $Y = (y_1, y_2, \ldots . y_n)_*$ is a vector that satisfies a system of differential equations of the form (1.10), such that y and Y are related by a reduction of the form (1.11). Then, the polynomials $e_{j,\ell}(t)$ for $1 \le \ell + 1 < j \le n$ are uniquely defined, and they satisfy:*

$$\begin{aligned} &(t - \lambda_j)\,(e_{j,\,j-\ell-3}(t) + e'_{j,\,j-\ell-2}(t)) \\ &\qquad\qquad = e_{j+1,\,j-\ell-2}(t) + \sum_{h=0}^{\ell+1} d_{j,\,j-h}(t)\, e_{j-h,\,j-\ell-2}(t) \\ &(j = 1, 2, \cdots, n\,;\ \ell = -1, 0, \cdots, j-2), \end{aligned} \tag{1.12}$$

where

$$e_{n+1,\,k}(t) = -P_k(t) \quad (k = 0, 1, \cdots, n-1),$$

$$e_{1,\,0}(t) \equiv 1, \quad e_{j,\,j-1}(t) \equiv \varphi_{j-1}, \quad e_{j,\,-k}(t) \equiv 0 \quad (k > 0).$$

1.2 Determination of $e_{j,\,i}$ and $d_{j,\,i}$

In this section, we shall prove Theorem 1.3. In the paper [AK], we have verified that the above transformation leads to a desired system in the distinct case, where $q = n$ and $n_j = 1$ $(j = 1, 2, ..., n)$.

We shall also use this transformation to reduce (1.3) to (1.10) in the general case just considered. For that purpose, we introduce the following notation:

$$\begin{cases} f_k^i = \displaystyle\prod_{\nu=1}^{k}(t-\lambda_\nu)^{n_\nu - i}, \\ f_k^i = f_k^{i+1}\,\psi_k, \\ \psi_k = \displaystyle\prod_{\nu=1}^{k}(t-\lambda_\nu), \\ (f_k^i)' = f_k^{i+1} g_k^i, \\ g_k^i = \displaystyle\sum_{\nu=1}^{k}(n_\nu - i) \prod_{\substack{\nu'=1 \\ \nu'\neq\nu}}^{k}(t-\lambda_{\nu'}) \\ \quad (k = 1, 2, \cdots, q). \end{cases}$$

Then, we can rewrite the coefficients and φ_j as follows :

$$\begin{cases} P_{n-i}(t) = f_{k-1}^i\,\widehat{P}_{n-i}(t) \quad (n_k < i \le n_{k-1}\,;\, k = q+1, q, \cdots, 2), \\ P_{n-i}(t) = \widehat{P}_{n-i}(t) \quad (n_1 < i \le n), \end{cases}$$

and for j in $N_{k-1} < j \le N_k$ $(k = 1, 2, \cdots, q)$:

$$\begin{cases} \varphi_j = f_k^0\,(t-\lambda_k)^{j-N_k} = f_k^1\,\psi_k\,(t-\lambda_k)^{j-N_k} = f_k^1\,\psi_{k-1}\,(t-\lambda_k)^{j+1-N_k}, \\ \varphi_j' = f_k^1\,(t-\lambda_k)^{j-N_k}\,\left\{(t-\lambda_k)\,g_{k-1}^0 + (j-N_{k-1})\,\psi_{k-1}\right\}. \end{cases}$$

We set

$$\begin{cases} d_{j,j}(t) = c_j\,t + a_j = c_j\,(t-\lambda_k) + a_j', \\ d_{j,i}(t) = c_{j,i}\,t + a_{j,i} = c_{j,i}\,(t-\lambda_k) + a_{j,i}' \\ (N_{k-1} < j \le N_k\,;\, k = q, q-1, \cdots, 1). \end{cases}$$

Then, we write $D(t)$ in the form

$$D(t) = \mathrm{diag}\,(\,(t-\lambda_1)\mathcal{E}_{n_1},\,(t-\lambda_2)\mathcal{E}_{n_2},\,\cdots,\,(t-\lambda_q)\mathcal{E}_{n_q}\,)\,C + A',$$

where $\mathcal{E}_\nu$ denotes $\nu\times\nu$ identity matrix and the first matrix in the right hand side is a block-diagonal matrix, and C is a lower triangular matrix, and A' is of the form

$$A' = \begin{pmatrix} a'_1 & 1 & & & & \\ a'_{2,1} & a'_2 & 1 & & & \mathbf{0} \\ & \ddots & \ddots & \ddots & & \\ & a'_{j,i} & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ a'_{n,1} & a'_{n,2} & \cdots & \cdots & \cdots & a'_n \end{pmatrix}.$$

First step (Relation between C and A')

[I] For the diagonal elements $d_{j,i}(t)$, the argument is identical to the one given in [AK], so we shall show only the conclusion for the following two cases. For the first case, with $j = N_k (k = 1, 2, \ldots, q)$:

$$a'_j = n_k - 1 - \frac{\widehat{e}_{j+1,\,j-1}(\lambda_k)}{\psi_{k-1}(\lambda_k)},$$

and

$$\begin{cases} d_{j,j}(t) = c_j\,(t - \lambda_k) + n_k - 1 - \dfrac{\widehat{e}_{j+1,\,j-1}(\lambda_k)}{\psi_{k-1}(\lambda_k)}, \\ e_{j,\,j-2}(t) = f^1_k\,\widehat{e}_{j,\,j-2}(t) \qquad (j = N_k), \end{cases}$$

where

$$\widehat{e}_{j,\,j-2}(t) = (t - \lambda_k)^{-1}\{\,\widehat{e}_{j+1,\,j-1}(t) + (a'_j - n_k + 1)\psi_{k-1}\,\} + c_j\psi_{k-1} - g^0_{k-1}.$$

For the second case, where $N_{k-1} < j \leq N_k - 1 (k = 1, 2, \ldots, q)$:

$$a'_j = j - 1 - N_{k-1},$$

$$\begin{cases} d_{j,j}(t) = c_j\,(t - \lambda_k) + j - 1 - N_{k-1}, \\ e_{j,\,j-2}(t) = f^1_k\,(t - \lambda_k)^{j-N_k}\,\widehat{e}_{j,\,j-2}(t) \qquad (N_{k-1} < j \leq N_k - 1), \end{cases}$$

where

$$\begin{aligned} \widehat{e}_{j,\,j-2}(t) &= \widehat{e}_{j+1,\,j-1}(t) + c_j\,\psi_{k-1} - g^0_{k-1} \\ &= \widehat{e}_{N_k,\,N_k-2}(t) + (c_{N_k-1} + c_{N_k-2} + \cdots + c_j)\psi_{k-1} - (N_k - j)g^0_{k-1}. \end{aligned}$$

For $j = 1$, the formula (1.12) becomes

$$e_{2,0}(t) + d_{1,1}(t) = 0 \qquad (a'_1 = 0). \tag{1.13}$$

Thus we have determined all diagonal elements a'_j of A' and the form of $d_{j,j}(t)$ and $e_{j,j-2}(t)$ ($j = n, n-1, \cdots, 2$) from $P_{n-1}(t)$.

We remark that the $e_{j,j-2}(t)$ are polynomials of degree at most $j-1$. It is easily seen that $e_{n,n-2}(t)$ has degree at most $n-1$, because it is derived by taking the quotient of $P_{n-1}(t)$ by powers of the factor $(t-\lambda_q)$. Then, each $e_{j,j-2}(t)$ is derived by taking the quotient of $e_{j+1,j-1}(t)$ by powers of the factor $(t-\lambda_k)$.

[II] Now we shall proceed to the investigation of the subdiagonal elements $d_{j,j-i}(t)$ ($i = 1, 2, \cdots, j-1$) and the coefficients of transformation $e_{j,j-i}(t)$ ($i = 1, 2, \cdots, j-1$).

We wish to prove that for $k = 1, 2, \cdots, q$:

$$e_{j,j-i}(t) = f_k^{i-1}(t-\lambda_k)^{j-N_k}\,\widehat{e}_{j,j-i}(t) \qquad (N_{k-1} < j \le N_k). \tag{1.14}$$

From now on, the factor to the power of a non-positive integer is understood to be equal to 1, that is, $(t-\lambda_k)^p \equiv 1$ $(p \le 0)$.

For $i = 2$, we have already seen

$$e_{j,j-2}(t) = f_k^1(t-\lambda_k)^{j-N_k}\,\widehat{e}_{j,j-2}(t),$$

and for $j = n+1$, we have

$$e_{n+1,n+1-i}(t) = P_{n-i+1}(t) = f_q^{i-1}\,\widehat{P}_{n-i+1}.$$

Taking account of these facts, we shall prove the formula (1.14) by mathematical induction in the subdiagonal order i. Setting $\ell = i-2$ ($i = 1, 2, \cdots, j$) in (1.12), we have for $N_{k-1} < j \le N_k$ ($k = 1, 2, \cdots, q$) :

$$\begin{aligned}&(t-\lambda_k)(e_{j,j-i-1}(t) + e'_{j,j-i}(t))\\ &= e_{j+1,j-i}(t) + \sum_{h=0}^{i-2} d_{j,j-h}(t)\,e_{j-h,j-i}(t) + d_{j,j-i+1}(t)\,\varphi_{j-i}.\end{aligned} \tag{1.15}$$

Suppose that the same formulas as (1.14) are valid for $j = N_k + 1$, for which one can express

$$e_{j,\,j-2}(t) = f_k^1\,(t\,-\,\lambda_k)^{j-N_k}\,\widehat{e}_{j,\,j-2}(t) \;=\; f_k^2\,\psi_{k-1}(t\,-\,\lambda_k)^{j-N_k+1}\,\widehat{e}_{j,\,j-2}(t),$$

$$e'_{j,\,j-2}(t) = f_k^2\,(t\,-\,\lambda_k)^{j-N_k}\Big[\,\{g_k^1\,+\,(j\,-\,N_k)\,\psi_{k-1}\}\,\widehat{e}_{j,\,j-2}(t)\,+\,\psi_k\widehat{e}'_{j,\,j-2}(t)\}\,\Big],$$

$$\varphi_{j-2} = f_k^0\,(t\,-\,\lambda_k)^{j-N_k-2} \;=\; f_k^2\,\psi_{k-1}^2\,(t\,-\,\lambda_k)^{j-N_k}.$$

Now set $i = 2$ in (1.15) and let $j = N_k$. Substituting above expressions into (1.15), we obtain

$$\begin{aligned}&(t\,-\,\lambda_k)\,e_{j,\,j-3}(t)\,-\,e_{j+1,\,j-2}(t)\\ &=\; f_k^2\,(t\,-\,\lambda_k)^{j-N_k+1}\,\Big[\,\{(d_{j,\,j}(t)\,-\,j\,+\,N_k)\,\psi_{k-1}\,-\,g_k^1\}\,\widehat{e}_{j,\,j-2}(t)\\ &-\,\psi_k\,\widehat{e}'_{j,\,j-2}(t)\Big]\,+\,d_{j,\,j-1}(t)\,f_k^2\,(t\,-\,\lambda_k)^{j-N_k}\,\psi_{k-1}^2.\end{aligned}$$

Moreover, since $e_{j+1,\,j-2}(t) = f_k^2\,\widehat{e}_{j+1,\,j-2}(t)$, we consequently obtain

$$\begin{aligned}e_{j,\,j-3}(t) = f_k^2\,(t\,-\,\lambda_k)^{-1}\,\{\,\widehat{e}_{j+1,\,j-2}(t)\,+\,a'_{j,\,j-1}\,\psi_{k-1}^2\,\}&\\ +\,f_k^2\,\Big[\,\{\,(d_{j,\,j}(t)\,-\,j\,+\,N_k)\,\psi_{k-1}\,-\,g_k^1\,\}\,\widehat{e}_{j,\,j-2}(t)&\\ +\,c_{j,\,j-1}\,\psi_{k-1}^2\,-\,\psi_k\,\widehat{e}'_{j,\,j-2}(t)\Big].&\end{aligned}$$

This immediately leads to

$$a'_{j,\,j-1} \;=\; -\frac{\widehat{e}_{j+1,\,j-2}(\lambda_k)}{\psi_{k-1}^2(\lambda_k)} \qquad (j = N_k), \tag{1.16}$$

and

$$\begin{cases} d_{j,\,j-1}(t) = c_{j,\,j-1}\,(t\,-\,\lambda_k)\,-\,\dfrac{\widehat{e}_{j+1,\,j-2}(\lambda_k)}{\psi_{k-1}^2(\lambda_k)}, \\ e_{j,\,j-3}(t) = f_k^2\,\widehat{e}_{j,\,j-3}(t) \qquad (j = N_k). \end{cases}$$

Then, substituting the above form of $e_{j+1,\,j-2}(t)$ into (1.15), we have

$$\begin{aligned} e_{j,\,j-3}(t) = f_k^2\,(t\,-\,\lambda_k)^{j-N_k} & \Big[\{(d_{j,\,j}(t)\,-\,j\,+\,N_k)\psi_{k-1}\,-\,g_k^1\}\widehat{e}_{j,\,j-2}(t) \\ & -\,\psi_k\,\widehat{e}'_{j,\,j-2}(t)\,+\,\widehat{e}_{j+1,\,j-2}(t)\,+\,c_{j,\,j-1}\,\psi_{k-1}^2\Big] \\ & +\,f_k^2\,(t\,-\,\lambda_k)^{j-N_k-1}\,a'_{j,\,j-1}\,\psi_{k-1}^2. \end{aligned}$$

From this, for $j\,=\,N_k-1,\,N_k-2,\,\cdots,\,N_{k-1}+2$, successively, we can conclude that there hold

$$\begin{cases} a'_{j,\,j-1} = 0, \\ d_{j,\,j-1}(t) = c_{j,\,j-1}\,(t\,-\,\lambda_k), \\ e_{j,\,j-3}(t) = f_k^2\,(t\,-\,\lambda_k)^{j-N_k}\,\widehat{e}_{j,\,j-3}(t). \end{cases}$$

In case $j = N_{k-1}+1$, the polynomials $e_{j,\,j-2}(t) = f_{k-1}^1\,\widehat{e}_{j,\,j-2}(t)$ and $\varphi_{j-2} = f_{k-1}^1\psi_{k-2}$ do not include the factor $(t-\lambda_k)$. Then, in this case, $a'_{j,\,j-1}$ can be determined by putting $t=\lambda_k$ in (1.15). In fact, in this case the formula (1.15) becomes

$$\begin{aligned} e_{j,\,j-3}(t) = \ & f_{k-1}^2\,(t\,-\,\lambda_k)^{-1} \\ & \times\,\Big\{\widehat{e}_{j+1,\,j-2}(t)\,+\,a'_j\,\psi_{k-1}\,\widehat{e}_{j,\,j-2}(t)\,+\,a'_{j,\,j-1}\,\psi_{k-2}^2(t\,-\,\lambda_{k-1})\Big\} \\ & +\,f_{k-1}^2\Big[c_j\,\psi_{k-1}\,\widehat{e}_{j,\,j-2}(t)\,+\,c_{j,\,j-1}\,\psi_{k-2}^2\,(t\,-\,\lambda_{k-1}) \\ & -\,\{\,g_{k-1}^1\,\widehat{e}_{j,\,j-2}(t)\,+\,\psi_{k-1}\,\widehat{e}'_{j,\,j-2}(t)\,\}\Big]. \end{aligned}$$

From this, we immediately obtain

$$\begin{cases} a'_{j,\,j-1} = -\dfrac{\widehat{e}_{j+1,\,j-2}(\lambda_k)}{\psi_{k-2}^2(\lambda_k)\,(\lambda_k\,-\,\lambda_{k-1})}\,-\,\dfrac{a'_j\,\widehat{e}_{j,\,j-2}(\lambda_k)}{\psi_{k-2}(\lambda_k)}, \\ d_{j,\,j-1}(t) = c_{j,\,j-1}(t\,-\,\lambda_k)\,-\,\dfrac{\widehat{e}_{j+1,\,j-2}(\lambda_k)}{\psi_{k-2}^2(\lambda_k)\,(\lambda_k\,-\,\lambda_{k-1})}\,-\,\dfrac{a'_j\,\widehat{e}_{j,\,j-2}(\lambda_k)}{\psi_{k-2}(\lambda_k)}, \\ e_{j,\,j-3}(t) = f_{k-1}^2\,\widehat{e}_{j,\,j-3}(t). \end{cases}$$

Consecutive calculations for $k\,=\,q,\,q-1,\,\cdots,\,1$ finally leads to the determination of all subdiagonal elements $a'_{j,\,j-1}$ and the form of $d_{j,\,j-1}(t)$ and $e_{j,\,j-3}(t)$ in terms of $P_{n-2}(t)$.

[III] We shall sketch the proof of (1.14) by mathematical induction.

Suppose the formulas (1.14) are valid up to $(i-2)$-th subdiagonal elements, that is,

$$e_{j,\,j-\ell-2}(t), \qquad d_{j,\,j-\ell}(t) \qquad (\,\ell\,=\,0,\,1,\,\cdots\,,\,i-2\,)$$

are known. Then, we can prove that the formula (1.14) holds for the $(i-1)$st subdiagonal elements $e_{j,\,j-i-1}(t)$, together with the determination of the form of $d_{j,\,j-i+1}(t)$ in terms of $P_{n-i}(t)$.

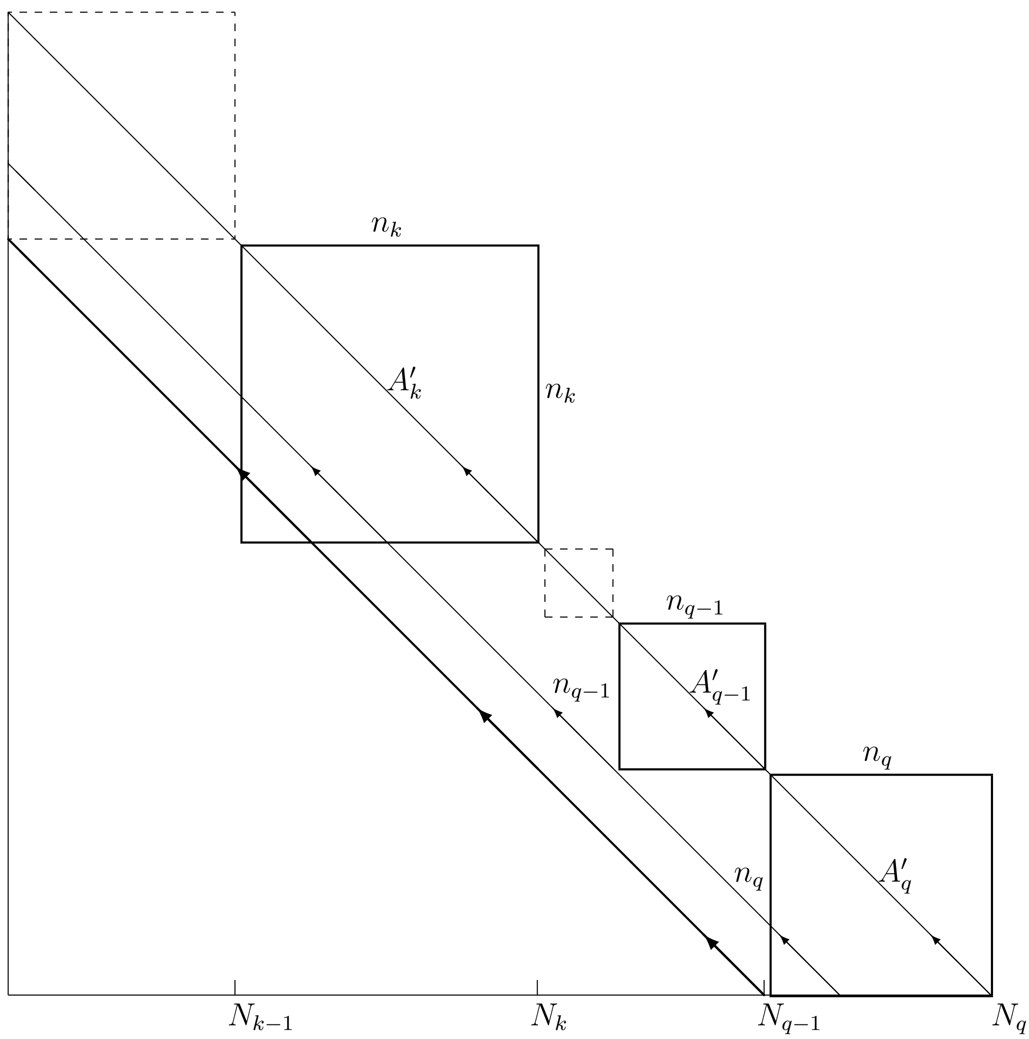

Figure 1: Procedure of Mathematical Induction

The remainder of the calculation is similar to [II], so we shall show only the conclusion:

$$\begin{cases} a'_{j,\,j-i+1} \;=\; 0, \\ \\ d_{j,\,j-i+1}(t) \;=\; c_{j,\,j-i+1}\,(t\,-\,\lambda_k) \\ \\ \qquad\qquad (j\,=\,N_k\,-\,1,\,N_k\,-\,2,\,\cdots\,,\,N_{k-1}+i, 2 \le i \le n_k), \end{cases}$$

and also see that $e_{j,\,j-i-1}(t)$ has the form (1.14). This result implies that the sub-matrix $D_k(\lambda_k)$ corresponding to the regular singular point $t = \lambda_k$ has the following form :

$$D_k(\lambda_k) \;=\; \begin{pmatrix} a'_{N_{k-1}+1,\,N_{k-1}+1} & 1 & & & \mathbf{0} \\ & a'_{N_{k-1}+2,\,N_{k-1}+2} & 1 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & \ddots & 1 \\ a'_{N_k,\,N_{k-1}+1} & a'_{N_k,\,N_{k-1}+2} & \cdots & \cdots & a'_{N_k,\,N_k} \end{pmatrix}.$$

For the case $N_{k-1} < j \le N_{k-1} + i - 1,\ 2 \le i \le n_k$ or $N_{k-1} < j \le N_k,\ i > n_k$, we can also see that $e_{j,j-i-1(t)}$ has the form (1.14) by same way to [II].

Continuing the above procedure of calculations for all blocks ($N_{k-1} < j \le N_k$: $k = q,\, q-1,\, \cdots\,,\, 1$), one can determine $(i-1)$st subdiagonal elements $a'_{j,\,j-i+1}$ from $P_{n-i}(t)$. Thus the proof of mathematical induction is completed. □

Second step (Determination of C)

In the first step, we showed that the constant matrix A' is expressed in the form

$$
A' = \left(\begin{array}{c|c|ccc}
A'_1 & & & & \mathbf{0} \\
 & 1 & & & \\ \hline
 & A'_2 & & & \\
 & & 1 & & \\ \cline{2-3}
a'_{j,i} & & \ddots & \ddots & \\
 & & & & 1 \\ \cline{3-5}
 & & & & A'_q
\end{array} \right),
$$

where each diagonal block is an $n_k \times n_k$ companion matrix :

$$
A'_k = \begin{pmatrix}
0 & 1 & & & \mathbf{0} \\
 & 1 & 1 & & \\
\mathbf{0} & & \ddots & \ddots & \\
 & & & n_k - 2 & 1 \\
a'_{N_k,\, N_{k-1}+1} & a'_{N_k,\, N_{k-1}+2} & \cdots & \cdots & a'_{N_k,\, N_k}
\end{pmatrix}.
$$

Then, the matrix A' includes

$$\frac{1}{2}\, n^2 \;-\; \frac{1}{2} \sum_{k=1}^{q} n_k^2 \;+\; n$$

constants to be determined. Since C is a triangular matrix, it includes

$$\frac{1}{2}\, n(n+1)$$

constants. Hence, the reduced system of differential equations includes

$$\frac{n(2n+3)}{2} \;-\; \frac{1}{2} \sum_{k=1}^{q} n_k^2$$

constants to be determined, the number of which is equal to $\mathcal{N}$ in Section 1.1.

Now we shall prove that the above $\mathcal{N}$ constants are determined by the constants of $P_{n-j}(t)$ $(j = 1, 2, \cdots, n)$. Since we have clarified the relations between $a'_{j,i}$ and $c_{j,i}$, we have only to determine the constants $c_{j,i}$.

For that purpose, we first investigate the degrees of the polynomials of transformation $e_{j,i}(t)$. We have already shown that each $e_{j,j-2}(t)$ is a polynomial of degree at most $(j-1)$. Here we also prove by induction that the degree of the polynomial $e_{j,i}(t)$ is at most $j-1$.

We rewrite (1.12) in the form

$$\begin{aligned}(t - \lambda_j)\,(e_{j,j-k-1}(t) + e'_{j,j-k}(t)) \\ = e_{j+1,j-k}(t) + \sum_{h=0}^{k-1} d_{j,j-h}(t)\,e_{j-h,j-k}(t),\end{aligned} \tag{1.17}$$

where $e_{j-k+1,j-k}(t) = \varphi_{j-k}$, and, in particular, for $j = n$, we have

$$\begin{aligned}(t - \lambda_q)(e_{n,n-k-1}(t) + e'_{n,n-k}(t)) \\ = -P_{n-k}(t) + \sum_{h=0}^{k-1} d_{n,n-h}(t)\,e_{n-h,n-k}(t).\end{aligned} \tag{1.18}$$

In the formula (1.18) for $k = 2$, the right hand side is easily seen to be a polynomial of degree at most n, and hence $e_{n,n-3}(t)$ becomes a polynomial of degree at most $n-1$. And then, from the formula (1.17) for $j = n-1, n-2, \cdots, 3$, successively, one can verify that each $e_{j,j-3}(t)$ is a polynomial of degree at most $j-1$, because polynomials with the highest degree in the right hand side of (1.17) are $e_{j+1,j-2}(t)$ and $d_{j,j}(t)\,e_{j,j-2}(t)$.

In general, suppose that for $h = 2, 3, \cdots, k$, the polynomials

$$e_{\ell,\ell-h}(t) \qquad (\ell = n, n-1, \cdots, h)$$

are known and each $e_{\ell,\ell-h}(t)$ is of degree at most $\ell - 1$. Then, the right hand side of (1.18) is known as a polynomial of degree at most n. Dividing it by the factor $(t - \lambda_q)$, we obtain $e_{n,n-k-1}(t)$ as a polynomial of degree $(n-1)$. Furthermore, suppose that each $e_{\ell,\ell-k-1}(t)$ $(\ell = n, n-1, \cdots, j+1)$ is known as a polynomial of degree $\ell - 1$. Then, from (1.17) we can immediately see that $e_{j,j-k-1}(t)$ is a polynomial of degree at most $j-1$, because the degree of the right hand side is j. Thus, we have verified that $e_{j,i}(t)$ is a polynomial of degree at most $j-1$. □

We are now in a position to determine the constants $c_{j,\,i}$.
For $N_{k-1} \,<\, j \,\leq\, N_k$ and $i \,=\, j$, we have

$$(t \,-\, \lambda_k)\, e'_{j,\,0}(t) \,-\, e_{j+1,\,0}(t) \,-\, \sum_{h=0}^{j-2} d_{j,\,j-h}(t)\, e_{j-h,\,0}(t) \,=\, d_{j,\,1}(t). \tag{1.19}$$

Since $e_{j,\,i}(t)$ is a polynomial of degree at most $j-1$, the degree of the left hand side of (1.19) is at most j. However, the degree of $d_{j,\,1}(t)$ is 1. Then, the coefficients of t^ℓ $(\ell \,=\, j,\, j-1,\, \cdots,\, 2)$ must be vanishing. Assigning zero to them, the formulas (1.19), together with (1.13), determine $d_{j,\,1}(t)$, i.e., the constants $c_{j,\,1}$.

Consequently, we obtain

$$\sum_{j=2}^{n} (j-1) \,+\, n \,=\, \frac{n(n+1)}{2}$$

equations determining the same number of $c_{j,\,i}$. This is sufficient, because the system determined by comparing coefficients in (1.19) is triangular given a suitable ordering of variables.

The next result is useful because, as Kohno mentions in [K3], one may solve the multiple point connection problem for the generalized Schlesinger system.

Proposition 1.4. *The system (1.10) can be reduced to a generalized Schlesinger system:*

$$\frac{dX}{dt} = \left(\sum_{j=1}^{q} \frac{\bar{A}_j}{t-\lambda_j} + C \right) X,$$

with $\bar{A}_j (j = 1, 2, \ldots, q)$ being n by n constant matrices.

Proof. Recall that we set $D(t) := A + Ct$ and $D(t)$ can be also rewritten

$$D(t) \,=\, \mathrm{diag}\,(\,(t-\lambda_1)I_{n_1},\, (t-\lambda_2)I_{n_2},\, \cdots,\, (t-\lambda_q)I_{n_q}\,)\,C \,+\, A',$$

where I_ν denotes the $\nu \times \nu$ identity matrix and the first matrix in the right hand side is a block-diagonal matrix. We set

$$A' = \begin{pmatrix} A'_{11} & \cdots & A'_{1q} \\ \cdots & \cdots & \cdots \\ A'_{q1} & \cdots & A'_{qq} \end{pmatrix}, \quad A_{ij} \in M(n_i, n_j; \mathbb{C}).$$

For each $j = 1, 2, \ldots, q$, setting

$$\bar{A}_j = \begin{pmatrix} & 0 & \\ A'_{j1} & \cdots & A'_{jp} \\ & 0 & \end{pmatrix},$$

and multiplying both sides of (1.10) by $(tI - B)^{-1}$ from the left, we obtain a generalized Schlesinger system. □

1.3 General case: different degrees

We consider the more general case where the degree m of the polynomial may be strictly more than the differential degree n of the differential equation.
Let $m < n$. Then, we consider a linear differential equation of the form

$$\sum_{j=0}^{m} Q_j(t)\, y^{(j)} \;=\; 0, \tag{1.15}$$

where all the coefficients $Q_j(t)$ are polynomials of degree at most n and $Q_m(t)$ has the same form (1.4) as follows:

$$Q_m(t) = (t - \lambda_1)^{n_1}(t - \lambda_2)^{n_2} \cdots (t - \lambda_q)^{n_q},$$

with $1 \le n_\nu \le m$ $(\nu = 1, 2, \cdots, q)$ and $n_1 + n_2 + \cdots + n_q = n$. Moreover, it is assumed that for each ν the functions

$$\frac{(t - \lambda_\nu)^i Q_{m-i}(t)}{Q_m(t)} \qquad (i = 1, 2, \cdots, n_\nu)$$

are holomorphic at $t = \lambda_\nu$.

Hence the linear differential equation (1.15) has regular singular points at $t = \lambda_\nu\, (\nu = 1, 2, \cdots, q)$ and an irregular singularity of rank 1 at $t = \infty$.

Now, in order to apply our method of reduction, we first have to rewrite (1.15) in the form (1.3), that is, by differentiating in $(n - m)$ times, we derive the n - th order differential equation

$$\sum_{j=0}^{m} (\, Q_j(t)\, y^{(j)}\,)^{(n-m)} \;=\; 0. \tag{1.16}$$

By the Leibniz rule, we have

$$\sum_{j=0}^{m}\left\{\sum_{k=0}^{n-m} {}_{n-m}C_{n-m-k}\ Q_j(t)^{(n-m-k)}\, y^{(j+k)}\right\} \;=\; 0.$$

Then, setting $j+k=n-\ell$, we obtain

$$\sum_{\ell=0}^{n}\left\{\sum_{j=0}^{m} {}_{n-m}C_{\ell-m+j}\ Q_j(t)^{(\ell-m+j)}\right\} y^{(n-\ell)} \;=\; 0, \tag{1.17}$$

where the binomial coefficient is interpreted as follows:

$${}_pC_q \;\equiv\; 0 \qquad (\,q\;<\;0,\;q\;>\;p\,).$$

Moreover, setting

$$P_n(t) \;=\; Q_m(t),$$

and

$$\begin{aligned} P_{n-\ell}(t) &= -\sum_{j=0}^{m} {}_{n-m}C_{\ell-m+j}\ Q_j(t)^{(\ell-m+j)} \\ &= -\sum_{h=0}^{\ell} {}_{n-m}C_h\ Q_{m-\ell+h}(t)^{(h)} \end{aligned}$$

for $\ell = 1,\,2,\,\cdots,\,n$, we obtain exactly the same linear differential equation as (1.3).

We shall now investigate the singularities of the linear differential equation (1.17). From the expression, it is easy to see that all $P_\ell(t)\;\;(0\,\le\,\ell\,\le n)$ are polynomials of degree at most n. More precisely, we verify that for $0\le\ell\le m$, the $P_{n-\ell}(t)$ are polynomials of degree at most n , however, for $\ell\ge m{+}1$, the $P_{n-\ell}(t)$ are polynomials of degree less than n, the fact of which can be seen directly from the expression

$$P_{n-\ell}(t) \;=\; -\sum_{h=\ell-m\,\ge 1}^{\ell} {}_{n-m}C_h\ Q_{m-\ell+h}(t)^{(h)}.$$

Hence, $t=\infty$ is an irregular singularity of rank 1 and then it is not difficult to derive the characteristic constant μ as roots of the characteristic equation

$$\mu^{n-m}\left\{\sum_{i=0}^{m} q_{m-i}\,\mu^{m-i}\right\} \;=\; 0,$$

the q_{m-i} being coefficients of the highest degree t^n in Q_{m-i}.

On the other hand, for each ν $(1 \le \nu \le q)$, the polynomials $Q_{m-\ell}(t)$ $(0 \le \ell \le n_\nu)$ have the factor $(t-\lambda_\nu)^{n_\nu-\ell}$ and hence their derivatives $Q_{m-\ell}(t)^{(h)}$ have at least the factor $(t-\lambda_\nu)^{n_\nu-\ell-h}$. Taking account of this fact, together with the above expression of $P_{n-\ell}(t)$, we can see that for $1 \le \ell \le n_\nu$ the polynomial $P_{n-\ell}(t)$ includes the factor $(t-\lambda_\nu)^{n_\nu-\ell}$.

Therefore, the points $t = \lambda_\nu$ $(1 \le \nu \le q)$ are regular singularities.

We shall investigate the characteristic exponents of convergent power series solutions near regular singular points $t = \lambda_\nu$ $(\nu = 1, 2, \cdots, q)$.

For each ν, denoting

$$\begin{cases} O_{m-i}(t) \;=\; (t-\lambda_\nu)^{n_\nu-i}\,\widehat{Q}^{\nu}_{m-i}(t), \\ r^{\nu}_{m-i} \;=\; \widehat{Q}^{\nu}_{m-i}(\lambda_\nu) \qquad (i = 1, 2, \cdots, n_\nu), \end{cases}$$

we rewrite the linear differential equation (1.15) in the form

$$\sum_{i=0}^{n_\nu} (t-\lambda_\nu)^{n_\nu-i}\,\widehat{Q}^{\nu}_{m-i}(t)\,y^{(m-i)} \;+\; \sum_{i=n_\nu+1}^{m} Q_{m-i}(t)\,y^{(m-i)} \;=\; 0. \tag{1.18}$$

Substituting a power series of the form

$$y(t) = (t-\lambda_\nu)^{\rho}\,\{\, g_0 \;+\; g_1\,(t-\lambda_\nu) \;+\; \cdots \;\}$$

into (1.18), we have

$$(t-\lambda_\nu)^{\rho+n_\nu-m}\,f_\nu(\rho)\,g_0 \;+\; d_1\,(t-\lambda_\nu)^{\rho+n_\nu-m+1} \;+\; \cdots \;\;=\; 0,$$

where we have set

$$f_\nu(\rho) \;\equiv\; \sum_{i=0}^{n_\nu} r^{\nu}_{m-i}\,[\,\rho\,]_{m-i}.$$

Hence, we see that the characteristic exponents are given by roots of

$$f_\nu(\rho) \;=\; 0,$$

that is, by integral roots of $[\,\rho\,]_{m-n_\nu} \;=\; 0$ and roots of

$$f_\nu(\rho - m + n_\nu) \;=\; 0.$$

Next, we consider the characteristic exponents of the linear differential equation (1.17). In this case, we have only to substitute the above power series into (1.15), obtaining

$$(t-\lambda_\nu)^{\rho+n_\nu-n}\, f_\nu(\rho)\,[\rho+n_\nu-m]_{n-m}\, g_0 \;+\; \widehat{d_1}\,(t-\lambda_\nu)^{\rho+n_\nu-n+1} \;+\; \cdots \;=\; 0.$$

Hence, the characteristic exponents are given by the characteristic equation

$$f_\nu(\rho)\,[\rho+n_\nu-m]_{n-m} \;=\; 0,$$

that is, by the same roots as above and integral roots of $[\rho+n_\nu-m]_{n-m} \;=\; 0$.

As we see, the transformation of (1.15) to (1.17) yields no essential changes of behaviors of solutions.

1.4 Example of the reduction

In this section, we consider an example of the reduction of a fourth order linear differential equation. We apply our theory, with unknown function y to the following equation:

$$t\,(t\,-\,1)^3\, y^{(4)} \;=\; P_3(t)\, y^{(3)} \;+\; P_2(t)\, y'' \;+\; P_1(t)\, y' \;+\; P_0(t)\, y, \qquad (1.19)$$

where

$$\begin{cases} P_3(t) \;=\; -\,(t\,-\,1)^2\,(5\,t\,+\,1), \\ P_2(t) \;=\; -\,(t\,-\,1)^2\,(t\,-\,4), \\ P_1(t) \;=\; t^4\,-\,t^3\,-\,t^2\,+\,21\,t\,-\,8, \\ P_0(t) \;=\; 4\,t^3\,-\,3\,t^2\,+\,4\,t\,+\,3. \end{cases}$$

This linear differential equation has regular singularities at $t \;=\; 0,\; 1$ and an irregular singular point of rank one at infinity. Now, we consider the reduction of (1.19) to a system of linear differential equations of the form $(t\,I\,-\,B)\,Y' \;=\; (A\,+\,C\,t)\,Y$ by the transformation

$$\begin{cases} y_1 \;=\; y, \\ y_2 \;=\; \varphi_1\, y' \;+\; e_{2,0}(t)\, y, \\ y_3 \;=\; \varphi_2\, y'' \;+\; e_{3,1}(t)\, y' \;+\; e_{3,0}(t)\, y, \\ y_4 \;=\; \varphi_3\, y^{(3)} \;+\; e_{4,2}(t)\, y'' \;+\; e_{4,1}(t)\, y' \;+\; e_{4,0}(t)\, y, \end{cases}$$

where I is the 4×4 identity matrix, A and C are 4 by 4 constant matrices with C lower triangular, B is a diagonal matrix:

$$B \;=\; \mathrm{diag}(1,\,1,\,1,\,0),$$

$\{y_j\}_{j=1}^4$ are defined using the unknown function y from (1.19), $\varphi_1 = t - 1$, $\varphi_2 = (t - 1)^2$, $\varphi_3 = (t - 1)^3$, and the coefficients $e_{i,j}(t)$ ($i = 2, 3, 4$, $j = 0, \dots, i - 2$) are polynomials in t.

Then, we have the following relations:

$$
\begin{array}{ll}
① & t\,(e_{4,2}(t) + \varphi_3') \;=\; -P_3 + d_{4,4}\,\varphi_3, \\
② & t\,(e_{4,1}(t) + e_{4,2}'(t)) \;=\; -P_2 + d_{4,4}\,e_{4,2}(t) + d_{4,3}\,\varphi_2, \\
③ & t\,(e_{4,0}(t) + e_{4,1}'(t)) \;=\; -P_1 + d_{4,4}\,e_{4,1}(t) + d_{4,3}\,e_{3,1}(t) + d_{4,2}\,\varphi_1, \\
④ & t\,e_{4,0}'(t) \;=\; -P_0 + d_{4,4}\,e_{4,0}(t) + d_{4,3}\,e_{3,0}(t) + d_{4,2}\,e_{2,0}(t) + d_{4,1}, \\
⑤ & (t - 1)\,(e_{3,1}(t) + \varphi_2') \;=\; e_{4,2}(t) + d_{3,3}\,\varphi_2, \\
⑥ & (t - 1)\,(e_{3,0}(t) + e_{3,1}'(t)) \;=\; e_{4,1}(t) + d_{3,3}\,e_{3,1}(t) + d_{3,2}\,\varphi_1, \\
⑦ & (t - 1)\,e_{3,0}'(t) \;=\; e_{4,0}(t) + d_{3,3}\,e_{3,0}(t) + d_{3,2}\,e_{2,0}(t) + d_{3,1}, \\
⑧ & (t - 1)\,(e_{2,0}(t) + \varphi_1') \;=\; e_{3,1}(t) + d_{2,2}\,\varphi_1, \\
⑨ & (t - 1)\,e_{2,0}'(t) \;=\; e_{3,0}(t) + d_{2,2}\,e_{2,0}(t) + d_{2,1}, \\
⑩ & -\,e_{2,0}(t) \;=\; d_{1,1}.
\end{array}
$$

In the above, $d_{j,i}$ are polynomials of degree 1 in t. Avoiding the difficult calculation, we shall show just the order of steps in our reduction algorithm. First, we calculate the principal diagonal elements $d_{i,i}(t)$ ($i = 4, 3, 2, 1$). We follow the order of calculation ①→ ⑤→ ⑧ → ⑩.

Next, we shall proceed to the calculation of the first subdiagonal elements $d_{i,i-1}(t)$ ($i = 4, 3, 2$) by following ②→ ⑥→ ⑨.

In order to determine the second subdiagonal elements $d_{i,i-2}(t)$ ($i = 4, 3$), we follow the order of calculations ③→ ⑦.

Lastly, from ④ we obtain the value of $d_{4,1}(t)$.

We have thus determined all coefficients $e_{i,j}(t)$ of the transformation and the elements of $d_{j,i}(t)$ as follows:

$$
\left\{
\begin{array}{lcl}
e_{2,0}(t) & = & -\omega^2\,(t - 1), \\
e_{3,0}(t) & = & (t - 1)\,(t - \omega), \\
e_{3,1}(t) & = & (t - 1)^2, \\
e_{4,0}(t) & = & -\,t^3 + 3\,t^2 + (\omega - 3)\,t - (\omega + 5), \\
e_{4,1}(t) & = & 3(t - 1)(t - 3), \\
e_{4,2}(t) & = & 3(t - 1)^2,
\end{array}
\right.
$$

$$\begin{cases} d_{1,1} &= \omega^2 (t - 1), \\ d_{2,1} &= (\omega - 1)(t - 1), \\ d_{2,2} &= \omega (t - 1) + 1, \\ d_{3,1} &= -(9\omega + 4)(t - 1) + 6, \\ d_{3,2} &= -(\omega - 1)(t - 1) + 6, \\ d_{3,3} &= (t - 1) - 1, \\ d_{4,1} &= -(17\omega + 12)\, t + (18\omega + 26), \\ d_{4,2} &= (\omega + 2)\, t + 18, \\ d_{4,3} &= 2t + 1, \\ d_{4,4} &= 1, \end{cases}$$

where ω is a non-real root of $\omega^3 - 1 = 0$.

Consequently, we can reduce the single linear differential equation (1.19) to a system of linear differential equations of the form

$$(t\,I - B)\frac{dY}{dt} = (A + C\,t)\,Y = \{\mathrm{diag}\,(\,(t - 1)I_{n_1},\, t\,)\,C + \bar{A}\}Y, \tag{1.20}$$

where for the rest of this paper, I_ν denotes the ν by ν identity matrix, and A, $\bar{A}$ and C are the constant matrices given as follows:

$$A = \begin{pmatrix} -\omega^2 & 1 & 0 & 0 \\ -(\omega - 1) & -(\omega - 1) & 1 & 0 \\ 9\omega + 10 & \omega + 5 & -2 & 1 \\ 18\omega + 26 & 18 & 1 & 1 \end{pmatrix},$$

$$\bar{A} = \begin{pmatrix} \bar{a}_1 & 1 & 0 & 0 \\ \bar{a}_{2,1} & \bar{a}_2 & 1 & 0 \\ \bar{a}_{3,1} & \bar{a}_{3,2} & \bar{a}_3 & 1 \\ \bar{a}_{4,1} & \bar{a}_{4,3} & \bar{a}_{4,3} & \bar{a}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 6 & 6 & -1 & 1 \\ 18\omega + 26 & 18 & 1 & 1 \end{pmatrix},$$

and

$$C = \begin{pmatrix} \omega^2 & 0 & 0 & 0 \\ \omega - 1 & \omega & 0 & 0 \\ -(9\omega + 4) & -(\omega - 1) & 1 & 0 \\ -17\omega - 12 & \omega + 2 & 2 & 0 \end{pmatrix}.$$

We will describe how to compute the entries of $\bar{A}$ in the next section.

2 Restricted Fuchsian relation

In Section 2.1, we shall compute the difference between characteristic exponents of a single differential equation and of the corresponding system. In Section 2.2, we will show that the nonholomorphic solutions for the single differential equation satisfy the restricted Fuchsian relation, by employing a method of Kohno. To this end, we compute the restricted Fuchsian relations of the single differential equation and the system of differential equations, and we will compare them (cf.[A1]). That is, we shall take a sum of the characteristic exponents obtained in Section 2.1, and the characteristic exponents for irregular singular points.

2.1 Characteristic exponents and constants

We shall investigate the characteristic exponents for the regular singular points and the characteristic constants for the irregular singular point for the differential equation (1.3) and the output (1.10) of the reduction. In order to show that our reduction preserves monodromic properties of the solutions for differential equations at the regular singular points, we will show that the difference between the sum of the characteristic exponents for the regular singular points of (1.3) and the corresponding sum for (1.10) is an integer. The calculation of this difference will appear in the proposition at the end of this section. In the reduction from (1.3) to (1.10), all entries $e_{j,i}(t)$ of the transformation matrix are polynomials in t. We shall show here that the characteristic exponents at each regular singular point of both (1.3) and (1.10) are invariant modulo integers.

We recall the notation of [A1] for obtaining the restricted Fuchsian relation of (1.3) and (1.10). We introduce the notation N_k, f_k^i and ψ_k where $k = 1, 2, \cdots, q$

and i is an integer index (not an exponent).

$$\begin{cases} N_k = n_1 + n_2 + \cdots + n_k \quad (k = 1, 2, \ldots, q), \\ f_k^i = \displaystyle\prod_{\nu=1}^{k} (t - \lambda_\nu)^{n_\nu - i}, \\ \psi_k = \displaystyle\prod_{\nu=1}^{k} (t - \lambda_\nu), \end{cases}$$

where $N_0 \equiv 0$, then we know that

$$\begin{cases} N_q = n, \\ f_k^i = f_k^{i+1} \, \psi_k. \end{cases}$$

Then, we can rewrite the coefficients (1.5) of (1.3) as follows:

$$\begin{cases} P_{n-i}(t) = f_k^i \, \widehat{P}_{n-i}(t) \quad (n_{k+1} < i \leq n_k \, ; \, k = 1, 2, \ldots, q), \\ P_{n-i}(t) = \widehat{P}_{n-i}(t) \quad (n_1 < i \leq n), \end{cases}$$

where $n_{q+1} \equiv 0$.

As we saw in the introduction, in the punctured disc $0 < |t - \lambda_\nu| < r(\nu = 1, 2, \ldots, q)$, there exists at least one solution of (1.3)

$$y(t) = (t - \lambda_\nu)^\rho \sum_{m=0}^{\infty} g(m) \, (t - \lambda_\nu)^m.$$

Substituting it into (1.3), we find that the characteristic exponent ρ is a root of the equation

$$[\rho]_n = \sum_{i=1}^{n_\nu} \gamma_i \, [\rho]_{n-i},$$

where $[\rho]_k$ is the Pochhammer symbol for $k = 0, 1, 2, \ldots,$ defined by the following recursion:

$$[\rho]_k = \rho(\rho - 1) \cdots (\rho - k + 1), \quad [\rho]_0 \equiv 1,$$

and the coefficients γ_i are given by

$$\gamma_i = \left[\frac{P_{n-i}(t)}{P_n(t)}(t-\lambda_\nu)^i\right]_{t=\lambda_\nu}$$

$$= \frac{\widehat{P}_{n-i}(\lambda_\nu)}{\displaystyle\prod_{\substack{\ell=1\\ \ell\neq\nu}}^{k-1}(\lambda_\nu-\lambda_\ell)^i \prod_{\ell=k}^{q}(\lambda_\nu-\lambda_\ell)^{n_\ell}} \qquad (n_k < i \leq n_{k-1} \leq n_\nu).$$

Then, ρ is a root of

$$[\rho]_{n-n_\nu} = 0, \tag{2.1}$$

or a root of

$$[\rho - n + n_\nu]_{n_\nu} = \sum_{i=1}^{n_\nu} \gamma_i\, [\rho - n + n_\nu]_{n_\nu - i}. \tag{2.2}$$

For (1.3), (2.1) and (2.2) imply that there exist $n - n_\nu$ holomorphic solutions and n_ν possibly nonholomorphic solutions in the punctured disc $0 < |t - \lambda_\nu| < r$.

Next, we shall consider the characteristic exponents and constants for (1.10). For (1.10), there also exist $(n-n_\nu)(\nu = 1, 2, \ldots, q)$ holomorphic solutions and n_ν possibly nonholomorphic solutions near each singular point $t = \lambda_\nu$. For the calculation, we shall rewrite (1.10) by setting $\bar{a}_j := a_j - \lambda_k c_j$, and $\bar{a}_{j,i} := a_{j,i} - \lambda_k c_{j,i}$ where a_j and $a_{j,i}$ are entries of A and c_j and $c_{j,i}$ are entries of C of (1.10). $a_{j,i}$ and $c_{j,i}$ are the (j,i)-entries of A and C, respectively, and $a_j := a_{j,j}$ and $c_j := c_{j,j}$ are j-th diagonal entries. That is, we obtain the formula:

$$\begin{cases} d_{j,j}(t) = c_j\, t + a_j = c_j\,(t-\lambda_k) + \bar{a}_j, \\ d_{j,i}(t) = c_{j,i}\, t + a_{j,i} = c_{j,i}\,(t-\lambda_k) + \bar{a}_{j,i} \\ (N_{k-1} < j \leq N_k\,;\; k = q,\, q-1,\, \ldots,\, 1, i = 1, 2, \ldots, j-1). \end{cases}$$

Then, we rewrite the right hand side of (1.10) in the form

$$\{\mathrm{diag}((t-\lambda_1)I_{n_1}, (t-\lambda_2)I_{n_2}, \cdots, (t-\lambda_q)I_{n_q})C + \bar{A}\}Y,$$

where C is a lower triangular constant matrix, and $\bar{A}$ is of the form

$$\bar{A} = \begin{pmatrix} \bar{a}_1 & 1 & & & \\ \bar{a}_{2,1} & \bar{a}_2 & 1 & & \mathbf{0} \\ \bar{a}_{3,1} & \bar{a}_{3,2} & \ddots & \ddots & \\ \vdots & \vdots & \ddots & \ddots & 1 \\ \bar{a}_{n,1} & \bar{a}_{n,2} & \cdots & \bar{a}_{n,n-1} & \bar{a}_n \end{pmatrix}.$$

In [A1], to find the order of $e_{j,\,j-i}(t)$, we showed that for $k = 1, 2, \dots, q$

$$e_{j,\,j-i}(t) = f_k^{i-1}\,(t - \lambda_k)^{j-N_k}\,\widehat{e}_{j,\,j-i}(t) \qquad (\,N_{k-1} < j \le N_k, i = 1, 2, \dots, j-1),$$

where $(t - \lambda_k)^p \equiv 1$ $(p \le 0)$ and $\widehat{e}_{j,\,j-i}(t)$ is a polynomial of t, and we obtained an explicit form of the entries of $\bar{A}$ near the diagonal. Concretely, the $n_\nu \times n_\nu$ diagonal block $\bar{A}_\nu$ is given as follows:

$$\bar{A}_\nu = \begin{pmatrix} 0 & 1 & & & \text{\Large 0} \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ \text{\Large 0} & & & n_\nu - 2 & 1 \\ \bar{a}_{N_\nu,\,N_{\nu-1}+1} & \bar{a}_{N_\nu,\,N_{\nu-1}+2} & \cdots\ \cdots & \bar{a}_{N_\nu,\,N_\nu - 1} & \bar{a}_{N_\nu} \end{pmatrix},$$

where for $1 \le i \le n_\nu$,

$$\bar{a}_{N_\nu,\,N_\nu - i+1} = -\frac{\widehat{e}_{N_\nu+1,\,N_\nu - i}(\lambda_\nu)}{(\psi_{\nu-1}(\lambda_\nu))^i}, \tag{2.3}$$

$$\bar{a}_{N_\nu} = n_\nu - 1 + \bar{a}_{N_\nu, N_\nu}.$$

The characteristic exponents of nonholomorphic solutions of (1.10) are given by eigenvalues of the constant matrix $\bar{A}_\nu$. Since the matrix $\bar{A}_\nu$ is a companion matrix, the eigenvalues are roots of the equation

$$[\,\widehat{\rho}\,]_{n_\nu} = \sum_{i=1}^{n_\nu} \bar{a}_{N_\nu,\,N_\nu - i+1}\,[\,\widehat{\rho}\,]_{n_\nu - i}. \tag{2.4}$$

Now we shall show that the reduction described above preserves characteristic properties.

Proposition 2.1. *The sum of the characteristic exponents of nonholomorphic solutions of (1.10) differ from the sum of the characteristic exponents of nonholomorphic solutions of (1.3) at each regular singular point* $t = \lambda_\nu$ *only by the integers* $n_\nu N_{\nu-1}$.

Proof. We define

$$\rho' := \widehat{\rho} - n + N_\nu.$$

Substituting

$$\rho' + n - N_\nu$$

into $\widehat{\rho}$ of (2.4), we find that from (2.3), for $\nu = 1, 2, \ldots, q$

$$[\rho']_{n_\nu} = \sum_{i=1}^{n_\nu} \gamma_i [\rho']_{n_\nu - i}, \tag{2.5}$$

but the proof is an induction argument that we omit. By the relation between the coefficients of a polynomial and its roots for (2.5), (2.2), and (2.4) we have thus verified that:

$$\sum_{\ell=1}^{n_\nu} \rho_{\nu,\ell} - \sum_{\ell=1}^{n_\nu} \rho'_{\nu,\ell} = n_\nu(n - n_\nu),$$

and

$$\sum_{\ell=1}^{n_\nu} \rho'_{\nu,\ell} - \sum_{\ell=1}^{n_\nu} \widehat{\rho}_{\nu,\ell} - = n_\nu(-n + N_\nu).$$

By adding the two formulas above, we obtain the desired equation:

$$\sum_{\ell=1}^{n_\nu} \rho_{\nu,\ell} - \sum_{\ell=1}^{n_q} \widehat{\rho}_{\nu,\ell} = n_\nu N_{\nu-1}.$$

□

This proposition means that the transformation treated between (1.3) and (1.10) in this paper preserves the monodromic properties. We remark that there is a fundamental set of solutions to (1.10) with the form (1.6) in the punctured disk $0 < |t - \lambda_\nu| < r := \min\{|\lambda_\nu - \lambda_i| : i \neq \nu, i = 1, 2, \ldots, q\}$ ($\nu = 1, 2, \ldots, q$) where $g(m)$ is a nonzero n-entry column vector. There also exists a formal solution of (1.10) at the irregular singular point with the form (1.8) where $h(s)$ is a nonzero n-entry column vector. Like the case of a single differential equation, we call the numbers ρ, η characteristic exponents and μ characteristic constants.

2.2 Restricted Fuchsian relation

We shall explain an important identity, which necessarily exists among characteristic exponents for (1.3) and plays an essential role in the global analysis of linear differential equations with regular or irregular singularities.

As in the previous section, in the punctured disc $0 < |t - \lambda_\nu| < r (\nu = 1, 2, \ldots, q)$, there exist n_ν non-holomorphic solutions of (1.3)

$$y_{\nu,\ell}(t) = (t - \lambda_\nu)^{\rho_{\nu,\ell}} \sum_{m=0}^{\infty} g_{\nu,\ell}(m)\,(t - \lambda_\nu)^m \qquad (\ell = 1, 2, \ldots, n_\nu),$$

where the characteristic exponent $\rho_{\nu,\ell}$ are roots of the characteristic equation

$$[\rho - n + n_\nu]_{n_\nu} = \sum_{i=1}^{n_\nu} \gamma_{\nu,i}\,[\rho - n + n_\nu]_{n_\nu - i}. \tag{2.6}$$

The coefficients $\gamma_{\nu,i}$ are given by

$$\gamma_{\nu,i} = \left[\frac{P_{n-i}(t)}{P_n(t)}(t - \lambda_\nu)^i\right]_{t=\lambda_\nu}. \tag{2.7}$$

We now consider the sum of all characteristic exponents $\rho_{\nu,\ell}$. From the characteristic equation (2.6), we immediately obtain

$$\sum_{\ell=1}^{n_\nu} \rho_{\nu,\ell} = \sum_{\ell=1}^{n_\nu} (n - \ell) + \gamma_{\nu,1}$$

$$= n\,n_\nu - \frac{n_\nu(n_\nu + 1)}{2} + \gamma_{\nu,1}.$$

Then, we have

$$\begin{aligned} \sum_{\nu=1}^{q} \sum_{\ell=1}^{n_\nu} \rho_{\nu,\ell} &= \left(n - \frac{1}{2}\right) \sum_{\nu=1}^{q} n_\nu - \frac{1}{2} \sum_{\nu=1}^{q} n_\nu^2 + \sum_{\nu=1}^{q} \gamma_{\nu,1} \\ &= \left(n - \frac{1}{2}\right) n - \frac{1}{2} \sum_{\nu=1}^{q} n_\nu^2 + \sum_{\nu=1}^{q} \gamma_{\nu,1}. \end{aligned} \tag{2.8}$$

In order to calculate the last sum in the above formula, we apply the fact

$$\gamma_{\nu,1} = \left[\frac{P_{n-1}(t)}{P_n(t)}(t - \lambda_\nu)\right]_{t=\lambda_\nu}.$$

The right hand side is the residue of $\frac{P_{n-1}(t)}{P_n(t)}dt$ at $t = \lambda_\nu$. We can then express the sum of $\gamma_{\nu,1}$ in the form

$$\sum_{\nu=1}^{q} \gamma_{\nu,1} = \frac{1}{2\pi i} \int_{|t|=R} \frac{P_{n-1}(t)}{P_n(t)}\,dt, \tag{2.9}$$

where all $t = \lambda_\nu$ $(\nu = 1, 2, \cdots, q)$ are included in the disk $|t| < R$ for sufficiently large R, and the path of integration is oriented counterclockwise. According to the theory of residues for rational functions, the right hand side of (2.9) is equal to minus one times the value of the residue at infinity. If we denote the polynomial $P_{n-\ell}(t)$ of degree n by

$$P_{n-\ell}(t) = \sum_{j=0}^{n} p_{n-\ell, j}\, t^j \qquad (\ell = 0, 1, \ldots, n),$$

then for sufficiently large values of t the integrand can be written as follows:

$$\begin{aligned}\frac{P_{n-1}(t)}{P_n(t)} &= \frac{p_{n-1,n} + p_{n-1,n-1}\, t^{-1} + \cdots + p_{n-1,0}\, t^{-n}}{1 + p_{n,n-1}\, t^{-1} + \cdots + p_{n,0}\, t^{-n}} \\ &= (p_{n-1,n} + p_{n-1,n-1}\, t^{-1} + \cdots + p_{n-1,0}\, t^{-n})(1 - p_{n,n-1}\, t^{-1} + \cdots) \\ &= p_{n-1,n} + (p_{n-1,n-1} - p_{n-1,n}\, p_{n,n-1})\, t^{-1} + \cdots .\end{aligned}$$

Consequently, we have

$$\sum_{\nu=1}^{q} \gamma_{\nu,1} = p_{n-1,n-1} - p_{n-1,n}\, p_{n,n-1}. \tag{2.10}$$

Now we shall investigate the characteristic exponents of formal solutions for (1.3) at the irregular singularity $t = \infty$, which are expressed in the form

$$y(t) = e^{\mu t}\, t^{\eta} \sum_{s=0}^{\infty} h(s)\, t^{-s},$$

where we assume that $h(0) \neq 0$, and we define $h(-s) \equiv 0$ when s is a positive integer. We begin with some preparative calculations for finding the characteristic exponent η, following the method in the paper [K1]. We define $y_k(t)(k = 0, 1, \ldots, n)$ to be the kth derivative of $y(t)$ with respect to t:

$$y_k(t) = \frac{d^k y(t)}{dt^k},$$

and we shall denote by $h_k(s)$ the coefficients of the formal series, that is,

$$y_k(t) = e^{\mu t}\, t^{\eta} \sum_{s=0}^{\infty} h_k(s)\, t^{-s},$$

where

$$y_0(t) \equiv y(t).$$

Then, we have the following result:

Lemma 2.2. *For* $k = 1, 2, \ldots, n$, *the relation*

$$h_k(s) = \mu\, h_{k-1}(s) \;+\; (\eta - s + 1)\, h_{k-1}(s-1) \quad (s = 0, 1, \ldots) \tag{2.11}$$

holds, where $h_k(-s) = 0$ *for* $s > 0$.

Proof.

$$\begin{aligned}
& y_k(t) = y'_{k-1}(t) \\
& \Leftrightarrow e^{\mu\, t}\, t^{\eta} \sum_{s=0}^{\infty} h_k(s)\, t^{-s} = e^{\mu t} t^{\eta} \left\{ \mu \sum_{s=0}^{\infty} h_{k-1}(s) t^{-s} + \sum_{s=0}^{\infty} (\eta - s) h_{k-1}(s) t^{-s-1} \right\} \\
& \Leftrightarrow \sum_{s=0}^{\infty} h_k(s)\, t^{-s} = \left\{ \mu \sum_{s=0}^{\infty} h_{k-1}(s) t^{-s} + \sum_{s=0}^{\infty} (\eta - s) h_{k-1}(s) t^{-s-1} \right\}.
\end{aligned}$$

Comparing the coefficients of t^{-s}, we have the above formula. □

Moreover, substituting

$$t^j\, y_k(t) = e^{\mu\, t}\, t^{\eta} \sum_{s=-j}^{\infty} h_k(s+j)\, t^{-s}$$

into (1.3), we have

$$\sum_{j=0}^{n} \left\{ p_{n,\,j}\, h_n(s+j) \;-\; \sum_{k=0}^{n-1} p_{k,\,j}\, h_k(s+j) \right\} \;=\; 0 \quad (s = -j, -j+1, \ldots). \tag{2.12}$$

With this preparation complete, we are now in a position to calculate the value of the characteristic constants μ and the characteristic exponents η of the formal solutions of (1.3). To this end, we iteratively apply (2.11) to get the $k+1$-term sum (see [K1]):

$$h_k(s) = \mu^k\, h(s) \;+\; \{\mu^k \;+\; (\eta - s + 1) k\, \mu^{k-1}\}\, h(s-1) \;+\; \cdots\;. \tag{2.13}$$

We then apply (2.12) with the substitution $s = -n$. Then, from (2.13), we obtain

$$\left(\mu^n \;-\; \sum_{k=0}^{n-1} p_{k,\,n}\, \mu^k \right)\, h(0) \;=\; 0.$$

Hence, the equation

$$J(\mu) \equiv \mu^n - \sum_{k=0}^{n-1} p_{k,n}\,\mu^k = 0$$

determines the n characteristic constants, which we denote by $\mu_\ell (\ell = 1, 2, \ldots, n)$. From here, we assume that they are mutually distinct, i.e., $\mu_\ell \neq \mu_i$ $(\ell \neq i)$. Without this assumption, the argument becomes more complicated, because we can no longer use the assumption that $J'(\mu_\ell) \neq 0$ to produce (2.14).

Next, we substitute $s = 1$ into (2.13) to obtain

$$h_k(1) = \mu^k\, h(1) + \{\mu^k + \eta\, k\, \mu^{k-1}\}\, h(0).$$

Then, combining this with what we get from substituting $s = -n+1$ into (2.12), we obtain

$$J(\mu)\, h(1) + \{J(\mu) + \eta\, J'(\mu)\}\, h(0)$$

$$+ \left\{ p_{n,n-1}\, \mu^n - \sum_{k=0}^{n-1} p_{k,n-1}\, \mu^k \right\} h(0) = 0,$$

whence the characteristic exponent corresponding to μ_ℓ is given by the formula

$$\eta_\ell = -\frac{p_{n,n-1}\, \mu_\ell^n - \sum_{k=0}^{n-1} p_{k,n-1}\, \mu_\ell^k}{J'(\mu_\ell)}. \tag{2.14}$$

By exactly the same consideration as in the case (2.9), we can express the sum of the characteristic exponents (2.14) in the form of the integral

$$\sum_{i=1}^{n} \eta_i = -\frac{1}{2\pi\, i} \int_{|\mu|=R} \frac{p_{n,n-1}\, \mu^n - \sum_{k=0}^{n-1} p_{k,n-1}\, \mu^k}{J(\mu)}\, d\mu$$

for sufficiently large R, with the path of integration oriented counterclockwise. From the residue theorem we obtain

$$\sum_{i=1}^{n} \eta_i = p_{n-1,n-1} - p_{n-1,n}\, p_{n,n-1}. \tag{2.15}$$

Combining this formula with (2.10) and (2.8), we consequently obtain the restricted Fuchs relation.

Theorem 2.3. *Consider (1.3), and set* $\rho_{\nu,\ell}(\nu = 1, 2, \ldots, q, \ell = 1, 2, \ldots, n_\nu)$ *to be the characteristic exponents at regular singular points* $t = \lambda_\nu$ *and* $\eta_i(i = 1, 2, \ldots, n)$ *to be the characteristic exponents at the irregular singular point at infinity. Assume the characteristic constants of the formal solutions at the irregular singular point are mutually distinct, i.e.,* $\mu_\ell \neq \mu_i$ $(\ell \neq i)$*. Then, the restricted Fuchs relation for non-holomorphic solutions for* (1.3) *is the following:*

$$\sum_{\nu=1}^{q}\sum_{\ell=1}^{n_\nu} \rho_{\nu,\ell} - \sum_{i=1}^{n} \eta_i = \left(n - \frac{1}{2}\right) n - \frac{1}{2}\sum_{\nu=1}^{q} n_\nu^2. \tag{2.16}$$

Remark 2.4. *For the special case* $q = 1$*, we have* $n_1 = n$ *and hence the right hand side of (2.16) is equal to* $n(n-1)/2$*. In particular, the above restricted Fuchs relation is a generalization of the Lemma 3.1 in [K1].*

3 Toward the multi-point connection problem

In Section 3.1, we will explain how the two-point connection problem is useful for analyzing the Stokes phenomenon. In Section 3.2, we will introduce an associated fundamental function which was introduced by K. Okubo in the 1960's [O1]. In Section 3.3, we will give an example of the two-point connection problem.

3.1 What is the connection problem?

We assume that t is a complex variable. We consider an n-th order single differential equation which has one irregular singular point of rank one at infinity and a regular singular point at the origin, with unknown function y, of the form:

$$t^n \frac{d^n y}{dt^n} = \sum_{\ell=1}^{n} a_{n-\ell}(t) t^{n-\ell} \frac{d^{n-\ell} y}{dt^{n-\ell}}, \tag{3.1}$$

where $a_\ell(t)(\ell = 0, 1, \ldots, n-1)$ are holomorphic functions at the origin. There exists a fundamental set of solutions expressed in terms of convergent power series:

$$y_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^m \quad (j = 1, 2, \ldots, n),$$

in a punctured disc around the regular singular point $t = 0$, where $\rho_i - \rho_j \notin \mathbb{Z}$ $(i \neq j)$. We can calculate formal solutions:

$$y^k(t) = e^{\lambda_k t} t^{\mu_k} \sum_{s=0}^{\infty} h^k(s) t^{-s} \quad (k = 1, 2, \ldots, n)$$

at infinity, where $\lambda_k, \mu_k \in \mathbb{C}$. On each sector S with vertex at the origin and central angle not exceeding π, there exists a fundamental set of solutions $y_S^k(t)$ $(k = 1, 2, \ldots, n)$, such that

$$y_S^k(t) \sim y^k(t) \quad (|t| \to \infty \ \mathit{in}\ S).$$

We write $Y_0(t)$ to denote a vector function whose components are given by a fundamental set of solutions $y_j(t)$ near the origin, and $Y_S(t)$ to denote a vector function whose components are given by a fundamental set of solutions $y_S^k(t)$ near infinity on S;

$$Y_0(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad Y_S(t) = \begin{pmatrix} y_S^1(t) \\ y_S^2(t) \\ \vdots \\ y_S^n(t) \end{pmatrix}.$$

Let us denote the analytic continuation of the $y_S^k(t)$ into a sector S' by the same notation $y_S^k(t)$. Then, we have a linear relation between $y_S^k(t)$ and $y_{S'}^k(t)$:

$$Y_S(t) = T(S : S')Y_{S'}(t) \quad T(S : S') \in \mathcal{M}_n(\mathbb{C}) \quad \mathit{in} \quad S'. \tag{3.2}$$

We call this constant matrix $T(S : S')$ the *Stokes matrix* or the *lateral connection matrix*. If we can find the exact value of the matrix $T(S : S')$, then the asymptotic behavior of $y_S^k(t)$ as t tends to infinity in S' will be immediately understood.

On the other hand, a linear relation between two fundamental sets of solutions $y_j(t)$ and $y_S^k(t)$ in S clearly holds:

$$Y_0(t) = W(S)Y_S(t) \quad \mathit{in} \quad S, \quad W(S) \in GL_n(\mathbb{C}). \tag{3.3}$$

We call this coefficients matrix the *central connection matrix*. Its derivation is often called *the central connection problem*.

If we can solve such a central connection problem (3.3) for every sector S, then after the analytic continuation of the $y_S^k(t)$ across a domain near $t = 0$ and then into the sector S', we can directly obtain the lateral connection formula (3.2). That is, once the central connection problem is solved, the Stokes phenomenon will be completely understood.

3.2 Associated fundamental function

We will give here a short sketch of a method for the establishment of the asymptotic expansion $y_j(t)$ as t tends to infinity, together with the determination of the lateral connection matrices $T(S : S')$ for every sector S.

Assume that the central connection problem were solved. There exists a fundamental set of solutions of (3.1) expanded in terms of convergent power series in a punctured disc around the regular singular point $t = 0$:

$$y_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^m \quad (j = 1, 2, \ldots, n),$$

where $\rho_i - \rho_j \notin \mathbb{Z} (i \neq j)$. The fundamental solutions $y^k_S(t) (k = 1, 2, \ldots, n)$ of (3.1) are characterized by formal solutions at the irregular singular point:

$$y^k_S(t) \sim y^k(t) \quad (|t| \to \infty \ in \ S).$$

Then, $y_j(t)$ can be expressed as:

$$y_j(t) = t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^m = \sum_{k=1}^{n} W^k_j(S) y^k_S(t),$$

where $W^k_j(S)$ are entries of the matrix $W(S)$:

$$W(S) = \begin{pmatrix} W^1_1(S) & W^2_1(S) & \cdots & W^n_1(S) \\ W^1_2(S) & W^2_2(S) & \cdots & W^n_2(S) \\ \vdots & \vdots & & \vdots \\ W^1_n(S) & W^2_n(S) & \cdots & W^n_n(S) \end{pmatrix}.$$

We shall introduce a set of functions $x^k_j(s;t)$, distinguished by the property that they admit the same local behavior as $y_j(t)$ in a punctured disc around the origin and $y^k(t)$ near infinity. We call the functions $x^k_j(s;t)$ the associated fundamental functions and we will work out the expansion of $y_j(t)$ in terms of $x^k_j(s;t)$:

$$x^k_j(s;t) \sim \begin{cases} t^{\rho_j} (|t| \to 0), \\ e^{\lambda_k t} t^{\mu_k} (|t| \to \infty). \end{cases}$$

Now we consider a first order non homogeneous differential equation:

$$t \frac{dx^k_j(s;t)}{dt} = (\lambda_k t + \mu_k - s) x^k_j(s;t) + t^{\rho_j} \lambda_k g^k_j(s-1) \quad (s = 0, 1, 2, \ldots)$$

which has the particular solutions:

$$x^k_j(s;t) = t^{\rho_j} \sum_{m=0}^{\infty} g^k_j(m+s) t^m.$$

By quadrature, from the first order non homogeneous differential equation, we obtain the integral representation:

$$x_j^k(s;t) = \lambda_k g_j^k(s-1)t^{\rho_j} \int_0^1 e^{\lambda_k t(1-\tau)} \tau^{s+\rho_j-\mu_k-1} d\tau.$$

We remark that the integral is well-defined for all integers s satisfying $s+\rho-\mu > 0$, and if $\rho - \mu \notin \mathbb{Z}$, it can be regularized by analytic continuation for all integers s.

It is known that asymptotic behavior of $x(s;t)$ is

$$x_j^k(s;t) \sim e^{2\pi i(\rho_j-\mu_k)\ell} e^{\lambda_k t} t^{\mu_k - s} + t^{\rho_j}\{g_j^k(s-1)t^{-1} + g_j^k(s-2)t^{-2} + \cdots\}$$

as $|t| \to \infty$ in $|\arg(\lambda_k t) - 2\pi\ell| < 3\pi/2$, where ℓ is an integer. This concludes our introduction of the associated fundamental functions $x_j^k(s;t)(k,j = 1,2,\ldots,n)$, and our analysis of the asymptotic behavior of $x_j^k(s;t)(k,j = 1,2,\ldots,n)$.

Next, we shall define additional functions:

$$f_j^k(m) = \sum_{m=0}^{\infty} h^k(s) g_j^k(m+s) \quad (k = 1,2,\ldots,n).$$

We can show that $f_j^k(m)(k = 1,2,\ldots,n)$ satisfies the same reccurances which $G_j(m)$ satisfies, but the proof is omitted. From these facts, we can analyze the asymptotic expansion of $y_j(t)$:

$$\begin{aligned}
y_j(t) &= t^{\rho_j} \sum_{m=0}^{\infty} G_j(m) t^m \\
&= \sum_{m=0}^{\infty} \left(\sum_{k=1}^{n} W_j^k f_j^k(m) \right) t^{m+\rho_j} \\
&= \sum_{k=1}^{n} W_j^k \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} h^k(s) g_j^k(m+s) t^{m+\rho_j} \\
&= \sum_{k=1}^{n} W_j^k \sum_{s=0}^{\infty} h^k(s) x_j^k(s;t).
\end{aligned}$$

The asymptotic behavior of the associated fundamental function $x_j^k(s;t)$ is the same as that of $y^k(t)$. We will see more detail in the next section, where we work out an example.

3.3 Example

In this section, we apply the Okubo-Kohno method to describe the global behavior of solutions of Airy's differential equation:

$$t^2 y'' + \frac{1}{3} t y' - t^2 y = 0. \tag{3.4}$$

This equation has one regular singular point at the origin, and one irregular singular point at infinity in the complex projective line. In [K2], Kohno computes some entries of the central connection matrix of (3.4). Here, we shall compute the remaining entries, and furthermore, we shall determine the Stokes matrix.

To begin, we find a fundamental set of solutions of (3.4) in a punctured disc around the regular singular point $t = 0$. These solutions have the form

$$y(t) = t^\rho \sum_{m=0}^{\infty} G(m) t^m \qquad (G(0) \neq 0). \tag{3.5}$$

By substituting this expansion into (3.4), we obtain the linear difference equation

$$\begin{cases} (m+\rho)(m+\rho-\frac{2}{3})G(m) = G(m-2), \\ G(0) \neq 0, \qquad G(r) = 0 \quad (r < 0). \end{cases} \tag{3.6}$$

In order for negative terms to vanish, it is necessary that ρ is equal to 0 or 2/3, and that $G(1) = 0$. By induction, $G(2m+1) = 0$ for all $m \geq 0$. If we set $G(0) = 1$, we obtain

$$\begin{cases} G(2m) = \dfrac{\Gamma(\frac{\rho}{2})\Gamma(\frac{\rho}{2}+\frac{2}{3})}{4^m \Gamma(m+\frac{\rho}{2}+1)\Gamma(m+\frac{\rho}{2}+\frac{2}{3})}, \\ G(2m+1) = 0. \end{cases} \tag{3.7}$$

Consequently, the two values of ρ yield a fundamental set of solutions in a punctured disc around the regular singular point $t = 0$ as follows:

$$\begin{cases} y_1(t) = \displaystyle\sum_{m=0}^{\infty} \frac{\Gamma(\frac{2}{3})}{\Gamma(m+1)\Gamma(m+\frac{2}{3})} \left(\frac{t}{2}\right)^{2m}, \\ y_2(t) = \displaystyle\sum_{m=0}^{\infty} \frac{2^{2/3}\Gamma(\frac{4}{3})}{\Gamma(m+1)\Gamma(m+\frac{4}{3})} \left(\frac{t}{2}\right)^{2m+2/3}. \end{cases} \tag{3.8}$$

By the asymptotic properties of Γ, the first series has infinite radius of convergence, and the second series is $t^{2/3}$ times a series with infinite radius of convergence.

We now consider solutions of (3.4) near $t = \infty$. Because the singularity is irregular, the solutions do not have the convergent expansions of the form (3.5). However, there are formal power series solutions of the form

$$y(t) = e^{\lambda t} t^{\mu} \sum_{s=0}^{\infty} h(s) t^{-s} \quad (h(0) \neq 0). \tag{3.9}$$

In order to seek the value of the characteristic constant λ and the characteristic exponent μ, we follow the method in the paper [K1]. We define $y^{(\kappa)}(t)$ $(\kappa = 0, 1, 2)$ to be the κth derivative of $y(t)$ with respect to t:

$$y^{(\kappa)}(t) = \frac{d^{\kappa} y(t)}{dt^{\kappa}},$$

and we shall write the coefficients of the formal series $h^{\kappa}(s)$, that is

$$y^{(\kappa)}(t) = e^{\lambda\, t}\, t^{\mu} \sum_{s=0}^{\infty} h^{\kappa}(s)\, t^{-s}, \tag{3.10}$$

with $h^{0}(s) \equiv h(s)$.
We substitute (3.10) into (3.4) to find that our initial terms satisfy:

$$(\lambda^2 - 1)h(0) = 0, \tag{3.11}$$

$$(\lambda^2 - 1)h(1) + 2\lambda \left(\mu + \frac{1}{6}\right) h(0) = 0, \tag{3.12}$$

and the remaining terms satisfy the following recursion for $s \geq 0$:

$$(\lambda^2 - 1)h(s+2) + 2\lambda \left(-s - 1 + \mu + \frac{1}{6}\right) h(s+1) + (s - \mu) \left(s - \mu + \frac{2}{3}\right) h(s) = 0.$$

Because we assumed $h(0) \neq 0$, we see from the initial term equations that λ must be equal to ± 1 and μ must be equal to $-\frac{1}{6}$. Then, from the recursion, we obtain the linear difference equation in s:

$$h(s) = \frac{(s - \mu - 1)(s - \mu - \frac{1}{3})}{2\lambda s} h(s-1).$$

Setting $h(0) = 1$, we obtain the explicit formula:

$$h(s) = \left(\frac{1}{2\lambda}\right)^{s} \frac{\Gamma(s - \mu)\Gamma(s - \mu + \frac{2}{3})}{\Gamma(s+1)\Gamma(-\mu)\Gamma(-\mu + \frac{2}{3})}.$$

Using the two possible values of λ, we obtain two formal solutions near $t = \infty$:

$$\begin{cases} y^1(t) = e^t t^{-\frac{1}{6}} \displaystyle\sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{6})\Gamma(s+\frac{5}{6})}{\Gamma(s+1)\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})} \left(\frac{1}{2\lambda}\right)^s \qquad (\lambda = 1), \\ y^2(t) = e^{-t} t^{-\frac{1}{6}} \displaystyle\sum_{s=0}^{\infty} \frac{\Gamma(s+\frac{1}{6})\Gamma(s+\frac{5}{6})}{\Gamma(s+1)\Gamma(\frac{1}{6})\Gamma(\frac{5}{6})} \left(-\frac{1}{2\lambda}\right)^s \qquad (\lambda = -1). \end{cases} \tag{3.13}$$

It is straightforward to see that these formal solutions diverge wildly, but they are useful because they are in fact asymptotic expansions of holomorphic solutions in sectors near infinity.

We shall now apply the Okubo-Kohno method.

Suppose that we are given a convergent power series solution of the form (3.5) near $t = 0$, and suppose we have an additional expansion as a combination of holomorphic functions $\{x(s;t) : s = 0, 1, \ldots\}$ as follows:

$$y(t) = \sum_{s=0}^{\infty} h(s) x(s;t).$$

The solution $y(t)$ behaves near infinity like

$$y(t) \sim T e^{\lambda t} t^{\mu} \left\{ 1 + O\left(\frac{1}{t}\right) \right\} \qquad (|t| \to \infty),$$

where T is a Stokes multiplier. If our functions $\{x(s;t) : s = 0, 1, \ldots\}$ admit the following asymptotic behavior

$$\begin{cases} x(s;t) \sim t^{\rho} \qquad (|t| \to 0), \\ x(s;t) \sim e^{\lambda t} t^{\mu - s} \qquad (|t| \to \infty), \end{cases} \tag{3.14}$$

we can reasonably expect them to combine to form y, and satisfy convenient uniqueness properties.

We will construct functions $\{x(s;t) : s = 0, 1, \ldots\}$ of the form :

$$x(s;t) = t^{\rho} \sum_{m=0}^{\infty} g(m+s) t^m \tag{3.15}$$

that satisfy the first order non-homogeneous linear differential equations

$$t x'(s;t) = (\lambda t + \mu - s) x(s;t) + \lambda g(s-1) t^{\rho} \qquad (s = 0, 1, \ldots), \tag{3.16}$$

and the asymptotics given in (3.14). We will see that $x(s;t)$ is uniquely defined by these properties once we have chosen $g(0)$.

By substituting (3.15) into (3.16) and isolating powers of t, we see that the coefficient $g(m+s)$ satisfies the first order linear difference equation

$$(m+s+\rho-\mu)g(m+s) = \lambda g(m+s-1). \tag{3.17}$$

This linear difference equation therefore uniquely determines $x(s;t)$ once the initial term is specified. We set:

$$g(m+s) = \frac{\lambda^{m+s+\rho-\mu}}{\Gamma(m+s+\rho-\mu+1)} \tag{3.18}$$

as a particular solution of (3.17). By quadrature, the non-homogeneous equation (3.16) has solution given by the integral representation

$$x(s;t) = \lambda g(s-1)t^{\rho}\int_0^1 \exp\{\lambda t(1-\tau)\}\tau^{s+\rho-\mu-1}d\tau. \tag{3.19}$$

We therefore have our sequence of associated fundamental functions $\{x(s;t) : s = 0, 1, \ldots\}$, and they have the expected asymptotic behavior in sectors. Indeed, for arbitrarily small positive ε, and any integer ℓ, we have:

$$x(s;t) \sim e^{2\pi i(\rho-\mu)\ell}e^{\lambda t}t^{\mu-s} + t^{\rho}\{g(s-1)t^{-1} + g(s-2)t^{-2} + \cdots\} \tag{3.20}$$

as $t \longrightarrow \infty$ in $|\arg(\lambda t) - 2\pi\ell| \leq 3\pi/2 - \varepsilon$.

We return to our example, where our solutions were determined by the values of $\rho \in \{0, 2/3\}$ and $\lambda = \pm 1$. Here, we consider the cases where $\rho = 2/3$, $\lambda = \pm 1$ and $\mu = -\frac{1}{6}$. Then, the associated fundamental functions are defined by

$$\left(m+s+\frac{5}{6}\right)g_2^k(m+s) = \lambda_k g_2^k(m+s-1) \qquad (k=1,2; \lambda_1 = 1, \lambda_2 = e^{\pi i}), \tag{3.21}$$

and using the explicit formula for $g_2^k(m)$ from (3.18), we have

$$\begin{cases} x_2^k(s;t) = \displaystyle\sum_{m=0}^{\infty} g_2^k(m+s)t^{m+\frac{2}{3}}, \\ \qquad\qquad = \displaystyle\sum_{m=0}^{\infty} \frac{(\lambda_k)^{m+s+\frac{5}{6}}}{\Gamma(m+s+\frac{11}{6})}t^m \qquad (k=1,2). \end{cases} \tag{3.22}$$

If we write $h^k(s)(k = 1, 2)$ to denote the coefficients in the formal power series expansion (3.13) of $y^k(t)$, we may define the functions $f_2^k(m)(k = 1, 2)$ by

$$f_2^k(m) = \sum_{s=0}^{\infty} h^k(s) g_2^k(m+s) \qquad (k = 1, 2). \tag{3.23}$$

Because our explicit formula for $g_2^k(m)$ from (3.18) yields a holomorphic function on the right half m-plane, the same is true for $f_2^k(m)$. Indeed, we have the asymptotic relations:

$$f_2^k(m) \sim \frac{(\lambda_k)^{m+\frac{5}{6}}}{\Gamma(m+\frac{11}{6})} \left\{ 1 + O\left(\frac{1}{m}\right) \right\}. \tag{3.24}$$

Here the proof is omitted.

We claim that $f_2^k(m)(k = 1, 2)$ satisfies the same recurrence that defines $G_2(m)$, but we omit the proof. Therefore, $G_2(m)$ can be expressed as a linear combination of the $f_2^k(m)(k = 1, 2)$ as follows :

$$G_2(m) = W_2^1 f_1^1(m) + W_2^2 f_2^2(m), \tag{3.25}$$

where the $W_2^k(k = 1, 2)$ are, in general, periodic functions of m with period 1, however, they may be considered to be constants for integral values of m. From this, we consequently obtain the expansion of $y_2(t)$ in terms of sequences of associated fundamental functions $\{x_2^k(s; t) : s = 0, 1, \dots (k = 1, 2)\}$:

$$\begin{aligned} y_2(t) &= \sum_{m=0}^{\infty} G_2(m) t^{m+\frac{2}{3}} \\ &= W_2^1 \sum_{m=0}^{\infty} f_2^1(m) t^{m+\frac{2}{3}} + W_2^2 \sum_{m=0}^{\infty} f_2^2(m) t^{m+\frac{2}{3}} \\ &= W_2^1 \sum_{s=0}^{\infty} h^1(s) \left(\sum_{m=0}^{\infty} g_2^1(m+s) t^{m+\frac{2}{3}} \right) + W_2^2 \sum_{s=0}^{\infty} h^2(s) \left(\sum_{m=0}^{\infty} g_2^2(m+s) t^{m+\frac{2}{3}} \right) \\ &= W_2^1 \sum_{s=0}^{\infty} h^1(s) x_2^1(s; t) + W_2^2 \sum_{s=0}^{\infty} h^2(s) x_2^2(s; t). \end{aligned} \tag{3.26}$$

We conclude that for each nonnegative integer m, $f_2^k(m)(k=1,2)$ is the coefficient attached to $t^{m+\frac{2}{3}}$, when $y_2(t)$ is expanded as a power series. We may now use the asymptotic behavior (3.20) of the associated fundamental functions to analyze the asymptotic behavior of the original solutions. We derive from (3.26)

$$y_2(t) \sim W_2^1 \sum_{m=0}^{\infty} h^1(s) \left\{ e^t t^{-\frac{1}{6}-s} + \sum_{r=0}^{\infty} g_2^1(s-r) t^{-r} \right\}$$

$$+ W_2^2 \sum_{s=0}^{\infty} h^s(s) \left\{ e^{-t} t^{-\frac{1}{6}-s} + \sum_{r=0}^{\infty} g_2^2(s-r) t^{-r} \right\}$$

$$\sim W_2^1 y^1(t) + W_2^2 y^2(t)$$

$$+ \sum_{r=0}^{\infty} \left(W_2^1 f_2^1(-r) + W_2^2 f_2^2(-r) \right) t^{-r}$$

$$\sim W_2^1 y^1(t) + W_2^2 y^2(t) + \sum_{r=0}^{\infty} G_2(-r) t^{-r}$$

$$\sim W_2^1 y^1(t) + W_2^2 y^2(t)$$

as $t \longrightarrow \infty$ in the sector

$$\widehat{S} = \bigcap_{k=1}^{2} \left\{ |\arg(\lambda_k t)| < \frac{3}{2}\pi \right\} = \left\{ -\frac{3}{2}\pi < \arg t < \frac{\pi}{2} \right\}.$$

Now that we have all of the necessary asymptotic information in hand, we can determine $W_2^k(k=1,2)$ by combining the fact that $G_j(m)(j=1,2)$ vanishes on odd inputs with our knowledge of the asymptotic behavior on even inputs. Explicitly, we combine (3.13) and (3.22) to get

$$f_2^2(m) = e^{\pi i (m+\frac{5}{6})} f_2^1(m)$$

for all $m \geq 0$, and from that, we apply

$$0 = G_2(2m+1) = W_2^1 f_2^1(2m+1) + f_2^2(2m+1)$$

$$= (W_2^1 - W_2^2 e^{\frac{5}{6}\pi i}) f_2^1(2m+1)$$

to deduce one relation:

$$W_2^1 = W_2^2 e^{\frac{5}{6}\pi i}. \tag{3.27}$$

For the second relation, we consider the formula:

$$G_2(2m) = W_2^1 f_2^1(2m) + W_2^2 f_2^2(2m).$$

From (3.7) and using the asymptotic behavior of $f_2^k(2m)$ given in (3.24), we may divide by $f_2^1(2m)$ to find that for sufficiently large m,

$$W_2^1 + W_2^2 e^{\frac{5}{6}\pi i}$$

$$= \frac{\Gamma(\frac{4}{3})\Gamma(2m+\frac{11}{6})}{4^m \Gamma(m+1)\Gamma(m+\frac{4}{3})} \left\{1 + O\left(\frac{1}{m}\right)\right\}$$

$$= \frac{\Gamma(\frac{4}{3}) 2^{2m+\frac{11}{6}} \Gamma(m+\frac{11}{12})\Gamma(m+\frac{17}{12})}{\sqrt{2\pi}\, 4^m \Gamma(m+1)\Gamma(m+\frac{4}{3})} \left\{1 + O\left(\frac{1}{m}\right)\right\}$$

$$= \frac{\Gamma(\frac{4}{3}) 2^{\frac{4}{3}}}{\sqrt{2\pi}} \left\{1 + O\left(\frac{1}{m}\right)\right\}.$$

However, $W_2^1 + W_2^2 e^{\frac{5}{6}\pi i}$ is constant, so the $O(1/m)$ terms vanish:

$$W_2^1 + W_2^2 e^{\frac{5}{6} i} = \frac{2^{\frac{4}{3}}\Gamma(\frac{4}{3})}{\sqrt{2\pi}}. \tag{3.28}$$

By combining this with (3.27), we find that the connection coefficients $W_2^k (k = 1, 2)$ are:

$$W_2^1 = W_2^2 e^{\frac{5}{6}\pi i} = \frac{2^{\frac{7}{6}}}{\sqrt{3}} \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}.$$

Therefore, we obtain the connection formula:

$$y_2(t) \sim \begin{cases} W_2^2 y^2(t) & (S_1 : -\frac{3}{2}\pi < \arg t < -\frac{\pi}{2}), \\ \\ W_2^1 y^1(t) & (S_2 : -\frac{\pi}{2} < \arg t < \frac{\pi}{2}), \\ \\ W_2^2 e^{\frac{5}{3}\pi i} y^2(t) & (S_3 : \frac{\pi}{2} < \arg t < \frac{3}{2}\pi). \end{cases}$$

For $y_1(t)$, in [K2], Kohno employed a similar calculation to find the following connection formula:

$$y_1(t) \sim \begin{cases} W_1^2 y^2(t) & (S_1 : -\frac{3}{2}\pi < \arg t < -\frac{\pi}{2}), \\ \\ W_1^1 y^1(t) & (S_2 : -\frac{\pi}{2} < \arg t < \frac{\pi}{2}), \\ \\ W_1^2 e^{\frac{\pi}{3} i} y^2(t) & (S_3 : \frac{\pi}{2} < \arg t < \frac{3}{2}\pi), \end{cases}$$

where $W_1^1 = W_1^2 e^{\frac{\pi}{6} i} = \left(\frac{1}{2}\right)^{\frac{1}{6}} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})}$. Even without the exact value of W_1^1 and W_1^2, we can compute the Stokes coefficients. For example, the analytic continuation of Y_{S_2} from S_2 to S_3:

$$T(S_2 : S_3) = W^{-1}(S_2) W(S_3) = \begin{pmatrix} 0 & i \\ \\ i & 1 \end{pmatrix},$$

with

$$Y_{S_2} = \begin{pmatrix} y_{S_2}^1 \\ y_{S_2}^2 \end{pmatrix}, W(S_1) = W(S_2) = \begin{pmatrix} W_1^1 & W_1^2 \\ \\ W_2^1 & W_2^2 \end{pmatrix} = \begin{pmatrix} W_1^2 e^{\frac{\pi}{6} i} & W_1^2 \\ \\ W_2^2 e^{\frac{5}{6}\pi i} & W_2^2 \end{pmatrix},$$

$$W(S_3) = \begin{pmatrix} W_1^1 e^{\frac{\pi}{3} i} & W_1^2 e^{\frac{\pi}{3} i} \\ \\ W_2^1 e^{\frac{5}{3}\pi i} & W_2^2 e^{\frac{5}{3}\pi i} \end{pmatrix} = \begin{pmatrix} W_1^2 e^{\frac{\pi}{2} i} & W_1^2 e^{\frac{\pi}{3} i} \\ \\ W_2^2 e^{\frac{\pi}{2} i} & W_2^2 e^{\frac{5}{3}\pi i} \end{pmatrix}.$$

4 Numerical Computation of Stokes Multipliers

In Section 4.1, we introduce some notation. In Section 4.2, we introduce the Sibuya-Malgrange Isomorphism. In Section 4.3, we shall give formulas for Stokes multipliers for three cases. In Section 4.4, we shall show applications of calculations of Stokes multipliers for three cases. In the appendix, we provide tools for understanding the proof of the Sibuya-Malgrange Isomorphism theorem.

4.1 Notation

Definition 4.1. (**Sector**) *We define sectors* $S(\alpha, \beta, R) = \{t : \alpha < \arg t < \beta, 0 < |t| < R\}$. *If* $R = +\infty$, *then we write* $S(\alpha, \beta)$.

Definition 4.2. (**Asymptotic expansion**) *Let* S *be an open sector of the complex plane whose vertex is at the origin. Let* $\widehat{f}(t) = \sum_{m\geq 0} \widehat{f}_m t^m \in \mathbb{C}[[t]]$ *be a formal power series. Let* f *be an analytic function on the sector* S. *We say that* f *is asymptotic to* $\widehat{f}(t) = \sum_{m\geq 0} \widehat{f}_m t^m$ *on the sector* S *if for every closed subsector* S' *of* $S \cup \{0\}$ *and every positive integer* $N \in \mathbb{N}^*$, *there exists a positive constant* $C_{S',N}$ *such that*

$$\forall t \in S', \quad |f(t) - \sum_{m=0}^{N-1} \widehat{f}_m t^m| \leq C_{S',N}|t|^N.$$

We denote it by

$$f(t) \sim \widehat{f}(t) \quad (t \in S).$$

Definition 4.3. *(types of asymptotic expansion)*

We describe three notions of asymptotic expansion of a holomorphic function, in increasing complexity. Let f *be a holomorphic function on a sector* S.

1. *We write* $f(t) \sim \widehat{f}(t) \quad (t \in S)$ *if* f *is asymptotic to the formal series* $\hat{f}$.

2. *If* $\varphi(t) \neq 0$ *is holomorphic on* S, *and* $f/\varphi \sim \widehat{f}$ *on* S *in the first sense, we write* $f(t) \sim \varphi(t)\widehat{f}(t) \quad (t \in S)$.

3. *If* $\varphi(t) \neq 0$ *and* $\psi(t) \neq 0$ *are holomorphic on* S, *and there exist* f *and* g *such that*

$$\begin{cases} f \sim \varphi(t)\widehat{f}, \\ g \sim \psi(t)\widehat{g}, \end{cases}$$

then we write $h(t) \sim \varphi(t)\widehat{f}(t) + \psi(t)\widehat{g}(t) \quad (t \in S)$, *where* $h(t) = f(t) + g(t)$.

Definition 4.4. *(Flat function) We say that an analytic function on S is flat at the origin, if its asymptotic expansion at the origin is*

$$0 + 0\,t + 0\,t^2 + \cdots .$$

We denote the space of the flat functions on S by $\bar{\mathcal{A}}^{<0}(S)$.

4.2 Sibuya-Malgrange Isomorphism

Definition 4.5. *(Good covering) Let $\mathcal{U} := \{I_1, I_2, \cdots I_n\}$ be a set of open intervals in S^1. $\mathcal{U}$ is a good covering of S^1 if it satisfies the following conditions:*

(1) $\alpha_\ell < \alpha_{\ell+1}$, $(\ell = 1, 2, \cdots, n)$ with $\alpha_{n+1} = \alpha_1 + 2\pi$,

(2) $\beta_\ell - \alpha_\ell < \pi$, $(\ell = 1, 2, \cdots, n)$,

(3) $I_\ell \cap I_{\ell+1} \neq \phi$, $(\ell = 1, 2, \cdots, n)$ and $I_\ell \cap I_k = \phi$ otherwise if $\ell \neq \{k \pm 1, k\}$ with $I_{n+1} = I_1$.

Definition 4.6. *We define the following presheaves on S^1:*

$\bar{\mathcal{A}}$: The presheaf that assigns to any sector S the space of holomorphic functions on S with asymptotic expansion.

$\mathcal{A}$: The sheaf associated to $\bar{\mathcal{A}}$.

$\bar{\mathcal{A}}^{<0}$: The presheaf that assigns to any sector S the space of flat functions on S.

$\mathcal{A}^{<0}$: The sheaf associated to $\bar{\mathcal{A}}^{<0}$.

Now, we shall introduce the Cauchy-Heine's integral and theorem.

Definition 4.7. *(Cauchy-Heine's integral) Let $V = S(\alpha, \beta + 2\pi, R)$ be a sector that overlaps itself around 0. Let $\bar{V} = S(\alpha, \beta, R)$ denote its self intersection. Let $\gamma =]0, t_0] \subset \bar{V}$ be a straight line path and $\varphi \in \mathcal{A}^{<0}(\bar{V})$ be a flat function on $\bar{V}$. We call the function*

$$f(t) = \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\xi)}{\xi - t} d\xi$$

the Cauchy-Heine integral associated with φ and γ.

The function f is well-defined and analytic on $V \backslash \gamma$ and it can be analytically continued to the domain $U = V \cap \{|t| < |t_0|\}$. The analytic continuation is given by integrating the same integrand over paths that are homotopic to γ with endpoints fixed. The difference between two Cauchy-Heine integrals associated to different $t_0 \in \bar{V}$ is analytic on their common domain of the definition.

Theorem 4.8. *(Cauchy-Heine theorem) The Cauchy-Heine integral*

$$f(t) = \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\xi)}{\xi - t} \, d\xi$$

satisfies the following properties:

- $f \in \bar{\mathcal{A}}(V)$,
- *Its Taylor series at the origin reads*

$$T_V \, f(t) \; = \sum_{n \geq 0} a_n \, t^n, \quad with \quad a_n \; = \frac{1}{2 \, \pi \, i} \int_\gamma \frac{\varphi(\xi)}{\xi^{n+1}} \, d\xi,$$

- *The variation* $var(f(t)) = f(t) - f(te^{2\pi i})$ *equals* $\varphi(t)$ *for all* $t \in \bar{U} \; = \; \bar{V} \cap \{|t| < |t_0|\}$.

Theorem 4.9. *(Borel-Ritt, see [W, section 9]) For any sector* $S \subset \tilde{\mathbb{C}}^*$*, where* $\tilde{\mathbb{C}}^*$ *is the universal cover of* $\mathbb{C}^*$*, the following map defined by Taylor expansion is surjective:*

$$\bar{\mathcal{A}}(S) \longrightarrow \mathbb{C}[[t]].$$

This theorem gives the short exact sequence of the sheaves:

$$0 \longrightarrow \mathcal{A}^{<0} \hookrightarrow \mathcal{A} \longrightarrow \mathbb{C}[[t]] \longrightarrow 0. \tag{4.1}$$

Therefore, by taking cohomology, we obtain the exact sequence:

$$\begin{aligned} 0 \longrightarrow \Gamma(S^1, \mathcal{A}) \;&\longrightarrow\; \Gamma(S^1, \mathbb{C}[[t]]_{S^1}) \\ &\longrightarrow\; H^1(S^1, \mathcal{A}^{<0}) \;\xrightarrow{\;\Phi\;}\; H^1(S^1, \mathcal{A}) \longrightarrow 0. \end{aligned} \tag{4.2}$$

Lemma 4.10. *The map*

$$H^1(S^1, \mathcal{A}^{<0}) \xrightarrow{\;\Phi\;} H^1(S^1, \mathcal{A})$$

is the 0*-map.*

Proof. We consider the following piece of diagram (A.17):

$$\begin{array}{ccc} & & \mathcal{C}^0(\mathcal{U},\mathcal{A}) \\ & & \Big\downarrow \delta_0 \\ \mathcal{C}^1(\mathcal{U},\mathcal{A}^{<0}) & \xrightarrow{a} & \mathcal{C}^1(\mathcal{U},\mathcal{A}) \\ \Big\downarrow & & \Big\downarrow b \\ H^1(S^1,\mathcal{A}^{<0}) & \xrightarrow{\Phi} & H^1(S^1,\mathcal{A}), \end{array} \tag{4.3}$$

where $\mathcal{U}$ is a good covering, $\mathcal{C}^k(\mathcal{U},\mathcal{A})$ denotes the group of Čech k-cochains ($k = 0, 1$) with coefficients in the sheaf $\mathcal{A}$, and the other symbols are defined similarly. For each j ($j = 1, 2, \dots, n$), let $u_j \in \Gamma(I_j \cap I_{j+1}, \mathcal{A}^{<0})$. It is enough that for any cohomology class $[u] \in H^1(S^1, \mathcal{A}^{<0})$, any cocycle $\oplus u_j \in \mathcal{C}^1(\mathcal{U}, \mathcal{A}^{<0})$ that represents $[u]$ (where we may need to refine $\mathcal{U}$ depending on $[u]$) is sent to zero in $H^1(S^1, \mathcal{A})$ by Φ.

By using the Cauchy-Heine Theorem, we obtain sections w_j of $\mathcal{A}$ on each sector that overlaps itself at $I_j \cap I_{j+1}$. Concretely,

$$w_j(t) = \frac{1}{2\pi i}\int_\gamma \frac{u_j(\xi)}{\xi - t}d\xi,$$

where $\gamma \subset I_j \cap I_{j+1}$. Let

$$v_j := \sum_{i=1}^{n} w_i|_{I_j} + h(t) \in \Gamma(I_j, \mathcal{A}),$$

with $h(t) \in \mathbb{C}\{t\}$. Then, $\oplus v_j \in \mathcal{C}^0(\mathcal{U},\mathcal{A})$. Therefore $b(\delta_0(\oplus v_j)) = 0$ in $H^1(S^1,\mathcal{A})$. On the other hand, we can prove the variation $\oplus\mathrm{var}(w_j)$ is equal to $\delta_0(\oplus v_j)$, so $b(a(\oplus u_j)) = 0$. See Appendix A.2 for the proof.

By the commutativity of (4.3), we conclude $\Phi([\oplus u_j]) = 0$.

□

By Lemma 4.10, we obtain the short exact sequence:

$$0 \longrightarrow \Gamma(S^1,\mathcal{A}) \longrightarrow \Gamma(S^1,\mathbb{C}[[t]]_{S^1}) \longrightarrow H^1(S^1,\mathcal{A}^{<0}) \longrightarrow 0. \tag{4.4}$$

We remark that $\Gamma(S^1,\mathcal{A}) \simeq \mathbb{C}\{t\}$ and $\Gamma(S^1,\mathbb{C}[[t]]_{S^1}) \simeq \mathbb{C}[[t]]$. The short exact sequence (4.4) implies the next theorem.

Theorem 4.11. *(Sibuya-Malgrange Isomorphism, see [S], [M])*
We call the following isomorphism Ψ the Sibuya-Malgrange Isomorphism:

$$\mathbb{C}[[t]]/\mathbb{C}\{t\} \xrightarrow{\Psi} H^1(S^1,\mathcal{A}^{<0}).$$

4.3 Formula for Stokes multipliers

We consider the differential equation:

$$t^{r+1}\frac{dX}{dt} = A(t)X, \quad t \in \mathbb{C}, \quad r \geq 1. \tag{4.5}$$

Assume $X(t)$ is an $n \times n$ matrix function, and $A(t)$ is an $n \times n$ matrix whose entries are polynomials in t. Since $r \geq 1$, $t = 0$ is an irregular singular point. The equation has a formal solution

$$\hat{X}(t) = \hat{F}(t)t^L e^{Q(1/t)},$$

where

1. $\hat{F}(t) \in M_n(\mathbb{C}[[t]])$ has expansion $\sum_{m=0}^{\infty} F(m)t^m$ with $F(0) = I$,
2. L is a constant matrix in the Jordan form,
3. $Q(1/t) = \mathrm{diag}(q_1(1/t), \ldots, q_n(1/t))$, where $q_j(1/t) = \sum_{k=1}^{r} \frac{\sigma_{j,k}}{t^k}$.

We define sectors $S_j(\alpha_j, \beta_j)$ to be where we have fundamental solutions:

$$X_j(t) = F_j(t)t^L e^{Q(1/t)}$$

that are asymptotic to the formal solution. On $S_j \cap S_{j+1}$, we have the relation:

$$X_j(t) = X_{j+1}(t)C_j.$$

Because of $F_j(t) \sim \hat{F}(t)$ on $S_j (j = 1, 2, \ldots, m)$, $\mathrm{diag}(C_j) = I$, then we can separate C_j to $C_j = I + C'_j$. Then, substituting the fundamental solutions into the above formula we obtain on $S_j \cap S_{j+1}$:

$$F_{j+1} - F_j = F_j t^L e^{Q(1/t)} C'_j e^{-Q(1/t)} t^{-L}. \tag{4.6}$$

We shall call C'_j the Stokes matrix, and entries of C'_j the Stokes multipliers.
We shall define the Stokes direction and the anti-Stokes direction for (4.6).

Definition 4.12. *To simplify, we consider the case:*

$$q_j(1/t) - q_\ell(1/t) = \frac{\sigma_{j,\ell}}{t^\nu} \quad (j, \ell = 1, 2, \ldots, r),$$

where each $\sigma_{j,\ell}$ is different. The function $\exp(q_j(1/t) - q_\ell(1/t))$ *has ν decreasing sectors. All sectors have an argument $\frac{\pi}{\nu}$. These are separated by 2ν open rays. We call these rays* **Stokes directions**. *And we call the bisectors of the Stokes directions* **anti-Stokes directions**.

We shall write here a formula for the Stokes direction θ:

$$\frac{\sigma_{j,\ell}}{|t|^\nu e^{i\nu\theta}} \in i\,\mathbb{R}.$$

In (4.6), the sums F_j and F_{j+1} are asymptotic to $\widehat{F}$ near the anti-Stokes direction $\arg t = \alpha_j$ for $F_{j+1} - F_j$. Since $\{F_{j+1} - F_j\}_j$ is a 1−cocycle, then we obtain that $[F_{j+1} - F_j] \in H^1(S^1, \mathcal{A}^{<0})$.

Then, by the Cauchy-Heine's theorem and the Sibuya-Malgrange Isomorphism, we obtain the following formula:

$$\widehat{F}(k) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{F_{j+1} - F_j}{\xi^{k+1}} d\xi,$$

with $\widehat{F}(t) = \sum_{k=0}^{\infty} \widehat{F}(k) t^k$.

We shall show the formulas for three examples. We consider a formal series;

$$X(x) = \widehat{F}(x) x^L \exp(Q(1/x)),$$

where L is a Jordan matrix, $Q(1/x)$ is a diagonal matrix whose entries are polynomials in $1/x$:

$$Q(1/x) := \mathrm{diag}(q_1(1/x), \cdots, q_1(1/x), q_2(1/x), \cdots, q_2(1/x)),$$

and we define a matrix-valued formal series:

$$\widehat{F} = I + \sum_{m=1}^{\infty} F_m x^m,$$

$$\widehat{F} = \begin{pmatrix} \widehat{G} & \widehat{H} \end{pmatrix} = \begin{pmatrix} \widehat{G}_{n_1} & \widehat{H}_{n_1, n_2} \\ \\ \widehat{G}_{n_2, n_1} & \widehat{H}_{n_2} \end{pmatrix},$$

and we let F_α denote a matrix-valued function whose asymptotic expansion in the anti-Stokes direction α is $\widehat{F}$;

$$F_\alpha = (G_\alpha \quad H_\alpha) = \begin{pmatrix} G_{\alpha_{n_1}} & H_{\alpha_{n_1, n_2}} \\ \\ G_{\alpha_{n_2, n_1}} & H_{\alpha_{n_2}} \end{pmatrix}.$$

4.3.1 Case of 2 diagonal matrices

We consider the following block-diagonal matrices:

$$\begin{cases} L = d_1\, I_{n_1} \,\oplus\, d_2\, I_{n_2}, \quad d_1,\, d_2 \in \mathbb{R}, \\ Q(1/x) = 0_{n_1} \,\oplus\, \left(-\dfrac{1}{x}\right) I_{n_2}, \end{cases}$$

where $n_i\,(i \;=\; 1,\,2)$ are the dimensions of the matrix, the anti-Stokes direction is $\theta \;=\; 0$, and the Stokes matrix is following:

$$I \,+\, C_0 \;=\; I \,+\, \begin{pmatrix} 0_{n_1} & 0_{n_1 \times n_2} \\ & \\ C_{n_2,\,n_1} & 0_{n_2} \end{pmatrix},$$

where $C_{n_2,\,n_1}$ is an $n_2 \,\times\, n_1$ matrix and $0_{n_1 \times n_2}$ is an $n_1 \,\times\, n_2$ zero matrix.

By the Sibuya-Malgrange Isomorphism we obtain the next formula:

$$\begin{aligned} \widehat{G}_{n_2,\,n_1} &= \sum_{m\,=\,0}^{\infty} \left(\frac{1}{2\,\pi\, i} \int_0^{\infty} \widehat{H}_{0 - n_2}(\xi)\, \xi^{d_2\,-\,d_1\,-\,m\,-\,1}\, e^{-\,\frac{1}{\xi}}\, d\,\xi \right) C_{n_2,\,n_1}\, x^m \\ &= \sum_{m\,=\,0}^{\infty} \frac{1}{2\,\pi\, i}\, \Gamma(m \,+\, d_1 \,-\, d_2)\,(1 \,+\, O(m^{-1}))\, C_{n_2,\,n_1}\, x^m. \end{aligned}$$

Comparing the coefficients of x^m, we obtain

$$\begin{aligned} C_{n_2,\,n_1} &= 2\,\pi\, i\, \frac{\widehat{G}_{n_2,\,n_1}(m)}{\Gamma(m \,+\, d_1 \,-\, d_2)}\,(1 \,+\, O(m^{-1})) \\ &= 2\,\pi\, i\, m^{d_2\,-\,d_1}\, \frac{\widehat{G}_{n_2,\,n_1}(m)}{\Gamma(m)}\,(1 \,+\, O(m^{-1})), \end{aligned}$$

Next we consider the Stokes matrix of the direction $\theta \;=\; \pi$;

$$I \,+\, C_\pi \;=\; I \,+\, \begin{pmatrix} 0_{n_1} & C_{n_1,\,n_2} \\ & \\ 0_{n_2 \times n_1} & 0_{n_2} \end{pmatrix},$$

where $C_{n_1,\,n_2}$ is a $(n_1 \,\times\, n_2)-$matrix, and $0_{n_2 \times n_1}$ is a $(n_2 \,\times\, n_1)-$ zero matrix. By the Sibuya-Malgrange Isomorphism, we obtain $\widehat{H}_{n_2,\,n_1}$:

$$\widehat{H}_{n_2,\,n_1} \;=\; \sum_{m\,=\,0}^{\infty} \left(\frac{1}{2\,\pi\, i} \int_0^{\infty\, e^{\pi\, i}} \widehat{G}_{\pi - n_1}(\xi)\, \xi^{d_1\,-\,d_2}\, e^{\frac{1}{\xi}}\, \frac{1}{\xi^{m\,+\,1}}\, d\,\xi \right) C_{n_1,\,n_2}\, x^m,$$

and the Stokes matrix;

$$C_\pi = \frac{1}{2\,\pi\,i}\,(-1)^{d_1\,-\,d_2\,-\,m}\,\frac{\widehat{H}_{n_1,\,n_2}(m)}{\Gamma(m\,-\,d_1\,+\,d_2)}\,(1\,+\,O(m^{-1}))$$
$$= \frac{1}{2\,\pi\,i}\,(-1)^{d_1\,-\,d_2\,-\,m}\,m^{d_2\,-\,d_1}\,\frac{\widehat{H}_{n_1,\,n_2}(m)}{\Gamma(m)}\,(1\,+\,O(m^{-1})).$$

4.3.2 Case of 1 diagonal and 1 Jordan matrix

We again consider a pair of block-diagonal matrices:

$$\begin{cases} L = d_1\,I_{n_1}\,\oplus\,D_{n_2}, \quad d_1 \in \mathbb{R}, \\ Q(1/x) = 0_{n_1}\,\oplus\,(-\dfrac{1}{x})\,I_{n_2}, \end{cases}$$

where $D_{N_2} := d_2\,I_{n_2} + N_{n_2}$ and N_{n_2} are nilpotent matrices. By the Sibuya-Malgrange Isomorphism we obtain:

$$\widehat{G}_{n_2,\,n_1} \mod \mathbb{C}\{x\} = \sum_{m=0}^{\infty}\left(\frac{1}{2\pi i}\int_0^\infty \widehat{H}_{0-n_2}(\xi)\xi^{d_2-d_1-m-1}e^{-\frac{1}{\xi}}\xi^{N_{n_2}}d\xi\right)C_{n_2,n_1}x^m,$$

with

$$x^{N_{n_2}} = I + \frac{N_{n_2}}{1\,!}\log\,x + \frac{N_{n_2}^2}{2\,!}(\log x)^2 + \cdots\,.$$

We note the following identities concerning derivatives of the Gamma function:

$$\begin{cases} \Gamma^{(n)}(m) &= \displaystyle\int_0^\infty e^{-\frac{1}{\xi}}\,\frac{1}{\xi^{m\,+\,1}}\,\left(\log\frac{1}{\xi}\right)^n\,d\,\xi, \\ \Gamma^{(n)}(m\,+\,d) &= m^d\,\Gamma^{(n)}(m)\,(1\,+\,O(m^{-1})), \\ \Gamma^{(n)}(m) &= (\log m)^n\Gamma(m)\,(1\,+\,O(\frac{1}{m\,\log m})). \end{cases}$$

$$\widehat{G}_{n_2,\,n_1} \mod \mathbb{C}\{x\} = \sum_{m\,=\,0}^{\infty}\frac{1}{2\,\pi\,i}\,m^{d_1\,-\,d_2}\,\Gamma(m)\,(1\,+\,O(m^{-1}))\,m^{-\,N_{n_2}}\,C_{n_2,\,n_1}\,x^m$$

Comparing the coefficient of x^m, we obtain the Stokes matrix for the anti-Stokes direction $\theta\,=\,0$

$$C_{n_2,\,n_1} = 2\,\pi\,i\,m^{D_2}\,\frac{\widehat{G}_{n_2,\,n_1}(m)}{\Gamma(m)}\,m^{-\,d_1}\,(1\,+\,O(m^{-1})),$$

By the Sibuya-Malgrange Isomorphism, we obtain the formula for the anti-Stokes direction $\theta = \pi$:

$$\begin{aligned}
&\widehat{H}_{n_1, n_2} \mod \mathbb{C}\{x\} \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{2\pi i} \int_0^{\infty e^{\pi i}} \widehat{G}_{0-n_2}(\xi)\, \xi^{d_1 - d_2 - m - 1}\, e^{-\frac{1}{\xi}}\, \xi^{-N_{n_2}}\, d\xi \right) C_{n_1, n_2}\, x^m \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{2\pi i} \Gamma(m + d_2 - d_1)\,(1 + O(m^{-1}))\, m^{N_{n_2}}\, C_{n_2, n_1}\, x^m \right).
\end{aligned}$$

Then, we obtain the Stokes matrix for the anti-Stokes direction $\theta = \pi$

$$C_{n_1, n_2} = 2\pi i\, m^{-D_2} \frac{\widehat{H}_{n_1, n_2}(m)}{\Gamma(m)}\, m^{d_1}\,(1 + O(m^{-1})).$$

4.3.3 Case of 2 Jordan matrices

$$\begin{cases} L = D_{n_1} \oplus D_{n_2}, \\ Q(1/x) = 0_{n_1} \oplus (-\dfrac{1}{x})\, I_{n_2}, \end{cases}$$

where $D_{N_1} := d_1 I_{n_1} + N_{n_1}$, $D_{N_2} := d_2 I_{n_2} + N_{n_2}$, $d_1, d_2 \in \mathbb{R}$ et $N_{n_i}\,(i = 1, 2)$ are nilpotent matrices. By the Sibuya-Malgrange Isomorphism, we obtain

$$\begin{aligned}
&\widehat{G}_{n_2, n_1} \mod \mathbb{C}\{x\} \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{2\pi i} \int_0^{\infty} \widehat{H}_{0-n_2}(\xi)\, \xi^{d_2 - d_1 - m - 1}\, e^{-\frac{1}{\xi}}\, \xi^{N_{n_2}}\, C_{n_2, n_1}\, \xi^{-N_{n_1}}\, d\xi \right) x^m \\
&= \sum_{m=0}^{\infty} \frac{1}{2\pi i} \Gamma(m + d_1 - d_2)\,(1 + O(m^{-1}))\, m^{-N_{n_2}}\, C_{n_2, n_1}\, m^{N_{n_1}}\, x^m.
\end{aligned}$$

Comparing of the coefficient of x^m, we obtain the Stokes matrix for the anti-Stokes direction $\theta = 0$:

$$C_{n_2, n_1} = 2\pi i\, m^{D_2} \frac{G_{n_2, n_1}(m)}{\Gamma(m)}\, m^{-D_1}\,(1 + O(m^{-1})).$$

4.4 Application

In this section we shall calculate the Stokes multipliers for an example.

4.4.1 Example (dimension 2 and level 1)

We shall calculate the Stokes multipliers for the linear differential system:

$$t^2 \frac{dX}{dt} = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} \alpha & 0 \\ 0 & \alpha + \mu \end{pmatrix} t \right\} X, \tag{4.7}$$

with X a $(2 \times 2)-$matrix. The system (4.7) has an irregular singular point of Poincaré rank 1 at $t = 0$. By a change of variables

$$X = Q\,Y, \quad Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

we transform the constant matrix of (4.7) into a diagonal matrix. We obtain

$$\begin{aligned} t^2 \frac{dY}{dt} &= A(t)\,Y \\ &= (A_0 + A_1\,t)\,Y \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -\alpha - \frac{1}{2}\mu & \frac{1}{2}\mu \\ \frac{1}{2}\mu & -\alpha - \frac{1}{2}\mu \end{pmatrix} t \right\} Y. \end{aligned} \tag{4.8}$$

We apply a gauge transformation to the formal solution

$$Y = \widehat{F}(t)\,Y_1, \quad \widehat{F}(t) = \sum_{m=0}^{\infty} F(m)\,t^m = \sum_{m=0}^{\infty} \begin{pmatrix} f_m & g_m \\ h_m & k_m \end{pmatrix}, F(0) = I$$

to obtain a differential system having the diagonal matrix:

$$\begin{aligned} \frac{dY_1}{dt} &= B(t)\,Y_1 \\ &= (B_0 + B_1\,t)\,Y_1 \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} t \right\} Y_1, \end{aligned} \tag{4.9}$$

with

$$\widehat{F}' = A(t)\,\widehat{F} - \widehat{F}\,B(t).$$

By comparing coefficients, we find an equation as follows :

$$-(m-1)\,F(m-1) \;=\; A_0\,F(m) + A_1\,F(m-1) - F(m-1)\,B_1 - F(m)\,B_0, \quad (m \;\geq\; 1).$$

Setting $m \;=\; 1$, we find B_1

$$B_1 \;=\; \begin{pmatrix} -\alpha - \frac{1}{2}\,\mu & 0 \\ & \\ 0 & -\alpha - \frac{1}{2}\,\mu \end{pmatrix}, \quad F_1 \;=\; \begin{pmatrix} f_1 & \frac{1}{4}\,\mu \\ & \\ -\frac{1}{4}\,\mu & k_1 \end{pmatrix}.$$

And we can obtain a recurrence as follows:

$$\begin{cases} g_m \;=\; (-\frac{1}{8}\frac{\mu^2}{m-1} + \frac{m-1}{2})\,g_{m-1} \quad (m \;\geq\; 2), \\ \\ g_0 \;=\; 0, \quad g_1 \;=\; \frac{1}{4}\,\mu, \end{cases}$$

$$\begin{cases} h_m \;=\; (\frac{1}{8}\frac{\mu^2}{m-1} - \frac{m-1}{2})h_{m-1} \quad (m \;\geq\; 2), \\ \\ h_0 \;=\; 0, \quad h_1 = -\frac{1}{4}\,\mu. \end{cases}$$

Then, we can find the formal solution for (4.9)

$$Y_1 \;=\; \begin{pmatrix} t^{-\alpha-\frac{1}{2}\,\mu}\;\exp(-\frac{1}{t}) & 0 \\ & \\ 0 & t^{-\alpha-\frac{1}{2}\,\mu}\;\exp(\frac{1}{t}) \end{pmatrix}.$$

Therefore, we will find the Stokes and anti-Stokes directions. We set

$$q_1(1/t) = -\frac{1}{t}, \quad q_2(1/t) = \frac{1}{t}.$$

Then, the anti-Stokes direction for $q_1(1/t) \;-\; q_2(1/t)$ is $\alpha_1 = 0$, and the anti-Stokes direction for $q_2(1/t) \;-\; q_1(1/t)$ is $\alpha_2 = \pi$.

By examining the entry $(1,2)$, we find that C_{α_1} is given as follows:

$$\mathrm{pp}\,(g_m) = \frac{C_{\alpha_1}}{2\,\pi\,i}\int_0^{\infty} e^{-\frac{2}{\xi}}\,\xi^{-m}\frac{d\xi}{\xi}$$

$$= \frac{C_{\alpha_1}}{2\,\pi\,i}2^{-m}\Gamma(m)$$

$$\Leftrightarrow C_{\alpha_1} = \mathrm{pp}(2^m\,\frac{1}{\Gamma(m)}\,2\pi\,i\,g_m).$$

Similarly by examining the entry $(2,1)$, we calculate the Stokes multiplier C_{α_2} as follows.

$$\mathrm{pp}\,(h_m) = \frac{C_{\alpha_2}}{2\,\pi\,i}\int_0^{e^{\pi i}\,\infty} e^{\frac{2}{\xi}}\,\xi^{-m}\frac{d\xi}{\xi}$$

$$= \frac{C_{\alpha_2}}{2\,\pi\,i}(-2)^{-m}\Gamma(m)$$

$$\Leftrightarrow C_{\alpha_2} = \mathrm{pp}((-2)^m\,\frac{1}{\Gamma(m)}\,2\pi\,i\,h_m).$$

We denote the principal part of this formula by $\mathrm{pp}(\mathrm{CH}_{1,\,2})(m)$. Then, the Stokes matrix is:

$$\begin{pmatrix} 1 & C_{\alpha_1} \\ C_{\alpha_2} & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}.$$

4.4.2 Two Jordan blocks of dimension 1 and 3

We consider the following system of differential equations;

$$x^2\,X' \,=\, A(x)\,X \,=\, \begin{pmatrix} 0 & 0 & 0 & 0 \\ x^2 & 1 & x & 0 \\ x^2 & 0 & 1 & x \\ x^2 & 0 & 0 & 1 \end{pmatrix} X.$$

where $X(x)$ is a matrix-valued function. It admits the formal fundamental solution $\widehat{X}(x) = \widehat{F}(x)\, x^L\, e^{Q(1/x)}$ with

$$\widehat{F}(x) = I + \sum_{m=1}^{\infty} F_m\, x^m = \begin{pmatrix} 1 & \mathbf{0}_{n_1 \times n_2} \\ \widehat{G}_{n_2,\,n_1} & \mathbf{0}_{n_2} \end{pmatrix},$$

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q(1/x) = \operatorname{diag}\left(0, -\frac{1}{x}, -\frac{1}{x}, -\frac{1}{x}\right),$$

where $\mathbf{0}_{n_1 \times n_2}$ is a 1×3 matrix, $\widehat{G}_{n_2,\,n_1}$ is a 3×1 matrix, and $n_2 = 3$. Note

$$x^L\, e^{Q(1/x)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-\frac{1}{x}} & e^{-\frac{1}{x}} \log x & \frac{1}{2}\, e^{-\frac{1}{x}} (\log x)^2 \\ 0 & 0 & e^{-\frac{1}{x}} & e^{-\frac{1}{x}} \log x \\ 0 & 0 & 0 & e^{-\frac{1}{x}} \end{pmatrix}.$$

The system of differential equations admits two anti-Stokes directions: $\alpha = \mathbb{R}_{\pm}$, and the Stokes matrices are defined by

$$C_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ c_0^1 & 0 & 0 & 0 \\ c_0^2 & 0 & 0 & 0 \\ c_0^3 & 0 & 0 & 0 \end{pmatrix}, \quad C_\pi = \begin{pmatrix} 0 & c_\pi^1 & c_\pi^2 & c_\pi^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{0}.$$

By the Sibuya-Malgrange Isomorphism, we obtain a formula for the Stokes matrix in the direction $\mathbb{R}_+$:

$$\widehat{F}(x) \mod \mathbb{C}\{x\} = \sum_{m=0}^{\infty} \left(\frac{1}{2\,\pi\,i} \int_0^{\infty} F_\pi(\xi)\,\xi^{-n-1}\,e^{-\frac{1}{\xi}}\,C_0\,e^{-\frac{1}{\xi}}\,\xi^{-L}\,d\,\xi \right) x^m.$$

We compare the m-th term of the $(4,1)$ entry:

$$\widehat{f_4}(m) \approx \frac{1}{2\,\pi\,i} \int_0^{\infty} \xi^{-n-1}\,e^{-\frac{1}{\xi}} d\xi\, c_0^3$$

$$= \frac{1}{2\,\pi\,i}\,\Gamma\,(m)\,c_0^3.$$

By the Sibuya-Malgrange theorem, the left hand side has an estimate of Gevrey order 1. Therefore, we obtain the Stokes multiplier c_3^0 by dividing both sides by the Γ function.

Using

$$\Gamma^{(p)}(m + d) = m^d\,\Gamma^{(p)}(m) \left(1 + O\left(\frac{1}{m}\right)\right),$$

$$\frac{\Gamma^{(p)}(m)}{\Gamma(m)} = (\log\, m)^p \left(1 + O\left(\frac{1}{m\,\log\, m}\right)\right),$$

we compare the coefficients of the $(3,1)$ and $(2,1)$ entries:

$$\widehat{f_3}(m) \approx \frac{1}{2\,\pi\,i} \int_0^{\infty} \frac{c_0^2 + c_0^3\,\log\,\xi}{\xi^{m+1}}\,e^{-\frac{1}{\xi}} d\xi$$

$$= \frac{1}{2\,\pi\,i} \left(\Gamma\,(m)\,c_0^2 - \frac{d}{dm}\,\Gamma(m)\,c_0^3\right)$$

$$= \frac{1}{2\,\pi\,i}\,\Gamma(m) \left\{c_0^2 - \log\, m\,c_0^3 \left(1 + O\left(\frac{1}{m\,\log\, m}\right)\right)\right\},$$

and

$$\widehat{f_2}(m) \approx \frac{1}{2\,\pi\, i}\int_0^{\infty} \frac{c_0^1\, +\, c_0^2\,\log\,\xi\, +\, c_0^3\,(\log\,\xi)^2}{\xi^{n\,+\,1}}\, e^{-\frac{1}{\xi}} d\xi$$

$$= \frac{1}{2\,\pi\, i}\left(\Gamma\,(m)\, c_0^1\, -\, \frac{d}{dm}\,\Gamma(m)\, c_0^2\, +\, \left(\frac{d}{dm}\right)^2\, \Gamma(m)\, c_0^3\right)$$

$$= \frac{1}{2\,\pi\, i}\,\Gamma(m)\,\left\{c_0^1\, -\, \log\, m\, c_0^2\, +\, (\log m)^2\, c_0^3\right\}\left(1\, +\, O\left(\frac{1}{m\,\log\, m}\right)\right).$$

We obtain the following formulas;

$$\begin{cases} \widehat{f_2}(m) = \dfrac{1}{2\,\pi\, i}\,\left\{c_0^1\, -\, c_0^2\,\log m\, +\, c_0^3\,\dfrac{1}{2}\,(\log m)^2\right\}\,\Gamma(m)\,\left(1\, +\, O\left(\dfrac{1}{m\,\log m}\right)\right), \\ \widehat{f_3}(m) = \dfrac{1}{2\,\pi\, i}\,(c_0^2\, -\, c_0^3\,\log m)\,\Gamma(m)\,\left(1\, +\, O\left(\dfrac{1}{m\,\log m}\right)\right), \\ \widehat{f_4}(m) = \dfrac{1}{2\,\pi\, i}\, c_0^3\,\Gamma(m). \end{cases}$$

Next we find recurrences which $\widehat{f_j}\,(j\; =\; 2,\, 3,\, 4)$ satisfy, by using the differential equations. By the transformation equation, we obtain some initial terms;

$$F(2)\; =\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},\quad F(3)\; =\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix},$$

$$F(4)\; =\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 \end{pmatrix},\quad F(5)\; =\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ -7 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 \end{pmatrix},$$

$$F(6) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -37 & 0 & 0 & 0 \\ 34 & 0 & 0 & 0 \\ -120 & 0 & 0 & 0 \end{pmatrix}.$$

We obtain the recurrences of the first column of $\widehat{F}(x)$;

$$\begin{cases} f_2(m) = \Gamma(m)\,\Big(-3 + 3\gamma - \gamma^2 \\ \qquad\qquad + (2 - \gamma)\,\Psi(m) - \displaystyle\sum_{n=0}^{m-3} \frac{\Psi(n+2)}{n+2}\Big), \\ f_3(m) = \Gamma(m)\,(\gamma - 2 + \Psi(m)), \\ f_4(m) = -\Gamma(m), \end{cases}$$

where $\gamma \sim 0.577216\cdots$ is Euler's constant, and $\Psi(m)$ is the polygamma function. Then, we can obtain the Stokes multipliers;

$$\begin{cases} c_0^1 = -2\pi i\,(\gamma^2 - 3\gamma + 3), \\ c_0^2 = 2\pi i\,(\gamma - 2), \\ c_0^3 = -2\pi i. \end{cases}$$

We recall that the polygamma function is defined by;

$$\Psi^{(m)}(x) = \left(\frac{d}{dz}\right)^m \Psi(z) = \left(\frac{d}{dz}\right)^{m+1} \log\Gamma(z).$$

In particular,

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

4.4.3 Two Jordan blocks of dimension 2

We consider the following system of differential equations;

$$x^2\,X' \;=\; A(x)\,X \;=\; \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1+\frac{x}{2} & x \\ x^2 & 0 & 0 & 1+\frac{x}{2} \end{pmatrix} X,$$

where X is a matrix variable. It admits a formal fundamental solution: $\widehat{X}(x) = \widehat{F}(x)\,x^L\,e^{Q(1/x)}$ with

$$\widehat{F}(x) = I + \sum_{m=1}^{\infty} F_m\,x^m \;=\; \begin{pmatrix} \widehat{G}_{n_1} & \widehat{H}_{1,\,2} \\ \widehat{G}_{n_2,\,n_1} & \widehat{H}_{n_2} \end{pmatrix},$$

$$L = N_1 \,\oplus\, (\frac{1}{2}\,I_2 + N_2) \;=\; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$Q(1/x) = \operatorname{diag}\left(0,\,0,\,-\frac{1}{x},\,-\frac{1}{x}\right),$$

where $N_1,\ N_2$, are nilpotents of a dimension 2, and I_2 is the unit matrix of dimension 2. $\widehat{H}_{n_1,\,n_2}$ is a $(2\,\times\,2)-$matrix, $\widehat{G}_{n_2,\,n_1}$ is a $(2\,\times\,2)-$matrix, and $\widehat{H}_{n_2}$, $\widehat{G}_{n_1}$ have a dimension 2 with $n_1 = n_2 = 2$. Note that:

$$x^L\, e^{Q(1/x)} = \begin{pmatrix} 1 & \log x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x^{\frac{1}{2}}\, e^{-\frac{1}{x}} & x^{\frac{1}{2}} e^{-\frac{1}{x}} \log x \\ 0 & 0 & 0 & x^{\frac{1}{2}}\, e^{-\frac{1}{x}} \end{pmatrix}.$$

The system of differential equations admits two anti-Stokes directions: $\mathbb{R}_{\pm}$, and the Stokes matrices are defined by

$$C_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c_0^1 & c_0^3 & 0 & 0 \\ c_0^2 & c_0^4 & 0 & 0 \end{pmatrix}, \quad C_\pi = \begin{pmatrix} 0 & 0 & c_\pi^1 & c_\pi^3 \\ 0 & 0 & c_\pi^2 & c_\pi^4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0.$$

By the Sibuya-Malgrange Isomorphism, we obtain the following formula for the Stokes matrix in the direction $\mathbb{R}_+$:

$$\widehat{G}_{n_2,\, n_1}(x) \mod \mathbb{C}\{x\} = \sum_{m=0}^{\infty} \left(\frac{1}{2\,\pi\, i} \int_0^{\infty} \widehat{H}_{\pi - n_2}(\xi) \frac{1}{\xi^{m+1}} \xi^{\frac{1}{2}} e^{-N_2} C_{n_2, n_1}\, e^{N_1} d\xi \right) x^m,$$

where we set

$$C_{n_2,\, n_1} = \begin{pmatrix} c_0^1 & c_0^3 \\ c_0^2 & c_0^4 \end{pmatrix},$$

and we set

$$\widehat{G}_{n_2,\, n_1}(x) = \begin{pmatrix} \widehat{f}_3(x) & \widehat{g}_3(x) \\ \widehat{f}_4(x) & \widehat{g}_3(x) \end{pmatrix}.$$

We obtain the recurrences of the first column of $\widehat{F}(x)$:

$$\begin{cases} \widehat{f}_3(m) = -\dfrac{2}{\sqrt{\pi}}\,\Gamma\left(m - \dfrac{1}{2}\right)\left\{2 - \gamma - 2\log 2 - \psi\left(m - \dfrac{1}{2}\right)\right\}, \\ \widehat{f}_4(m) = -\dfrac{2}{\sqrt{\pi}}\,\Gamma\left(m - \dfrac{1}{2}\right), \\ \widehat{g}_4(m) = \dfrac{2}{\sqrt{\pi}}\,\Gamma\left(m - \dfrac{1}{2}\right)\left\{2 - \gamma - 2\log 2 - \psi\left(m - \dfrac{1}{2}\right)\right\}, \end{cases}$$

$$\begin{aligned} \widehat{g}_3(m) = {} & \frac{1}{32(m-4)(m-3)^3} \\ & \big\{(16(m-4))(6m^4 - 77m^3 + 409m^2 - 1086m + 1200)g_3(m-1) \\ & + (-637056 + 820944\,m - 22672m^4 - 444912m^2 + 131376m^3 \\ & + 2208m^5 - 96m^6)g_3(m-2) \\ & + (-2719776 + 3857328m - 150696m^4 - 2322640m^2 + 768280m^3 \\ & + 17568m^5 - 1136m^6 + 32m^7)g_3(m-3) \\ & + 2(64m^5 - 1388m^4 + 12092m^3 - 53037m^2 + 117531m - 105732) \\ & (m-5)^2 g_3(m-4) \\ & + (m-5)(2\,m-11)^3(m-6)^3\, g_3(m-5)\big\}, \end{aligned}$$

where $\gamma \sim 0.577216\cdots$ is Euler's constant γ and $\psi(m)$ is the polygamma function. By the Stokes matrix formula, we can obtain the Stokes multipliers:

$$C_{n_1,n_2} \approx \begin{pmatrix} -4\sqrt{\pi}\,i\,(2 - 2\log 2 - \gamma) & -11.55438125\,i \\ -4\sqrt{\pi}\,i & 4\sqrt{\pi}\,i\,(2 - 2\log 2 - \gamma) \end{pmatrix}.$$

A Appendix

A.1 Proof of the Poincaré-Perron theorem

We shall consider the following difference equation:

$$y(s+n) + a_{n-1}(s)y(s+n-1) + \cdots + a_0(s)y(s) = 0 \tag{A.1}$$

where we assume that s is a non negative integer,

$$a_0(s) \neq 0,$$

and the coefficients $a_j(s)$ admit the behavior

$$\lim_{s\to\infty} a_j(s) = a_j \quad (j = 0, 1, \ldots, n-1).$$

The algebraic equation

$$f(\rho) = \rho^n + a_{n-1}\rho^{n-1} + \cdots + a_0 = 0 \tag{A.2}$$

is called the characteristic equation of (A.1), whose roots are denoted by $\rho_j (j = 1, 2, \ldots, n)$.

Proposition A.1. *(H.Poincaré 1885) If the absolute values of the roots of the characteristic equation (A.2) are mutually distinct, then for every nontrivial solution $y(s)$ of (A.1), there holds*

$$\lim_{s\to\infty} \frac{y(s+1)}{y(s)} = \rho_j,$$

where ρ_j is one of the roots of (A.2).

Proof. We shall set

$$x_j(s) = y(s+j-1) \quad (j = 1, 2, \ldots, n). \tag{A.3}$$

Then, we can rewrite (A.1) as an equivalent system of difference equations

$$\begin{cases} x_j(s+1) = x_{j+1}(s), \\ x_n(s+1) = -\displaystyle\sum_{k=1}^{n} a_{k-1}(s) x_k(s). \end{cases}$$

Here, we set

$$X(s) = {}^t(x_1(s), x_2(s), \ldots, x_n(s)),$$

where $X(s) = {}^t(x_1(s), x_2(s), \ldots, x_n(s))$ means the transpose of the indicated row vector. Then, we can rewrite (A.1) as

$$X(s+1) = A(s)X(s), \tag{A.4}$$

where $A(s) = (a_{j,k}(s))$ is the following $n \times n$ matrix:

$$A(s) = (a_{j,k}(s)) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_0(s) & -a_1(s) & \cdots & \cdots & -a_{n-1}(s) \end{pmatrix}.$$

Concretely, $a_{j,k}(s)$ $(j, k = 1, 2, \ldots, n)$ is the (j, k)-element of $A(s)$ with $a_{n,k} = -a_{k-1}$ $(k = 1, 2, \ldots, n), a_{j,j+1} = 1$ $(j = 1, 2, \ldots, n-1)$, and $a_{j,k} = 0$ $(k \neq j+1, j = 1, 2, \ldots, n-1, k = 1, 2, \ldots, n)$. By our assumption on $a_j(s)$, we have

$$\lim_{s\to\infty} a_{j,k}(s) = a_{j,k} \quad (j, k = 1, 2, \ldots, n).$$

We let A be the $n \times n$ $(a_{j,k})$ matrix. For simplicity, we assume that $\det A(s) \neq 0$ for $s \geq 0$. A and $A(s)$ are companion matrix, then the eigenvalues of A and $A(s)$ are roots of the characteristic equation:

$$\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_0 = 0,$$

and

$$\rho^n + a_{n-1}(s)\rho^{n-1} + \cdots + a_0(s) = 0.$$

We set $\rho_j(s)$ to be the eigenvalue of $A(s)$ with jth largest absolute value, where we choose an arbitrary ordering for those cases where eigenvalues have the same absolute value. This choice introduces no ambiguity for large s, because eigenvalues depend continuously on matrix entries.

Now, we assume that the eigenvalues $\rho_j (j = 1, 2, \ldots, n)$ of A are mutually distinct. Then, for sufficiently large $s \geq s_1$, the eigenvalues $\rho_j(s)(j = 1, 2, \ldots, n)$ of $A(s)$ are also mutually distinct and

$$\lim_{s\to\infty} \rho_j(s) = \rho_j \quad (j = 1, 2, \ldots, n).$$

By our assumption on ρ_j, we may diagonalize A and $A(s)$:

$$\Omega(s) = C(s)A(s)C^{-1}(s),$$
$$\Omega = CAC^{-1},$$

where

$$\Omega(s) = \mathrm{diag}(\rho_1(s), \rho_2(s), \ldots, \rho_n(s)),$$
$$\Omega = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_n),$$

and

$$\lim_{s\to\infty} \Omega(s) = \Omega, \tag{A.5a}$$
$$\lim_{s\to\infty} C(s) = C, \tag{A.5b}$$
$$\lim_{s\to\infty} C^{-1}(s) = C^{-1}.$$

We shall consider the linear transform:

$$U(s) = C(s)X(s). \tag{A.6}$$

We substitute $X(s+1) = A(s)X(s)$ into (A.6), then we obtain the following:

$$\begin{aligned} U(s+1) &= C(s+1)X(s+1) \\ &= C(s+1)A(s)X(s) \\ &= C(s)A(s)X(s) - (C(s+1) - C(s))A(s)X(s) \\ &= \Omega U(s) + \left\{\Omega(s) - \Omega + (C(s+1) - C(s))A(s)C^{-1}\right\} U(s). \end{aligned}$$

Here, we set

$$E(s) = \Omega(s) - \Omega + (C(s+1) - C(s))A(s)C^{-1},$$

where $e_{j,k}(s)$ $(j, k = 1, 2, \ldots, n)$ is the (j, k)-element of $E(s)$. Then, we have

$$U(s+1) = (\Omega + E(s))U(s). \tag{A.7}$$

From (A.5a) and (A.5b), we can see that

$$\lim_{s\to\infty} E(s) = 0.$$

Here, we set

$$U(s) = {}^t(u_1(s), u_2(s), \ldots, u_n(s)),$$

where $U(s) = {}^t(u_1(s), u_2(s), \ldots, u_n(s))$ means the transpose of the indicated row vector. We can write the j-th element of (A.7) following:

$$u_j(s+1) = \rho_j u_j(s) + \sum_{i=1}^{n} e_{j,i}(s)u_i(s) \quad (j = 1, 2, \ldots, n), \tag{A.8}$$

with

$$\lim_{s\to\infty} e_{j,i}(s) = 0 \quad (j, i = 1, 2, \ldots, n). \tag{A.9}$$

Now, we assume that the absolute values of $\rho_j (j = 1, 2, \ldots, n)$ are mutually distinct:

$$|\rho_1| > |\rho_2| > \cdots > |\rho_n|. \tag{A.10}$$

As $s \geq s_1$ tends to infinity, some one of n absolute values $|u_j(s)| (j = 1, 2, \ldots, n)$ will be as large as or lager than the remaining for an infinite number of values of s.

Lemma A.2. *Let J satisfy*

$$|u_J(s)| \geq |u_j(s)| \quad (j = 1, 2, \ldots, n) \tag{A.11}$$

for infinitly many s hold for a set $S = \{s_k \mid k = 1, 2, \ldots; s_k \to \infty\}$. Then, $u_J(s_k) \neq 0$ for all $k \geq 1$.

Proof. Suppose $u_J(s_N) = 0$ for some s_N. Then, from (A.11) and (A.8) we have:

$$\begin{aligned} |u_J(s_N + 1)| &= |\rho_j u_j(s_N) + \sum_{i=1}^{n} e_{j,i}(s) u_j(s_N)| \\ &\leq |\rho_j||u_J(s_N)| + n|e_{j,i}(s_N)||u_J(s_N)| \\ &= 0. \end{aligned}$$

Then, by induction on $s \geq s_N$, we have

$$u_j(s) = 0,$$

for all $j = 1, 2, \ldots, n$. Hence, $X(s) = 0$ for all values of s because $\det A(s) \neq 0$ for $s \geq 0$. This contradicts the assumption that $y(s)$ is nontrivial. □

Lemma A.3. *Let J satisfy (A.11) for infinitly many s hold for a set $S = \{s_k \mid k = 1, 2, \ldots; s_k \to \infty\}$. Then, there exists some s_{N_0} such that (A.11) holds for all values of $s \geq s_{N_0}$.*

Proof. We divide both sids of the (A.8) by $u_J(s)$ when $j = J$ to obtain

$$\frac{u_J(s+1)}{u_J(s)} = \rho_J + d_J(s), \tag{A.12}$$

and we divide both sids of (A.8) by $u_J(s+1)$ to obtain

$$\frac{u_j(s+1)}{u_J(s+1)} = \frac{\rho_j}{\rho_J + d_J(s)} \left(\frac{u_j(s)}{u_J(s)} \right) + \frac{d_j(s)}{\rho_J + d_J(s)} \quad (j = 1, 2, \ldots, n), \tag{A.13}$$

where we set

$$d_j(s) = \sum_{i=1}^{n} e_{ji}(s) \left(\frac{u_i(s)}{u_J(s)} \right) \quad (j = 1, 2, \ldots, n).$$

From (A.9) and (A.11), for $j = 1, 2, \ldots, n$ it follows that

$$\begin{aligned} |d_j(s_k)| &\leq \sum_{i=1}^{n} |e_{ji}(s_k)| \left| \frac{u_i(s_k)}{u_J(s_k)} \right| \\ &\leq \sum_{i=1}^{n} |e_{ji}(s_k)| \\ &\leq n\bar{e}(s_k), \end{aligned}$$

where we set

$$\bar{e}(s_k) = \max_{1 \leq j, i \leq n} |e_{ji}(s_k)| \longrightarrow 0 \ (k \to \infty).$$

We shall prove the Lemma A.3 for $j = J+1, J+2, \ldots, n$ by induction. From (A.13), we have

$$\begin{aligned} \left| \frac{u_j(s_k+1)}{u_J(s_k+1)} \right| &\leq \left| \frac{\rho_j}{\rho_J + d_J(s_k)} \right| \left| \frac{u_j(s_K)}{u_J(s_k)} \right| + \left| \frac{d_j(s_k)}{\rho_J + d_J(s_k)} \right| \\ &\leq \left| \frac{\rho_j}{\rho_J + d_J(s_k)} \right| + \left| \frac{n\bar{e}(s_k)}{\rho_J + d_J(s_k)} \right|. \end{aligned}$$

Then, from (A.10), because $d_J(s_k) \to 0$ and $\bar{e}(s_k) \to 0$, we find that for arbitrarily small $\varepsilon > 0$ there exists s_{N_1} such that for all $s_k \geq s_{N_1}$

$$\left| \frac{u_j(s_k+1)}{u_J(s_k+1)} \right| \leq \left| \frac{\rho_j}{\rho_J} \right| + \varepsilon < 1 \quad (j = J+1, J+2, \ldots, n).$$

This implies that $s_k+1 \in S$ and hence the inequalities (A.11) for $j = J+1, J+2, \ldots, n$ always hold for $s \geq s_{N_1}$.

Next, we shall prove that

$$|u_j(s_k+1)| < |u_J(s_k+1)| \quad (j = 1, 2, \ldots, J-1)$$

for all $s_k \geq s_{N_2}$. We shall prove it by contradiction. We assume that for some number of j $(1 \leq j \leq J-1)$

$$|u_j(s_k+1)| \geq |u_J(s_k+1)| \quad (j = 1, 2, \ldots, J-1)$$

hold for an infinite number of values s_k. As above, we can prove that

$$\left|\frac{u_J(s_k+2)}{u_j(s_k+2)}\right| < 1 \quad (s_k \geq s_{N_1}).$$

It follows that for all $s_k \geq s_{N_1}$

$$|u_j(s_k)| \geq |u_J(s_k)| \quad (j = 1, 2, \ldots, J-1).$$

It contradicts the definition of $u_J(s)$. It concludes that if we assume (A.11) for $S = \{s_k \mid k = 1, 2, \ldots; s_k \to \infty\}$, then there exists $s \geq s_{N_0}$ such that (A.11). This contradicts (A.11) □

Lemma A.4. *Let J satisfy (A.11) for infinitly many s hold for a set $S = \{s_k \mid k = 1, 2, \ldots; s_k \to \infty\}$. Then,*

$$\lim_{s\to\infty} \frac{u_j(s)}{u_J(s)} = 0 \quad (j \neq J,\ j = 1, 2, \ldots, n). \tag{A.14}$$

Proof. First, we shall prove (A.14) for $j = J+1, J+2, \ldots, n$. To the contrary, we assume that there exists $\ell > J$ and some fixed $\eta > 0$ such that

$$\left|\frac{u_\ell(s)}{u_J(s)}\right| \geq \eta \tag{A.15}$$

for an infinite number of values $\{s'_k \mid s_{N_0} \leq s'_k \to \infty\ (k \to \infty)\}$. Then, as the proof of Lemma A.3, we obtain the following:

$$\left|\frac{u_\ell(s'_k+1)}{u_J(s'_k+1)}\right| \leq \left(\left|\frac{\rho_\ell}{\rho_J}\right| + \varepsilon\right)\left|\frac{u_\ell(s'_k)}{u_J(s'_k)}\right|.$$

We remark that

$$\left|\frac{\rho_\ell}{\rho_J}\right| + \varepsilon < 1$$

because of our assumption (A.10). Then, we can take such a large positive integer m_k that

$$\left|\frac{u_\ell(s'_k+m_k)}{u_J(s'_k+m_k)}\right| \leq \left(\left|\frac{\rho_\ell}{\rho_J}\right| + \varepsilon\right)^{m_k}\left|\frac{u_\ell(s'_k)}{u_J(s'_k)}\right| < \eta.$$

This, in turn, implies that there is an infinite set $S'' = \{s''_k\}$ for which there holds

$$\left|\frac{u_\ell(s''_k)}{u_J(s''_k)}\right| < \eta.$$

Then, from (A.13), we have

$$\left|\frac{u_\ell(s_k''+1)}{u_J(s_k''+1)}\right| \leq \left|\frac{\rho_\ell}{\rho_J + d_J(s_k'')}\right|\eta + \left|\frac{d_\ell(s_k'')}{\rho_J + d_J(s_k'')}\right| < \eta$$

for all $s \geq s_{N_3}$. This is the contradiction for the assumption (A.15).

Next, we shall prove (A.14) for $j = 1, 2, \ldots, J-1$. To the contrary, we assume that there exists $\ell < J$ and some fixed $\eta > 0$ such that (A.15) for an infinite number of values $\{s_k' \mid s_{N_0} \leq s_k' \to \infty\ (k \to \infty)\}$. In this case, se have for sufficiently large s_k'

$$\left|\frac{u_\ell(s_k'+m_k)}{u_J(s_k'+m_k)}\right| \geq \left(\left|\frac{\rho_\ell}{\rho_J}\right| - \varepsilon\right)^{m_k} \eta > 1,$$

that is, we have an infinite number of values of s for which the above inequality holds. □

We shall return the proof of Proposition A.1. From (A.12), we immediately obtain

$$\lim_{s\to\infty} \frac{u_J(s+1)}{u_J(s)} = \rho_J.$$

We conclude it and Lemma A.11. For any solution $U(s) = (u_1(s), u_2(s), \ldots, u_n(s))$ of (A.8), there necessarily exists such an element $u_J(d)$ that

$$\begin{cases} \lim\limits_{s\to\infty} \dfrac{u_J(s+1)}{u_J(s)} = \rho_J, \\ \lim\limits_{s\to\infty} \dfrac{u_j(s)}{u_J(s)} = 0 \quad (j \neq J,\ j = 1, 2, \ldots, n). \end{cases}$$

We now again consider (A.4). Setting $C^{-1}(s) = (\hat{c}_{ji}(s))$ we have

$$x_j(s) = \sum_{i=1}^{n} \hat{c}_{ji}(s) u_i(s) \quad (j = 1, 2, \ldots, n),$$

with

$$\lim_{s\to\infty} \hat{c}_{ji}(s) = \hat{c}_{ji} \quad (j, i = 1, , 2 \ldots, n).$$

For each i, not all of $\hat{c}_{ji}$ $(j = 1, , 2 \ldots, n)$ are vanishing. So, for J, let $\hat{c}_{KJ} \neq 0$.

$$\begin{cases} \lim\limits_{s\to\infty} \dfrac{x_K(s+1)}{x_K(s)} = \lim\limits_{s\to\infty} \dfrac{\hat{c}_{KJ}(s+1)u_J(s+1)}{\hat{c}_{KJ}(s)u_J(s)} = \rho_J, \\ \lim\limits_{s\to\infty} \dfrac{x_j(s)}{x_K(s)} = \lim\limits_{s\to\infty} \dfrac{\hat{c}_{jJ}(s)}{\hat{c}_{KJ}(s)u_J(s)} = \dfrac{\hat{c}_{jJ}}{\hat{c}_{KJ}} \quad (j \neq K), \end{cases}$$

that is, one of the elements of the solution $X(s) = (x_1(s), x_2(s), \ldots, x_n(s))$ tends to some one of eigenvalues of A as $s \longrightarrow \infty$. From (A.3), we obtain

$$\lim_{s\to\infty} \frac{y(s+K)}{y(s+K-1)} = \rho_J.$$

Substituting $K = 1$ into it, we have

$$\lim_{s\to\infty} \frac{y(s+1)}{y(s)} = \rho_J,$$

with

$$\hat{c}_{1J} \neq 0.$$

Thus we have completed the proof of Proposition A.1. □

We shall introduce the Poincaré-Perron theorem.

Theorem A.5. *(Poincaré-Perron) We consider the following deference equation:*

$$\Phi(s+q) + a_1(s)\Phi(s+q-1) + \cdots + a_q(x)\Phi(s) = 0, \tag{A.16}$$

where the coefficients $a_j(s)(j = 1, 2, \ldots, q)$ have two following properties:

- *There is a constant $\widehat{a}_j(s)$ such that*

$$\lim_{k\to\infty} \frac{a_j(s)}{s^{k_j}} = \widehat{a}_j \quad (j = 1, 2, \ldots, q).$$

- *The points*

$$(0, 0), (1, k_1), \ldots, (q, k_q)$$

are on a straight line or below the Newton polygon.

Then, we have

$$\lim_{s\to\infty} \sup \left(\frac{|\Phi(s)|}{|\Gamma(s+1)^{\tau}|} \right)^{1/s} = |\gamma_j|,$$

where $\tau = \frac{1}{q}$, and γ_j is one of the roots of the equation

$$t^q + \widehat{a}_{i_1} t^{q-i_1} + \widehat{a}_{i_2} t^{q-i_2} + \cdots + \widehat{a}_q = 0.$$

A.2 Proof of the Sibuya-Malgrange Isomorphism

In this section, we shall review some properties of exact sequences of cohomology and sheaves on S^1.

We shall make a remark about (4.4).

Remark A.6. *We obtain a commutative diagram from the short exact sequence (4.1):*

$$\begin{array}{ccccc}
0 \to \Gamma(S^1, \mathcal{A}^{<0}) & \longrightarrow & \Gamma(S^1, \mathcal{A}) & \longrightarrow & \Gamma(S^1, \mathbb{C}[[t]]_{S^1}) \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \mathcal{C}^0(\mathcal{U}, \mathcal{A}^{<0}) & \longrightarrow & \mathcal{C}^0(\mathcal{U}, \mathcal{A}) & \longrightarrow & \mathcal{C}^0(\mathcal{U}, \mathbb{C}[[t]]) \to 0 \\
\delta_0 \downarrow & & \downarrow \delta_0 & & \downarrow \delta_0 \\
0 \to \mathcal{C}^1(\mathcal{U}, \mathcal{A}^{<0}) & \longrightarrow & \mathcal{C}^1(\mathcal{U}, \mathcal{A}) & \longrightarrow & \mathcal{C}^1(\mathcal{U}, \mathbb{C}[[t]]) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
H^1(S^1, \mathcal{A}^{<0}) & \xrightarrow{\Phi} & H^1(S^1, \mathcal{A}) & \longrightarrow & 0
\end{array} \tag{A.17}$$

Applying the snake lemma, and the fact that $\Gamma(S^1, \mathcal{A}^{<0}) = 0$, *we obtain the exact sequence of cohomology (4.2):*

$$0 \longrightarrow \Gamma(S^1, \mathcal{A}) \longrightarrow \Gamma(S^1, \mathbb{C}[[t]]_{S^1}) \longrightarrow H^1(S^1, \mathcal{A}^{<0}) \longrightarrow 0.$$

We shall prove the equality $\delta_0(\oplus v_j)$ and $\oplus \mathrm{var}(w_j)$ of Lemma 4.10.

Proof. We recall that we set $v_j(t) \in \Gamma(I_j, \mathcal{A})$ following,

$$v_j := \sum_{i=1}^{k} w_i|_{I_j} + h(t)$$

with $h(t) \in \mathbb{C}\{t\}$.

Let us set

$$I_{j,j+1} := I_j \cap I_{j+1} \ (j = 1, 2, \ldots, n-1), I_{n,1} := I_n \cap I_1 \quad (j = 1, 2, \ldots, n).$$

Setting for $(j = 1, 2, \ldots, n-1)$

$$v_j|_{I_j \cap I_{j+1}} := \sum_{k=1}^{n} w_k(t)$$

$$v_{j+1}|_{I_j \cap I_{j+1}} := \sum_{k=1}^{n} w_k(t) - w_j(t) + w_j(te^{2\pi i}),$$

and

$$v_n|_{I_n \cap I_1} := \sum_{k=1}^{n} w_k(t)$$

$$v_1|_{I_n \cap I_1} := \sum_{k=1}^{n-1} w_k(t) + w_n(te^{2\pi i}),$$

we obtain

$$\delta_0(v_j) = v_j|_{I_j \cap I_{j+1}} - v_{j+1}|_{I_j \cap I_{j+1}} = w_j(t) - w_j(te^{2\pi i}) = \mathrm{var}(w_j),$$
$$\delta_0(v_n) = v_n|_{I_n \cap I_1} - v_1|_{I_n \cap I_1} = w_n(t) - w_n(te^{2\pi i}) = \mathrm{var}(w_n).$$

By these formulas, we can conclude

$$\delta_0(\oplus v_j) = \oplus \mathrm{var}(w_j).$$

□

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