ON A TRANSFORM OF AN ACYCLIC COMPLEX AND ITS APPLICATION

January 2015

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（千葉大学審査学位論文）

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2015年1月

千葉大学大学院　理学研究科

基盤理学専攻　数学・情報数理学コース

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Preface

Let I and Q be two ideals of a commutative ring R. We set

$$I :_R Q = \{a \in R \mid aQ \subseteq I\}$$

and call it the ideal quotient of I by Q. This is an ideal of R which contains I. The ideal quotient is a very important notion in the theory of commutative algebra. For example, if $(R, \mathfrak{m})$ is a Noetherian local ring and I is an $\mathfrak{m}$-primary ideal of R, then $\operatorname{depth} R/I = \dim R/I = 0$, so the Gorensteinness of R/I is characterized by the socle of R/I $\operatorname{Soc} R/I = (I :_R \mathfrak{m})/I$. Also, when $(R, \mathfrak{m})$ is a local ring and I is an ideal of R, we define the saturation of I $(I)^{\mathrm{sat}}$ using $I :_R \mathfrak{m}^i$, where i is a positive integer.

The $*$-transform of an acyclic complex of length 3 is introduced in [8] for the purpose of composing an R-free resolution of the ideal quotient of a certain ideal whose R-free resolution is given, and its generalization is explained in [18]. Here, let us recall the outline of the generalized $*$-transform.

Let $(R, \mathfrak{m})$ be an n-dimensional Cohen-Macaulay local ring, where $2 \leq n \in \mathbb{Z}$, and let $x_1, x_2, \ldots, x_n$ be an sop for R. We put $Q = (x_1, x_2, \ldots, x_n)R$. Suppose that an acyclic complex

$$F_\bullet \quad : \quad 0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

of finitely generated free R-modules such that $\operatorname{Im} \varphi_n \subseteq QF_{n-1}$ is given. We put $M = \operatorname{Im} \varphi_1$. M is an R-submodule of F_0, and $F_\bullet$ is an R-free resolution of F_0/M. Then, transforming $F_\bullet$, we can get an acyclic complex

$$ {}^*F_\bullet \quad : \quad 0 \longrightarrow {}^*F_n \xrightarrow{{}^*\varphi_n} {}^*F_{n-1} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0 $$

of finitely generated free R-modules such that $\operatorname{Im} {}^*\varphi_1 = M :_{F_0} Q = \{x \in F_0 \,|\, Qx \subseteq M\}$ and $\operatorname{Im} {}^*\varphi_n \subseteq \mathfrak{m} \cdot {}^*F_{n-1}$. ${}^*F_\bullet$ is an R-free resolution of $F_0/(M :_{F_0} Q)$. We call ${}^*F_\bullet$ the $*$-transform of $F_\bullet$ with respect to $x_1, x_2, \ldots, x_n$. If $F_0 = R$, then M is an ideal of R, so $M :_{F_0} Q$ is an ideal quotient.

We give a little more detailed explanation of this operation. We use the Koszul complex $K_\bullet = K_\bullet(x_1, x_2, \ldots, x_n)$. We denote the boundary map of $K_\bullet$ by $\partial_\bullet$. Let $e_1, e_2, \ldots, e_n$ be an R-free basis of K_1 such that $\partial_1(e_i) = x_i$ for $i = 1, 2, \ldots, n$. Moreover, we use the following notation about $K_\bullet$:

- $N := \{1, 2, \ldots, n\}$.
- $N_p := \{I \subseteq N \,|\, \sharp\, I = p\}$ for $1 \leq p \leq n$ and $N_0 := \{\emptyset\}$. Here, if S is a finite set, $\sharp\, S$ denotes the number of elements of S.
- If $1 \leq p \leq n$ and $I = \{i_1, i_2, \ldots, i_p\} \in N_p$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq n$, we set

$$ e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \in K_p. $$

  In particular, for $i = 1, 2, \ldots, n$, $\check{e}_i := e_{N \setminus \{i\}} \in K_{n-1}$. Furthermore, $e_\emptyset$ denotes the identity element 1_R of $R = K_0$.

- If $1 \leq p \leq n$, $I \in N_p$ and $i \in N$, we set

$$ s(i, I) = \sharp\, \{j \in I \,|\, j < i\}. $$

We define $\sharp\,\emptyset = 0$, so $s(i, I) = 0$ if $i \le \min I$.

Then, for any $p = 0, 1, \dots, n$, $\{e_I\}_{I \in N_p}$ is an R-free basis of K_p and

$$\partial_p(e_I) = \sum_{i \in I} (-1)^{s(i,I)} \cdot x_i \cdot e_{I \setminus \{i\}}.$$

We explain subsequently. Let us fix an R-free basis of F_n, say $\{v_\lambda\}_{\lambda \in \Lambda}$. We set $\widetilde{\Lambda} = \Lambda \times N$ and take a family $\{v_{(\lambda,i)}\}_{(\lambda,i) \in \widetilde{\Lambda}}$ of elements in F_{n-1} so that

$$\varphi_n(v_\lambda) = \sum_{i \in N} x_i \cdot v_{(\lambda,i)}$$

for all $\lambda \in \Lambda$. This is possible as $\operatorname{Im} \varphi_n \subseteq QF_{n-1}$. The next theorem is the essential part of the process to get ${}^*F_\bullet$.

Theorem. *There exists a chain map $\sigma_\bullet : F_n \otimes_R K_\bullet \longrightarrow F_\bullet$*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & F_n \otimes_R K_n & \xrightarrow{F_n \otimes \partial_n} & F_n \otimes_R K_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_n \otimes_R K_1 & \xrightarrow{F_n \otimes \partial_1} & F_n \otimes_R K_0 \\
 & & \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_{n-1}} & & & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_0} \\
0 & \longrightarrow & F_n & \xrightarrow{\varphi_n} & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \xrightarrow{\varphi_1} & F_0
\end{array}$$

satisfying the following four conditions.

(1) $\sigma_0^{-1}(\operatorname{Im} \varphi_1) = \operatorname{Im}(F_n \otimes \partial_1)$.

(2) $\operatorname{Im} \sigma_0 + \operatorname{Im} \varphi_1 = M :_{F_0} Q$.

(3) $\sigma_{n-1}(v_\lambda \otimes \check{e}_i) = (-1)^{n+i-1} \cdot v_{(\lambda,i)}$ *for all* $(\lambda, i) \in \widetilde{\Lambda}$.

(4) $\sigma_n(v_\lambda \otimes e_N) = (-1)^n \cdot v_\lambda$ *for all* $\lambda \in \Lambda$.

In the rest, $\sigma_\bullet : F_n \otimes_R K_\bullet \longrightarrow F_\bullet$ is the chain map constructed in the above theorem. We take the mapping cone of $\sigma_\bullet$. We notice that it gives an R-free resolution of length

$n+1$ of $F_0/(M :_{F_0} Q)$, that is,

$$0 \longrightarrow F_n \otimes_R K_n \xrightarrow{\psi_{n+1}} \begin{matrix} F_n \otimes_R K_{n-1} \\ \oplus \\ F_n \end{matrix} \xrightarrow{\psi_n} \begin{matrix} F_n \otimes_R K_{n-2} \\ \oplus \\ F_{n-1} \end{matrix} \xrightarrow{'\varphi_{n-1}} \begin{matrix} F_n \otimes_R K_{n-3} \\ \oplus \\ F_{n-2} \end{matrix}$$
$$\xrightarrow{^*\varphi_{n-2}} \begin{matrix} F_n \otimes_R K_{n-4} \\ \oplus \\ F_{n-3} \end{matrix} \longrightarrow \cdots \longrightarrow \begin{matrix} F_n \otimes_R K_1 \\ \oplus \\ F_2 \end{matrix} \xrightarrow{^*\varphi_2} \begin{matrix} F_n \otimes_R K_0 \\ \oplus \\ F_1 \end{matrix} \xrightarrow{^*\varphi_1} F_0$$

is acyclic and $\operatorname{Im} {}^*\varphi_1 = M :_{F_0} Q$, where

$$\psi_{n+1} = \begin{pmatrix} F_n \otimes \partial_n \\ (-1)^n \cdot \sigma_n \end{pmatrix}, \ \psi_n = \begin{pmatrix} F_n \otimes \partial_{n-1} & 0 \\ (-1)^{n-1} \cdot \sigma_{n-1} & \varphi_n \end{pmatrix}, \ '\varphi_{n-1} = \begin{pmatrix} F_n \otimes \partial_{n-2} & 0 \\ (-1)^{n-2} \cdot \sigma_{n-2} & \varphi_{n-1} \end{pmatrix},$$
$$^*\varphi_p = \begin{pmatrix} F_n \otimes \partial_{p-1} & 0 \\ (-1)^{p-1} \cdot \sigma_{p-1} & \varphi_p \end{pmatrix} \text{ for } 2 \le p \le n-2 \text{ and } {}^*\varphi_1 = \begin{pmatrix} \sigma_0 & \varphi_1 \end{pmatrix}.$$

Here, since $\sigma_n : F_n \otimes_R K_n \longrightarrow F_n$ is an isomorphism by (4) of the above theorem, ψ_{n+1} splits, and therefore, removing $F_n \otimes_R K_n$ and F_n from

$$0 \longrightarrow F_n \otimes_R K_n \xrightarrow{\psi_{n+1}} \begin{matrix} F_n \otimes_R K_{n-1} \\ \oplus \\ F_n \end{matrix} \xrightarrow{\psi_n} \begin{matrix} F_n \otimes_R K_{n-2} \\ \oplus \\ F_{n-1} \end{matrix} \longrightarrow \cdots,$$

we get the acyclic complex

$$0 \longrightarrow {}'F_n \xrightarrow{'\varphi_n} {}'F_{n-1} \xrightarrow{'\varphi_{n-1}} {}^*F_{n-2} \xrightarrow{^*\varphi_{n-2}} {}^*F_{n-3} \longrightarrow \cdots \longrightarrow {}^*F_2 \xrightarrow{^*\varphi_2} {}^*F_1 \xrightarrow{^*\varphi_1} {}^*F_0 = F_0,$$

where

$$'F_n = F_n \otimes_R K_{n-1}, \ 'F_{n-1} = \begin{matrix} F_n \otimes_R K_{n-2} \\ \oplus \\ F_{n-1} \end{matrix}, \ {}^*F_p = \begin{matrix} F_n \otimes_R K_{p-1} \\ \oplus \\ F_p \end{matrix} \text{ for } 1 \le p \le n-2$$
$$\text{and } '\varphi_n = \begin{pmatrix} F_n \otimes \partial_{n-1} \\ (-1)^{n-1} \cdot \sigma_{n-1} \end{pmatrix}.$$

This complex is an R-free resolution of length n of $F_0/(M :_{F_0} Q)$, but $\operatorname{Im} {}'\varphi_n \subseteq \mathfrak{m} \cdot {}'F_{n-1}$ may not hold. Thus, it is necessary to remove non-minimal components from $'F_n$ and $'F_{n-1}$. Going through this operation, we get free R-modules $^*F_n$ and $^*F_{n-1}$ such that

$$0 \longrightarrow {}^*F_n \xrightarrow{^*\varphi_n} {}^*F_{n-1} \xrightarrow{^*\varphi_{n-1}} {}^*F_{n-2} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{^*\varphi_1} {}^*F_0 = F_0$$

is acyclic and $\operatorname{Im}{}^*\varphi_n \subseteq \mathfrak{m} \cdot {}^*F_{n-1}$, where ${}^*\varphi_n$ and ${}^*\varphi_{n-1}$ are the restrictions of ${}'\varphi_n$ and ${}'\varphi_{n-1}$, respectively.

Here, we give a supplementary explanation about the length of ${}^*F_\bullet$. If $\{v_{(\lambda,i)}\}_{(\lambda,i)\in\widetilde{\Lambda}}$ is a subset of a certain R-free basis of F_{n-1}, then ${}'\varphi_n$ splits. Therefore, in this case, we can remove $F_n \otimes_R K_{n-1}$ itself and an unnecessary component of F_{n-1} from

$$0 \longrightarrow F_n \otimes_R K_{n-1} \xrightarrow{{}'\varphi_n} \begin{array}{c} F_n \otimes_R K_{n-2} \\ \oplus \\ F_{n-1} \end{array} \xrightarrow{{}'\varphi_{n-1}} {}^*F_{n-2} \longrightarrow \cdots \, ,$$

and we get the free R-module ${}^*F_{n-1}$ such that

$$0 \longrightarrow {}^*F_{n-1} \xrightarrow{{}^*\varphi_{n-1}} {}^*F_{n-2} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0$$

is acyclic, where ${}^*\varphi_{n-1}$ is the restriction of ${}'\varphi_{n-1}$. This complex is an R-free resolution of length $n-1$ of $F_0/(M :_{F_0} Q)$, and so we have $\operatorname{depth}_R F_0/(M :_{F_0} Q) > 0$. This condition is very important for analyzing symbolic powers of ideals through ideal quotients.

If R is regular, for any finitely generated free R-module F_0 and any R-submodule M of F_0, we can take $\mathfrak{m}$ and the minimal R-free resolution of F_0/M as Q and $F_\bullet$, respectively, and then ${}^*F_\bullet$ gives an R-free resolution of $F_0/(M :_{F_0} \mathfrak{m})$. Here, we notice that we can take the $*$-transform of ${}^*F_\bullet$ again because $\operatorname{Im}{}^*\varphi_n \subseteq \mathfrak{m} \cdot {}^*F_{n-1}$, and an R-free resolution of $F_0/(M :_{F_0} \mathfrak{m}^2)$ is induced. Repeating this procedure, we get an R-free resolution of $F_0/(M :_{F_0} \mathfrak{m}^k)$ for any positive integer k, and it contains complete information about the 0-th local cohomology module of F_0/M with respect to $\mathfrak{m}$. This method is very useful for computing the symbolic powers of the ideal generated by the maximal minors of a certain matrix. In fact, in [8], the symbolic powers of the case of a 2×3 matrix are computed using this method.

In the first half of this paper, we describe substance of the generalized $*$-transform and its proof in detail.

In the second half, we compute the saturation of the powers of a certain determinantal ideal, applying the theory of $*$-transform. We assume that $(R, \mathfrak{m})$ is an $(m+1)$-dimensional Cohen-Macaulay local ring, where $2 \leq m \in \mathbb{Z}$. Let $x_1, x_2, \ldots, x_m, x_{m+1}$ be an sop for R, and let $\{\alpha_{i,j}\}$ be a family of positive integers, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m, m+1$. We set

$$a_{i,j} = \begin{cases} x_{i+j-1}^{\alpha_{i,j}} & \text{if } i+j \leq m+2 \\ \\ x_{i+j-m-2}^{\alpha_{i,j}} & \text{if } i+j > m+2 \end{cases}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m, m+1$, and consider the matrix $A = (a_{i,j})$ of size $m \times (m+1)$. If $\alpha_{i,j} = 1$ for all i and j, the matrix A looks

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_m & x_{m+1} \\ x_2 & x_3 & x_4 & \cdots & x_{m+1} & x_1 \\ x_3 & x_4 & x_5 & \cdots & x_1 & x_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_m & x_{m+1} & x_1 & \cdots & x_{m-2} & x_{m-1} \end{pmatrix}.$$

In this situation, we denote the ideal generated by the maximal minors of A by I, and study the saturation of the m-th power of I.

Contents

Chapter 1

The $*$-transforms of acyclic complexes

1.1 Introduction to Chapter 1

Let I and J be ideals of a commutative ring R. The ideal quotient

$$I :_R J = \{a \in R \,|\, aJ \subseteq I\}$$

is an important notion in the theory of commutative algebra. For example, if $(R, \mathfrak{m})$ is a Noetherian local ring and I is an $\mathfrak{m}$-primary ideal of R, the Gorenstein property of R/I is characterized by the socle $\mathrm{Soc}(R/I) = (I :_R \mathfrak{m})/I$. The $*$-transform of an acyclic complex of length 3 is introduced in [8] for the purpose of composing an R-free resolution of the ideal quotient of a certain ideal I whose R-free resolution is given. Here, let us recall its outline.

Let $(R, \mathfrak{m})$ be a 3-dimensional Cohen-Macaulay local ring and Q a parameter ideal of R. Suppose that an acyclic complex

$$F_\bullet \quad : \quad 0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 = R$$

of finitely generated free R-modules such that $\operatorname{Im}\varphi_3 \subseteq QF_2$ is given. Then, taking the $*$-transform of $F_\bullet$, we get an acyclic complex

$$^*F_\bullet \quad : \quad 0 \longrightarrow {}^*F_3 \xrightarrow{^*\varphi_3} {}^*F_2 \xrightarrow{^*\varphi_2} {}^*F_1 \xrightarrow{^*\varphi_1} {}^*F_0 = R$$

of finitely generated free R-modules such that $\operatorname{Im}{}^*\varphi_1 = \operatorname{Im}\varphi_1 :_R Q$ and $\operatorname{Im}{}^*\varphi_3 \subseteq \mathfrak{m}\cdot {}^*F_2$. If R is regular, for any ideal I of R, we can take $\mathfrak{m}$ and the minimal R-free resolution of R/I as Q and $F_\bullet$, respectively, and then ${}^*F_\bullet$ gives an R-free resolution of $R/(I :_R \mathfrak{m})$. Here, let us notice that we can take the $*$-transform of ${}^*F_\bullet$ again since $\operatorname{Im}{}^*\varphi_3 \subseteq \mathfrak{m}\cdot {}^*F_2$, and an R-free resolution of $R/(I :_R \mathfrak{m}^2)$ is induced. Repeating this procedure, we get an R-free resolution of $R/(I :_R \mathfrak{m}^k)$ for any $k > 0$, and it contains complete information about the 0-th local cohomology module of R/I with respect to $\mathfrak{m}$. This method is very useful for computing the symbolic powers of the ideal generated by the maximal minors of a certain 2×3 matrix as is described in [8].

Thus, in [8], the theory of $*$-transform is developed for only acyclic complexes of length 3 on a 3-dimensional Cohen-Macaulay local ring. The purpose of this chapter is to generalize the machinery of $*$-transform so that we can apply it to acyclic complexes of length n as follows. Let $(R,\mathfrak{m})$ be an n-dimensional Cohen-Macaulay local ring, where $2 \le n \in \mathbb{Z}$, and let Q be a parameter ideal of R. Suppose that an acyclic complex

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

of finitely generated free R-modules such that $\operatorname{Im}\varphi_n \subseteq QF_{n-1}$ is given. We aim to give a concrete procedure to get an acyclic complex

$$0 \longrightarrow {}^*F_n \xrightarrow{{}^*\varphi_n} {}^*F_{n-1} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0$$

of finitely generated free R-modules such that $\mathrm{Im}\,{}^*\varphi_1 = \mathrm{Im}\,\varphi_1 :_{F_0} Q$ and $\mathrm{Im}\,{}^*\varphi_n \subseteq \mathfrak{m}\cdot{}^*F_{n-1}$. Let us notice that we don't need any restriction on the rank of F_0, so there may be some application to the study of $M :_F Q$, where F is a finitely generated free R-module and M is an R-submodule of F. Moreover, as the generalized $*$-transform works for acyclic complexes of length $n \geq 2$, we can apply it to the study of some ideal quotients in n-dimensional Cohen-Macaulay local rings. In fact, in the subsequent paper [9], setting I to be the m-th power of the ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} x_1^{\alpha_{1,1}} & x_2^{\alpha_{1,2}} & x_3^{\alpha_{1,3}} & \cdots & x_m^{\alpha_{1,m}} & x_{m+1}^{\alpha_{1,m+1}} \\ x_2^{\alpha_{2,1}} & x_3^{\alpha_{2,2}} & x_4^{\alpha_{2,3}} & \cdots & x_{m+1}^{\alpha_{2,m}} & x_1^{\alpha_{2,m+1}} \\ x_3^{\alpha_{3,1}} & x_4^{\alpha_{3,2}} & x_5^{\alpha_{3,3}} & \cdots & x_1^{\alpha_{3,m}} & x_2^{\alpha_{3,m+1}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_m^{\alpha_{m,1}} & x_{m+1}^{\alpha_{m,2}} & x_1^{\alpha_{m,3}} & \cdots & x_{m-2}^{\alpha_{m,m}} & x_{m-1}^{\alpha_{m,m+1}} \end{pmatrix}$$

and setting $Q = (x_1, x_2, x_3, \ldots, x_m, x_{m+1})R$, where $x_1, x_2, x_3, \ldots, x_m, x_{m+1}$ is an sop for an $(m+1)$-dimensional Cohen-Macaulay local ring R and $\{\alpha_{i,j}\}_{1\leq i\leq m, 1\leq j\leq m+1}$ is a family of positive integers, the ideal quotient $I :_R Q$ is computed, and it is proved that $I :_R Q$ coincides with the saturation of I, that is, the depth of $R/(I :_R Q)$ is positive.

Throughout this chapter, R is a commutative ring, and in Section 1.3, we assume that R is an n-dimensional Cohen-Macaulay local ring. For R-modules G and H, the elements of $G \oplus H$ are denoted by column vectors;

$$\begin{pmatrix} g \\ h \end{pmatrix} \quad (g \in G,\ h \in H).$$

In particular, the elements of the forms

$$\begin{pmatrix} g \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ h \end{pmatrix}$$

are denoted by $[g]$ and $\langle h \rangle$, respectively. Moreover, if V is a subset of G, then the family $\{[v]\}_{v \in V}$ is denoted by $[V]$. Similarly $\langle W \rangle$ is defined for a subset W of H. If T is a subset of an R-module, we denote by $R \cdot T$ the R-submodule generated by T. If S is a finite set, $\sharp\, S$ denotes the number of elements of S.

1.2 Preliminaries for Chapter 1

In this section, we summarize preliminary results. Let R be a commutative ring.

Lemma 1.2.1. *Let $G_\bullet$ and $F_\bullet$ be acyclic complexes, whose boundary maps are denoted by $\partial_\bullet$ and $\varphi_\bullet$, respectively. Suppose that a chain map $\sigma_\bullet : G_\bullet \longrightarrow F_\bullet$ is given and $\sigma_0^{-1}(\operatorname{Im} \varphi_1) = \operatorname{Im} \partial_1$ holds. Then the mapping cone* $\operatorname{Cone}(\sigma_\bullet)$:

$$\cdots \longrightarrow \begin{matrix} G_{p-1} \\ \oplus \\ F_p \end{matrix} \xrightarrow{\psi_p} \begin{matrix} G_{p-2} \\ \oplus \\ F_{p-1} \end{matrix} \longrightarrow \cdots \longrightarrow \begin{matrix} G_1 \\ \oplus \\ F_2 \end{matrix} \xrightarrow{\psi_2} \begin{matrix} G_0 \\ \oplus \\ F_1 \end{matrix} \xrightarrow{\psi_1} F_0 \longrightarrow 0$$

is acyclic, where

$$\psi_p = \begin{pmatrix} \partial_{p-1} & 0 \\ (-1)^{p-1} \cdot \sigma_{p-1} & \varphi_p \end{pmatrix} \textit{ for all } p \geq 2 \textit{ and } \psi_1 = \begin{pmatrix} \sigma_0 & \varphi_1 \end{pmatrix}.$$

Hence, if $G_\bullet$ and $F_\bullet$ are complexes of finitely generated free R-modules, then $\operatorname{Cone}(\sigma_\bullet)$ *gives an R-free resolution of $F_0/(\operatorname{Im} \varphi_1 + \operatorname{Im} \sigma_0)$.*

Proof. See [8, 2.1]. $\square$

Lemma 1.2.2. *Let $2 \leq n \in \mathbb{Z}$ and $C_{\bullet\bullet}$ be a double complex such that $C_{p,q} = 0$ unless $0 \leq p, q \leq n$. For any $p, q \in \mathbb{Z}$, we denote the boundary maps $C_{p,q} \longrightarrow C_{p-1,q}$ and $C_{p,q} \longrightarrow C_{p,q-1}$ by $d'_{p,q}$ and $d''_{p,q}$, respectively. We assume that $C_{p\bullet}$ and $C_{\bullet q}$ are acyclic for $0 \leq p, q \leq n$. Let $T_\bullet$ be the total complex of $C_{\bullet\bullet}$ and let $d_\bullet$ be its boundary map, that is, if $\xi \in C_{p,q} \subseteq T_r$ $(p + q = r)$, then*

$$d_r(\xi) = (-1)^p \cdot d''_{p,q}(\xi) + d'_{p,q}(\xi) \in C_{p,q-1} \oplus C_{p-1,q} \subseteq T_{r-1}.$$

Then the following assertions hold.

(1) *Suppose that $\xi_n \in C_{n,0}$ and $\xi_{n-1} \in C_{n-1,1}$ such that $d'_{n,0}(\xi_n) = (-1)^n \cdot d''_{n-1,1}(\xi_{n-1})$ are given. Then there exist elements $\xi_p \in C_{p,n-p}$ for all $p = 0, 1, \ldots, n-2$ such that*

$$\begin{aligned}\xi_n + \xi_{n-1} + \xi_{n-2} + \cdots + \xi_0 &\in \operatorname{Ker} d_n \\ &\subseteq T_n = C_{n,0} \oplus C_{n-1,1} \oplus C_{n-2,2} \oplus \cdots \oplus C_{0,n}.\end{aligned}$$

(2) *Suppose that $\xi_n + \xi_{n-1} + \cdots + \xi_1 + \xi_0 \in \operatorname{Ker} d_n \subseteq T_n = C_{n,0} \oplus C_{n-1,1} \oplus \cdots \oplus C_{1,n-1} \oplus C_{0,n}$ and $\xi_0 \in \operatorname{Im} d'_{1,n}$. Then*

$$\xi_n + \xi_{n-1} + \cdots + \xi_1 + \xi_0 \in \operatorname{Im} d_{n+1}.$$

In particular, we have $\xi_n \in \operatorname{Im} d''_{n,1}$.

Proof. (1) It is enough to show that if $1 \leq p \leq n-1$ and two elements $\xi_{p+1} \in C_{p+1,n-p-1}$, $\xi_p \in C_{p,n-p}$ such that

$$d'_{p+1,n-p-1}(\xi_{p+1}) = (-1)^{p+1} \cdot d''_{p,n-p}(\xi_p)$$

are given, then we can take $\xi_{p-1} \in C_{p-1,n-p+1}$ so that

$$d'_{p,n-p}(\xi_p) = (-1)^p \cdot d''_{p-1,n-p+1}(\xi_{p-1}).$$

In fact, if the assumption of the claim stated above is satisfied, we have

$$\begin{aligned} d''_{p-1,n-p}(d'_{p,n-p}(\xi_p)) &= d'_{p,n-p-1}(d''_{p,n-p}(\xi_p)) \\ &= d'_{p,n-p-1}((-1)^{p+1} \cdot d'_{p+1,n-p-1}(\xi_{p+1})) \\ &= 0, \end{aligned}$$

and so

$$d'_{p,n-p}(\xi_p) \in \operatorname{Ker} d''_{p-1,n-p} = \operatorname{Im} d''_{p-1,n-p+1},$$

which means the existence of the required element ξ_{p-1}.

(2) We set $\eta_0 = 0$. By the assumption, there exists $\eta_1 \in C_{1,n}$ such that

$$\xi_0 = d'_{1,n}(\eta_1) = d'_{1,n}(\eta_1) + d''_{0,n+1}(\eta_0).$$

Here we assume $0 \leq p \leq n-1$ and two elements $\eta_p \in C_{p,n-p+1}$, $\eta_{p+1} \in C_{p+1,n-p}$ such that

$$\xi_p = d'_{p+1,n-p}(\eta_{p+1}) + (-1)^p \cdot d''_{p,n-p+1}(\eta_p)$$

are fixed. We would like to find $\eta_{p+2} \in C_{p+2,n-p-1}$ such that

$$\xi_{p+1} = d'_{p+2,n-p-1}(\eta_{p+2}) + (-1)^{p+1} \cdot d''_{p+1,n-p}(\eta_{p+1}).$$

Now $d'_{p+1,n-p-1}(\xi_{p+1}) = (-1)^{p+1} \cdot d''_{p,n-p}(\xi_p)$ holds, since $\xi_n + \xi_{n-1} + \cdots + \xi_1 + \xi_0 \in \operatorname{Ker} d_n$.

Hence, we have

$$
\begin{aligned}
& d'_{p+1,n-p-1}(\xi_{p+1} + (-1)^p \cdot d''_{p+1,n-p}(\eta_{p+1})) \\
&= d'_{p+1,n-p-1}(\xi_{p+1}) + (-1)^p \cdot d'_{p+1,n-p-1}(d''_{p+1,n-p}(\eta_{p+1})) \\
&= (-1)^{p+1} \cdot d''_{p,n-p}(\xi_p) + (-1)^p \cdot d''_{p,n-p}(d'_{p+1,n-p}(\eta_{p+1})) \\
&= (-1)^{p+1} \cdot d''_{p,n-p}(\xi_p - d'_{p+1,n-p}(\eta_{p+1})) \\
&= (-1)^{p+1} \cdot d''_{p,n-p}((-1)^p \cdot d''_{p,n-p+1}(\eta_p)) \\
&= 0,
\end{aligned}
$$

and it follows that

$$\xi_{p+1} + (-1)^p \cdot d''_{p+1,n-p}(\eta_{p+1}) \in \operatorname{Ker} d'_{p+1,n-p-1} = \operatorname{Im} d'_{p+2,n-p-1}.$$

Thus we see the existence of the required element η_{p+2}. □

Lemma 1.2.3. *Suppose that*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \xrightarrow{\rho} L$$

is an exact sequence of R-modules. Then the following assertions hold.

(1) *If there exists a homomorphism $\phi : G \longrightarrow F$ of R-modules such that $\phi \circ \varphi = \mathrm{id}_F$, then*

$$0 \longrightarrow {}^*G \xrightarrow{{}^*\psi} H \xrightarrow{\rho} L$$

is exact, where ${}^*G = \operatorname{Ker} \phi$ and ${}^*\psi$ is the restriction of ψ to ${}^*G$.

(2) *If* $F = {}'F \oplus {}^*F$, $G = {}'G \oplus {}^*G$, $\varphi({}'F) = {}'G$ *and* $\varphi({}^*F) \subseteq {}^*G$, *then*

$$0 \longrightarrow {}^*F \xrightarrow{{}^*\varphi} {}^*G \xrightarrow{{}^*\psi} H \xrightarrow{\rho} L$$

is exact, where ${}^*\varphi$ *and* ${}^*\psi$ *are the restrictions of* φ *and* ψ *to* ${}^*F$ *and* ${}^*G$*, respectively.*

Proof. See [8, 2.3]. □

1.3 $*$-transform

Let $2 \leq n \in \mathbb{Z}$ and let R be an n-dimensional Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$. Suppose that an acyclic complex

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

of finitely generated free R-modules such that $\operatorname{Im}\varphi_n \subseteq QF_{n-1}$ is given, where $Q = (x_1, x_2, \ldots, x_n)R$ is a parameter ideal of R. We put $M = \operatorname{Im}\varphi_1$, which is an R-submodule of F_0. In this section, transforming $F_\bullet$ suitably, we aim to construct an acyclic complex

$$0 \longrightarrow {}^*F_n \xrightarrow{{}^*\varphi_n} {}^*F_{n-1} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0$$

of finitely generated free R-modules such that $\operatorname{Im}{}^*\varphi_n \subseteq \mathfrak{m} \cdot {}^*F_{n-1}$ and $\operatorname{Im}{}^*\varphi_1 = M :_{F_0} Q$. Let us call ${}^*F_\bullet$ the $*$-transform of $F_\bullet$ with respect to $x_1, x_2, \ldots, x_n$.

In this operation, we use the Koszul complex $K_\bullet = K_\bullet(x_1, x_2, \ldots, x_n)$. We denote the boundary map of $K_\bullet$ by $\partial_\bullet$. Let $e_1, e_2, \ldots, e_n$ be an R-free basis of K_1 such that $\partial_1(e_i) = x_i$ for all $i = 1, 2, \ldots, n$. Moreover, we use the following notation:

- $N := \{1, 2, \ldots, n\}$.

- $N_p := \{I \subseteq N \,|\, \sharp\, I = p\}$ for $1 \leq p \leq n$ and $N_0 := \{\emptyset\}$.

- If $1 \leq p \leq n$ and $I = \{i_1, i_2, \ldots, i_p\} \in N_p$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq n$, we set

$$e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \in K_p.$$

  In particular, for $1 \leq i \leq n$, $\check{e}_i := e_{N \setminus \{i\}}$. Furthermore, $e_\emptyset$ denotes the identity element 1_R of $R = K_0$.

- If $1 \leq p \leq n$, $I \in N_p$ and $i \in N$, we set

$$s(i, I) = \sharp\, \{j \in I \,|\, j < i\}.$$

  We define $\sharp\, \emptyset = 0$, so $s(i, I) = 0$ if $i \leq \min I$.

Then, for any $p = 0, 1, \ldots, n$, $\{e_I\}_{I \in N_p}$ is an R-free basis of K_p and

$$\partial_p(e_I) = \sum_{i \in I} (-1)^{s(i,I)} \cdot x_i \cdot e_{I \setminus \{i\}}.$$

Theorem 1.3.1. $(M :_{F_0} Q)/M \cong F_n/QF_n$.

Proof. We put $L_0 = F_0/M$. Moreover, for $1 \leq p \leq n-1$, we put $L_p = \operatorname{Im} \varphi_p \subseteq F_{p-1}$ and consider the exact sequence

$$0 \longrightarrow L_p \longrightarrow F_{p-1} \xrightarrow{\varphi_{p-1}} L_{p-1} \longrightarrow 0,$$

where $\varphi_0 : F_0 \longrightarrow L_0$ is the canonical surjection. Because

$$\operatorname{Ext}_R^{p-1}(R/Q, F_{p-1}) = \operatorname{Ext}_R^{p}(R/Q, F_{p-1}) = 0,$$

we get

$$\mathrm{Ext}_R^p(R/Q, L_p) \cong \mathrm{Ext}_R^{p-1}(R/Q, L_{p-1}).$$

Therefore $\mathrm{Ext}_R^{n-1}(R/Q, L_{n-1}) \cong \mathrm{Hom}_R(R/Q, F_0/M) \cong (M :_{F_0} Q)/M$. Now, we see that

$$\mathrm{Ext}_R^n(R/Q, F_n) \cong \mathrm{Hom}_R(R/Q, F_n/QF_n) \cong F_n/QF_n$$

and

$$\mathrm{Ext}_R^n(R/Q, F_{n-1}) \cong \mathrm{Hom}_R(R/Q, F_{n-1}/QF_{n-1}) \cong F_{n-1}/QF_{n-1}$$

hold, because $x_1, x_2, \ldots, x_n$ is an R-regular sequence. Furthermore, we look at the exact sequence

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} L_{n-1} \longrightarrow 0.$$

Then, we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}_R^{n-1}(R/Q, L_{n-1}) & \longrightarrow & \mathrm{Ext}_R^n(R/Q, F_n) & \xrightarrow{\widetilde{\varphi_n}} & \mathrm{Ext}_R^n(R/Q, F_{n-1}) \quad (\mathrm{ex}) \\ & & & & \downarrow\cong & & \downarrow\cong \\ & & & & F_n/QF_n & \xrightarrow{\overline{\varphi_n}} & F_{n-1}/QF_{n-1}, \end{array}$$

where $\widetilde{\varphi_n}$ and $\overline{\varphi_n}$ denote the maps induced from φ_n. Let us notice $\overline{\varphi_n} = 0$ as $\mathrm{Im}\,\varphi_n \subseteq QF_{n-1}$. Hence

$$\mathrm{Ext}_R^{n-1}(R/Q, L_{n-1}) \cong F_n/QF_n,$$

and so the required isomorphism follows. $\square$

Let us fix an R-free basis of F_n, say $\{v_\lambda\}_{\lambda\in\Lambda}$. We set $\widetilde{\Lambda} = \Lambda \times N$ and take a family $\{v_{(\lambda,i)}\}_{(\lambda,i)\in\widetilde{\Lambda}}$ of elements in F_{n-1} so that

$$\varphi_n(v_\lambda) = \sum_{i\in N} x_i \cdot v_{(\lambda,i)}$$

for all $\lambda \in \Lambda$. This is possible as $\operatorname{Im} \varphi_n \subseteq QF_{n-1}$. The next result is the essential part of the process to get ${}^*F_\bullet$.

Theorem 1.3.2. *There exists a chain map $\sigma_\bullet : F_n \otimes_R K_\bullet \longrightarrow F_\bullet$*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & F_n \otimes_R K_n & \xrightarrow{F_n \otimes \partial_n} & F_n \otimes_R K_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_n \otimes_R K_1 & \xrightarrow{F_n \otimes \partial_1} & F_n \otimes_R K_0 \\
 & & \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \sigma_{n-1}} & & & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_0} \\
0 & \longrightarrow & F_n & \xrightarrow{\varphi_n} & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \xrightarrow{\varphi_1} & F_0
\end{array}$$

satisfying the following conditions.

(1) $\sigma_0^{-1}(\operatorname{Im} \varphi_1) = \operatorname{Im}(F_n \otimes \partial_1)$.

(2) $\operatorname{Im} \sigma_0 + \operatorname{Im} \varphi_1 = M :_{F_0} Q$.

(3) $\sigma_{n-1}(v_\lambda \otimes \check{e}_i) = (-1)^{n+i-1} \cdot v_{(\lambda,i)}$ *for all* $(\lambda, i) \in \widetilde{\Lambda}$.

(4) $\sigma_n(v_\lambda \otimes e_N) = (-1)^n \cdot v_\lambda$ *for all* $\lambda \in \Lambda$.

Proof. Let us notice that, for any $p = 0, 1, \ldots, n$, $\{v_\lambda \otimes e_I\}_{(\lambda,I) \in \Lambda \times N_p}$ is an R-free basis of $F_n \otimes_R K_p$, so $\sigma_p : F_n \otimes_R K_p \longrightarrow F_p$ can be defined by choosing suitable element $w_{(\lambda,I)} \in F_p$ that corresponds to $v_\lambda \otimes e_I$ for $(\lambda, I) \in \Lambda \times N_p$. We set $w_{(\lambda,N)} = (-1)^n \cdot v_\lambda$ for $\lambda \in \Lambda$ and $w_{(\lambda,N \setminus \{i\})} = (-1)^{n+i-1} \cdot v_{(\lambda,i)}$ for $(\lambda, i) \in \widetilde{\Lambda}$. Then

$$\begin{aligned}
\varphi_n(w_{(\lambda,N)}) &= (-1)^n \cdot \varphi_n(v_\lambda) \\
&= (-1)^n \cdot \sum_{i \in N} x_i \cdot v_{(\lambda,i)} \\
&= \sum_{i \in N} (-1)^{s(i,N)} \cdot x_i \cdot w_{(\lambda,N \setminus \{i\})}.
\end{aligned}$$

Moreover, we can take families $\{w_{(\lambda,I)}\}_{(\lambda,I)\in\Lambda\times N_p}$ of elements in F_p for any $p = 0, 1, \ldots, n-2$ so that

$$\varphi_p(w_{(\lambda,I)}) = \sum_{i\in I}(-1)^{s(i,I)} \cdot x_i \cdot w_{(\lambda,I\setminus\{i\})} \tag{$\sharp$}$$

for all $p = 1, 2, \ldots, n$ and $(\lambda, I) \in \Lambda\times N_p$. If this is true, an R-linear map $\sigma_p : F_n\otimes_R K_p \longrightarrow F_p$ is defined by setting $\sigma_p(v_\lambda \otimes e_I) = w_{(\lambda,I)}$ for $(\lambda, I) \in \Lambda \times N_p$ and $\sigma_\bullet : F_n \otimes_R K_\bullet \longrightarrow F_\bullet$ becomes a chain map satisfying (3) and (4).

In order to see the existence of $\{w_{(\lambda,I)}\}_{(\lambda,I)\in\Lambda\times N_p}$, let us consider the double complex $F_\bullet \otimes_R K_\bullet$.

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & F_p \otimes_R K_q & \xrightarrow{\varphi_p\otimes K_q} & F_{p-1} \otimes_R K_q & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F_p\otimes\partial_q} & & \downarrow{\scriptstyle F_{p-1}\otimes\partial_q} & & \\
\cdots & \longrightarrow & F_p \otimes_R K_{q-1} & \xrightarrow{\varphi_p\otimes K_{q-1}} & F_{p-1} \otimes_R K_{q-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

We can take it as $C_{\bullet\bullet}$ of 1.2.2. Let $T_\bullet$ be the total complex and $d_\bullet$ be its boundary map. In particular, we have

$$T_n = (F_n \otimes_R K_0) \oplus (F_{n-1} \otimes_R K_1) \oplus \cdots \oplus (F_1 \otimes_R K_{n-1}) \oplus (F_0 \otimes_R K_n).$$

For $I \subseteq N$, we define

$$t(I) = \begin{cases} \displaystyle\sum_{i\in I}(i-1) & \text{if } I \neq \emptyset, \\ 0 & \text{if } I = \emptyset. \end{cases}$$

For a while, we fix $\lambda \in \Lambda$ and set

$$\xi_n(\lambda) = (-1)^{\frac{n(n+1)}{2}} \cdot (-1)^{t(N)} \cdot w_{(\lambda,N)} \otimes e_\emptyset \in F_n \otimes_R K_0,$$

$$\xi_{n-1}(\lambda) = (-1)^{\frac{(n-1)n}{2}} \cdot \sum_{i \in N} (-1)^{t(N \setminus \{i\})} \cdot w_{(\lambda, N \setminus \{i\})} \otimes e_i \in F_{n-1} \otimes_R K_1.$$

It is easy to see that

$$\xi_n(\lambda) = v_\lambda \otimes e_\emptyset$$

since $t(N) = (n-1)n/2$ and $n^2 + n \equiv 0 \pmod 2$. Moreover, we have

$$\xi_{n-1}(\lambda) = (-1)^n \cdot \sum_{i \in N} v_{(\lambda,i)} \otimes e_i$$

since $t(N \setminus \{i\}) = (n-1)n/2 - (i-1)$. Then

$$\begin{aligned}
(\varphi_n \otimes K_0)(\xi_n(\lambda)) &= \varphi_n(v_\lambda) \otimes e_\emptyset \\
&= (\sum_{i \in N} x_i \cdot v_{(\lambda,i)}) \otimes e_\emptyset \\
&= \sum_{i \in N} v_{(\lambda,i)} \otimes x_i \\
&= (F_{n-1} \otimes \partial_1)(\sum_{i \in N} v_{(\lambda,i)} \otimes e_i) \\
&= (-1)^n \cdot (F_{n-1} \otimes \partial_1)(\xi_{n-1}(\lambda)).
\end{aligned}$$

Hence, by (1) of 1.2.2 there exist elements $\xi_p(\lambda) \in F_p \otimes K_{n-p}$ for all $p = 0, 1, \ldots, n-2$ such that

$$\xi_n(\lambda) + \xi_{n-1}(\lambda) + \xi_{n-2}(\lambda) + \cdots + \xi_0(\lambda) \in \operatorname{Ker} d_n \subseteq T_n,$$

which means

$$(\varphi_p \otimes K_{n-p})(\xi_p(\lambda)) = (-1)^p \cdot (F_{p-1} \otimes \partial_{n-p+1})(\xi_{p-1}(\lambda))$$

for any $p = 1, 2, \ldots, n$. Let us denote $N \setminus I$ by I^{c} for $I \subseteq N$. Because $\{e_{I^{\mathrm{c}}}\}_{I \in N_p}$ is an R-free basis of K_{n-p}, it is possible to write

$$\xi_p(\lambda) = (-1)^{\frac{p(p+1)}{2}} \cdot \sum_{I \in N_p} (-1)^{t(I)} \cdot w_{(\lambda, I)} \otimes e_{I^{\mathrm{c}}}$$

for any $p = 0, 1, \ldots, n-2$ (Notice that $\xi_n(\lambda)$ and $\xi_{n-1}(\lambda)$ are defined so that they satisfy the same equalities), where $w_{(\lambda, I)} \in F_p$. Then we have

$$(\varphi_p \otimes K_{n-p})(\xi_p(\lambda)) = (-1)^{\frac{p(p+1)}{2}} \cdot \sum_{I \in N_p} (-1)^{t(I)} \cdot \varphi_p(w_{(\lambda, I)}) \otimes e_{I^{\mathrm{c}}}.$$

On the other hand,

$$\begin{aligned}&(-1)^p \cdot (F_{p-1} \otimes \partial_{n-p+1})(\xi_{p-1}(\lambda)) \\ &\qquad = (-1)^p \cdot (-1)^{\frac{(p-1)p}{2}} \cdot \sum_{J \in N_{p-1}} \{(-1)^{t(J)} \cdot w_{(\lambda, J)} \otimes (\sum_{i \in J^{\mathrm{c}}} (-1)^{s(i, J^{\mathrm{c}})} \cdot x_i \cdot e_{J^{\mathrm{c}} \setminus \{i\}})\}.\end{aligned}$$

Here we notice that if $I \in N_p$, $J \in N_{p-1}$ and $i \in N$, then

$$I^{\mathrm{c}} = J^{\mathrm{c}} \setminus \{i\} \quad \Longleftrightarrow \quad I = J \cup \{i\}.$$

Hence we get

$$\begin{aligned}&(-1)^p \cdot (F_{p-1} \otimes \partial_{n-p+1})(\xi_{p-1}(\lambda)) \\ &\qquad = (-1)^{\frac{p(p+1)}{2}} \cdot \sum_{I \in N_p} \{(\sum_{i \in I} (-1)^{t(I \setminus \{i\}) + s(i, I^{\mathrm{c}} \cup \{i\})} \cdot x_i \cdot w_{(\lambda, I \setminus \{i\})}) \otimes e_{I^{\mathrm{c}}}\}.\end{aligned}$$

For $I \in N_p$ and $i \in I$, we have

$$t(I \setminus \{i\}) = t(I) - (i-1),$$

$$s(i, I) + s(i, I^{\mathrm{c}} \cup \{i\}) = s(i, N) = i - 1,$$

and so

$$
\begin{aligned}
t(I \setminus \{i\}) + s(i, I^{c} \cup \{i\}) &= t(I) - s(i, I) \\
&\equiv t(I) + s(i, I) \pmod 2.
\end{aligned}
$$

Therefore we see that the required equality (♯) holds for all $I \in N_p$.

Let us prove (1). We have to show $\sigma_0^{-1}(\operatorname{Im}\varphi_1) \subseteq \operatorname{Im}(F_n \otimes \partial_1)$. Take any $\eta_n \in F_n \otimes_R K_0$ such that $\sigma_0(\eta_n) \in \operatorname{Im}\varphi_1$. As $\{\xi_n(\lambda)\}_{\lambda\in\Lambda}$ is an R-free basis of $F_n \otimes_R K_0$, we can express

$$
\eta_n = \sum_{\lambda\in\Lambda} a_\lambda \cdot \xi_n(\lambda) = \sum_{\lambda\in\Lambda} a_\lambda \cdot (v_\lambda \otimes e_\emptyset),
$$

where $a_\lambda \in R$ for $\lambda \in \Lambda$. Then we have

$$
\sum_{\lambda\in\Lambda} a_\lambda \cdot w_{(\lambda,\emptyset)} = \sum_{\lambda\in\Lambda} a_\lambda \cdot \sigma_0(v_\lambda \otimes e_\emptyset) = \sigma_0(\eta_n) \in \operatorname{Im}\varphi_1.
$$

Now we set

$$
\eta_p = \sum_{\lambda\in\Lambda} a_\lambda \cdot \xi_p(\lambda) \in F_p \otimes_R K_{n-p}
$$

for $0 \le p \le n-1$. Then

$$
\begin{aligned}
\eta_n + \eta_{n-1} + \cdots + \eta_1 + \eta_0 &= \sum_{\lambda\in\Lambda} a_\lambda \cdot (\xi_n(\lambda) + \xi_{n-1}(\lambda) + \cdots + \xi_1(\lambda) + \xi_0(\lambda)) \\
&\in \operatorname{Ker} d_n \subseteq T_n.
\end{aligned}
$$

Because

$$
\begin{aligned}
\eta_0 &= \sum_{\lambda \in \Lambda} a_\lambda \cdot \xi_0(\lambda) \\
&= \sum_{\lambda \in \Lambda} a_\lambda \cdot (w_{(\lambda,\emptyset)} \otimes e_N) \\
&= (\sum_{\lambda \in \Lambda} a_\lambda \cdot w_{(\lambda,\emptyset)}) \otimes e_N \\
&\in \mathrm{Im}(\varphi_1 \otimes K_n),
\end{aligned}
$$

we get $\eta_n \in \mathrm{Im}(F_n \otimes \partial_1)$ by (2) of 1.2.2.

Finally we prove (2). Let us consider the following commutative diagram

$$
\begin{array}{ccccccc}
F_n \otimes_R K_1 & \xrightarrow{F_n \otimes \partial_1} & F_n \otimes_R K_0 & \longrightarrow & F_n/QF_n & \longrightarrow & 0 \quad (\mathrm{ex}) \\
\downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_0} & & \downarrow{\scriptstyle \overline{\sigma_0}} & & \\
F_1 & \xrightarrow{\varphi_1} & F_0 & \longrightarrow & F_0/M & \longrightarrow & 0 \quad (\mathrm{ex}),
\end{array}
$$

where $\overline{\sigma_0}$ is the map induced from σ_0. For all $\lambda \in \Lambda$ and $i \in N$, we have

$$
x_i \cdot w_{(\lambda,\emptyset)} = \varphi_1(w_{(\lambda,\{i\})}) \in M,
$$

which means $w_{(\lambda,\emptyset)} \in M :_{F_0} Q$. Hence $\mathrm{Im}\,\sigma_0 \subseteq M :_{F_0} Q$, and so $\mathrm{Im}\,\overline{\sigma_0} \subseteq (M :_{F_0} Q)/M$. On the other hand, as $\sigma_0^{-1}(\mathrm{Im}\,\varphi_1) = \mathrm{Im}(F_n \otimes \partial_1)$, we see that $\overline{\sigma_0}$ is injective. Therefore we get $\mathrm{Im}\,\overline{\sigma_0} = (M :_{F_0} Q)/M$ since $(M :_{F_0} Q)/M \cong F_n/QF_n$ by 1.3.1 and F_n/QF_n has a finite length. Thus the assertion (2) follows and the proof is complete. □

In the rest, $\sigma_\bullet : F_n \otimes_R K_\bullet \longrightarrow F_\bullet$ is the chain map constructed in 1.3.2. Then, by 1.2.1 the mapping cone $\mathrm{Cone}(\sigma_\bullet)$ gives an R-free resolution of $F_0/(M :_{F_0} Q)$, that is,

$$
0 \longrightarrow F_n \otimes_R K_n \xrightarrow{\psi_{n+1}} \begin{array}{c} F_n \otimes_R K_{n-1} \\ \oplus \\ F_n \end{array} \xrightarrow{\psi_n} \begin{array}{c} F_n \otimes_R K_{n-2} \\ \oplus \\ F_{n-1} \end{array} \xrightarrow{'\varphi_{n-1}} \begin{array}{c} F_n \otimes_R K_{n-3} \\ \oplus \\ F_{n-2} \end{array}
$$

$$
\xrightarrow{^*\varphi_{n-2}} \begin{array}{c} F_n \otimes_R K_{n-4} \\ \oplus \\ F_{n-3} \end{array} \longrightarrow \cdots \longrightarrow \begin{array}{c} F_n \otimes_R K_1 \\ \oplus \\ F_2 \end{array} \xrightarrow{^*\varphi_2} \begin{array}{c} F_n \otimes_R K_0 \\ \oplus \\ F_1 \end{array} \xrightarrow{^*\varphi_1} F_0
$$

is acyclic and $\operatorname{Im} {}^*\varphi_1 = M :_{F_0} Q$, where

$$\psi_{n+1} = \begin{pmatrix} F_n \otimes \partial_n \\ (-1)^n \cdot \sigma_n \end{pmatrix},\ \psi_n = \begin{pmatrix} F_n \otimes \partial_{n-1} & 0 \\ (-1)^{n-1} \cdot \sigma_{n-1} & \varphi_n \end{pmatrix},\ {}'\varphi_{n-1} = \begin{pmatrix} F_n \otimes \partial_{n-2} & 0 \\ (-1)^{n-2} \cdot \sigma_{n-2} & \varphi_{n-1} \end{pmatrix},$$
$${}^*\varphi_p = \begin{pmatrix} F_n \otimes \partial_{p-1} & 0 \\ (-1)^{p-1} \cdot \sigma_{p-1} & \varphi_p \end{pmatrix} \text{ for } 2 \leq p \leq n-2 \text{ and } {}^*\varphi_1 = \begin{pmatrix} \sigma_0 & \varphi_1 \end{pmatrix}.$$

Because $\sigma_n : F_n \otimes_R K_n \longrightarrow F_n$ is an isomorphism by (4) of 1.3.2, we can define

$$\phi = \begin{pmatrix} 0 & (-1)^n \cdot \sigma_n^{-1} \end{pmatrix} : \begin{array}{c} F_n \otimes_R K_{n-1} \\ \oplus \\ F_n \end{array} \longrightarrow F_n \otimes_R K_n.$$

Then $\phi \circ \psi_{n+1} = \mathrm{id}_{F_n \otimes_R K_n}$ and $\operatorname{Ker} \phi = F_n \otimes_R K_{n-1}$. Hence, by (1) of 1.2.3, we get the acyclic complex

$$0 \longrightarrow {}'F_n \xrightarrow{{}'\varphi_n} {}'F_{n-1} \xrightarrow{{}'\varphi_{n-1}} {}^*F_{n-2} \xrightarrow{{}^*\varphi_{n-2}} {}^*F_{n-3} \longrightarrow \cdots \longrightarrow {}^*F_2 \xrightarrow{{}^*\varphi_2} {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0,$$

where

$${}'F_n = F_n \otimes_R K_{n-1},\ {}'F_{n-1} = \begin{array}{c} F_n \otimes_R K_{n-2} \\ \oplus \\ F_{n-1} \end{array},\ {}^*F_p = \begin{array}{c} F_n \otimes_R K_{p-1} \\ \oplus \\ F_p \end{array} \text{ for } 1 \leq p \leq n-2$$
$$\text{and } {}'\varphi_n = \begin{pmatrix} F_n \otimes \partial_{n-1} \\ (-1)^{n-1} \cdot \sigma_{n-1} \end{pmatrix}.$$

Although $\operatorname{Im} {}'\varphi_n$ may not be contained in $\mathfrak{m} \cdot {}'F_{n-1}$, removing non-minimal components from ${}'F_n$ and ${}'F_{n-1}$, we get free R-modules ${}^*F_n$ and ${}^*F_{n-1}$ such that

$$0 \longrightarrow {}^*F_n \xrightarrow{{}^*\varphi_n} {}^*F_{n-1} \xrightarrow{{}^*\varphi_{n-1}} {}^*F_{n-2} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0$$

is acyclic and $\operatorname{Im} {}^*\varphi_n \subseteq \mathfrak{m} \cdot {}^*F_{n-1}$, where ${}^*\varphi_n$ and ${}^*\varphi_{n-1}$ are the restrictions of ${}'\varphi_n$ and ${}'\varphi_{n-1}$, respectively. In the rest of this section, we describe a concrete procedure to get ${}^*F_n$ and ${}^*F_{n-1}$. For that purpose, we use the following notation. As described in Introduction, for any $\xi \in F_n \otimes_R K_{n-2}$ and $\eta \in F_{n-1}$,

$$[\xi] := \begin{pmatrix} \xi \\ 0 \end{pmatrix} \in {}'F_{n-1} \text{ and } \langle \eta \rangle := \begin{pmatrix} 0 \\ \eta \end{pmatrix} \in {}'F_{n-1}.$$

In particular, for any $(\lambda, I) \in \Lambda \times N_{n-2}$, we denote $[v_\lambda \otimes e_I]$ by $[\lambda, I]$. Moreover, for a subset U of F_{n-1}, $\langle U \rangle := \{\langle u \rangle\}_{u \in U}$.

Now, let us choose a subset ${}'\Lambda$ of $\widetilde{\Lambda}$ and a subset U of F_{n-1} so that

$$\{v_{(\lambda,i)}\}_{(\lambda,i)\in {}'\Lambda} \cup U$$

is an R-free basis of F_{n-1}. We would like to choose ${}'\Lambda$ as big as possible. The following almost obvious fact is useful to find ${}'\Lambda$ and U.

Lemma 1.3.3. *Let V be an R-free basis of F_{n-1}. If a subset ${}'\Lambda$ of $\widetilde{\Lambda}$ and a subset U of V satisfy*

(i) $\sharp\, {}'\Lambda + \sharp\, U \leq \sharp\, V$, *and*

(ii) $V \subseteq R \cdot \{v_{(\lambda,i)}\}_{(\lambda,i)\in {}'\Lambda} + R \cdot U + \mathfrak{m} F_{n-1}$,

then $\{v_{(\lambda,i)}\}_{(\lambda,i)\in {}'\Lambda} \cup U$ is an R-free basis of F_{n-1}.

Let us notice that

$$\{[\lambda, I]\}_{(\lambda,I)\in \Lambda \times N_{n-2}} \cup \{\langle v_{(\lambda,i)} \rangle\}_{(\lambda,i)\in {}'\Lambda} \cup \langle U \rangle$$

is an R-free basis of ${}'F_{n-1}$. We define ${}^*F_{n-1}$ to be the direct summand of ${}'F_{n-1}$ generated by

$$\{[\lambda, I]\}_{(\lambda,I)\in \Lambda \times N_{n-2}} \cup \langle U \rangle.$$

Let ${}^*\varphi_{n-1}$ be the restriction of ${}'\varphi_{n-1}$ to ${}^*F_{n-1}$.

Theorem 1.3.4. *If we can take $\widetilde{\Lambda}$ itself as ${}'\Lambda$, then*

$$0 \longrightarrow {}^*F_{n-1} \xrightarrow{{}^*\varphi_{n-1}} {}^*F_{n-2} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0$$

is acyclic. Hence we have $\operatorname{depth}_R F_0/(M :_{F_0} Q) > 0$.

Proof. If ${}'\Lambda = \widetilde{\Lambda}$, there exists a homomorphism $\phi : {}'F_{n-1} \longrightarrow {}'F_n$ such that

$$\phi([\lambda, I]) = 0 \quad \text{for any } (\lambda, I) \in \Lambda \times N_{n-2},$$

$$\phi(\langle v_{(\lambda,i)} \rangle) = (-1)^i \cdot v_\lambda \otimes \check{e}_i \quad \text{for any } (\lambda, i) \in \widetilde{\Lambda},$$

$$\phi(\langle u \rangle) = 0 \quad \text{for any } u \in U.$$

Then $\phi \circ {}'\varphi_n = \mathrm{id}_{{}'F_n}$ and $\operatorname{Ker} \phi = {}^*F_{n-1}$. Hence, by (1) of 1.2.3 we get the required assertion. □

In the rest of this section, we assume ${}'\Lambda \subsetneq \widetilde{\Lambda}$ and put ${}^*\Lambda = \widetilde{\Lambda} \setminus {}'\Lambda$. Then, for any $(\mu, j) \in {}^*\Lambda$, it is possible to write

$$v_{(\mu,j)} = \sum_{(\lambda,i) \in {}'\Lambda} a^{(\mu,j)}_{(\lambda,i)} \cdot v_{(\lambda,i)} + \sum_{u \in U} b^{(\mu,j)}_u \cdot u,$$

where $a^{(\mu,j)}_{(\lambda,i)}, b^{(\mu,j)}_u \in R$. Here, if ${}'\Lambda$ is big enough, we can choose every $b^{(\mu,j)}_u$ from $\mathfrak{m}$. In fact, if $b^{(\mu,j)}_u \notin \mathfrak{m}$ for some $u \in U$, then we can replace ${}'\Lambda$ and U by ${}'\Lambda \cup \{(\mu, j)\}$ and $U \setminus \{u\}$, respectively. Furthermore, because of a practical reason, let us allow that some terms of $v_{(\lambda,i)}$ for $(\lambda, i) \in {}^*\Lambda$ with non-unit coefficients appear in the right hand side, that is, for any $(\mu, j) \in {}^*\Lambda$, we write

$$v_{(\mu,j)} = \sum_{(\lambda,i)\in\widetilde{\Lambda}} a^{(\mu,j)}_{(\lambda,i)} \cdot v_{(\lambda,i)} + \sum_{u\in U} b^{(\mu,j)}_u \cdot u,$$

where

$$a^{(\mu,j)}_{(\lambda,i)} \in \begin{cases} R & \text{if } (\lambda,i) \in {}'\Lambda, \\ \\ \mathfrak{m} & \text{if } (\lambda,i) \in {}^*\Lambda \end{cases} \quad \text{and } b^{(\mu,j)}_u \in \mathfrak{m}.$$

Using this expression, for any $(\mu, j) \in {}^*\Lambda$, the following element in ${}'F_n$ can be defined.

$${}^*v_{(\mu,j)} := (-1)^j \cdot v_\mu \otimes \check{e}_j + \sum_{(\lambda,i)\in\widetilde{\Lambda}} (-1)^{i-1} \cdot a^{(\mu,j)}_{(\lambda,i)} \cdot v_\lambda \otimes \check{e}_i.$$

Lemma 1.3.5. *For any $(\mu, j) \in {}^*\Lambda$, we have*

$${}'\varphi_n({}^*v_{(\mu,j)}) = (-1)^j \cdot [v_\mu \otimes \partial_{n-1}(\check{e}_j)] + \sum_{(\lambda,i)\in\widetilde{\Lambda}} (-1)^{i-1} \cdot a^{(\mu,j)}_{(\lambda,i)} \cdot [v_\lambda \otimes \partial_{n-1}(\check{e}_i)] + \sum_{u\in U} b^{(\mu,j)}_u \cdot \langle u\rangle.$$

As a consequence, we have ${}'\varphi_n({}^*v_{(\mu,j)}) \in \mathfrak{m} \cdot {}^*F_{n-1}$ for any $(\mu, j) \in {}^*\Lambda$.

Proof. By the definition of ${}'\varphi_n$, for any $(\mu, j) \in {}^*\Lambda$, we have

$${}'\varphi_n({}^*v_{(\mu,j)}) = [(F_n \otimes \partial_{n-1})({}^*v_{(\mu,j)})] + \langle(-1)^{n-1} \cdot \sigma_{n-1}({}^*v_{(\mu,j)})\rangle.$$

Because

$$(F_n \otimes \partial_{n-1})({}^*v_{(\mu,j)}) = (-1)^j \cdot v_\mu \otimes \partial_{n-1}(\check{e}_j) + \sum_{(\lambda,i)\in\widetilde{\Lambda}} (-1)^{i-1} \cdot a^{(\mu,j)}_{(\lambda,i)} \cdot v_\lambda \otimes \partial_{n-1}(\check{e}_i)$$

and

$$\begin{aligned} \sigma_{n-1}({}^*v_{(\mu,j)}) &= (-1)^j \cdot \sigma_{n-1}(v_\mu \otimes \check{e}_j) + \sum_{(\lambda,i)\in\widetilde{\Lambda}} (-1)^{i-1} \cdot a^{(\mu,j)}_{(\lambda,i)} \cdot \sigma_{n-1}(v_\lambda \otimes \check{e}_i) \\ &= (-1)^{n-1} \cdot v_{(\mu,j)} + (-1)^n \cdot \sum_{(\lambda,i)\in\widetilde{\Lambda}} a^{(\mu,j)}_{(\lambda,i)} \cdot v_{(\lambda,i)} \\ &= (-1)^{n-1} \cdot (v_{(\mu,j)} - \sum_{(\lambda,i)\in\widetilde{\Lambda}} a^{(\mu,j)}_{(\lambda,i)} \cdot v_{(\lambda,i)}) \\ &= (-1)^{n-1} \cdot \sum_{u\in U} b^{(\mu,j)}_u \cdot u, \end{aligned}$$

we get the required equality. $\square$

Let ${}^*F_n$ be the R-submodule of ${}'F_n$ generated by $\{{}^*v_{(\mu,j)}\}_{(\mu,j)\in{}^*\Lambda}$ and let ${}^*\varphi_n$ be the restriction of ${}'\varphi_n$ to ${}^*F_n$. By 1.3.5 we have $\operatorname{Im}{}^*\varphi_n \subseteq {}^*F_{n-1}$. Thus we get a complex

$$0 \longrightarrow {}^*F_n \xrightarrow{{}^*\varphi_n} {}^*F_{n-1} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0.$$

This is the complex we desire. In fact, the following result holds.

Theorem 1.3.6. $({}^*F_\bullet, {}^*\varphi_\bullet)$ *is an acyclic complex of finitely generated free R-modules with the following properties.*

(1) $\operatorname{Im}{}^*\varphi_1 = M :_{F_0} Q$ *and* $\operatorname{Im}{}^*\varphi_n \subseteq \mathfrak{m}\cdot{}^*F_{n-1}$.

(2) $\{{}^*v_{(\mu,j)}\}_{(\mu,j)\in{}^*\Lambda}$ *is an R-free basis of* ${}^*F_n$.

(3) $\{[\lambda, I]\}_{(\lambda,I)\in\Lambda\times N_{n-2}} \cup \langle U\rangle$ *is an R-free basis of* ${}^*F_{n-1}$.

Proof. First, let us notice that $\{v_\lambda \otimes \check{e}_i\}_{(\lambda,i)\in\widetilde{\Lambda}}$ is an R-free basis of ${}'F_n$ and

$$v_\mu \otimes \check{e}_j \in R\cdot{}^*v_{(\mu,j)} + R\cdot\{v_\lambda\otimes\check{e}_i\}_{(\lambda,i)\in{}'\Lambda} + \mathfrak{m}\cdot{}'F_n$$

for any $(\mu,j)\in{}^*\Lambda$. Hence, by Nakayama's lemma it follows that ${}'F_n$ is generated by

$$\{v_\lambda\otimes\check{e}_i\}_{(\lambda,i)\in{}'\Lambda} \cup \{{}^*v_{(\mu,j)}\}_{(\mu,j)\in{}^*\Lambda},$$

which must be an R-free basis since $\operatorname{rank}_R{}'F_n = \sharp\,\widetilde{\Lambda} = \sharp\,{}'\Lambda + \sharp\,{}^*\Lambda$. Let ${}''F_n$ be the R-submodule of ${}'F_n$ generated by $\{v_\lambda\otimes\check{e}_i\}_{(\lambda,i)\in{}'\Lambda}$. Then ${}'F_n = {}''F_n \oplus {}^*F_n$.

Next, let us recall that

$$\{[\lambda, I]\}_{(\lambda,I)\in\Lambda\times N_{n-2}} \cup \{\langle v_{(\lambda,i)}\rangle\}_{(\lambda,i)\in'\Lambda} \cup \langle U\rangle$$

is an R-free basis of $'F_{n-1}$. Because

$$'\varphi_n(v_\lambda \otimes \check{e}_i) = [v_\lambda \otimes \partial_{n-1}(\check{e}_i)] + (-1)^i \cdot \langle v_{(\lambda,i)}\rangle,$$

we see that

$$\{[\lambda, I]\}_{(\lambda,I)\in\Lambda\times N_{n-2}} \cup \{'\varphi_n(v_\lambda \otimes \check{e}_i)\}_{(\lambda,i)\in'\Lambda} \cup \langle U\rangle$$

is also an R-free basis of $'F_{n-1}$. Let $''F_{n-1} = R \cdot \{'\varphi_n(v_\lambda \otimes \check{e}_i)\}_{(\lambda,i)\in'\Lambda}$. Then $'F_{n-1} = {}''F_{n-1} \oplus {}^*F_{n-1}$.

It is obvious that $'\varphi_n(''F_n) = {}''F_{n-1}$. Moreover, by 1.3.5 we get $'\varphi_n({}^*F_n) \subseteq {}^*F_{n-1}$. Therefore, by (2) of 1.2.3, it follows that ${}^*F_\bullet$ is acyclic. We have already seen (3) and the first assertion of (1). The second assertion of (1) follows from 1.3.5. Moreover, the assertion (2) is now obvious. □

1.4 Computing symbolic powers

Let x, y, z be an sop for R and I an ideal of R generated by the maximal minors of the matrix

$$\Phi = \begin{pmatrix} x^\alpha & y^\beta & z^\gamma \\ y^{\beta'} & z^{\gamma'} & x^{\alpha'} \end{pmatrix},$$

where $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are positive integers. As is well known, R/I is a Cohen-Macaulay ring with $\dim R/I = 1$. In this section, we give a minimal free resolution of I^n for any

$n > 0$, and consider its $*$-transform in order to compute the symbolic power $I^{(n)}$. We put

$$a = z^{\gamma+\gamma'} - x^{\alpha'}y^{\beta}\,,\ b = x^{\alpha+\alpha'} - y^{\beta'}z^{\gamma}\,,\ c = y^{\beta+\beta'} - x^{\alpha}z^{\gamma'}\,.$$

Then, $I = (a,b,c)R$ and we have the next result (See [17] for the definition of d-sequences).

Lemma 1.4.1. *The following assertions hold.*

(1) $x^{\alpha}a + y^{\beta}b + z^{\gamma}c = 0$ *and* $y^{\beta'}a + z^{\gamma'}b + x^{\alpha'}c = 0$.

(2) *Let* $\mathfrak{p} \in \operatorname{Ass}_R R/I$. *Then* $IR_{\mathfrak{p}}$ *is generated by any two elements of* a,b,c.

(3) *Any two elements of* a,b,c *form an ssop for* R.

(4) a,b,c *is an unconditioned d-sequence.*

Proof. (1) These equalities can be checked directly.

(2) Let us prove $IR_{\mathfrak{p}} = (a,b)R_{\mathfrak{p}}$. If $x \in \mathfrak{p}$, then $y,z \in \sqrt{(a,c,x)R} \subseteq \mathfrak{p}$, and so $\mathfrak{p} = \mathfrak{m}$, which contradicts to the Cohen-Macaulayness of R/I. Hence $x \notin \mathfrak{p}$. Then

$$c = -(y^{\beta'}a + z^{\gamma'}b)/x^{\alpha'} \in (a,b)R_{\mathfrak{p}}\,,$$

which means $IR_{\mathfrak{p}} = (a,b)R_{\mathfrak{p}}$.

(3) For example, as $x,z \in \sqrt{(a,b,y)R}$, it follows that a,b is an ssop for R.

(4) Let us prove that a,b,c is a d-sequence. As a,b is a regular sequence by (3), it is enough to show $(a,b)R :_R c^2 = (a,b)R :_R c$. We obviously have $(a,b)R :_R c^2 \supseteq (a,b)R :_R c$. Take any $\mathfrak{q} \in \operatorname{Ass}_R R/(a,b)R :_R c$. As $R/(a,b)R :_R c \hookrightarrow R/(a,b)R$, we have $\operatorname{ht}_R \mathfrak{q} = 2$.

If $c \in \mathfrak{q}$, then $\mathfrak{q} \in \mathrm{Min}_R R/I$, and so $IR_{\mathfrak{q}} = (a,b)R_{\mathfrak{q}}$ by (2), which means

$$(a,b)R_{\mathfrak{q}} :_{R_{\mathfrak{q}}} c^2 = (a,b)R_{\mathfrak{q}} :_{R_{\mathfrak{q}}} c = R_{\mathfrak{q}} .$$

If $c \notin \mathfrak{q}$, we have

$$(a,b)R_{\mathfrak{q}} :_{R_{\mathfrak{q}}} c^2 = (a,b)R_{\mathfrak{q}} :_{R_{\mathfrak{q}}} c = (a,b)R_{\mathfrak{q}} .$$

Therefore we get the required equality. $\square$

We take an indeterminate t and consider the Rees algebra $R[It]$. Moreover, we take three indeterminates A, B, C and put $S = R[A, B, C]$. We regard S as a $\mathbb{Z}$-graded ring by setting $\deg A = \deg B = \deg C = 1$. Let $\pi : S \longrightarrow R[It]$ be the graded homomorphism of R-algebras such that $\pi(A) = at$, $\pi(B) = bt$ and $\pi(C) = ct$. By (4) of 1.4.1 it follows that $\mathrm{Ker}\,\pi$ is generated by linear forms (cf. [16, Theorem 3.1]). On the other hand,

$$0 \longrightarrow R^{\oplus 2} \xrightarrow{{}^t\Phi} R^{\oplus 3} \xrightarrow{(a\ b\ c)} R \longrightarrow R/I \longrightarrow 0$$

is a minimal free resolution of R/I. Hence $\mathrm{Ker}\,\pi$ is generated by

$$f := x^{\alpha}A + y^{\beta}B + z^{\gamma}C \quad \text{and} \quad g := y^{\beta'}A + z^{\gamma'}B + x^{\alpha'}C .$$

Thus we get $S/(f,g)S \cong R[It]$. Then, as f, g is a regular sequence of S,

$$0 \longrightarrow S(-2) \xrightarrow{\binom{-g}{f}} S(-1) \oplus S(-1) \xrightarrow{(f\ g)} S \xrightarrow{\pi} R[It] \longrightarrow 0$$

is a graded S-free resolution of $R[It]$. Now we take its homogeneous part of degree n, and get the next result.

Theorem 1.4.2. *For any $n \geq 2$,*

$$0 \longrightarrow S_{n-2} \xrightarrow{\binom{-g}{f}} S_{n-1} \oplus S_{n-1} \xrightarrow{(f\ \ g)} S_n \xrightarrow{\epsilon} R$$

is acyclic and it is a minimal free resolution of I^n , where S_d $(d \in \mathbb{Z})$ is the R-submodule of S consisting of homogeneous elements of degree d and ϵ is the R-linear map defined by substituting a, b, c for A, B, C , respectively.

Let us denote the complex in 1.4.2 by $(F_\bullet^1, \varphi_\bullet^1)$, that is, we set

$$F_3^1 = S_{n-2}\,,\ F_2^1 = S_{n-1} \oplus S_{n-1}\,,\ F_1^1 = S_n\,,\ F_0^1 = R\,,$$
$$\varphi_3^1 = \binom{-g}{f}\,,\ \varphi_2^1 = (f\ \ g) \quad \text{and} \quad \varphi_1^1 = \epsilon\,.$$

Then $F_\bullet^1$ is an acyclic complex of finitely generated free R-modules and $\operatorname{Im}\varphi_1^1 = I^n$. The number "1" of $F_\bullet^1$ means that it is the first acyclic complex we need for computing $I^{(n)}$. Our strategy is as follows. Taking the $*$-transform of $F_\bullet^1$ with respect to suitable powers of x, y, z , we get ${}^*F_\bullet^1$, which is denoted by $F_\bullet^2$. If its length is still 3, we again take some $*$-transform of $F_\bullet^2$ and get $F_\bullet^3$. By repeating this operation successively, we eventually get an acyclic complex $F_\bullet^k$ of length 2 . Then the family $\{ F_\bullet^i \}_{1 \leq i \leq k}$ of acyclic complexes has complete information about $I^{(n)}$.

Let $\alpha'' = \min\{\alpha, \alpha'\}$, $\beta'' = \min\{\beta\,, \beta'\}$, $\gamma'' = \min\{\gamma\,, \gamma'\}$ and $Q = (x^{\alpha''},\, y^{\beta''},\, z^{\gamma''})R$. Because f and g are elements of Q , we have $\operatorname{Im}\varphi_3^1 \subseteq QF_2^1$, and so by 1.3.1 we get the following.

Theorem 1.4.3. $(I^n :_R Q)/I^n \cong (R/Q)^{\oplus\binom{n}{2}}$.

Now we are going to take the $*$-transform of $F_\bullet^1$ with respect to $x^{\alpha''}$, $y^{\beta''}$, $z^{\gamma''}$. At first, we have to fix Λ^1 , which is an R-free basis of F_3^1 . For any $0 \leq d \in \mathbb{Z}$, let us denote by

$\mathrm{m}^d_{A,B,C}$ the set $\{A^iB^jC^k \mid 0 \le i,j,k \in \mathbb{Z} \text{ and } i+j+k=d\}$, which is an R-free basis of S_d. We take $\mathrm{m}^{n-2}_{A,B,C}$ as Λ^1. Then, for any $M \in \mathrm{m}^{n-2}_{A,B,C}$, we have to write

$$\varphi^1_3(M) = x^{\alpha''}\cdot v^1_{(M,1)} + y^{\beta''}\cdot v^1_{(M,2)} + z^{\gamma''}\cdot v^1_{(M,3)}\,,$$

where $v^1_{(M,i)} \in F^1_2$ for $i=1,2,3$. As is described at the end of Introduction, for $h \in S_{n-1}$, let us denote the elements

$$\begin{pmatrix} h \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ h \end{pmatrix} \in F^1_2 = S_{n-1} \oplus S_{n-1}$$

by $[h]$ and $\langle h\rangle$, respectively. Then, for any $M \in \mathrm{m}^{n-2}_{A,B,C}$, we have

$$\begin{aligned}
\varphi^1_3(M) &= \begin{pmatrix} -gM \\ fM \end{pmatrix} \\
&= \begin{pmatrix} -y^{\beta'}AM - z^{\gamma'}BM - x^{\alpha'}CM \\ x^{\alpha}AM + y^{\beta}BM + z^{\gamma}CM \end{pmatrix} \\
&= -y^{\beta'}\cdot[AM] - z^{\gamma'}\cdot[BM] - x^{\alpha'}\cdot[CM] \\
&\qquad\quad + x^{\alpha}\cdot\langle AM\rangle + y^{\beta}\cdot\langle BM\rangle + z^{\gamma}\cdot\langle CM\rangle \\
&= x^{\alpha''}\cdot v^1_{(M,1)} + y^{\beta''}\cdot v^1_{(M,2)} + z^{\gamma''}\cdot v^1_{(M,3)}\,,
\end{aligned}$$

where

$$\begin{aligned}
v^1_{(M,1)} &= x^{\alpha-\alpha''}\cdot\langle AM\rangle - x^{\alpha'-\alpha''}\cdot[CM]\,, \\
v^1_{(M,2)} &= y^{\beta-\beta''}\cdot\langle BM\rangle - y^{\beta'-\beta''}\cdot[AM]\,, \\
v^1_{(M,3)} &= z^{\gamma-\gamma''}\cdot\langle CM\rangle - z^{\gamma'-\gamma''}\cdot[BM]\,.
\end{aligned}$$

We set $\widetilde{\Lambda^1} = \Lambda^1 \times \{1,2,3\}$ and we have to choose its subset $'\Lambda^1$ as big as possible so that $\{v^1_{(M,i)}\}_{(M,i)\,\in\,'\Lambda^1}$ is a part of an R-free basis of F^1_2. For that purpose, we need to fix a canonical R-free basis of F_2^1 . For a subset H of S_{n-1} , we denote the families $\{[h]\}_{h\in H}$ and $\{\langle h\rangle\}_{h\in H}$ by $[H]$ and $\langle H\rangle$, respectively. Let us notice that $[\mathrm{m}_{A,B,C}^{n-1}]\cup\langle\mathrm{m}_{A,B,C}^{n-1}\rangle$ is an R-free basis of F_2^1 .

Setting $n=2$, we get the next result.

Theorem 1.4.4. (cf. [12]) *$I^{(2)}=I^2:_R Q$ and $I^{(2)}/I^2\cong R/Q$.*

Proof. By replacing rows and columns of Φ and by replacing x,y,z if necessary, we may assume that one of the following conditions are satisfied;

$$\text{(i)}\,\alpha\le\alpha'\,,\ \beta\le\beta'\,,\ \gamma\le\gamma'\,;\quad \text{(ii)}\,\alpha\ge\alpha'\,,\ \beta\le\beta'\,,\ \gamma\le\gamma'\,.$$

Let $n=2$. Then $\Lambda^1=\{\,1\,\}$. In the case (i), we have $\alpha''=\alpha,\beta''=\beta,\gamma''=\gamma$ and

$$v_{(1,1)}^1=\langle A\rangle-x^{\alpha'-\alpha}\cdot[C]\,,\ v_{(1,2)}^1=\langle B\rangle-y^{\beta'-\beta}\cdot[A]\,,\ v_{(1,3)}^1=\langle C\rangle-z^{\gamma'-\gamma}\cdot[B]\,.$$

Then, as $[\mathrm{m}_{A,B,C}^1]\cup\langle\mathrm{m}_{A,B,C}^1\rangle$ is an R-free basis, by 1.3.3 we see that $\{v_{(1,i)}^1\}_{i=1,2,3}\cup[\mathrm{m}_{A,B,C}^1]$ is an R-free basis of F_2^1 . On the other hand, in the case (ii), we have $\alpha''=\alpha',\beta''=\beta,\gamma''=\gamma$ and

$$v_{(1,1)}^1=x^{\alpha-\alpha'}\cdot\langle A\rangle-[C]\,,\ v_{(1,2)}^1=\langle B\rangle-y^{\beta'-\beta}\cdot[A]\,,\ v_{(1,3)}^1=\langle C\rangle-z^{\gamma'-\gamma}\cdot[B]\,.$$

Then, $\{v_{(1,i)}^1\}_{i=1,2,3}\cup\{\langle A\rangle,[A],[B]\}$ is an R-free basis of F_2^1 . In either case, we can take $\widetilde{\Lambda}^1$ as ${}'\Lambda^1$. Hence, by 1.3.4 we see depth $R/(I^2:_R Q)>0$, and so $I^{(2)}=I^2:_R Q$. The second assertion follows from 1.4.3. $\square$

Similarly as the proof of 1.4.4, in order to study $I^{(n)}$ for $n \geq 3$, we have to consider dividing the situation into several cases. In the rest of this section, let us assume

$$\alpha = 1\,,\ \alpha' = 2\,,\ 2\beta \leq \beta'\,,\ 2\gamma \leq \gamma'\,,$$

and explain how to compute $I^{(3)}$ using $*$-transforms. We have $\alpha'' = 1, \beta'' = \beta$ and $\gamma'' = \gamma$. Let $n = 3$. Then $\Lambda^1 = \{A, B, C\}$ and we have

$$v^1_{(A,1)} = \langle A^2 \rangle - x{\cdot}[AC]\,,\quad v^1_{(A,2)} = \langle AB \rangle - y^{\beta'-\beta}{\cdot}[A^2]\,,\quad v^1_{(A,3)} = \langle AC \rangle - z^{\gamma'-\gamma}{\cdot}[AB]\,,$$

$$v^1_{(B,1)} = \langle AB \rangle - x{\cdot}[BC]\,,\ v^1_{(B,2)} = \langle B^2 \rangle - y^{\beta'-\beta}{\cdot}[AB]\,,\quad v^1_{(B,3)} = \langle BC \rangle - z^{\gamma'-\gamma}{\cdot}[B^2]\,,$$

$$v^1_{(C,1)} = \langle AC \rangle - x{\cdot}[C^2]\,,\quad v^1_{(C,2)} = \langle BC \rangle - y^{\beta'-\beta}{\cdot}[AC]\,,\ v^1_{(C,3)} = \langle C^2 \rangle - z^{\gamma'-\gamma}{\cdot}[BC]\,.$$

We set ${}'\Lambda^1 = \{(A,1)\,, (A,2)\,, (A,3)\,, (B,2)\,, (B,3)\,, (C,3)\} \subseteq \widetilde{\Lambda}^1 = \{A, B, C\} \times \{1,2,3\}$. Then we have the following.

Lemma 1.4.5. $\{v^1_{(M,i)}\}_{(M,i)\,\in\,{}'\Lambda^1} \cup [\mathrm{m}^2_{A,B,C}]$ *is an R-free basis of F^1_2.*

Proof. Let us recall that $[\mathrm{m}^2_{A,B,C}] \cup \langle \mathrm{m}^2_{A,B,C} \rangle$ is an R-free basis of F^1_2. Because $\sharp\,{}'\Lambda^1 = \sharp\,\mathrm{m}^2_{A,B,C} = 6$ and $\langle \mathrm{m}^2_{A,B,C} \rangle \subseteq R{\cdot}\{v^1_{(M,i)}\}_{(M,i)\,\in\,{}'\Lambda^1} + R{\cdot}[\mathrm{m}^2_{A,B,C}]$, we get the required assertion by 1.3.3. $\square$

Let $K_\bullet = K_\bullet(x, y^\beta, z^\gamma)$. By 1.3.2 there exists a chain map $\sigma^1_\bullet : F^1_3 \otimes_R K_\bullet \longrightarrow F^1_\bullet$ such that $\operatorname{Im}\sigma^1_0 + \operatorname{Im}\varphi^1_1 = I^3 :_R Q$ and $\sigma^1_2(M \otimes \check{e}_i) = (-1)^i{\cdot}v^1_{(M,i)}$ for any $(M,i) \in \widetilde{\Lambda}^1 = \{A, B, C\} \times \{1,2,3\}$. Moreover, we get an acyclic complex

$$0 \longrightarrow {}'F^1_3 \xrightarrow{{}'\varphi^1_3} {}'F^1_2 \xrightarrow{{}'\varphi^1_2} {}^*F^1_1 \xrightarrow{{}^*\varphi^1_1} {}^*F^1_0 = R\,,$$

where

$$'F_3^1 = F_3^1 \otimes_R K_2\,, \quad 'F_2^1 = \begin{array}{c} F_3^1 \otimes_R K_1 \\ \oplus \\ F_2^1 \end{array}\,, \quad {}^*F_1^1 = \begin{array}{c} F_3^1 \otimes_R K_0 \\ \oplus \\ F_1^1 \end{array}\,, \quad '\varphi_3^1 = \begin{pmatrix} \mathrm{id}_{F_3^1} \otimes \partial_2 \\ \sigma_2^1 \end{pmatrix}\,,$$

and ${}^*\varphi_1^1 = (\,\sigma_0^1 \;\; \varphi_1^1\,)$. Let us recall our notation introduced in Section 1.3. For any $(M,i) \in \widetilde{\Lambda}^1$ we set

$$[M,i] = [M \otimes e_i] = \begin{pmatrix} M \otimes e_i \\ 0 \end{pmatrix} \in {}'F_3^1\,.$$

On the other hand, for any $\eta \in F_2^1$, we set

$$\langle \eta \rangle = \begin{pmatrix} 0 \\ \eta \end{pmatrix} \in {}'F_2^1\,.$$

In particular, as $F_2^1 = S_2 \oplus S_2$, $\langle [h] \rangle \in {}'F_2^1$ is defined for any $h \in S_2$. We set $\langle [\mathrm{m}^2_{A,B,C}] \rangle = \{\langle [M] \rangle\}_{M \in \mathrm{m}^2_{A,B,C}}$. Let ${}^*F_2^1$ be the R-submodule of ${}'F_2^1$ generated by

$$\{[M,i]\}_{(M,i)\,\in\,\widetilde{\Lambda}^1} \cup \langle [\mathrm{m}^2_{A,B,C}] \rangle\,,$$

and let ${}^*\varphi_2^1$ be the restriction of ${}'\varphi_2^1$ to ${}^*F_2^1$. In order to define ${}^*F_3^1$, we set ${}^*\Lambda^1 = \widetilde{\Lambda}^1 \setminus {}'\Lambda^1 = \{(B,1)\,,\,(C,1)\,,\,(C,2)\}$. We need the next result which can be checked directly.

Lemma 1.4.6. *The following equalities hold;*

$$\begin{aligned} v^1_{(B,1)} &= v^1_{(A,2)} - x\cdot[BC] + y^{\beta'-\beta}\cdot[A^2]\,, \\ v^1_{(C,1)} &= v^1_{(A,3)} - x\cdot[C^2] + z^{\gamma'-\gamma}\cdot[AB]\,, \\ v^1_{(C,2)} &= v^1_{(B,3)} - y^{\beta'-\beta}\cdot[AC] + z^{\gamma'-\gamma}\cdot[B^2]\,. \end{aligned}$$

So, we define the elements ${}^{*}w^1_{(M,i)} \in {}'F^1_3$ for $(M,i) \in {}^{*}\Lambda^1$ as follows;

$$\begin{aligned} {}^{*}w^1_{(B,1)} &= -B \otimes \check{e}_1 - A \otimes \check{e}_2 \,, \\ {}^{*}w^1_{(C,1)} &= -C \otimes \check{e}_1 + A \otimes \check{e}_3 \,, \\ {}^{*}w^1_{(C,2)} &= C \otimes \check{e}_2 + B \otimes \check{e}_3 \,. \end{aligned}$$

Let ${}^{*}F^1_3$ be the R-submodule of ${}'F^1_3$ generated by $\{{}^{*}w^1_{(M,i)}\}_{(M,i)\in {}^{*}\Lambda^1}$ and let ${}^{*}\varphi^1_3$ be the restriction of ${}'\varphi^1_3$ to ${}^{*}F^1_3$. Thus we get a complex

$$0 \longrightarrow {}^{*}F^1_3 \xrightarrow{{}^{*}\varphi^1_3} {}^{*}F^1_2 \xrightarrow{{}^{*}\varphi^1_2} {}^{*}F^1_1 \xrightarrow{{}^{*}\varphi^1_1} {}^{*}F^1_0 = R \,.$$

Let us denote $({}^{*}F^1_\bullet \,, {}^{*}\varphi^1_\bullet)$ by $(F^2_\bullet \,, \varphi^2_\bullet)$. Moreover, we put $w^2_{(M,i)} = {}^{*}w^1_{(M,i)}$ for $(M,i) \in {}^{*}\Lambda^1$. Then, by 1.3.5 and 1.3.6 we have the next result.

Lemma 1.4.7. $(F^2_\bullet \,, \varphi^2_\bullet)$ *is an acyclic complex of finitely generated free R-modules satisfying the following conditions.*

(1) $\operatorname{Im} \varphi^2_1 = I^3 :_R Q$.

(2) $\{w^2_{(B,1)} \,, w^2_{(C,1)} \,, w^2_{(C,2)}\}$ *is an R-free basis of* F^2_3 .

(3) $\{[M,i]\}_{(M,i) \in \widetilde{\Lambda^1}} \cup \langle [\mathrm{m}^2_{A,B,C}] \rangle$ *is an R-free basis of* F^2_2 .

(4) *The following equalities hold;*

$$\varphi_3^2(w^2_{(B,1)}) = -y^{\beta}\cdot[B,3] + z^{\gamma}\cdot[B,2] - x\cdot[A,3] + z^{\gamma}\cdot[A,1] - x\cdot\langle[BC]\rangle + y^{\beta'-\beta}\cdot\langle[A^2]\rangle\,,$$

$$\varphi_3^2(w^2_{(C,1)}) = -y^{\beta}\cdot[C,3] + z^{\gamma}\cdot[C,2] + x\cdot[A,2] - y^{\beta}\cdot[A,1] - x\cdot\langle[C^2]\rangle + z^{\gamma'-\gamma}\cdot\langle[AB]\rangle\,,$$

$$\varphi_3^2(w^2_{(C,2)}) = x\cdot[C,3] - z^{\gamma}\cdot[C,1] + x\cdot[B,2] - y^{\beta}\cdot[B,1] - y^{\beta'-\beta}\cdot\langle[AC]\rangle + z^{\gamma'-\gamma}\cdot\langle[B^2]\rangle\,.$$

We put $\Lambda^2 = {}^*\Lambda^1 = \{(B,1)\,,(C,1)\,,(C,2)\}$ and $\widetilde{\Lambda}^2 = \Lambda^2 \times \{1,2,3\}$. We simply denote $((M,i),j) \in \widetilde{\Lambda}^2$ by (M,i,j). Then

$$\widetilde{\Lambda}^2 = \left\{ \begin{array}{lll} (B,1,1)\,, & (B,1,2)\,, & (B,1,3)\,, \\ (C,1,1)\,, & (C,1,2)\,, & (C,1,3)\,, \\ (C,2,1)\,, & (C,2,2)\,, & (C,2,3) \end{array} \right\}.$$

As we are assuming $2\beta \le \beta'$ and $2\gamma \le \gamma'$, by (4) of 1.4.7 we get

$$\varphi_3^2(w^2_{(M,i)}) = x\cdot v^2_{(M,i,1)} + y^{\beta}\cdot v^2_{(M,i,2)} + z^{\gamma}\cdot v^2_{(M,i,3)}$$

for any $(M,i) \in \Lambda^2$, where

$v^2_{(B,1,1)} = -[A,3] - \langle[BC]\rangle\,,\quad v^2_{(B,1,2)} = -[B,3] + y^{\beta'-2\beta}\cdot\langle[A^2]\rangle\,,\quad v^2_{(B,1,3)} = [B,2] + [A,1]\,,$

$v^2_{(C,1,1)} = [A,2] - \langle[C^2]\rangle\,,\quad v^2_{(C,1,2)} = -[C,3] - [A,1]\,,\quad v^2_{(C,1,3)} = [C,2] + z^{\gamma'-2\gamma}\cdot\langle[AB]\rangle\,,$

$v^2_{(C,2,1)} = [C,3] + [B,2]\,,\quad v^2_{(C,2,2)} = -[B,1] - y^{\beta'-2\beta}\cdot\langle[AC]\rangle$ and

$$v^2_{(C,2,3)} = -[C,1] + z^{\gamma'-2\gamma}\cdot\langle[B^2]\rangle\,.$$

Thus a family $\{v^2_{(M,i,j)}\}_{(M,i,j)\in\widetilde{\Lambda}^2}$ of elements in F_2^2 is fixed and we see $\mathrm{Im}\,\varphi_3^2 \subseteq QF_2^2$.

Because $\operatorname{Im}\varphi_1^2 :_R Q = (I^3 :_R Q) :_R Q = I^3 :_R Q^2$ and $F_3^2 \cong R^{\oplus 3}$, by 1.3.1 we get the next result.

Theorem 1.4.8. *Let $\alpha = 1$, $\alpha' = 2$, $2\beta \leq \beta'$ and $2\gamma \leq \gamma'$. Then we have*

$$(I^3 :_R Q^2)/(I^3 :_R Q) \cong (R/Q)^{\oplus 3}.$$

The following relation, which can be checked directly, is very important.

Lemma 1.4.9. $v^2_{(B,1,3)} + v^2_{(C,1,2)} + v^2_{(C,2,1)} = 2\cdot[B,2]$.

By 1.3.2 there exists a chain map $\sigma^2_\bullet : F_3^2 \otimes_R K_\bullet \longrightarrow F^2_\bullet$ such that $\operatorname{Im}\sigma_0^2 + \operatorname{Im}\varphi_1^2 = I^3 :_R Q^2$ and $\sigma_2^2(w^2_{(M,i)} \otimes \check{e}_j) = (-1)^j \cdot v^2_{(M,i,j)}$ for any $(M,i,j) \in \widetilde{\Lambda}^2$. Moreover, we get an acyclic complex

$$0 \longrightarrow {}'F_3^2 \xrightarrow{{}'\varphi_3^2} {}'F_2^2 \xrightarrow{{}'\varphi_2^2} {}^*F_1^2 \xrightarrow{{}^*\varphi_1^2} {}^*F_0^2 = R\,,$$

where

$${}'F_3^2 = F_3^2 \otimes_R K_2\,, \quad {}'F_2^2 = \begin{array}{c} F_3^2 \otimes_R K_1 \\ \oplus \\ F_2^2 \end{array}, \quad {}^*F_1^2 = \begin{array}{c} F_3^2 \otimes_R K_0 \\ \oplus \\ F_1^2 \end{array}, \quad {}'\varphi_3^2 = \begin{pmatrix} \mathrm{id}_{F_3^2} \otimes \partial_2 \\ \sigma_2^2 \end{pmatrix},$$

and ${}^*\varphi_1^2 = (\sigma_0^2 \;\; \varphi_1^2)$. In order to remove non-minimal components from ${}'F_3^2$ and ${}'F_2^2$, we would like to choose a subset ${}'\Lambda^2$ of $\widetilde{\Lambda}^2$ as big as possible so that $\{v^2_{(M,i,j)}\}_{(M,i,j)\in {}'\Lambda^2}$ is a part of an R-free basis of F_2^2.

Theorem 1.4.10. *Let $\alpha = 1$, $\alpha' = 2$, $2\beta \leq \beta'$ and $2\gamma \leq \gamma'$. Suppose that 2 is a unit in R. Then, we can take $\widetilde{\Lambda}^2$ itself as ${}'\Lambda^2$. Hence* $\operatorname{depth} R/(I^3 :_R Q^2) > 0$*, and so $I^{(3)} = I^3 :_R Q^2$. Moreover, we have $\ell_R(\, I^{(3)}/I^3\,) = 6\cdot\ell_R(\, R/Q\,)$.*

Proof. We would like to show that

$$\{v^2_{(M,i,j)}\}_{(M,i,j)\,\in\,\widetilde{\Lambda^2}} \cup \langle[\mathrm{m}^2_{A,B,C}]\rangle$$

is an R-free basis of F^2_2. Let us recall that

$$\{[M,i]\}_{(M,i)\,\in\,\widetilde{\Lambda^1}} \cup \langle[\mathrm{m}^2_{A,B,C}]\rangle$$

is an R-free basis of F^2_2. Because $\sharp\,\widetilde{\Lambda^1} = \sharp\,\widetilde{\Lambda^2}$, we have to prove

$$\{[M,i]\}_{(M,i)\,\in\,\widetilde{\Lambda^1}} \subseteq G := R{\cdot}\{v^2_{(M,i,j)}\}_{(M,i,j)\,\in\,\widetilde{\Lambda^2}} + R{\cdot}\langle[\mathrm{m}^2_{A,B,C}]\rangle\,.$$

In fact, we have $[A,2] = v^2_{(C,1,1)} + \langle[C^2]\rangle \in G$. Similarly, we can easily see that $[A,3]$, $[B,1]$, $[B,3]$, $[C,1]$ and $[C,2]$ are included in G. Moreover, as 2 is a unit in R, we have $[B,2] \in G$ by 1.4.9. Then $[A,1] = v^2_{(B,1,3)} - [B,2] \in G$ and $[C,3] = v^2_{(C,2,1)} - [B,2] \in G$. The last assertion holds since

$$\begin{aligned}
\ell_R(\,I^{(3)}/I^3\,) &= \ell_R(\,(I^3 :_R Q^2)/I^3\,)\\
&= \ell_R(\,(I^3 :_R Q^2)/(I^3 :_R Q)\,) + \ell_R(\,(I^3 :_R Q)/I^3\,)\\
&= 3{\cdot}\ell_R(\,R/Q\,) + 3{\cdot}\ell_R(\,R/Q\,)
\end{aligned}$$

by 1.4.3 and 1.4.8. Thus the proof is complete. □

In the rest of this section, let us consider the case where $\mathrm{ch}\,R = 2$. In this case, we have

$$v^2_{(B,1,3)} + v^2_{(C,1,2)} + v^2_{(C,2,1)} = 0\,.$$

We set ${}'\Lambda^2 = \widetilde{\Lambda}^2 \setminus \{ (B,1,3) \}$. Then, it is easy to see that

$$\{v^2_{(M,i,j)}\}_{(M,i,j)\in {}'\Lambda^2} \cup \{ [B,2] \} \cup \langle [\mathrm{m}^2_{A,B,C}] \rangle$$

is an R-free basis of F^2_2. For any $(M,i,j) \in \widetilde{\Lambda}^2$, let us simply denote $[(M,i),j] = [w^2_{(M,i)} \otimes e_j] \in {}'F^2_2$ by $[M,i,j]$. Then

$$\{ [M,i,j] \}_{(M,i,j)\in \widetilde{\Lambda}^2} \cup \{ \langle v^2_{(M,i,j)} \rangle \}_{(M,i,j)\in {}'\Lambda^2} \cup \{ \langle [B,2] \rangle \} \cup \langle \langle [\mathrm{m}^2_{A,B,C}] \rangle \rangle$$

is an R-free basis of ${}'F^2_2$. Let ${}^*F^2_2$ be the R-submodule of ${}'F^2_2$ generated by

$$\{ [M,i,j] \}_{(M,i,j)\in \widetilde{\Lambda}^2} \cup \{ \langle [B,2] \rangle \} \cup \langle \langle [\mathrm{m}^2_{A,B,C}] \rangle \rangle$$

and let ${}^*\varphi^2_2$ be the restriction of ${}'\varphi^2_2$ to ${}^*F^2_2$. In order to define ${}^*F^2_3$, we set ${}^*\Lambda^2 = \widetilde{\Lambda}^2 \setminus {}'\Lambda^2 = \{(B,1,3)\}$. Because

$$v^2_{(B,1,3)} = -v^2_{(C,1,2)} - v^2_{(C,2,1)}\,,$$

we define ${}^*w^2_{(B,1,3)} \in {}'F^2_3$ to be

$$-w^2_{(B,1)} \otimes \check{e}_3 + w^2_{(C,1)} \otimes \check{e}_2 - w^2_{(C,2)} \otimes \check{e}_1\,.$$

Let ${}^*F^2_3$ be the R-submodule of ${}'F^2_3$ generated by ${}^*w^2_{(B,1,3)}$ and let ${}^*\varphi^2_3$ be the restriction of ${}'\varphi^2_3$ to ${}^*F^2_3$. Thus we get a complex

$$0 \longrightarrow {}^*F^2_3 \xrightarrow{{}^*\varphi^2_3} {}^*F^2_2 \xrightarrow{{}^*\varphi^2_2} {}^*F^2_1 \xrightarrow{{}^*\varphi^2_1} {}^*F^2_0 = R\,.$$

Let us denote $({}^*F^2_\bullet\,, {}^*\varphi^2_\bullet)$ by $(F^3_\bullet\,, \varphi^3_\bullet)$. Moreover, we put $w^3_{(B,1,3)} = {}^*w^2_{(B,1,3)}$. Then, by 1.3.5 and 1.3.6 we have the next result.

Lemma 1.4.11. $(F^3_\bullet, \varphi^3_\bullet)$ *is an acyclic complex of finitely generated free R-modules satisfying the following conditions.*

(1) $\operatorname{Im}\varphi^3_1 = I^3 :_R Q^2$.

(2) $w^3_{(B,1,3)}$ *is an R-free basis of* F^3_3 .

(3) $\{ [M,i,j] \}_{(M,i,j) \in \widetilde{\Lambda^2}} \cup \{ \langle [B,2] \rangle \} \cup \langle\langle [\mathrm{m}^2_{A,B,C}] \rangle\rangle$ *is an R-free basis of* F^3_2 .

(4) *The following equality holds* ;

$$\varphi^3_3(w^3_{(B,1,3)}) = -x\cdot[B,1,2] + y^\beta\cdot[B,1,1] + x\cdot[C,1,3]$$
$$- z^\gamma\cdot[C,1,1] - y^\beta\cdot[C,2,3] + z^\gamma\cdot[C,2,2]\,.$$

We put $\Lambda^3 = {}^*\Lambda^2 = \{(B,1,3)\}$ and $\widetilde{\Lambda^3} = \Lambda^3 \times \{1,2,3\}$. We simply denote $((B,1,3),i) \in \widetilde{\Lambda^3}$ by $(B,1,3,i)$. Then $\widetilde{\Lambda^3} = \{ (B,1,3,i) \}_{i=1,2,3}$. By (4) of 1.4.11 we have

$$\varphi^3_3(w^3_{(B,1,3)}) = x\cdot v^3_{(B,1,3,1)} + y^\beta\cdot v^3_{(B,1,3,2)} + z^\gamma\cdot v^3_{(B,1,3,3)}\,,$$

where

$$v^3_{(B,1,3,1)} = [C,1,3] - [B,1,2]\,, \quad v^3_{(B,1,3,2)} = [B,1,1] - [C,2,3] \quad \text{and}$$
$$v^3_{(B,1,3,3)} = [C,2,2] - [C,1,1]\,.$$

Thus a family $\{v^3_{(B,1,3,i)}\}_{i=1,2,3}$ of elements in F^3_2 is fixed and we see $\operatorname{Im}\varphi^3_3 \subseteq QF^3_2$. Because

$$\operatorname{Im}\varphi^3_1 :_R Q = (I^3 :_R Q^2) :_R Q = I^3 :_R Q^3$$

and $F^3_3 \cong R$, by 1.3.1 we get the next result.

Theorem 1.4.12. *Let* $\alpha = 1$, $\alpha' = 2$, $2\beta \le \beta'$, $2\gamma \le \gamma'$ *and* $\operatorname{ch} R = 2$. *Then we have*

$$(I^3 :_R Q^3)/(I^3 :_R Q^2) \cong R/Q\,.$$

It is easy to see that

$$\{v^2_{(B,1,3,i)}\}_{i=1,2,3} \cup \left\{ \begin{array}{ll} [B,1,2]\,, & [B,1,3]\,, \\ [C,1,1]\,, & [C,1,2]\,, \\ [C,2,1]\,, & [C,2,3] \end{array} \right\} \cup \langle\langle [\mathrm{m}^2_{A,B,C}] \rangle\rangle$$

is an R-free basis of F_2^3. Therefore, by 1.3.4 we get the following result.

Theorem 1.4.13. *Let* $\alpha = 1$, $\alpha' = 2$, $2\beta \leq \beta'$, $2\gamma \leq \gamma'$ *and* $\operatorname{ch} R = 2$. *Then, it follows that* $\operatorname{depth} R/(I^3 :_R Q^3) > 0$, *and so* $I^{(3)} = I^3 :_R Q^3$. *Moreover, we have* $\ell_R(\, I^{(3)}/I^3\,) = 7\cdot\ell_R(\, R/Q\,) = 7\beta\gamma\cdot\ell_R(\, R/(x,y,z)R\,)$.

The last assertion of 1.4.13 holds since $\ell_R(\, I^{(3)}/I^3\,)$ coincides with

$$\begin{aligned} & \ell_R(\,(I^3 :_R Q^3)/(I^3 :_R Q^2)\,) + \ell_R(\,(I^3 :_R Q^2)/(I^3 :_R Q)\,) + \ell_R(\,(I^3 :_R Q)/I^3\,) \\ = \; & \ell_R(\,R/Q\,) + 3\cdot\ell_R(\,R/Q\,) + 3\cdot\ell_R(\,R/Q\,) \end{aligned}$$

by 1.4.3, 1.4.8 and 1.4.12.

1.5 Computing ϵ-multiplicity

Let R be a 3-dimensional regular local ring with the maximal ideal $\mathfrak{m} = (x,y,z)R$. Let I be the ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} x & y & z \\ y & z & x^2 \end{pmatrix}.$$

We will compute the length of $I^{(n)}/I^n$ for all n using the $*$-transforms. As a consequence of our result, we get $\epsilon(I) = 1/2$, where

$$\epsilon(I) := \lim_{n\to\infty} \frac{3!}{n^3}\cdot\ell_R(\,I^{(n)}/I^n\,),$$

which is the invariant called ϵ-multiplicity of I (cf. [6], [26]). Let us maintain the same notations as in Section 1.4. So, $a = z^2 - x^2y$, $b = x^3 - yz$, $c = y^2 - xz$, $S = R[A, B, C]$, $f = xA + yB + zC$ and $g = yA + zB + x^2C$. Furthermore, we need the following notations which is not used in Section 1.4. For any $0 \leq d \in \mathbb{Z}$, we denote the set $\{B^\beta C^\gamma \mid 0 \leq \beta, \gamma \in \mathbb{Z} \text{ and } \beta + \gamma = d\}$ by $\mathrm{m}^d_{B,C}$, and for a monomial L and a set $\mathcal{S}$ consisting of monomials, we denote the set $\{LM \mid M \in \mathcal{S}\}$ by $L \cdot \mathcal{S}$.

Let $n \geq 2$. Then the complex

$$0 \longrightarrow S_{n-2} \xrightarrow{\binom{-g}{f}} S_{n-1} \oplus S_{n-1} \xrightarrow{(f\ g)} S_n \xrightarrow{\epsilon} R$$

is acyclic and gives a minimal free resolution of I^n . We denote this complex by $(F^1_\bullet, \varphi^1_\bullet)$.

As Λ^1, which is an R-free basis of $F^1_3 = S_{n-2}$, we take $\mathrm{m}^{n-2}_{A,B,C}$. Then, for any $M \in \Lambda^1$, we have

$$\varphi^1_3(M) = x \cdot v^1_{(M,1)} + y \cdot v^1_{(M,2)} + z \cdot v^1_{(M,3)} \,,$$

where

$$v^1_{(M,1)} = \langle AM \rangle - x \cdot [CM] \,, \;\; v^1_{(M,2)} = \langle BM \rangle - [AM] \,, \;\; v^1_{(M,3)} = \langle CM \rangle - [BM] \,.$$

Hence, by 1.3.1 we get the following.

Lemma 1.5.1. $(I^n :_R \mathfrak{m})/I^n \cong (R/\mathfrak{m})^{\oplus \binom{n}{2}}$.

The next result can be checked directly.

Lemma 1.5.2. *Suppose $n \geq 4$. Then, for any $N \in \mathrm{m}^{n-4}_{A,B,C}$, we have*

$$v^1_{(A^2N,3)} = v^1_{(ABN,2)} - v^1_{(B^2N,1)} + v^1_{(ACN,1)} - x \cdot v^1_{(C^2N,2)} + x \cdot v^1_{(BCN,3)} \,.$$

Let $\widetilde{\Lambda}^1 = \Lambda^1 \times \{1,2,3\}$. Then the following result holds.

Lemma 1.5.3. *As an R-module, $F_2^1 = S_{n-1} \oplus S_{n-1}$ is generated by*

$$\{v^1_{(M,i)}\}_{(M,i)\in\widetilde{\Lambda}^1} \cup \{[AC^{n-2}], [BC^{n-2}], [C^{n-1}]\}.$$

Proof. Let G be the sum of the R-submodule of F_2^1 generated by the elements stated above and $\mathfrak{m}F_2^1$. It is enough to show that $[\mathrm{m}^{n-1}_{A,B,C}]$ and $\langle \mathrm{m}^{n-1}_{A,B,C}\rangle$ are contained in G.

First, let us prove $\langle L\rangle \in G$ for any $L \in \mathrm{m}^{n-1}_{A,B,C}$. We write $L = A^\alpha B^\beta C^\gamma$, where $0 \le \alpha,\beta,\gamma \in \mathbb{Z}$ and $\alpha+\beta+\gamma = n-1$. If $\alpha > 0$,

$$\langle L\rangle = \langle A\cdot A^{\alpha-1}B^\beta C^\gamma\rangle = v^1_{(A^{\alpha-1}B^\beta C^\gamma,1)} + x\cdot[C\cdot A^{\alpha-1}B^\beta C^\gamma] \in G.$$

Hence, we have to consider the case where $\alpha = 0$. However, as

$$\langle C^{n-1}\rangle = \langle C\cdot C^{n-2}\rangle = v^1_{(C^{n-2},3)} + [B\cdot C^{n-2}] \in G \quad \text{and}$$

$$\langle BC^{n-2}\rangle = \langle B\cdot C^{n-2}\rangle = v^1_{(C^{n-2},2)} + [A\cdot C^{n-2}] \in G,$$

we may assume $\beta \ge 2$. Then, as

$$\begin{aligned}
\langle L\rangle &= \langle B\cdot B^{\beta-1}C^\gamma\rangle \\
&= v^1_{(B^{\beta-1}C^\gamma,2)} + [A\cdot B^{\beta-1}C^\gamma] \\
&= v^1_{(B^{\beta-1}C^\gamma,2)} + [B\cdot AB^{\beta-2}C^\gamma] \\
&= v^1_{(B^{\beta-1}C^\gamma,2)} - v^1_{(AB^{\beta-2}C^\gamma,3)} + \langle C\cdot AB^{\beta-2}C^\gamma\rangle
\end{aligned}$$

and as $\langle C\cdot AB^{\beta-2}C^\gamma\rangle = \langle A\cdot B^{\beta-2}C^{\gamma+1}\rangle$, we get $\langle L\rangle \in G$.

Next, we prove $[\mathrm{m}^{n-1}_{A,B,C}] \subseteq G$. Let us notice $\mathrm{m}^{n-1}_{A,B,C} = A\cdot\mathrm{m}^{n-2}_{A,B,C} \cup B\cdot\mathrm{m}^{n-2}_{B,C} \cup \{C^{n-1}\}$. For any $M \in \mathrm{m}^{n-2}_{A,B,C}$ and any $X \in \mathrm{m}^{n-2}_{B,C}$, we have

$$[AM] = -v^1_{(M,2)} + \langle BM\rangle \in G \quad \text{and} \quad [BX] = -v^1_{(X,3)} + \langle CX\rangle \in G.$$

Hence, the proof is complete as $[C^{n-1}] \in G$ holds obviously. $\square$

Now, let q be the largest integer such that $q \leq n/2$. For any $1 \leq k \leq q$, we would like to construct an acyclic complex

$$0 \longrightarrow F_3^k \xrightarrow{\varphi_3^k} F_2^k \xrightarrow{\varphi_2^k} F_1^k \xrightarrow{\varphi_1^k} F_0^k = R$$

of finitely generated free R-modules satisfying the following conditions.

($\sharp_1^k$) $\operatorname{Im} \varphi_1^k = I^n : \mathfrak{m}^{k-1}$.

($\sharp_2^k$) F_3^k has an R-free basis indexed by $\Lambda^k := \mathrm{m}_{A,B,C}^{n-2k}$, say $\{w_M^k\}_{M \in \Lambda^k}$.

($\sharp_3^k$) Let $\widetilde{\Lambda^k} = \Lambda^k \times \{1, 2, 3\}$. Then, there exists a family $\{v_{(M,i)}^k\}_{(M,i) \in \widetilde{\Lambda^k}}$ of elements in F_2^k satisfying the following conditions.

(i) For any $M \in \Lambda^k$, $\varphi_3^k(w_M^k) = x \cdot v_{(M,1)}^k + y \cdot v_{(M,2)}^k + z \cdot v_{(M,3)}^k$.

(ii) If $k < q$, for any $N \in \Lambda^{k+1} := \mathrm{m}_{A,B,C}^{n-2k-2}$,

$$v_{(A^2N,3)}^k = v_{(ABN,2)}^k - v_{(B^2N,1)}^k + v_{(ACN,1)}^k - x \cdot v_{(C^2N,2)}^k + x \cdot v_{(BCN,3)}^k.$$

(iii) There exists a subset U^k of F_2^k such that $\{v_{(M,i)}^k\}_{(M,i) \in \widetilde{\Lambda^k}} \cup U^k$ generates F_2^k and

$$\sharp\, U^k = \operatorname{rank} F_2^k - 3 \cdot \binom{n-2k+2}{2} + \binom{n-2k}{2},$$

where the last binomial coefficient is regarded as 0 if $k = q$.

Let us notice that the acyclic complex $(F_\bullet^1, \varphi_\bullet^1)$, which is already constructed, satisfies ($\sharp_1^1$), ($\sharp_2^1$) (w_M^1 is M itself for $M \in \Lambda^1$) and ($\sharp_3^1$). So, we assume $1 \leq k < q$ and an acyclic complex $(F_\bullet^k, \varphi_\bullet^k)$ satisfying the required conditions is given. Taking the $*$-transform of $(F_\bullet^k, \varphi_\bullet^k)$ with respect to x, y, z, we would like to construct $(F_\bullet^{k+1}, \varphi_\bullet^{k+1})$.

First, we have the following result since the conditions $(\sharp_1^k)$, $(\sharp_2^k)$ and (i) of $(\sharp_3^k)$ are satisfied and $(I^n :_R \mathfrak{m}^{k-1}) :_R \mathfrak{m} = I^n :_R \mathfrak{m}^k$.

Lemma 1.5.4. $(I^n :_R \mathfrak{m}^k)/(I^n :_R \mathfrak{m}^{k-1}) \cong F_3^k/\mathfrak{m}F_3^k$, *so*

$$\ell_R(\,(I^n :_R \mathfrak{m}^k)/(I^n :_R \mathfrak{m}^{k-1})\,) = \binom{n-2k+2}{2}.$$

If Γ is a subset of Λ^k and $1 \le i \le 3$, we denote by (Γ, i) the subset $\{(M,i) \mid M \in \Gamma\}$ of $\widetilde{\Lambda^k}$. Let us notice that Λ^k is a disjoint union of $A^2{\cdot}\Lambda^{k+1}$, $A{\cdot}\mathrm{m}_{B,C}^{n-2k-1}$ and $\mathrm{m}_{B,C}^{n-2k}$. We set

$${}'\Lambda^k = (\Lambda^k, 1) \cup (\Lambda^k, 2) \cup (A{\cdot}\mathrm{m}_{B,C}^{n-2k-1} \cup \mathrm{m}_{B,C}^{n-2k}, 3).$$

Then the next result holds.

Lemma 1.5.5. $\{v_{(M,i)}^k\}_{(M,i)\in {}'\Lambda^k} \cup U^k$ *is an R-free basis of F_2^k.*

Proof. Because

$$\begin{aligned}
\sharp\,{}'\Lambda^k &= 2\cdot\sharp\,\Lambda^k + \sharp\,(A{\cdot}\mathrm{m}_{B,C}^{n-2k-1} \cup \mathrm{m}_{B,C}^{n-2k}) \\
&= 2\cdot\sharp\,\Lambda^k + \sharp\,(\Lambda^k \setminus A^2{\cdot}\Lambda^{k+1}) \\
&= 3\cdot\sharp\,\Lambda^k - \sharp\,\Lambda^{k+1} \\
&= 3\cdot\binom{n-2k+2}{2} - \binom{n-2k}{2},
\end{aligned}$$

by (iii) of $(\sharp_3^k)$ we have $\sharp\,{}'\Lambda^k + \sharp\,U^k = \operatorname{rank} F_2^k$. Hence, by 1.3.3 it is enough to show that, for any $N \in \Lambda^{k+1}$, $v_{(A^2N,3)}^k$ is contained in the sum of the R-submodule of F_3^k generated by $\{v^k_{(M,i)}\}_{(M,i)\in{}'\Lambda^k} \cup U^k$ and $\mathfrak{m}F^k_2$. We write $N = A^\alpha X$, where $X \in \mathrm{m}^{n-2k-2-\alpha}_{B,C}$. Then, using the equalities in (ii) of $(\sharp^k_3)$, the required containment can be proved by induction on α. $\square$

Let ${}^*F^k_2$ be the R-submodule of ${}'F^k_2$ generated by

$$\{[M,i]\}_{(M,i)\in\widetilde{\Lambda^k}} \cup \langle U^k\rangle\,,$$

where $[M,i] = [w^k_M \otimes e_i]$ for any $(M,i)\in\widetilde{\Lambda^k}$, and let ${}^*\varphi^k_2$ be the restriction of ${}'\varphi^k_2$ to ${}^*F^k_2$. In order to define ${}^*F^k_3$, we notice $\widetilde{\Lambda^k}\setminus{}'\Lambda^k = \{(A^2N,3) \mid N\in\Lambda^{k+1}\}$. Looking at (ii) of $(\sharp^k_3)$, we define ${}^*w^k_{(A^2N,3)} \in {}'F^k_3$ to be

$$-w^k_{A^2N}\otimes\check{e}_3 - w^k_{ABN}\otimes\check{e}_2 - w^k_{B^2N}\otimes\check{e}_1 + w^k_{ACN}\otimes\check{e}_1 + x\cdot w^k_{C^2N}\otimes\check{e}_2 + x\cdot w^k_{BCN}\otimes\check{e}_3$$

for any $N\in\Lambda^{k+1}$. Let ${}^*F^k_3$ be the R-submodule of ${}'F^k_3$ generated by $\{{}^*w^k_{(A^2N,3)}\}_{N\in\Lambda^{k+1}}$ and let ${}^*\varphi^k_3$ be the restriction of ${}'\varphi^k_3$ to ${}^*F^k_3$. Thus we get a complex

$$0 \longrightarrow {}^*F^k_3 \xrightarrow{{}^*\varphi^k_3} {}^*F^k_2 \xrightarrow{{}^*\varphi^k_2} {}^*F^k_1 \xrightarrow{{}^*\varphi^k_1} {}^*F^k_0 = R\,.$$

Let us denote $({}^*F^k_\bullet, {}^*\varphi^k_\bullet)$ by $(F^{k+1}_\bullet, \varphi^{k+1}_\bullet)$. Moreover, we put $w^{k+1}_N = {}^*w^k_{(A^2N,3)}$ for any $N\in\Lambda^{k+1}$. Then, by 1.3.5 and 1.3.6 we have the next result.

Lemma 1.5.6. $(F^{k+1}_\bullet, \varphi^{k+1}_\bullet)$ *is an acyclic complex satisfying the following conditions.*

(1) $\operatorname{Im}\varphi^{k+1}_1 = I^n :_R \mathfrak{m}^k$.

(2) $\{w^{k+1}_N\}_{N\in\Lambda^{k+1}}$ *is an R-free basis of* F^{k+1}_3.

(3) $\{[M,i]\}_{(M,i)\in\widetilde{\Lambda^k}} \cup \langle U^k\rangle$ *is an R-free basis of* F^{k+1}_2.

(4) *For any $N \in \Lambda^{k+1}$, the following equality holds ;*

$$\begin{aligned}\varphi_3^{k+1}(w_N^{k+1}) = -x\cdot[A^2N,2] + y\cdot[A^2N,1] - x\cdot[ABN,3] + z\cdot[ABN,1] \\ - y\cdot[B^2N,3] + z\cdot[B^2N,2] + y\cdot[ACN,3] - z\cdot[ACN,2] \\ + x^2[C^2N,3] - xz\cdot[C^2N,1] + x^2[BCN,2] - xy\cdot[BCN,1]\,.\end{aligned}$$

The assertions (1) and (2) of the lemma above imply that $(F_\bullet^{k+1}, \varphi_\bullet^{k+1})$ satisfies $(\sharp_1^{k+1})$ and $(\sharp_2^{k+1})$, respectively. Moreover, by (4) we have

$$\varphi_3^{k+1}(w_N^{k+1}) = x\cdot v_{(N,1)}^{k+1} + y\cdot v_{(N,2)}^{k+1} + z\cdot v_{(N,3)}^{k+1}$$

for any $N \in \Lambda^{k+1}$, where

$$\begin{aligned} v_{(N,1)}^{k+1} &= -[A^2N,2] - [ABN,3] + x\cdot[C^2N,3] + x\cdot[BCN,2]\,, \\ v_{(N,2)}^{k+1} &= [A^2N,1] - [B^2N,3] + [ACN,3] - x\cdot[BCN,1]\,, \\ v_{(N,3)}^{k+1} &= [ABN,1] + [B^2N,2] - [ACN,2] - x\cdot[C^2N,1]\,. \end{aligned}$$

Thus a family $\{v_{(N,i)}^{k+1}\}_{(N,i)\in\widetilde{\Lambda^{k+1}}}$ of elements in F_2^{k+1} satisfying (i) of $(\sharp_3^{k+1})$ is fixed, where $\widetilde{\Lambda^{k+1}} = \Lambda^{k+1}\times\{1,2,3\}$. The next result, which can be checked directly, insists that (ii) of $(\sharp_3^{k+1})$ is satisfied if $k+1<q$.

Lemma 1.5.7. *Suppose $k+1<q$. Then $n-2k-4\geq 0$ and we have*

$$v_{(A^2L,3)}^{k+1} = v_{(ABL,2)}^{k+1} - v_{(B^2L,1)}^{k+1} + v_{(ACL,1)}^{k+1} - x\cdot v_{(C^2L,2)}^{k+1} + x\cdot v_{(BCL,3)}^{k+1}$$

for any $L \in \Lambda^{k+2} := \mathrm{m}_{A,B,C}^{n-2k-4}$.

If Γ is a subset of Λ^k and $1\leq i\leq 3$, we denote by $[\,\Gamma, i\,]$ the family $\{[M,i]\}_{M\in\Gamma}$ of elements in $F_3^k\otimes_R K_1$. The next result means that (iii) of $(\sharp_3^{k+1})$ is satisfied.

Lemma 1.5.8. *We set*

$$U^{k+1} = [A\cdot \mathrm{m}_{B,C}^{n-2k-1} \cup \mathrm{m}_{B,C}^{n-2k}, 1] \cup [\,\Lambda^k, 3\,] \cup$$

$$\{[ABC^{n-2k-2}, 2]\,, [AC^{n-2k-1}, 2]\,, [BC^{n-2k-1}, 2]\,, [C^{n-2k}, 2]\} \cup \langle U^k \rangle\,.$$

Then $\{v_{(N,i)}^{k+1}\}_{(N,i)\,\in\,\widetilde{\Lambda^{k+1}}} \cup U^{k+1}$ *generates* F_2^{k+1} *and*

$$\sharp\, U^{k+1} = \operatorname{rank} F_2^{k+1} - 3\cdot \binom{n-2k}{2} + \binom{n-2k-2}{2}.$$

Proof. Let G be the sum of the R-submodule of F_2^{k+1} generated by $\{v_{(N,i)}^{k+1}\}_{(N,i)\,\in\,\widetilde{\Lambda^{k+1}}} \cup U^{k+1}$ and $\mathfrak{m}F_2^{k+1}$. We would like to show $G = F_2^{k+1}$. Let us recall that

$$[\,\Lambda^k, 1\,] \cup [\,\Lambda^k, 2\,] \cup [\,\Lambda^k, 3\,] \cup \langle U^k \rangle$$

is an R-free basis of F_2^{k+1} and notice that Λ^k is a disjoint union of $A^2\cdot\Lambda^{k+1}$, $A\cdot \mathrm{m}_{A,B,C}^{n-2k-1}$ and $\mathrm{m}_{B,C}^{n-2k}$. Because $[\Lambda^k, 3] \subseteq U^{k+1}$, it is enough to show $[\,A^2\cdot\Lambda^{k+1}, 1\,] \cup [\,\Lambda^k, 2\,] \subseteq G$.

First, we prove $[\,A^2\cdot\Lambda^{k+1}, 1\,] \subseteq G$. Let us take any $N \in \Lambda^{k+1}$. Then

$$[A^2N, 1] = v_{(N,2)}^{k+1} + [B^2N, 3] - [ACN, 3] + x\cdot[BCN, 1] \in G\,,$$

and so the required inclusion follows.

Next, we prove $[\,\Lambda^k, 2\,] \subseteq G$. Because

$$[A^2N, 2] = -v_{(N,1)}^{k+1} - [ABN, 3] + x\cdot[C^2N, 3] + x\cdot[BCN, 2] \in G$$

for any $N \in \Lambda^{k+1}$, we have $[A^2\cdot\Lambda^{k+1}, 2] \subseteq G$. Furthermore, for any $B^\beta C^\gamma \in \mathrm{m}_{B,C}^{n-2k-1}$, we get $[AB^\beta C^\gamma, 2] \in G$. In fact, $[AB^\beta C^\gamma, 2] \in U^{k+1}$ if $\beta = 0$ or 1, and if $\beta \geq 2$, we have

$$\begin{aligned}[AB^\beta C^\gamma, 2] &= [B^2\cdot AB^{\beta-2}C^\gamma, 2] \\ &= v_{(AB^{\beta-2}C^\gamma,3)}^{k+1} - [A^2B^{\beta-1}C^\gamma, 1] + [A^2B^{\beta-2}C^{\gamma+1}, 2] + \\ &\qquad x\cdot[AB^{\beta-2}C^{\gamma+2}, 1] \in G\,.\end{aligned}$$

Hence $[A\cdot \mathrm{m}_{B,C}^{n-2k-1},2] \subseteq G$. In order to prove $[\mathrm{m}_{B,C}^{n-2k},2] \subseteq G$, we newly take any $B^\beta C^\gamma \in \mathrm{m}_{B,C}^{n-2k}$. We have $[B^\beta C^\gamma,2] \in U^{k+1}$ if $\beta = 0$ or 1, and if $\beta \geq 2$, we have

$$\begin{aligned}
[B^\beta C^\gamma,2] &= [B^2\cdot B^{\beta-2}C^\gamma,2] \\
&= v^{k+1}_{(B^{\beta-2}C^\gamma,3)} - [AB^{\beta-1}C^\gamma,1] + [AB^{\beta-2}C^{\gamma+1},2] + x\cdot[B^{\beta-2}C^{\gamma+2},1] \in G\,.
\end{aligned}$$

Hence the required inclusion follows, and we have seen the first assertion of the theorem.

By (3) of 1.5.6 we have $\operatorname{rank} F_2^{k+1} = 3\cdot\sharp\,\Lambda^k + \sharp\,U^k$. On the other hand,

$$\begin{aligned}
\sharp\,U^{k+1} &= \sharp\,(\Lambda^k \setminus A^2\cdot\Lambda^{k+1}) + \sharp\,\Lambda^k + 4 + \sharp\,U^k \\
&= 2\cdot\sharp\,\Lambda^k - \sharp\,\Lambda^{k+1} + 4 + \sharp\,U^k\,.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\operatorname{rank} F_2^{k+1} - \sharp\,U^{k+1} &= \sharp\,\Lambda^k + \sharp\,\Lambda^{k+1} - 4 \\
&= \binom{n-2k+2}{2} + \binom{n-2k}{2} - 4 \\
&= 3\cdot\binom{n-2k}{2} - \binom{n-2k-2}{2},
\end{aligned}$$

and so the second assertion holds. $\square$

Thus we have constructed an acyclic complex

$$0 \longrightarrow F_3^q \xrightarrow{\varphi_3^q} F_2^q \xrightarrow{\varphi_2^q} F_1^q \xrightarrow{\varphi_1^q} F_0^q = R$$

of finitely generated free R-modules satisfying $(\sharp_1^q)$, $(\sharp_2^q)$ and $(\sharp_3^q)$. Of course, $n-2q = 0$ or 1, and

$$\Lambda^q = \begin{cases} \{\,1\,\} & \text{if } n-2q = 0\,, \\ \{A,B,C\} & \text{if } n-2q = 1\,. \end{cases}$$

The second condition of (iii) of ($\sharp_3^q$) implies $\operatorname{rank} F_2^q = \sharp\,\widetilde{\Lambda^q} + \sharp\,U^q$. Hence, by the first condition of (iii) of ($\sharp_3^q$), we see that $\{v_{(M,i)}^q\}_{(M,i)\,\in\,\widetilde{\Lambda^q}} \cup \langle U^q \rangle$ must be an R-free basis of F_2^q. Therefore, by 1.3.4 we get the next result.

Theorem 1.5.9. $\operatorname{depth} R/(I^n :_R \mathfrak{m}^q) > 0$, *and so* $I^{(n)} = I^n :_R \mathfrak{m}^q$.

Let us compute $\ell_R(\, I^{(n)}/I^n\,)$. By 1.5.9 and 1.5.4 we have

$$\ell_R(\, I^{(n)}/I^n\,) = \sum_{k=1}^{q} \ell_R(\,(I^n : \mathfrak{m}^k)/(I^n : \mathfrak{m}^{k-1})\,) = \sum_{k=1}^{q} \binom{n-2k+2}{2}.$$

As a consequence, we get the next result.

Theorem 1.5.10. *The following equality holds ;*

$$\ell_R(\, I^{(n)}/I^n\,) = \begin{cases} \dfrac{1}{2}\dbinom{n+2}{3} - \dfrac{1}{4}\dbinom{n+1}{2} - \dfrac{1}{8}\dbinom{n}{1} - \dfrac{1}{8} & \textit{if } n \textit{ is even,} \\[2ex] \dfrac{1}{2}\dbinom{n+2}{3} - \dfrac{1}{4}\dbinom{n+1}{2} - \dfrac{1}{8}\dbinom{n}{1} & \textit{if } n \textit{ is odd.} \end{cases}$$

Chapter 2

Saturations of powers of certain determinantal ideals

2.1 Introduction to Chapter 2

Let R be a Noetherian ring and m an integer with $m \geq 2$. Let $x_1, x_2, \ldots, x_{m+1}$ be a sequence of elements of R generating a proper ideal of height $m+1$ and let $\{\alpha_{ij}\}$ be a family of positive integers, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m+1$. We set

$$a_{ij} = \begin{cases} x_{i+j-1}^{\alpha_{ij}} & \text{if } i+j \leq m+2 \\ x_{i+j-m-2}^{\alpha_{ij}} & \text{if } i+j > m+2 \end{cases}$$

for any $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m+1$, and consider the matrix $A = (a_{ij})$ of size $m \times (m+1)$. If $\alpha_{ij} = 1$ for all i and j, the matrix A looks

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_m & x_{m+1} \\ x_2 & \cdots & x_m & x_{m+1} & x_1 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ x_m & x_{m+1} & x_1 & \cdots & x_{m-1} \end{pmatrix}.$$

However, we may put any exponents to each entries. In this chapter, we study the ideal generated by the maximal minors of A. If $m = 2$, this kind of ideals are known as ideals of Herzog-Northcott type in the recent literature [22], and it is a well known result of Herzog [14] that the defining ideal of a space monomial curves is Herzog-Northcott type.

Because the ideals of Herzog-Northcott type provide interesting examples of symbolic Rees algebras, a lot of authors studied the symbolic powers of those ideals (cf. [5], [10], [11], [13], [15], [20], [25]). Although the symbolic powers of ideals usually behave very wild, if the ideal is Herzog-Northcott type, its second symbolic power can be controlled well (cf. [12], [19], [21]). The purpose of this chapter is to generalize this fact for ideals stated above replacing "symbolic power" by "saturation". In order to explain our main result, let us recall the definitions of the symbolic power and the saturation of an ideal.

Let $(R, \mathfrak{m})$ be a local ring and I an ideal of R such that $\dim R/I > 0$. Let r be a positive integer. We set

$$(I^r)^{\mathrm{sat}} = \{ x \in R \mid \mathfrak{m}^i \cdot x \subseteq I^r \text{ for some integer } i \geq 0 \}$$

and call it the saturation of I^r. As $(I^r)^{\mathrm{sat}}/I^r \cong \mathrm{H}^0_{\mathfrak{m}}(R/I^r)$, where $\mathrm{H}^0_{\mathfrak{m}}(\,\cdot\,)$ denotes the 0-th local cohomology functor, we have $(I^r)^{\mathrm{sat}} = I^r$ if and only if $\operatorname{depth} R/I^r > 0$. Moreover, if J is an $\mathfrak{m}$-primary ideal such that $\operatorname{depth} R/(I^r :_R J) > 0$, we have $(I^r)^{\mathrm{sat}} = I^r :_R J$. On the other hand, the r-th symbolic power of I is defined by

$$I^{(r)} = \{ x \in R \mid sx \in I^r \text{ for some } s \in R \text{ such that } s \notin \mathfrak{p} \text{ for any } \mathfrak{p} \in \operatorname{Min}_R R/I \}.$$

In order to compare $(I^r)^{\mathrm{sat}}$ and $I^{(r)}$, let us take a minimal primary decomposition of I^r;

$$I^r = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R R/I^r} \mathrm{Q}(\mathfrak{p}),$$

where $\mathrm{Q}(\mathfrak{p})$ denotes the $\mathfrak{p}$-primary component. It is easy to see that

$$(I^r)^{\mathrm{sat}} = \bigcap_{\mathfrak{m} \neq \mathfrak{p} \in \operatorname{Ass}_R R/I^r} \mathrm{Q}(\mathfrak{p}) \quad \text{and} \quad I^{(r)} = \bigcap_{\mathfrak{p} \in \operatorname{Min}_R R/I} \mathrm{Q}(\mathfrak{p}).$$

Hence we have $(I^r)^{\mathrm{sat}} \subseteq I^{(r)}$ and the equality holds if and only if $\mathrm{Ass}_R\, R/I^r$ is a subset of $\{\,\mathfrak{m}\,\} \cup \mathrm{Min}_R\, R/I$. Therefore, if $\dim R/I = 1$ then $(I^r)^{\mathrm{sat}} = I^{(r)}$. If $\dim R/I \geq 2$, $(I^r)^{\mathrm{sat}}$ may be different from $I^{(r)}$, but even in that case, $(I^r)^{\mathrm{sat}}$ has meaning as an approximation of $I^{(r)}$.

If $(R, \mathfrak{m})$ is a 3-dimensional Cohen-Macaulay local ring and I is an ideal of Herzog-Northcott type, then $I^{(2)}/I^2$ is a cyclic R-module and its generator can be described precisely (cf. [12, (2.2) and (2.3)]). This fact can be generalized as follows, which is the main result of this chapter.

Theorem 2.1.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of $\dim R = m + 1$, where $m \geq 2$. Let $x_1, x_2, \ldots, x_{m+1}$ be an sop for R and set I to be the ideal generated by the maximal minors of A. Then the following assertions hold.*

(1) $(I^r)^{\mathrm{sat}} = I^r$ *for any* $r = 1, \ldots, m - 1$.

(2) $(I^m)^{\mathrm{sat}}/I^m$ *is a cyclic R-module.*

The proof of this theorem is given in Section 2.4. Moreover, for I of 2.1.1, we can describe a generator of $(I^m)^{\mathrm{sat}}/I^m$ assuming suitable condition on $\{\alpha_{ij}\}$. In order to compare $(I^r)^{\mathrm{sat}}$ with $I^{(r)}$ for I of 2.1.1, we have to compare $\mathrm{Ass}_R\, R/I^r$ with $\{\mathfrak{m}\} \cup \mathrm{Min}_R\, R/I$. For that purpose, we study the associated primes of powers of ideals in a more general situation in Section 2.3. Our results are closely related to [3, Lemma 3.3 and Corollary 3.5] and the frameworks for the proofs are similar. Anyway, as a corollary of the results stated in Section 2.3, we see that the ideal I of 2.1.1 satisfies $(I^r)^{\mathrm{sat}} \subsetneq I^{(r)}$ if $r > m \geq 3$

and $\alpha_{ij} = 1$ for any i, j .

Throughout this chapter R is a commutative ring, and we often assume that R is a Noetherian local ring with the maximal ideal $\mathfrak{m}$. For positive integers m, n and an ideal $\mathfrak{a}$ of R, we denote by $\mathrm{Mat}(m, n\,; \mathfrak{a})$ the set of $m \times n$ matrices with entries in $\mathfrak{a}$. For any $A \in \mathrm{Mat}(m, n\,; R)$ and any $k \in \mathbb{Z}$ we denote by $\mathrm{I}_k(A)$ the ideal generated by the k-minors of A . In particular, $\mathrm{I}_k(A)$ is defined to be R (resp. (0)) for $k \leq 0$ (resp. $k > \min\{m, n\}$). If $A, B \in \mathrm{Mat}(m, n\,; R)$ and the (i, j) entries of A and B are congruent modulo a fixed ideal $\mathfrak{a}$ for any (i, j) , we write $A \equiv B \bmod \mathfrak{a}$.

2.2 Preliminaries for Chapter 2

In this section, we assume that R is just a commutative ring. Let m, n be positive integers with $m \leq n$ and $A = (\, a_{ij}\,) \in \mathrm{Mat}(m, n\,; R)$. Let us recall the following rather well-known fact.

Lemma 2.2.1. *Suppose* $\mathrm{I}_m(A) \subseteq \mathfrak{p} \in \mathrm{Spec}\, R$ *and put* $\ell = \max\{\, 0 \leq k \in \mathbb{Z} \mid \mathrm{I}_k(A) \not\subseteq \mathfrak{p}\,\}$ *. Then* $\ell < m$ *and there exists* $B \in \mathrm{Mat}(m - \ell, n - \ell\,; \mathfrak{p}R_{\mathfrak{p}})$ *such that* $\mathrm{I}_k(A)_{\mathfrak{p}} = \mathrm{I}_{k-\ell}(B)$ *for any* $k \in \mathbb{Z}$ *.*

Proof. We prove by induction on ℓ . The assertion is obvious if $\ell = 0$. So, let us consider the case where $\ell > 0$. Then $\mathrm{I}_1(A) \not\subseteq \mathfrak{p}$, and so some entry of A is a unit in $R_{\mathfrak{p}}$. Hence, applying elementary operations to A in $\mathrm{Mat}(m, n\,; R_{\mathfrak{p}})$, we get a matrix of the form

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{array}\right),$$

where $A' \in \mathrm{Mat}(m-1, n-1\,; R_{\mathfrak{p}})$. It is easy to see that $\mathrm{I}_k(A)_{\mathfrak{p}} = \mathrm{I}_{k-1}(A')$ for any $k \in \mathbb{Z}$. Hence $\mathrm{I}_{m-1}(A') \subseteq \mathfrak{p}R_{\mathfrak{p}}$ and $\ell - 1 = \max\{\, 0 \leq k \in \mathbb{Z} \mid \mathrm{I}_k(A') \not\subseteq \mathfrak{p}R_{\mathfrak{p}} \,\}$. By the hypothesis of induction, there exists

$$B \in \mathrm{Mat}(\,(m-1)-(\ell-1)\,,\,(n-1)-(\ell-1)\,;\mathfrak{p}R_{\mathfrak{p}}) = \mathrm{Mat}(m-\ell, n-\ell\,; \mathfrak{p}R_{\mathfrak{p}})$$

such that $\mathrm{I}_t(A') = \mathrm{I}_{t-(\ell-1)}(B)$ for any $t \in \mathbb{Z}$. Then we have $\mathrm{I}_k(A)_{\mathfrak{p}} = \mathrm{I}_{k-\ell}(B)$ for any $k \in \mathbb{Z}$. $\square$

In the rest of this section, we assume $n = m+1$. For any $j = 1, 2, \ldots, m+1$, A_j denotes the $m \times m$ submatrix of A determined by removing the j-th column. We set $d_j = (-1)^{j-1} \cdot \det A_j$ and $I = (d_1, d_2, \ldots, d_{m+1})R = \mathrm{I}_m(A)$. Let us take an indeterminate t over R and consider the Rees algebra of I;

$$\mathrm{R}(I) := R[\,d_1t, d_2t, \ldots, d_{m+1}t\,] \subseteq R[\,t\,],$$

which is a graded ring such that $\deg d_jt = 1$ for all $j = 1, 2, \ldots, m+1$. On the other hand, let $S = R[\,T_1, T_2, \ldots, T_{m+1}\,]$ be a polynomial ring over R with $m+1$ variables. We regard S as a graded ring by setting $\deg T_j = 1$ for all $j = 1, 2, \ldots, m+1$. Let $\pi : S \longrightarrow \mathrm{R}(I)$ be the homomorphism of R-algebras such that $\pi(T_j) = d_jt$ for any j. Then π is a surjective graded homomorphism. Now we set

$$f_i = \sum_{j=1}^{m+1} x_{ij}T_j \in S_1$$

for any $i = 1, \ldots, m$. It is easy to see $(f_1, f_2, \ldots, f_m)S \subseteq \mathrm{Ker}\,\pi$. For our purpose, the following result due to Avramov [2] is very important (Another elementary proof is given in [7]).

Proposition 2.2.2. *Suppose that R is a Noetherian ring. If* $\operatorname{grade} \mathrm{I}_k(A) \geq m - k + 2$ *for all $k = 1, \ldots, m$, then* $\operatorname{Ker} \pi = (f_1, f_2, \ldots, f_m)S$ *and $f_1, f_2, \ldots, f_m$ is an S-regular sequence.*

As the last preliminary result, we describe a technique using determinants of matrices. Suppose that $y_1, y_2, \ldots, y_{m+1}$ are elements of R such that

$$A \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We put $y = y_1 + y_2 + \cdots + y_{m+1}$.

Lemma 2.2.3. *If y, y_k form a regular sequence for some $k = 1, 2, \ldots, m+1$, then there exists $\delta \in R$ such that $y_j \cdot \delta = d_j$ for any $j = 1, 2, \ldots, m+1$.*

Proof. We put $d = d_1 + d_2 + \cdots + d_{m+1}$. Then the following assertion holds:

Claim $y \cdot d_j = y_j \cdot d$ for all $j = 1, 2, \ldots, m+1$.

In order to prove the claim above, let us consider the following $(m+1) \times (m+1)$ matrix:

$$B = \left(\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ \hline & & & \\ & & A & \\ & & & \end{array} \right).$$

Expanding $\det B$ along the first row, we get $\det B = d$. Let us fix $j = 1, 2, \ldots, m+1$.

Multiplying the j-th column of B by y_j , we get

$$B' = \begin{array}{c} \begin{array}{ccccc} \phantom{a_{11}} & & \overset{j}{\smile} & & \phantom{a_{1,m+1}} \end{array} \\ \begin{pmatrix} 1 & \cdots & y_j & \cdots & 1 \\ a_{11} & \cdots & a_{1j}y_j & \cdots & a_{1,m+1} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj}y_j & \cdots & a_{m,m+1} \end{pmatrix} \end{array} .$$

Then $\det B' = y_j \cdot \det B = y_j \cdot d$. Next, for any $\ell \in \{ 1, 2, \ldots, m+1 \} \setminus \{ j \}$, we add the ℓ-th column of B' multiplied by y_ℓ to the j-th column, and get

$$B'' = \begin{array}{c} \begin{array}{ccccc} \phantom{a_{11}} & & \overset{j}{\smile} & & \phantom{a_{1,m+1}} \end{array} \\ \begin{pmatrix} 1 & \cdots & y & \cdots & 1 \\ a_{11} & \cdots & 0 & \cdots & a_{1,m+1} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & 0 & \cdots & a_{m,m+1} \end{pmatrix} \end{array} ,$$

since our assumption means

$$a_{i1}y_1 + \cdots + a_{ij}y_j + \cdots + a_{i,m+1}y_{m+1} = 0$$

for all $i = 1, \ldots, m$. Then $\det B'' = \det B' = y_j \cdot d$. Finally, replacing the first j columns of B'' , we get

$$B''' = \left(\begin{array}{c|ccc} y & 1 & \cdots & 1 \\ \hline 0 & & & \\ \vdots & & A_j & \\ 0 & & & \end{array} \right) .$$

Then $y \cdot d_j = y \cdot (-1)^{j-1} \cdot \det A_j = (-1)^{j-1} \cdot \det B''' = \det B'' = y_j \cdot d$. Thus we get the equalities of the claim.

Now we take $k = 1, 2, \ldots, m+1$ so that y, y_k form a regular sequence. Because $y \cdot d_k = y_k \cdot d$, there exists $\delta \in R$ such that $d = y\delta$. Then $y \cdot d_j = y_j \cdot y\delta$ for any $j = 1, 2, \ldots, m+1$. As y is an R-NZD, we get $d_j = y_j \cdot \delta$ for any $j = 1, 2, \ldots, m+1$, and the proof is complete. $\square$

Lemma 2.2.4. *If R is a Cohen-Macaulay local ring and $y_1, y_2, \ldots, y_{m+1}$ is an ssop for R, then y, y_k form a regular sequence for any $k = 1, 2, \ldots, m+1$.*

Proof. It is enough to show for $k = 1$. Because $(y_1, y_2, \ldots, y_{m+1})R = (y, y_1, \ldots, y_m)R$, it follows that $y, y_1, \ldots, y_m$ is an ssop for R, too. Hence y, y_1 is R-regular. $\square$

Lemma 2.2.5. *Suppose that $\mathfrak{a}$ is an ideal of R and $a_{ij} \in \mathfrak{a}$ for all i, j. We put $Q = (y_1, y_2, \ldots, y_{m+1})R$. Then δ of 2.2.3 is an element of $\mathfrak{a}^m :_R Q$.*

Proof. We get this assertion since $d_j \in \mathfrak{a}^m$ for any $j = 1, 2, \ldots, m+1$. $\square$

2.3 Associated primes of R/I^n

Let R be a Noetherian ring and $A = (a_{ij}) \in \mathrm{Mat}(m, m+1\,; R)$, where $1 \le m \in \mathbb{Z}$. Let $I = \mathrm{I}_m(A)$. Throughout this section, we assume that I is a proper ideal and grade $\mathrm{I}_k(A) \ge m - k + 2$ for all $k = 1, \ldots, m$. Let us keep the notations of Section 2.2.

Let $K_\bullet$ be the Koszul complex of $f_1, f_2, \ldots, f_m$, which is a complex of graded free S-modules. We denote its boundary map by $\partial_\bullet$. Let $e_1, e_2, \ldots, e_m$ be an S-basis of K_1 consisting of homogeneous elements of degree 1 such that $\partial_1(e_i) = f_i$ for any $i = 1, \ldots, m$.

Then, for any $s = 1, \dots, m$,

$$\{\, e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_s} \mid 1 \le i_1 < i_2 < \cdots < i_s \le m \,\}$$

is an S-basis of K_s consisting of homogeneous elements of degree s, and we have

$$\partial_s(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_s}) = \sum_{p=1}^{s} (-1)^{p-1} \cdot f_{i_p} \cdot e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s}\,,$$

where $\widehat{e_{i_p}}$ means that e_{i_p} is omitted from the exterior product. Let $1 \le r \in \mathbb{Z}$. Taking the homogeneous part of degree r of $K_\bullet$, we get a complex

$$[K_\bullet]_r \,:\, 0 \longrightarrow [K_m]_r \xrightarrow{\partial_m} [K_{m-1}]_r \longrightarrow \cdots \longrightarrow [K_1]_r \xrightarrow{\partial_1} [K_0]_r \longrightarrow 0$$

of finitely generated free R-modules. It is obvious that $[K_s]_r = 0$ if $r < s$. On the other hand, if $r \ge s$, then

$$\left\{ T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{m+1}^{\alpha_{m+1}} \cdot e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_s} \;\middle|\; \begin{array}{l} 0 \le \alpha_1, \alpha_2, \dots, \alpha_{m+1} \in \mathbb{Z}\,, \\ \alpha_1 + \alpha_2 + \cdots + \alpha_{m+1} = r - s\,, \\ 1 \le i_1 < i_2 < \cdots < i_s \le m \end{array} \right\}$$

is an R-basis of $[K_s]_r$.

Proposition 2.3.1. *If* $(R, \mathfrak{m})$ *is a local ring and* $A \in \mathrm{Mat}(m, m+1\,; \mathfrak{m})$, *we have*

$$\mathrm{proj.\,dim}_R\, R/I^r = \begin{cases} r+1 & \text{if } r < m\,, \\ m+1 & \text{if } r \ge m\,. \end{cases}$$

Proof. By 2.2.2 and [4, 1.6.17], we see that

$$0 \longrightarrow K_m \xrightarrow{\partial_m} K_{m-1} \longrightarrow \cdots \longrightarrow K_1 \xrightarrow{\partial_1} K_0 \xrightarrow{\pi} \mathrm{R}(I) \longrightarrow 0$$

is a graded S-free resolution of $\mathrm{R}(I)$. Hence, for any integer $r \ge 0$,

$$0 \longrightarrow [K_m]_r \xrightarrow{\partial_m} [K_{m-1}]_r \longrightarrow \cdots \longrightarrow [K_1]_r \xrightarrow{\partial_1} [K_0]_r \xrightarrow{\pi} I^r t^r \longrightarrow 0$$

is an R-free resolution of the R-module $I^r t^r$. Let us notice $I^r t^r \cong I^r$ as R-modules. Suppose $1 \leq s \leq m$ and $r \geq s$. Then, for any non-negative integers $\alpha_1, \alpha_2, \dots, \alpha_{m+1}$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_{m+1} = r - s$ and positive integers $i_1, i_2, \dots, i_s$ with $1 \leq i_1 < i_2 < \cdots < i_\ell \leq m$, we have

$$\begin{aligned}
& \partial_s(T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{m+1}^{\alpha_{m+1}} \cdot e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_s}) \\
= \ & T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{m+1}^{\alpha_{m+1}} \cdot \sum_{p=1}^{s} (-1)^{p-1} \cdot (\sum_{j=1}^{m+1} a_{i_p,j} T_j) \cdot e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s} \\
= \ & \sum_{p=1}^{s} \sum_{j=1}^{m+1} (-1)^{p-1} a_{i_p,j} \cdot T_1^{\alpha_1} \cdots T_j^{1+\alpha_j} \cdots T_{m+1}^{\alpha_{m+1}} \cdot e_{i_1} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_s} \\
\in \ & \mathfrak{m} \cdot [K_{s-1}]_r \, .
\end{aligned}$$

Hence $[K_\bullet]_r$ gives a minimal R-free resolution of I^r. If $r < m$, we have $[K_r]_r \neq 0$ and $[K_s]_r = 0$ for any $s > r$, and so $\operatorname{proj.dim}_R I^r = r$. On the other hand, if $r \geq m$, we have $[K_m]_r \neq 0$ and $[K_s]_r = 0$ for any $s > m$, and so $\operatorname{proj.dim}_R I^r = m$. Thus we get the required equality as $\operatorname{proj.dim}_R R/I^r = \operatorname{proj.dim}_R I^r + 1$. □

By Auslander-Buchsbaum formula (cf. [4, 1.3.3]), we get the following.

Corollary 2.3.2. *If $(R, \mathfrak{m})$ is local and $A \in \operatorname{Mat}(m, m+1\,; \mathfrak{m})$, we have*

$$\operatorname{depth} R/I^r = \begin{cases} \operatorname{depth} R - r - 1 & \text{if } r < m\,, \\ \operatorname{depth} R - m - 1 & \text{if } r \geq m\,. \end{cases}$$

Here we remark that $\operatorname{depth} R \geq \operatorname{grade} \mathrm{I}_1(A) \geq m + 1$ by our assumption of this section. As a consequence of 2.3.2, we see that the next assertion holds.

Corollary 2.3.3. *Suppose that $(R, \mathfrak{m})$ is a local ring and $A \in \operatorname{Mat}(m, m+1\,; \mathfrak{m})$. Then we have $\mathfrak{m} \in \operatorname{Ass}_R R/I^r$ if and only if $r \geq m$ and* $\operatorname{depth} R = m + 1$.

The next result is a generalization of 2.3.3 .

Proposition 2.3.4. *Let* $I \subseteq \mathfrak{p} \in \operatorname{Spec} R$ *and* $1 \leq r \in \mathbb{Z}$ *. We put*

$$\ell = \max\{\, 0 \leq k < m \mid \mathrm{I}_k(A) \not\subseteq \mathfrak{p} \,\}\,.$$

Then the following conditions are equivalent.

(1) $\mathfrak{p} \in \operatorname{Ass}_R R/I^r$ *.*

(2) $r \geq m - \ell$ *and* $\operatorname{depth} R_{\mathfrak{p}} = m - \ell + 1$ *.*

When this is the case, $\operatorname{grade} \mathrm{I}_{\ell+1}(A) = m - \ell + 1$ *.*

Proof. By 2.2.1, there exists $B \in \operatorname{Mat}(m-\ell, m-\ell+1\,; \mathfrak{p}R_{\mathfrak{p}})$ such that $\mathrm{I}_k(B) = \mathrm{I}_{k+\ell}(A)_{\mathfrak{p}}$ for any $k \in \mathbb{Z}$. Hence, for any $k = 1, \ldots, m - \ell$, we have

$$\operatorname{grade} \mathrm{I}_k(B) = \operatorname{grade} \mathrm{I}_{k+\ell}(A)_{\mathfrak{p}} \geq m - (k + \ell) + 2 = (m - \ell) - k + 2\,.$$

Therefore, by 2.3.3, we see that $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/\mathrm{I}_{m-\ell}(B)^r$ if and only if $r \geq m - \ell$ and $\operatorname{depth} R_{\mathfrak{p}} = m - \ell + 1$. Let us notice $\mathrm{I}_{m-\ell}(B) = I_{\mathfrak{p}}$. Because $\mathfrak{p} \in \operatorname{Ass}_R R/I^r$ if and only if $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Ass}_{R_{\mathfrak{p}}} R_{\mathfrak{p}}/{I_{\mathfrak{p}}}^r$, we see (1) $\Leftrightarrow$ (2) . Furthermore, as $\mathrm{I}_{\ell+1}(A) \subseteq \mathfrak{p}$, we have $\operatorname{grade} \mathrm{I}_{\ell+1}(A) \leq \operatorname{depth} R_{\mathfrak{p}}$, and so we get $\operatorname{grade} \mathrm{I}_{\ell+1}(A) = m - \ell + 1$ if the condition (2) is satisfied. $\square$

For any positive integer r , let Λ_A^r be the set of integers i such that $\max\{1, m - r + 1\} \leq i \leq m$ and $\operatorname{grade} \mathrm{I}_i(A) = m - i + 2$. We denote by $\operatorname{Assh}_R R/\mathrm{I}_i(A)$ the set of $\mathfrak{p} \in \operatorname{Ass}_R R/\mathrm{I}_i(A)$ such that $\dim R/\mathfrak{p} = \dim R/\mathrm{I}_i(A)$. Then the following assertion holds.

Proposition 2.3.5. *Let R be a Cohen-Macaulay ring. Then, for any positive integer r, we have*

$$\mathrm{Ass}_R\, R/I^r = \bigcup_{i\in\Lambda_A^r} \mathrm{Assh}_R\, R/\mathrm{I}_i(A)\,.$$

Proof. Let us take any $\mathfrak{p} \in \mathrm{Ass}_R\, R/I^r$ and put $\ell = \max\{0 \leq k < m \mid \mathrm{I}_k(A) \not\subseteq \mathfrak{p}\}$. Then $\mathrm{I}_{\ell+1}(A) \subseteq \mathfrak{p}$. Moreover, by 2.3.4 we have $r \geq m - \ell$, $\operatorname{depth} R_{\mathfrak{p}} = m - \ell + 1$ and $\operatorname{grade} \mathrm{I}_{\ell+1}(A) = m - \ell + 1$. Hence $\ell + 1 \in \Lambda_A^r$. Let us notice that $\operatorname{ht} \mathfrak{p} = \operatorname{depth} R_{\mathfrak{p}}$ and $\operatorname{ht} \mathrm{I}_{\ell+1}(A) = \operatorname{grade} \mathrm{I}_{\ell+1}(A)$ as R is Cohen-Macaulay. Therefore $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathrm{I}_{\ell+1}(A)$, which means $\mathfrak{p} \in \mathrm{Assh}_R\, R/\mathrm{I}_{\ell+1}(A)$.

Conversely, let us take any $i \in \Lambda_A^r$ and $\mathfrak{q} \in \mathrm{Assh}_R\, R/\mathrm{I}_i(M)$. Then $\operatorname{ht} \mathfrak{q} = \operatorname{ht} \mathrm{I}_i(A) = \operatorname{grade} \mathrm{I}_i(A) = m - i + 2$. As our assumption implies $\operatorname{ht} \mathrm{I}_{i-1}(A) \geq m - i + 3$, it follows that $i - 1 = \max\{0 \leq k < m \mid \mathrm{I}_k(A) \not\subseteq \mathfrak{q}\}$. Let us notice $r \geq m - (i-1)$ as $m - r + 1 \leq i$, which is one of the conditions for $i \in \Lambda_A^r$. Moreover, we have $\operatorname{depth} R_{\mathfrak{q}} = \operatorname{ht} \mathfrak{q} = m - (i-1) + 1$. Thus we get $\mathfrak{q} \in \mathrm{Ass}_R\, R/I^r$ by 2.3.4, and the proof is complete. $\square$

As a natural question, one may ask whether the results stated above can be extended to the case where $A = (a_{ij}) \in \mathrm{Mat}(m, n\,; R)$ with $m < n$. As far as the authors know, the following two kinds of generalizations seem to be possible.

First, we would like to suggest considering the powers of modules. We set M to be the cokernel of the R-linear map $R^m \longrightarrow R^n$ defined by ${}^tA$. Let us assume $\operatorname{grade} \mathrm{I}_k(A) \geq m - k + 2$ for all $k = 1, \ldots, m$. Then, by [2, Proposition 4], M has rank $n - m$ and the r-th symmetric power $\mathcal{S}^r(M)$ is torsion-free over R for any $r \geq 1$. In this case, M can be embedded into a finitely generated free R-module F and the r-th power M^r of M is defined to be the image of $\mathcal{S}^r(M)$ in $\mathcal{S}^r(F)$. Similarly as the case of $m \times (m+1)$ matrix, we consider the polynomial ring $S = R[\,T_1, T_2, \ldots, T_n\,]$ and set $f_i = a_{i1}T_1 + a_{i2}T_2 + \cdots + a_{in}T_n \in S$ for all $i = 1, \ldots, m$. Then, by [2, Propsosition 1 and Proposition 4], we get an R-free resolution of $\mathcal{S}^r(F)/M^r$ by taking the homogeneous part of the Koszul complex of $f_1, f_2, \ldots, f_m$ over S. In this way, detailed information about the associated primes of $\mathcal{S}^r(F)/M^r$ could be deduced.

On the other hand, by using the free resolution of Akin-Buchsbaum-Weyman [1], another generalization seems to be possible. This idea was suggested by the referee. Similarly as the case of $m \times (m+1)$ matrix, we set I to be the ideal generated by the maximal minors of A. Let us assume $\operatorname{grade} \mathrm{I}_k(A) \geq (m-k+1)(n-m)+1$ for any $k = 1, \ldots, m$. Then, by [1, Theorem 5.4], we get an R-free resolution of R/I^r for any $r \geq 1$, and it could be used in place of 2.3.1 to deduce the $m \times n$ matrix version of 2.3.4 and 2.3.5.

Proposition 2.3.6. *Let R be an $(m+1)$-dimensional Cohen-Macaulay local ring, where $2 \leq m \in \mathbb{Z}$. Let A be the matrix given in Introduction. Then the following assertions hold.*

(1) $\operatorname{ht} \mathrm{I}_k(A) \geq m-k+2$ *for all* $k = 1, 2, \ldots, m$.

(2) $$\operatorname{proj.dim}_R R/I^r = \begin{cases} r+1 & \text{if } r < m\,, \\ m+1 & \text{if } r \geq m\,. \end{cases}$$

(3) $\operatorname{depth} R/I^r = \begin{cases} m-r & \text{if } r < m\,, \\ 0 & \text{if } r \geq m\,. \end{cases}$

Furthermore, if $\alpha_{ij} = 1$ for all i and j, the following assertions hold.

(4) $\operatorname{ht} \mathrm{I}_2(A) = m$ *and* $\operatorname{Assh}_R R/\mathrm{I}_2(A) \subseteq \operatorname{Ass}_R R/I^r$ *for any* $r \geq m-1$.

(5) *If m is an odd integer with $m \geq 3$, then* $\operatorname{ht} \mathrm{I}_3(A) = m-1$ *and* $\operatorname{Assh}_R R/\mathrm{I}_3(A) \subseteq \operatorname{Ass}_R R/I^r$ *for any* $r \geq m-2$.

(6) *If $m \geq 3$, then* $(I^r)^{\mathrm{sat}} \subsetneq I^{(r)}$ *for any* $r \geq m-1$.

Proof. (1) We aim to prove the following.

Claim $J_{k-1} + \mathrm{I}_k(A)$ is an $\mathfrak{m}$-primary ideal for any $k = 1, 2, \ldots, m$, where $J_{k-1} = (x_1, x_2, \ldots, x_{k-1})R$.

If this is true, we have $\dim R/\mathrm{I}_k(A) \leq k-1$, and so $\operatorname{ht} \mathrm{I}_k(A) \geq \dim R - (k-1) = m-k+2$, which is the required inequality.

In order to prove Claim, we take any $\mathfrak{p} \in \operatorname{Spec} R$ containing $J_{k-1} + \mathrm{I}_k(A)$. It is enough to show $J_{m+1} = (x_1, x_2, \ldots, x_{m+1})R \subseteq \mathfrak{p}$. For that purpose, we prove $J_\ell \subseteq \mathfrak{p}$ for any $\ell = k-1, k, \ldots, m+1$ by induction on ℓ. As we obviously have $J_{k-1} \subseteq \mathfrak{p}$, let us assume $k \leq \ell \leq m+1$ and $J_{\ell-1} \subseteq \mathfrak{p}$. Because the k-minor of A with respect to the first k rows and the columns $\ell-k+1, \ldots, \ell-1, \ell$ is congruent with

$$\det \begin{pmatrix} & 0 & & & x_\ell^{\alpha_{1,\ell}} \\ & & & x_\ell^{\alpha_{2,\ell-1}} & \\ & & \iddots & & * \\ x_\ell^{\alpha_{k,\ell-k+1}} & & & & \end{pmatrix}$$

mod $J_{\ell-1}$, it follows that $J_{\ell-1} + \mathrm{I}_k(A)$ includes some power of x_ℓ. Hence $x_\ell \in \mathfrak{p}$, and so we get $J_\ell \subseteq \mathfrak{p}$.

(2) and (3) follow from 2.3.1 and 2.3.2, respectively.

In the rest of this proof, we assume $\alpha_{ij} = 1$ for any i and j.

(4) Let $\mathfrak{q} = (x_1 - x_2, x_2 - x_3, \ldots, x_m - x_{m+1})R$. Then $x_1 \equiv x_j \mod \mathfrak{q}$ for any $j = 1, 2, \ldots, m+1$. Hence, any 2-minor of A is congruent with

$$\det \begin{pmatrix} x_1 & x_1 \\ x_1 & x_1 \end{pmatrix} = 0$$

mod $\mathfrak{q}$. This means $\mathrm{I}_2(A) \subseteq \mathfrak{q}$, and so $\operatorname{ht} \mathrm{I}_2(A) \leq \mu_R(\mathfrak{q}) = m$. On the other hand, $\operatorname{ht} \mathrm{I}_2(A) \geq m$ by (1). Thus we get $\operatorname{ht} \mathrm{I}_2(A) = m$. Then, for any $r \geq m-1$, we have $2 \in \Lambda_A^r$, and so $\operatorname{Assh}_R R/\mathrm{I}_2(A) \subseteq \operatorname{Ass}_R R/I^r$ by 2.3.5.

(5) Let $\mathfrak{p}$ be the ideal of R generated by $\{x_i - x_{i+2}\}$, where i runs all odd integers with $1 \leq i \leq m-2$. Similarly, we set $\mathfrak{q}$ to be the ideal of R generated by $\{x_j - x_{j+2}\}$, where j runs all even integers with $2 \leq j \leq m-1$. Let A' be the submatrix of A with the rows i_1, i_2, i_3 and the columns j_1, j_2, j_3, where $1 \leq i_1 < i_2 < i_3 \leq m$ and $1 \leq j_1 < j_2 < j_3 \leq m+1$. We can choose p, q with $1 \leq p < q \leq 3$ so that $i_p \equiv i_q \mod 2$. Then, for any $t = 1, 2, 3$, we have $i_p + j_t \equiv i_q + j_t \mod 2$, and so, if $i_p + j_t$ is odd (resp. even), it follows that $a_{i_p, j_t} \equiv a_{i_q, j_t} \mod \mathfrak{q}$ (resp. $\mathfrak{p}$). Hence, we see that the p-th row of A' is congruent with the q-th row of A' mod $\mathfrak{p} + \mathfrak{q}$, which means $\det A' \equiv 0 \mod \mathfrak{p} + \mathfrak{q}$. As a consequence, we get $\mathrm{I}_3(A) \subseteq \mathfrak{p} + \mathfrak{q}$. Therefore $\operatorname{ht} \mathrm{I}_3(A) \leq \mu_R(\mathfrak{p}) + \mu_R(\mathfrak{q}) = (m-1)/2 + (m-1)/2 = m-1$.

(6) Let us take any $\mathfrak{p} \in \operatorname{Assh}_R R/\mathrm{I}_2(A)$ and $r \geq m-1$. Then, by (4) we have

ht $\mathfrak{p} = m \geq 3$ and $\mathfrak{p} \in \operatorname{Ass}_R R/I^r$. Hence $\operatorname{Ass}_R R/I^r$ is *not* a subset of $\{\,\mathfrak{m}\,\} \cup \operatorname{Min}_R R/I$. Therefore, by the observation stated in Introduction, we get $(I^r)^{\mathrm{sat}} \subsetneq I^{(r)}$ and the proof is complete. $\square$

2.4 Computing $(I^m)^{\mathrm{sat}}$

In this section, we assume that $(R, \mathfrak{m})$ is an $(m+1)$-dimensional Cohen-Macaulay local ring, where $2 \leq m \in \mathbb{Z}$. Let $x_1, x_2, \ldots, x_{m+1}$ be an sop for R and let A be the matrix given in Introduction. We put $I = \mathrm{I}_m(A)$. Then, by (3) of 2.3.6, we get the assertion (1) of 2.1.1. Let us prove (2) of 2.1.1.

For any $j = 1, 2, \ldots, m+1$, we set $d_j = (-1)^{j-1} \cdot \det A_j$, where A_j is the submatrix of A determined by removing the j-th column. Then $I = (d_1, d_2, \ldots, d_{m+1})R$. Furthermore, for any $k = 1, 2, \ldots, m+1$, we denote by β_k the minimum of the exponents of x_k that appear in the entries of A . Let us notice that A's entries which are powers of x_k appear as follows:

$$\left(\begin{array}{cccc|c|cccc}
 & & & x_k^{\alpha_{1,k}} & & & & & \\
 & & x_k^{\alpha_{2,k-1}} & & & & & & \\
 & \iddots & & & & & & & \\
x_k^{\alpha_{k,1}} & & & & & & & & \\ \hline
 & & & & & & & & x_k^{\alpha_{k+1,m+1}} \\
 & & & & & & & x_k^{\alpha_{k+2,m}} & \\
 & & & & & & \iddots & & \\
 & & & & & x_k^{\alpha_{m,k+2}} & & &
\end{array}\right)$$

if $1 \leq k < m$, and

$$\left(\begin{array}{cccc|c}
 & & & x_m^{\alpha_{1,m}} & \\
 & & x_m^{\alpha_{2,m-1}} & & \\
 & \iddots & & & \\
x_m^{\alpha_{m,1}} & & & &
\end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|cccc}
 & & & & x_{m+1}^{\alpha_{1,m+1}} \\
 & & & x_{m+1}^{\alpha_{2,m}} & \\
 & & \iddots & & \\
 & x_{m+1}^{\alpha_{m,2}} & & &
\end{array}\right)$$

if $k = m$ or $m+1$, respectively. So, we have

$$\beta_k = \begin{cases} \min\{\alpha_{i,k-i+1}\}_{1 \le i \le k} \cup \{\alpha_{i,k-i+m+2}\}_{k<i\le m} & \text{if } 1 \le k < m\,, \\ \min\{\alpha_{i,k-i+1}\}_{1\le i\le m} \quad \text{if } k = m \text{ or } m+1\,. \end{cases}$$

Then, for any $k = 1, 2, \ldots, m+1$, we can choose i_k so that one of the following conditions is satisfied:

(i) $1 \le i_k \le k$ and $\beta_k = \alpha_{i_k, k-i_k+1}$ or (ii) $k < i_k \le m$ and $\beta_k = \alpha_{i_k, k-i_k+m+2}$.

Now, for any $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m+1$, we set

$$a'_{ik} = \begin{cases} x_k^{\alpha_{i,k-i+1}-\beta_k} & \text{if } i \le k\,, \\ x_k^{\alpha_{i,k-i+m+2}-\beta_k} & \text{if } i > k\,. \end{cases}$$

Then $a'_{i_k,k} = 1$ for all $k = 1, 2, \ldots, m+1$. The next assertion can be verified easily.

Lemma 2.4.1. *Suppose* $1 \le i \le m$ *and* $1 \le j \le m+1$.

(1) *If* $i + j \le m+2$, *setting* $k = i+j-1$, *we have* $1 \le k \le m+1$, $i \le k$ *and* $a_{ij} = x_k^{\beta_k} \cdot a'_{ik}$.

(2) *If* $i + j > m+2$, *setting* $k = i+j-m-2$, *we have* $1 \le k < m$, $i > k$ *and* $a_{ij} = x_k^{\beta_k} \cdot a'_{ik}$.

Let Q be the ideal of R generated by $x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_{m+1}^{\beta_{m+1}}$. Then $A \in \mathrm{Mat}(m, m+1; Q)$ by 2.4.1. The assertion (2) of 2.1.1 follows from the next

Proposition 2.4.2. $(I^m)^{\mathrm{sat}} = I^m :_R Q$ *and* $(I^m)^{\mathrm{sat}}/I^m \cong R/Q$.

Proof. Let S be the polynomial ring over R with variables $T_1, T_2, \dots, T_{m+1}$. We regard S as a graded ring by setting $\deg T_j = 1$ for any $j = 1, 2, \dots, m+1$. Let

$$f_i = \sum_{j=1}^{m+1} a_{ij} T_j \in S_1$$

for any $i = 1, 2, \dots, m$ and let $K_\bullet$ be the Koszul complex of $f_1, f_2, \dots, f_m$. Then $K_\bullet$ is a graded complex. Let $\partial_\bullet$ be the boundary map of $K_\bullet$ and let $e_1, e_2, \dots, e_m$ be an S-basis of K_1 consisting of homogeneous elements of degree 1 such that $\partial_1(e_i) = f_i$ for all $i = 1, 2, \dots, m$. As is stated in the proof of 2.3.1,

$$\begin{array}{ccccccccccccc} 0 & \to & [K_m]_m & \xrightarrow{\partial_m} & [K_{m-1}]_m & \to & \cdots & \to & [K_1]_m & \xrightarrow{\partial_1} & [K_0]_m & \xrightarrow{\epsilon} & R & \to & 0 \\ & & & & & & & & & & \| & & & & \\ & & & & & & & & & & S_m & & & & \end{array}$$

is an acyclic complex, where ϵ is the R-linear map such that

$$\epsilon(T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{m+1}^{\alpha_{m+1}}) = d_1^{\alpha_1} d_2^{\alpha_2} \cdots d_{m+1}^{\alpha_{m+1}}$$

for any non-negative integers $\alpha_1, \alpha_2, \dots, \alpha_{m+1}$ with $\alpha_1 + \alpha_2 + \cdots + \alpha_{m+1} = m$. We obviously have $\operatorname{Im} \epsilon = I^m$. We set $e = e_1 \wedge e_2 \wedge \cdots \wedge e_m$ and $\check{e}_i = e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_m$ for any $i = 1, 2, \dots, m$. Let us take $\{ e \}$ and $\{T_j \check{e}_i \mid 1 \le i \le m, 1 \le j \le m+1\}$ as R-basis of $[K_m]_m$ and $[K_{m-1}]_m$, respectively. Because

$$(\sharp) \qquad \partial_m(e) = \sum_{i=1}^{m} (-1)^{i-1} f_i \cdot \check{e}_i = \sum_{i=1}^{m} \sum_{j=1}^{m+1} (-1)^{i-1} a_{ij} \cdot T_j \check{e}_i \,,$$

we have $\partial_m([K_m]_m) \subseteq Q \cdot [K_{m-1}]_m$. Hence, by 1.3.1 we get

$$(I^m :_R Q)/I^m \cong [K_m]_m / Q[K_m]_m \cong R/Q \,.$$

Here, for any $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m+1$, we set

$$T_{ik} = \begin{cases} T_{k-i+1} & \text{if } i \leq k\,, \\ T_{k-i+m+2} & \text{if } i > k\,. \end{cases}$$

Then the following assertion holds:

Claim 1 *Suppose* $1 \leq k, \ell \leq m+1$ *and* $k \neq \ell$, *then* $T_{ik} \neq T_{i\ell}$ *for any* $i = 1, 2, \ldots, m$.

In order to prove the claim above, we may assume $k < \ell$. Then the following three cases can happen: (i) $i \leq k < \ell$, (ii) $k < i \leq \ell$ or (iii) $k < \ell < i$. Because $k-i+1 < \ell-i+1$ and $k-i+m+2 < \ell-i+m+2$, we get $T_{ik} \neq T_{i\ell}$ in the cases of (i) and (iii). Furthermore, as $m+1 > \ell-k$, we get $k-i+m+2 > \ell-i+1$, and so $T_{ik} \neq T_{i\ell}$ holds also in the case of (ii). Thus we have seen Claim 1.

Now, for any $k = 1, 2, \ldots, m+1$, we set

$$v_{(k,e)} = \sum_{i=1}^{m} (-1)^{i-1} a'_{ik} \cdot T_{ik} \check{e}_i \in [K_{m-1}]_m\,.$$

Then the following equality holds:

Claim 2 $\partial_m(e) = \sum_{k=1}^{m+1} x_k^{\beta_k} \cdot v_{(k,e)}$.

In fact, by (♯) and 2.4.1 we have

$$\begin{aligned} \partial_m(e) &= \sum_{i=1}^{m} \Big(\sum_{j=1}^{m-i+2} (-1)^{i-1} a_{ij} \cdot T_j \check{e}_i + \sum_{j=m-i+3}^{m+1} (-1)^{i-1} a_{ij} \cdot T_j \check{e}_i \Big) \\ &= \sum_{i=1}^{m} \Big(\sum_{k=i}^{m+1} (-1)^{i-1} x_k^{\beta_k} a'_{ik} \cdot T_{k-i+1} \check{e}_i + \sum_{k=1}^{i-1} (-1)^{i-1} x_k^{\beta_k} a'_{ik} \cdot T_{k-i+m+2} \check{e}_i \Big) \\ &= \sum_{i=1}^{m} \sum_{k=1}^{m+1} (-1)^{i-1} x_k^{\beta_k} a'_{ik} \cdot T_{ik} \check{e}_i\,, \end{aligned}$$

and so the equality of Claim 2 follows.

Finally, we need the following:

Claim 3 $\{v_{(k,e)}\}_{1\le k\le m}$ is a part of an R-basis of $[K_{m-1}]_m$.

If this is true, by 1.3.4 (See [8, 3.4] for the case where $m=2$) we get depth $R/(I^m :_R Q) > 0$, which means $(I^m)^{\mathrm{sat}} = I^m :_R Q$. So, let us prove Claim 3. By Claim 1, we see that $T_{i_1,1}\check{e}_{i_1}, T_{i_2,2}\check{e}_{i_2}, \ldots, T_{i_m,m}\check{e}_{i_m}$ are different to each other. We set

$$U = \{\, T_j\check{e}_i \mid 1\le i\le m\,,\ 1\le j\le m+1\} \setminus \{T_{i_k,k}\check{e}_{i_k} \mid 1\le k\le m\}$$

and aim to prove that $U\cup\{v_{(k,e)}\}_{1\le k\le m}$ is an R-basis of $[K_{m-1}]_m$. By 1.3.3, it is enough to show that the submodule of $[K_{m-1}]_m$ generated by $U\cup\{v_{(k,e)}\}_{1\le k\le m}$ includes $T_{i_k,k}\check{e}_{i_k}$ for any $k=1,2,\ldots,m$. This can be easily seen since

$$v_{(k,e)} = (-1)^{k-1}\cdot T_{i_k,k}\check{e}_{i_k} + \sum_{i\ne i_k}(-1)^{i-1}a'_{ik}\cdot T_{ik}\check{e}_i$$

and $T_{ik}\check{e}_i \in U$ if $i\ne i_k$, which follows from Claim 1. Thus the assertion of Claim 3 follows, and the proof of 2.4.2 is complete. □

If we assume a suitable condition on $\{\alpha_{ij}\}$, we can describe a generator of $(I^m)^{\mathrm{sat}}/I^m$. For any $i=1,2,\ldots,m$ and $k=1,2,\ldots,m+1$, we set

$$b_{ik} = \begin{cases} a'_{ik}d_{k-i+1} & \text{if } i\le k \\ a'_{ik}d_{k-i+m+2} & \text{if } i>k\,, \end{cases}$$

and $B=(b_{ik})\in \mathrm{Mat}(m,m+1\,;I)$. Then the next equality holds:

Lemma 2.4.3. $B\begin{pmatrix} x_1^{\beta_1} \\ x_2^{\beta_2} \\ \vdots \\ x_{m+1}^{\beta_{m+1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$

Proof. For all $i = 1, 2, \ldots, m$, we have

$$\sum_{j=1}^{m+1} a_{ij} d_j = 0\,.$$

Let us divide the left side of this equality as follows:

$$\sum_{j=1}^{m-i+2} a_{ij} d_j + \sum_{j=m-i+3}^{m+1} a_{ij} d_j = 0\,.$$

If $1 \leq j \leq m-i+2$, setting $k = i+j-1$, we have $i \leq k \leq m+1$ and

$$a_{ij} d_j = x_k^{\beta_k} a'_{ik} \cdot d_{k-i+1} = x_k^{\beta_k} \cdot b_{ik}\,.$$

On the other hand, if $m-i+3 \leq j \leq m+1$, setting $k = i+j-m-2$, we have $1 \leq k < i$ and

$$a_{ij} d_j = x_k^{\beta_k} a'_{ik} \cdot d_{k-i+m+2} = x_k^{\beta_k} \cdot b_{ik}\,.$$

Thus we get

$$\sum_{k=1}^{m+1} b_{ik} \cdot x_k^{\beta_k} = 0$$

for all $i = 1, 2, \cdots, m$, which means the required equality. $\square$

For any $k = 1, 2, \ldots, m+1$, we denote by B_k the submatrix of B determined by removing the k-th column. We set $b_k = (-1)^{k-1} \det B_k \in I^m$.

Proposition 2.4.4. *Suppose $\beta_k = \alpha_{k,1}$ for any $k = 1, 2, \ldots, m$ (For example, this holds if $\alpha_{k,1} = 1$ for any $k = 1, 2, \ldots, m$). Then, there exists $\delta \in R$ such that $x_k^{\beta_k} \cdot \delta = b_k$ for any $k = 1, 2, \ldots, m+1$ and $(I^m)^{\mathrm{sat}} = I^m + (\delta)$.*

Proof. The existence of δ such that $x_k^{\beta_k} \cdot \delta = b_k$ for any $k = 1, 2, \ldots, m+1$ follows from 2.2.3 and 2.4.3. Then $\delta \in I^m :_R Q \subseteq (I^m)^{\mathrm{sat}}$. We put $Q' = (x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_m^{\beta_m})R$. Then

$$A_1 \equiv \begin{pmatrix} & 0 & & x_{m+1}^{\alpha_{1,m+1}} \\ & & x_{m+1}^{\alpha_{2,m}} & \\ & \iddots & & 0 \\ x_{m+1}^{\alpha_{m,2}} & & & \end{pmatrix} \bmod Q',$$

and so $d_1 \equiv \pm x_{m+1}^{\alpha} \bmod Q'$, where $\alpha := \alpha_{1,m+1} + \alpha_{2,m} + \cdots + \alpha_{m,2}$. Furthermore, if $2 \le k \le m+1$, we have $d_k \in Q'$ since the entries of the first column of M_k are $x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_m^{\beta_m}$. Hence $Q' + I = Q' + (x_{m+1}^{\alpha})$. On the other hand, the assumption of 2.4.4 implies that, for all $k = 1, 2, \ldots, m$, we can take k itself as i_k, and then $a'_{kk} = 1$. Hence

$$B = \begin{pmatrix} d_1 & a'_{12}d_2 & a'_{13}d_3 & \cdots & a'_{1m}d_m & a'_{1,m+1}d_{m+1} \\ a'_{21}d_{m+1} & d_1 & a'_{23}d_2 & \cdots & a'_{2m}d_{m-1} & a'_{2,m+1}d_m \\ a'_{31}d_m & a'_{32}d_{m+1} & d_1 & \cdots & a'_{3m}d_{m-2} & a'_{3,m+1}d_{m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a'_{m1}d_3 & a'_{m2}d_4 & a'_{m3}d_5 & \cdots & d_1 & a'_{m,m+1}d_2 \end{pmatrix},$$

and so

$$B_{m+1} \equiv \begin{pmatrix} \pm x_{m+1}^{\alpha} & & & \\ & \pm x_{m+1}^{\alpha} & & 0 \\ & 0 & \ddots & \\ & & & \pm x_{m+1}^{\alpha} \end{pmatrix} \bmod Q',$$

which means $b_{m+1} \equiv \pm x_{m+1}^{m\alpha} \bmod Q'$. Thus we get

$$x_{m+1}^{\beta_{m+1}} \cdot \delta \equiv x_{m+1}^{m\alpha} \bmod Q'.$$

Here we notice $\beta_{m+1} \le \alpha_{1,m+1} < \alpha$. Because $x_1^{\beta_1}, x_2^{\beta_2}, \ldots, x_{m+1}^{\beta_{m+1}}$ is an R-regular sequence, it follows that

$$\delta \equiv \pm x_{m+1}^{m\alpha-\beta_{m+1}} \bmod Q' ,$$

and so

$$Q' + (\delta) = Q' + (x_{m+1}^{m\alpha-\beta_{m+1}}) \supseteq Q' + (x_{m+1}^{m\alpha}) = Q' + I^m .$$

Now we consider the R-linear map

$$f : R \longrightarrow \frac{Q' + (x_{m+1}^{m\alpha-\beta_{m+1}})}{Q' + (x_{m+1}^{m\alpha})} = \frac{Q' + (\delta)}{Q' + I^m}$$

such that $f(1)$ is the class of $x_{m+1}^{m\alpha-\beta_{m+1}}$. Then we have the following:

Claim $\operatorname{Ker} f = Q$.

If this is true, then $R/Q \cong (Q' + (\delta))/(Q' + I^m)$, and so

$$\ell_R(\, R/Q\,) = \ell_R(\, \frac{Q' + (\delta)}{Q' + I^m}\,) .$$

Because $(Q' + (\delta))/(Q' + I^m)$ is a homomorphic image of $(I^m + (\delta))/I^m$ and $I^m + (\delta) \subseteq (I^m)^{\mathrm{sat}}$, we have

$$\ell_R(\, \frac{Q' + (\delta)}{Q' + I^m}\,) \le \ell_R(\, \frac{I^m + (\delta)}{I^m}\,) \le \ell_R(\, (I^m)^{\mathrm{sat}}/I^m\,) = \ell_R(\, R/Q\,) ,$$

where the last equality follows from 2.4.2 . Thus we see

$$\ell_R(\, \frac{I^m + (\delta)}{I^m}\,) = \ell_R(\, (I^m)^{\mathrm{sat}}/I^m\,) ,$$

and so $I^m + (\delta) = (I^m)^{\mathrm{sat}}$ holds.

Proof of Claim. Let us take any $z \in \operatorname{Ker} f$. Then, there exists $w \in R$ such that

$$z \cdot x_{m+1}^{m\alpha - \beta_{m+1}} \equiv w \cdot x_{m+1}^{m\alpha} \bmod Q' .$$

This congruence implies

$$x_{m+1}^{m\alpha-\beta_{m+1}}(z - w \cdot x_{m+1}^{\beta_{m+1}}) \in Q' .$$

Because $x_1^{\beta_1}, \ldots, x_m^{\beta_m}, x_{m+1}^{m\alpha-\beta_{m+1}}$ is an R-regular sequence, we have $z - w \cdot x_{m+1}^{\beta_{m+1}} \in Q'$, which means $z \in Q$. Hence $\operatorname{Ker} f \subseteq Q$. As the converse inclusion is obvious, we get the equality of the claim, and the proof of 2.4.4 is complete. □

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