On the KPZ fixed point in asymmetric exclusion processes

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Preface

In probability theory, finding universal structures in random systems is a matter of central focus. In a system consisting of independent random variables, the universality is characterized by the law of large numbers and central limit theorem. However in general they are not applicable to a system with correlated random variables, and thus the studies on universality in correlated random systems are one of the most challenging problems in mathematics and physics. Among the universalities, the Kardar-Parisi-Zhang (KPZ) universality has been studied actively for a long time. To study universal properties on fluctuations in physical phenomena such as burning paper and spreading coffee stains, Kardar, Parisi, and Zhang introduced a stochastic partial differential equation which is now called the KPZ equation [21]. Let $h(t,x) \in \mathbb{R}$ be the height of the interfacial at position $x \in \mathbb{R}$ and time $t \in \mathbb{R}_{\geq 0}$. Then the one-dimensional KPZ equation is written as follows:

$$\partial_t h(x,t) = \frac{1}{2}\lambda \left(\partial_x h(x,t)\right)^2 + \nu \partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where $\eta(x,t)$ is the Gaussian white noise with covariance $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$ and $\lambda, \nu, D > 0$. According to the analysis of the dynamic renormalization group, the scale of the fluctuation of the height function $h(t,x)$ is $\mathcal{O}(t^{\frac{1}{3}})$ as $t \to \infty$. This index $\frac{1}{3}$ represents non-Gaussian fluctuation and appears in many interfacial growth models. We call the collection of models with a few indices including $\frac{1}{3}$ above the KPZ universality class. Examples of models belonging to the KPZ universality class include random growth models, last passage percolation, directed polymers and random stirred fluids.

Our understanding of the KPZ equation and universality has been substantially developed in a last few decades. The above KPZ equation has been known to be ill-defined as it is due to the non-linear term. Hence giving a proper mathematical meaning of the KPZ equation has been an important problem. Bertini and Giacomin [5] showed that some quantity in terms of the height function of the discretized model called the weakly asymmetric exclusion process (WASEP) converges to the (well-posed) stochastic heat equation (SHE) and they defined the solution of the KPZ equation as an inverse Cole-Hopf transform of that of the SHE. This notion of the solution is called the Cole-Hopf solution to the KPZ equation. Recently, Hairer developed a general theory for regularizing a class of stochastic partial differential equation using rough path theory without using the Cole-Hopf transformation. Applying the theory to the KPZ equation, he succeeded in having a solution to the KPZ equation [17]. Furthermore in [1] and [38], the exact solution of the distribution function of the height has been obtained by using an integrable structures in the discretized model called he asymmetric exclusion process.

In the studies on the KPZ universality class, there are a few models having nice mathematical structures, which allows us to access a detailed information of fluctuation properties of some quantities [14]. The study on the solvable stochastic models is called the integrable probability nowadays. The totally asymmetric simple exclusion process (TASEP), which we analyze mainly in this thesis, is one of the most basic models in the integral probability. It is a typical stochastic particle system and can be interpreted as a stochastic growth model of an interface.

On a macroscopic level, the particle density evolves deterministically according to the Burgers equation [35, 36]. Therefore, a natural question arises: what kind of characteristic does the fluctuation around the deterministic growth have? It has been known that it exhibits universal properties characterizing the KPZ class. There are many important results in the literature of the integrable probability. First, for the step initial condition, the one-point limiting distribution for the particle current has been obtained by Johansson [19] by converting the problem to the last passage percolation and then using the RSK correspondence. It turned out that the limiting distribution is the GUE Tracy-Widom distribution. In [24, 34], this result has also been obtained by using an explicit determinantal form of the transition probability in the TASEP [39]. For the last passage problems with symmetries, similar results have been found by Baik-Rains [4]. The results include the one-point limiting distribution of the particle current for the alternating initial condition in the language of the TASEP or equivalently, the height distribution for the flat initial condition in the language of the growth process called the polynuclear growth (PNG) model [29]. In this case, the limiting distribution turned out to be the GOE Tracy-Widom distribution.

These results on the one-point fluctuations have been generalized to the case of the multi-point fluctuations. For the case corresponding to the step initial condition, a Fredholm determinant formula for the limiting multi-point distributions has been first obtained in the PNG model with space-time continuous setting [30] by using the technique related to the RSK correspondence. A similar result has been obtained for the space-time discretized PNG model [20]. The limiting process characterized by the multi-point distribution is called the Airy_2 process. On the other hand, for the other initial conditions, the first important result has been given in [37]. Sasamoto has developed the technique for obtaining the multi-point function in terms of the transition probability in the TASEP [39] not only for the step initial condition but also for the alternating one and has obtained a Fredholm determinant formula for the limiting functions in the alternating case. The process characterized by the multi-point distribution is now called the Airy_1 process. This approach in [37] has been further studied and been applied to the TASEP and the PNG model with different settings [8, 10, 11, 12].

We have been interested in the entire structure of the universal limiting process for more general initial data. Our understanding of this problem has been advanced by the recent result by Matetski, Quastel, and Remenik [23]. Their result is based on the approach in [11, 37]: A Fredholm determinant formula for the distribution functions with an arbitrary initial data has already been obtained in [11] based on the approach developed in [37]. The correlation kernel for the Fredholm determinant can be expressed in terms of the biorthogonal functions, which are written as $\Phi_k(x)$ and $\Psi_k(x)$ in this thesis. The problem is that one of them, say $\Phi_k(x)$ does not have an explicit representation while $\Psi_k(x)$ does. Thus it had not been clear how to take the KPZ scaling limit of this kernel. [23] has overcome this situation. They represent the function $\Phi_k(x)$ in terms of a stopping time of the random walk with jumps obeying the geometric distributions. This expression allows us to take the KPZ scaling limit since by Donsker's invariance principle, we easily find the stopping time converges to the one for the Brownian motion in the limit. Based on this technique, the limiting multi-point distribution functions for the particle positions in the arbitrary initial condition has been obtained. The process with this multi-point distribution is called the KPZ fixed point. The results on the limiting distributions of TASEP stated so far are summarized in Table 1. Recently various interesting progresses on this problem have been made for example in [25, 27, 32].

name	year	initial	continuous or discrete time	one or multi point
Johanson[19]	2000	step	both	one
BR[3], PS[29]	2000	periodic	both	one
S[37], BFPS[11]	2005, 2007	periodic	continuous	multi
BFP[10]	2007	periodic	discrete (Bernoulli)	multi
MQR[23]	2017	general	continuous	multi
A[2]	2020	general	discrete (Bernoulli)	multi
A[2]	2020	general	discrete (Geometric)	multi

Table 1: The table above summarizes the references on the limiting distributions of TASEPs. Due to the limitation of space, we used the abbreviated notations for the names of authors. They stands for the following authors: BR $\to$ J. Baik and E.M. Rains, PS$\to$ M. Prähofer and H. Spohn, S $\to$ T. Sasamoto, BFP $\to$ A. Borodin, P. L. Ferrari and M. Prähofer, MQR $\to$ K. Matetski, J. Quastel and D. Remenik, A $\to$ Y. Arai.

In this thesis, we show that the technique in [23] can be applicable to different versions of the TASEPs besides the usual continuous time one. In particular, we focus on two versions of the discrete time TASEPs: the case where the random jump at each time step follows the (truncated) geometric distribution and the parallel update is applied and also the case where the random jump follows the Bernoulli distribution and the (backward) sequential update is applied. Furthermore, in both cases, we consider the situation where the hopping probabilities are time-dependent. For the step initial condition, these dynamics have appear as a special case of the higher spin vertex model and have been recently studied in [22]. To the best of our knowledge, however, the analyses for the arbitrary initial condition has not been studied yet. We obtain Schütz's type determinantal formulas for transition probabilities for both the geometric and Bernoulli TASEP with time dependent hopping probabilities. Combining these with Schütz's formula for the continuous time TASEP, we get the determinantal transition probability for the system mixed with the three types of dynamics. Using this, we obtain a Fredholm determinant formula for the multi-point distribution for the particle positions, in which we can take the KPZ scaling limit. This is a generalized formula to the one [23]: When we vanish the whole parameters of the mixed dynamics except the part of the continuous time TASEP, the determinantal formula is reduced to the result in [23]. Finally taking the KPZ scaling limit for both discrete time geometric and Bernoulli TASEP, we see that the multi-point distribution functions converge to the one describing the KPZ fixed point.

The thesis is organized as follows. In Chap. 1, we state the three versions of the TASEPs, continuous time and two types of discrete time versions: geometric and Bernoulli hopping. Their mixed version is also stated. We also give our main result in [2]: the Fredholm determinant formula for the mixed TASEP (Theorem 1.2.4) and the KPZ scaling limit in two cases of the geometric and Bernoulli TASEPs (Theorem 1.2.11, and Propositions 1.2.15 and 1.2.16). In Chap. 2, after giving the determinantal formulas for the transition probabilities for the above three types of TASEPs, we give the proof of Theorem 1.2.4 using the framework developed in [23]. In Chap. 3, we give proofs of Theorem 1.2.11, and Propositions 1.2.15 and 1.2.16. The crucial step is the saddle point analysis for the kernels.

Chapter 1

Models and results

In this chapter we define three versions of the TASEP and introduce our main results in [2].

1.1 Models

In this thesis we consider the TASEPs on $\mathbb{Z}$. Each particle jumps only to the right independently and stochastically if the target site is empty. If the site is occupied by the other particle, it cannot move, which represents the exclusion interaction.

In the TASEPs we mainly focus on the position of each particle. Let $X_t(i) \in \mathbb{Z}$ be a position of the ith particle at time t. We set $t \in \mathbb{Z}$ or $t \in \mathbb{R}$ according to the version. Since the dynamics of the TASEPs preserves the order of the particles, we can always assume

$$\cdots < X_t(2) < X_t(1) < X_t(0) < X_t(-1) < X_t(-2) < \cdots.$$

The particles at $\pm\infty$ are playing no role in the dynamics when adding $\pm\infty$ into the state space.

In this thesis, we deal with the following three versions of the TASEP. As written in Lemmas 2.1.1, 2.1.2, and 2.1.3 in Chap. 2.1, they have a common feature that the transition probability for each model is written as a single determinant form.

1.1.1 Continuous time TASEP

The continuous time TASEP on $\mathbb{Z}$ was introduced in [41] in the literature of mathematics. In this case $t \in \mathbb{R}_{\geq 0}$ and each particle independently attempts to jump to the right neighboring site at rate $\gamma \in \mathbb{R}_{\geq 0}$ provided this site is empty. It is a continuous time Markov process with the generator L defined as follows: Let $\eta = \{\eta(x) : x \in \mathbb{Z}\} \in \{0,1\}^{\mathbb{Z}}$ be a particle configuration. For $x \in \mathbb{Z}$, $\eta(x) = 1$ means the site x is occupied by a particle while $\eta(x) = 0$ means it is empty. The generator L acting on cylinder functions $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ is defined by

$$(Lf)(\eta) = \gamma \sum_{x \in \mathbb{Z}} \eta(x)(1 - \eta(x+1))(f(\eta^{x,x+1}) - f(\eta))$$

where

$$\eta(x) = \begin{cases} 1, & \text{if the site is occupied by a particle,} \\ 0, & \text{if the site } x \text{ is empty,} \end{cases}$$

and $\eta^{x,x+1}$ denotes the configuration η with the occupations at site x and $x+1$ have been interchanged, that is,

$$\eta^{x,x+1}(y) = \begin{cases} \eta(x+1) & \text{for } y = x, \\ \eta(x) & \text{for } y = x+1, \\ \eta(y) & \text{otherwise.} \end{cases}$$

1.1.2 Discrete time Bernoulli TASEP with sequential update

We define the discrete time Bernoulli TASEP with sequential update on $\mathbb{Z}$. This version was studied previously in [9] as a marginal of dynamics on Gelfand-Tsetlin patterns which preserve the class of Schur processes and more recently in [7, 22] in the studies of the integrable probability.

Let us assume the particle configurations at time $t \in \mathbb{Z}_{\geq 0}$ as $X_t(j) = a_j, j \in \mathbb{Z}$. The particle positions at time $t+1$ are determined by the following update rule: We update the position of the ith particle $X_{t+1}(i)$ in increasing order. Suppose that we already updated the $i-1$th particle and its position is b_{i-1} i.e. $X_{t+1}(i-1) = b_{i-1}$. Then the update rule is given as follows:

- When $X_{t+1}(i-1) - X_t(i) = b_{i-1} - a_i > 1$,

$$\mathbb{P}(X_{t+1}(i) = a | X_t(i) = a_i, X_{t+1}(i-1) = b_{i-1}) = \begin{cases} 1 - p_{t+1} & \text{for } a = a_i, \\ p_{t+1} & \text{for } a = a_i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

- When $X_{t+1}(i-1) - X_t(i) = b_{i-1} - a_i = 1$,

$$\mathbb{P}(X_{t+1}(i) = a | X_t(i) = a_i, X_{t+1}(i-1) = b_{i-1}) = \begin{cases} 1 & \text{for } a = a_i, \\ 0 & \text{otherwise.} \end{cases}$$

This dynamics mean that starting from right to left, for the time step $t \to t+1$, the ith particle jumps to the right neighboring site with probability $p_{t+1} \in (0,1)$ provided this site is empty. Since the update is sequential from right to left, during a time step, a block of consecutive particles can jump. For later use, we define $\beta_t, \ t = 0, 1, 2, \ldots$ by

$$p_t = \frac{\beta_t}{1+\beta_t}, \quad \left(\beta_t = \frac{p_t}{1-p_t}\right). \tag{1.1.1}$$

Remark 1.1.1. In the case of discrete time Bernoulli TASEP with *parallel* update, some integrable structures have also been studied for example in [12, 18, 28]. In [28], an explicit form of the transition probability was obtained by using the Bethe ansatz. However, it is written as a ratio of two determinants not a single determinant. To study the KPZ fixed point in this case is an interesting future problem.

1.1.3 Discrete time geometric TASEP with parallel update

We define the discrete time geometric TASEP with parallel update on $\mathbb{Z}$. This was studied previously in [42] as a marginal of dynamics on Gelfand-Tsetlin patterns which preserve the class of Schur processes. More recently it has been also investigated in [7, 22].

Let us assume that for $t \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}$, $X_t(j) = a_j$. The update rule of the positions at time $t+1$ are given as follows: For each $1 \leq i \leq N$,

$$\mathbb{P}(X_{t+1}(i) = a_i + a | X_t(i) = a_i,\ X_t(i-1) = a_{i-1}) \\ = \begin{cases} \alpha_{t+1}^a (1-\alpha_{t+1}) & \text{for } a = 0, 1, \ldots, a_{i-1} - a_i - 2, \\ \alpha_{t+1}^a & \text{for } a = a_{i-1} - a_i - 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1.2}$$

where the update is independent for each i and t.

Note that in this dynamics, the jth particle can jump with multiple cites according to the truncated geometric distribution defined in (1.1.2) with parameter α_t.

Remark 1.1.2. As shown in Lemma 2.1.3 below, we have a determinantal formula for the transition probability in this model. In the discrete time geometric TASEP with *sequential* update, it is not clear if it has any solvable structures via Bethe ansatz or an explicit formula for the transition probability.

1.1.4 TASEP$_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$: TASEP mixed with the continuous time TASEP and the discrete time TASEPs

In this thesis, we consider the TASEP combined with the above three versions. First we take three time parameters $t_1,\ t_2 \in \mathbb{Z}_{\geq 0}$ and $t_3 \in \mathbb{R}_{\geq 0}$. Then particles evolve according to the discrete time geometric TASEP with parameter $\boldsymbol{\alpha} := \{\alpha_1, \alpha_2, \ldots, \alpha_{t_1}\}$(Chap. 1.1.3) from time 0 to t_1, the discrete time Bernoulli TASEP with parameter $\boldsymbol{\beta} = \{\beta_{t_1+1}, \beta_{t_1+2}, \ldots, \beta_{t_1+t_2}\}$ (Chap. 1.1.2) from time t_1 to $t_1 + t_2$, and the continuous time TASEP with parameter γ (Chap. 1.1.1) from $t_1 + t_2$ to $t_1 + t_2 + t_3$. In this thesis we denote this mixed TASEP as TASEP$_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$.

This type of the mixed TASEP with $t_3 = 0$ has been introduced in [22, 26]. A related process has been studied in [16]. We decided the order of the three dynamics as above. In fact the distribution of the particles' positions is invariant even if we freely exchange order of these dynamics since the semigroups of all the three dynamics are shown to be exchangeable thanks to the Yang-Baxter relations [13, 15].

Remark 1.1.3. The motivation of introducing the TASEP$_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$ is that we can treat the above three models in a unified way. As stated in Proposition 2.1.4 below, one can see that the transition probability of the TASEP$_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$ is also written as a single determinant combining the determinantal formulas (Lemmas 2.1.1, 2.1.2 and 2.1.3) for the above three TASEPs in Chap. 1.1.1-1.1.3. Starting from the determinantal formula, one can generalize the approach to the continuous time TASEP in [23] to the TASEP$_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$. We will explain it in Secs. 2.2-2.4.

1.2 Results

In this section, we give our main results.

1.2.1 Joint distribution of the particle positions

Here we give a single Fredholm determinant formula for joint distribution of the particle position in TASEP$_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$ defined in Chap. 1.1.4. For the descriptions of the results below including the following one, we state some definitions.

Definition 1.2.1. *For a real single-valued function, $\widehat{f} : \mathbb{A} \to (-\infty, \infty]$ with (in general an uncountable) domain $\mathbb{A}$, the epigraph* $\mathrm{epi}(\widehat{f})$ *and the hypograph* $\mathrm{hypo}(\widehat{f})$ *are defined as follows.*

$$\mathrm{epi}(\widehat{f}) = \{(x,y) : y \geq \widehat{f}(x)\}, \ \mathrm{hypo}(\widehat{f}) = \{(x,y) : y \leq \widehat{f}(x)\}.$$

Definition 1.2.2. *Let RW_m, $m = 0, 1, 2\dots$ be the position of a random walker with Geom$[\frac{1}{2}]$ jumps strictly to the left starting at some fixed site c, i.e.,*

$$RW_m = c - \chi_1 - \chi_2 - \cdots - \chi_m,$$

where χ_i, $i = 1, 2, \dots$ are the i.i.d. random variable with $\mathbb{P}(\chi_i = k) = 1/2^{k+1}$, $k = 0, 1, 2, \dots$.

We also define the stopping time

$$\tau = \min\{m \geq 0 : RW_m > X_0(m+1)\}, \tag{1.2.1}$$

where τ is the hitting time of the strict epigraph of the curve $(X_0(k+1))_{k=0,\dots,n-1}$ by the random walk RW_k. When the number of particles is N, $X_0(m)$ is constant and defined only $m \leq N$.

At last we define the multiplication operators.

Definition 1.2.3. *For a fixed vector $a \in \mathbb{R}^m$ and indices $n_1 < \cdots < n_m$, we define χ_a and $\bar{\chi}_a$ by the multiplication operators acting on the space $\ell^2(\{n_1, \dots, n_m\} \times \mathbb{Z})$(or acting on the space $L^2(\{x_1, \dots, x_m\} \times \mathbb{R})$) with*

$$\chi_a(n_j, x) = \mathbf{1}_{x > a_j}, \quad \bar{\chi}_a(n_j, x) = \mathbf{1}_{x \leq a_j}. \tag{1.2.2}$$

We obtain the following result.

Theorem 1.2.4 ([2])**.** *We consider the* $\mathrm{TASEP}_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$ *introduced in Chap. 1.1.4. Let $t = t_1 + t_2 + t_3$ be the final time, and $X_t(j)$, $j \in \mathbb{Z}$ be the the position of the particle labeled j at $t(= t_1 + t_2 + t_3)$. Assume that the initial positions $X_0(j) \in \mathbb{Z}$ for $j = 1, 2, \dots$ are arbitrary constants satisfying $X_0(1) > X_0(2) > \cdots$ while $X_0(j) = \infty$ for $j \leq 0$.*

For $n_j \in \mathbb{Z}_{\geq 1}$ $j = 1, 2, \dots, M$ with $1 \leq n_1 < n_2 < \cdots < n_M$, and $a = (a_1, a_2, \dots, a_M) \in \mathbb{Z}^M$ we have

$$\mathbb{P}(X_t(n_j) > a_j, j = 1, \dots, M) = \det(I - \bar{\chi}_a K_t \bar{\chi}_a)_{\ell^2(\{n_1,\dots,n_M\}\times\mathbb{Z})}, \tag{1.2.3}$$

where $\bar{\chi}_{\boldsymbol{a}}(n_j, x)$ is defined in (1.2.2) and the kernel K_t is given by

$$K_t(n_i, x; n_j, y) = -Q^{n_j - n_i}(x,y)\mathbf{1}_{n_i < n_j} + (S_{-t,-n_i})^* \bar{S}^{\mathrm{epi(X_0)}}_{-t,n_j}(x,y), \tag{1.2.4}$$

$$Q^m(x,y) = \frac{1}{2^{x-y}} \binom{x-y-1}{m-1} \mathbf{1}_{x \geq y+m}, \tag{1.2.5}$$

$$S_{-t,-n}(z_1, z_2) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^n}{2^{z_2 - z_1} w^{n+1+z_2-z_1}} \mathfrak{F}_{\alpha,\beta,\gamma}(w,t), \tag{1.2.6}$$

$$\bar{S}_{-t,n}(z_1, z_2) = \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^{z_2 - z_1 + n - 1}}{2^{z_1 - z_2} w^n} \bar{\mathfrak{F}}_{\alpha,\beta,\gamma}(w,t), \tag{1.2.7}$$

$$\bar{S}^{\mathrm{epi(X_0)}}_{-t,n}(z_1, z_2) = \mathbb{E}_{RW_0 = z_1}\left[\bar{S}_{-t,n-\tau}(RW_\tau, z_2)\mathbf{1}_{\tau < n}\right], \tag{1.2.8}$$

$$\mathfrak{F}_{\alpha,\beta,\gamma}(w,t) = \prod_{j=1}^{t_1} \frac{1}{1 - \frac{2\alpha_j}{2-\alpha_j}\left(w - \frac{1}{2}\right)} \cdot \prod_{j=t_1+1}^{t_1+t_2} \left\{1 + \frac{2\beta_j}{2+\beta_j}\left(w - \frac{1}{2}\right)\right\} \cdot e^{\gamma t_3 (w - \frac{1}{2})}, \tag{1.2.9}$$

$$\bar{\mathfrak{F}}_{\alpha,\beta,\gamma}(w,t) = \prod_{j=1}^{t_1} \left\{1 + \frac{2\alpha_j}{2-\alpha_j}\left(w - \frac{1}{2}\right)\right\} \cdot \prod_{j=t_1+1}^{t_1+t_2} \frac{1}{1 - \frac{2\beta_j}{2+\beta_j}\left(w - \frac{1}{2}\right)} \cdot e^{\gamma t_3 (w - \frac{1}{2})}, \tag{1.2.10}$$

where Γ_0 is a simple counterclockwise loop around 0 not enclosing any other poles. The superscript $\mathrm{epi(X_0)}$ *in* (1.2.8) *refers to the fact that τ is the hitting time of the strict epigraph of the curve $(X_0(k+1))_{k=0,\dots,n-1}$ by the random walk RW_k (see Def. 1.2.2).*

Remark 1.2.5. In the case of continuous time TASEP, i.e. the special case $\alpha_i = \beta_j = 0$ with $1 \le i \le t_1$, $t_1 + 1 \le j \le t_1 + t_2$, this formula has been obtained in Theorem 2.6 in [23]. Theorem 1.2.4 above is the generalization of the result in [23] to the $\mathrm{TASEP}_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$, which includes the two types of discrete time TASEPs as well as the continuous time one.

1.2.2 The Kardar-Parisi-Zhang (KPZ) scaling limit

Here we state our result on the scaling limit of the joint distribution function in Theorem 1.2.4. Although we expect that the scaling limit can be taken for the general $\mathrm{TASEP}_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$, we analyze two simpler cases, the discrete time Bernoulli TASEP with sequential update and the discrete time geometric TASEP with parallel update in this thesis since the asymptotic analysis in the general case would be somewhat involved.

We focus on the following two cases:

- The discrete time Bernoulli TASEP (Chap. 1.1.2)

  In the $\mathrm{TASEP}_{\alpha,\beta,\gamma}$ introduced in Chap. 1.1.4, the case is realized by the specialization

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{t_1} = \gamma = 0,\ \beta_{t_1+1} = \beta_{t_1+2} = \cdots = \beta_{t_1+t_2} = \beta = \frac{p}{1-p}.$$

- The discrete time geometric TASEP (Chap. 1.1.3)

  As above, it is realized by

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{t_1} = \alpha,\ \gamma = \beta_{t_1+1} = \beta_{t_1+2} = \cdots = \beta_{t_1+t_2} = 0.$$

To see the universal behavior of the fluctuations, we focus on the height function defined as follows.

Definition 1.2.6. *For $z \in \mathbb{Z}$, the TASEP height function related to X_t is given by*

$$h_t(z) = -2(X_t^{-1}(z-1) - X_0^{-1}(-1)) - z \tag{1.2.11}$$

where

$$X_t^{-1}(u) = \min\{k \in \mathbb{Z} : X_t(k) \le u\} \tag{1.2.12}$$

denote the label of the rightmost particle which sits to the left of, or at, u at time t and we fix $h_0(0) = 0$.

Note that it can be represented as

$$h_t(z+1) = h_t(z) + \widehat{\eta}_t(z). \tag{1.2.13}$$

where

$$\widehat{\eta}_t(z) = \begin{cases} 1 & \text{if there is a particle at } z \text{ at time } t, \\ -1 & \text{if there is no particle at } z \text{ at time } t. \end{cases}$$

We can extend the height function to a continuous function of $x \in \mathbb{R}$ by linearly interpolating between the integer points.

It is well known that the TASEP belongs to the Kardar-Parisi-Zhang (KPZ) universality class. Thus we expect that the proper scaling of the height function is

$$\frac{h_t(x) - At}{Ct^{\frac{1}{3}}}, \text{ with } x = Bt^{2/3}. \tag{1.2.14}$$

On average the height of the TASEP grows as t^1 with speed A, which is a constant. On the other hand the fluctuation of the height around the average is of order $t^{1/3}$ contrary to the $t^{1/2}$ of the usual scaling in the central limit theorem. The scaling exponent of the x-direction is $2/3$, the twice of the one in h-direction $1/3$, which suggest that the path of the height function becomes the Brownian motion like. The exponents $(1/3, 2/3)$ are known to be universal and characterizing the KPZ universality class while the constants A, B, C are not universal and depend on the models. As shown in Chap. 3, we have

- the discrete time Bernoulli TASEP case

$$A = \frac{p-2}{2}, B = 2, C = 1, \tag{1.2.15}$$

- the discrete time geometric TASEP case

$$A = \frac{\alpha - 2}{2(1-\alpha)}, B = 2, C = 1. \tag{1.2.16}$$

Based on the property of the height function, we define the scaled height, which is equivalent to (1.2.15) and (1.2.16) but a slightly different form appearing as the "1:2:3 scaling" in [33].

Definition 1.2.7. *For $\mathbf{t} \in \mathbb{R}_{\geq 0}$ and $\mathbf{x} \in \mathbb{R}$, we define the scaling height function as the following.*

- *The discrete time Bernoulli TASEP*

$$\widehat{h}^{\varepsilon}(\mathbf{t}, \mathbf{x}) = \varepsilon^{\frac{1}{2}} \left[h_t(x) + \frac{2-p}{2} \varepsilon^{-\frac{3}{2}} \mathbf{t} \right], \tag{1.2.17}$$

where t and x are scaled as

$$t = \frac{(2-p)^3}{4p(1-p)} \varepsilon^{-\frac{3}{2}} \mathbf{t}, \ \ x = 2\varepsilon^{-1} \mathbf{x}. \tag{1.2.18}$$

- *The discrete time geometric TASEP*

$$\widehat{h}^{\varepsilon}(\mathbf{t}, \mathbf{x}) = \varepsilon^{\frac{1}{2}} \left[h_t(x) + \frac{2-\alpha}{2(1-\alpha)} \varepsilon^{-\frac{3}{2}} \mathbf{t} \right], \tag{1.2.19}$$

where t and x are scaled as

$$t = \frac{(2-\alpha)^3}{4\alpha(1-\alpha)} \varepsilon^{-\frac{3}{2}} \mathbf{t}, \ \ x = 2\varepsilon^{-1} \mathbf{x}. \tag{1.2.20}$$

Our goal is to compute the $\varepsilon \to 0$ limit of the joint distribution function,

$$\lim_{\varepsilon \to 0} \mathbb{P}_{\widehat{h}_0^\varepsilon}(\widehat{h}^\varepsilon(\mathbf{t}, \mathbf{x}_1) \le \mathbf{a}_1, \dots, \widehat{h}^\varepsilon(\mathbf{t}, \mathbf{x}_m) \le \mathbf{a}_m) \tag{1.2.21}$$

for $\mathbf{x}_1 < \mathbf{x}_2 < \cdots < \mathbf{x}_m \in \mathbb{R}$ and $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{R}$. Here $\mathbb{P}_{\widehat{h}_0^\varepsilon}(\cdot)$ represents the probability measure in which the initial height profile is $\widehat{h}^\varepsilon(0, x)$. We will show that the limit converges to the joint distribution function characterizing the KPZ fixed point introduced in [23].

Here we introduce the KPZ fixed point. First we define UC and LC as follows.

Definition 1.2.8. (UC *and* LC [23]).
We define UC *as the space of upper semicontinuous functions* $\widehat{h} : \mathbb{R} \to [-\infty, \infty)$ *with* $\widehat{h}(x) \le C_1 + C_2|x|$ *for some* $C_1, C_2 < \infty$ *and* $\widehat{h}(x) > -\infty$ *for some* x *and* LC *as* $\mathrm{LC} = \{\widehat{g} : -\widehat{g} \in \mathrm{UC}\}$.

Now we are ready to state the KPZ fixed point. For more detailed information, see [23].

Definition 1.2.9 (The KPZ fixed point [23])**.** *The KPZ fixed point is the unique Markov process on* UC, $(\widehat{h}(\mathbf{t}, \cdot))_{\mathbf{t}>0}$ *with transition probabilities given by*

$$\mathbb{P}_{\widehat{h}_0}(\widehat{h}(\mathbf{t}, \mathbf{x}_1) \le \mathbf{a}_1, \dots, \widehat{h}(\mathbf{t}, \mathbf{x}_m) \le \mathbf{a}_m) = \det\left(\mathbf{I} - \chi_{\mathbf{a}} \mathbf{K}^{\mathrm{hypo}(\widehat{h}_0)}_{\mathbf{t},\mathrm{ext}} \chi_{\mathbf{a}}\right)_{L^2(\{\mathbf{x}_1, \dots, \mathbf{x}_m\} \times \mathbb{R})}. \tag{1.2.22}$$

Here in LHS, $\mathbf{x}_1 < \mathbf{x}_2 < \cdots < \mathbf{x}_m \in \mathbb{R}$ *and* $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{R}$, $\widehat{h}_0 \in \mathrm{UC}$ *and* $\mathbb{P}_{\widehat{h}_0}$ *means the measure on the process with initial data* $\widehat{h}_0$*. In RHS, the kernel is given by*

$$\begin{aligned}
&\mathbf{K}^{\mathrm{hypo}(\widehat{h}_0)}_{\mathbf{t},\mathrm{ext}}(\mathbf{x}_i, v; \mathbf{x}_j, u) \\
&= -\frac{1}{\sqrt{4\pi(x_j - x_i)}} \exp\left(-\frac{(u-v)^2}{4(x_j - x_i)}\right) \mathbf{1}_{\mathbf{x}_i < \mathbf{x}_j} + \left(\mathbf{S}^{\mathrm{hypo}(\widehat{h}_0^-)}_{\mathbf{t}, -\mathbf{x}_i}\right)^* \mathbf{S}_{\mathbf{t}, \mathbf{x}_j}(v, u), &(1.2.23)\\
&\mathbf{S}_{\mathbf{t},\mathbf{x}}(v, u) = \mathbf{t}^{-\frac{1}{3}} e^{\frac{2\mathbf{x}^3}{3\mathbf{t}^2} - \frac{(v-u)\mathbf{x}}{\mathbf{t}}} \mathrm{Ai}(-\mathbf{t}^{-\frac{1}{3}}(v-u) + \mathbf{t}^{-\frac{4}{3}}\mathbf{x}^2), &(1.2.24)\\
&\mathbf{S}^{\mathrm{hypo}(\widehat{h})}_{\mathbf{t},\mathbf{x}}(v, u) = \mathbb{E}_{B(0)=v}[\mathbf{S}_{\mathbf{t}, \mathbf{x} - \boldsymbol{\tau}'}(B(\boldsymbol{\tau}'), u) \mathbf{1}_{\boldsymbol{\tau}' < \infty}], &(1.2.25)
\end{aligned}$$

where $(A)^*$ *represents the adjoint of an integral operator* A*, and* $B(x)$ *is a Brownian motion with diffusion coefficient* 2 *and* $\boldsymbol{\tau}'$ *is the hitting time of the hypograph of the function* $\widehat{h}$.

Remark 1.2.10. (1.2.23) and (1.2.24) can be written in terms of the differential operators $\mathbf{S}_{\mathbf{t},\mathbf{x}} = \exp\{\mathbf{x}\partial^2 + \mathbf{t}\partial^3/3\}$,

$$\mathbf{K}^{\mathrm{hypo}(\widehat{h}_0)}_{\mathbf{t},\mathrm{ext}}(\mathbf{x}_i, \cdot; \mathbf{x}_j, \cdot) = -e^{(\mathbf{x}_j - \mathbf{x}_i)\partial^2} \mathbf{1}_{\mathbf{x}_i < \mathbf{x}_j} + \left(\mathbf{S}^{\mathrm{hypo}(\widehat{h}_0^-)}_{\mathbf{t}, -\mathbf{x}_i}\right)^* \mathbf{S}_{\mathbf{t}, \mathbf{x}_j}.$$

In addition using the integral representation for the Airy function

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_{\langle} dw \; e^{\frac{1}{3}w^3 - zw},$$

where $\langle$ is the positively oriented contour going the straight lines from $e^{-\frac{i\pi}{3}}\infty$ to $e^{\frac{i\pi}{3}}\infty$ through 0, we find that $\mathbf{S}_{\mathbf{t},\mathbf{x}}(v, u)$ (1.2.24) can be expressed as

$$\mathbf{S}_{\mathbf{t},\mathbf{x}}(v, u) = \frac{1}{2\pi i} \int_{\langle} dw \; e^{\frac{\mathbf{t}}{3}w^3 + xw^2 - (v-u)w}. \tag{1.2.26}$$

Now we assume that the limit

$$\widehat{h}_0 = \lim_{\varepsilon \to 0} \widehat{h}^\varepsilon(0,\cdot) \tag{1.2.27}$$

exists. Note that by(1.2.11) and (1.2.27), (1.2.29), this assumption is rewritten as

$$\varepsilon^{\frac{1}{2}}[(X_0^\varepsilon)^{-1}(\mathbf{x}) + 2\varepsilon^{-1}\mathbf{x} - 2] \xrightarrow[\varepsilon\to 0]{} -\widehat{h}_0(-\mathbf{x}), \tag{1.2.28}$$

where $(X_0^\varepsilon)^{-1}(\mathbf{x}) := 2X_0^{-1}(-2\varepsilon^{-1}\mathbf{x} - 1)$ and the left hand side is interpreted as a linear interpolation to make it a continuous function of $x \in \mathbb{R}$ and we chose the frame of reference by

$$X_0^{-1}(-1) = 1, \tag{1.2.29}$$

i.e. the particle labeled 1 is initially the rightmost in $\mathbb{Z}_{<0}$.

Under this assumption, we have the following result for the limiting joint distribution function (1.2.21).

Theorem 1.2.11. (One-sided fixed point formula for the discrete time TASEPs [2]). *Let $\widehat{h}_0 \in$ UC with $\widehat{h}_0(\mathbf{x}) = -\infty$ for $\mathbf{x} > 0$. Then given $\mathbf{x}_1 < \mathbf{x}_2 < \cdots < \mathbf{x}_m \in \mathbb{R}$ and $\mathbf{a}_1, \dots, \mathbf{a}_m \in \mathbb{R}$, we have*

$$\lim_{\varepsilon\to 0} \mathbb{P}_{\widehat{h}_0^\varepsilon}(\widehat{h}^\varepsilon(\mathbf{t},\mathbf{x}_1) \le \mathbf{a}_1, \dots, \widehat{h}^\varepsilon(\mathbf{t},\mathbf{x}_m) \le \mathbf{a}_m) = \det\left(\mathbf{I} - \chi_{\mathbf{a}} \mathbf{K}^{\mathrm{hypo}(\widehat{h}_0)}_{\mathbf{t},\mathrm{ext}} \chi_{\mathbf{a}}\right)_{L^2(\{\mathbf{x}_1,\dots,\mathbf{x}_m\}\times\mathbb{R})}, \tag{1.2.30}$$

where RHS is equivalent to that of (1.2.22).

Remark 1.2.12. We only give pointwise convergence of the kernels. In principle, one expects the convergence could be upgraded to trace class (see [23], [31] and [40]) which would give a full proof of Theorem 1.2.11.

Remark 1.2.13. The One-sided fixed point formula for the continuous time TASEP has been given in Proposition 3.6 in [23]. Our theorem 1.2.11 indicates that Bernoulli TASEP and geometric TASEP settle into the same class "KPZ fixed point". The KPZ fixed point is believed to be the universal process for the KPZ class with arbitrary fixed initial data. Our result supports this universality.

Remark 1.2.14. In fact we can remove the assumption $\widehat{h}_0(\mathbf{x}) = -\infty$ for $\mathbf{x} > 0$ in the above theorem by using the similar argument in Theorem 3.8. in [23].

To prove Theorem 1.2.11, we use the following relationship between the particle positions $X_t(j)$ and the height function $h_t(z)$ (1.2.11). Let $s_1, \dots, s_k, m_1, \dots, m_k \in \mathbb{R}$ and $z_1, \dots, z_k, n_1, \dots, n_k \in \mathbb{Z}$. We have

$$\mathbb{P}(h_t(z_1) \le s_1, \dots, h_t(z_k) \le s_k) = \mathbb{P}(X_t(n_1) \ge m_1, \dots, X_t(n_k) \ge m_k), \tag{1.2.31}$$

which follows from the definitions of $h_t(x)$ (1.2.11). By this relation, we see

$$\lim_{\varepsilon\to 0} \mathbb{P}_{\widehat{h}_0^\varepsilon}(\widehat{h}^\varepsilon(\mathbf{t},\mathbf{x}_1) \le \mathbf{a}_1, \dots, \widehat{h}^\varepsilon(\mathbf{t},\mathbf{x}_m) \le \mathbf{a}_m) = \lim_{\varepsilon\to 0} \mathbb{P}_{X_0^\varepsilon}\left(X_t^\varepsilon(n_1) > a_1, \dots, X_t^\varepsilon(n_m) > a_m\right), \tag{1.2.32}$$

where $a_1, \dots, a_m \in \mathbb{R}$ and $t,\ n_j, x_j$ are scaled as

- the discrete time Bernoulli TASEP case

$$t=\frac{(2-p)^3}{4p(1-p)}\varepsilon^{-\frac{3}{2}}\mathbf{t},\ n_i=\frac{2-p}{4}\varepsilon^{-\frac{3}{2}}\mathbf{t}-\varepsilon^{-1}\mathbf{x}_i-\frac{1}{2}\varepsilon^{-\frac{1}{2}}\mathbf{a}_i+1,\ a_i=2\varepsilon^{-1}\mathbf{x}_i-2, \quad (1.2.33)$$

- the discrete time geometric TASEP case

$$t=\frac{(2-\alpha)^3}{4\alpha(1-\alpha)}\varepsilon^{-\frac{3}{2}}\mathbf{t},\ n_i=\frac{2-\alpha}{4(1-\alpha)}\varepsilon^{-\frac{3}{2}}\mathbf{t}-\varepsilon^{-1}\mathbf{x}_i-\frac{1}{2}\varepsilon^{-\frac{1}{2}}\mathbf{a}_i+1,\ a_i=2\varepsilon^{-1}\mathbf{x}_i-2. \quad (1.2.34)$$

Thus we find that our goal, LHS of (1.2.32), can be obtained by taking the $\varepsilon\to 0$ limit of the expression (1.2.3) in Theorem 1.2.4 under the scaling (1.2.33) or (1.2.34). The critical step of this problem is the following propositions about the pointwise convergences. First, we state the result for the discrete time Bernoulli TASEP.

Proposition 1.2.15. (Pointwise convergence for the discrete time Bernoulli TASEP [2]). *Under the scaling (1.2.33),(dropping the i subscripts) and assuming that (1.2.28) holds in* LC, *if we set* $z=\frac{p(2-p)}{4(1-p)}\varepsilon^{-\frac{3}{2}}\mathbf{t}+2\varepsilon^{-1}\mathbf{x}+\varepsilon^{-\frac{1}{2}}(u+\mathbf{a})-2$ *and* $y'=\varepsilon^{-\frac{1}{2}}v$, *then we have for* $\mathbf{t}>0$ *as* $\varepsilon\to 0$,

$$\mathbf{S}^{\varepsilon}_{-t,x}(v,u):=\varepsilon^{-\frac{1}{2}}S^{\mathrm{Ber}}_{-t,-n}(y',z)\to\mathbf{S}_{-\mathbf{t},\mathbf{x}}(v,u) \quad (1.2.35)$$

$$\bar{\mathbf{S}}^{\varepsilon}_{-t,-x}(v,u):=\varepsilon^{-\frac{1}{2}}\bar{S}^{\mathrm{Ber}}_{-t,n}(y',z)\to\mathbf{S}_{-\mathbf{t},-\mathbf{x}}(v,u) \quad (1.2.36)$$

$$\bar{\mathbf{S}}^{\varepsilon,\mathrm{epi}(-h_0^{\varepsilon,-})}_{-t,-x}(v,u):=\varepsilon^{-\frac{1}{2}}\bar{S}^{\mathrm{Ber},\mathrm{epi}(X_0)}_{-t,n}(y',z)\to\mathbf{S}^{\mathrm{epi}(-\widehat{h}_0^-)}_{-\mathbf{t},-\mathbf{x}}(v,u) \quad (1.2.37)$$

pointwise, where $\widehat{h}_0^-(x)=\widehat{h}_0(-x)$ *for* $x\geq 0$, $\mathbf{S}_{\mathbf{t},\mathbf{x}}(v,u)$ *is given by* (1.2.24) *and for* $\widehat{g}\in$ LC,

$$\mathbf{S}^{\mathrm{epi}(\widehat{g})}_{\mathbf{t},\mathbf{x}}(v,u)=\mathbb{E}_{B(0)=v}[\mathbf{S}_{\mathbf{t},\mathbf{x}-\boldsymbol{\tau}'}(B(\boldsymbol{\tau}'),u)\mathbf{1}_{\boldsymbol{\tau}'<\infty}]$$

and

$$S^{\mathrm{Ber}}_{-t,-n}(z_1,z_2)=\frac{1}{2\pi i}\oint_{\Gamma_0}dw\frac{(1-w)^n}{2^{z_2-z_1}w^{n+1+z_2-z_1}}\left(1+\frac{2p}{2-p}\left(w-\frac{1}{2}\right)\right)^t, \quad (1.2.38)$$

$$\bar{S}^{\mathrm{Ber}}_{-t,n}(z_1,z_2)=\frac{1}{2\pi i}\oint_{\Gamma_0}dw\frac{(1-w)^{z_2-z_1+n-1}}{2^{z_1-z_2}w^n}\left(1-\frac{2p}{2-p}\left(w-\frac{1}{2}\right)\right)^{-t}. \quad (1.2.39)$$

$$\bar{S}^{\mathrm{Ber},\mathrm{epi}(X_0)}_{-t,n}(z_1,z_2)=\mathbb{E}_{RW_0=z_1}\left[\bar{S}^{\mathrm{Ber}}_{-t,n-\tau}(RW_\tau,z_2)\mathbf{1}_{\tau<n}\right] \quad (1.2.40)$$

with Γ_0 *being a simple counterclockwise loop around* 0 *not enclosing* 1, $1/p$ *and* $(1-p)/p$.

Next, we state that the point wise convergence for the discrete time geometric TASEP is obtained as the following.

Proposition 1.2.16. (Pointwise convergence for the discrete time geometric TASEP [2]). *Under the scaling (1.2.34),(dropping the i subscripts) and assuming that (1.2.28) holds in* LC, *if we set* $z=-\frac{\alpha(2-\alpha)}{4(1-\alpha)}\varepsilon^{-\frac{3}{2}}\mathbf{t}+2\varepsilon^{-1}\mathbf{x}+\varepsilon^{-\frac{1}{2}}(u+\mathbf{a})-2$ *and* $y'=\varepsilon^{-\frac{1}{2}}v$, *then we have for* $\mathbf{t}>0$ *as* $\varepsilon\to 0$,

$$\mathbf{S}^{\varepsilon}_{-t,x}(v,u):=\varepsilon^{-\frac{1}{2}}S^{\mathrm{geo}}_{-t,-n}(y',z)\to\mathbf{S}_{-\mathbf{t},\mathbf{x}}(v,u) \quad (1.2.41)$$

$$\bar{\mathbf{S}}^{\varepsilon}_{-t,-x}(v,u):=\varepsilon^{-\frac{1}{2}}\bar{S}^{\mathrm{geo}}_{-t,n}(y',z)\to\mathbf{S}_{-\mathbf{t},-\mathbf{x}}(v,u) \quad (1.2.42)$$

$$\bar{\mathbf{S}}^{\varepsilon,\mathrm{epi}(-h_0^{\varepsilon,-})}_{-t,-x}(v,u):=\varepsilon^{-\frac{1}{2}}\bar{S}^{\mathrm{geo},\mathrm{epi}(X_0)}_{-t,n}(y',z)\to\mathbf{S}^{\mathrm{epi}(-\widehat{h}_0^-)}_{-\mathbf{t},-\mathbf{x}}(v,u) \quad (1.2.43)$$

pointwise, where $\widehat{h}_0^-(x) = \widehat{h}_0(-x)$ *for* $x \geq 0$, $\mathbf{S}_{\mathbf{t},\mathbf{x}}(v,u)$ *is given by* (1.2.24) *and for* $\widehat{g} \in \mathrm{LC}$,

$$\mathbf{S}_{\mathbf{t},\mathbf{x}}^{\mathrm{epi}(\widehat{g})}(v,u) = \mathbb{E}_{B(0)=v}[\mathbf{S}_{\mathbf{t},\mathbf{x}-\boldsymbol{\tau}'}(B(\boldsymbol{\tau}'),u)\mathbf{1}_{\boldsymbol{\tau}'<\infty}]$$

and

$$S_{-t,-n}^{\mathrm{geo}}(z_1,z_2) = \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{(1-w)^n}{2^{z_2-z_1}w^{n+1+z_2-z_1}}\left(1-\frac{2\alpha}{2-\alpha}\left(w-\frac{1}{2}\right)\right)^{-t}, \tag{1.2.44}$$

$$\bar{S}_{-t,n}^{\mathrm{geo}}(z_1,z_2) = \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{(1-w)^{z_2-z_1+n-1}}{2^{z_1-z_2}w^n}\left(1+\frac{2\alpha}{2-\alpha}\left(w-\frac{1}{2}\right)\right)^{t}, \tag{1.2.45}$$

$$\bar{S}_{-t,n}^{\mathrm{geo},\mathrm{epi}(X_0)}(z_1,z_2) = \mathbb{E}_{RW_0=z_1}\left[\bar{S}_{-t,n-\tau}^{\mathrm{geo}}(RW_\tau,z_2)\mathbf{1}_{\tau<n}\right] \tag{1.2.46}$$

with Γ_0 *being a simple counterclockwise loop around* 0 *not enclosing* 1, $1/\alpha$ *and* $(1-\alpha)/\alpha$.

Chapter 2

Distribution function of the TASEP

2.1 Transition probabilities for TASEPs

Let
$$\Omega_N = \{\vec{x} = (x_N, x_{N-1}, \cdots, x_1) \in \mathbb{Z}^N : x_N < \cdots < x_2 < x_1\}$$
be the Weyl chamber, whose elements express the particle positions of the TASEPs.

The main object of this section is the transition probability of the TASEP: For $\vec{x}, \vec{y} \in \Omega_N$, we define
$$G_t(x_N, \dots, x_1) = \mathbb{P}(X_t = \vec{x} | X_0 = \vec{y}), \tag{2.1.1}$$
which means the probability that at time t the particles are at positions $x_N < \cdots < x_2 < x_1$ provided that initially they are at positions $y_N < \cdots < y_2 < y_1$.

For all the three types of the TASEPs introduced in Chap. 1.1.1-1.1.3, the transition probabilities are obtained using Bethe ansatz (See [39]) and represented as determinants.

First, we give the result of the continuous time TASEP introduced in Chap. 1.1.1.

Lemma 2.1.1. ([39])
For the continuous time TASEP with $N \in \{1, 2, 3, \dots\}$ particles and rate $\gamma \geq 0$ introduced in Chap.1.1.1, the transition probability has the following determinantal form
$$G_t^{(\gamma)}(x_N, \dots, x_1) = \det[F_{i-j}^{(\gamma)}(x_{N+1-i} - y_{N+1-j})]_{1 \leq i,j \leq N} \tag{2.1.2}$$
with
$$F_n^{(\gamma)}(x, t) = \frac{(-1)^n}{2\pi i} \oint_{\Gamma_{0,1}} dw \frac{(1-w)^{-n}}{w^{x-n+1}} e^{\gamma t(w-1)} \tag{2.1.3}$$
where $\Gamma_{0,1}$ is any simple loop oriented anticlockwise which includes $w = 0$ and $w = 1$.

Next we introduce the result on the discrete time Bernoulli TASEP as follows.

Lemma 2.1.2 ([2])**.** *For the discrete time Bernoulli TASEP with $N \in \{1, 2, \dots\}$ particles and parameters $\beta_i \geq 0,\ i = 1, 2, \dots, t$ introduced in Chap. 1.1.2 , the transition probability has the following determinantal form*
$$G_t^{(\boldsymbol{\beta})}(x_N, \dots, x_1) = \det[F_{i-j}^{(\boldsymbol{\beta})}(x_{N+1-i} - y_{N+1-j}, t)]_{1 \leq i,j \leq N} \tag{2.1.4}$$

with

$$F_n^{(\boldsymbol{\beta})}(x,t)=\frac{(-1)^n}{2\pi i}\oint_{\Gamma_{0,1}}dw\frac{(1-w)^{-n}}{w^{x-n+1}}\prod_{j=1}^{t}\frac{1+\beta_j w}{1+\beta_j} \tag{2.1.5}$$

where $\Gamma_{0,1}$ *is any simple loop oriented anticlockwise which includes* $w=0$ *and* $w=1$*.*

Proof. This determinantal formula has been obtained for the time homogeneous case $\beta_1=\beta_2=\cdots=\dfrac{p}{1-p}$ in [10] and [34]. If we confirm that the following two equations hold, one easily find that the result can be extended to the time inhomogeneous case:

$$F_n^{(\boldsymbol{\beta})}(x,t+1)=\frac{1}{1+\beta_{t+1}}F_n^{(\boldsymbol{\beta})}(x,t)+\frac{\beta_{t+1}}{1+\beta_{t+1}}F_n^{(\boldsymbol{\beta})}(x-1,t) \tag{2.1.6}$$

and

$$F_{n-1}^{(\boldsymbol{\beta})}(x,t)=F_n^{(\boldsymbol{\beta})}(x,t)-F_n^{(\boldsymbol{\beta})}(x+1,t). \tag{2.1.7}$$

It is easy to see that the above two equations hold. □

We also give the result on the discrete time geometric TASEP introduced in Chap.1.1.3.

Lemma 2.1.3 ([2])**.** *For the discrete time geometric TASEP with* $N\in\{1,2,\dots\}$ *particles and parameters* $0\le\alpha_i\le 1,\ i=1,2,\dots,t$ *introduced in Chap. 1.1.3 , the transition probability has the following determinantal form*

$$G_t^{(\boldsymbol{\alpha})}(x_N,\dots,x_1)=\det[F_{i-j}^{(\boldsymbol{\alpha})}(x_{N+1-i}-y_{N+1-j},t)]_{1\le i,j\le N} \tag{2.1.8}$$

with

$$F_n^{(\boldsymbol{\alpha})}(x,t)=\frac{(-1)^n}{2\pi i}\oint_{\Gamma_{0,1}}dw\frac{(1-w)^{-n}}{w^{x-n+1}}\prod_{j=1}^{t}\frac{1-\alpha_j}{1-\alpha_j w}$$

where $\Gamma_{0,1}$ *is any simple loop oriented anticlockwise which includes* $w=0$ *and* $w=1$*.*

Proof. We will check that the determinantal representation (2.1.8) satisfies the Kolmogorov forward equation

$$G_{t+1}^{(\boldsymbol{\alpha})}(x_N,\dots,x_1)=\sum_{\mu\subset\{1,\dots,N-1\}}(1-\alpha_{t+1})^{|\bar{\mu}|+1}\prod_{i\in\overline{\mu}\cup\{N\}}\sum_{a_i=0}^{k_i-2}\alpha_{t+1}^{a_i}\cdot\prod_{j\in\mu}\alpha_{t+1}^{k_j-1}\cdot G_t^{(\boldsymbol{\alpha})}\left(\vec{x}^{(\mu)}\right) \tag{2.1.9}$$

where μ can take the empty set ϕ, $\bar{\mu}:=\{1,\dots,N-1\}\setminus\mu$, $|\bar{\mu}|$ means the number of elements in $\bar{\mu}$, and we define k_i and $\vec{x}^{(\mu)}:=(x_N^{\mu},\dots,x_1^{\mu})$ by

$$k_i=\begin{cases}x_i-x_{i+1} & \text{for } i=1,\dots,N-1,\\ \infty & \text{for } i=N,\end{cases}\qquad x_i^{\mu}=\begin{cases}x_{i+1}+1 & \text{for } i\in\mu,\\ x_i-a_i & \text{for } i\in\overline{\mu}\cup\{N\}.\end{cases} \tag{2.1.10}$$

RHS in (2.1.9) consists of 2^{N-1} terms and each element j in the subset μ represents the label of the particle which is on $x_{j+1}+1$ at time t. Taking the hopping probability (1.1.2) in the geometric TASEP into account, we see that when $j\in\mu$, we should assign the weight $\alpha_{t+1}^{\sharp\text{jump}}$ without the factor $1-\alpha_{t+1}$ for the jump of the $j+1$th particle. Thus for the jumps of the

$N-|\mu|=|\bar{\mu}|+1$ particles, we put the factor $(1-\alpha_{t+1})^{|\bar{\mu}|+1}$. In Appendix A, we explain (2.1.9) in the case of $N=3$. We see that (2.1.9) is equivalent to the following two conditions

$$G_{t+1}^{(\boldsymbol{\alpha})}(x_N,\dots,x_1)=\sum_{a_1,\dots,a_n\in\{0,\dots,\infty\}}(1-\alpha_{t+1})^N\alpha_{t+1}^{a_1+\cdots+a_N}G_t^{(\boldsymbol{\alpha})}(x_N-a_N,\dots,x_1-a_1) \quad (2.1.11)$$

$$\begin{aligned}\sum_{m,n=0}^{\infty}&(1-\alpha_{t+1})\alpha_{t+1}^{m+n}G_t^{(\boldsymbol{\alpha})}(x_N,\dots,x_k-m-1,x_k-n,x_{k-1},\dots,x_1)\\&=\sum_{m=0}^{\infty}\alpha_{t+1}^{m}G_t^{(\boldsymbol{\alpha})}(x_N,\dots,x_k-m-1,x_k,x_{k-1},\dots,x_1)\end{aligned} \quad (2.1.12)$$

for $k=1,\dots,N-1$. Now we show that (2.1.11) and (2.1.12) imply (2.1.9). First, we show below the equivalence

$$\begin{aligned}&\prod_{i=1}^{N}\sum_{a_i=0}^{\infty}(1-\alpha_{t+1})\alpha_{t+1}^{a_i}G_t^{(\boldsymbol{\alpha})}(x_N-a_N,\dots,x_1-a_1)\\&=\sum_{\mu\subset\{1,\dots,N-1\}}(1-\alpha_{t+1})^{|\bar{\mu}|+1}\prod_{i\in\bar{\mu}\cup\{N\}}\sum_{a_i=0}^{k_i-2}\alpha_{t+1}^{a_i}\cdot\prod_{j\in\mu}\alpha_{t+1}^{k_j-1}\cdot G_t^{(\boldsymbol{\alpha})}\left(\vec{x}^{(\mu)}\right)\end{aligned} \quad (2.1.13)$$

by using the equation (2.1.12) with $k=1$ and the version of $N-1$ particles in (2.1.13),

$$\begin{aligned}&\prod_{i=2}^{N}\sum_{a_i=0}^{\infty}(1-\alpha_{t+1})\alpha_{t+1}^{a_i}G_t^{(\boldsymbol{\alpha})}(x_N-a_N,\dots,x_2-a_2,x_1)\\&=\sum_{\nu\subset\{2,\dots,N-1\}}(1-\alpha_{t+1})^{|\bar{\nu}|+1}\prod_{i\in\bar{\nu}\cup\{N\}}\sum_{a_i=0}^{k_i-2}\alpha_{t+1}^{a_i}\cdot\prod_{j\in\nu}\alpha_{t+1}^{k_j-1}\cdot G_t^{(\boldsymbol{\alpha})}\left(\vec{x}^{(\nu)},x_1\right)\end{aligned} \quad (2.1.14)$$

where $\bar{\nu}:=\{2,\dots,N-1\}\setminus\nu$, $\vec{x}^{(\nu)}:=(x_N^{\nu},\dots,x_2^{\nu})$ with

$$x_i^{\nu}=\begin{cases}x_{i+1}+1 & \text{for } i\in\nu,\\ x_i-a_i & \text{for } i\in\bar{\nu}\cup\{N\}.\end{cases} \quad (2.1.15)$$

In LHS of (2.1.13), we divide the sum of a_1 as follows.

$$\begin{aligned}\text{LHS of (2.1.13)}&=\left(\prod_{i=2}^{N}\sum_{a_i=0}^{\infty}(1-\alpha_{t+1})\alpha_{t+1}^{a_i}\right)\left\{\left(\sum_{a_1=0}^{k_1-2}(1-\alpha_{t+1})\alpha_{t+1}^{a_1}\right)G_t^{(\boldsymbol{\alpha})}(x_N-a_N,\dots,x_1-a_1)\right.\\&\quad\left.+\left(\sum_{a_1=k_1-1}^{\infty}(1-\alpha_{t+1})\alpha_{t+1}^{a_1}\right)G_t^{(\boldsymbol{\alpha})}(x_N-a_N,\dots,x_1-a_1)\right\}\\&=\left(\prod_{i=2}^{N}\sum_{a_i=0}^{\infty}(1-\alpha_{t+1})\alpha_{t+1}^{a_i}\right)\left\{\left(\sum_{a_1=0}^{k_1-2}(1-\alpha_{t+1})\alpha_{t+1}^{a_1}\right)G_t^{(\boldsymbol{\alpha})}(x_N-a_N,\dots,x_1-a_1)\right.\\&\quad\left.+\left(\sum_{a_1=0}^{\infty}(1-\alpha_{t+1})\alpha_{t+1}^{a_1+k_1-1}\right)G_t^{(\boldsymbol{\alpha})}(x_N-a_N,\dots,x_2-a_2,x_2+1-a_1)\right\}.\end{aligned} \quad (2.1.16)$$

By using (2.1.12) with $k=1$, we find

$$
\begin{aligned}
(2.1.16) = \left(\prod_{i=2}^{N}\sum_{a_i=0}^{\infty}(1-\alpha_{t+1})\alpha_{t+1}^{a_i}\right)\Bigg\{ \left(\sum_{a_1=0}^{k_1-2}(1-\alpha_{t+1})\alpha_{t+1}^{a_1}\right) G_t^{(\boldsymbol{\alpha})}(x_N-a_N,\dots,x_1-a_1) \\
+ \alpha_{t+1}^{k_1-1}G_t^{(\boldsymbol{\alpha})}(x_N-a_N,\dots,x_2-a_2,x_2+1)\Bigg\}.
\end{aligned} \tag{2.1.17}
$$

By applying (2.1.14) to (2.1.17),

$$
\begin{aligned}
(2.1.17) &= \left\{ \sum_{\nu\subset\{2,\dots,N-1\}}(1-\alpha_{t+1})^{|\bar{\nu}|+1}\prod_{i\in\overline{\nu}\cup\{N\}}\sum_{a_i=0}^{k_i-2}\alpha_{t+1}^{a_i}\cdot\prod_{j\in\nu}\alpha_{t+1}^{k_j-1}\right\} \\
&\quad\times\left\{\left(\sum_{a_1=0}^{k_1-2}(1-\alpha_{t+1})\alpha_{t+1}^{a_1}\right)G_t^{(\boldsymbol{\alpha})}(\vec{x}^{(\nu)},x_1-a_1)+\alpha_{t+1}^{k_1-1}G_t^{(\boldsymbol{\alpha})}(\vec{x}^{(\nu)},x_2+1)\right\} \\
&\quad= \text{RHS of (2.1.13)}.
\end{aligned}
$$

Thus we have shown (2.1.13) by using (2.1.14). Similarly, we can show (2.1.14) by using the equation (2.1.12) with $k=2$ and

$$
\begin{aligned}
&\prod_{i=3}^{N}\sum_{a_i=0}^{\infty}(1-\alpha_{t+1})\alpha_{t+1}^{a_i}G_t^{(\boldsymbol{\alpha})}(x_N-a_N,\dots,x_3-a_3,x_2,x_1) \\
&\quad=\sum_{\lambda\subset\{3,\dots,N-1\}}(1-\alpha_{t+1})^{|\bar{\lambda}|+1}\prod_{i\in\bar{\lambda}\cup\{N\}}\sum_{a_i=0}^{k_i-2}\alpha_{t+1}^{a_i}\cdot\prod_{j\in\lambda}\alpha_{t+1}^{k_j-1}\cdot G_t^{(\boldsymbol{\alpha})}\left(\vec{x}^{(\lambda)},x_2,x_1\right)
\end{aligned}
$$

where $\overline{\lambda}:=\{3,\dots,N-1\}\setminus\lambda$, $\vec{x}^{(\lambda)}:=(x_N^{\lambda},\dots,x_3^{\lambda})$ and for $i=3,\dots,N$

$$
x_i^{\lambda}=\begin{cases} x_{i+1}+1 & \text{for } i\in\lambda, \\ x_i-a_i & \text{for } i\in\overline{\lambda}\cup\{N\}. \end{cases}
$$

Therefore, by repeatedly using the similar calculation, we can show the equivalence (2.1.13) by using conditions can be obtained from (2.1.12) for $k=1,\dots,N-1$, which leads to the equivalence between the Kolmogorov forward equation (2.1.9) and two conditions (2.1.11) and (2.1.12).

Now we will check (2.1.11) and (2.1.12). For convenience, we put $F_n^j(x,t)=F_n^{(\boldsymbol{\alpha})}(x-y_{N+1-j},t)$. Inserting (2.1.8) into RHS of (2.1.11) and using the multilinearity of the determinant, we find that RHS of (2.1.11) becomes

$$
\begin{aligned}
&\sum_{a_1,\dots,a_n\in\{0,\dots,\infty\}}(1-\alpha_{t+1})^N\alpha_{t+1}^{a_1+\dots+a_N}\det[F_{i-j}^{j}(x_{N+1-i}-a_{N+1-j},t)]_{1\le i,j\le N} \\
&=\det\left[(1-\alpha_{t+1})\sum_{a_{N+1-i}=0}^{\infty}\alpha_{t+1}^{a_{N+1-i}}F_{i-j}^{j}(x_{N+1-i}-a_{N+1-j},t)\right]_{1\le i,j\le N}.
\end{aligned} \tag{2.1.18}
$$

Thus we see that if the functions $F_n^{(\boldsymbol{\alpha})}$ satisfies

$$F_n^{(\boldsymbol{\alpha})}(x,t+1) = \sum_{y=0}^{\infty} \alpha_{t+1}^y (1-\alpha_{t+1}) F_n^{(\boldsymbol{\alpha})}(x-y,t), \tag{2.1.19}$$

then (2.1.18) is equal to LHS of (2.1.11) $G_{t+1}^{(\boldsymbol{\alpha})}(x_N,\dots,x_1) = \det\left[F_{i-j}^j(x_{N+1-i},t+1)\right]_{1\le i,j\le N}$,

We also consider the condition (2.1.12). It can be written as

$$0 = \det \begin{bmatrix} \vdots \\ \frac{1}{1-\alpha_{t+1}} F_{N-k-j}^j(x_k-1,t+1) \\ F_{N+1-k-j}^j(x_k,t+1) - F_{N+1-k-j}^j(x_k,t) \\ \vdots \end{bmatrix}_{1\le j\le N.} \tag{2.1.20}$$

One easily sees that it holds if the functions $F_n^{(\boldsymbol{\alpha})}$ satisfy

$$F_{n-1}^{(\boldsymbol{\alpha})}(x-1,t+1) = c(F_n^{(\boldsymbol{\alpha})}(x,t+1) - F_n^{(\boldsymbol{\alpha})}(x,t)) \tag{2.1.21}$$

for arbitrary c. Here we choose $c = (1-\alpha_{t+1})/\alpha_{t+1}$.

Therefore the function $F_n^{(\boldsymbol{\alpha})}$ are determined by the two relations (2.1.19) and (2.1.21), as well as the initial condition

$$G_0^{(\boldsymbol{\alpha})}(x_N,\dots,x_1) = \delta_{y_N,x_N}\cdots\delta_{y_1,x_1}. \tag{2.1.22}$$

$F_0^{(\boldsymbol{\alpha})}(x,t)$ is already determined by one particle configurations. In fact, in this case, $G_t^{(\boldsymbol{\alpha})}(x) = \mathbb{P}(x(t)=x|x(0)=y) = F_0^{(\boldsymbol{\alpha})}(x-y,t)$. Therefore

$$F_0^{(\boldsymbol{\alpha})}(x-y,t) = \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{1}{w^{x-y+1}} \prod_{j=0}^{t} \frac{1-\alpha_j}{1-\alpha_j w} \tag{2.1.23}$$

where Γ_0 is any simple loop around 0 oriented anticlockwise. This result is consistent with (2.1.19) and (2.1.22). Denote by Δ the discrete derivative $\Delta_{\alpha_t} f(x,t) := \dfrac{1-\alpha_t}{\alpha_t}(f(x+1,t) - f(x+1,t-1))$. Then by (2.1.21),

$$F_{-n}^{(\boldsymbol{\alpha})}(x,t) = (-1)^n(\Delta_{\alpha_t}^n F_0^{(\boldsymbol{\alpha})})(x,t) \tag{2.1.24}$$

holds. Therefore to obtain $F_{-n}^{(\boldsymbol{\alpha})}$ we simply apply

$$\Delta_{\alpha_t}^n \frac{1}{w^x}\prod_{j=0}^{t}\frac{1-\alpha_j}{1-\alpha_j w} = (-1)^n \frac{(1-w)^n}{w^{x+n}}\prod_{j=0}^{t}\frac{1-\alpha_j}{1-\alpha_j w}. \tag{2.1.25}$$

From the above, for $n \ge 0$,

$$F_{-n}^{(\boldsymbol{\alpha})}(x,t) = \frac{(-1)^n}{2\pi i}\oint_{\Gamma_0} dw \frac{(1-w)^n}{w^{x+n+1}}\prod_{j=0}^{t}\frac{1-\alpha_j}{1-\alpha_j w}. \tag{2.1.26}$$

In this case, there is no pole at $w = 1$, and therefore replacing Γ_0 by $\Gamma_{0,1}$ leaves the result unchanged.

For $n > 0$, $F_n^{(\boldsymbol{\alpha})}$ is determined by the recurrence relation

$$F_{n+1}^{(\boldsymbol{\alpha})}(x,t) = \sum_{y \geq x} F_n^{(\boldsymbol{\alpha})}(y,t) \tag{2.1.27}$$

together with the property that $F_0^{(\boldsymbol{\alpha})}(x,t) = 0$ for x large enough.

In order for (2.1.27) to be satisfied for all n, we need to take the poles both at 0 and 1. $\square$

Finally, combining the above three formulas in Lemmas 2.1.1–2.1.3, we obtain the transition probability of the $\text{TASEP}_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$ introduced in Chap. 1.1.4.

Proposition 2.1.4 ([2]). *For the* $\text{TASEP}_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$ *with* $N \in \{1, 2, \dots\}$ *particles and parameters* $\boldsymbol{\alpha}_{t_1} := (\alpha_1, \dots, \alpha_{t_1}) \in [0,1]^{t_1}$, $\boldsymbol{\beta}_{t_2} := (\beta_{t_1+1}, \dots, \beta_{t_1+t_2}) \in \mathbb{R}_{\geq 0}^{t_2}$, $\gamma > 0$ *and* $t_3 > 0$ *introduced in Chap. 1.1.4 , the transition probability to* $t = t_1 + t_2 + t_3$ *has the following determinantal form*

$$G_t^{\alpha,\beta,\gamma}(x_N, \dots, x_1) = \det[F_{i-j}^{\alpha,\beta,\gamma}(x_{N+1-i} - y_{N+1-j}, t)]_{1 \leq i,j \leq N} \tag{2.1.28}$$

with

$$F_n^{\alpha,\beta,\gamma}(x,t) = \frac{(-1)^n}{2\pi i} \oint_{\Gamma_{0,1}} dw \frac{(1-w)^{-n}}{w^{x-n+1}} f_{\alpha,\beta,\gamma}(w,t)$$

where $\Gamma_{0,1}$ *is any simple loop oriented anticlockwise which includes* $w = 0$ *and* $w = 1$ *and*

$$f_{\alpha,\beta,\gamma}(w,t) = \prod_{j=1}^{t_1} \frac{1-\alpha_j}{1-\alpha_j w} \cdot \prod_{j=t_1+1}^{t_1+t_2} \frac{1+\beta_j w}{1+\beta_j} \cdot e^{\gamma t_3 (w-1)}. \tag{2.1.29}$$

Proof. From Lemma 2.1.1, Lemma 2.1.2, and Lemma 2.1.3, we can show that (2.1.28) satisfies the Kolmogorov forward equation of the discrete time geometric TASEP from time 0 to t_1, and satisfies the Kolmogorov forward equation of the discrete time Bernoulli TASEP from time t_1 to $t_1 + t_2$, and satisfies the Kolmogorov forward equation of the continuous time TASEP from time $t_1 + t_2$ to $t_1 + t_2 + t_3$. $\square$

Remark 2.1.5. In the case of the step initial condition, $y_j = -j,\ j = 1, 2, \dots$, it has been known that the TASEP has a connection to the Schur measures and processes [18, 20, 22, 42]. It is natural to ask the corresponding Schur measure to the $\text{TASEP}_{\boldsymbol{\alpha},\boldsymbol{\beta},\gamma}$ with the step initial condition. Combining the findings in [22, 42], we expect that the position of the Nth particle from the right is equivalent in distribution to the marginal λ_N of the Schur measure,

$$s_\lambda(\underbrace{1, 1, \dots, 1}_{N\text{times}}) s_\lambda(\rho)/Z, \tag{2.1.30}$$

where $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}_{\geq 0}$ with $\lambda_1 \geq \cdots \geq \lambda_N$ is a partition, Z is the normalization constant, $s_\lambda(x_1, \dots, x_N)$ is the Schur symmetric polynomial and $s_\lambda(\rho)$ is the Schur function with the Schur positive specialization ρ defined by the relation of the specialization of the complete symmetric functions $h_k, k = 0, 1, 2 \dots$

$$\sum_{z=0}^{\infty} z^k h_k(\rho) = \prod_{i=1}^{t_1} \frac{1}{1 - z\alpha_i} \cdot \prod_{j=t_1+1}^{t_1+t_2} (1 + z\beta_j) \cdot e^{\gamma t_3 z}. \tag{2.1.31}$$

2.2 Biorthogonal ensembles

In the following we consider the joint distribution function of the particle positions in the $\text{TASEP}_{\alpha,\beta,\gamma}$ introduced in Chap. 1.1.4. We will give a formula in terms of a Fredholm determinant whose kernel can be written in an explicit form.

Proposition 2.2.1. *We consider the* $\text{TASEP}_{\alpha,\beta,\gamma}$ *introduced in Chap. 1.1.4. Let* $1 \leq n_1 < n_2 < \cdots < n_m \leq N$ *and* $(a_1, a_2, \ldots, a_m) \in \mathbb{Z}^m$*. Then we have*

$$\mathbb{P}(X_t(n_j) > a_j, j = 1, \ldots, m) = \det(I - \bar{\chi}_{\boldsymbol{a}} K_t \bar{\chi}_{\boldsymbol{a}})_{\ell^2(\{n_1,\ldots,n_m\}\times\mathbb{Z})}. \tag{2.2.1}$$

Here the kernel K_t *is given by*

$$K_t(n_i, x_i; n_j, x_j) = -\phi^{n_j - n_i}(x_i, x_j)\mathbf{1}_{n_i<n_j} + \sum_{k=1}^{n_j} \psi^{n_i}_{n_i-k}(x_i)\varphi^{n_j}_{n_j-k}(x_j) \tag{2.2.2}$$

where $\phi(x, y) = \mathbf{1}_{x>y}$*. The functions* $\psi^n_k(x)$ *and* $\varphi^n_k(x), k = 0, \ldots, n-1$ *are defined as the following: For* $\psi^n_k(x)$ *with* $k \leq n-1$*, we define*

$$\psi^n_k(x) := \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{(1-w)^k}{w^{x+k+1-X_0(n-k)}} f_{\alpha,\beta,\gamma}(w, t) \tag{2.2.3}$$

where Γ_0 *is any positively oriented simple loop including the pole at* $w = 0$ *and* $f_{\alpha,\beta,\gamma}(w,t)$ *is defined by* (2.1.29)*. The functions* $\varphi^n_k(x), k = 0, \ldots, n-1$*, are defined implicitly by*
(1) The biorthogonality relation $\sum_{x\in\mathbb{Z}} \psi^n_k(x)\varphi^n_l(x) = \mathbf{1}_{k=l}$*;*
(2) $\varphi^n_k(x)$ *is a polynomial of degree at most* $n-1$ *in* x *for each* k*.*

Remark 2.2.2. The fact that the joint distribution of particle positions can be expressed by the Fredholm determinant is proved by [10] for the discrete time Bernoulli TASEP and by [11] for the continuous time TASEP. The above result includes a generalization of the initial conditions for particle position in the results of distribution of particle position of [10] and [11], and is the result when the continuous time TASEP, the discrete time Bernoulli TASEP and the discrete time geometric TASEP are mixed.

Proof. We denote by $M_{n,m}$ the set of all $n \times m$ matrices with real entries. Then for $1 \leqslant n < m \leqslant N$ we put $W_{[n,m)} : \Lambda_N \to M_{n,m}$ for a fixed particles configuration $\mathbf{z} = (z^n_i) \in \Lambda_N$ where

$$\Lambda_N = \left\{ \mathbf{z} = (z^n_i)_{n,i} : z^n_i \in \mathbb{Z}, 1 \leqslant i \leqslant n \leqslant N \right\} \tag{2.2.4}$$

and the matrix $W_{[n,m)}(\mathbf{z})$ is given by

$$\left[W_{[n,m)}(\mathbf{z})\right]_{i,j} = \phi^{m-n}\left(z^n_i, z^m_j\right) 1_{(n<m)}, \quad 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m. \tag{2.2.5}$$

Similarly we set $\psi^{(N)} : \Lambda_N \to M_{N,N}$ with

$$\left[\psi^{(N)}(\mathbf{z})\right]_{i,j} = \psi^N_{N-j}\left(z^N_i\right), \quad 1 \leqslant i, j \leqslant N \tag{2.2.6}$$

Also, for $0 \leqslant m < N$, let us define the function $E_m : \Lambda_N \to M_{N,m+1}$ with

$$[E_m(\mathbf{z})]_{i,j} = \begin{cases} \phi\left(z_{m+1}^m, z_j^{m+1}\right), & \text{if } i = m+1, 1 \leqslant j \leqslant m+1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.7}$$

By Appendix B, we can rewrite (2.1.28) in a form involving transition probabilities of non-intersecting random walks whose configurations form a Gelfand-Tsetlin pattern and distribution forms a determinantal point process with correlation kernel K_t. Therefore, we derive kernel K_t. By Appendix C, the (n,m) -block of K_t is

$$[K_t]_{(n,\cdot),(m,\cdot)} = -W_{[n,m)}\mathbf{1}_{\{n<m\}} + W_{[n,N)}\psi^{(N)}M^{-1}\left(\sum_{k=1}^{m-1} E_{k-1}W_{[k,m)} + E_{m-1}\right) \tag{2.2.8}$$

where the matrix M is

$$[M]_{i,j} = \begin{cases} \left(\phi^{N-i}\psi_{N-j}^N\right)\left(z_i^{i-1}\right), & i<j \\ 1, & i=j \\ 0, & i>j. \end{cases} \tag{2.2.9}$$

Since

$$\psi_{N-k}^N(x) = \sum_{y<x} \psi_{N+1-k}^{N+1}(x) \tag{2.2.10}$$

holds for $x, y \in \mathbb{Z}$, we obtain

$$\left[W_{[n,N)}\psi^{(N)}(\mathbf{z})\right]_{i,j} = \left(\phi^{N-n}\psi_{N-j}^N\right)\left(z_i^n\right) = \psi_{n-j}^n\left(z_i^n\right). \tag{2.2.11}$$

Also, it is easy to see that

$$\left[\left(\sum_{k=1}^{m-1} E_{k-1}W_{[k,m)} + E_{m-1}\right)(\mathbf{z})\right]_{i,j} = \begin{cases} \phi^{m+1}\left(z_i^{i-1}, z_j^m\right), & 1 \leqslant i \leqslant m \\ 0, & m < i \leqslant N. \end{cases} \tag{2.2.12}$$

By (2.2.10), (2.2.11), we have

$$[K_t(\mathbf{z})]_{(n,i),(m,j)} = -\phi^{m-n}\left(z_i^n, z_j^m\right)\mathbf{1}_{\{n<m\}} + \sum_{\ell,k=1}^{m} \psi_{n-\ell}^n\left(z_i^n\right)\left[M^{-1}\right]_{\ell,k}\phi^{m+1}\left(z_k^{k-1}, z_j^m\right). \tag{2.2.13}$$

Note that for $i = 1, \dots, m$, $\phi^{m+1}\left(z_i^{i-1}, x\right)$ form a basis of span$\left\{1, x, \dots, x^{m-1}\right\}$. From the assumption of Proposition 2.2.1, $\left\{\varphi_{m-1}^m(x), \dots, \varphi_0^m(x)\right\}$ also form a basis of this space, so let us define a matrix $A_m \in M_{m,m}$ which does a change of basis to $\left\{\varphi_{m-1}^m(x), \dots, \varphi_0^m(x)\right\}$ such that

$$\phi^{m+1}\left(z_i^{i-1}, x\right) = \sum_{\ell=1}^{m} [A_m]_{i,\ell}\,\varphi_{m-\ell}^m(x). \tag{2.2.14}$$

By the biorthogonality assumption of Proposition 2.2.1, we get

$$[A_m]_{i,j} = \left(\phi^{m-i+1}\psi_{m-j}^m\right)\left(z_i^{i-1}\right).$$

Note that $A_N = M$ and M is invertible. From the assumption of Proposition 2.2.1, we can define φ_k^n uniquely. Therefore we have

$$\sum_{k=1}^{m} \left[M^{-1}\right]_{\ell,k} \phi^{m+1}\left(z_k^{k-1}, z_j^m\right) = \sum_{k=1}^{m} \left[A_N^{-1}\right]_{\ell,k} \sum_{i=1}^{m} [A_m]_{k,i} \varphi_{m-i}^m\left(z_j^m\right). \tag{2.2.15}$$

Because

$$[A_m]_{k,i} = \left(\phi^{m-k+1}\psi_{m-i}^m\right)\left(z_k^{k-1}\right) = \left(\phi^{N-k+1}\psi_{N-i}^N\right)\left(z_k^{k-1}\right) = [A_N]_{k,i}$$

holds for $1 \leqslant k, i \leqslant m$, by the fact that A_N is upper-triangular, we get

$$\sum_{k=1}^{m} \left[A_N^{-1}\right]_{\ell,k} [A_m]_{k,i} = \sum_{k=1}^{m} \left[A_N^{-1}\right]_{\ell,k} [A_N]_{k,i} = \sum_{k=1}^{N} \left[A_N^{-1}\right]_{\ell,k} [A_N]_{k,i} = \delta_{\ell,i}$$

for $1 \leqslant i \leqslant m$. Because

$$(2.2.15) = \varphi_{m-\ell}^m(z_j^m),$$

we obtain the (n, m) -th block of K_t

$$[K_t(\mathbf{z})]_{(n,i),(m,j)} = -\phi^{\mathfrak{m}-n}\left(z_i^n, z_j^m\right)\mathbf{1}_{\{n<m\}} + \sum_{\ell=1}^{m} \psi_{n-\ell}^n\left(z_i^n\right)\varphi_{m-\ell}^m\left(z_j^m\right).$$

From the above, we have (2.2.2).

This completes the proof. □

Theorem 2.2.3 ([2]). *We consider the* $\mathrm{TASEP}_{\alpha,\beta,\gamma}$ *introduced in Chap. 1.1.4. For* $(n_1, n_2, \dots, n_m) \in \mathbb{Z}^m$ *with* $1 \le n_1 < n_2 < \cdots < n_m \le N$ *and* $(a_1, a_2, \dots, a_m) \in \mathbb{Z}^m$*, we have*

$$\mathbb{P}(X_t(n_j) > a_j, j = 1, \dots, m) = \det(I - \bar{\chi}_{\boldsymbol{a}} K_t \bar{\chi}_{\boldsymbol{a}})_{\ell^2(\{n_1,\dots,n_m\}\times\mathbb{Z})}. \tag{2.2.16}$$

Here the right hand side is a Fredholm determinant with the kernel

$$K_t(n_i, x_i; n_j, x_j) = -Q^{n_j-n_i}(x_i, x_j)\mathbf{1}_{n_i<n_j} + \sum_{k=1}^{n_j} \Psi_{n_i-k}^{n_i}(x_i)\Phi_{n_j-k}^{n_j}(x_j) \tag{2.2.17}$$

where $Q^n(x_i, x_j)$ *represents* n*-times convolution of* $Q(x, y) = 1/2^{x-y} \cdot \mathbf{1}_{x>y}$*. The functions* $\Psi_k^n(x)$ *and* $\Phi_k^n(x), k = 0, \dots, n-1$ *are defined as follows: For* $\Psi_k^n(x)$ *with* $k \le n-1$*, we define*

$$\Psi_k^n(x) := \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{(1-w)^k}{2^{x-X_0(n-k)} w^{x+k+1-X_0(n-k)}} f_{\alpha,\beta,\gamma}(w, t) \tag{2.2.18}$$

where Γ_0 *is any positively oriented simple loop including the pole at* $w = 0$ *and* $f_{\alpha,\beta,\gamma}(w, t)$ *is defined by* (2.1.29)*. The functions* $\Phi_k^n(x), k = 0, \dots, n-1$*, are defined implicitly by*
(1) The biorthogonality relation $\sum_{x\in\mathbb{Z}} \Psi_k^n(x)\Phi_l^n(x) = \mathbf{1}_{k=l}$*;*
(2) $2^{-x}\Phi_k^n(x)$ *is a polynomial of degree at most* $n-1$ *in* x *for each* k*.*

Proof. We can prove by conjugating the kernel of Proposition 2.2.1 by a power of 2.

□

In the following, we will write $\Phi_k^n(x)$ that was not explicitly written in previous research [10] in an explicit form.

First, we prepare the tools to use. Q^m can easily be taken from definition Q;

$$Q^m(x,y) = \frac{1}{2^{x-y}} \binom{x-y-1}{m-1} \mathbf{1}_{x \geq y+m}. \tag{2.2.19}$$

As operators on $\ell^2(\mathbb{Z})$, Q and Q^m are invertible;

$$Q^{-1}(x,y) = 2 \cdot \mathbf{1}_{x=y-1} - \mathbf{1}_{x=y}, \quad Q^{-m}(x,y) = (-1)^{y-x+m} 2^{y-x} \binom{m}{y-x}. \tag{2.2.20}$$

Now we define

$$R_{\alpha,\beta,\gamma,t}(x,y) := \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{f_{\alpha,\beta,\gamma}(w,t)}{2^{x-y} w^{x-y+1}}, \tag{2.2.21}$$

where

$$f_{\alpha,\beta,\gamma}(w,t) = \prod_{j=1}^{t_1} \frac{1-\alpha_j}{1-\alpha_j w} \cdot \prod_{j=t_1+1}^{t_1+t_2} \frac{1+\beta_j w}{1+\beta_j} \cdot e^{\gamma t_3 (w-1)}.$$

Note that $\Psi_0^n = R_{\alpha,\beta,\gamma,t} \delta_{X_0(n)}$ with $\delta_y(x) = \mathbf{1}_{x=y}$.

Then, the following lemma holds.

Lemma 2.2.4 ([2]). *For $n \in \mathbb{Z}$,*

$$\Psi_k^n = R_{\alpha,\beta,\gamma,t} Q^{-k} \delta_{X_0(n-k)}. \tag{2.2.22}$$

Proof. By $Q^{n-m} \Psi_{n-k}^n = \Psi_{m-k}^m$ and (2.2.21),

$$\Psi_k^n = Q^{-k} R_{\alpha,\beta,\gamma,t} \delta_{X_0(n-k)}$$

holds.

Now, note that Q and $R_{\alpha,\beta,\gamma,t}$ commute, because the kernels $Q(x,y)$ and $R_{\alpha,\beta,\gamma,t}(x,y)$ only depend on $x-y$. Therefore, we obtain

$$\Psi_k^n = R_{\alpha,\beta,\gamma,t} Q^{-k} \delta_{X_0(n-k)}.$$

$\square$

From the expression of $R_{\alpha,\beta,\gamma,t}$, we define

$$R^{-1}_{\alpha,\beta,\gamma,t}(x,y) := \frac{1}{2\pi i} \oint_{\Gamma_0} dw \frac{f^{-1}_{\alpha,\beta,\gamma}(w,t)}{2^{x-y} w^{x-y+1}}, \tag{2.2.23}$$

It is not hard to check that $R_{\alpha,\beta,\gamma,t} R^{-1}_{\alpha,\beta,\gamma,t} = R^{-1}_{\alpha,\beta,\gamma,t} R_{\alpha,\beta,\gamma,t} = I$. At this time, the following theorem holds.

Theorem 2.2.5 ([2])**.** *Fix $0 \le k < n$ and consider particles at $X_0(1) > X_0(2) > \cdots > X_0(n)$. Let $h_k^n(l,z)$ be the unique solution to the initial-boundary value problem for the backwards heat equation*

$$\begin{cases} (Q^*)^{-1}h_k^n(l,z) = h_k^n(l+1,z) & l < k, z \in \mathbb{Z}, \quad (2.2.24a) \\ h_k^n(k,z) = 2^{z-X_0(n-k)} & z \in \mathbb{Z}, \quad (2.2.24b) \\ h_k^n(l, X_0(n-l)) = 0 & l < k. \quad (2.2.24c) \end{cases}$$

Then the functions Φ_k^n from Theorem 2.2.3 are given by

$$\Phi_k^n(z) = (R^*_{\alpha,\beta,\gamma,t})^{-1}h_k^n(0,\cdot)(z) = \sum_{y\in\mathbb{Z}} h_k^n(0,y)R^{-1}_{\alpha,\beta,\gamma,t}(y,z). \tag{2.2.25}$$

Here $Q^*(x,y) = Q(y,x)$ is the kernel of the adjoint of Q (and likewise for $R^*_{\alpha,\beta,\gamma,t}$).

Remark 2.2.6. It is not true that in general $Q^*h_k^n(l+1,z) = h_k^n(l,z)$. In fact, $Q^*h_k^n(k,z)$ is divergent from the following.

$$\begin{aligned} Q^*h_k^n(k,\cdot)(z) &= \sum_{y\in\mathbb{Z}} h_k^n(k,y)Q(y,z) \\ &= \sum_{y\in\mathbb{Z}} 2^{y-X_0(n-k)}\frac{1}{2^{y-z}}\mathbf{1}_{y>z} \\ &= \sum_{y\in\mathbb{Z},y>z} 2^{z-X_0(n-k)} \\ &= \infty. \end{aligned}$$

Proof. This proof is almost the same as [23]. We show that the same can be proved in the mixed TASEP.

The existence and uniqueness of solutions of (2.2.24a)-(2.2.24c) is elementary consequence of the fact that $ker(Q^*)^{-1}$ has dimension 1 and it is spanned by the function 2^z, which allows us to march forwards from the initial condition $h_k^n(k,z) = 2^{z-X_0(n-k)}$ uniquely solving the boundary value problem $h_k^n(l, X_0(n-k)) = 0$ at each step.
First, we prove that $2^{-x}h_k^n(0,x)$ is a polynomial of degree at most k. We use the mathematical induction. By (2.2.24b),

$$2^{-x}h_k^n(k,x) = 2^{-x}2^{x-X_0(n-k)} = 2^{-X_0(n-k)}. \tag{2.2.26}$$

Therefore, $2^{-x}h_k^n(k,x)$ is polynomial of degree 0.

Now, assume that $\widehat{h}_k^n(l,x) := 2^{-x}h_k^n(l,x)$ is a polynomial of degree at most $k-l$ for some $0 < l \le k$. By (2.2.20) and (2.2.24a),

$$\begin{aligned} \widehat{h}_k^n(l,y) &= 2^{-y}(Q^*)^{-1}h_k^n(l-1,y) \\ &= 2^{-y}(2\cdot h_k^n(l-1,y-1) - h_k^n(l-1,y)) \\ &= 2^{-(y-1)}h_k^n(l-1,y-1) - 2^{-y}h_k^n(l-1,y) \\ &= \widehat{h}_k^n(l-1,y-1) - \widehat{h}_k^n(l-1,y). \end{aligned}$$

Taking the sum over $x \geq X_0(n-l+1)$, one sees

$$\begin{aligned}\sum_{y=X_0(n-l+1)+1}^{x} 2^{-y} h_k^n(l,y) &= \sum_{y=X_0(n-l+1)+1}^{x} \widehat{h}_k^n(l,y) \\ &= \sum_{y=X_0(n-l+1)+1}^{x} (\widehat{h}_k^n(l-1,y-1) - \widehat{h}_k^n(l-1,y)) \\ &= \widehat{h}_k^n(l-1, X_0(n-l+1)) - \widehat{h}_k^n(l-1,x).\end{aligned}$$

Therefore, using (2.2.24c), we have $\widehat{h}_k^n(l-1,x) = -\sum_{y=X_0(n-l+1)+1}^{x} 2^{-y} h_k^n(l,y)$.

By the induction hypothesis, $\widehat{h}_k^n(l-1,x)$ is a polynomial of degree at most $k-l+1$ because $\widehat{h}_k^n(l,y)$ is a polynomial of degree at most $k-l$*1.

Similarly, taking the sum $x < X_0(n-l+1)$, we get $\widehat{h}_k^n(l-1,x) = \sum_{y=x+1}^{X_0(n-l+1)} \widehat{h}_k^n(l,y)$,which is again a polynomial of degree at most $k-l+1$. From the above, it was shown that $2^{-x}h_k^n(0,x)$ is a polynomial of degree at most k.

Now, we show that $\sum_{y\in\mathbb{Z}} h_k^n(0,y) R^{-1}_{\alpha,\beta,\gamma,t}(y,z)$, which is the rhs of (2.2.25), satisfies the condition (2) in Theorem 2.2.3. By (2.2.23), we have

$$\begin{aligned}2^{-z}\sum_{y\in\mathbb{Z}} h_k^n(0,y) R^{-1}_{\alpha,\beta,\gamma,t}(y,z) &= 2^{-z}\sum_{y\geq z}\left(\frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{f^{-1}_{\alpha,\beta,\gamma}(w,t)}{2^{y-z}w^{y-z+1}}\right) h_k^n(0,y) \\ &= \sum_{y\geq z}\left(\frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{f^{-1}_{\alpha,\beta,\gamma}(w,t)}{w^{y-z+1}}\right) 2^{-y} h_k^n(0,y) \\ &= \sum_{x\geq 0}\left(\frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{f^{-1}_{\alpha,\beta,\gamma}(w,t)}{w^{x+1}}\right) 2^{-(x+z)} h_k^n(0,x+z).\end{aligned} \tag{2.2.27}$$

Because $2^{-z}h_k^n(0,z)$ is a polynomial of degree at most k, it is enough to note that the sum is a polynomial of degree at most k in z as well. Next, we check the biorthogonality relation (1) of Theorem 2.2.3. Using (2.2.22), we get

$$\begin{aligned}\sum_{z\in\mathbb{Z}} \Psi_l^n(z)\Phi_k^n(z) &= \sum_{z_1,z_2\in\mathbb{Z}}\sum_{z\in\mathbb{Z}} R_{\alpha,\beta,\gamma,t}(z,z_1) Q^{-l}(z_1, X_0(n-l)) h_k^n(0,z_2) R^{-1}_{\alpha,\beta,\gamma,t}(z_2,z) \\ &= \sum_{z\in\mathbb{Z}} Q^{-l}(z, X_0(n-l)) h_k^n(0,z) = (Q^*)^{-l} h_k^n(0, X_0(n-l)),\end{aligned}$$

where in the first equality we have used the decay of $R_{\alpha,\beta,\gamma,t}$ and the fact that $2^{-x}h_k^n(0,x)$ is a polynomial together with the fact that the z_1 sum is finite to apply Fubini.

*1This can be understood from Faulhaber's formula : $\sum_{j=1}^{n} j^k = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j} B_j n^{k+1-j}$ where B_j is Bernoulli number.

For $l \leq k$, from (2.2.24b) and (2.2.24c), we have the boundary condition

$$h_k^n(l, X_0(n-l)) = \mathbf{1}_{l=k}. \tag{2.2.28}$$

Thus, we get

$$(Q^*)^{-l} h_k^n(0, X_0(n-l)) = h_k^n(l, X_0(n-l)) = \mathbf{1}_{l=k}.$$

For $l > k$, we use (2.2.24a) and (2.2.24b), $2^z \in ker(Q^*)^{-1}$,

$$(Q^*)^{-l} h_k^n(0, X_0(n-l)) = (Q^*)^{-(l-k-1)}(Q^*)^{-1} h_k^n(k, X_0(n-l)) = 0.$$

This completes the proof. □

2.3 Representation of the TASEP kernel by using a hitting probability

Combining Theorem 2.2.3 with (2.2.22) and (2.2.25), we have obtained the following expression of the kernel K_t (2.2.17),

$$K_t(n_i, \cdot; n_j, \cdot) = -Q^{n_j - n_i} \mathbf{1}_{n_i < n_j} + R_{\alpha,\beta,\gamma,t} Q^{-n_i} G_{0,n_j} R^{-1}_{\alpha,\beta,\gamma,t}. \tag{2.3.1}$$

Here Q, $R_{\alpha,\beta,\gamma,t}$ and $R^{-1}_{\alpha,\beta,\gamma,t}$ are given by (2.2.19), (2.2.21) and (2.2.23) respectively and G_{0,n_j} is defined by

$$G_{0,n}(z_1, z_2) = \sum_{k=0}^{n-1} Q^{n-k}(z_1, X_0(n-k)) h_k^n(0, z_2), \tag{2.3.2}$$

where h_k^n is the solution of (2.2.24a)-(2.2.24c).

In this section, following the method in [23], we further rewrite the kernel in order to take the KPZ scaling limit. We use the fact that h_k^n can be written as hitting probabilities of random walk. Let RW_m^* with $RW_{-1}^* = c$ be the position of the random walk with Geom$[\frac{1}{2}]$ jumps strictly to the right starting from $c \in \mathbb{Z}$, i.e.

$$RW_m^* = c + \chi_0 + \cdots + \chi_m,$$

where $\chi_j,\ j = 0, 1, \ldots, m$ are i.i.d. random variables with $\mathbb{P}(\chi_j = k) = 1/2^{k+1},\ k \in \mathbb{Z}_{\geq 0}$. Note that Q^* defined below (2.2.25) represents the transition kernel of the random walk: for $m = -1, 0, \ldots,$ we have

$$Q^*(x, y) = \mathbb{P}(RW_{m+1}^* = x | RW_m^* = y). \tag{2.3.3}$$

For $0 \leq l \leq k \leq n-1$ we define the stopping times

$$\tau^{l,n} = \min\{m \in \{l, \ldots, n-1\} : RW_m^* > X_0(n-m)\}, \tag{2.3.4}$$

where we set $\min \varnothing = \infty$.

Then, we have the following.

Lemma 2.3.1. ([23])
For $z \le X_0(n-l)$, the function h_k^n can be written by

$$h_k^n(l,z) = \mathbb{P}_{RW_{l-1}^*=z}(\tau^{l,n}=k) \tag{2.3.5}$$

which is the probability of the walk starting at $z \in \mathbb{Z}$ at time $l-1 \in \mathbb{Z}$ and hitting $X_0(n-k)$ at time $k \in \mathbb{Z}$.

Remark 2.3.2. Eq. (2.3.5) is written in [23] and the proof is left to the readers as Exercise 5.17 in [33]. Here we give an answer.

Proof. By (2.2.24a)-(2.2.24c), it is enough to check $(Q^*)^{-1}\mathbb{P}_{RW_{l-1}^*=z}(\tau^{l,n}=k) = \mathbb{P}_{RW_l^*=z}(\tau^{l+1,n}=k)$. Now, we assume that $X_0(n-k) = x$[*2] for convenience. Then, by(2.2.20)

$$\begin{aligned}(Q^*)^{-1}\mathbb{P}_{RW_{l-1}^*=z}(\tau^{l,n}=k) &= 2\ \mathbb{P}_{RW_{l-1}^*=z-1}(\tau^{l,n}=k) - \mathbb{P}_{RW_{l-1}^*=z}(\tau^{l,n}=k) \\ &= 2\left(\mathbb{P}_{RW_{l-1}^*=z-1}(\tau^{l,n}=k) - \frac{1}{2}\ \mathbb{P}_{RW_{l-1}^*=z}(\tau^{l,n}=k)\right).\end{aligned} \tag{2.3.6}$$

By the memoryless property of geometric distribution, for $\forall y > X_0(n-k)$,

$$\mathbb{P}_{RW_{l-1}^*=z-1}(\tau^{l,n}=k) = 2^{y-X_0(n-k)}\mathbb{P}_{RW_{l-1}^*=z-1}(\tau^{l,n}=k, RW_k^*=y). \tag{2.3.7}$$

Also, by (2.3.4),

$$\begin{aligned}&\mathbb{P}_{RW_{l-1}^*=z-1}(\tau^{l,n}=k, RW_k^*=y) \\ &= \sum_{z-1<y_l<\cdots<y_{k-1}<x}\left(\frac{1}{2}\right)^{y_l-(z-1)}\left(\frac{1}{2}\right)^{y_{l+1}-y_l}\cdots\left(\frac{1}{2}\right)^{y-y_{k-1}} \times \mathbf{1}_{y_l\le X_0(n-l)}\times\cdots\times\mathbf{1}_{y_{k-1}\le X_0(n-k+1)} \\ &= \left(\frac{1}{2}\right)^{y-(z-1)}\sum_{z-1<y_l<\cdots<y_{k-1}<x}\mathbf{1}_{y_l\le X_0(n-l)}\times\cdots\times\mathbf{1}_{y_{k-1}\le X_0(n-k+1)}.\end{aligned} \tag{2.3.8}$$

Note that $\displaystyle\sum_{z-1<y_l<\cdots<y_{k-1}<x} = \sum_{y_l=z}^{x-1}\sum_{y_{l+1}=y_l+1}^{x-1}\cdots\sum_{y_{k-1}=y_{k-2}+1}^{x-1}$, by (2.3.7) and (2.3.8),

$$\begin{aligned}(2.3.6) = \left(\frac{1}{2}\right)^{x-z}&\left\{\sum_{y_l=z}^{x-1}\sum_{y_{l+1}=y_l+1}^{x-1}\cdots\sum_{y_{k-1}=y_{k-2}+1}^{x-1}\mathbf{1}_{y_l\le X_0(n-l)}\times\cdots\times\mathbf{1}_{y_{k-1}\le X_0(n-k+1)}\right. \\ &\left.-\sum_{y_l=z+1}^{x-1}\sum_{y_{l+1}=y_l+1}^{x-1}\cdots\sum_{y_{k-1}=y_{k-2}+1}^{x-1}\mathbf{1}_{y_l\le X_0(n-l)}\times\cdots\times\mathbf{1}_{y_{k-1}\le X_0(n-k+1)}\right\} \\ &= \left(\frac{1}{2}\right)^{x-z}\sum_{y_{l+1}=z+1}^{x-1}\cdots\sum_{y_{k-1}=y_{k-2}+1}^{x-1}\mathbf{1}_{z\le X_0(n-l)}\times\mathbf{1}_{y_{l+1}\le X_0(n-l-1)}\times\cdots\times\mathbf{1}_{y_{k-1}\le X_0(n-k+1)}.\end{aligned} \tag{2.3.9}$$

[*2] Since we start from arbitrary fixed right finite initial configuration, we can write like this.

Since $z \leq X_0(n-l)$ was assumed,

$$
\begin{aligned}
(2.3.9) &= \left(\frac{1}{2}\right)^{x-z} \sum_{z<y_{l+1}<\cdots<y_{k-1}<x} \mathbf{1}_{y_{l+1}\leq X_0(n-l-1)} \times \cdots \times \mathbf{1}_{y_{k-1}\leq X_0(n-k+1)} \\
&= \mathbb{P}_{RW_l^*=z}(\tau^{l+1,n}=k).
\end{aligned} \tag{2.3.10}
$$

This completes the proof. □

From the memoryless property of geometric distribution we get for all $y > X_0(n-k)$,

$$
\mathbb{P}_{RW_{-1}^*=z}(\tau^{0,n}=k, RW_k^*=y) = 2^{X_0(n-k)-y}\mathbb{P}_{RW_{-1}^*=z}(\tau^{0,n}=k) \tag{2.3.11}
$$

and as a consequence we get for $z_2 \leq X_0(n)$, $G_{0,n}(z_1,z_2)$ can be expressed as

$$
\begin{aligned}
G_{0,n}(z_1,z_2) &= \sum_{k=0}^{n-1} \mathbb{P}_{RW_{-1}^*=z_2}(\tau^{0,n}=k)(Q^*)^{n-k}(X_0(n-k),z_1) \\
&= \sum_{k=0}^{n-1}\sum_{z>X_0(n-k)} \mathbb{P}_{RW_{-1}^*=z_2}(\tau^{0,n}=k, RW_k^*=z)(Q^*)^{n-k-1}(z,z_1) \\
&= \mathbb{P}_{RW_{-1}^*=z_2}(\tau^{0,n}<n, RW_{n-1}^*=z_1),
\end{aligned} \tag{2.3.12}
$$

where in the second equality we used (2.2.19) and (2.3.11). while in the third one we used (2.3.3). Note that RHS of the above equation represents the probability for the walk starting at $z_2 \in \mathbb{Z}$ at time -1 to end up at $z_1 \in \mathbb{Z}$ after n steps, having hit the curve $(X_0(n-m))_{m=0,\ldots,n-1}$ in between.

The next step is to extend the region $z_2 \leq X_0(n)$ in (2.3.12) to $z_2 \in \mathbb{Z}$. We begin by observing that for each fixed y_1 and $n \geq 1$, $2^{-y_2}Q^n(y_1,y_2)$ extends in y_2 to a polynomial $2^{-y_2}\bar{Q}^{(n)}(y_1,y_2)$ of degree $n-1$ with

$$
\bar{Q}^{(n)}(y_1,y_2) = \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{(1+w)^{y_1-y_2-1}}{2^{y_1-y_2}w^n}. \tag{2.3.13}
$$

Now, for $y_1 - y_2 \geq 1$, we note that

$$
\bar{Q}^{(n)}(y_1,y_2) = Q^n(y_1,y_2). \tag{2.3.14}
$$

By (2.2.20) and (2.3.13), for $n > 1$, we get

$$
Q^{-1}\bar{Q}^{(n)} = \bar{Q}^{(n)}Q^{-1} = \bar{Q}^{(n-1)}. \tag{2.3.15}
$$

Also, we get

$$
Q^{-1}\bar{Q}^{(1)} = \bar{Q}^{(1)}Q^{-1} = 0. \tag{2.3.16}
$$

Remark 2.3.3. We note that

$$
\begin{aligned}
\bar{Q}^{(n)}\bar{Q}^{(m)}(x,y) &= \sum_{z\in\mathbb{Z}} \bar{Q}^{(n)}(x,z)\bar{Q}^{(m)}(z,y) \\
&= \sum_{z\in\mathbb{Z}} \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{(1+w)^{x-z-1}}{2^{x-z}w^n} \cdot \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{(1+w)^{z-y-1}}{2^{z-y}w^n} \\
&= \infty.
\end{aligned}
$$

Using the extension of Q^m, we have the following lemma.

Lemma 2.3.4. ([23])
For all $z_1, z_2 \in \mathbb{Z}$, we have

$$G_{0,n}(z_1, z_2) = \mathbb{E}_{RW_0=z_1}\left[\bar{Q}^{(n-\tau)}(RW_\tau, z_2)\mathbf{1}_{\tau<n}\right], \tag{2.3.17}$$

where RW_m and τ are defined by Definition. 1.2.2.

Proof. Although the proof is given in Lemma 2.4 of [23], we give its outline for self-containedness.

For $z_2 \leq X_0(n)$, (2.3.12) can be written as

$$\begin{aligned} G_{0,n}(z_1, z_2) &= \mathbb{P}_{RW^*_{-1}=z_2}(\tau^{0,n} \leq n-1, RW^*_{n-1} = z_1) = \mathbb{P}_{RW_0=z_1}(\tau \leq n-1, RW_n = z_2) \\ &= \sum_{k=0}^{n-1} \sum_{z>X_0(k+1)} \mathbb{P}_{RW_0=z_1}(\tau = k, RW_k = z)Q^{n-k}(z, z_2) \\ &= \mathbb{E}_{RW_0=z_1}\left[Q^{(n-\tau)}(RW_\tau, z_2)\mathbf{1}_{\tau<n}\right], \end{aligned}$$

where in the second equality we used the fact $Q^n(x, y)$ (2.2.19) represents the n-step transition probability of RW_m. Let

$$\bar{G}_{0,n}(z_1, z_2) = \mathbb{E}_{RW_0=z_1}\left[\bar{Q}^{(n-\tau)}(RW_\tau, z_2)\mathbf{1}_{\tau<n}\right]. \tag{2.3.18}$$

From the relation $z_2 < \mathrm{RW}_\tau < z_1$ for $\forall \tau < n$ and (2.3.14) we see that for $\bar{G}_{0,n}(z_1, z_2) = G_{0,n}(z_1, z_2)$ for $z_2 \leq X_0(n)$. Furthermore we find that $2^{-z_2}G_{0,n}(z_1, z_2)$ is polynomial in z_2 with degree at most k, and similarly $2^{-z_2}\bar{G}_{0,n}(z_1, z_2)$ is polynomial in z_2 with degree at most k since $2^{-y_2}h^n_k(0, y_2)$ is polynomial in y_2 with degree at most k. From the above, we find that the equality $\bar{G}_{0,n}(z_1, z_2) = G_{0,n}(z_1, z_2)$ holds for the all $z_2 \in \mathbb{Z}$. □

Thus from (2.3.1) and (2.3.17), we see that the kernel K_t (2.2.17) can be expressed as

$$\begin{aligned} K_t(n_1, x_1; n_2, x_2) = &-Q^{n_2-n_1}(x_1, x_2)\mathbf{1}_{n_1<n_2} \\ &+ \sum_{x,y\in\mathbb{Z}} (R_{\alpha,\beta,\gamma,t}Q^{-n_1})(x_1, x)\mathbb{E}_{RW_0=x}\left[\bar{Q}^{(n_2-\tau)}(RW_\tau, y)R^{-1}_{\alpha,\beta,\gamma,t}(y, x_2)\mathbf{1}_{\tau<n_2}\right]. \end{aligned} \tag{2.3.19}$$

2.4 Formulas for the mixed TASEP: Proof of Theorem 1.2.4

To show Theorem 1.2.4, we have the following relations.

Proposition 2.4.1 ([2])**.**

$$A^{-1}_{\alpha,\beta,\gamma}(t)(R_{\alpha,\beta,\gamma,t}Q^{-n})^*(z_1, z_2) = S_{-t,-n}(z_1, z_2), \tag{2.4.1}$$

$$A_{\alpha,\beta,\gamma}(t)\bar{Q}^{(n)}R^{-1}_{\alpha,\beta,\gamma,t}(z_1, z_2) = \bar{S}_{-t,n}(z_1, z_2). \tag{2.4.2}$$

Here $S_{-t,-n}(z_1, z_2)$ and $\bar{S}_{-t,n}(z_1, z_2)$ are defined by (1.2.6) *and* (1.2.7) *respectively and $A_{\alpha,\beta,\gamma}(t)$ is defined by*

$$A_{\alpha,\beta,\gamma}(t) := e^{-\frac{\gamma t_3}{2}} \prod_{j=1}^{t_1} \frac{1-\alpha_j}{2-\alpha_j} \prod_{j=t_1+1}^{t_1+t_2} \frac{2+\beta_j}{1+\beta_j}. \tag{2.4.3}$$

Proof. By (2.2.22), the lhs of (2.4.1) becomes

$$\begin{aligned} &A^{-1}_{\alpha,\beta,\gamma}(t)(\Psi^n_n)^*(z_1)\mid_{X_0(0)=z_2} = A^{-1}_{\alpha,\beta,\gamma}(t)(\Psi^n_n)(z_2)\mid_{X_0(0)=z_1} \\ &= \frac{1}{2\pi i}\oint_{\Gamma_0} dw \frac{(1-w)^n}{2^{z_2-z_1}w^{n+1+z_2-z_1}}\mathfrak{F}_{\alpha,\beta,\gamma}(w,t) = S_{-t,-n}(z_1,z_2). \end{aligned}$$

By (2.2.23) and (2.3.13), the lhs of (2.4.2) is written as

$$\begin{aligned} &A_{\alpha,\beta,\gamma}(t)\sum_{z\in\mathbb{Z}}\bar{Q}^{(n)}(z_1,z)R^{-1}_{\alpha,\beta,\gamma,t}(z,z_2) \\ &= A_{\alpha,\beta,\gamma}(t)\frac{1}{2^{z_1-z_2}}\sum_{z\in\mathbb{Z}}\frac{1}{2\pi i}\oint_{\Gamma_0} dw\frac{(1+w)^{z_1-z-1}}{w^n}\cdot\frac{1}{2\pi i}\oint_{\Gamma_0} d\bar{w}\frac{f^{-1}_{\alpha,\beta,\gamma}(\bar{w},t)}{\bar{w}^{z-z_2+1}} \\ &= A_{\alpha,\beta,\gamma}(t)\frac{1}{2^{z_1-z_2}}\sum_{z\in\mathbb{Z}}\frac{1}{2\pi i}\oint_{\Gamma_0} dw\frac{(1-w)^{z_1-z-1}}{w^n}(-1)^{n-1}\frac{1}{2\pi i}\oint_{\Gamma_0} d\bar{w}\frac{f^{-1}_{\alpha,\beta,\gamma}(\bar{w},t)}{\bar{w}^{z-z_2+1}} \\ &= A_{\alpha,\beta,\gamma}(t)\frac{1}{2^{z_1-z_2}}\frac{(-1)^{n-1}}{2\pi i}\oint_{\Gamma_0} dw\frac{(1-w)^{z_1-z_2-1}}{w^n}f^{-1}_{\alpha,\beta,\gamma}\left(\frac{1}{1-w},t\right). \end{aligned}$$

By changing variables $w \mapsto \dfrac{-w}{1-w}$, we have

$$\frac{A_{\alpha,\beta,\gamma}(t)}{2^{z_1-z_2}}\frac{1}{2\pi i}\oint_{\Gamma_0} dw\frac{(1-w)^{z_2-z_1+n-1}}{w^n}f^{-1}_{\alpha,\beta,\gamma}(1-w,t)$$

$$= \frac{1}{2\pi i}\oint_{\Gamma_0} dw\frac{(1-w)^{z_2-z_1+n-1}}{2^{z_1-z_2}w^n}\bar{\mathfrak{F}}_{\alpha,\beta,\gamma}(w,t) = \bar{S}_{-t,n}(z_1,z_2)$$

This completes the proof. □

Proof of Theorem 1.2.4. First, we consider right finite initial data. If $X_0(1) < \infty$ then we are in the setting of the above chapter. Formula (1.2.4) follow directly from above definition.

Now, we check (1.2.3). To check (1.2.3), it is enough to check

$$Q^{n_j-n_i}K^{(n_j)}_t = (S_{-t,-n_i})^*\bar{S}^{\mathrm{epi(X_0)}}_{-t,n_j},$$

where $K^{(n_j)}_t = R_{\alpha,\beta,\gamma,t}Q^{-n_j}G_{0,n_j}R^{-1}_{\alpha,\beta,\gamma,t}$.

Because Q and $R_{\alpha,\beta,\gamma,t}$ commute, by lemma 2.3.4,

$$\begin{aligned} Q^{n_j-n_i}K^{(n_j)}_t &= R_{\alpha,\beta,\gamma,t}Q^{-n_i}G_{0,n_j}R^{-1}_{\alpha,\beta,\gamma,t} \\ &= A^{-1}_{\alpha,\beta,\gamma}(t)R_{\alpha,\beta,\gamma,t}Q^{-n_i}G_{0,n_j}R^{-1}_{\alpha,\beta,\gamma,t}A_{\alpha,\beta,\gamma}(t) \\ &= (S_{-t,-n_i})^*\bar{S}^{\mathrm{epi(X_0)}}_{-t,n_j}. \end{aligned}$$

If $X_0(j) = \infty$ for $j = 1,\dots l$ and $X_0(l+1) < \infty$ then

$$\mathbb{P}_{X_0}(X_t(n_j) > a_j, j = 1,\dots,M) = \det(I - \bar{\chi}_a K^{(l)}_t\bar{\chi}_a)_{\ell^2(\{n_1,\dots,n_M\}\times\mathbb{Z}}$$

with the correlation kernel

$$K_t^{(l)}(n_i,\cdot;n_j,\cdot) = -Q^{n_j-n_i}\mathbf{1}_{n_i<n_j} + (S_{-t,-n_i})^*\bar{S}^{\mathrm{epi}(\theta_l \mathrm{X}_0)}_{-t,n_j-l},$$

where $\theta_l X_0(j) = X_0(l+j)$. Now, using the fact that $Q^l\bar{S}^{\mathrm{epi}(\theta_l \mathrm{X}_0)}_{-t,n_j-l} = \bar{S}^{\mathrm{epi}(\mathrm{X}_0)}_{-t,n_j}$ and (2.4.1), we have that (1.2.4) still holds in this case. $\square$

Chapter 3

Asymptotics for the discrete time TASEPs

In this chapter we take the KPZ scaling limit for the discrete time Bernoulli and geometric TASEP and prove Proposition 1.2.15 and 1.2.16.

3.1 Proof of Proposition 1.2.15

First, we prove (1.2.35). By changing variables $w = \frac{1}{2}(1-\varepsilon^{\frac{1}{2}}y)$, we have

$$\begin{aligned}(1.2.38) &= \frac{1}{2\pi i}\oint_{C_\varepsilon} \frac{1}{2}\varepsilon^{\frac{1}{2}}dy \frac{\{\frac{1}{2}(1+\varepsilon^{\frac{1}{2}}y)\}^n}{2^{z-y'}\{\frac{1}{2}(1-\varepsilon^{\frac{1}{2}}y)\}^{n+1+z-y'}}\left(1-\frac{p}{2-p}\varepsilon^{\frac{1}{2}}y\right)^t \\ &= \frac{1}{2\pi i}\oint_{C_\varepsilon} \varepsilon^{\frac{1}{2}}dy \frac{(1+\varepsilon^{\frac{1}{2}}y)^n}{(1-\varepsilon^{\frac{1}{2}}y)^{n+1+z-y'}}\left(1-\frac{p}{2-p}\varepsilon^{\frac{1}{2}}y\right)^t\end{aligned} \tag{3.1.1}$$

where C_ε is a circle of radius $\varepsilon^{-\frac{1}{2}}$ centred at $\varepsilon^{-\frac{1}{2}}$. In order to apply the saddle point method, we rewrite (3.1.1) as

$$\frac{1}{2\pi i}\oint_{C_\varepsilon} \varepsilon^{\frac{1}{2}}\ e^{f(\varepsilon^{\frac{1}{2}}y)+\varepsilon^{-1}F_2(\varepsilon^{\frac{1}{2}}y)+\varepsilon^{-\frac{1}{2}}F_1(\varepsilon^{\frac{1}{2}}y)+F_0(\varepsilon^{\frac{1}{2}}y)}dy, \tag{3.1.2}$$

where the functions $f(x)$ and $F_i(x),\ i=0,1,2$ are defined by

$$f(x) = \frac{2-p}{4}\widehat{t}\log(1+x) - \frac{2-p}{4(1-p)}\widehat{t}\log(1-x) + \frac{(2-p)^3}{4p(1-p)}\widehat{t}\log\left(1-\frac{p}{2-p}x\right), \tag{3.1.3}$$

$$F_2(x) = -\mathbf{x}\log(1-x^2),\ F_1(x) = (v-u-\frac{1}{2}\mathbf{a})\log(1-x) - \frac{1}{2}\mathbf{a}\log(1+x),\ F_0(x) := \log 2(1+x) \tag{3.1.4}$$

where $\widehat{t} := \varepsilon^{-\frac{3}{2}}\mathbf{t}$, t and n are defined by (1.2.33), z and y' are defined above equation (1.2.35). Calculating the derivatives of $f(x)$ up to the third order, we have

$$f'(x) = \frac{x^2}{(1-x^2)(1-\frac{p}{2-p}x)}\widehat{t},\ f''(x) = \frac{2x - \frac{p}{2-p}x^2 - \frac{p}{2-p}x^4}{(1-\frac{p}{2-p}x - x^2 + \frac{p}{2-p}x^3)^2}\widehat{t},$$
$$f^{(3)}(x) = \frac{2\left(1+3x^2 - 8\frac{p}{2-p}x^3 + 3\left(\frac{p}{2-p}\right)^2 x^4 + \left(\frac{p}{2-p}\right)^2 x^6\right)}{(1-\frac{p}{2-p}x - x^2 + \frac{p}{2-p}x^3)^3}\widehat{t}. \tag{3.1.5}$$

Thus we see that $f(x)$ has the double saddle point at $x=0$,

$$f(0) = 0,\ f'(0) = 0,\ f''(0) = 0 \text{ and } f^{(3)}(0) = 2\widehat{t}. \tag{3.1.6}$$

Therefore, for small ε, $f(x)$ is expanded as

$$f(\varepsilon^{\frac{1}{2}}y) \approx \frac{\mathbf{t}}{3}y^3. \tag{3.1.7}$$

For small ε, we also have

$$\varepsilon^{-1}F_2(\varepsilon^{\frac{1}{2}}y) \approx \mathbf{x}y^2,\ \varepsilon^{-\frac{1}{2}}F_1(\varepsilon^{\frac{1}{2}}y) \approx (u-v)y,\ F_0(\varepsilon^{\frac{1}{2}}y) \approx \log 2. \tag{3.1.8}$$

Now, we see the convergence of the integration path. First, we deform C_ε to the contour $\langle_\varepsilon \cup C_\varepsilon^{\frac{\pi}{3}}$ where $\langle_\varepsilon$ is the part of Airy contour $\langle$ within the ball of radius $\varepsilon^{-\frac{1}{2}}$ centred at $\varepsilon^{-\frac{1}{2}}$, and $C_\varepsilon^{\frac{\pi}{3}}$ is the part of C_ε to the right of $\langle$. From (3.1.2), (3.1.7), and (3.1.8), we have

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\langle_\varepsilon} \varepsilon^{\frac{1}{2}}\ e^{f(\varepsilon^{\frac{1}{2}}y)+\varepsilon^{-1}F_2(\varepsilon^{\frac{1}{2}}y)+\varepsilon^{-\frac{1}{2}}F_1(\varepsilon^{\frac{1}{2}}y)+F_0(\varepsilon^{\frac{1}{2}}y)}dy = \mathbf{S}_{\mathbf{t},\mathbf{x}}(y), \tag{3.1.9}$$

where $\mathbf{S}_{\mathbf{t},\mathbf{x}}(y)$ is defined by (1.2.24). Thus the remaining part is to show that the integral over $C_\varepsilon^{\frac{\pi}{3}}$ converges to 0. To see this note that the real part of the exponent of the integral over C_ε in (3.1.1), parametrized as $y = \varepsilon^{-\frac{1}{2}}(1-e^{i\theta})$, is given by

$$\varepsilon^{-\frac{3}{2}}\mathbf{t}\left[\frac{(2-p)^3}{8p(1-p)}\log\left(1+\frac{4p(1-p)}{(2-p)^2}(\cos\theta - 1)\right) + \left(\frac{2-p}{8} + \mathcal{O}(\varepsilon^{\frac{1}{2}})\right)\log(5-4\cos\theta)\right].$$

Because the $y \in C_\varepsilon^{\frac{\pi}{3}}$ correspond to $\frac{\pi}{3} < |\theta| \leq \pi$[*1], using $\log(1+x) < x$[*2] for $x \in (-1,\infty)\setminus\{0\}$, we get

$$\varepsilon^{-\frac{3}{2}}\mathbf{t}\left[\frac{(2-p)^3}{8p(1-p)}\log\left(1+\frac{4p(1-p)}{(2-p)^2}(\cos\theta-1)\right)\right] < \frac{2-p}{2}\varepsilon^{-\frac{3}{2}}\mathbf{t}\left[\cos\theta - 1\right] \tag{3.1.10}$$

and

$$\varepsilon^{-\frac{3}{2}}\mathbf{t}\left[\frac{2-p}{8}\log(5-4\cos\theta)\right] < \frac{2-p}{2}\varepsilon^{-\frac{3}{2}}\mathbf{t}\left[1-\cos\theta\right]. \tag{3.1.11}$$

[*1]Since $\theta = 0$ corresponds to the origin $\mathbf{0} \in C_\varepsilon$ and $\langle$ is the positively oriented contour going the straight lines from $e^{-\frac{i\pi}{3}}\infty$ to $e^{\frac{i\pi}{3}}\infty$ through 0, the domain of θ can be written by this domain.

[*2]This inequality comes from $\log(1+x) \leq x$ for $x > -1$, but since $x = 0$ corresponds to $\theta = 0$ in (3.1.10) and (3.1.11), we use this inequality to correspond to the calculations of (3.1.10) and (3.1.11).

Therefore, for sufficiently small ε, the exponent there is less than $-\varepsilon^{-\frac{3}{2}}\kappa\mathbf{t}$ for some $\kappa > 0$. Hence we see that the part $C_\epsilon^{\frac{\pi}{3}}$ of the integral vanishes and this completes the proof of (1.2.35). We can also prove(1.2.36) in the similar way to (1.2.35) thus omit the proof.

For the proof of (1.2.37), we define the scaled walk $\mathbf{B}^\varepsilon(x) = \varepsilon^{\frac{1}{2}}(\mathrm{RW}_{\varepsilon^{-1}x} + 2\varepsilon^{-1}x - 1)$ for $x \in \varepsilon\mathbb{Z}_{\geq 0}$, interpolated linearly in between, and let $\boldsymbol{\tau}^\varepsilon$ be the hitting time by $\mathbf{B}^\varepsilon$ of $\mathrm{epi}(-\widehat{h}^\varepsilon(0,\cdot)^-)$, where $\widehat{h}^\varepsilon(\mathbf{t},\mathbf{x})$ is defined by (1.2.17) and $\widehat{h}^\varepsilon(\mathbf{t},\mathbf{x})^- = \widehat{h}^\varepsilon(\mathbf{t},-\mathbf{x})$. By Donsker's invariance principle [6], $\mathbf{B}^\varepsilon(x)$ converges locally uniformly in distribution to a Brownian motion $\mathbf{B}(x)$ with diffusion coefficient 2. Combining this with (1.2.28), one finds the hitting time $\boldsymbol{\tau}^\varepsilon$ converges to $\boldsymbol{\tau}$. (For more detailed proof, see Proposition 3.2 in [23]).) This leads to (1.2.37).

3.2 Proof of Proposition 1.2.16

Proposition 1.2.16 can be shown in a similar manner to Proposition 1.2.15. Here we give only the proof of (1.2.41). (1.2.42) can be obtained in a parallel way to (1.2.41) whereas (1.2.43) follows from (1.2.42) and the Donsker's invariance principle as in the case of (1.2.37) in Proposition 1.2.15. As for (3.1.1) and (3.1.2), we rewrite (1.2.44) by changing variables $w = (1-\varepsilon^{\frac{1}{2}}y)/2$,

$$\begin{aligned}(1.2.44) &= \frac{1}{2\pi i}\oint_{C_\varepsilon} \varepsilon^{\frac{1}{2}}dy \frac{(1+\varepsilon^{\frac{1}{2}}y)^n}{(1-\varepsilon^{\frac{1}{2}}y)^{n+1+z-y}}\left(1+\frac{\alpha}{2-\alpha}\varepsilon^{\frac{1}{2}}y\right)^{-t} \\ &= \frac{1}{2\pi i}\oint_{C_\varepsilon} \varepsilon^{\frac{1}{2}}\ e^{g(\varepsilon^{\frac{1}{2}}y)+\varepsilon^{-1}G_2(\varepsilon^{\frac{1}{2}}y)+\varepsilon^{-\frac{1}{2}}G_1(\varepsilon^{\frac{1}{2}}y)+G_0(\varepsilon^{\frac{1}{2}}y)}dy \end{aligned} \tag{3.2.1}$$

where C_ε is a circle of radius $\varepsilon^{-\frac{1}{2}}$ centred at $\varepsilon^{-\frac{1}{2}}$ and $g(x)$, $G_j(x)$, $j = 0, 1, 2$ are defined by

$$\begin{aligned} &g(x) = \frac{2-\alpha}{4(1-\alpha)}\widehat{t}\log(1+x) - \frac{2-\alpha}{4}\widehat{t}\log(1-x) - \frac{(2-\alpha)^3}{4\alpha(1-\alpha)}\widehat{t}\log\left(1+\frac{\alpha}{2-\alpha}x\right) \\ &G_2(x) = -\mathbf{x}\log(1-x^2),\ G_1(x) = (v-u-\frac{1}{2}\mathbf{a})\log(1-x) - \frac{1}{2}\mathbf{a}\log(1+x),\ G_0(x) = \log 2(1+x) \end{aligned} \tag{3.2.2}$$

where $\widehat{t} := \varepsilon^{-\frac{3}{2}}\mathbf{t}$, t and n are defined by (1.2.34), z and y' are defined above equation (1.2.41). Here we apply the saddle point method to (3.2.1). Noting

$$\begin{aligned} &g'(x) = \frac{x^2}{(1-x^2)(1+\frac{\alpha}{2-\alpha}x)}\widehat{t},\ g''(x) = \frac{2x+\frac{\alpha}{2-\alpha}x^2+\frac{\alpha}{2-\alpha}x^4}{(1+\frac{\alpha}{2-\alpha}x-x^2-\frac{\alpha}{2-\alpha}x^3)^2}\widehat{t}, \\ &g^{(3)}(x) = \frac{2\left(1+3x^2+8\frac{\alpha}{2-\alpha}x^3+3\left(\frac{\alpha}{2-\alpha}\right)^2x^4+\left(\frac{\alpha}{2-\alpha}\right)^2x^6\right)}{(1+\frac{\alpha}{2-\alpha}x-x^2-\frac{\alpha}{2-\alpha}x^3)^3}\widehat{t}, \end{aligned} \tag{3.2.3}$$

we find $g(x)$ has a double saddle point at $x = 0$,

$$g(0) = 0,\ g'(0) = 0,\ g''(0) = 0 \text{ and } g^{(3)}(0) = 2\widehat{t}. \tag{3.2.4}$$

Therefore, for small ε, we have

$$g(\varepsilon^{\frac{1}{2}}y) \approx \frac{\mathbf{t}}{3}y^3. \tag{3.2.5}$$

For $G_i(x),\ i=0,1,2$, we easily see

$$\varepsilon^{-1}G_2(\varepsilon^{\frac{1}{2}}y)\approx \mathbf{x}y^2,\ \varepsilon^{-\frac{1}{2}}G_1(\varepsilon^{\frac{1}{2}}y)\approx (u-v)y,\ G_0(\varepsilon^{\frac{1}{2}}y)\approx \log 2. \tag{3.2.6}$$

As discussed above (3.1.9), we divide the contour C_ε in (3.2.1) into two parts $\langle_\varepsilon \ \cup\ C_\varepsilon^{\frac{\pi}{3}}$. From (3.2.1), (3.2.4), and (3.2.6), we have

$$\lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\langle_\varepsilon}\varepsilon^{\frac{1}{2}}\ e^{g(\varepsilon^{\frac{1}{2}}y)+\varepsilon^{-1}G_2(\varepsilon^{\frac{1}{2}}y)+\varepsilon^{-\frac{1}{2}}G_1(\varepsilon^{\frac{1}{2}}y)+G_0(\varepsilon^{\frac{1}{2}}y)}dy=\mathbf{S}_{\mathbf{t},\mathbf{x}}(y), \tag{3.2.7}$$

where $\mathbf{S}_{\mathbf{t},\mathbf{x}}(y)$ is defined by (1.2.24).

Finally we show that the part coming from $C_\varepsilon^{\frac{\pi}{3}}$ vanishes as $\varepsilon\to 0$ To see this note that the real part of the exponent of the integral over C_ε in (3.2.1), parametrized as $y=\varepsilon^{-\frac{1}{2}}(1-e^{i\theta})$, is given by

$$\begin{aligned}\varepsilon^{-\frac{3}{2}}\mathbf{t}\Bigg[&\left(\frac{2-\alpha}{8(1-\alpha)}+\mathcal{O}(\varepsilon^{\frac{1}{2}})\right)\log\left(1+\frac{4(4-\alpha)(1-\alpha)}{(2-\alpha)^2+4\alpha(1-\cos\theta)}(1-\cos\theta)\right)\\ &+\left(\frac{(2-\alpha)(4-\alpha)}{8\alpha}+\mathcal{O}(\varepsilon^{\frac{1}{2}})\right)\log\left(1+\frac{4\alpha}{(2-\alpha)^2+4\alpha(1-\cos\theta)}(\cos\theta-1)\right)\Bigg].\end{aligned}$$

Note that we used an expression transform

$$\begin{aligned}&\frac{2-\alpha}{8(1-\alpha)}\log(5-4\cos\theta)-\frac{(2-\alpha)^3}{8\alpha(1-\alpha)}\log\left(1+\frac{4\alpha(1-\cos\theta)}{(2-\alpha)^2}\right)\\ &=\frac{2-\alpha}{8(1-\alpha)}\left[\log(5-4\cos\theta)-\log\left(1+\frac{4\alpha(1-\cos\theta)}{(2-\alpha)^2}\right)\right]-\frac{(2-\alpha)(4-\alpha)}{8\alpha}\log\left(1+\frac{4\alpha(1-\cos\theta)}{(2-\alpha)^2}\right)\\ &=\frac{2-\alpha}{8(1-\alpha)}\log\left(1+\frac{4(4-\alpha)(1-\alpha)}{(2-\alpha)^2+4\alpha(1-\cos\theta)}(1-\cos\theta)\right)\\ &+\frac{(2-\alpha)(4-\alpha)}{8\alpha}\log\left(1+\frac{4\alpha}{(2-\alpha)^2+4\alpha(1-\cos\theta)}(\cos\theta-1)\right).\end{aligned}$$

Because the $y\in C_\varepsilon^{\frac{\pi}{3}}$ correspond to $\frac{\pi}{3}<|\theta|\le\pi$, using $\log(1+x)<x$ for $x\in(-1,\infty)\setminus\{0\}$(See *1 and *2), we get

$$\begin{aligned}&\varepsilon^{-\frac{3}{2}}\mathbf{t}\left[\frac{2-\alpha}{8(1-\alpha)}\log\left(1+\frac{4(4-\alpha)(1-\alpha)}{(2-\alpha)^2+4\alpha(1-\cos\theta)}(1-\cos\theta)\right)\right]\\ &\quad<\frac{(2-\alpha)(4-\alpha)}{2\left\{(2-\alpha)^2+4\alpha(1-\cos\theta)\right\}}\varepsilon^{-\frac{3}{2}}\mathbf{t}\left[1-\cos\theta\right]\end{aligned}$$

and

$$\begin{aligned}&\varepsilon^{-\frac{3}{2}}\mathbf{t}\left[\frac{(2-\alpha)(4-\alpha)}{8\alpha}\log\left(1+\frac{4\alpha}{(2-\alpha)^2+4\alpha(1-\cos\theta)}(\cos\theta-1)\right)\right]\\ &\quad<\frac{(2-\alpha)(4-\alpha)}{2\left\{(2-\alpha)^2+4\alpha(1-\cos\theta)\right\}}\varepsilon^{-\frac{3}{2}}\mathbf{t}\left[\cos\theta-1\right].\end{aligned}$$

Therefore, for sufficiently small ε, the exponent there is less than $-\varepsilon^{-\frac{3}{2}}\kappa\mathbf{t}$ for some $\kappa>0$. Hence this part of the integral vanishes.

3.3 Proof of Theorem 1.2.11

By using Propositions 1.2.15 or 1.2.16, we can prove Theorem 1.2.11 as following. This proof is almost the same as [23]. First, we change variables in the kernel as in Proposition 1.2.15 (resp. Proposition 1.2.16), so that for $z_i = \frac{p(2-p)}{4(1-p)}\varepsilon^{-\frac{3}{2}}\mathbf{t} + 2\varepsilon^{-1}\mathbf{x}_i + \varepsilon^{-\frac{1}{2}}(u_i + \mathbf{a}_i) - 2$ (resp. $z_i = -\frac{\alpha(2-\alpha)}{4(1-\alpha)}\varepsilon^{-\frac{3}{2}}\mathbf{t}+2\varepsilon^{-1}\mathbf{x}_i+\varepsilon^{-\frac{1}{2}}(u_i+\mathbf{a}_i)-2$) we need to compute the limit of $\varepsilon^{-\frac{1}{2}}(\bar{\chi}_{2\varepsilon^{-1}\mathbf{x}-2}K_t\bar{\chi}_{2\varepsilon^{-1}\mathbf{x}-2})(z_i, z_j)$. Note that the change of variables turns $\bar{\chi}_{2\varepsilon^{-1}\mathbf{x}-2}(z)$ into $\bar{\chi}_{-\mathbf{a}}(u)$. We have $n_i < n_j$ for small ε if and only if $\mathbf{x}_j < \mathbf{x}_i$ and in this case we have, under our scaling,

$$\varepsilon^{-\frac{1}{2}}Q^{n_j-n_i}(z_i, z_j) \to e^{(\mathbf{x}_i-\mathbf{x}_j)\partial^2}(u_i, u_j), \tag{3.3.1}$$

as $\varepsilon \to 0$. For the second term in (1.2.4), by Proposition 1.2.15 we get

$$\begin{aligned}
\varepsilon^{-\frac{1}{2}}(S_{-t,-n_i})^*\bar{S}^{\mathrm{epi}(\mathrm{X}_0)}_{-t,n_j}(z_i, z_j) &= \varepsilon^{-\frac{1}{2}}\int_{-\infty}^{\infty} d\nu(S_{-t,-n_i})^*(z_i, \nu)\bar{S}^{\mathrm{epi}(\mathrm{X}_0)}_{-t,n_j}(\nu, z_j)\\
&= \varepsilon^{-1}\int_{-\infty}^{\infty} d\nu(S_{-t,-n_i})^*(z_i, \varepsilon^{-\frac{1}{2}}\nu)\bar{S}^{\mathrm{epi}(\mathrm{X}_0)}_{-t,n_j}(\varepsilon^{-\frac{1}{2}}\nu, z_j)\\
&= \int_{-\infty}^{\infty} d\nu(\mathbf{S}^{\varepsilon}_{-t,x_i})^*(u_i, \nu)\bar{\mathbf{S}}^{\varepsilon,\mathrm{epi}(-\mathrm{h}_0^{\varepsilon,-})}_{-t,-x_j}(\nu, u_j)\\
&= (\mathbf{S}^{\varepsilon}_{-t,x_i})^*\bar{\mathbf{S}}^{\varepsilon,\mathrm{epi}(-\mathrm{h}_0^{\varepsilon,-})}_{-t,-x_j}(u_i, u_j)\\
&\xrightarrow[\varepsilon\to 0]{} (\mathbf{S}_{-t,x_i})^*\mathbf{S}^{\mathrm{epi}(-\widehat{\mathrm{h}}_0^-)}_{-t,-x_j}(u_i, u_j).
\end{aligned} \tag{3.3.2}$$

Therefore, we have a limiting kernel

$$\mathbf{K}_{\mathrm{lim}}(x_i, u_i; x_j, u_j) = -e^{(\mathbf{x}_i-\mathbf{x}_j)\partial^2}(u_i, u_j)\mathbf{1}_{x_i>x_j} + (\mathbf{S}_{-t,x_i})^*\mathbf{S}^{\mathrm{epi}(-\widehat{\mathrm{h}}_0^-)}_{-t,-x_j}(u_i, u_j) \tag{3.3.3}$$

surrounded by projection $\bar{\chi}_{-\mathbf{a}}$. It is nicer to have projection $\chi_{\mathbf{a}}$, so we change variables $u_i \mapsto -u_i$ and replace the Fredholm determinant of the kernel by that of its adjoint to get $\det\left(\mathbf{I} - \chi_{\mathbf{a}}K^{\mathrm{hypo}(\widehat{h}_0)}_{\mathbf{t},\mathrm{ext}}\chi_{\mathbf{a}}\right)$ with $K^{\mathrm{hypo}(\widehat{h}_0)}_{\mathbf{t},\mathrm{ext}}(u_i, u_j) = \mathbf{K}_{\mathrm{lim}}(x_j, -u_j; x_i, -u_i)$.

By using $(\mathbf{S}_{\mathbf{t},\mathbf{x}})^*\mathbf{S}_{\mathbf{t},-\mathbf{x}} = I$ and $\mathbf{S}^{\mathrm{epi}(\widehat{h})}_{-\mathbf{t},\mathbf{x}}(v, u) = \mathbf{S}^{\mathrm{hypo}(-\widehat{h})}_{\mathbf{t},\mathbf{x}}(-v, -u)$ (see [23] for more information on these equations), we get the following:

$$\mathbf{K}^{\mathrm{hypo}(\widehat{h}_0)}_{\mathbf{t},\mathrm{ext}}(\mathbf{x}_i, \cdot; \mathbf{x}_j, \cdot) = -e^{(\mathbf{x}_j-\mathbf{x}_i)\partial^2}\mathbf{1}_{\mathbf{x}_i<\mathbf{x}_j} + \left(\mathbf{S}^{\mathrm{hypo}(\widehat{h}_0^-)}_{\mathbf{t},-\mathbf{x}_i}\right)^*\mathbf{S}_{\mathbf{t},\mathbf{x}_j}.$$

Appendix A

The master equation for the discrete time geometric TASEP with $N = 3$

Here, we explain (2.1.9) in more detail in the case of $N = 3$. In this case (2.1.9) can be decomposed into four terms,

$$G_{t+1}^{(\boldsymbol{\alpha})}(x_3, x_2, x_1) = G_t^{(\boldsymbol{\alpha},1)}(x_3, x_2, x_1) + G_t^{(\boldsymbol{\alpha},2)}(x_3, x_2, x_1) + G_t^{(\boldsymbol{\alpha},3)}(x_3, x_2, x_1) + G_t^{(\boldsymbol{\alpha},4)}(x_3, x_2, x_1) \tag{A.0.1}$$

where for $k_1 := x_1 - x_2$, $k_2 := x_2 - x_3$, and

$$G_t^{(\boldsymbol{\alpha},1)}(x_3, x_2, x_1) = \sum_{a_3=0}^{\infty} \sum_{a_2=0}^{k_2-2} \sum_{a_1=0}^{k_1-2} (1 - \alpha_{t+1})^3 \alpha_{t+1}^{a_1+a_2+a_3} G_t^{(\boldsymbol{\alpha})}(x_3 - a_3, x_2 - a_2, x_1 - a_1) \tag{A.0.2}$$

$$G_t^{(\boldsymbol{\alpha},2)}(x_3, x_2, x_1) = \sum_{a_3=0}^{\infty} \sum_{a_2=0}^{k_2-2} (1 - \alpha_{t+1})^2 \alpha_{t+1}^{a_2+a_3+k_1-1} G_t^{(\boldsymbol{\alpha})}(x_3 - a_3, x_2 - a_2, x_2 + 1) \tag{A.0.3}$$

$$G_t^{(\boldsymbol{\alpha},3)}(x_3, x_2, x_1) = \sum_{a_3=0}^{\infty} \sum_{a_1=0}^{k_1-2} (1 - \alpha_{t+1})^2 \alpha_{t+1}^{a_1+a_3+k_2-1} G_t^{(\boldsymbol{\alpha})}(x_3 - a_3, x_3 + 1, x_1 - a_1) \tag{A.0.4}$$

$$G_t^{(\boldsymbol{\alpha},4)}(x_3, x_2, x_1) = \sum_{a_3=0}^{\infty} (1 - \alpha_{t+1}) \alpha_{t+1}^{a_3+k_1+k_2-2} G_t^{(\boldsymbol{\alpha})}(x_3 - a_3, x_3 + 1, x_2 + 1) \tag{A.0.5}$$

The four equations (A.0.2) through (A.0.5) correspond to the case $\mu = \phi$, $\mu = \{1\}$, $\mu = \{2\}$, and $\mu = \{1, 2\}$ respectively and the situations for all the equations are illustrated in Fig. A.1(a)-(d) below.

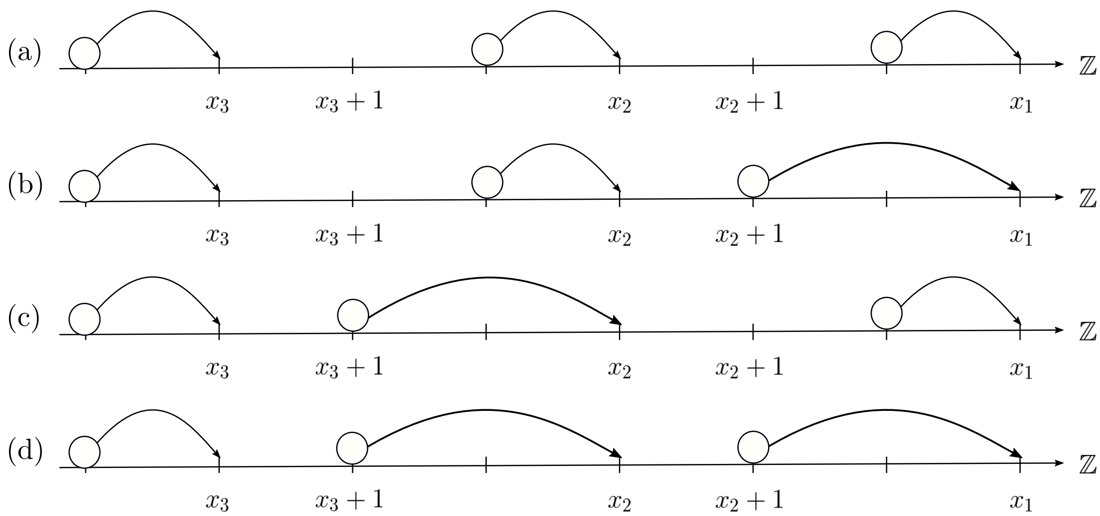

Fig. A.1: The evolutions of the geometric TASEP with 3 particles. The circles correspond to the particles and they move to the directions of arrows during time step $t \to t+1$. (a) The case $\mu = \phi$. Neither of the particles are blocked by each other. (b) The case $\mu = \{1\}$. At time t, the first particle (from the right) is at $x_2 + 1$ which leads to the blocking of the second particle. (c) The case $\mu = \{2\}$. At time t, the second particle is at $x_3 + 1$ which leads to the blocking of the third particle. (d) The case $\mu = \{1, 2\}$. At time t, the first and second particles are at $x_2 + 1$ and $x_3 + 1$ respectively which leads to the blockings of both the second and the third particles.

Appendix B

Rewriting formula with non-intersecting random walks

In this appendix, we provide that (2.1.28) can be rewritten by using the transition probabilities of non-intersecting random walks whose configurations form a Gelfand-Tsetlin pattern and distribution forms a determinantal point process with correlation kernel. Note that the content of this appendix is written in [33].

First, we start by reviewing the following proposition.

Proposition B.0.1 ([33])**.** *For $\vec{x}, \vec{y} \in \Omega_N$, we have the following identity:*

$$\mathbb{P}\left(X_t = \overrightarrow{x} \mid X_0 = \overrightarrow{y}\right) = \sum_{\mathbf{z} \in \mathrm{GT}_N : z_1^n = x_n} \det\left[\psi_{N-j}^N\left(z_i^N\right)\right]_{1 \leqslant i,j \leqslant N'} \tag{B.0.1}$$

where the functions ψ are defined in (2.2.3) *and the sum runs over the domain* GT_N *of triangular arrays given by a Gelfand-Tsetlin pattern*

$$\mathrm{GT}_N = \left\{\mathbf{z} = \left(z_i^N\right)_{n,i} : z_i^N \in \mathbb{Z}, 1 \leqslant i \leqslant n < N, z_i^{n+1} < z_i^n \leqslant z_{i+1}^{n+1}\right\}$$

with fixed values $z_1^N = x_n$ for all $N = 1, \cdots, N$.

Proof. We use only the following equation for this decomposition

$$F_{n+1}(x,t) = \sum_{y \geqslant x} F_n(y,t). \tag{B.0.2}$$

By Schütz's formula, we get

$$\mathbb{P}\left(X_t = \overrightarrow{x} \mid X_0 = \overrightarrow{y}\right) = \det\begin{bmatrix} F_0\left(z_1^N - y_N, t\right) & \cdots & F_{-N+1}\left(z_1^N - y_1, t\right) \\ \vdots & \ddots & \vdots \\ F_{N-1}\left(z_1^1 - y_N, t\right) & \cdots & F_0\left(z_1^1 - y_1, t\right) \end{bmatrix}. \tag{B.0.3}$$

Now, by (B.0.2), we have

$$\begin{aligned} \left[F_{N-1}\left(z_1^1 - y_N, t\right) \quad \cdots \quad F_0\left(z_1^1 - y_1, t\right)\right] &= \sum_{z_2^2 \geqslant z_1^1} \left[F_{N-2}\left(z_2^2 - y_N, t\right) \quad \cdots \quad F_{-1}\left(z_2^2 - y_1, t\right)\right] \\ &= \sum_{z_2^2 \geqslant z_1^1} \sum_{z_3^3 \geqslant z_2^2} \left[F_{N-3}\left(z_3^3 - y_N, t\right) \quad \cdots \quad F_{-2}\left(z_3^3 - y_1, t\right)\right]. \end{aligned} \tag{B.0.4}$$

Applying (B.0.2) to the penultimate row in (B.0.3) we derive

$$\sum_{z_2^3 \geqslant z_1^2} \left[F_{N-3}\left(z_2^3 - y_N, t\right) \quad \cdots \quad F_{-2}\left(z_2^3 - y_1, t\right) \right]. \tag{B.0.5}$$

By (B.0.4) and (B.0.5), we obtain

$$(\text{B.0.3}) = \sum_{z_2^2 \geqslant z_1^1} \sum_{z_3^3 \geqslant z_2^2} \sum_{z_2^3 \geqslant z_1^2} \det \begin{bmatrix} F_0\left(z_1^N - y_N, t\right) & \cdots & F_{-N+1}\left(z_1^N - y_1, t\right) \\ \vdots & \ddots & \vdots \\ F_{N-3}\left(z_1^3 - y_N, t\right) & \cdots & F_{-2}\left(z_1^3 - y_1, t\right) \\ F_{N-3}\left(z_2^3 - y_N, t\right) & \cdots & F_{-2}\left(z_2^3 - y_1, t\right) \\ F_{N-3}\left(z_3^3 - y_N, t\right) & \cdots & F_{-2}\left(z_3^3 - y_1, t\right) \end{bmatrix}. \tag{B.0.6}$$

Because the determinant is antisymmetric in the variables z_2^3 and z_3^3,

$$(\text{B.0.6}) = \sum_{z_2^2 \geqslant z_1^1} \sum_{z_3^3 \geqslant z_2^2} \sum_{z_2^3 \in \left[z_1^2, z_2^2\right)} \det \begin{bmatrix} F_0\left(z_1^N - y_N, t\right) & \cdots & F_{-N+1}\left(z_1^N - y_1, t\right) \\ \vdots & \ddots & \vdots \\ F_{N-3}\left(z_1^3 - y_N, t\right) & \cdots & F_{-2}\left(z_1^3 - y_1, t\right) \\ F_{N-3}\left(z_2^3 - y_N, t\right) & \cdots & F_{-2}\left(z_2^3 - y_1, t\right) \\ F_{N-3}\left(z_3^3 - y_N, t\right) & \cdots & F_{-2}\left(z_3^3 - y_1, t\right) \end{bmatrix}. \tag{B.0.7}$$

By repeating the same procedure, we have the formula

$$\mathbb{P}\left(X_t = \overrightarrow{x} \mid X_0 = \overrightarrow{y}\right) = \sum_{\mathbf{z} \in \mathrm{GT}_N : z_1^n = x_n} \det \left[F_{1-j}\left(z_i^N - y_{N+1-j}, t\right)\right]_{1 \leqslant i,j \leqslant N}. \tag{B.0.8}$$

Here, by using the identity

$$\psi_k^N(x) = (-1)^k F_{-k}\left(x - y_{N-k}, t\right), \tag{B.0.9}$$

we get

$$\begin{aligned} \det \left[F_{1-j}\left(z_i^N - y_{N+1-j}, t\right)\right]_{1 \leqslant i,j \leqslant N} &= \det \left[(-1)^{j-1} \psi_{j-1}^N\left(z_i^N\right)\right]_{1 \leqslant i,j \leqslant N} \\ &= (-1)^{(1+2+\cdots+N)-N} \det \left[\psi_{j-1}^N\left(z_i^N\right)\right]_{1 \leqslant i,j \leqslant N}. \end{aligned}$$

Also, we transform the formula as the following:

$$\det \left[\psi_{j-1}^N\left(z_i^N\right)\right]_{1 \leqslant i,j \leqslant N} = (-1)^{\lfloor N/2 \rfloor} \det \left[\psi_{N-j}^N\left(z_i^N\right)\right]_{1 \leqslant i,j \leqslant N}. \tag{B.0.10}$$

Since it is easy to see that $(1 + 2 + \cdots + N) - N + \lfloor N/2 \rfloor$ is an even integer, we get

$$\det \left[F_{1-j}\left(z_i^N - y_{N+1-j}, t\right)\right]_{1 \leqslant i,j \leqslant N} = \det \left[\psi_{N-j}^N\left(z_i^N\right)\right]_{1 \leqslant i,j \leqslant N}.$$

□

Now, note that the weight of a configuration $\mathbf{z} \in \mathrm{GT}_N$ in (B.0.1) is given by

$$\mathrm{W}_N(\mathbf{z}) = \left(\prod_{n=1}^{N} \det \left[\phi\left(z_i^{n-1}, z_j^n\right)\right]_{1 \leqslant i,j \leqslant n} \right) \det \left[\psi_{N-j}^N\left(z_i^N\right)\right]_{1 \leqslant i,j \leqslant N}. \tag{B.0.11}$$

where $\phi(x,y)=1_{\{x>y\}}$ and $z_n^{n-1}=+\infty$. Then for $\mathbf{z}=(z_i^n)_{n,i}\in\Lambda_N$ we get

$$\prod_{n=1}^{N}\det\left[\phi\left(z_i^{n-1},z_j^n\right)\right]_{1\leqslant i,j\leqslant n}=\mathbf{1}_{\{\mathbf{z}\in\mathrm{GT}_N\}}\tag{B.0.12}$$

where Λ_N is defined in (2.2.4). By (B.0.1), (B.0.11) and (B.0.12), we derive

$$\mathbb{P}\left(X_t=\overrightarrow{x}\mid X_0=\overrightarrow{y}\right)=\sum_{\substack{\mathbf{z}\in\Lambda_N\\ z_1^n=x_n}}\mathrm{W}_N.\tag{B.0.13}$$

From the above, we have been understood that (2.1.28) can be rewritten by the transition probabilities of non-intersecting random walks whose configurations form a Gelfand-Tsetlin pattern. Thus, we prove that the measure W_N is a determinantal point process. First, we show that the measure W_N is a L-ensemble. Let $\mathfrak{X}$ be a discrete space. Also, we put $\mathfrak{Z}\subset\mathfrak{X}$ such that $\mathfrak{X}=\{1,2,\ldots,N\}\cup\mathfrak{Z}$. Then we define $L:\mathfrak{X}\times\mathfrak{X}\to\mathbb{R}$ such that

$$L_{\{\mathbf{z}\}\cup\mathfrak{Z}^c}=\begin{bmatrix}0&E_0&E_1&E_2&\ldots&E_{N-1}\\0&0&-W_{[1,2)}&0&\cdots&0\\0&0&0&-W_{[2,3)}&\ddots&\vdots\\\vdots&\vdots&\vdots&\ddots&\ddots&0\\0&0&0&0&\cdots&-W_{[N-1,N)}\\\psi^{(N)}&0&0&0&\cdots&0\end{bmatrix}(\mathbf{z})\tag{B.0.14}$$

where $\mathbf{z}=(z_i^n)_{n,i}\in\Lambda_N$, $W_{[n,m)}(\mathbf{z})$, $\psi^{(N)}$ and E_m are defined in (2.2.5)-(2.2.7). Here we set the square matrices

$$[T_m]_{i,j}=\phi\left(z_i^m,z_j^{m+1}\right),\quad 1\leqslant i,j\leqslant m+1.\tag{B.0.15}$$

Then we obtain

$$(\text{B.0.14})=\begin{bmatrix}0&T_0&0&0&\ldots&0\\0&0&T_1&0&\cdots&0\\0&0&0&T_2&\ddots&\vdots\\\vdots&\vdots&\vdots&\ddots&\ddots&0\\0&0&0&0&\cdots&T_{N-1}\\\psi^{(N)}(\mathbf{z})&0&0&0&\cdots&0\end{bmatrix}.\tag{B.0.16}$$

Therefore we have

$$\mathrm{W}_N(\mathbf{z})=\det\left(L_{\{\mathbf{z}\}\cup\mathfrak{Z}^c}\right)\tag{B.0.17}$$

which means that W_N is a conditional L-ensemble. By Proposition 3.24 in [33], the measure W_N is a determinantal point process with correlation kernel.

Appendix C

Correlation kernel

Because we found that the measure W_N is a determinant point process in Appendix B, we derive its correlated kernel in this appendix whose content is written in [33].

Let $\mathfrak{X}$ be a discrete space. Then, we set $\mathfrak{Z} \subset \mathfrak{X}$ such that $\mathfrak{X} = \{1, 2, \ldots, N\} \cup \mathfrak{Z}$. By Proposition 3.24 in [33], the measure W_N on $\mathfrak{Z}$ is the determinantal point process with correlation kernel $K_t : \mathfrak{Z} \times \mathfrak{Z} \to \mathbb{R}$ written by

$$K_t = 1_{\mathfrak{Z}} - (1_{\mathfrak{Z}} + L)^{-1}\Big|_{\mathfrak{Z}\times\mathfrak{Z}} \tag{C.0.1}$$

where L is defined in (B.0.14). Then we can rewrite L as

$$L = \begin{bmatrix} 0 & B \\ C & D_0 \end{bmatrix}. \tag{C.0.2}$$

Here in RHS,

$$B = [E_0, \ldots, E_{N-1}] : \Lambda_N \to M_{N,N(N+1)/2}, \quad C = \left[0, \ldots, 0, \left(\psi^{(N)}\right)'\right]' : \Lambda_N \to M_{N(N+1)/2,N}$$

and $D_0 : \Lambda_N \to M_{N(N+1)/2,N(N+1)/2}$ such that

$$D_0 = \begin{bmatrix} 0 & -W_{[1,2)} & 0 & \cdots & 0 \\ 0 & 0 & W_{[2,3)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -W_{[N-1,N)} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $W_{[n,m)}$, $\psi^{(N)}$ and E_m are defined in (2.2.5)-(2.2.7). Now we put

$$D = \begin{bmatrix} 1 & W_{[1,2)} & 0 & \cdots & 0 \\ 0 & 1 & -W_{[2,3)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & -W_{[N-1,N)} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then the following lemma holds.

Lemma C.0.1. *The operators D and $M = BD^{-1}C$ are invertible. Moreover we can write the correlation kernel K_t given by* (C.0.1) *as*

$$K_t = 1_3 - D^{-1} + D^{-1}CM^{-1}BD^{-1}. \tag{C.0.3}$$

Proof. We prove only (C.0.3) because it is easy to see that D and $M = BD^{-1}C$ are invertible. Note that the following holds:

$$\begin{aligned}
&(1_3 + L)\begin{bmatrix} -M^{-1} & M^{-1}BD^{-1} \\ D^{-1}CM^{-1} & D^{-1} - D^{-1}CM^{-1}BD^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}\begin{bmatrix} -M^{-1} & M^{-1}BD^{-1} \\ D^{-1}CM^{-1} & D^{-1} - D^{-1}CM^{-1}BD^{-1} \end{bmatrix} \\
&= \begin{bmatrix} BD^{-1}CM^{-1} & BD^{-1} - BD^{-1}CM^{-1}BD^{-1} \\ -CM^{-1} + CM^{-1} & CM^{-1}BD^{-1} + 1 - CM^{-1}BD^{-1} \end{bmatrix} \\
&= \begin{bmatrix} MM^{-1} & BD^{-1} - MM^{-1}BD^{-1} \\ -CM^{-1} + CM^{-1} & CM^{-1}BD^{-1} + 1 - CM^{-1}BD^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Therefore we get

$$(1_3 + L)^{-1} = \begin{bmatrix} -M^{-1} & M^{-1}BD^{-1} \\ D^{-1}CM^{-1} & D^{-1} - D^{-1}CM^{-1}BD^{-1} \end{bmatrix}.$$

From the above we have (C.0.3). □

Now, note that we can easily check that

$$D^{-1} = (1 + D_0)^{-1} = \sum_{k\geqslant 0} D_0^k = \begin{bmatrix} 1 & W_{[1,2)} & \cdots & W_{[1,N)} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & W_{[N-1,N)} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D^{-1}C = \begin{bmatrix} W_{[1,N)}\psi^{(N)} \\ \vdots \\ W_{[N-1,N)}\psi^{(N)} \\ \psi^{(N)} \end{bmatrix}$$

and

$$BD^{-1} = \begin{bmatrix} E_0 & E_0W_{[1,2)} + E_1 & \cdots & \sum_{k=1}^{N-1} E_{k-1}W_{[k,N)} + E_{N-1} \end{bmatrix}.$$

Form the above we derive the (n, m) -block of the correlation kernel

$$[K_t]_{(n,\cdot),(m,\cdot)} = -W_{[n,m)}\mathbf{1}_{\{n<m\}} + W_{[n,N)}\psi^{(N)}M^{-1}\left(\sum_{k=1}^{m-1} E_{k-1}W_{[k,m)} + E_{m-1}\right).$$

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