Mathematical Analysis of Optimal Control Problems in Phase Transitions

January 2022

Shodai Kubota

Graduate School of
Science and Engineering

CHIBA UNIVERSITY

(千葉大学審査学位論文)

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Contents

Acknowledgments

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Chapter 1

Introduction

1.1 Optimal control problem

Recently, the focuses of scientific and technological researches have been on predictions and controls of physical phenomena (or technological process), based on the scientific understanding. In view of such background, this thesis is devoted to the mathematical analysis of optimal control problems governed by phase transition phenomena.

In general, the optimal control problem is known as a minimization problem for the following type of functional, called *cost*:

$$\mathcal{J} : u \in \mathscr{U}_{\mathrm{ad}} \mapsto \mathcal{J}(u) := \mathcal{M}(w, w_{\mathrm{ad}}) + \mathcal{J}_0(u) \in (-\infty, \infty], \text{ with } w = \mathcal{S}u \text{ in } \mathfrak{X}, \tag{1.1.1}$$

and the cost is defined on an admissible (constrained) class $\mathscr{U}_{\mathrm{ad}}$ in a topological vector space $\mathfrak{X}$. The minimizers of the cost are collectively called *optimal control*.

In the optimal control problem, the cost is usually considered with a phenomenon which is aimed to be controlled, and the mathematical model of the phenomenon is called *state-system*. In the context, $u \in \mathfrak{X}$ is a variable, called *control*, which is associated with a controllable factor, such as forcing terms in the state-system. $\mathcal{S} : \mathfrak{X} \longrightarrow \mathfrak{X}$ is the solution operator in the state-system, i.e. $\mathcal{S}$ maps any control $u \in \mathfrak{X}$ to the solution $w = \mathcal{S}u \in \mathfrak{X}$ to the state-system. Also, $w_{\mathrm{ad}} \in \mathfrak{X}$ is a fixed *admissible target profile*, which is desired to be realized via the controlling process. The functional:

$$\mathcal{M} : w \in \mathfrak{X} \mapsto \mathcal{M}(w, w_{\mathrm{ad}}) \in (-\infty, \infty],$$

is a convex function on $\mathfrak{X}$ which is to measure the *gap* between the admissible target profile $w_{\mathrm{ad}} \in \mathfrak{X}$, and the profile w of the solution to the state-system. Meanwhile, the functional:

$$\mathcal{J}_0 : u \in \mathfrak{X} \mapsto \mathcal{J}_0(u) \in (-\infty, \infty],$$

is another convex function on $\mathfrak{X}$ which is to indicate a generation cost $\mathcal{J}_0$ spent for $u \in \mathfrak{X}$, such as energy, economical cost, and so on.

From the definition (1.1.1), we can figure out that the optimal control, i.e. minimizer of $\mathcal{J}$ can be said as the equilibrium point of the trade-off relationship between the smallness of the gap $\mathcal{M}(\cdot, w_{\mathrm{ad}})$ from the target profile, and the smallness of the general cost $\mathcal{J}_0(\cdot)$.

So, mathematically, it would be easy to deal with if the state-system, i.e. the solution operator $\mathcal{S}$ is linear and smooth. In fact, if $\mathcal{S}$ is linear and smooth, then since we can suppose the convexity and smoothness of the cost $\mathcal{J}$, we can make a forecast for the mathematical analysis on the basis of the standard theory of convex analysis.

Meanwhile, in the field of optimal control problem, there are a lot of previous works, which adapted PDE models of *phase transitions* as the state-systems (cf. [4–6, 20, 22, 23, 28, 29, 34, 45–47, 57, 58, 62, 66–68, 72, 75, 77]). Here, "phase transition" is a collective word of dramatic changes (transitions) between some different situations (phases), and a lot of physical phenomena/technological processes belong to the category of phase transition: e.g. evaporation-melting-freezing of material, spinodal decomposition of alloy, grain boundary motion of polycrystal, tumor growth, and so on. However, since most of phase transition models include some nonlinearity, the standard theory of convex analysis would not be applicable to the associated optimal control problems, any more. Indeed, in the line of previous works, the researchers racked their brains for a rigorous theory to treat nonconvexity of the cost, caused by the nonlinearity of each phenomenon. Furthermore, in some phase transitions, such as grain boundary in polycrystal, the PDE model often contains singular term to reproduce *discontinuity* of spatial phase-distribution, and such singularity brings the non-smoothness of the associated optimization cost. Hence, in the optimization (minimization) of nonconvex and nonsmooth cost, the mathematicians still have a lot of open issues and the developments of new mathematical methods are needed as theoretical backbones of advanced sciences and technologies.

Based on these, we focus on the following three nonlinear and singular PDE models of phase transitions, as our state-systems:

(0) Micro-mesoscopic Allen–Cahn equation.

(I) Micro-mesoscopic solid-liquid phase transition models of Fix–Caginalp type.

(II) Kobayashi–Warren–Carter type model of grain boundary motion.

The derivations and mathematical formulas of the phase transition models (0)–(II) are demonstrated in the following Sections 1.2–1.4, respectively.

1.2 Micro-mesoscopic Allen–Cahn equation with singularities

Allen–Cahn equation is known as a PDE model of solid-liquid phase transition, proposed by S. M. Allen and J. W. Cahn [2]. The unknown variable is a kind of *order parameter*, and the order parameter of Allen–Cahn equation is supposed to reproduce the spatial-time distribution of solid-liquid phases.

In what follows, we fix a spatial dimension $N \in \mathbb{N}$ and a constant of time $T > 0$, and we suppose the solid-liquid phase transition takes place in a product space $Q := (0, T) \times \Omega$ of a time-interval $(0, T)$, and a bounded domain $\Omega \subset \mathbb{R}^N$ with a smooth boundary $\Gamma := \partial\Omega$.

Also, we denote by $w = w(t,x)$ the order parameter of Allen–Cahn equation, and assign the value as follows:

$$\begin{cases} w(t,x) = 1, & \text{if the phase is liquid,} \\ w(t,x) = -1, & \text{if the phase is solid,} \\ -1 < w(t,x) < 1, & \text{if the phase is intermediate,} \end{cases} \quad \text{at } (t,x) \in Q. \tag{1.2.1}$$

On this basis, the Allen–Cahn equation is derived as the gradient flow of a governing functional $\mathscr{F} : L^2(\Omega) \longrightarrow (-\infty, \infty]$, called *free energy*. Roughly summarized, the free energy is defined in a form of a sum:

$$w \in L^2(\Omega) \mapsto \mathscr{F}(w) := \mathscr{F}_B(w) + \mathscr{F}_I(w) \in (-\infty, \infty],$$

of two functionals $\mathscr{F}_B : L^2(\Omega) \longrightarrow (-\infty, \infty]$ and $\mathscr{F}_I : L^2(\Omega) \longrightarrow (-\infty, \infty]$, which are called the *bulk energy* and *interfacial energy*, respectively.

The bulk energy $\mathscr{F}_B = \mathscr{F}_B(w)$ is to reproduce the bistability of solid-liquid phases, and it is defined as:

$$w \in L^2(\Omega) \mapsto \mathscr{F}_B(w) := \int_\Omega W_B(w)\, dx \in (-\infty, \infty],$$

with use of a *double-well* function $W_B : \mathbb{R} \longrightarrow (-\infty, \infty]$, such as:

(# 0) **(polynomial type)** $W_B(w) = \dfrac{c}{4}(w^4 - 2w^2)$, for $w \in \mathbb{R}$,

(# 1) **(regular constraint type)** $W_B(w) = (1+w)\log(1+w) + (1-w)\log(1-w) - \dfrac{c}{2}w^2$, for $w \in \mathbb{R}$,

(# 2) **(singular constraint type)** $W_B(w) = I_{[-1,1]}(w) - \dfrac{c}{2}w^2$, for $w \in \mathbb{R}$;

where $c > 0$ is a constant, and $I_{[-1,1]}$ is the indicator function on the closed interval $[-1,1]$, i.e.:

$$w \in \mathbb{R} \mapsto I_{[-1,1]}(w) := \begin{cases} 0, & \text{if } w \in [-1,1], \\ \infty, & \text{otherwise .} \end{cases}$$

Incidentally, the interest in this thesis is in the Allen–Cahn equation, adopting the double-well function of (# 2) (singular constraint type).

On the other hand, the interfacial energy $\mathscr{F}_I = \mathscr{F}_I(w)$ is to control the motion of the *interface*, i.e. the free boundary between solid-liquid phases, and in this thesis we focus on the interfacial energy, defined as:

$$w \in L^2(\Omega) \mapsto \mathscr{F}_I(w) := \begin{cases} \kappa \displaystyle\int_\Omega |Dw| + \frac{\nu^2}{2} \int_\Omega |\nabla w|^2\, dx, \\ \qquad \text{if the both integrals make sense,} \\ \infty, \ \text{otherwise,} \end{cases} \tag{1.2.2}$$

where $\kappa \geq 0$ and $\nu \geq 0$ are fixed constants, and $\int_\Omega |Dw|$ denotes the *total variation* of each function $w \in L^2(\Omega)$. The setting (1.2.2) can be said as a mixed interfacial energy under the *microscopic* and *mesoscopic settings*, which were classified by A. Visintin [85, Chapter VI].

For instance, when $\kappa = 0$ and $\nu > 0$, the setting (1.2.2) defines the microscopic interfacial energy as in [85, Chapter VI], and also, it coincides with the original setting by Allen–Cahn [2]. Incidentally, the phrase "microscopic" is derived from the property, brought by $\nu > 0$, such that the solution to Allen–Cahn equation admits only spatially smooth phase change, as in the microscopic scale.

Meanwhile, when $\kappa > 0$ and $\nu = 0$, the setting (1.2.2) provides the mesoscopic interfacial energy, proposed by Visintin [85, Chapter VI]. Here, the phrase "mesoscopic" implies the property, brought by $\nu = 0$, such that the solution to Allen–Cahn equation admits the both situations: the discontinuous phase change as in the macroscopic scale; and the smooth one as in the microscopic scale. In addition, Visintin suggests that the mesoscopic interfacial energy brings the singular diffusion $-\kappa \mathrm{div}\bigl(\frac{Dw}{|Dw|}\bigr)$ in the mathematical expression of the corresponding Allen–Cahn equation.

Otherwise, when $\kappa = \nu = 0$, the setting (1.2.2) is classified as the *macroscopic setting* of interfacial energy (cf. [85, Chapter VI]).

In view of these, we call the Allen–Cahn equation under { (#2), (1.2.2) } *micro-mesoscopic Allen–Cahn equation.* Mathematically, the micro-mesoscopic Allen–Cahn type equation is described in a form of quasilinear PDE:

$$\begin{cases} \partial_t w - \mathrm{div}\left(\kappa \dfrac{Dw}{|Dw|} + \nu^2 \nabla w\right) + \partial I_{[-1,1]}(w) \ni cw + u(t,x), \quad (t,x) \in Q, \\ \text{(B.C.)} + \text{(I.C.)}, \end{cases} \tag{1.2.3}$$

where $u = u(t,x)$ is the given relative temperature with zero-degree of *critical temperature,* e.g. melting/freezing point. We note that our Allen–Cahn type equation (1.2.3) contains the following singularities:

- the singularity as in the diffusion term $-\mathrm{div}\left(\kappa \frac{Dw}{|Dw|} + \nu^2 \nabla w\right)$;
- the singularity of the set-valued subdifferential $\partial I_{[-1,1]}$ of the indicator function $I_{[-1,1]}$, i.e.:

$$w \in \mathbb{R} \mapsto \partial I_{[-1,1]}(w) := \begin{cases} 0, & \text{if } w \in (-1,1), \\ [0,\infty), & \text{if } w = 1, \\ (-\infty, 0], & \text{if } w = -1, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{1.2.4}$$

Also, we note the set-valued term forces to use the notation "$\ni$" of set theory instead of equality "$=$" of the equation (1.2.3).

Now, in this thesis, we consider a class of optimal control problems governed by:

- time-discrete scheme of the micro-meso Allen–Cahn equation (1.2.3) under one-dimensional setting of Ω (in Chapter 3).

1.3 Micro-mesoscopic solid-liquid phase transition models of Fix–Caginalp type

Almost simultaneously, a modeling method of solid-liquid phase transition was proposed by G. J. Fix [27] (1983) and G. Caginalp [19] (1986). The proposed mathematical model is described as a system of PDEs, and it is based on the *weak formulation of Stefan problem:*

$$\partial_t(u + Lw) + \Delta u = f(t, x), \ \ (t, x) \in Q, \tag{1.3.1}$$

$$u \in \partial I_{[-1,1]}(w), \quad \text{in } Q, \tag{1.3.2}$$

subject to suitable initial-boundary condition. In the context, $u = u(t, x)$ is the relative temperature with the zero-degree of critical temperature, and $w = w(t, x)$ is the order parameter as in (1.2.1). $\partial I_{[-1,1]}$ is the subdifferential of the indicator function $I_{[-1,1]}$ on the closed interval $[-1, 1]$ as in (1.2.4). Besides, $L > 0$ is a fixed constant, and $f = f(t, x)$ is a given heat source.

The first equation (1.3.1) is a kinetic equation of heat exchange, and second one (1.3.2) is to reproduce the *temperature-phase rule:*

$$w(t,x) \begin{cases} = 1 \text{ (liquid)}, & \text{if } u(t,x) > 0, \\ = -1 \text{ (solid)}, & \text{if } u(t,x) < 0, \ \text{ for } (t,x) \in Q. \\ \in (-1,1) \text{ (intermediate)}, & \text{if } u(t,x) = 0, \end{cases} \tag{1.3.3}$$

In the first equation (1.3.1), the term

$$e := u + Lw \tag{1.3.4}$$

is called *enthalpy*, i.e. the heat quantity, including the effect of *latent heat*. Indeed, from (1.3.3) and (1.3.4), we can observe that the enthalpy e and temperature u are governed by the relationship as in Figures 1.1 and 1.2, and in particular, Figure 1.2 reproduces the effect of *heat of melting (resp. heat of solidification)* in the melting (resp. solidification) process, with the constant L of *latent heat*.

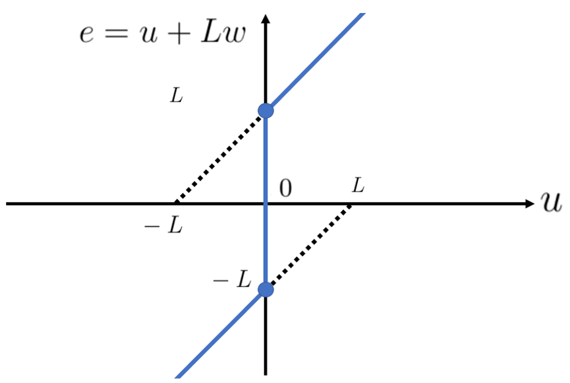

Figure 1.1:

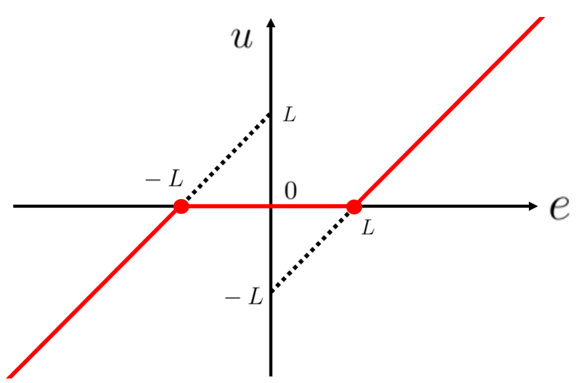

Figure 1.2:

Furthermore, it should be noted that the *Stefan condition* is contained in the first equation (1.3.1), in a weak sense. More precisely, in the original Stefan problem, it is supposed that the free boundary S is a surface, and S is formed iff. the temperature u is at the critical temperature $u = 0$, i.e.:

$$S = u^{-1}(0) = \left\{ \ (t, x) \in Q \ \middle| \ u(t, x) = 0 \ \right\}.$$

Under such situation, the Stefan condition can be derived, in the distributional sense, by applying the integration by part onto the open set $Q \setminus S$.

The modeling idea of Fix–Caginalp is to describe the time-evolution of the phase-field dynamics, by replacing the temperature-phase rule (1.3.2) (or (1.3.3)) with the Allen–Cahn equation. In fact, the original model of Fix–Caginalp [19, 27] is based on the case when:

- the setting of bulk energy $\mathscr{F}_B$ is given by using the double-well function W_B as in (# 0) (polynomial type);
- the setting of interfacial energy $\mathscr{F}_I$ is given by (1.2.2) with $\kappa = 0$ and $\nu > 0$ (microscopic setting).

Now, in this thesis, we deal with a mathematical model, which corresponds to the case when:

- the setting of bulk energy $\mathscr{F}_B$ is given by using the double-well function W_B as in (# 2) (singular constraint type);
- the setting of interfacial energy $\mathscr{F}_I$ is given by (1.2.2) (micro-mesoscopic setting).

Then, the mathematical model can be called as *micro-meso solid-liquid phase transition model of Fix–Caginalp type*, and as a system of PDEs, it is formulated as follows:

$$\begin{cases} \partial_t(u + Lw) - \Delta u = f(t, x), \quad (t, x) \in Q, \\ \partial_t w - \mathrm{div}\left(\kappa \dfrac{Dw}{|Dw|} + \nu^2 \nabla w\right) + \partial I_{[-1,1]}(w) \ni cw + u \quad \text{in } Q, \\ \text{(B.C.)} + \text{(I.C.)}. \end{cases} \tag{1.3.5}$$

We note that the mathematical model (1.3.5) can be regarded as a generalized time-evolving version of the weak formulation of Stefan problem. Additionally, we stress that our mathematical model is to inherent the singularities of the micro-meso Allen–Cahn equation, as in the previous Section 1.2.

Now, in this thesis, we consider a class of optimal control problems governed by:

- micro-mesoscopic Fix–Caginalp type (1.3.5) under one-dimensional setting of Ω (in Chapter 4).

1.4 Kobayashi–Warren–Carter type model of grain boundary motion

In recent years, the phase-field models of grain boundary motions were proposed by by R. Kobayashi, J. A. Warren, and W. C. Carter [46, 47]. The line of the phase-field models are collectively called *Kobayashi–Warren–Carter type model*, or *K.W.C. model* in short.

In the K.W.C. model, we consider a situation such that in a time-interval $(0, T)$ with a constant $T > 0$, a spatial domain $\Omega \subset \mathbb{R}^N$, with a spatial dimension $N \in \mathbb{N}$, is occupied by a polycrystal, like Ceramics. Figure 1.3 in the below is to see such situation

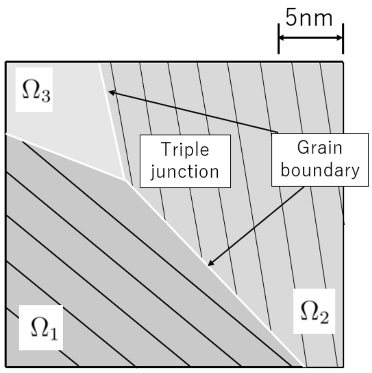

Figure 1.3: Sample illustration

As is illustrated in Figure 1.3, the polycrystal is supposed to be divided by a number of small masses Ω_1, Ω_2, and Ω_3 of crystals, called *grains*. Also, some grains Ω_1 and Ω_2 show the characteristic phases of orientations by stripe patterns, and the other one Ω_3 has no such phase. The former phase is called *oriented phase*, and the latter one is called *disoriented phase*. The free boundary between these phases are called *grain boundary*.

Based on this, the K.W.C. model is described as a system of parabolic PDEs with two unknown order parameters, denoted by η and θ. Also, the crystalline orientation is reproduced by using a vectorial function consisting of η, $\cos\theta$ and $\sin\theta$, as follows:

$$\varpi = \begin{bmatrix} \eta\cos\theta \\ \eta\sin\theta \end{bmatrix}.$$

Here, η is the length of this vector, and in physical, it indicates the orientation order of grain. η is supposed to satisfy the range constraint property on the closed interval $[0, 1]$. More precisely, if η is equal to 1, then it means the completely oriented phase. Also, if η is equal to 0, then it means the disoriented phase. Besides, if η holds $0 < \eta < 1$, then it means the intermediate phase. On the other hand, θ is the angle of the crystalline orientation.

Based on these, the model of K.W.C. is derived as a gradient flow of the following function $\mathscr{F}$ on $[L^2(\Omega)]^2$, called *free energy*:

$$[\eta, \theta] \in [L^2(\Omega)]^2 \mapsto \mathscr{F}(\eta, \theta) := \Psi(\eta) + \Phi(\eta, \theta);$$

which is defined as a sum of two functionals $\Psi : L^2(\Omega) \longrightarrow [0, \infty]$ and $\Phi : [L^2(\Omega)]^2 \longrightarrow [0, \infty]$, defined as follows:

$$\eta \in L^2(\Omega) \mapsto \Psi(\eta) := \begin{cases} \frac{1}{2}\int_\Omega |\nabla \eta|^2 \, dx + \frac{1}{2}\int_\Omega G(\eta)\, dx, & \text{if } \eta \in H^1(\Omega), \\ \infty, & \text{otherwise,} \end{cases} \tag{1.4.1}$$

and

$$[\eta, \theta] \in [L^2(\Omega)]^2 \mapsto \Phi(\eta, \theta) := \begin{cases} \int_\Omega \alpha(\eta)|\nabla \theta| \, dx + \frac{\nu^2}{2}\int_\Omega |\nabla \theta|^2 \, dx, \\ \qquad \text{if } [\eta, \theta] \in L^2(\Omega) \times H^1(\Omega), \\ \infty, \ \text{otherwise.} \end{cases} \tag{1.4.2}$$

Here, Ψ is to describe the dynamics of η, with a fixed (smooth) function $G : \mathbb{R} \longrightarrow [0, \infty)$, and it consists of the potential of Laplacian $-\Delta \eta$ of η, and the potential $\int_\Omega G(\eta)\, dx$ of the perturbation of η. On the other hand, Φ is called *interfacial energy*, and it is to describe the dynamics of the grain boundary interfaces. The interfacial energy is given by the sum of the total variation $\int_\Omega \alpha(\eta)|D\theta|$ of θ with unknown dependent weight $\alpha(\eta)$, and the relaxation term $\frac{\nu^2}{2}\int_\Omega |\nabla \theta|^2 \, dx$ of θ with a fixed positive constant ν^2. The weight $\alpha : \mathbb{R} \longrightarrow (0, \infty)$ is called *mobility* and it causes the interaction between η and θ.

Now, K.W.C. model is derived as a gradient systems with respect to η and θ, as follows:

$$\begin{cases} -\partial_t \eta = \nabla_\eta \mathscr{F}(\eta, \theta) \ \text{ in } Q := (0, T) \times \Omega, \\ -\alpha_0(t, x, \eta)\partial_t \theta = \nabla_\theta \mathscr{F}(\eta, \theta) \ \text{ in } Q, \\ \text{(B.C.)} + \text{(I.C.)}. \end{cases} \tag{1.4.3}$$

Here, due to the non-smoothness of the total variation, the equation for θ has some singularities. In addition, this system has another weight $\alpha_0(t, x, \eta)$ in front of the time-derivative of θ. This weight $\alpha_0 : Q \times \mathbb{R} \longrightarrow (0, \infty)$ is called the *mobility for time-variation*, and it can be different with the spatial mobility $\alpha(\eta)$. Incidentally, in the original paper of Kobayashi–Warren–Carter [46], α and α_0 are given by the same quadratic function as follows:

$$\alpha(\eta) = \alpha_0(t, x, \eta) = \frac{1}{2}\eta^2 + \delta_\alpha \text{ for all } (t, x) \in Q \text{ and } \eta \in \mathbb{R}, \text{ with a constant } \delta_\alpha \geq 0.$$

In view of such background, we set the following system of PDEs, as a state-system of our optimal control problem:

$$\begin{cases} \partial_t \eta - \Delta \eta + g(\eta) + \alpha'(\eta)|\nabla \theta| = u(t, x), \quad (t, x) \in Q, \\ \alpha_0(t, x)\partial_t \theta - \mathrm{div}\left(\alpha(\eta)\frac{D\theta}{|D\theta|} + \nu^2 \nabla \theta\right) = v(t, x), \quad (t, x) \in Q, \\ \text{(B.C.)} + \text{(I.C.)}. \end{cases} \tag{1.4.4}$$

The system (1.4.4) can be said as a modified version of K.W.C. model (1.4.3). In the context, $g = \frac{d}{d\eta}G$ and $\alpha' = \frac{d}{d\eta}\alpha$ are the derivatives of functions G and α, as in (1.4.1) and

(1.4.2), respectively. $u = u(t, x)$ and $v = v(t, x)$ are forcing terms for the orientation order η and orientation angle θ, respectively, and in particular, the forcing u can be regarded as the temperature source (control) of the grain boundary formulation. Additionally, $\alpha_0 : Q \longrightarrow (0, \infty)$ is a mobility for time-variation that is simplified by ignoring the η-dependence of α_0, as in (1.4.3).

We note that the singular diffusion $-\mathrm{div}\big(\alpha(\eta)\frac{D\theta}{|D\theta|} + \nu^2\nabla\theta\big)$ as in (1.4.4) is said to be effective to reproduce the *facet*, i.e. the locally uniform (constant) phase in each oriented grain (cf. [6, 20, 28, 29, 34, 45–47, 57, 58, 68, 72, 75, 77]). Furthermore, nowadays, there are vast numbers of literatures (cf. [6, 20, 28, 29, 34, 45–47, 57, 58, 68, 72, 75, 77]), which study the K.W.C. type systems, kindred to (1.4.2). In most of these previous works, the boundary conditions have been considered under homogeneous settings of zero-Dirichlet/zero-Neumann types. However, as a distinct case, we can quote [62], which imposed boundary conditions containing the *dynamic boundary type*.

Hence, in this thesis, we deal with the optimal control problem, governed by the following state-systems:

- K.W.C. models under one-dimensional setting of Ω, subject to:
  - (a) the zero-Neumann boundary condition for η, and zero-Dirichlet boundary condition for θ (in Chapter 5);
  - (b) the dynamic boundary condition for η, and zero-Dirichlet boundary condition for θ (in Chapter 6);
- K.W.C. models under higher-dimensional setting of Ω, subject to:
  - (c) the boundary condition (a), and various range-constraints for the temperature u (temperature constraints) (in Chapter 7).

1.5 Outline of this thesis

This thesis consists of 7 Chapters. We state the results, concerned with the optimal control problems, in each Chapter. These results are based on our submitted papers, listed below:

- the result of Chapter 2 is based on my single authorship published in 2021 (cf. [49]);
- the results of Chapter 3 are based on my single authorship published in 2021 (cf. [48]);
- the results of Chapter 4 are based on the joint-work with Ken Shirakawa and Noriaki Yamazaki published in 2020 (cf. [51]);
- the results of Chapter 5 are based on the joint-work with Harbir Antil, Ken Shirakawa, and Noriaki Yamazaki published in 2021 (cf. [7]);

- the results of Chapter 6 are based on the joint-work with Ryota Nakayashiki and Ken Shirakawa published in 2020 (cf. [50]);

- the results of Chapter 7 are based on the joint-work with Harbir Antil, Ken Shirakawa, and Noriaki Yamazaki submitted in 2021 (cf. [8]).

In Chapter 2, we consider a convex function defined as a 1D-regularized total variation with nonhomogeneous coefficients. Let $\Omega := (-L, L) \subset \mathbb{R}$ be a one-dimensional spatial domain with a constant $0 < L < \infty$, and let us define $H := L^2(\Omega)$ and $V := H^1(\Omega)$. Let $0 \leq \alpha \in V$ and $0 < \beta \in V$ be fixed functions.

In this Chapter, we consider the following convex function on H:

$$\theta \in H \mapsto \Phi_{\alpha,\beta}(\theta) := V_\alpha(\theta) + W_\beta(\theta); \tag{1.5.1}$$

which is defined as a sum of two convex functions on H, defined as follows:

$$\theta \in H \mapsto V_\alpha(\theta) := \sup \left\{ \int_\Omega \theta \partial_x \varphi \, dx, \left| \begin{array}{l} \varphi \in V \cap C_c(\Omega), \text{ such} \\ \text{that } |\varphi| \leq \alpha \text{ on } \overline{\Omega} \end{array} \right. \right\}, \tag{1.5.2}$$

and

$$\theta \in H \mapsto W_\beta(\theta) := \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega \beta |\partial_x \theta|^2 dx, & \text{if } \theta \in V, \\ \infty, & \text{otherwise.} \end{cases} \tag{1.5.3}$$

The functional V_α, defined in (1.5.2), is a kind of generalized total variation, so that the functional $\Phi_{\alpha,\beta}$, defined in (1.5.1), can be called a *regularized total variation* with nonhomogeneous coefficients α and β.

On this basis, we set the goal to prove the following Main Theorem.

Main Theorem (Decomposition of the subdifferential)**.** The subdifferential $\partial \Phi_{\alpha,\beta} \subset H \times H$ of the convex function $\Phi_{\alpha,\beta}$ is decomposed as follows:

$$\partial \Phi_{\alpha,\beta} = \partial V_\alpha + \partial W_\beta \text{ in } H \times H, \tag{1.5.4}$$

i.e. $\partial \Phi_{\alpha,\beta}$ is represented as the sum the subdifferentials $\partial V_\alpha \subset H \times H$ and $\partial W_\beta \subset H \times H$ of the respective convex functions V_α and W_β.

The equation (1.5.4) leads to the H^2-regularity of the following nonhomogeneous quasilinear equation with singularity:

$$\begin{cases} -\partial_x \left(\alpha(x) \dfrac{D\theta}{|D\theta|} + \beta(x) \partial_x \theta \right) = \theta^* \text{ with } \theta^* \in H, \\ \partial_x \theta(\pm L) = 0. \end{cases} \tag{1.5.5}$$

When the both α and β are homogeneous (constants), we can obtain the H^2-regularity by using the mathematical method, developed in [60], which is based on the general theory of PDEs (e.g. [53]). However, when α and β are nonhomogeneous, the extra error terms brought by α and β make it difficult to see $\theta \in H^2(\Omega)$ in (1.5.5), by referring to the existing method. Hence, it can be said that our Main Theorem will be to enhance the previous method of [60], and moreover, to report another variational approach based on the subdifferential.

In the meantime, the Main Theorem is motivated by the mathematical analysis of grain boundary motion, studied in [80, 89], and especially, the convex function $\Phi_{\alpha,\beta}$ is based on the *free energy*, proposed by Kobayashi–Warren–Carter [46, 47]. In this context, the variable θ is the order parameter of crystalline orientation, and the nonhomogeneous coefficients α and β are associated with another order parameter, such as the orientation order of grain in a polycrystal. In this light, our Main Theorem can be expected to provide useful information for some advanced problems that require smoothness of the system while including singularity, such as the optimal control problem governed by the K.W.C. model.

Note that the result of this Chapter is based on the recent work [49].

In Chapter 3, let $0 < L < \infty$ and $0 < T < \infty$ be fixed positive constants, let $\Omega := (-L, L) \subset \mathbb{R}$ be a one-dimensional spatial domain. Besides, we set $X := L^2(\Omega)$ and $Y := H^1(\Omega)$.

The aim of this Chapter is to give some advanced observations for the optimal control problem, governed by the following micro-mesoscopic Allen–Cahn type equation with singularities:

$$\begin{cases} \partial_t w - \partial_x \left(\dfrac{\partial_x w}{|\partial_x w|} + \nu^2 \partial_x w \right) + \partial I_{[-1,1]}(w) + g(w) \ni M_u u, \quad \text{in } (0,T) \times \Omega, \\ \partial_x w(t, \pm L) = 0, \;\; t \in (0,T), \\ w(0,x) = w_0(x), \;\; x \in \Omega. \end{cases} \tag{1.5.6}$$

In this context, the unknown $w = w(t,x)$ is the nonconserved order parameter as in (1.2.1). Besides, $w_0 \in Y$ is a given initial data of w. The forcing term $u = u(t,x)$ denotes the control variable that controls the profile of the solution w, and physically, u is the relative temperature. Furthermore, g is a semi-monotone C^1-function on $\mathbb{R}$, and $\nu > 0$ and $M_u \geq 0$ are fixed constants. In addition, $\partial I_{[-1,1]}$ is the subdifferential of the indicator function $I_{[-1,1]}$ on the closed interval $[-1, 1]$, that is defined as:

$$I_{[-1,1]}(z) := \begin{cases} 0, & \text{if } z \in [-1,1], \\ \infty, & \text{otherwise.} \end{cases}$$

Recently, Allen–Cahn type equations kindred to (1.5.6) have been studied by a lot of mathematicians (cf. [22, 66–68, 77]), and in many previous works, the singular term $-\partial_x\big(\frac{\partial_x w}{|\partial_x w|}\big)$ is approximated by quasilinear diffusions:

$$-\partial_x \left(\frac{\partial_x w}{\sqrt{\varepsilon^2 + |\partial_x w|^2}} + \nu^2 \partial_x w \right) \text{ with a constant } \varepsilon > 0, \tag{1.5.7}$$

and the singular term $\partial I_{[-1,1]}(w)$ of the set-valued subdifferential is approximated by means of the *Yosida regularizations,* or the following C^1-regularizations:

$$\partial \widetilde{I}^{\delta}_{[-1,1]}(r) := \begin{cases} 0, \text{ if } -1 \leq r \leq 1, \\ \dfrac{1}{2\delta^2} \dfrac{r}{|r|}(r-1)^2, \text{ if } 1 < |r| \leq 1+\delta, \\ \dfrac{1}{\delta} \dfrac{r^2}{|r|} - \left(\dfrac{1}{\delta} + \dfrac{1}{2} \right), \text{ if } |r| > 1+\delta, \end{cases} \quad \text{for } r \in \mathbb{R}, \text{ and } \delta > 0. \tag{1.5.8}$$

The setting {(1.5.7),(1.5.8)} was adopted as a representative effective range of approximating method for the singular terms. Indeed, the previous researches of [66–68] adopted the setting {(1.5.7),(1.5.8)} to deal with the optimal control problems governed by (1.5.6), and as a consequence, they obtained a distributional characterizations of the first necessary optimality condition for the optimal control. However, in the previous results, we still have two unfinished issues.

The first unfinished issue is to make clear the effective range of approximating methods, besides {(1.5.7),(1.5.8)}. From the viewpoint of application, it is beneficial if we could choose a matching approximating method for specific property of each scientific/ technological background. In this light, it could be said that the current mathematical results would not be sufficient to respond to the requirements for such flexibility.

In the meantime, the second unfinished one is concerned with the precise observation for the distribution, which appears in the optimality condition, obtained in [66–68], and is roughly expressed as:

$$\zeta^\circ \sim \partial_x \bigl[\boldsymbol{\eth}(\partial_x w^\circ) \partial_x u^\circ \bigr] - \bigl[\partial I_{[-1,1]} \bigr]'(w^\circ) u^\circ \text{ in } \mathfrak{D}'\bigl((0,T) \times \Omega\bigr), \tag{1.5.9}$$

with use of Dirac's delta $\boldsymbol{\eth}$, some operator $\bigl[\partial I_{[-1,1]}\bigr]'$ corresponding to derivative of the subdifferential $\partial I_{[-1,1]}$, an optimal control u°, and a solution w° to the singular Allen–Cahn type equation (1.5.6) when $u = u^\circ$. The expression (1.5.9) looks ill-posed as a variational formula. Nevertheless, it would be expectable that the support $\mathrm{spt}(\zeta^\circ)$ of the distribution ζ° would be somehow associated with $\{\partial_x w^\circ = 0\} \cup \{|w^\circ| = 1\}$ ($\approx \{\partial_x w^\circ = 0\}$), and ζ° would have some functional expression on the outside of the support:

$$\mathrm{spt}(\zeta^\circ)^{\mathrm{C}} \approx \{\partial_x w^\circ \neq 0\} \ \bigl(= \{\partial_x w^\circ \neq 0\} \cap \{|w^\circ| < 1\}\bigr).$$

For developments of the unfinished issues, we now let $n \in \mathbb{N}$ be a fixed number, define $\mathbb{X} := [X]^n$, and consider the time-discrete scheme for the singular Allen–Cahn type equation (1.5.6), with the time step size $\tau := T/n$, as a simplified state-equation of our optimal control problem. On this basis, we set our goal to study a class of optimal control problems, denoted by $(\mathrm{OP})^{(\varepsilon,\delta)}$, with constant parameters $\varepsilon, \delta \in [0,1]$.

$(\mathrm{OP})^{(\varepsilon,\delta)}$: to find a sequence of functions $u^* = [u_1^*, \dots, u_n^*] \in \mathbb{X}$, called *optimal control*, which minimizes the following cost functional $\mathcal{J}^{(\varepsilon,\delta)} = \mathcal{J}^{(\varepsilon,\delta)}(u)$ on $\mathbb{X}$, defined as

$$\mathcal{J}^{(\varepsilon,\delta)} : u \in \mathbb{X} \mapsto \mathcal{J}^{(\varepsilon,\delta)}(u) := \frac{M_w}{2} |(w - w^{\mathrm{ad}})|_{\mathbb{X}}^2 + \frac{M_u}{2} |u|_{\mathbb{X}}^2 \in [0, \infty), \tag{1.5.10}$$

i.e.

$$\mathcal{J}^{(\varepsilon,\delta)}(u^*) = \min \left\{ \ \mathcal{J}^{(\varepsilon,\delta)}(u) \ \middle| \ u \in \mathbb{X} \ \right\},$$

where for any $u = [u_1, \dots, u_n] \in \mathbb{X}$, the sequence of functions $w = [w_1, \dots, w_n] \in \mathbb{X}$ is a solution to the following time-discrete quasilinear equations, denoted by $(\mathrm{AC})^{(\varepsilon,\delta)}$.

$(\mathrm{AC})^{(\varepsilon,\delta)}$: to find a sequence of functions $w = [w_1, \cdots, w_n] \in \mathbb{X}$, which fulfills

$$\begin{cases} \frac{1}{\tau}(w_i - w_{i-1}) - \partial_x\big((f^\varepsilon)'(\partial_x w_i) + \nu^2 \partial_x w_i\big) + K^\delta(w_i) + g(w_i) \ni M_u u_i \ \text{ in } \Omega, \\ \partial_x w_i(\pm L) = 0, \quad i = 1, 2, 3, \dots, n, \end{cases}$$

starting from the initial value $w_0 \in Y$.

In this context, the equation $(\mathrm{AC})^{(0,0)}$, i.e. the case when $\varepsilon = \delta = 0$, corresponds to the time-discrete scheme of the Allen–Cahn type equation (1.5.6), while the equations $(\mathrm{AC})^{(\varepsilon,\delta)}$, for positive ε, δ, are its approximating equations. In this regard, the effective range of approximation, brought in the first unfinished issue, will be presented in forms of assumptions for:

– the regularization sequence $\{f^\varepsilon\}_{\varepsilon \in (0,1]} \subset C^2(\mathbb{R})$ for the function $f^0 : r \in \mathbb{R} \mapsto |r| \in [0, \infty)$ of the absolute value;

– the regularization sequence $\{K^\delta\}_{\delta \in (0,1]} \subset C^1(\mathbb{R})$ for the set-valued function $K^0 : r \in \mathbb{R} \mapsto \partial I_{[-1,1]}(r) \in 2^{\mathbb{R}}$ of the subdifferential of indicator function.

In addition, we will obtain a positive answer for the second unfinished issue, as a consequence of six results. Here, we give their assertions, briefly.

- Existence, uniqueness, and H^2-regularity of the solution $w = [w_1, \dots, w_n]$ to the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$, for every $\varepsilon \in [0,1]$, $\delta \in [0,1]$, initial data $w_0 \in Y$, and forcing term $u = [u_1, \dots, u_n] \in \mathbb{X}$.
- Continuous dependence of solutions to the state-systems $(\mathrm{AC})^{(\varepsilon,\delta)}$, with respect to the constants $\varepsilon \in [0,1]$, $\delta \in [0,1]$, initial data $w_0 \in Y$, and forcing term $u = [u_1, \dots, u_n] \in \mathbb{X}$, including the (strong) convergence in $[C(\overline{\Omega})]^n$.
- Existence for the optimal control problem $(\mathrm{OP})^{(\varepsilon,\delta)}$, for every constants $\varepsilon \in [0,1]$, $\delta \in [0,1]$, and initial data $w_0 \in Y$ with $\hat{K}^\delta(w_0) \in L^1(\Omega)$.
- Some semi-continuous dependence of the optimal controls, with respect to the constants $\varepsilon \in [0,1]$, $\delta \in [0,1]$, and initial data $w_0 \in Y$ with $\hat{K}^\delta(w_0) \in L^1(\Omega)$.
- Derivation of the first order necessary optimality conditions for $(\mathrm{OP})^{(\varepsilon,\delta)}$, via adjoint method, under $\varepsilon \in (0,1]$ and $\delta \in (0,1]$.
- The optimality conditions which are obtained as approximation limits of the necessary conditions as $\varepsilon, \delta \downarrow 0$, and a precise characterization of the limiting optimality conditions.

The results of this Chapter are based on the recent work [48].

In Chapter 4, we consider a class of approximate problems for the following one dimensional phase-filed system with singularity:

Problem (P;f, h, ℓ)^0.

$$[u + w]_t - u_{xx} = a_0 f(t,x) \quad (t,x) \in Q := (0,T) \times (0,L), \tag{1.5.11}$$

$$w_t - \kappa\left(\frac{w_x}{|w_x|}\right)_x + \partial I_{[-1,1]}(w) + g(w) \ni u \quad \text{in } Q, \tag{1.5.12}$$

$$-u_x(t,0) + n_0(u(t,0) - b_1) = a_1 h(t), \quad t \in (0,T), \tag{1.5.13}$$

$$u_x(t,L) + n_0(u(t,L) - b_2) = a_2 \ell(t), \quad t \in (0,T), \tag{1.5.14}$$

$$w_x(t,0) = w_x(t,L) = 0, \quad t \in (0,T), \tag{1.5.15}$$

$$u(0,x) = u_0(x), \quad w(0,x) = w_0(x), \quad x \in (0,L), \tag{1.5.16}$$

where $0 < T < \infty$ and $0 < L < \infty$ are fixed positive constants, a_0, a_1, a_2 are given nonnegative constants, $\kappa > 0$, $n_0 > 0$, b_1, b_2 are given constants, g is a given continuous function on $\mathbb{R}$, $[f, h, \ell]$ is a triplet of given functions, and u_0, w_0 are given initial data. In addition, $\partial I_{[-1,1]}$ is the subdifferential of an indicator function $I_{[-1,1]}$ on the closed interval $[-1,1]$, that is defined as:

$$I_{[-1,1]}(z) := \begin{cases} 0, & \text{if } z \in [-1,1], \\ \infty, & \text{otherwise.} \end{cases} \tag{1.5.17}$$

In the physical context, the unknown function $u = u(t,x)$ is the relative temperature, and $w = w(t,x)$ is the nonconserved order parameter as in (1.2.1).

Many mathematicians studied the singular diffusion equation (1.5.12) with or without constraint $\partial I_{[-1,1]}(w)$ from the various point of view (cf. [4–6, 22, 28–32, 41, 42, 45, 52, 63, 65–68, 73, 77, 85]). For instance, Kenmochi–Shirakawa studied in [41] the precise structure of steady-state solution, and characterized in [42] the asymptotic stability of steady-states, by means of an original concept named "local stability". Furthermore, the line of results [41, 42] was enhanced by Shirakawa–Kimura [77], under the higher dimensional setting of spatial domain.

In addition, Ohtsuka–Shirakawa–Yamazaki [66–68] considered the optimal control problem of (1.5.12) with respect to the temperature control u in the case when $g(w) = -w$.

The system (P;f, h, ℓ)^0 was considered by Kenmochi–Shirakawa [43] and Shirakawa [74, 76]. In particular, Kenmochi–Shirakawa [43] discussed the large-time behavior of solutions to (P;$0, 0, 0$)^0 on the basis of the previous work [41, 42] of stability analysis. In addition, Shirakawa–Yamazaki [81] considered the optimal control problem and its optimality condition for (P;f, h, ℓ)^0 with $g(w) = cw^3 - w$ for some small constant $c \geq 0$ via the limiting observation of approximate problems : in such approximate problems, the singular diffusion term $\left(\frac{w_x}{|w_x|}\right)_x$ and the constraint $\partial I_{[-1,1]}(w)$ as in (1.5.12) were approximated by

$$\left(\frac{w_x^\varepsilon}{\sqrt{|w_x^\varepsilon|^2 + \varepsilon^2}} + \varepsilon w_x^\varepsilon\right)_x \quad \text{and} \quad K^\varepsilon(w^\varepsilon), \tag{1.5.18}$$

respectively, for given small parameter $\varepsilon \in (0,1]$. Here, K^ε is a nondecreasing function on $\mathbb{R}$ defined by

$$K^\varepsilon(r) := \mathrm{sign}(r) \int_0^{|r|} \min\left\{\frac{1}{\varepsilon}, \frac{[s-1]^+}{\varepsilon^2}\right\} ds \quad \text{for } r \in \mathbb{R}, \tag{1.5.19}$$

where $[\,\cdot\,]^+$ denotes the positive part of a function and $\mathrm{sign}(\cdot)$ is a signal function so that $\mathrm{sign}(0) = 0$.

In this present paper, we consider a class of approximate functions for singular diffusion term $\left(\frac{w_x}{|w_x|}\right)_x$ in (P;f,h,ℓ)^0. Then we investigate the following approximate problems, denoted by (P;f,h,ℓ)$^\varepsilon$, with small parameter $\varepsilon \in (0,1]$:

Problem (P;f,h,ℓ)$^\varepsilon$.

$$[u^\varepsilon + w^\varepsilon]_t - u^\varepsilon_{xx} = a_0 f(t,x) \quad (t,x) \in Q, \tag{1.5.20}$$

$$w^\varepsilon_t - \kappa\left(a^\varepsilon(w^\varepsilon_x) + \varepsilon w^\varepsilon_x\right)_x + K^\varepsilon(w^\varepsilon) + g(w^\varepsilon) = u^\varepsilon \ \text{ in } Q, \tag{1.5.21}$$

$$-u^\varepsilon_x(t,0) + n_0(u^\varepsilon(t,0) - b_1) = a_1 h(t), \quad t \in (0,T), \tag{1.5.22}$$

$$u^\varepsilon_x(t,L) + n_0(u^\varepsilon(t,L) - b_2) = a_2 \ell(t), \quad t \in (0,T), \tag{1.5.23}$$

$$w^\varepsilon_x(t,0) = w^\varepsilon_x(t,L) = 0, \quad t \in (0,T), \tag{1.5.24}$$

$$u^\varepsilon(0,x) = u_0(x), \quad w^\varepsilon(0,x) = w_0(x), \quad x \in (0,L), \tag{1.5.25}$$

where a^ε is a given function on $\mathbb{R}$ with $a^\varepsilon(r) \to a^0(r) := \frac{r}{|r|}$ in an appropriate sense as $\varepsilon \to 0$. The typical example is $a^\varepsilon(r) = \frac{r}{\sqrt{r^2+\varepsilon^2}}$ (cf. (1.5.18)). Then, we clarify the class of approximate functions a^ε so that (P;f,h,ℓ)$^\varepsilon$ is the approximate problem for (P;f,h,ℓ)^0 as $\varepsilon \to 0$.

In addition, we consider a class of approximate optimal control problems, denoted by (OP)$^\varepsilon$, as follows:

Problem (OP)$^\varepsilon$: Find a triplet of control functions $[f^\varepsilon_*, h^\varepsilon_*, \ell^\varepsilon_*] \in \mathcal{U}$, called *optimal control*, such that

$$J^\varepsilon(f^\varepsilon_*, h^\varepsilon_*, \ell^\varepsilon_*) = \inf_{[f,h,\ell]\in\mathcal{U}} J^\varepsilon(f,h,\ell).$$

Here, we set $\mathcal{U} := L^2(0,T;L^2(0,L)) \times L^2(0,T) \times L^2(0,T)$ as a control space, and $J^\varepsilon(f,h,\ell)$ is the cost functional defined by

$$\begin{aligned} J^\varepsilon(f,h,\ell) := {}& \frac{c_0}{2}\int_0^T |(u^\varepsilon - u_d)(t)|^2_{L^2(0,L)}\,dt + \frac{c_1}{2}\int_0^T |(w^\varepsilon - w_d)(t)|^2_{L^2(0,L)}\,dt \\ &+ \frac{m_0}{2}\int_0^T a_0^2 |f(t)|^2_{L^2(0,L)}\,dt + \frac{m_1}{2}\int_0^T a_1^2|h(t)|^2\,dt + \frac{m_2}{2}\int_0^T a_2^2|\ell(t)|^2\,dt, \end{aligned} \tag{1.5.26}$$

where $|\cdot|_{L^2(0,L)}$ is a standard norm of $L^2(0,L)$, c_0, c_1, m_0, m_1, m_2 are given nonnegative constants, and u_d, w_d are the given desired target profiles in $L^2(0,T;L^2(0,L))$. In addition, a couple of functions $[u^\varepsilon, w^\varepsilon]$ is a unique solution to the initial-boundary value state problem (P;f,h,ℓ)$^\varepsilon$ with the control parameter $[f,h,\ell] \in \mathcal{U}$.

Note that (OP)$^\varepsilon$ can be regarded as an optimal control problem in solid-liquid phase transition phenomena. Indeed, if the constant a_0 is equal to 0, then (OP)$^\varepsilon$ is a boundary control problem. Similarly, if $a_1 = a_2 = 0$, then (OP)$^\varepsilon$ reduces to a distributed control problem with the heat source as control. Note that b_1 (resp. b_2) denotes the outside temperature at $x = 0$ (resp. $x = L$). There is a vast amount of literature on optimal control of phase transitions problems. In particular, we refer to the contributions [1, 12, 21, 33, 64–69, 71, 81, 82, 84].

In addition, note that (OP)^0 is an optimal control problem for our original phase-filed system (P;f,h,ℓ)^0 with singularity. Therefore, in this present paper, we show the relationship between (OP)$^\varepsilon$ and its limiting problem (OP)^0 as $\varepsilon \to 0$. Furthermore, by using necessary conditions for $(\mathrm{OP})^\varepsilon$, we propose the numerical scheme to find the stationary point of the cost functional $J^\varepsilon(\cdot,\cdot,\cdot)$ to $(\mathrm{OP})^\varepsilon$, and show the convergence of our numerical algorithm. Moreover, we give some numerical experiments for $(\mathrm{OP})^\varepsilon$ under the simple situations.

On this basis, the main focus of the results is summarized as follows:

- Solvability and continuous dependence the systems $(\mathrm{P};f,h,\ell)^\varepsilon$.
- Solvability and ε-dependence of optimal control problems $(\mathrm{OP})^\varepsilon$.
- Necessary optimality conditions in the cases of $\varepsilon > 0$, and limiting optimality conditions as $\varepsilon \downarrow 0$.
- Numerical scheme for $(\mathrm{OP})^\varepsilon$ with small parameter $\varepsilon > 0$, and numerical experiments.

The results of this Chapter are based on the joint-work with Ken Shirakawa and Noriaki Yamazaki [51].

In Chapter 5, we consider a class of optimal control problems governed by the following state-systems, which are denoted by $(\mathrm{S})_\varepsilon$, with $\varepsilon \geq 0$:

$(\mathrm{S})_\varepsilon$

$$\begin{cases} \partial_t\eta - \partial_x^2\eta + g(\eta) + \alpha'(\eta)\sqrt{\varepsilon^2 + |\partial_x\theta|^2} = M_u u \quad \text{in } Q := (0,T)\times\Omega, \\ \partial_x\eta(t,x) = 0, \quad (t,x) \in \Sigma := (0,T)\times\Gamma, \\ \eta(0,x) = \eta_0(x), \quad x \in \Omega; \end{cases} \tag{1.5.27}$$

$$\begin{cases} \alpha_0(t,x)\partial_t\theta - \partial_x\left(\alpha(\eta)\dfrac{\partial_x\theta}{\sqrt{\varepsilon^2+|\partial_x\theta|^2}} + \nu^2\partial_x\theta\right) = M_v v \quad \text{in } Q, \\ \theta(t,x) = 0, \quad (t,x)\in\Sigma, \\ \theta(0,x) = \theta_0(x), \quad x \in \Omega. \end{cases} \tag{1.5.28}$$

For each $\varepsilon \geq 0$, we denote the optimal control problem by $(\mathrm{OP})_\varepsilon$, and prescribe the problem as follows:

$(\mathrm{OP})_\varepsilon$ Find a pair of functions $[u^*, v^*] \in [L^2(0,T;L^2(\Omega))]^2$, called *optimal control*, which minimizes a cost functional $\mathcal{J}_\varepsilon = \mathcal{J}_\varepsilon(u,v)$, defined as:

$$\begin{aligned} \mathcal{J}_\varepsilon : [u,v] \in [L^2(0,T;L^2(\Omega))]^2 &\mapsto \mathcal{J}_\varepsilon(u,v) \\ := \frac{M_\eta}{2}\int_0^T |(\eta-\eta_{\mathrm{ad}})(t)|_H^2\,dt &+ \frac{M_\theta}{2}\int_0^T |(\theta-\theta_{\mathrm{ad}})(t)|_H^2\,dt \\ + \frac{M_u}{2}\int_0^T |u(t)|_H^2\,dt &+ \frac{M_v}{2}\int_0^T |v(t)|_H^2\,dt \in [0,\infty), \end{aligned} \tag{1.5.29}$$

where $[\eta,\theta] \in [L^2(0,T;L^2(\Omega))]^2$ solves the state-system $(\mathrm{S})_\varepsilon$.

Here, $(0,T)$ is a time-interval with a positive constant $T > 0$, $\Omega := (0,1) \subset \mathbb{R}$ is a one-dimensional spatial domain with a boundary $\Gamma := \{0,1\}$. The order parameters, $\eta \in L^2(0,T;L^2(\Omega))$ and $\theta \in L^2(0,T;L^2(\Omega))$ indicate the orientation order and orientation angle of the polycrystal body, respectively. Moreover, $[\eta_0, \theta_0] \in H^1(\Omega) \times H^1_0(\Omega)$ is an initial pair, i.e. a pair of initial data of $[\eta, \theta]$. The forcing pair $[u, v] \in [L^2(0,T;L^2(\Omega))]^2$ denotes the control variables that can control the profile of solution $[\eta, \theta] \in [L^2(0,T;L^2(\Omega))]^2$ to $(\mathrm{S})_\varepsilon$. Additionally, $0 < \alpha_0 \in W^{1,\infty}(Q)$ and $0 < \alpha \in C^2(\mathbb{R})$ are given functions to reproduce the mobilities of grain boundary motions. Besides, we set α_0 is independent of the order parameter η in our study. Finally, $g \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ is a perturbation for the orientation order η, and $\nu > 0$ is a fixed constant to relax the diffusion of the orientation angle θ.

In the state-system $(\mathrm{S})_\varepsilon$, the PDE part of the first initial-boundary value problem (1.5.27) is a type of Allen–Cahn equation, so that the forcing term u can be regarded as a *temperature control* of the grain boundary formation. Also, the second problem (1.5.28) is to reproduce crystalline micro-structure of polycrystal, and the case of $\varepsilon = 0$ is the closest to the original setting adopted by Kobayashi–Warren–Carter [46, 47]. Indeed, when $\varepsilon = 0$, the quasilinear diffusion as in (1.5.28) is described in a singular form $-\partial_x \left(\alpha(\eta) \frac{\partial_x \theta}{|\partial_x \theta|} + \nu^2 \partial_x \theta \right)$, and it is known that this type of singularity is effective to reproduce the *facet*, i.e. the locally uniform (constant) phase in each oriented grain (cf. [6, 20, 28, 29, 34, 45–47, 57, 58, 68, 72, 75, 77]). Hence, the systems $(\mathrm{S})_\varepsilon$, for positive ε, can be said as regularized approximating systems, that are to approach to the physically realistic situation, reproduced by the limiting system $(\mathrm{S})_0$, as $\varepsilon \downarrow 0$.

On the other hand, the pair of functions $[\eta_{\mathrm{ad}}, \theta_{\mathrm{ad}}] \in [L^2(0,T;L^2(\Omega))]^2$, in the optimal control problem $(\mathrm{OP})_\varepsilon$, is a given *admissible target profile* of $[\eta, \theta] \in [L^2(0,T;L^2(\Omega))]^2$. Moreover, $M_\eta \geq 0$, $M_\theta \geq 0$, $M_u \geq 0$, and $M_v \geq 0$ are fixed constants, that are to adjust the meaning of optimality in the problem $(\mathrm{OP})_\varepsilon$.

This chapter focuses on two issues:

♯1) key-properties of the state-systems $(\mathrm{S})_\varepsilon$, for $\varepsilon \geq 0$;

♯2) mathematical analysis of the optimal control problem $(\mathrm{OP})_\varepsilon$, for $\varepsilon \geq 0$.

With regard to the first issue ♯1), various singular systems, related to $(\mathrm{S})_\varepsilon$, have been studied by several authors, e.g. [35–37, 44, 58, 62, 72, 78–80, 87, 88]. In particular, the mathematical theories developed in [35, Theorems 2.1 and 2.2] and [62, Main Theorems 1 and 2] are applicable for the well-posedness and ε-dependence of the system $(\mathrm{S})_\varepsilon$. However, since the previous works dealt with only homogeneous case, i.e., the case of $[u, v] = [0, 0]$, some extension of the existing theories is needed for the application to our optimal control problem $(\mathrm{OP})_\varepsilon$. Meanwhile, for issue ♯2), the important point will be how to compute the Gâteaux differential of the cost $\mathcal{J}_\varepsilon$. This will be carried via a linearization of the state-system $(\mathrm{S})_\varepsilon$. When $\varepsilon > 0$, the problem $(\mathrm{OP})_\varepsilon$ admits sufficient regularity, and we can address the issue ♯2) by using the standard linearization method. Although such linearization method does not work for the problem $(\mathrm{OP})_0$, i.e. the case of $\varepsilon = 0$, it is possible to obtain some partial results by considering the limit as $\varepsilon \downarrow 0$ for $(\mathrm{OP})_\varepsilon$.

On this basis, the main focus of the results is summarized as follows:

- Solvability and continuous dependence of state-systems.
- Solvability and ε-dependence of optimal control problems.
- Necessary optimality conditions in the cases of $\varepsilon > 0$, and limiting optimality conditions as $\varepsilon \downarrow 0$.

The results of this Chapter are based on the joint-work with Harbir Antil, Ken Shirakawa, and Noriaki Yamazaki [7].

In Chapter 6, we consider a class of optimal control problems governed by the one-dimensional K.W.C. models with dynamic boundary conditions, which are denoted by $(\mathrm{S})_\varepsilon$, with $\varepsilon \geq 0$:

$(\mathrm{S})_\varepsilon$

$$
\begin{cases}
\partial_t \eta - \partial_x^2 \eta + g(\eta) + \alpha'(\eta)\sqrt{\varepsilon^2 + |\partial_x \theta|^2} = Lu \text{ in } Q := (0,T) \times \Omega, & (1.5.30\text{a}) \\
\partial_t \eta_\Gamma(t,\ell) + (-1)^{\ell-1} \partial_x \eta_{|_\Gamma}(t,\ell) = L_\Gamma u_\Gamma(t,\ell), \quad (t,\ell) \in \Sigma := (0,T) \times \Gamma, & (1.5.30\text{b}) \\
\eta_{|_\Gamma} = \eta_\Gamma \text{ on } \Sigma & (1.5.30\text{c}) \\
\eta(0,x) = \eta_0(x), \ x \in \Omega, \quad \eta_\Gamma(0,\ell) = \eta_{\Gamma,0}(\ell), \ \ell \in \Gamma; & (1.5.30\text{d})
\end{cases}
$$

$$
\begin{cases}
\alpha_0(t,x)\partial_t \theta - \partial_x \left(\alpha(\eta) \dfrac{\partial_x \theta}{\sqrt{\varepsilon^2 + |\partial_x \theta|^2}} + \nu^2 \partial_x \theta \right) = Mv \text{ in } Q, & (1.5.31\text{a}) \\
\theta(t,\ell) = 0, \quad (t,\ell) \in \Sigma, & (1.5.31\text{b}) \\
\theta(0,x) = \theta_0(x), \ x \in \Omega. & (1.5.31\text{c})
\end{cases}
$$

For each $\varepsilon \geq 0$, we denote the optimal control problem by $(\mathrm{OP})_\varepsilon$, and prescribe the problem as follows:

$(\mathrm{OP})_\varepsilon$ Find a triplet $[\boldsymbol{u}^*, v^*] = [u^*, u_\Gamma^*, v^*] \in (L^2(0,T;L^2(\Omega)) \times L^2(0,T;H_\Gamma)) \times L^2(0,T;L^2(\Omega))$ with $\boldsymbol{u}^* = [u^*, u_\Gamma^*]$, called *optimal control*, which minimizes a cost functional $\mathcal{J}_\varepsilon$, defined as:

$$
\begin{aligned}
\mathcal{J}_\varepsilon : [\boldsymbol{u}, v] = [u, u_\Gamma, v] &\in (L^2(0,T;L^2(\Omega)) \times L^2(0,T;H_\Gamma)) \times L^2(0,T;L^2(\Omega)) \\
&\mapsto \mathcal{J}_\varepsilon(\boldsymbol{u}, v) = \mathcal{J}_\varepsilon(u, u_\Gamma, v) \\
:= &\frac{K}{2}\int_0^T |(\eta - \eta_{\mathrm{ad}})(t)|_H^2 \, dt + \frac{K_\Gamma}{2}\int_0^T |(\eta_\Gamma - \eta_{\Gamma,\mathrm{ad}})(t)|_{H_\Gamma}^2 \, dt \\
&+ \frac{\Lambda}{2}\int_0^T |(\theta - \theta_{\mathrm{ad}})(t)|_H^2 \, dt \\
&+ \frac{L}{2}\int_0^T |u(t)|_H^2 \, dt + \frac{L_\Gamma}{2}\int_0^T |u_\Gamma(t)|_{H_\Gamma}^2 \, dt \\
&+ \frac{M}{2}\int_0^T |v(t)|_H^2 \, dt \in [0,\infty),
\end{aligned}
\tag{1.5.32}
$$

where $[\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in (L^2(0,T;L^2(\Omega)) \times L^2(0,T;H_\Gamma)) \times L^2(0,T;L^2(\Omega))$ with $\boldsymbol{\eta} = [\eta, \eta_\Gamma]$ is a solution to the system $(\mathrm{S})_\varepsilon$.

Here, $(0,T)$ is a time-interval with a positive constant $T > 0$, $\Omega := (0,1) \subset \mathbb{R}$ is a one-dimensional spatial domain, and let $\Gamma := \{0,1\}$ be the boundary of Ω. The components $\eta \in L^2(0,T;L^2(\Omega))$ and $\theta \in L^2(0,T;L^2(\Omega))$ are order parameters which indicate the *orientation order* and *orientation angle* of a polycrystal body Ω, respectively. Also, the component $\eta_\Gamma \in L^2(0,T;H_\Gamma)$, where

$$H_\Gamma := \left\{ \ \tilde{w} \ \middle| \ \tilde{w} : \Gamma \longrightarrow \mathbb{R} \ \right\} \ (\sim \mathbb{R}^2),$$

is an order parameter, which influences the dynamics of η, as an external factor exchanged via the boundary Γ of polycrystal. $[\eta_0, \eta_{\Gamma,0}] \in L^2(\Omega) \times H_\Gamma$ and $\theta_0 \in L^2(\Omega)$ are initial data of $\boldsymbol{\eta} = [\eta, \eta_\Gamma]$ and θ, respectively, and for simplicity, these initial data are written in a form of a *initial triplet* $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \in (L^2(\Omega) \times H_\Gamma) \times L^2(\Omega)$ with the *initial pair* $\boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}]$. The forcing triplet $[\boldsymbol{u}, v] \in (L^2(0,T;L^2(\Omega)) \times L^2(0,T;H_\Gamma)) \times L^2(0,T;L^2(\Omega))$ with the forcing pair $\boldsymbol{u} = [u, u_\Gamma]$, denotes the control variables that can control the profile of solution $[\boldsymbol{\eta}, \theta] \in (L^2(0,T;L^2(\Omega)) \times L^2(0,T;H_\Gamma)) \times L^2(0,T;L^2(\Omega))$ to $(\mathrm{S})_\varepsilon$. $0 < \alpha_0 \in W^{1,\infty}(Q)$ and $0 < \alpha \in C^2(\mathbb{R})$ are given functions to reproduce the mobilities of grain boundary motions. Besides, "$|_\Gamma$" denotes the trace on Γ for a Sobolev function. In addition, we set α_0 is independent of the order parameter η in our study. Finally, $g \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ is a perturbation for the orientation order η, and $\nu > 0$ is a fixed constant to relax the diffusion of the orientation angle θ.

As a mathematical model of grain boundary motion, $(\mathrm{S})_\varepsilon$ can be said as a coupled system of an Allen–Cahn type equation (1.5.30a) subject to the dynamic boundary condition {(1.5.30b),(1.5.30c)}, and a quasilinear diffusion equation (1.5.31a) subject to the homogeneous Dirichlet boundary condition (1.5.31b). However, it should be noted that the PDE (1.5.30b) can be regarded as a governing equation on Γ, and the initial-boundary value problem (1.5.30) also forms a kind of *transmission problem* between the PDE (1.5.30a) in Ω and the PDE (1.5.30b) on Γ, subject to the *transmission condition* (1.5.30c).

Since $\boldsymbol{u} = [u, u_\Gamma] \in \mathfrak{X}$ is a forcing term of the Allen-Cahn type equation (1.5.30a), the components $u \in \mathscr{H}$ and $u_\Gamma \in \mathscr{H}_\Gamma$ can be understood as the temperature controls on the interior Ω and the boundary Γ of polycrystal, respectively. Meanwhile, the quasilinear diffusion equation (1.5.31a) is to reproduce crystalline micro-structure of polycrystal, and the case when $\varepsilon = 0$ is the closest to the original setting adopted by Kobayashi–Warren–Carter [46,47]. Indeed, when $\varepsilon = 0$, the quasilinear diffusion as in (1.5.31) is described in a singular form $-\partial_x \left(\alpha(\eta) \frac{\partial_x \theta}{|\partial_x \theta|} + \nu^2 \partial_x \theta \right)$, and it is known that this type of singularity is effective to reproduce the *facet*, i.e. the locally uniform (constant) phase in each oriented grain (cf. [6,20,28,29,34,45–47,57,58,68,72,75,77]). Hence, the systems $(\mathrm{S})_\varepsilon$, for positive ε, can be said as regularized approximating systems, that are to approach to the physically realistic situation, reproduced by the limiting system $(\mathrm{S})_0$, as $\varepsilon \downarrow 0$.

Furthermore, in the optimal control problems $(\mathrm{OP})_\varepsilon$ for $\varepsilon \geq 0$, the functions $\eta_{\mathrm{ad}} \in L^2(0,T;L^2(\Omega))$, $\eta_{\Gamma,\mathrm{ad}} \in L^2(0,T;H_\Gamma)$, and $\theta_{\mathrm{ad}} \in L^2(0,T;L^2(\Omega))$ are given *admissible target profile* of the order parameters η, η_Γ, and θ, respectively, and the coefficients $K \geq 0$, $K_\Gamma \geq 0$, $\Lambda \geq 0$, $L \geq 0$, $L_\Gamma \geq 0$, and $M \geq 0$ are fixed constants which are to adjust the meaning of optimality in the problems $(\mathrm{OP})_\varepsilon$.

This chapter focuses on two issues:

♯1) key-properties of the state-systems $(\mathrm{S})_\varepsilon$, for $\varepsilon \geq 0$;

♯2) mathematical analysis of the optimal control problems $(\mathrm{OP})_\varepsilon$, for $\varepsilon \geq 0$.

With regard to the first issue ♯1), some kindred K.W.C. type systems have been studied by several mathematicians, e.g. [35,61,62], and in particular, the analytic ideas, as in [62, Main Theorems 1 and 2], would be effective for the well-posedness and ε-dependence of the system $(\mathrm{S})_\varepsilon$. However, the previous works [35, 61, 62] adopted the homogeneous setting of forcing, and imposed different types of boundary conditions with this study. In this light, we need to enhance the existing mathematical method before we deal with the study of our optimal control problems $(\mathrm{OP})_\varepsilon$. Meanwhile, for issue ♯2), there are now a number of previous works [16, 23, 81, 82, 84], which dealt with optimal control problems, governed by PDE systems kindred to $(\mathrm{S})_\varepsilon$. Hence, by integrating the previous works [16,23,35,61,62,81,82,84], we can expect to develop a certain mathematical control theory that enables to handle dynamically transmitted situations, as in the dynamic boundary condition of our state-system $(\mathrm{S})_\varepsilon$.

On this basis, the main focus of the results is summarized as follows:

- Solvability and continuous dependence of state-systems.
- Solvability and ε-dependence of optimal control problems.
- Necessary optimality conditions in the cases of $\varepsilon > 0$, and limiting optimality conditions as $\varepsilon \downarrow 0$.

The results of this Chapter are based on the joint-work with Ryota Nakayashiki and Ken Shirakawa [50].

In Chapter 7, let $(0, T)$ be a time-interval with a constant $0 < T < \infty$, and let $N \in \{2, 3, 4\}$ denote the spatial dimension. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\Gamma := \partial\Omega$, and let n_Γ be the unit outer normal on Γ. Besides, we set $Q := (0, T) \times \Omega$ and $\Sigma := (0, T) \times \Gamma$, and we define $H := L^2(\Omega)$ with norm $|\cdot|_H$, $V := H^1(\Omega)$, $V_0 := H^1_0(\Omega)$, and $\mathscr{H} := L^2(0, T; L^2(\Omega))$, as the base spaces for this Chapter. Moreover, we set:

$$[\![\kappa^0, \kappa^1]\!] := \left\{ \, \tilde{u} \in \mathscr{H} \;\middle|\; \kappa^0 \leq \tilde{u} \leq \kappa^1 \text{ a.e. in } Q \, \right\},$$
$$\text{for arbitrary measurable obstacles } \kappa^\ell : Q \longrightarrow [-\infty, \infty], \quad \ell = 0, 1,$$

and define a family of functional classes $\mathfrak{K} \subset 2^{\mathscr{H}}$, as follows:

$$\mathfrak{K} := \left\{ \, K \subset \mathscr{H} \;\middle|\; \begin{array}{l} K = [\![\kappa^0, \kappa^1]\!] \text{ for some measurable obstacles} \\ \kappa^\ell : Q \longrightarrow [-\infty, \infty], \; \ell = 0, 1, \text{ such that} \\ \kappa^0 \leq \kappa^1 \text{ a.e. in } Q \text{ (i.e. } K \neq \emptyset) \end{array} \right\}. \tag{1.5.33}$$

In this thesis, we consider a class of optimal control problems, denoted by $(\mathrm{OP})_\varepsilon^K$, which are labeled by constants $\varepsilon \geq 0$ and functional classes $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$, with the obstacles $\kappa^\ell : Q \longrightarrow [-\infty, \infty]$, $\ell = 0, 1$. For every $\varepsilon \geq 0$ and $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$, the optimal control problem $(\mathrm{OP})_\varepsilon^K$ is prescribed as follows:

$(\mathrm{OP})^K_\varepsilon$ Find a pair of functions $[u^*, v^*] \in [\mathscr{H}]^2$, called the *optimal control*, such that

$$[u^*, v^*] \in \mathscr{U}_{\mathrm{ad}}^K := \left\{ \ [\tilde{u}, \tilde{v}] \in [\mathscr{H}]^2 \ \middle| \ \tilde{u} \in K \ \right\},$$
$$\text{and} \ \ \mathcal{J}_\varepsilon(u^*, v^*) = \min \left\{ \ \mathcal{J}_\varepsilon(u, v) \ \middle| \ [u, v] \in \mathscr{U}_{\mathrm{ad}}^K \ \right\},$$

where $\mathcal{J}_\varepsilon = \mathcal{J}_\varepsilon(u, v)$ is a cost functional on $[\mathscr{H}]^2$, defined as follows:

$$\begin{aligned} \mathcal{J}_\varepsilon : [u, v] \in [\mathscr{H}]^2 \mapsto \mathcal{J}_\varepsilon(u, v) \\ := \frac{M_\eta}{2} \int_0^T |(\eta - \eta_{\mathrm{ad}})(t)|_H^2 \, dt + \frac{M_\theta}{2} \int_0^T |(\theta - \theta_{\mathrm{ad}})(t)|_H^2 \, dt \\ + \frac{M_u}{2} \int_0^T |u(t)|_H^2 \, dt + \frac{M_v}{2} \int_0^T |v(t)|_H^2 \, dt \quad \in [0, \infty), \end{aligned} \tag{1.5.34}$$

with $[\eta, \theta] \in [\mathscr{H}]^2$ solving the state-system, denoted by $(\mathrm{S})_\varepsilon$:

$(\mathrm{S})_\varepsilon$

$$\begin{cases} \partial_t \eta - \Delta \eta + g(\eta) + \alpha'(\eta) \sqrt{\varepsilon^2 + |\nabla \theta|^2} = M_u u \quad \text{in } Q, \\ \nabla \eta(t, x) \cdot n_\Gamma = 0, \quad (t, x) \in \Sigma, \\ \eta(0, x) = \eta_0(x), \quad x \in \Omega; \end{cases} \tag{1.5.35}$$

$$\begin{cases} \alpha_0(t, x) \partial_t \theta - \mathrm{div} \left(\alpha(\eta) \dfrac{\nabla \theta}{\sqrt{\varepsilon^2 + |\nabla \theta|^2}} + \nu^2 \nabla \theta \right) = M_v v \quad \text{in } Q, \\ \theta(t, x) = 0, \quad (t, x) \in \Sigma, \\ \theta(0, x) = \theta_0(x), \quad x \in \Omega. \end{cases} \tag{1.5.36}$$

The state-system $(\mathrm{S})_\varepsilon$ is based on a phase field model of grain boundary motion, known as *Kobayashi–Warren–Carter system* (cf. [46, 47]). In this context, the unknowns $\eta \in \mathscr{H}$ and $\theta \in \mathscr{H}$ are order parameters that indicate the *orientation order* and *orientation angle* of the polycrystal body, respectively. Besides, $[\eta_0, \theta_0] \in V \times V_0$ is an *initial pair*, i.e. a pair of initial data of $[\eta, \theta]$. The *forcing pair* $[u, v] \in [\mathscr{H}]^2$ denotes the control variables that can control the profile of solution $[\eta, \theta] \in [\mathscr{H}]^2$ to $(\mathrm{S})_\varepsilon$. Additionally, $0 < \alpha_0 \in W^{1,\infty}(Q)$ and $0 < \alpha \in C^2(\mathbb{R})$ are given functions to reproduce the mobilities of grain boundary motions. Finally, $g \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ is a perturbation for the orientation order η, and $\nu > 0$ is a fixed constant to relax the diffusion of the orientation angle θ.

The first part (1.5.35) of the state-system $(\mathrm{S})_\varepsilon$ is the initial-boundary value problem of an Allen–Cahn type equation, so that the forcing term u can be regarded as a *temperature control* of the grain boundary formation. Also, the second problem (1.5.36) is the initial-boundary value problem to reproduce crystalline micro-structure of polycrystal, and the case of $\varepsilon = 0$ is the closest to the original setting adopted by Kobayashi–Warren–Carter [46, 47]. Indeed, when $\varepsilon = 0$, the quasi-linear diffusion as in (1.5.36) is described in a singular form $-\mathrm{div}\big(\alpha(\eta) \frac{\nabla \theta}{|\nabla \theta|} + \nu^2 \nabla \theta\big)$, and it is known that this type of singularity is effective to reproduce the *facet*, i.e. the locally uniform (constant) phase in each oriented grain (cf. [6, 20, 28, 29, 34, 45–47, 57, 58, 68, 72, 75, 77]). Hence, the systems $(\mathrm{S})_\varepsilon$, for positive ε, can be regarded as *regularized approximating systems*, that are to approach to the physically realistic situation $(\mathrm{S})_0$, in the limit $\varepsilon \downarrow 0$.

Meanwhile, in the optimal control problem $(\mathrm{OP})_\varepsilon^K$, the class $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$ is to constrain the range of temperature control u, and the obstacles $\kappa^\ell : Q \longrightarrow [-\infty, \infty]$, $\ell = 0, 1$, indicate the *control bounds* of the temperature. The pair of functions $[\eta_{\mathrm{ad}}, \theta_{\mathrm{ad}}] \in [\mathscr{H}]^2$ is a given *admissible target profile* of $[\eta, \theta] \in [\mathscr{H}]^2$. Moreover, $M_\eta \geq 0$, $M_\theta \geq 0$, $M_u \geq 0$, and $M_v \geq 0$ are fixed constants.

The objective of this Chapter is to significantly extend the results of our previous work [7], which dealt with:

- ♯1) key-properties of the state-systems $(\mathrm{S})_\varepsilon$ with 1-dimensional domain $\Omega = (0, 1)$;
- ♯2) mathematical analysis of the optimal control problem $(\mathrm{OP})_\varepsilon^K$, for $\varepsilon \geq 0$, but with 1-dimensional domain $\Omega \subset (0, 1)$ without any control constraints, i.e., $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$ $(= \mathscr{H})$;

In light of this, the novelty of this Chapter is in:

- ♯3) the development of a mathematical analysis to obtain optimal controls of grain boundaries under the higher dimensional setting $N \in \{2, 3, 4\}$ of the spatial domain, and the temperature constraint $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$.

In addition, the presence of constraints $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$ makes the mathematical analysis further challenging. Notice that such constraints are meaningful from a practical point of view. We further emphasize that in the main part of this Chapter, the L^∞-boundedness of η will be essential, and the main results will be valid under the following assumption on the data:

(r.s.0) $\varepsilon > 0$, $[\eta_0, \theta_0] \in D_0 := \bigl(V \cap L^\infty(\Omega)\bigr) \times V_0$, and $K \in \mathfrak{K}_0$, where

$$\mathfrak{K}_0 := \mathfrak{K} \cap 2^{L^\infty(Q)} = \left\{ K \;\middle|\; \begin{array}{l} K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K} \text{ such that} \\ \kappa^\ell \in L^\infty(Q),\ \ell = 0, 1 \end{array} \right\}. \tag{1.5.37}$$

Hence, in general cases of constraints $K \in \mathfrak{K}$ (including no constraint case), we will be forced to adopt some limiting (approximating) approach on the basis of the results under the restricted situation (r.s.0).

On this basis, the main focus of the results is summarized as follows:

- Solvability and continuous dependence of state-systems.
- Solvability and parameter-dependence of optimal control problems.
- Necessary optimality conditions for control constraints and $\varepsilon > 0$, and specific parameter dependence of necessary optimality conditions.
- Limiting optimality conditions.

The results of this Chapter are based on the joint-work with Harbir Antil, Ken Shirakawa, and Noriaki Yamazaki [8].

Chapter 2

Subdifferential decomposition of 1D-regularized total variation with nonhomogeneous coefficients

Throughout this Chapter, we consider a convex function defined as a 1D-regularized total variation with nonhomogeneous coefficients, and prove the Main Theorem concerned with the decomposition of the subdifferential of this convex function to a weighted singular diffusion and a linear regular diffusion. The Main Theorem will be to enhance the previous regularity result for quasilinear equation with singularity, and moreover, it will be to provide some useful information in the advanced mathematical studies of grain boundary motion, based on K.W.C. type energy.

2.1 Preliminaries

We begin by prescribing the assumptions and notations used throughout this Chapter.

Assumptions. Throughout this paper, let $\Omega := (-L, L) \subset \mathbb{R}$ be a fixed spatial bounded domain with a constant $0 < L < \infty$, and let $\Gamma := \partial\Omega = \{-L, L\}$ be the boundary of Ω. Also, let ∂_x be the distributional spatial differential. On this basis, we define

$$H := L^2(\Omega), H_\Gamma := \left\{ \ \tilde{z} \mid \tilde{z} : \Gamma \longrightarrow \mathbb{R} \ \right\} \ (\sim \mathbb{R}^2), \text{ and } V := H^1(\Omega) \ (\subset C(\overline{\Omega})).$$

Let $\alpha \in V$ and $\beta \in V$ be fixed functions, such that:

$$\min \alpha(\overline{\Omega}) \geq 0, \text{ and } \min \beta(\overline{\Omega}) > 0. \tag{2.1.1}$$

Abstract notations. For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X. Let $I_X : X \longrightarrow X$ be the identity map from X onto X. In particular, when X is a Hilbert space, we denote by $(\cdot, \cdot)_X$ the inner product of X.

For any subset A of a Banach space X, let $\chi_A : X \longrightarrow \{0, 1\}$ be the characteristic function of A, i.e.:

$$\chi_A : w \in X \mapsto \chi_A(w) := \begin{cases} 1, \text{ if } w \in A, \\ 0, \text{ otherwise.} \end{cases}$$

Notations in convex analysis. (cf. [18, Chapter II]) Let X be an abstract Hilbert space X. For a proper, lower semi-continuous (l.s.c.), and convex function $\Psi : X \longrightarrow (-\infty, \infty]$ on a Hilbert space X, we denote by $D(\Psi)$ the effective domain of Ψ. Also, we denote by $\partial\Psi$ the subdifferential of Ψ. The subdifferential $\partial\Psi$ corresponds to a weak differential of convex function Ψ, and it is known as a maximal monotone graph in the product space $X \times X$. The set $D(\partial\Psi) := \bigl\{ z \in X \mid \partial\Psi(z) \neq \emptyset \bigr\}$ is called the domain of $\partial\Psi$. We often use the notation "$[z_0, z_0^*] \in \partial\Psi$ in $X \times X$", to mean that "$z_0^* \in \partial\Psi(z_0)$ in X for $z_0 \in D(\partial\Psi)$", by identifying the operator $\partial\Psi$ with its graph in $X \times X$.

Example 2.1 (Examples of the subdifferential)**.** For any $\varepsilon \geq 0$, let $f^\varepsilon : \mathbb{R} \longrightarrow [0, \infty)$ be a continuous and convex function, defined as follows:

$$f^\varepsilon : y \in \mathbb{R} \mapsto f^\varepsilon(y) := \sqrt{\varepsilon^2 + |y|^2} \in [0, \infty). \tag{2.1.2}$$

When $\varepsilon > 0$, $f^\varepsilon \in C^\infty(\mathbb{R})$, and hence the subdifferential $\partial f^\varepsilon \subset \mathbb{R} \times \mathbb{R}$ coincides with the single-valued function of the standard differential $(f^\varepsilon)' \in L^\infty(\mathbb{R})$, i.e.:

$$D(\partial f^\varepsilon) = \mathbb{R}, \text{ and } \partial f^\varepsilon(y) = (f^\varepsilon)'(y) = \frac{y}{\sqrt{\varepsilon^2 + |y|^2}}, \text{ for any } y \in \mathbb{R}.$$

Meanwhile, when $\varepsilon = 0$, the corresponding function f^0 coincides with the function of absolute value $|\cdot| : \mathbb{R} \longrightarrow [0, \infty)$. Hence, the subdifferential ∂f^0 of this case coincides with the set-valued signal function $\mathrm{Sgn} : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$, which is defined as follows:

$$\xi \in \mathbb{R} \mapsto \mathrm{Sgn}(\xi) := \begin{cases} \dfrac{\xi}{|\xi|}, \text{ if } \xi \neq 0, \\ [-1, 1], \quad \text{otherwise}, \end{cases} \tag{2.1.3}$$

i.e.:

$$D(\partial f^0) = D(\partial|\cdot|) = \mathbb{R}, \text{ and } \partial f^0(y) = \partial|\cdot|(y) = \mathrm{Sgn}(y), \text{ for any } y \in \mathbb{R}.$$

Next, we mention about a notion of functional convergence, known as "Mosco-convergence".

Definition 2.1 (Mosco-convergence: cf. [59])**.** Let X be an abstract Hilbert space. Let $\Psi : X \longrightarrow (-\infty, \infty]$ be a proper, l.s.c., and convex function, and let $\{\Psi_n\}_{n=1}^\infty$ be a sequence of proper, l.s.c., and convex functions $\Psi_n : X \longrightarrow (-\infty, \infty]$, $n = 1, 2, 3, \ldots$. Then, it is said that $\Psi_n \to \Psi$ on X, in the sense of Mosco, as $n \to \infty$, iff. the following two conditions are fulfilled.

(M1) The condition of lower-bound: $\displaystyle\varliminf_{n\to\infty} \Psi_n(\check{w}_n) \geq \Psi(\check{w})$, if $\check{w} \in X$, $\{\check{w}_n\}_{n=1}^\infty \subset X$, and $\check{w}_n \to \check{w}$ weakly in X, as $n \to \infty$.

(M2) The condition of optimality: for any $\hat{w} \in D(\Psi)$, there exists a sequence $\{\hat{w}_n\}_{n=1}^\infty \subset X$ such that $\hat{w}_n \to \hat{w}$ in X and $\Psi_n(\hat{w}_n) \to \Psi(\hat{w})$, as $n \to \infty$.

Remark 2.1. Let X, Ψ, and $\{\Psi_n\}_{n=1}^\infty$ be as in Definition 2.1. Then, the following facts hold.

(Fact 1) (cf. [10, Theorem 3.66]) Let us assume that

$$\Psi_n \to \Psi \text{ on } X, \text{ in the sense of Mosco, as } n \to \infty,$$

and

$$\begin{cases} [w, w^*] \in X \times X, \quad [w_n, w_n^*] \in \partial\Psi_n \text{ in } X \times X, \, n \in \mathbb{N}, \\ w_n \to w \text{ in } X, \text{ and } w_n^* \to w^* \text{ weakly in } X, \text{ as } n \to \infty. \end{cases}$$

Then, it holds that:

$$[w, w^*] \in \partial\Psi \text{ in } X \times X, \text{ and } \Psi_n(w_n) \to \Psi(w), \text{ as } n \to \infty.$$

(Fact 2) (cf. [22, Lemma 4.1] and [30, Appendix]) Let $N \in \mathbb{N}$ denote a constant of dimension, and let $S \subset \mathbb{R}^N$ be a bounded open set. Then, under the assumptions and notations as in (Fact 1), a sequence $\{\widehat{\Psi}_n^S\}_{n=1}^\infty$ of proper, l.s.c., and convex functions on $L^2(S; X)$, defined as:

$$z \in L^2(S; X) \mapsto \widehat{\Psi}_n^S(z) := \begin{cases} \displaystyle\int_S \Psi_n(z(y))\, dt, \\ \qquad \text{if } \Psi_n(z) \in L^1(S), \\ \infty, \quad \text{otherwise}, \end{cases} \text{for } n = 1, 2, 3, \ldots;$$

converges to a proper, l.s.c., and convex function $\widehat{\Psi}^S$ on $L^2(S; X)$, defined as:

$$z \in L^2(S; X) \mapsto \widehat{\Psi}^S(z) := \begin{cases} \displaystyle\int_S \Psi(z(y))\, dt, \text{ if } \Psi(z) \in L^1(S), \\ \infty, \quad \text{otherwise}; \end{cases}$$

on $L^2(S; X)$, in the sense of Mosco, as $n \to \infty$.

Example 2.2 (Example of Mosco-convergence)**.** Let $\{f^\varepsilon\}_{\varepsilon \geq 0} \subset C(\mathbb{R})$ be the sequence of nonexpansive convex functions, as in (2.1.2). Then, for any $\varepsilon_0 \geq 0$, $f^\varepsilon \to f^{\varepsilon_0}$, uniformly on $\mathbb{R}$, as $\varepsilon \to \varepsilon_0$, so that:

$$f^\varepsilon \to f^{\varepsilon_0} \text{ on } \mathbb{R}, \text{ in the sense of Mosco, as } \varepsilon \to \varepsilon_0.$$

Basic and specific notations. For arbitrary $r_0, s_0 \in [-\infty, \infty]$, we define:

$$r_0 \vee s_0 := \max\{r_0, s_0\} \text{ and } r_0 \wedge s_0 := \min\{r_0, s_0\},$$

and in particular, we set:

$$[r]^+ := r \vee 0 \text{ and } [r]^- := -(r \wedge 0), \text{ for any } r \in \mathbb{R}.$$

Finally, we remark on the specific functionals $V_\alpha : H \longrightarrow [0, \infty]$, $W_\beta : H \longrightarrow [0, \infty]$, and $\Phi_{\alpha,\beta} : H \longrightarrow [0, \infty]$, that are defined in (1.5.2), (1.5.3), and (1.5.1), respectively.

Remark 2.2. (cf. [3, 15]) The functional V_α coincides with the so-called *lower semi-continuous envelope* of the following convex function:

$$\theta \in W^{1,1}(\Omega) \mapsto \widetilde{V}_\alpha(\theta) := \int_\Omega \alpha |\partial_x \theta|\, dx \in [0, \infty),$$

more precisely,

$$V_\alpha(\theta) = \inf \left\{ \varliminf_{i \to \infty} \widetilde{V}_\alpha(\tilde{\vartheta}_i) \; \middle| \; \begin{array}{l} \{\tilde{\vartheta}_i\}_{i=1}^\infty \subset W^{1,1}(\Omega), \\ \text{and } \tilde{\vartheta}_i \to \theta \text{ in } H, \text{ as} \\ i \to \infty \end{array} \right\}, \tag{2.1.4}$$
for any $\theta \in H$.

In the light of (1.5.2) and (2.1.4), we can verify the following facts.

(Fact 3) V_α is a proper, l.s.c., and convex function on H, such that:

- the restriction $V_\alpha|_{W^{1,1}(\Omega)}$ coincides with $\widetilde{V}_\alpha$;
- $D(V_\alpha) \supset BV(\Omega)$, and $D(V_\alpha) = BV(\Omega)$ if $\min \alpha(\overline{\Omega}) > 0$.

(Fact 4) For any $\theta \in D(V_\alpha)$, there exists $\{\vartheta_i\}_{i=1}^\infty \subset W^{1,1}(\Omega)$ such that $\vartheta_i \to \theta$ in H, and $\widetilde{V}_\alpha(\vartheta_i) \to V_\alpha(\theta)$, as $i \to \infty$.

Remark 2.3. The functional W_β is a proper, l.s.c., and convex function on H, such that $D(W_\beta) = V$. Moreover, the subdifferential $\partial W_\beta \subset H \times H$ is a single valued operator, such that

$$[\theta, \theta^*] \in \partial W_\beta \text{ in } H \times H, \text{ iff. } \beta \partial_x \theta \in H_0^1(\Omega), \text{ and } \theta^* = -\partial_x(\beta \partial_x \theta) \text{ in } H.$$

Remark 2.4. Let us fix $\varepsilon \geq 0$ and let $\Phi_{\alpha,\beta}^\varepsilon$ be a function on H, defined as follows:

$$\Phi_{\alpha,\beta}^\varepsilon(\theta) := \begin{cases} \displaystyle\int_\Omega \alpha \sqrt{\varepsilon^2 + |\partial_x \theta|^2}\, dx + \frac{1}{2} \int_\Omega \beta |\partial_x \theta|^2 dx, & \text{if } \theta \in V, \\ \infty, \text{ otherwise.} \end{cases} \tag{2.1.5}$$

Under the assumption (2.1.1), the functions $\Phi_{\alpha,\beta}^\varepsilon$, for $\varepsilon \geq 0$, are proper, l.s.c., and convex on H. Especially, when $\varepsilon = 0$, the corresponding functional $\Phi_{\alpha,\beta}^0$ coincides with the convex function $\Phi_{\alpha,\beta}$, defined in (1.5.1).

Remark 2.5. Let us fix any $\varepsilon > 0$, and let us define a map $\mathcal{A}^\varepsilon : D(\mathcal{A}^\varepsilon) \subset H \longrightarrow H$, by putting:

$$D(\mathcal{A}^\varepsilon) := \left\{ \theta \in V \; \middle| \; \alpha (f^\varepsilon)'(\partial_x \theta) + \beta \partial_x \theta \in H_0^1(\Omega) \right\},$$

and

$$\theta \in D(\mathcal{A}^\varepsilon) \subset H \mapsto \mathcal{A}^\varepsilon \theta := -\partial_x \big(\alpha (f^\varepsilon)'(\partial_x \theta) + \beta \partial_x \theta \big).$$

Then, by applying the standard variational technique, we can observe that:

$$\mathcal{A}^\varepsilon = \partial \Phi_{\alpha,\beta}^\varepsilon \text{ in } H \times H.$$

2.2 Auxiliary lemma

In this Section, we prove an auxiliary lemma which is associated with the approximating approach to the Main Theorem.

Lemma 2.1. Let $\{\varepsilon_m\}_{m=1}^{\infty} \subset (0, \infty)$ be arbitrary sequence such that $\varepsilon_m \to 0$ as $m \to \infty$. Then, for the sequence $\{\Phi_{\alpha,\beta}^{\varepsilon_m}\}_{m=1}^{\infty}$, it holds that:

$$\Phi_{\alpha,\beta}^{\varepsilon_m} \to \Phi_{\alpha,\beta} \text{ on } H, \text{ in the sense of Mosco, as } m \to \infty.$$

Proof. First, we show the lower-bound condition (M1) in Definition 2.1. Let $\theta \in H$ and $\{\theta^m\}_{m=1}^{\infty} \subset H$ be such that:

$$\theta^m \to \theta \text{ weakly in } H, \text{ as } m \to \infty. \tag{2.2.1}$$

Then, it is sufficient to consider only the case when $\underline{\lim}_{m\to\infty} \Phi_{\alpha,\beta}^{\varepsilon_m}(\theta^m) < \infty$, since the other case is trivial. So, by taking a subsequence $\{m_k\}_{k=1}^{\infty} \subset \{m\}$, one can say that:

$$\varliminf_{m\to\infty} \Phi_{\alpha,\beta}^{\varepsilon_m}(\theta^m) = \lim_{k\to\infty} \Phi_{\alpha,\beta}^{\varepsilon_{m_k}}(\theta^{m_k}) < \infty. \tag{2.2.2}$$

With (2.1.5), (2.2.1), and (2.2.2) in mind, we further see that:

$$\begin{aligned} &\partial_x\theta^{m_k} \to \partial_x\theta \text{ weakly in } H, \\ \text{and } &\sqrt{\beta}\partial_x\theta^{m_k} \to \sqrt{\beta}\partial_x\theta \text{ weakly in } H, \text{ as } k \to \infty, \end{aligned} \tag{2.2.3}$$

by taking more one subsequence if necessary. In the light of (2.1.2), (2.2.1)–(2.2.3), Remark 2.3, weakly lower semi-continuity of $\Phi_{\alpha,\beta}$, the lower-bound condition can be verified (M1), as follows:

$$\lim_{k\to\infty} \Phi_{\alpha,\beta}^{\varepsilon_{m_k}}(\theta^{m_k}) \geq \varliminf_{k\to\infty} \Phi_{\alpha,\beta}(\theta^{m_k}) \geq \Phi_{\alpha,\beta}(\theta).$$

Next, we show the optimality condition (M2) in Definition2.1. Let us fix any $\theta \in D(\Phi_{\alpha,\beta})(= V)$, and let us take a sequence $\{\varphi^k\}_{k=1}^{\infty} \subset C^\infty(\overline{\Omega})$ such that:

$$\varphi^k \to \theta \text{ in } V, \text{ and in the pointwise sense, a.e. in } \Omega, \text{ as } k \to \infty. \tag{2.2.4}$$

By (2.2.4) and Lebesgue's dominated convergence theorem, we can configure a sequence $\{m_k\}_{k=0}^{\infty} \subset \mathbb{N}$ such that $1 =: m_0 < m_1 < m_2 < \cdots < m_k \uparrow \infty$, as $k \to \infty$, and for any $k \in \mathbb{N} \cup \{0\}$,

$$\sup_{m \geq m_k} \left| f^{\varepsilon_m}(\partial_x\varphi^k) - |\partial_x\varphi^k| \right|_{L^1(\Omega)} < \frac{1}{2^k(|\alpha|_{L^\infty(\Omega)} + 1)}. \tag{2.2.5}$$

Based on these, let us define:

$$\theta^m := \begin{cases} \varphi^k \text{ if } m_k \leq m < m_{k+1}, \text{ for } k \in \mathbb{N}, \\ \varphi^1 \text{ if } 1 \leq m < m_1, \end{cases} \quad \text{for any } m \in \mathbb{N}. \tag{2.2.6}$$

Taking into account (2.2.4)–(2.2.6) and Hölder's inequality, we obtain that:

$$\begin{aligned} &\left| \Phi_{\alpha,\beta}^{\varepsilon_m}(\theta^m) - \Phi_{\alpha,\beta}(\theta) \right| \\ &\leq \left| \int_\Omega \bigl(\alpha f^{\varepsilon_m}(\partial_x\theta^m) - \alpha|\partial_x\theta|\bigr)\, dx \right| + \frac{1}{2}\int_\Omega \beta \bigl| |\partial_x\theta^m|^2 - |\partial_x\theta|^2 \bigr|\, dx \end{aligned}$$

$$
\begin{aligned}
&\le |\alpha|_{L^\infty(\Omega)} \left(\int_\Omega \sup_{m \ge m_k} \left| f^{\varepsilon_m}(\partial_x \varphi^k) - |\partial_x \varphi^k| \right| dx + \int_\Omega \left| |\partial_x \varphi^k| - |\partial_x \theta| \right| dx \right) \\
&\qquad + \frac{|\beta|_{L^\infty(\Omega)}}{2} |\varphi^k - \theta|_V \left(\int_\Omega 2(|\partial_x \varphi^k|^2 + |\partial_x \theta|^2)\, dx \right)^{\frac{1}{2}} \\
&\le \frac{1}{2^k} + |\varphi^k - \theta|_V \cdot \\
&\qquad \cdot \left(\sqrt{2L} |\alpha|_{L^\infty(\Omega)} + \frac{|\beta|_{L^\infty(\Omega)}}{2} \left(\int_\Omega 2(|\partial_x \varphi^k|^2 + |\partial_x \theta|^2)\, dx \right)^{\frac{1}{2}} \right),
\end{aligned}
$$

for any $k \in \mathbb{N} \cup \{0\}$ and any $m \ge m_k$,

and therefore,

$$
\Phi_{\alpha,\beta}^{\varepsilon_m}(\theta^m) \to \Phi_{\alpha,\beta}(\theta), \text{ as } m \to \infty.
$$

Thus, we conclude this lemma. □

2.3 Proof of Main Theorem

In this Section, we give the proof of Main Theorem. Let us define a set-valued map $\mathcal{A}^0 : D(\mathcal{A}^0) \subset H \longrightarrow 2^H$, by putting:

$$
D(\mathcal{A}^0) := \left\{ \theta \in V \;\middle|\; \begin{array}{l} \text{there exists } \varpi^* \in L^\infty(\Omega) \text{ such that} \\ \bullet\ \varpi^* \in \mathrm{Sgn}(\partial_x \theta) \text{ a.e. in } \Omega \\ \bullet\ \alpha \varpi^* + \beta \partial_x \theta \in H_0^1(\Omega) \end{array} \right\}, \tag{2.3.1}
$$

and

$$
\begin{aligned}
&\theta \in D(\mathcal{A}^0) \subset H \\
&\mapsto \mathcal{A}^0 \theta := \left\{ \theta^* \in H \;\middle|\; \begin{array}{l} \theta^* = -\partial_x \bigl(\alpha \varpi^* + \beta \partial_x \theta \bigr) \text{ in } H, \\ \quad \text{for some } \varpi^* \in L^\infty(\Omega), \text{ satisfying} \\ \quad \varpi^* \in \mathrm{Sgn}(\partial_x \theta) \text{ a.e. in } \Omega \end{array} \right\}.
\end{aligned} \tag{2.3.2}
$$

We prove Main Theorem in accordance with the following two Steps.

Step 1: $\mathcal{A}^0 = \partial \Phi_{\alpha,\beta}$ in $H \times H$.

Step 2: $\partial \Phi_{\alpha,\beta} = \partial V_\alpha + \partial W_\beta$ in $H \times H$.

Verification of Step 1.

First, we show $\mathcal{A}^0 \subset \Phi_{\alpha,\beta}$ in $H \times H$. Let us assume $\theta \in D(\mathcal{A}^0)$ and $\theta^* \in \mathcal{A}^0 \theta$. Then, by (2.3.2), there exists $\varpi^* \in L^\infty(\Omega)$ such that:

$$
\varpi^* \in \mathrm{Sgn}(\partial_x \theta) \text{ a.e. in } \Omega \text{ and } \theta^* = -\partial_x(\alpha \varpi^* + \beta \partial_x \theta) \text{ in } H. \tag{2.3.3}
$$

From Remark 2.2, (2.1.3), (2.3.3), and Young's inequality, we can compute that:

$$
(\theta^*, \varphi - \theta)_H = \bigl(-\partial_x \bigl(\alpha \varpi^* + \beta \partial_x \theta \bigr), \varphi - \theta \bigr)_H
$$

$$
\begin{aligned}
&= \int_\Omega \alpha\varpi^* \partial_x(\varphi - \theta)\, dx + \int_\Omega \beta\, \partial_x\theta\, \partial_x(\varphi-\theta)\, dx \\
&\leq \int_\Omega \alpha\big(|\partial_x\varphi| - |\partial_x\theta|\big)\, dx + \frac{1}{2}\int_\Omega \beta(|\partial_x\varphi|^2 - |\partial_x\theta|^2)\, dx \\
&= \Phi_{\alpha,\beta}(\varphi) - \Phi_{\alpha,\beta}(\theta), \text{ for any } \varphi \in V.
\end{aligned}
$$

This implies that:

$$\theta \in D(\partial\Phi_{\alpha,\beta}) \text{ and } \theta^* \in \partial\Phi_{\alpha,\beta}(\theta) \text{ in } H.$$

Thus, the inclusion $\mathcal{A}^0 \subset \partial\Phi_{\alpha,\beta}$ in $H \times H$ is verified.

Next, we prove the equality $(\mathcal{A}^0 + I_H)H = H$. Since, the inclusion $(\mathcal{A}^0 + I_H)H \subset H$ is trivial, it is sufficient to prove the converse inclusion.

Let us take any $h \in H$. Then, by Remark 2.5 and Minty's theorem (cf. [14, Theorem 2.2]), we can configure a class of function $\{\theta^\varepsilon\}_{\varepsilon>0} \subset V$, by setting $\{\theta^\varepsilon := (\mathcal{A}^\varepsilon + I_H)^{-1}h\}_{\varepsilon>0}$ in H, i.e.

$$h - \theta^\varepsilon = \mathcal{A}^\varepsilon\theta^\varepsilon = \partial\Phi^\varepsilon_{\alpha,\beta}(\theta^\varepsilon) \text{ in } H, \text{ for any } \varepsilon > 0, \tag{2.3.4}$$

so that:

$$
\begin{aligned}
&\int_\Omega \big(\alpha(f^\varepsilon)'(\partial_x\theta^\varepsilon) + \beta\partial_x\theta^\varepsilon\big)\partial_x\varphi\, dx + \int_\Omega \theta^\varepsilon\varphi\, dx \\
&\qquad = \int_\Omega h\varphi\, dx, \text{ for any } \varphi \in V, \text{ and any } \varepsilon > 0.
\end{aligned} \tag{2.3.5}
$$

In the variational form (2.3.5), let us put $\varphi = \theta^\varepsilon$. Then, with (2.1.2) and Young's inequality in mind, we deduce that:

$$\frac{1}{2}|\theta^\varepsilon|_H^2 + |\sqrt{\beta}\partial_x\theta^\varepsilon|_H^2 \leq \frac{1}{2}|h|_H^2, \text{ for any } \varepsilon > 0. \tag{2.3.6}$$

The above (2.3.6) enable us to take a function $\theta \in V$ and a sequence $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \cdots > \varepsilon_m \downarrow 0$, as $m \to \infty$, such that:

$$
\begin{aligned}
&\theta^{\varepsilon_m} \to \theta \text{ in } H, \text{ weakly in } V, \\
&\text{and } \sqrt{\beta}\partial_x\theta^{\varepsilon_m} \to \sqrt{\beta}\partial_x\theta \text{ weakly in } H, \text{ as } m \to \infty.
\end{aligned} \tag{2.3.7}
$$

In the light of Lemma 2.1, (2.3.4), (2.3.7), and (Fact 1), it follows that:

$$h - \theta \in \partial\Phi_{\alpha,\beta}(\theta) \text{ in } H, \text{ and } \Phi^{\varepsilon_m}_{\alpha,\beta}(\theta^{\varepsilon_m}) \to \Phi_{\alpha,\beta}(\theta), \text{ as } m \to \infty. \tag{2.3.8}$$

Also, by Remark 2.2, (2.1.2), (2.3.7), (2.3.8), and weakly lower semi-continuity of the norm $|\cdot|_H$, we can compute that:

$$
\begin{aligned}
\frac{1}{2}\int_\Omega \beta|\partial_x\theta|^2\, dx &\leq \frac{1}{2}\varliminf_{m\to\infty}\int_\Omega \beta|\partial_x\theta^{\varepsilon_m}|^2\, dx \leq \frac{1}{2}\varlimsup_{m\to\infty}\int_\Omega \beta|\partial_x\theta^{\varepsilon_m}|^2\, dx \\
&\leq \lim_{m\to\infty}\Phi^{\varepsilon_m}_{\alpha,\beta}(\theta^{\varepsilon_m}) - \varliminf_{m\to\infty}\int_\Omega \alpha f^{\varepsilon_m}(\partial_x\theta^{\varepsilon_m})\, dx
\end{aligned}
$$

$$\leq \Phi_{\alpha,\beta}(\theta) - \int_\Omega \alpha|\partial_x\theta|\,dx = \frac{1}{2}\int_\Omega \beta|\partial_x\theta|^2\,dx. \tag{2.3.9}$$

Having in mind (2.3.7), (2.3.9), and the uniform convexity of L^2-based topologies, it is deduce that:

$$\sqrt{\beta}\partial_x\theta^{\varepsilon_m} \to \sqrt{\beta}\partial_x\theta \text{ in } H, \text{ as } m \to \infty. \tag{2.3.10}$$

Furthermore, by (2.1.1), (2.3.7), and (2.3.10), we obtain that:

$$\theta^{\varepsilon_m} \to \theta \text{ in } V, \text{ and } \partial_x\theta^{\varepsilon_m} \to \partial_x\theta \text{ in } H, \text{ as } m \to \infty. \tag{2.3.11}$$

In the meantime, by Example 2.1, $|(f^{\varepsilon_m})'(\partial_x\theta^{\varepsilon_m})| \leq 1$ a.e. in Ω, for any $m \in \mathbb{N}$, and one can say

$$\begin{aligned} &(f^{\varepsilon_m})'(\partial_x\theta^{\varepsilon_m}) \to \varpi^* \text{ weakly-}* \text{ in } L^\infty(\Omega), \text{ as } m \to \infty, \\ &\qquad\qquad\qquad\qquad \text{for some } \varpi^* \in L^\infty(\Omega), \end{aligned} \tag{2.3.12}$$

by taking a subsequence if necessary.

From (2.1.3), (2.3.11), (2.3.12), Example 2.2, (Fact 1), and [18, Proposition 2.16], it is inferred that:

$$\varpi^* \in \mathrm{Sgn}(\partial_x\theta) \text{ a.e. in } \Omega. \tag{2.3.13}$$

On account of (2.3.10)–(2.3.12), letting $m \to \infty$ in (2.3.5) yields that:

$$\int_\Omega \bigl(\alpha\varpi^* + \beta\partial_x\theta\bigr)\partial_x\varphi\,dx + \int_\Omega \theta\varphi\,dx = \int_\Omega h\varphi\,dx, \text{ for any } \varphi \in V. \tag{2.3.14}$$

In particular, putting $\varphi = \varphi_0 \in H_0^1(\Omega)$ in (2.3.14), we have:

$$(h-\theta, \varphi_0)_H = \int_\Omega \bigl(\alpha\varpi^* + \beta\partial_x\theta, \partial_x\varphi_0\bigr)\,dx, \text{ for any } \varphi_0 \in H_0^1(\Omega),$$

which implies:

$$-\partial_x\bigl(\alpha\varpi^* + \beta\partial_x\theta\bigr) = h - \theta \in H, \text{ in } \mathscr{D}'(\Omega). \tag{2.3.15}$$

In addition, we observe that:

$$\begin{aligned} \Bigl(\alpha\varpi^* + \beta\partial_x\theta, \psi\Bigr)_{H_\Gamma} &= \Bigl[\bigl(\alpha(x)\varpi^*(x) + \beta(x)\partial_x\theta(x)\bigr)\psi(x)\Bigr]_{-L}^{L} \\ &= \int_\Omega \partial_x\Bigl(\bigl(\alpha\varpi^* + \beta\partial_x\theta\bigr)[\psi]^{\mathrm{ex}}\Bigr)\,dx \\ &= -\int_\Omega (h-\theta)[\psi]^{\mathrm{ex}}\,dx + \int_\Omega \bigl(\alpha\varpi^* + \beta\partial_x\theta\bigr)\partial_x[\psi]^{\mathrm{ex}}\,dx \\ &= 0, \text{ for any } \psi \in H_\Gamma \text{ with any extension } [\psi]^{\mathrm{ex}} \in V. \end{aligned} \tag{2.3.16}$$

(2.3.15) and (2.3.16) lead to:

$$\alpha\varpi^* + \beta\partial_x\theta \in H_0^1(\Omega). \tag{2.3.17}$$

As a consequence of (2.3.1), (2.3.2), (2.3.13), and (2.3.17), we obtain that:

$$(\mathcal{A}^0 + I_H)\theta = h \text{ in } H, \text{ i.e. } h \in (\mathcal{A}^0 + I_H)H,$$

and we verify $H \subset (\mathcal{A}^0 + I_H)H$.

Finally, the inclusion $\mathcal{A}^0 \subset \partial\Phi_{\alpha,\beta}$ in $H \times H$, and the equality $(\mathcal{A}^0 + I_H)H = H$ enable us to apply Minty's theorem (cf. [14, Theorem 2.2]), and to verify that $\mathcal{A}^0$ is a maximal monotone. Moreover, the inclusion $\mathcal{A}^0 \subset \partial\Phi_{\alpha,\beta}$ and the maximality of $\mathcal{A}^0$ will lead to the coincidence $\mathcal{A}^0 = \partial\Phi_{\alpha,\beta}$ in $H \times H$.

Thus we finish the proof of Step 1.

Verification of Step 2.

By the general theory of the convex analysis [24, Chapter 1], we immediately have $\partial\Phi_{\alpha,\beta} \supset \partial V_\alpha + \partial W_\beta$ in $H \times H$. So, we prove the converse inclusion:

$$\partial\Phi_{\alpha,\beta} \subset \partial V_\alpha + \partial W_\beta \text{ in } H \times H. \tag{2.3.18}$$

Let us take any $[\theta, \theta^*] \in \partial\Phi_{\alpha,\beta}$ in $H \times H$, and apply the result of previous Step 1, to have a function $\varpi^* \in L^\infty(\Omega)$ as in (2.3.3). On this basis, we verify this Step 2, via the verifications of four Claims.

Claim ♯1). $\theta \in H^2(\Omega)$ and $\partial_x\theta \in H^1_0(\Omega)$.

For every $a \geq 0$ and $b > 0$, let $\rho_{(a,b)} : \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ be a set-valued function, defined as:

$$\rho_{(a,b)}(r) := a\,\mathrm{Sgn}(r) + br \subset \mathbb{R}, \text{ for any } r \in \mathbb{R}, \tag{2.3.19}$$

and let $\rho^*_{(a,b)}$ be the inverse of $\rho_{(a,b)}$. Then, as is easily checked from (2.1.3) and (2.3.19),

$$\rho^*_{(a,b)} : r \in \mathbb{R} \mapsto \frac{[r-a]^+ - [r+a]^-}{b} \in \mathbb{R}, \tag{2.3.20}$$

i.e. $(\rho_{(a,b)})^*$ is a single-valued Lipschitz function, such that

$$0 \leq (\rho^*_{(a,b)})' \leq \frac{1}{b} \text{ on } \mathbb{R}, \text{ for every } a \geq 0 \text{ and } b > 0.$$

Here, from (2.3.19), (2.3.20), and Step 1, we immediately see that:

$$\tilde{\theta} := \rho_{(\alpha(\cdot),\beta(\cdot))}(\partial_x\theta) = \alpha\varpi^* + \beta\partial_x\theta \in H^1_0(\Omega), \text{ and } \theta^* = -\partial_x\tilde{\theta} \text{ in } H. \tag{2.3.21}$$

Therefore, having in mind (2.3.20) and (2.3.21), and applying the generalized chain rule in BV-theory [3, Theorem 3.99], it is inferred that:

$$\begin{aligned}
\partial_x\theta &= (\rho^*_{(\alpha(\cdot),\beta(\cdot))})(\tilde{\theta}) = \frac{[\tilde{\theta} - \alpha]^+ - [\tilde{\theta} + \alpha]^-}{\beta} \in H^1_0(\Omega), \\
\partial_x^2\theta &= \partial_x\left[\frac{[\tilde{\theta} - \alpha]^+ - [\tilde{\theta} + \alpha]^-}{\beta}\right] \\
&= \frac{1}{\beta}\left[\partial_x(\tilde{\theta} - \alpha)\chi_{(\alpha(\cdot),\infty)}(\tilde{\theta}) + \partial_x(\tilde{\theta} + \alpha)\chi_{(-\infty,-\alpha(\cdot))}(\tilde{\theta})\right] \\
&\quad - \frac{\partial_x\beta}{\beta^2}\left([\tilde{\theta} - \alpha]^+ - [\tilde{\theta} + \alpha]^-\right) \in H.
\end{aligned}$$

Thus, Claim ♯1) is verified.

Claim ♯2). $\beta\partial_x\theta \in H_0^1(\Omega)$ and $[\theta, -\partial_x(\beta\partial_x\theta)] \in \partial W_\beta$ in $H \times H$.

This Claim ♯2) is immediately observed from Claim♯1) and Remark 2.3.

Claim ♯3). $\alpha\varpi^* \in H_0^1(\Omega)$ and $[\theta, -\partial_x(\alpha\varpi^*)] \in \partial V_\alpha$ in $H \times H$.

By using (2.3.21), Claim♯2), and the integration by part, we can observe that:

$$\alpha\varpi^* = \tilde{\theta} - \beta\partial_x\theta \in H_0^1(\Omega), \tag{2.3.22}$$

and

$$\begin{aligned}\int_\Omega -\partial_x(\alpha\varpi^*)(\varphi - \theta)\,dx &= \int_\Omega \alpha\varpi^*\partial_x(\varphi - \theta)\,dx \\ &\leq \int_\Omega \alpha|\partial_x\varphi|\,dx - \int_\Omega \alpha|\partial_x\theta|\,dx, \text{ for any } \varphi \in W^{1,1}(\Omega).\end{aligned} \tag{2.3.23}$$

Next, let us take any $z \in D(V_\alpha)$, and invoke (Fact 4) to take a sequence $\{\varphi_i\}_{i=1}^\infty \subset W^{1,1}(\Omega)$ such that:

$$\varphi_i \to z \text{ in } H, \text{ and } \widetilde{V}_\alpha(\varphi_i)\left(= \int_\Omega \alpha|\partial_x\varphi_i|\,dx\right) \to V_\alpha(z), \text{ as } i \to \infty. \tag{2.3.24}$$

Besides, putting $\varphi = \varphi_i$ in (2.3.23), with $i \in \mathbb{N}$, and using (2.3.24), we deduce that

$$\begin{aligned}(-\partial_x(\alpha\varpi^*), z - \theta)_H &+ V_\alpha(\theta) \\ &= \lim_{i\to\infty}\int_\Omega -\partial_x(\alpha\varpi^*)(\varphi_i - \theta)\,dx + V_\alpha(\theta) \\ &\leq \lim_{i\to\infty}\widetilde{V}_\alpha(\varphi_i) = V_\alpha(z), \text{ for any } z \in D(V_\alpha).\end{aligned} \tag{2.3.25}$$

(2.3.22) and (2.3.25) finish the verification of Claim ♯3).

Claim ♯4). $\theta^* \in \partial V_\alpha(\theta) + \partial W_\beta(\theta)$ in H.

This Claim ♯4) will be a straight forward consequence of (2.3.2), Step 1, Claim ♯1)–Claim ♯3), and the linearity of distributional differential:

$$\theta^* = -\partial_x(\alpha\varpi^* + \beta\partial_x\theta) = -\partial_x(\alpha\varpi^*) - \partial_x(\beta\partial_x\theta) \text{ in } \mathscr{D}'(\Omega).$$

Claim ♯1)–Claim ♯4) enable us to verify the inclusion (2.3.18), and to complete the proof of Main Theorem. □

Chapter 3

One-dimensional optimal control problems for time-discrete constrained quasilinear diffusion equations of Allen–Cahn types

In this Chapter, we recall the class of optimal control problems for a one-dimensional time-discrete constrained quasilinear diffusion state-systems of singular Allen–Cahn types and its regularized approximating problems. We note that the control parameter for each system is given by physical temperature. The principal part of this paper is started with the verification of a Key-Theorem dealing with the decompositions of the subdifferentials of the governing convex energies of the state-systems. On this basis, we will prove five Main Theorems, concerned with: the solvability and precise regularity results of state-systems; the continuous-dependence of the solutions to state-systems including convergences in spatially C^1-topologies; the existence and parameter-dependence of optimal controls; the necessary optimality conditions for approximate optimal controls; precise characterizations of the approximating limit of the optimality conditions.

3.1 Preliminaries

We begin by prescribing the notations used throughout this Chapter.

Basis notations. For arbitrary r_0, $s_0 \in [-\infty, \infty]$, we define:

$$r_0 \vee s_0 := \max\{r_0, s_0\} \text{ and } r_0 \wedge s_0 := \min\{r_0, s_0\},$$

and in particular, we set:

$$[r]^+ := r \vee 0 \text{ and } [r]^- := -(r \wedge 0), \text{ for any } r \in \mathbb{R}.$$

Abstract notations. For an abstract Banach space E, we denote by $|\cdot|_E$ the norm of E, and denote by $\langle \cdot, \cdot \rangle_E$ the duality pairing between E and its dual E^*. Let $I_d : E \longrightarrow E$ be the identity map from E onto E. In particular, when H is a Hilbert space, we denote by $(\cdot, \cdot)_H$ the inner product of H.

For any subset A of a Banach space E, let $\chi_A : E \longrightarrow \{0,1\}$ be the characteristic function of A, i.e.:

$$\chi_A : w \in E \mapsto \chi_A(w) := \begin{cases} 1, & \text{if } w \in A, \\ 0, & \text{otherwise.} \end{cases}$$

For two Banach spaces E and Ξ, we denote by $\mathscr{L}(E;\Xi)$ the Banach space of bounded linear operators from E into Ξ, and in particular, we let $\mathscr{L}(E) := \mathscr{L}(E;E)$.

For Banach spaces $E_1, \dots, E_N$, with $1 < N \in \mathbb{N}$, let $E_1 \times \cdots \times E_N$ be the product Banach space endowed with the norm $|\cdot|_{E_1 \times \cdots \times E_N} := |\cdot|_{E_1} + \cdots + |\cdot|_{E_N}$. However, when all $H_1, \dots, H_N$ are Hilbert spaces, $H_1 \times \cdots \times H_N$ denotes the product Hilbert space endowed with the inner product $(\cdot,\cdot)_{H_1 \times \cdots \times H_N} := (\cdot,\cdot)_{H_1} + \cdots + (\cdot,\cdot)_{H_N}$ and the norm $|\cdot|_{H_1 \times \cdots \times H_N} := \bigl(|\cdot|_{H_1}^2 + \cdots + |\cdot|_{H_N}^2\bigr)^{\frac{1}{2}}$. In particular, when all $E_1, \dots, E_N$ coincide with a Banach space Ξ, we write:

$$[\Xi]^N := \overbrace{\Xi \times \cdots \times \Xi}^{N \text{ times}}.$$

Additionally, for any transform (operator) $\mathcal{T} : E \longrightarrow \Xi$, we let:

$$\mathcal{T}[w_1, \dots, w_N] := \bigl[\mathcal{T}w_1, \dots, \mathcal{T}w_N\bigr] \text{ in } [\Xi]^N, \quad \text{for any } [w_1, \dots, w_N] \in [E]^N.$$

Specific notations of this Chapter. As is mentioned in the previous section, let $(0,T) \subset \mathbb{R}$ be a bounded time-interval with a finite constant $T > 0$. Let $\Omega := (-L, L) \subset \mathbb{R}$ be a fixed spatial bounded domain with a finite constant $L > 0$. Especially, we denote by ∂_x the distributional spatial derivative.

On this basis, we define

$$X := L^2(\Omega), Y := H^1(\Omega)\ (\subset C(\overline{\Omega})),\ \mathbb{X} := [X]^n, \text{ and } \mathbb{Y} := [Y]^n, \text{ for any } n \in \mathbb{N}.$$

Notations in convex analysis. (cf. [18, Chapter II]) Let H be an abstract Hilbert space. For a proper, lower semi-continuous (l.s.c.), and convex function $\psi : H \longrightarrow (-\infty, \infty]$, we denote by $D(\psi)$ the effective domain of ψ. Also, we denote by $\partial\psi$ the subdifferential of ψ. The subdifferential $\partial\psi$ corresponds to a weak differential of convex function ψ, and it is known as a maximal monotone graph in the product space $H \times H$. The set $D(\partial\psi) := \bigl\{z \in H \mid \partial\psi(z) \neq \emptyset\bigr\}$ is called the domain of $\partial\psi$. We often use the notation "$[z_0, z_0^*] \in \partial\psi$ in $H \times H$", to mean that "$z_0^* \in \partial\psi(z_0)$ in H for $z_0 \in D(\partial\psi)$", by identifying the operator $\partial\psi$ with its graph in $H \times H$.

Example 3.1 (Examples of the subdifferential)**.** As one of the representatives of the subdifferentials, we exemplify the following set-valued signal function $\mathrm{Sgn}^N : \mathbb{R}^N \longrightarrow 2^{\mathbb{R}^N}$, with $N \in \mathbb{N}$, which is defined as:

$$\begin{aligned} \xi = [\xi_1, \dots, \xi_N] \in \mathbb{R}^N \mapsto \mathrm{Sgn}^N(\xi) &= \mathrm{Sgn}^N(\xi_1, \dots, \xi_N) \\ &:= \begin{cases} \dfrac{\xi}{|\xi|} = \dfrac{[\xi_1, \dots, \xi_N]}{\sqrt{\xi_1^2 + \cdots + \xi_N^2}}, & \text{if } \xi \neq 0, \\ \mathbb{D}^N, & \text{otherwise,} \end{cases} \end{aligned} \tag{3.1.1}$$

where $\mathbb{D}^N$ denotes the closed unit ball in $\mathbb{R}^N$ centered at the origin. Indeed, the set-valued function Sgn^N coincides with the subdifferential of the Euclidean norm $|\cdot| : \xi \in \mathbb{R}^N \mapsto |\xi| = \sqrt{\xi_1^2 + \cdots + \xi_N^2} \in [0, \infty)$, i.e.:

$$\partial|\cdot|(\xi) = \mathrm{Sgn}^N(\xi), \text{ for any } \xi \in D(\partial|\cdot|) = \mathbb{R}^N,$$

and furthermore, it is observed that:

$$\partial|\cdot|(0) = \mathbb{D}^N \subsetneq [-1,1]^N = \bigl[\partial_{\xi_1}|\cdot| \times \cdots \times \partial_{\xi_N}|\cdot|\bigr](0).$$

Finally, we mention about a notion of functional convergence, known as "Mosco-convergence".

Definition 3.1 (Mosco-convergence: cf. [59])**.** Let H be an abstract Hilbert space. Let $\psi : H \longrightarrow (-\infty, \infty]$ be a proper, l.s.c., and convex function, and let $\{\psi_n\}_{n\in\mathbb{N}}$ be a sequence of proper, l.s.c., and convex functions $\psi_n : H \longrightarrow (-\infty, \infty]$, $n = 1, 2, 3, \dots$. Then, it is said that $\psi_n \to \psi$ on H, in the sense of Mosco, as $n \to \infty$, iff. the following two conditions are fulfilled:

(M1) The condition of lower-bound: $\varliminf_{n\to\infty} \psi_n(\check{w}_n) \geq \psi(\check{w})$, if $\check{w} \in H$, $\{\check{w}_n\}_{n\in\mathbb{N}} \subset H$, and $\check{w}_n \to \check{w}$ weakly in H, as $n \to \infty$;

(M2) The condition of optimality: for any $\hat{w} \in D(\psi)$, there exists a sequence $\{\hat{w}_n\}_{n\in\mathbb{N}} \subset H$ such that $\hat{w}_n \to \hat{w}$ in H and $\psi_n(\hat{w}_n) \to \psi(\hat{w})$, as $n \to \infty$.

As well as, if the sequence of convex functions $\{\widehat{\psi}_\varepsilon\}_{\varepsilon\in\Lambda}$ is labeled by a continuous argument $\varepsilon \in \Lambda$ with a range $\Lambda \subset \mathbb{R}$, then for any $\varepsilon_0 \in \Lambda$, the Mosco-convergence of $\{\widehat{\psi}_\varepsilon\}_{\varepsilon\in\Lambda}$, as $\varepsilon \to \varepsilon_0$, is defined by those of subsequences $\{\widehat{\psi}_{\varepsilon_n}\}_{n\in\mathbb{N}}$, for all sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \Lambda$, satisfying $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$.

Remark 3.1. Let H, ψ, and $\{\psi_n\}_{n\in\mathbb{N}}$ be as in Definition 3.1. Then, the following hold.

(Fact 1) (cf. [10, Theorem 3.66] and [39, Chapter 2]) Let us assume that

$$\psi_n \to \psi \text{ on } H, \text{ in the sense of Mosco, as } n \to \infty, \tag{3.1.2}$$

and

$$\begin{cases} [w, w^*] \in H \times H, \quad [w_n, w_n^*] \in \partial\psi_n \text{ in } H \times H,\ n \in \mathbb{N}, \\ w_n \to w \text{ in } H \text{ and } w_n^* \to w^* \text{ weakly in } H, \text{ as } n \to \infty. \end{cases}$$

Then, it holds that:

$$[w, w^*] \in \partial\psi \text{ in } H \times H, \text{ and } \psi_n(w_n) \to \psi(w), \text{ as } n \to \infty.$$

(Fact 2) (cf. [22, Lemma 4.1] and [30, Appendix]) Let $N \in \mathbb{N}$ denote the dimension constant, and let $S \subset \mathbb{R}^N$ be a bounded open set. Then, under the Mosco convergence as in (3.1.2), a sequence $\{\widehat{\psi}_n^S\}_{n\in\mathbb{N}}$ of proper, l.s.c., and convex functions on $L^2(S; H)$, defined as:

$$w \in L^2(S;H) \mapsto \widehat{\psi}_n^S(w) := \begin{cases} \displaystyle\int_S \psi_n(w(t))\,dt, \\ \qquad \text{if } \psi_n(w) \in L^1(S), \ \text{ for } n = 1, 2, 3, \dots; \\ \infty, \quad \text{otherwise,} \end{cases}$$

converges to a proper, l.s.c., and convex function $\widehat{\psi}^S$ on $L^2(S;H)$, defined as:

$$z \in L^2(S;H) \mapsto \widehat{\psi}^S(z) := \begin{cases} \displaystyle\int_S \psi(z(t))\,dt, & \text{if } \psi(z) \in L^1(S), \\ \infty, & \text{otherwise;} \end{cases}$$

on $L^2(S;H)$, in the sense of Mosco, as $n \to \infty$.

3.2 Auxiliary results

In this Section, we recall the previous work [49], and show some auxiliary results. Let $0 \leq \alpha \in Y$ and $0 < \beta \in Y$ be fixed functions, and let us set the following convex function on X:

$$z \in X \mapsto \Phi_{\alpha,\beta}(z) := V_\alpha(z) + W_\beta(z); \tag{3.2.1}$$

which is defined as a sum of two convex functions on X, defined as follows:

$$z \in X \mapsto V_\alpha(z) := \sup \left\{ \int_\Omega z\partial_x\varphi\,dx, \;\middle|\; \begin{array}{l} \varphi \in Y \cap C_c(\Omega), \text{ such} \\ \text{that } |\varphi| \leq \alpha \text{ on } \overline{\Omega} \end{array} \right\}, \tag{3.2.2}$$

and

$$z \in X \mapsto W_\beta(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \beta|\partial_x z|^2 dx, & \text{if } z \in Y, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.2.3}$$

The functional V_α, defined in (3.2.2), is a kind of generalized total variation, so that the functional $\Phi_{\alpha,\beta}$, defined in (3.2.1), can be called a *regularized total variation* with nonhomogeneous coefficients α and β.

Remark 3.2. (cf. [3, 15]) The functional V_α coincides with the so-called *lower semi-continuous envelope* of the following convex function:

$$z \in W^{1,1}(\Omega) \mapsto \widetilde{V}_\alpha(z) := \int_\Omega \alpha|\partial_x z|\,dx \in [0,\infty),$$

more precisely,

$$V_\alpha(z) = \inf \left\{ \varliminf_{i\to\infty} \widetilde{V}_\alpha(\tilde{z}_i) \;\middle|\; \{\tilde{z}_i\}_{i\in\mathbb{N}} \subset W^{1,1}(\Omega), \text{ and } \tilde{z}_i \to z \text{ in } X, \text{ as } i \to \infty \right\}.$$

Remark 3.3. The functional W_β, defined in (3.2.3), is a proper, l.s.c., and convex function on X, such that $D(W_\beta) = Y$. Moreover, the subdifferential $\partial W_\beta \subset X \times X$ is a single valued operator, such that

$$[z, z^*] \in \partial W_\beta \text{ in } X \times X, \text{ iff. } \beta\partial_x z \in H_0^1(\Omega), \text{ and } z^* = -\partial_x(\beta\partial_x z) \text{ in } X.$$

Now, we refer to the previous work [49], to recall the key-properties of $\Phi_{\alpha,\beta}$, in forms of Proposition.

Proposition 3.1. [49, Main Theorem] The subdifferential $\partial\Phi_{\alpha,\beta} \subset X \times X$ of the convex function $\Phi_{\alpha,\beta}$ is decomposed as follows:

$$\partial\Phi_{\alpha,\beta} = \partial V_\alpha + \partial W_\beta \text{ in } X \times X,$$

i.e. $\partial\Phi_{\alpha,\beta}$ is represented as the sum of the subdifferentials $\partial V_\alpha \subset X \times X$ and $\partial W_\beta \subset X \times X$ of the respective convex functions V_α and W_β.

3.3 Main Theorems

We begin by setting up the assumptions needed in our Main Theorems. All Main Theorems are discussed under the following assumptions.

(A1) Let $\nu > 0$ be a fixed constant.

(A2) We denote by f^0 the absolute value function on $\mathbb{R}$, i.e., $f^0(r) := |r|$ for all $r \in \mathbb{R}$. In addition, let $\{f^\varepsilon\}_{\varepsilon\in(0,1]} \subset C^2(\mathbb{R})$ be a sequence of convex C^2-regularizations of $f^0(\cdot) := |\cdot|$, such that:

$$f^\varepsilon(0) = 0 \text{ and } f^\varepsilon(r) \geq 0 \ \text{ for any } r \in \mathbb{R} \text{ and any } \varepsilon \in (0,1],$$

$$\begin{cases} f^\varepsilon(r) \to f^{\varepsilon_0}(r) & \text{for any } r \in \mathbb{R}, \\ f^\varepsilon(\cdot) \to f^{\varepsilon_0}(\cdot) & \text{on } \mathbb{R}, \ \ \text{in the sense of Mosco,} \end{cases} \quad \text{as } \varepsilon \to \varepsilon_0, \text{ for any } \varepsilon_0 \in [0,1],$$

and there exists a positive constant $C_0 > 0$, independent of $\varepsilon \in (0,1]$, satisfying:

$$|(f^\varepsilon)'(r)| \leq C_0(1 + |r|) \ \text{ for any } r \in \mathbb{R} \text{ and any } \varepsilon \in (0,1],$$

and

$$(f^\varepsilon)'' \to 0 \text{ uniformly on } \{|r| \geq \lambda\}, \ \text{for any } \lambda > 0, \ \text{as } \varepsilon \to 0. \tag{3.3.1}$$

(A3) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^1-function, which is a semi-monotone function on $\mathbb{R}$, i.e. there exists a positive constant $C_g > 0$ such that:

$$g(r) + C_g r \text{ is monotone in } r \in \mathbb{R}.$$

Also, g has a nonnegative primitive $0 \leq G \in C^2(\mathbb{R})$, i.e. the derivative G' coincides with g on $\mathbb{R}$.

(A4) We denote by $\hat{K}^0$, and K^0 the indicator function $I_{[-1,1]}$, and the subdifferential, i.e.

$$\hat{K}^0 = I_{[-1,1]} \text{ on } \mathbb{R}, \text{ and } K^0 := \partial I_{[-1,1]} \text{ in } \mathbb{R} \times \mathbb{R},$$

respectively. In addition, let $\{\hat{K}^\delta\}_{\delta\in(0,1]} \subset C^2(\mathbb{R})$, and $\{K^\delta\}_{\delta\in(0,1]} \subset C^1(\mathbb{R})$ be, respectively, the approximating sequence of $\hat{K}^0$, and K^0, such that:

- $K^\delta = (\hat{K}^\delta)'$ on $\mathbb{R}$, and $\hat{K}^\delta \equiv 0$ on $[-1, 1]$, for all $\delta \in (0, 1]$;
- for any $\delta \in (0, 1]$, there exists a constant $C_K^\delta > 0$ satisfying $0 \leq (K^\delta)' \leq C_K^\delta$ on $\mathbb{R}$;
- for any $\delta_0 \in [0, 1]$, $\hat{K}^\delta \to \hat{K}^{\delta_0}$ on $\mathbb{R}$, in the sense of Mosco, as $\delta \to \delta_0$, and in particular, if $\delta_0 \in (0, 1]$, then $\hat{K}^\delta \to \hat{K}^{\delta_0}$ in $C_{\mathrm{loc}}(\mathbb{R})$, as $\delta \to \delta_0$.

(A5) Let $T > 0$ be a fixed constant, and let τ^* be a small positive constant, such that:

$$\tau^* := \frac{1}{8\big(C_g + 1\big)}.$$

On this basis, we fix constants of the time-step number $n \in \mathbb{N}$ and the time-step size $\tau > 0$, to satisfy that:

$$0 < \tau := \frac{T}{n} < \tau^*.$$

(A6) Let $w^{\mathrm{ad}} = [w_1^{\mathrm{ad}}, \dots, w_n^{\mathrm{ad}}] \in \mathbb{X}$ be a fixed *admissible target profile.*

Remark 3.4. (cf. [22, 51, 58]) The assumption similar to (A2) was introduced in [22, Section 3], [51, Remark 3.1], and [58, Definition 3.1]. In this context, the typical examples of f^ε are as follows:

- (Hyperbola type) $f^\varepsilon(r) = \sqrt{r^2 + \varepsilon^2} - \varepsilon$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0, 1]$.
- (Hyperbolic-tangent type) $f^\varepsilon(r) = \varepsilon \log\Big(\cosh\Big(\dfrac{r}{\varepsilon}\Big)\Big)$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0, 1]$.
- (Arctangent type) $f^\varepsilon(r) = \dfrac{2\varepsilon}{\pi}\left[\dfrac{r}{\varepsilon}\tan^{-1}\Big(\dfrac{r}{\varepsilon}\Big) - \dfrac{1}{2}\log\Big(1 + \Big(\dfrac{r}{\varepsilon}\Big)^2\Big)\right]$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0, 1]$.

Remark 3.5. For any $\varepsilon \in [0, 1]$, and let us define a functional $\tilde{V}^\varepsilon : D(\tilde{V}^\varepsilon) \subset X \longrightarrow [0, \infty]$, as follows:

$$z \in X \mapsto \tilde{V}^\varepsilon(z) := \int_\Omega f^\varepsilon(z)\, dx \in [0, \infty]. \tag{3.3.2}$$

Then, from (A2) and (Fact 1), it is immediately seen that:

(♠1) $\tilde{V}^\varepsilon$ is continuous and convex on X, with $D(\tilde{V}^\varepsilon) = X$;

(♠2) for any $\varepsilon_0 \in [0, 1]$, $\tilde{V}^\varepsilon \to \tilde{V}^{\varepsilon_0}$ on X, in the sense of Mosco, as $\varepsilon \to \varepsilon_0$.

Remark 3.6. By the definition of f^ε, the function $(1/\nu^2)(f^\varepsilon)'$ is a maximal monotone graph on $\mathbb{R} \times \mathbb{R}$. Hence, its resolvent $((1/\nu^2)(f^\varepsilon)' + I_d)^{-1}$ is non-expansive. Therefore, we can verify that $\left((f^\varepsilon)' + \nu^2 I_d\right)^{-1}$ is a Lipschitz continuous function with a Lipschitz constant $1/\nu^2$.

In fact, let us fix $z_i \in \mathbb{R}$, for $i = 1, 2$, and let us set $z_i^* \in \mathbb{R}$, for $i = 1, 2$, as follows:

$$z_i^* := \left((f^\varepsilon)' + \nu^2 I_d\right)^{-1} (z_i), \text{ for any } i = 1, 2.$$

Then, we compute that:

$$z_i = \left((f^\varepsilon)' + \nu^2 I_d\right) (z_i^*) = \nu^2 \left(\frac{1}{\nu^2}(f^\varepsilon)' + I_d\right) (z_i^*),$$

i.e.

$$\frac{1}{\nu^2} z_i = \left(\frac{1}{\nu^2}(f^\varepsilon)' + I_d\right) (z_i^*), \text{ for any } i = 1, 2.$$

Thus, we obtain that:

$$z_i^* = \left(\frac{1}{\nu^2}(f^\varepsilon)' + I_d\right)^{-1} \left(\frac{1}{\nu^2} z_i\right), \text{ for any } i = 1, 2.$$

Based on these estimates, one can see that:

$$\begin{aligned}
&\left|\left((f^\varepsilon)' + \nu^2 I_d\right)^{-1} (z_1) - \left((f^\varepsilon)' + \nu^2 I_d\right)^{-1} (z_2)\right| \\
&= \left|\left(\frac{1}{\nu^2}(f^\varepsilon)' + I_d\right)^{-1} \left(\frac{1}{\nu^2} z_1\right) - \left(\frac{1}{\nu^2}(f^\varepsilon)' + I_d\right)^{-1} \left(\frac{1}{\nu^2} z_2\right)\right| \\
&\leq \left|\frac{z_1}{\nu^2} - \frac{z_2}{\nu^2}\right| = \frac{1}{\nu^2}|z_1 - z_2|, \text{ for any } z_1, z_2 \in \mathbb{R}.
\end{aligned}$$

Remark 3.7. The assumption (A3) leads to:

$$g' \geq -C_g \text{ on } \mathbb{R}, \text{ i.e. } |[g']^-|_{L^\infty(\mathbb{R})} \leq C_g.$$

Remark 3.8. The assumption (A4) guarantees that:

$$r \cdot K^\delta(r) \geq 0, \text{ for any } r \in \mathbb{R},$$

and hence,

$$(K^\delta(z), z)_X \geq 0, \text{ for any } z \in X.$$

Next, let us set the convex function on X as follows:

$$z \in X \mapsto \Phi^0(z) := V^0(z) + V^D(z); \tag{3.3.3}$$

which is defined as a sum of two convex functions on X, defined as follows:

$$z \in X \mapsto V^0(z) := \sup \left\{ \int_\Omega z \partial_x \varphi \, dx, \, \middle| \, \begin{array}{l} \varphi \in Y \cap C_{\rm c}(\Omega), \text{ such} \\ \text{that } |\varphi| \leq 1 \text{ on } \overline{\Omega} \end{array} \right\}, \tag{3.3.4}$$

and

$$z \in X \mapsto V^D(z) := \begin{cases} \dfrac{\nu^2}{2} \displaystyle\int_\Omega |\partial_x z|^2 dx, & \text{if } z \in Y, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.3.5}$$

Also, for any $\varepsilon \in (0, 1]$, let Φ^ε be a proper, l.s.c, and convex function

$$z \in X \mapsto \Phi^\varepsilon(z) := \begin{cases} \displaystyle\int_\Omega f^\varepsilon(\partial_x z) \, dx + \dfrac{\nu^2}{2} \displaystyle\int_\Omega |\partial_x z|^2 dx, & \text{if } z \in Y, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.3.6}$$

Moreover, for any $\delta \in [0, 1]$, let us denote by $\hat{\mathcal{K}}^\delta$ a proper, l.s.c, and convex function on X, defined as:

$$\hat{\mathcal{K}}^\delta : z \in X \mapsto \hat{\mathcal{K}}^\delta(z) := \int_\Omega \hat{K}^\delta(z) \, dx, \tag{3.3.7}$$

and let us denote by $\mathcal{K}^\delta$ the subdifferential $\partial \hat{\mathcal{K}}^\delta$ of $\hat{\mathcal{K}}^\delta$ in $X \times X$. As is easily seen,

$$\mathcal{K}^\delta(z) = \begin{cases} \left\{ z^* \in X \, \middle| \, z^* \in \partial I_{[-1,1]}(z) \text{ a.e. in } \Omega \right\} \text{ in } X, & \text{if } \delta = 0, \\ K^\delta(z), & \text{if } \delta \in (0, 1], \end{cases} \tag{3.3.8}$$
$$\text{for all } z \in X \text{ and } \delta \in [0, 1].$$

Based on these, let us denote by $\Psi^{(\varepsilon,\delta)}$ a function on X, defined as follows:

$$z \in X \mapsto \Psi^{(\varepsilon,\delta)}(z) := \begin{cases} \Phi^\varepsilon(z) + \hat{\mathcal{K}}^\delta(z), & \text{if } z \in Y, \\ \infty, & \text{otherwise,} \end{cases} \tag{3.3.9}$$

and this definition implies that $\Psi^{(\varepsilon,\delta)}$ is a proper, l.s.c, and convex on X, and

$$D(\Psi^{(\varepsilon,\delta)}) := D(\Phi^\varepsilon) \cap D(\hat{\mathcal{K}}^\delta) = \begin{cases} \left\{ \tilde{z} \in Y \, \middle| \, |\tilde{z}| \leq 1 \text{ on } \overline{\Omega} \right\}, & \text{if } \delta = 0, \\ Y, & \text{if } \delta \in (0, 1], \end{cases}$$
$$\text{for all } \varepsilon, \delta \in [0, 1].$$

The principal part of this paper is the verification of the following Key-Theorem, which is concerned with the decomposition property of the subdifferential $\partial \Psi^{(\varepsilon,\delta)} \subset X \times X$ of the convex function $\Psi^{(\varepsilon,\delta)}$.

Key-Theorem. Let us fix $\varepsilon \in [0,1]$ and $\delta \in [0,1]$. Then, $[z, z^*] \in \partial\Psi^{(\varepsilon,\delta)}$ in $X \times X$ if and only if there exist $\varpi^* \in Y \cap L^\infty(\Omega)$ and $\xi \in X$ such that

- $z \in H^2(\Omega)$ with $(\varpi^* + \nu^2\partial_x z)(\pm L) = 0$.
- $z^* = -\partial_x\varpi^* - \nu^2\partial_x^2 z + \xi$ in X, where $\varpi^* \in \partial f^\varepsilon(\partial_x z)$ and $\xi \in K^\delta(z)$ a.e. in Ω.

Key-Theorem plays an important role to lead the H^2-regularity of the solution $w = [w_1, \dots, w_n]$ to the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$ in Main Theorem 3.1.

Remark 3.9. In the proof of Key-Theorem, we will also use the following functional:

$$V^\varepsilon : z \in X \mapsto V^\varepsilon(z) := \begin{cases} \tilde{V}^\varepsilon(\partial_x z), \text{ if } z \in Y, \\ \infty, \text{ otherwise}, \end{cases} \text{for all } \varepsilon \in (0,1], \tag{3.3.10}$$

as extra notations, where $\tilde{V}^\varepsilon$ is the continuous and convex function on X, given in (3.3.2). As is easily seen from (A2), V^ε is proper and convex on X. But while, the assumption (A2) does not guarantee the lower semi-continuity of V^ε on X. Nevertheless, we can obtain:

$$\partial\Phi^\varepsilon = \partial V^\varepsilon + \partial V^D \text{ in } X \times X, \text{ for any } \varepsilon \in (0,1],$$

as a consequence of Key-Theorem (the Step 1 of the proof).

Remark 3.10. (cf. [49, Remark 5]) Let us fix any $\varepsilon \in (0,1]$, and let us define a map $\mathcal{A}^\varepsilon : D(\mathcal{A}^\varepsilon) \subset X \longrightarrow X$, by putting:

$$D(\mathcal{A}^\varepsilon) := \left\{ z \in Y \;\middle|\; \alpha(f^\varepsilon)'(\partial_x z) + \beta\partial_x z \in H_0^1(\Omega) \right\},$$

and

$$z \in D(\mathcal{A}^\varepsilon) \subset X \mapsto \mathcal{A}^\varepsilon(z) := -\partial_x\big(\alpha(f^\varepsilon)'(\partial_x z) + \beta\partial_x z\big).$$

Then, by applying the standard variational technique, we can observe that:

$$\mathcal{A}^\varepsilon = \partial\Phi^\varepsilon \text{ in } X \times X.$$

Now, the Main Theorems of this paper are stated as follows.

Main Theorem 3.1. Under the assumptions (A1)–(A6), let us fix a constant $\varepsilon \in [0,1]$ and $\delta \in [0,1]$, a forcing term $u = [u_1, \dots, u_n] \in \mathbb{X}$, an initial data $w_0 \in Y$ satisfying $\hat{K}^\delta(w_0) \in L^1(\Omega)$. Then, the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$ admits a unique solution $w = [w_1, \dots, w_n] \in \mathbb{Y}$, and moreover, the following items holds:

(I) $w_i \in H^2(\Omega)$ with $\partial_x w_i(\pm L) = 0$, for any $i = 1, 2, 3, \dots, n$.

(II) There exist $\varpi^* = [\varpi_i^*, \dots, \varpi_n^*] \in \mathbb{Y} \cap [L^\infty(\Omega)]^n$ and $\xi = [\xi_1, \dots, \xi_n] \in \mathbb{X}$ such that:

$$\frac{1}{\tau}(w_i - w_{i-1}) - \partial_x\varpi_i^* - \nu^2\partial_x^2 w_i + \xi_i + g(w_i) = M_u u_i \text{ in } X, i = 1, 2, 3, \dots, n, \tag{3.3.11}$$

with

$$\begin{cases} \bullet\, \varpi_i^* \in \partial f^\varepsilon(\partial_x w_i) \text{ in } \mathbb{R}, \text{ a.e. in } \Omega, \\ \bullet\, \xi_i \in K^\delta(w_i) \text{ in } \mathbb{R}, \text{ a.e. in } \Omega, \end{cases} i = 1, 2, 3, \dots, n.$$

(III)(Energy inequality)

$$\frac{1}{2\tau}|w_i - w_{i-1}|_X^2 + \mathscr{F}^{(\varepsilon,\delta)}(w_i) - \mathscr{F}^{(\varepsilon,\delta)}(w_{i-1}) \leq \tau M_u^2 |u_i|_X^2, i = 1, 2, 3, \dots, n,$$

where $\mathscr{F}^{(\varepsilon,\delta)}$ is a functional, called free energy, which is defined as follows:

$$\begin{aligned} z \in X \mapsto \mathscr{F}^{(\varepsilon,\delta)}(z) &:= \Psi^{(\varepsilon,\delta)}(z) + \int_\Omega G(z)\, dx, \\ &= \Phi^\varepsilon(z) + \hat{\mathcal{K}}^\delta(z) + \int_\Omega G(z) dx, \text{ for any } z \in X. \end{aligned} \tag{3.3.12}$$

Remark 3.11. By the definition of subdifferentials, we observe that the equation (3.3.11) is equivalent to the following variational inequality:

$$\begin{aligned} &\frac{1}{\tau}(w_i - w_{i-1}, w_i - z)_X + (g(w_i) - M_u u_i, w_i - z)_X \\ &\quad + \Psi^{(\varepsilon,\delta)}(w_i) - \Psi^{(\varepsilon,\delta)}(z) \leq 0, \text{ for any } z \in D(\Psi^{(\varepsilon,\delta)}) \text{ and } i = 1, 2, 3, \dots, n. \end{aligned}$$

Main Theorem 3.2. Let $\varepsilon \in [0,1]$, $\delta \in [0,1]$, $\{\varepsilon_m\}_{m\in\mathbb{N}} \subset (0,1]$, $\{\delta_m\}_{m\in\mathbb{N}} \subset (0,1]$, $u = [u_1, \dots, u_n] \in \mathbb{X}$, $\{u^m\}_{m\in\mathbb{N}} = \{[u_1^m, \dots, u_n^m]\}_{m\in\mathbb{N}} \subset \mathbb{X}$, $w_0 \in Y$, and $\{w_0^m\}_{m\in\mathbb{N}} \subset Y$ be given sequences such that:

$$\begin{cases} \varepsilon_m \to \varepsilon, \ \delta_m \to \delta, \ u^m \to u \text{ weakly in } \mathbb{X}, \\ \qquad \text{and } w_0^m \to w_0 \text{ weakly in } Y, \text{ as } m \to \infty, \\ \hat{K}_* := \sup_{m\in\mathbb{N}} \int_\Omega \hat{K}^{\delta_m}(w_0^m)\, dx < \infty. \end{cases} \tag{3.3.13}$$

In addition, let $w = [w_1, \dots, w_n] \in \mathbb{X}$ be the unique solution to $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the forcing term u and the initial data w_0, and for any $m \in \mathbb{N}$, let $w^m = [w_1^m, \dots, w_n^m] \in \mathbb{X}$ be the unique solution to $(\mathrm{AC})^{(\varepsilon_m,\delta_m)}$ for the forcing term u^m and the initial data w_0^m. Then, it holds that:

$$w^m \to w \text{ in } \mathbb{Y}, \text{ in } [C^1(\overline{\Omega})]^n, \text{ and weakly in } [H^2(\Omega)]^n, \text{ as } m \to \infty, \tag{3.3.14}$$

and in particular,

$$\begin{aligned} f^{\varepsilon_m}(\partial_x w_i^m) \to f^\varepsilon(\partial_x w_i) \text{ in } X, \text{ and in the pointwise sense on } \overline{\Omega}, \\ \text{for any } i = 1, 2, 3, \dots, n, \text{ as } m \to \infty. \end{aligned} \tag{3.3.15}$$

Main Theorem 3.3. Let us assume (A1)–(A6). Let us fix the constants $\varepsilon \in [0,1]$ and $\delta \in [0,1]$, and fix the initial data $w_0 \in Y$ satisfying $\hat{K}^\delta(w_0) \in L^1(\Omega)$. Then, the following two items hold.

(III-A) The problem $(\mathrm{OP})^{(\varepsilon,\delta)}$ has at least one optimal control $u^* = [u_1^*, \dots, u_n^*] \in \mathbb{X}$, so that:

$$\mathcal{J}^{(\varepsilon,\delta)}(u^*) = \min\left\{ \ \mathcal{J}^{(\varepsilon,\delta)}(u) \ \middle| \ u = [u_1, \dots, u_n] \in \mathbb{X} \ \right\}.$$

(III-B) Let us take the sequences $\{\varepsilon_m\}_{m\in\mathbb{N}} \subset (0,1]$, $\{\delta_m\}_{m\in\mathbb{N}} \subset (0,1]$, and take the sequence of initial data $\{w_0^m\}_{m\in\mathbb{N}} \subset Y$ as in (3.3.13). In addition, for any $m \in \mathbb{N}$, let $u^{(*,m)} = [u_1^{(*,m)}, \dots, u_n^{(*,m)}] \in \mathbb{X}$ be the optimal control of $(\mathrm{OP})^{(\varepsilon_m,\delta_m)}$ in the case when the initial data of corresponding state-system $(\mathrm{AC})^{(\varepsilon_m,\delta_m)}$ is given by w_0^m. Then, there exist a subsequence $\{m_k\}_{k\in\mathbb{N}} \subset \{m\}$ and a function $u^{**} = [u_1^{**}, \dots, u_n^{**}] \in \mathbb{X}$, such that:

$$\begin{cases} \bullet\ M_u u^{(*,m_k)} \to M_u u^{**} \text{ weakly in } \mathbb{X}, \text{ as } k \to \infty, \\ \bullet\ u^{**} \text{ is an optimal control of } (\mathrm{OP})^{(\varepsilon,\delta)}. \end{cases}$$

Main Theorem 3.4. (Necessary condition for $(\mathrm{OP})^{(\varepsilon,\delta)}$ under positive ε, δ) Let us assume (A1)–(A6). Let us fix the constants $\varepsilon \in (0,1]$ and $\delta \in (0,1]$, and fix the initial data $w_0 \in Y$ satisfying $\hat{K}^\delta(w_0) \in L^1(\Omega)$. Let $u^* = [u_1^*, \dots, u_n^*] \in \mathbb{X}$ be an optimal control of $(\mathrm{OP})^{(\varepsilon,\delta)}$, and let $w^* = [w_1^*, \dots, w_n^*] \in \mathbb{X}$ be the solution to $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the forcing term u^* and initial data w_0. Then, it holds that:

$$M_u\,(p_i^* + u_i^*) = 0 \text{ in } X, \text{ for any } i = 1, 2, 3, \dots, n.$$

In this context, $p^* = [p_1^*, \dots, p_n^*] \in \mathbb{X}$ is a unique solution to the following variational system:

$$\begin{aligned} &\frac{1}{\tau}(p_i^* - p_{i+1}^*, \varphi)_X + \big((f^\varepsilon)''(\partial_x w_i^*)\partial_x p_i^*, \partial_x \varphi\big)_X + \nu^2(\partial_x p_i^*, \partial_x \varphi)_X \\ &\quad + \big(g'(w_i^*)p_i^*, \varphi\big)_X + \big((K^\delta)'(w_i^*)p_i^*, \varphi\big)_X = \big(M_w(w_i^* - w_i^{\mathrm{ad}}), \varphi\big)_X, \\ &\quad \text{for any } \varphi \in Y, \text{ and } i = n, \dots, 3, 2, 1, \end{aligned}$$

subject to the terminal condition:

$$p_{n+1}^* = 0 \text{ in } X.$$

Main Theorem 3.5. Let us assume (A1)–(A6), and let us fix an initial data $w_0 \in Y$ satisfying $|w_0| \leq 1$ on $\overline{\Omega}$, i.e. $K^0(w_0) \in L^1(\Omega)$. Also, Let us define a duality map $F : Y \longrightarrow Y^*$, as follows:

$$\langle F\varphi, \psi \rangle_Y := (\varphi, \psi)_Y, \text{ for any } \varphi, \psi \in Y.$$

Then, there exists an optimal control $u^\circ = [u_1^\circ, \dots, u_n^\circ] \in \mathbb{X}$ of the problem $(\mathrm{OP})^{(0,0)}$, together with the solution $w^\circ = [w_1^\circ, \dots, w_n^\circ]$ to the state-system $(\mathrm{AC})^{(0,0)}$ for the forcing term u° and initial data w_0, and moreover, there exist $p^\circ = [p_1^\circ, \dots, p_n^\circ] \in \mathbb{X}$ and $\zeta^\circ = [\zeta_1^\circ, \dots, \zeta_n^\circ] \in \mathbb{Y}^*$, such that:

$$M_u\,(p_i^\circ + u_i^\circ) = 0, \text{ in } X, \text{ for any } i = 1, 2, 3, \dots, n, \tag{3.3.16}$$

$$p^\circ \in \mathbb{Y} \subset [C(\overline{\Omega})]^n, \tag{3.3.17}$$

and

$$\frac{1}{\tau}(p_i^\circ - p_{i+1}^\circ, \varphi)_X + \nu^2\langle F p_i^\circ, \varphi \rangle_Y - \nu^2(p_i^\circ, \varphi)_X + \big(g'(w_i^\circ)p_i^\circ, \varphi)_X + \langle \zeta_i^\circ, \varphi \rangle_Y$$

$$= \left(M_w(w_i^\circ - w_i^{\mathrm{ad}}), \varphi\right)_X, \text{ for any } \varphi \in Y, \text{ and } i = n, \dots, 3, 2, 1, \tag{3.3.18}$$

subject to $p_{n+1}^\circ = 0$ in X.

Moreover, for any $\gamma_0 \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ satisfying $\gamma_0(0) = \gamma_0'(0) = 0$, it follows that:

$$\gamma_0(\partial_x w_i^\circ) \left(\frac{1}{\tau}(p_i^\circ - p_{i+1}^\circ) - \nu^2 \partial_x^2 p_i^\circ + g'(w_i^\circ) p_i^\circ - M_w(w_i^\circ - w_i^{\mathrm{ad}}) \right) = 0 \text{ in } X,$$
$$\text{for any } i = n, \dots, 3, 2, 1, \tag{3.3.19}$$

and therefore,

$$\frac{1}{\tau}(p_i^\circ - p_{i+1}^\circ) - \nu^2 \partial_x^2 p_i^\circ + g'(w_i^\circ) p_i^\circ = M_w(w_i^\circ - w_i^{\mathrm{ad}}) \text{ a.e. in } \{\partial_x w_i^\circ \neq 0\},$$
$$\text{for any } i = n, \dots, 3, 2, 1. \tag{3.3.20}$$

Remark 3.12. From (3.3.19) and (3.3.20), it is observed that:

$$\mathrm{spt}(\zeta_i^\circ) \subset \{\partial_x w_i^\circ = 0\}, \text{ for any } i = 1, 2, 3, \dots, n.$$

and

$$\zeta_i^\circ = M_w(w_i^\circ - w_i^{\mathrm{ad}}) - \frac{1}{\tau}(p_i^\circ - p_{i+1}^\circ) + \nu^2 \partial_x^2 p_i^\circ - g'(w_i^\circ) p_i^\circ,$$
$$\text{a.e. in } \{\partial_x w_i^\circ \neq 0\} \ \left(\supset \Omega \setminus \mathrm{spt}(\zeta_i^\circ)\right), \text{ for any } i = n, \dots, 3, 2, 1.$$

These are to answer the functional expression of the distribution $\zeta^\circ \in \mathbb{Y}^*$, which somehow link to the second unfinished issue, mentioned in the Introduction.

3.4 Proof of Key-Theorem

In this Section, we give the proof of Key-Theorem. Before the proof, we prepare a Key-Lemma.

Key-Lemma. Let us fix $\varepsilon_0 \in [0,1]$ and $\delta_0 \in [0,1]$. Besides, let $\{\varepsilon_m\}_{m\in\mathbb{N}} \subset (0,1]$ and $\{\delta_m\}_{m\in\mathbb{N}} \subset (0,1]$ be given sequences such that $\varepsilon_m \to \varepsilon_0$ and $\delta_m \to \delta_0$, as $m \to \infty$, respectively. Then, for the sequence $\{\Psi^{(\varepsilon_m,\delta_m)}\}_{m\in\mathbb{N}}$, it holds that:

$$\Psi^{(\varepsilon_m,\delta_m)} \to \Psi^{(\varepsilon_0,\delta_0)} \text{ on } X, \text{ in the sense of Mosco, as } m \to \infty.$$

Proof. First, we show the lower-bound condition (M1) in Definition 3.1. Let $z \in X$ and $\{z^m\}_{m\in\mathbb{N}} \subset X$ such that:

$$z^m \to z \text{ weakly in } X, \text{ as } m \to \infty. \tag{3.4.1}$$

Then, we may suppose $\underline{\lim}_{m\to\infty} \Psi^{(\varepsilon_m,\delta_m)}(z^m) < \infty$, since the other case is trivial. So, by taking a subsequence $\{m_k\}_{k\in\mathbb{N}} \subset \{m\}$, one can also say that:

$$\varliminf_{m\to\infty} \Psi^{(\varepsilon_m,\delta_m)}(z^m) = \lim_{k\to\infty} \Psi^{(\varepsilon_{m_k},\delta_{m_k})}(z^{m_k}) < \infty. \tag{3.4.2}$$

Here, with (A2), (A4), (3.3.6), (3.3.7), (3.3.9), (3.4.1), and (3.4.2) in mind, we further see that:

$$\partial_x z^{m_k} \to \partial_x z \text{ weakly in } X, \text{ as } k \to \infty, \tag{3.4.3}$$

by taking a subsequence if necessary. In the light of Remark 3.5, (A2), (A4), (Fact 2), (3.4.1)–(3.4.3), and weakly lower semi-continuity of the norms, the lower-bound condition (M1) can be verified as follows:

$$\begin{aligned}
&\varliminf_{m\to\infty} \Psi^{(\varepsilon_m,\delta_m)}(z^m) = \lim_{k\to\infty} \Psi^{(\varepsilon_{m_k},\delta_{m_k})}(z^{m_k}) \\
&\geq \varliminf_{k\to\infty} \int_\Omega f^{\varepsilon_{m_k}}(\partial_x z^{m_k})\,dx + \frac{\nu^2}{2} \varliminf_{k\to\infty} \int_\Omega |\partial_x z^{m_k}|^2\,dx + \varliminf_{k\to\infty} \int_\Omega \hat{K}^{\delta_{m_k}}(z^{m_k})\,dx \\
&\geq \Psi^{(\varepsilon_0,\delta_0)}(z).
\end{aligned}$$

Next, we show the optimality condition (M2) in Definition 3.1. Let us fix any $z \in D(\Psi^{(\varepsilon_0,\delta_0)})$. In the light of (A2), we can say that:

$$\begin{aligned}
0 &\leq f^{\varepsilon_m}(r) \leq (f^{\varepsilon_m})'(r)r \\
&\leq C_0(1+|r|)|r|, \text{ for any } r \in \mathbb{R}, \text{ and } m \in \mathbb{N}.
\end{aligned} \tag{3.4.4}$$

With the assumption (A2) and (3.4.4) in mind, we can infer that:

$$f^{\varepsilon_m}(\partial_x z) \to f^{\varepsilon_0}(\partial_x z) \text{ in the pointwise sense a.e. in } \Omega, \text{ as } m \to \infty, \tag{3.4.5a}$$

$$f^{\varepsilon_m}(\partial_x z) \leq C_0(1+|\partial_x z|)|\partial_x z| \text{ a.e. in } \Omega, \text{ for any } m \in \mathbb{N}. \tag{3.4.5b}$$

By (3.4.5) and Lebesgue's dominated convergence theorem, it is verified that:

$$f^{\varepsilon_m}(\partial_x z) \to f^{\varepsilon_0}(\partial_x z) \text{ in } L^1(\Omega), \text{ as } m \to \infty. \tag{3.4.6}$$

Furthermore, we can show:

$$\hat{K}^{\delta_m}(z) \to \hat{K}^{\delta_0}(z) \text{ in } C(\overline{\Omega}), \text{ as } m \to \infty. \tag{3.4.7}$$

In fact, the case when $\delta_0 = 0$ is trivial since $\hat{K}^{\delta_m}(z) = \hat{K}^0(z)(= I_{[-1,1]}(z)) = 0$ a.e. in Ω. Meanwhile, when $\delta_0 \in (0,1]$, the uniform convergence (3.4.7) is obtained as a consequence of (A4) and the embedding $Y \subset C(\overline{\Omega})$.

Based on these, let us define $z^m := z$, for any $m \in \mathbb{N}$. Taking into account (3.4.6) and (3.4.7), we compute that:

$$\left|\Psi^{(\varepsilon_m,\delta_m)}(z^m) - \Psi^{(\varepsilon_0,\delta_0)}(z)\right| \leq \int_\Omega \left|f^{\varepsilon_m}(\partial_x z) - f^{\varepsilon_0}(\partial_x z)\right| dx + \int_\Omega \left|\hat{K}^{\delta_m}(z) - \hat{K}^{\delta_0}(z)\right| dx$$

$$\leq \int_{\Omega} \left| f^{\varepsilon_m}(\partial_x z) - f^{\varepsilon_0}(\partial_x z) \right| dx + 2L \left| \hat{K}^{\delta_m}(z) - \hat{K}^{\delta_0}(z) \right|_{C(\overline{\Omega})}$$
$$\to 0, \text{ as } m \to \infty,$$

and therefore,

$$\Psi^{(\varepsilon_m, \delta_m)}(z^m) \to \Psi^{(\varepsilon_0, \delta_0)}(z), \text{ as } m \to \infty.$$

□

For efficiency of explanation, we prove the Key-Theorem in accordance with the following two Steps.

Step 1: For arbitrary $\varepsilon \in (0,1]$ and $\delta \in (0,1]$, the subdifferential $\partial \Psi^{(\varepsilon,\delta)} \subset X \times X$ of the convex function $\Psi^{(\varepsilon,\delta)}$ is decomposed as follows:

$$\partial \Psi^{(\varepsilon,\delta)} = \partial V^{\varepsilon} + \partial V^{D} + K^{\delta} \text{ in } X \times X,$$

where $\Psi^{(\varepsilon,\delta)}$, V^{ε}, and V^{D} are convex functions given in (3.3.5), (3.3.9), and (3.3.10), respectively, and K^{δ} is the function as in (A4).

Step 2: For arbitrary $\varepsilon \in [0,1]$ and $\delta \in [0,1]$, the subdifferential $\partial \Psi^{(\varepsilon,\delta)} \subset X \times X$ of the convex function $\Psi^{(\varepsilon,\delta)}$ is decomposed as follows:

$$\partial \Psi^{(\varepsilon,\delta)} = \partial V^{\varepsilon} + \partial V^{D} + \mathcal{K}^{\delta} \text{ in } X \times X,$$

where $\mathcal{K}^{\delta}$ is the operator given in (3.3.8).

Verification of Step 1.

Let us fix $\varepsilon \in (0,1]$ and $\delta \in (0,1]$. The assumption (A4) guarantees $D(\hat{\mathcal{K}}^{\delta}) = X$, i.e.

$$\mathrm{int} D(\hat{\mathcal{K}}^{\delta}) = X. \tag{3.4.8}$$

Due to (3.4.8), the decomposition of the subdifferential $\partial \Psi^{(\varepsilon,\delta)} \subset X \times X$ of the convex function $\Psi^{(\varepsilon,\delta)} = \Phi^{\varepsilon} + \hat{\mathcal{K}}^{\delta}$ will be a straightforward consequence of [14, Theorem 2.10], and [18, Corollary 2.11], i.e.

$$\partial \Psi^{(\varepsilon,\delta)} = \partial \Phi^{\varepsilon} + K^{\delta} \text{ in } X \times X. \tag{3.4.9}$$

Hence, it is sufficient to prove that the subdifferential $\partial \Phi^{\varepsilon} \subset X \times X$ of the convex function Φ^{ε} is decomposed as follows:

$$\partial \Phi^{\varepsilon} = \partial V^{\varepsilon} + \partial V^{D} \text{ in } X \times X. \tag{3.4.10}$$

First, we verify the following inclusion:

$$\partial \Phi^{\varepsilon} \subset \partial V^{\varepsilon} + \partial V^{D} \text{ in } X \times X. \tag{3.4.11}$$

Let us take any $[z, z^*] \in \partial \Phi^{\varepsilon}$ in $X \times X$, and apply Remark 3.10 to the case when $\alpha \equiv 1$ and $\beta \equiv \nu^2$. Then, we have:

$$z^* = -\partial_x \left((f^{\varepsilon})'(\partial_x z) + \nu^2 \partial_x z \right) \text{ in } X, \text{ and}$$

$$(f^\varepsilon)'(\partial_x z) + \nu^2 \partial_x z \in H^1_0(\Omega). \tag{3.4.12}$$

Now, we define a function $\mu^\nu_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$\mu^\nu_\varepsilon(r) := (f^\varepsilon)'(r) + \nu^2 r, \text{ for any } r \in \mathbb{R}. \tag{3.4.13}$$

On this basis, we infer that:

$$\mu^\nu_\varepsilon(\partial_x z) = (f^\varepsilon)'(\partial_x z) + \nu^2 \partial_x z = \left((f^\varepsilon)' + \nu^2 I_d\right)(\partial_x z) \text{ in } Y. \tag{3.4.14}$$

Based on Remark 3.6 and (3.4.14), we can apply the generalized chain rule in BV-theory [3, Theorem 3.99], and can infer that:

$$\partial_x z = (\mu^\nu_\varepsilon)^{-1}(\mu^\nu_\varepsilon(\partial_x z)) \in Y, \text{ i.e. } z \in H^2(\Omega). \tag{3.4.15}$$

Furthermore, having in mind (3.4.12), (3.4.14), (3.4.15), and applying Remark 3.3 to the case when $\beta \equiv \nu^2$, one can see that:

$$\partial_x z \in H^1_0(\Omega) \text{ and } [z, -\nu^2 \partial_x^2 z] \in \partial V^D \text{ in } X \times X. \tag{3.4.16}$$

By using (A2), (3.4.12), (3.4.16), and the integration by part, we observe that:

$$(f^\varepsilon)'(\partial_x z) \in H^1_0(\Omega), \tag{3.4.17}$$

and

$$\begin{aligned}\int_\Omega -\partial_x\left((f^\varepsilon)'(\partial_x z)\right)(\varphi - z)\,dx &= \int_\Omega (f^\varepsilon)'(\partial_x z)\partial_x(\varphi - z)\,dx \\ &\leq \int_\Omega f^\varepsilon(\partial_x \varphi)\,dx - \int_\Omega f^\varepsilon(\partial_x z)\,dx, \text{ for any } \varphi \in Y,\end{aligned}$$

i.e.

$$[z, -\partial_x\left((f^\varepsilon)'(\partial_x z)\right)] \in \partial V^\varepsilon \text{ in } X \times X. \tag{3.4.18}$$

(3.4.15)–(3.4.18) enable us to verify the inclusion (3.4.11).

Now, from the maximality of $\partial\Phi^\varepsilon$ in $X \times X$, we can see the coincidence $\partial\Phi^\varepsilon = \partial V^\varepsilon + \partial V^D$ in $X \times X$.

Verification of Step 2.

Let us fix $\varepsilon \in [0,1]$ and $\delta \in [0,1]$. By the general theory of the convex analysis [24, Chapter 1], we immediately obtain that $\partial\Psi^{(\varepsilon,\delta)} \supset \partial V^\varepsilon + \partial V^D + \mathcal{K}^\delta$ in $X \times X$. This inclusion implies the monotonicity of $\partial V^\varepsilon + \partial V^D + \mathcal{K}^\delta$ in $X \times X$. So, we next see the maximality of $\partial V^\varepsilon + \partial V^D + \mathcal{K}^\delta$, by verifying:

$$(\partial V^\varepsilon + \partial V^D + \mathcal{K}^\delta + I_d)X = X,$$

and by applying Minty's theorem (cf. [14, Theorem 2.2]).

Since, the inclusion $(\partial V^\varepsilon + \partial V^D + \mathcal{K}^\delta + I_d)X \subset X$ is trivial, it is sufficient to prove the converse inclusion. Let us fix $h \in X$. Then, by Step 1, we can configure a class of functions $\{z^{(\tilde\varepsilon,\tilde\delta)}\}_{\tilde\varepsilon,\tilde\delta\in(0,1]} \subset Y$, by setting:

$$\{z^{(\tilde\varepsilon,\tilde\delta)} := (\partial V^{\tilde\varepsilon} + \partial V^D + K^{\tilde\delta} + I_d)^{-1}(h)\}_{\tilde\varepsilon,\tilde\delta\in(0,1]} \text{ in } X,$$

i.e.

$$h - z^{(\tilde{\varepsilon},\tilde{\delta})} = (\partial V^{\tilde{\varepsilon}} + \partial V^D + K^{\tilde{\delta}})(z^{(\tilde{\varepsilon},\tilde{\delta})}) = \partial \Psi^{(\tilde{\varepsilon},\tilde{\delta})}(z^{(\tilde{\varepsilon},\tilde{\delta})}) \text{ in } X, \text{ for any } \tilde{\varepsilon}, \tilde{\delta} \in (0,1]. \tag{3.4.19}$$

In the light of (3.4.16), (3.4.18), and (3.4.19), there exist $\varpi^{\tilde{\varepsilon}} \in Y \cap L^\infty(\Omega)$ and $\xi^{\tilde{\delta}} \in X$ such that

$$-\partial_x \varpi^{\tilde{\varepsilon}} - \nu^2 \partial_x^2 z^{(\tilde{\varepsilon},\tilde{\delta})} + \xi^{\tilde{\delta}} = h - z^{(\tilde{\varepsilon},\tilde{\delta})} \text{ in } X,$$
$$\text{where } \varpi^{\tilde{\varepsilon}} = \partial f^{\tilde{\varepsilon}}(\partial_x z^{(\tilde{\varepsilon},\tilde{\delta})}) \text{ and } \xi^{\tilde{\delta}} = K^{\tilde{\delta}}(z^{(\tilde{\varepsilon},\tilde{\delta})}) \text{ a.e. in } \Omega. \tag{3.4.20}$$

Using (3.4.16), (3.4.17), and the integration by part, we can see that:

$$\int_\Omega \varpi^{\tilde{\varepsilon}} \partial_x \varphi \, dx + \int_\Omega \nu^2 \partial_x z^{(\tilde{\varepsilon},\tilde{\delta})} \partial_x \varphi \, dx + \int_\Omega \xi^{\tilde{\delta}} \varphi + \int_\Omega z^{(\tilde{\varepsilon},\tilde{\delta})} \varphi \, dx$$
$$= \int_\Omega h\varphi \, dx, \text{ for any } \varphi \in Y, \text{ and any } \tilde{\varepsilon}, \tilde{\delta} \in (0,1]. \tag{3.4.21}$$

In the variational form (3.4.21), let us put $\varphi = z^{(\tilde{\varepsilon},\tilde{\delta})}$. Then, with Remark 3.8, (3.4.4), and Young's inequality in mind, we deduce that:

$$\frac{1}{2}|z^{(\tilde{\varepsilon},\tilde{\delta})}|_X^2 + \nu^2 |\partial_x z^{(\tilde{\varepsilon},\tilde{\delta})}|_X^2 \le \frac{1}{2}|h|_X^2, \text{ for any } \tilde{\varepsilon}, \tilde{\delta} \in (0,1], \tag{3.4.22}$$

so that

$$|z^{(\tilde{\varepsilon},\tilde{\delta})}|_Y^2 \le \frac{1}{1 \wedge (2\nu^2)} |h|_X^2, \text{ for any } \tilde{\varepsilon}, \tilde{\delta} \in (0,1]. \tag{3.4.23}$$

On account of (3.4.23), we find a function $z \in Y$ and sequences $\{\tilde{\varepsilon}_m\}_{m\in\mathbb{N}} \subset \{\tilde{\varepsilon}\}$ with $\tilde{\varepsilon}_m \to \varepsilon$ as $m \to \infty$, and $\{\tilde{\delta}_m\}_{m\in\mathbb{N}} \subset \{\tilde{\delta}\}$ with $\tilde{\delta}_m \to \delta$ as $m \to \infty$, such that:

$$z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)} \to z \text{ in } X, \text{ weakly in } Y, \text{ as } m \to \infty. \tag{3.4.24}$$

Furthermore, let us multiply $\xi^{\tilde{\delta}_m}$ the both sides of (3.4.19). Then, having in mind (A4), (3.4.4), (3.4.22), and Young's inequality, we deduce that:

$$\frac{1}{2}|\xi^{\tilde{\delta}_m}|_X^2 \le |h|_X^2 + |z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}|_X^2 \le 2|h|_X^2, \text{ for any } m \in \mathbb{N}, \tag{3.4.25}$$

via the following calculation:

$$\int_\Omega \left(-\partial_x \varpi^{\tilde{\varepsilon}_m} - \nu^2 \partial_x^2 z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}\right) \xi^{\tilde{\delta}_m} \, dx$$
$$= \int_\Omega \left(-\partial_x\big(\varpi^{\tilde{\varepsilon}_m} + \nu^2 \partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}\big)\right) \xi^{\tilde{\delta}_m} \, dx$$
$$= \int_\Omega \left(-\partial_x\big((f^{\tilde{\varepsilon}_m})'(\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}) + \nu^2 \partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}\big)\right) K^{\tilde{\delta}_m}(z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}) \, dx$$
$$= \int_\Omega \left((f^{\tilde{\varepsilon}_m})'(\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}) + \nu^2 \partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}\right) \partial_x K^{\tilde{\delta}_m}(z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}) \, dx$$

$$
\begin{aligned}
&= \int_\Omega (K^{\tilde{\delta}_m})'(z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)})(f^{\tilde{\varepsilon}_m})'(\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)})\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}\,dx \\
&\qquad + \int_\Omega \nu^2 (K^{\tilde{\delta}_m})'(\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)})|\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}|^2\,dx \\
&\geq 0, \ \text{for any } m \in \mathbb{N}.
\end{aligned} \tag{3.4.26}
$$

From the estimate (3.4.25), it is observed that:

$$
\xi^{\tilde{\delta}_m} \to \xi \text{ weakly in } X, \text{ as } m \to \infty, \text{ for some } \xi \in X, \tag{3.4.27}
$$

by taking a subsequence if necessary.

Additionally, by virtue of (3.4.20), (3.4.22), and (3.4.25), one can observe that :

($\Diamond$ 1) the sequence $\{\partial_x \varpi^{\tilde{\varepsilon}_m} + \nu^2 \partial_x^2 z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}\}_{m\in\mathbb{N}} = \{\partial_x(\varpi^{\tilde{\varepsilon}_m} + \nu^2 \partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)})\}_{m\in\mathbb{N}}$ is bounded in X.

Note that (3.4.13) leads to:

$$
\begin{aligned}
\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)} &= (\mu^\nu_{\tilde{\varepsilon}_m})^{-1}\big(\mu^\nu_{\tilde{\varepsilon}_m}(\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)})\big) \\
&= (\mu^\nu_{\tilde{\varepsilon}_m})^{-1}(\varpi^{\tilde{\varepsilon}_m} + \nu^2 \partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}), \text{ for any } m \in \mathbb{N}.
\end{aligned} \tag{3.4.28}
$$

With Remark 3.6 and (3.4.28) in mind, we apply the generalized chain rule in BV-theory [3, Theorem 3.99], and we infer that:

$$
\begin{aligned}
|\partial_x^2 z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}|_X &= \big|\big((\mu^\nu_{\tilde{\varepsilon}_m})^{-1}\big)'(\varpi^{\tilde{\varepsilon}_m} + \nu^2 \partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)})\partial_x(\varpi^{\tilde{\varepsilon}_m} + \nu^2 \partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)})\big|_X \\
&\leq \frac{1}{\nu^2}|\partial_x(\varpi^{\tilde{\varepsilon}_m} + \nu^2 \partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)})|_X, \text{ for any } m \in \mathbb{N}.
\end{aligned} \tag{3.4.29}
$$

From ($\Diamond$ 1) (3.4.23), and (3.4.29), it is deduced that:

($\Diamond$ 2) the sequence $\{\partial_x^2 z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}\}_{m\in\mathbb{N}}$ is bounded in X, and hence $\{z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}\}_{m\in\mathbb{N}}$ is bounded in $H^2(\Omega)$.

As a consequence of the one-dimensional compact embedding $H^2(\Omega) \subset C^1(\overline{\Omega})$, (3.4.24), and ($\Diamond$ 2), it is observed that:

$$
z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)} \to z \text{ in } Y, \text{ in } C^1(\overline{\Omega}), \text{ and weakly in } H^2(\Omega), \text{ as } m \to \infty, \tag{3.4.30}
$$

by taking a subsequence if necessary.

Note that by the assumption (A2) and (3.4.30), we can compute that:

$$
\begin{aligned}
|\varpi^{\tilde{\varepsilon}_m}|_X^2 &= |(f^{\tilde{\varepsilon}_m})'(\partial_x z^{(\tilde{\varepsilon}_m,\delta_m)})|_X^2 \\
&\leq \int_\Omega \big(C_0(1 + |\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}|)\big)^2 dx \\
&\leq 2C_0^2 \int_\Omega (1 + |\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}|^2) dx \\
&\leq 2C_0^2 \big(2L + \sup_{m\in\mathbb{N}} |\partial_x z^{(\tilde{\varepsilon}_m,\tilde{\delta}_m)}|_X^2\big) < \infty, \text{ for any } m \in \mathbb{N}.
\end{aligned}
$$

Based on these, it enables us to say

$$
\varpi^{\tilde{\varepsilon}_m} \to \varpi \text{ weakly in } X, \text{ as } m \to \infty, \text{ for some } \varpi \in X, \tag{3.4.31}
$$

by taking a subsequence if necessary. Besides, from ($\Diamond$ 1) and ($\Diamond$ 2), it is deduced that:

($\Diamond$ 3) the sequence $\{-\partial_x \varpi^{\tilde{\varepsilon}_m}\}_{m\in\mathbb{N}} = \{\omega^{\tilde{\varepsilon}_m}\}_{m\in\mathbb{N}}$ is bounded in X;

and we obtain that

$$\omega^{\tilde{\varepsilon}_m} \to \omega \text{ weakly in } X, \text{ as } m \to \infty, \text{ for some } \omega \in X, \tag{3.4.32}$$

by taking a subsequence if necessary.

In the light of (A2), (A4), (Fact 1), (Fact 2), (3.1.1), (3.4.27), and (3.4.30)–(3.4.32), it is inferred that:

$$\xi \in \mathcal{K}^\delta(z),\ \omega \in \partial V^\varepsilon(z), \text{ and } -\nu^2\partial_x^2 z \in \partial V^D(z) \text{ in } X. \tag{3.4.33a}$$

Additionally, taking into account Remark 3.5, (Fact 1), (3.4.30), (3.4.31), [14, Theorem 2.10], and [18, Corollary 2.11], one can obtain that:

$$\varpi \in \partial f^\varepsilon(\partial_x z) \text{ a.e. in } \Omega. \tag{3.4.33b}$$

In the meantime, letting $m \to \infty$ in (3.4.21) yields that:

$$\int_\Omega \varpi \partial_x \varphi\, dx + \int_\Omega \nu^2 \partial_x z \partial_x \varphi\, dx + \int_\Omega \xi\varphi + \int_\Omega z\varphi\, dx = \int_\Omega h\varphi\, dx, \text{ for any } \varphi \in Y.$$

This equation and (3.4.33) imply:

$$-\partial_x \varpi - \nu^2 \partial_x^2 z + \xi + z = h \text{ in } X,$$

i.e.

$$(\partial V^\varepsilon + \partial V^D + \mathcal{K}^\delta + I_d)(z) \ni h.$$

By applying Minty's theorem (cf. [14, Theorem 2.2]), $\partial V^\varepsilon + \partial V^D + \mathcal{K}^\delta$ is a maximal monotone. Now, from this maximality, we can see the coincidence $\partial\Psi^{(\varepsilon,\delta)} = \partial V^\varepsilon + \partial V^D + \mathcal{K}^\delta$ in $X \times X$.

Based on Step 1–Step 2, we conclude that for any $\varepsilon \in [0,1]$ and $\delta \in [0,1]$, $[z, z^*] \in \Psi^{(\varepsilon,\delta)}$ in $X \times X$ if and only if there exist $\varpi^* \in Y$ and $\xi \in X$ such that:

- $z \in H^2(\Omega)$ with $(\varpi^* + \nu^2 \partial_x z)(\pm L) = 0$;
- $z^* = -\partial_x \varpi^* - \nu^2 \partial_x^2 z + \xi$ in X, where $\varpi^* \in \partial f^\varepsilon(\partial_x z)$ and $\xi \in K^\delta(z)$ a.e. in Ω.

Hence, we finish the proof of Key-Theorem. □

Remark 3.13. The Key-Theorem is obtained on the basis of some previous works [49,77]. In fact, the proof of Step 1 is referred to the proving method of [49, Main Theorem], and the case when $\varepsilon = 0$ and $\delta \in (0,1]$ is verified as a consequence of [49, Main Theorem] and [77, Theorem 3.1]. However, the Key-Theorem covers various approximating sequences $\{f^\varepsilon\}_{\varepsilon\in(0,1]}$ and $\{K^\delta\}_{\delta\in(0,1]}$ in the range of the assumptions (A2) and (A4) which includes the previous setting adopted in [22, 66–68, 77].

In this light, it can be said that our Key-Theorem provides a general theory of approximating method for $-\partial_x(\frac{\partial_x w}{|\partial_x w|} + \nu^2 \partial_x w) + \partial I_{[-1,1]}(w)$, and also brings a key-answer for unfinished issue mentioned in Introduction.

3.5 Proof of Main Theorem 3.1

In this Section, we give the proof of Main Theorem 3.1. Let us fix $\varepsilon \in [0,1]$ and $\delta \in [0,1]$. Let us fix a forcing term $u = [u_1, \dots, u_n] \in \mathbb{X}$, and an initial data $w_0 \in Y$ satisfying $\hat{K}^\delta(w_0) \in L^1(\Omega)$. Let us fix $i \in \{1,2,3,\dots,n\}$.

On this basis, we define a functional $\mathscr{G} : X \longrightarrow (-\infty, \infty]$, by letting:

$$
\begin{aligned}
&w \in X \mapsto \mathscr{G}(w) \\
&:= \begin{cases} \dfrac{1}{2\tau}\displaystyle\int_\Omega |w - w_{i-1}|^2\, dx + \Psi^{(\varepsilon,\delta)}(w) + \int_\Omega G(w)\, dx - (M_u u_i, w)_X, \\ \qquad \text{if } w \in Y, \\ \infty, \quad \text{otherwise.} \end{cases}
\end{aligned}
\tag{3.5.1}
$$

Since the assumptions (A3), (A5), and Remark 3.7 imply

$$
\frac{d^2}{dw^2}\left(\frac{1}{4\tau}|w - \tilde{w}_0|^2 + G(w)\right) = \frac{1}{2\tau} + g'(w) > \frac{1}{2\tau^*} - C_g > \frac{C_g}{\tau^*}\left(\frac{1}{8(C_g+1)} - \tau^*\right) = 0,
$$
$$
\text{for any fixed } \tilde{w}_0 \in \mathbb{R},
$$

and the functional:

$$
\begin{aligned}
w \; &\in D(\mathscr{G}) = Y \subset X \mapsto \mathscr{G}(w) \\
&= \frac{1}{4\tau}|w - w_{i-1}|_X^2 + \int_\Omega \left(\frac{1}{4\tau}|w - w_{i-1}|^2 + G(w)\right) dx + \Psi^{(\varepsilon,\delta)}(w) - (M_u u_i, w)_X,
\end{aligned}
$$

is proper, l.s.c., strictly convex, and coercive on X.

Based on these, we find a unique minimizer of $\mathscr{G}$, denoted by $w^* \in X$, by applying [24, Proposition 1.2, Chapter II]. Since w^* is the minimizer of $\mathscr{G}$ and $\Psi^{(\varepsilon,\delta)}$ is a convex on X, we can compute that:

$$
\begin{aligned}
0 \; &\le \frac{1}{\lambda}\big(\mathscr{G}(w^* + \lambda(\varphi - w^*)) - \mathscr{G}(w^*)\big) \\
&\le \frac{1}{\tau}\int_\Omega (\varphi - w^*)(w^* - w_{i-1})\, dx + \frac{\lambda}{2\tau}\int_\Omega |\varphi - w^*|^2\, dx \\
&\quad + \frac{1}{\lambda}\int_\Omega \big(G(w^* + \lambda(\varphi - w^*)) - G(w^*)\big)\, dx \\
&\quad + \Psi^{(\varepsilon,\delta)}(\varphi) - \Psi^{(\varepsilon,\delta)}(w^*) - (M_u u_i, \varphi - w^*)_X, \text{ for any } \varphi \in Y, \text{ and } \lambda \in (0,1).
\end{aligned}
\tag{3.5.2}
$$

Letting $\lambda \downarrow 0$ in (3.5.2), it is inferred that:

$$
\Psi^{(\varepsilon,\delta)}(\varphi) - \Psi^{(\varepsilon,\delta)}(w^*) \le \left(-\left(\frac{1}{\tau}(w^* - w_{i-1}) + g(w^*) - M_u u_i\right), \varphi - w^*\right)_X,
$$
$$
\text{for any } \varphi \in Y,
$$

i.e.

$$
-\left(\frac{1}{\tau}(w^* - w_{i-1}) + g(w^*) - M_u u_i\right) \in \partial\Psi^{(\varepsilon,\delta)}(w^*) \text{ in } X. \tag{3.5.3}
$$

Here, on account of (3.5.3) and Key-Theorem, $w_i \in H^2(\Omega)$, and there exist $\varpi_i^* \in Y \cap L^\infty(\Omega)$ and $\xi_i \in X$ such that:

$$\frac{1}{\tau}(w^* - w_{i-1})_X - \partial_x \varpi_i^* - \nu^2 \partial_x^2 w^* + g(w^*) + \xi_i = M_u u_i, \text{ in } X, \tag{3.5.4}$$

with

$$\begin{cases} \bullet\, \varpi_i^* \in \partial f^\varepsilon(\partial_x w^*) \text{ in } \mathbb{R}, \text{ a.e. in } \Omega, \\ \bullet\, \xi_i \in K^\delta(w^*) \text{ in } \mathbb{R}, \text{ a.e. in } \Omega, \end{cases} \quad i = 1, 2, 3, \ldots, n.$$

The equation (3.5.4) implies that the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$ has a solution $w = [w_1, \ldots, w_n] \in [H^2(\Omega)]^n$.

Next, we prove the energy inequality as in (III). Let us fix $i \in \{1, 2, 3, \ldots, n\}$ and multiply $w_i - w_{i-1}$ the both sides of (3.3.11), we arrive at:

$$\begin{aligned} &\frac{1}{\tau}|w_i - w_{i-1}|_X^2 + (\varpi_i^*, \partial_x(w_i - w_{i-1}))_X + \nu^2(\partial_x w_i, \partial_x(w_i - w_{i-1}))_X \\ &\quad + (\xi_i, w_i - w_{i-1})_X + \big(g(w_i), w_i - w_{i-1}\big)_X = M_u(u_i, w_i - w_{i-1})_X, \end{aligned} \tag{3.5.5}$$

with

$$\begin{cases} \bullet\, \varpi_i^* \in \partial f^\varepsilon(\partial_x w_i) \text{ in } \mathbb{R}, \text{ a.e. in } \Omega, \\ \bullet\, \xi_i \in K^\delta(w_i) \text{ in } \mathbb{R}, \text{ a.e. in } \Omega. \end{cases}$$

Also, by applying the assumption (A3), Remark 3.7, and Taylor's theorem, we have:

$$\begin{aligned} G(w_{i-1}) &\geq G(w_i) + g(w_i)(w_{i-1} - w_i) - \frac{1}{2}|[g']^-|_{L^\infty(\mathbb{R})}(w_{i-1} - w_i)^2 \\ &\geq G(w_i) - g(w_i)(w_i - w_{i-1}) - \frac{C_g}{2}|w_i - w_{i-1}|^2, \text{ a.e. in } \Omega. \end{aligned} \tag{3.5.6}$$

By using (3.1.1), (3.5.6), the assumptions (A1)–(A4), and Young's inequality, we can deduce from (3.5.5) that:

$$\begin{aligned} &\frac{3}{4\tau}|w_i - w_{i-1}|_X^2 + \frac{\nu^2}{2}|\partial_x w_i|_X^2 - \frac{\nu^2}{2}|\partial_x w_{i-1}|_X^2 \\ &\quad + \int_\Omega f^\varepsilon(\partial_x w_i)\, dx - \int_\Omega f^\varepsilon(\partial_x w_{i-1})\, dx \\ &\quad + \int_\Omega G(w_i)\, dx - \int_\Omega G(w_{i-1})\, dx - \frac{C_g}{2}|w_i - w_{i-1}|_X^2 \\ &\quad + \int_\Omega \hat{K}^\delta(w_i)\, dx - \int_\Omega \hat{K}^\delta(w_{i-1})\, dx \leq \tau M_u^2 |u_i|_X^2, \end{aligned} \tag{3.5.7}$$

via

$$(\varpi_i^*, \partial_x(w_i - w_{i-1}))_X \geq \int_\Omega f^\varepsilon(\partial_x w_i)\, dx - \int_\Omega f^\varepsilon(\partial_x w_{i-1})\, dx,$$

$$\nu^2\big(\partial_x w_i, \partial_x(w_i - w_{i-1})\big)_X \geq \frac{\nu^2}{2}|\partial_x w_i|_X^2 - \frac{\nu^2}{2}|\partial_x w_{i-1}|_X^2,$$

$$(\xi_i, w_i - w_{i-1})_X \geq \int_\Omega \hat{K}^\delta(w_i)\,dx - \int_\Omega \hat{K}^\delta(w_{i-1})\,dx,$$

$$\big(g(w_i), w_i - w_{i-1}\big)_X \geq \int_\Omega G(w_i)\,dx - \int_\Omega G(w_{i-1})\,dx - \frac{C_g}{2}|w_i - w_{i-1}|_X^2,$$

and

$$M_u(u_i, w_i - w_{i-1})_X \leq \frac{1}{4\tau}|w_i - w_{i-1}|_X^2 + \tau M_u^2 |u_i|_X^2.$$

In view of (A5) and (3.5.7), we can see that:

$$\frac{1}{2\tau}|w_i - w_{i-1}|_X^2 + \mathscr{F}^{(\varepsilon,\delta)}(w_i) - \mathscr{F}^{(\varepsilon,\delta)}(w_{i-1}) \leq \tau M_u^2 |u_i|_X^2,\ i = 1, 2, 3, \ldots, n.$$

Thus we conclude Main Theorem 3.1. □

Remark 3.14. For every $\varepsilon, \delta \in (0, 1]$ and $i \in \{1, 2, 3, \ldots, n\}$, let us multiply the both sides of (3.3.11) by ξ_i. Then, by using the assumption (A3) and (3.4.26), one can observe that:

$$\frac{1}{\tau}(w_i - w_{i-1}, \xi_i)_X + |\xi_i|_X^2 - C_g(w_i, \xi_i)_X + (g(0), \xi_i)_X \leq M_u(u_i, \xi_i)_X, \tag{3.5.8}$$

via

$$\begin{aligned}(g(w_i), \xi_i)_X &= \big((g(w_i) + C_g w_i) - g(0), \xi_i\big)_X - (C_g w_i, \xi_i)_X + (g(0), \xi_i)_X \\ &\geq -C_g(w_i, \xi_i)_X + (g(0), \xi_i)_X.\end{aligned}$$

By using (3.5.8), Hölder's and Young's inequalities, we obtain that:

$$\frac{1}{4}|\xi_i|_X^2 \leq \frac{1}{\tau^2}|w_i - w_{i-1}|_X^2 + C_g^2 |w_i|_X^2 + 2|g(0)|_X^2 + 2M_u |u_i|_X^2. \tag{3.5.9}$$

The above estimate will be used later.

3.6 Proof of Main Theorem 3.2

Let us fix $m \in \mathbb{N}$. On account of the energy inequality, we can see that:

$$\frac{1}{2\tau}|w_i^m - w_{i-1}^m|_X^2 + \mathscr{F}^{(\varepsilon_m,\delta_m)}(w_i^m) - \mathscr{F}^{(\varepsilon_m,\delta_m)}(w_{i-1}^m) \leq \tau M_u^2 |u_i^m|_X^2,\ i = 1, 2, 3, \ldots, n.$$

Taking the sum of the above inequalities, for $i = 1, 2, 3, \ldots, n$, we have:

$$\begin{aligned}&\frac{1}{2\tau}\sum_{i=1}^{l}|w_i^m - w_{i-1}^m|_X^2 + \mathscr{F}^{(\varepsilon_m,\delta_m)}(w_i^m) \\ &\qquad \leq \mathscr{F}^{(\varepsilon_m,\delta_m)}(w_0^m) + \tau M_u^2 \sum_{i=1}^{l}|u_i^m|_X^2, l = 1, 2, 3, \ldots, n, \text{ and } m = 1, 2, 3, \ldots.\end{aligned}$$

Here, from (A3) and (3.3.13), we will find a positive constant R_0, independent of $m \in \mathbb{N}$, such that:

$$|u^m|_{\mathbb{X}} \leq R_0, |w_0^m|_{C(\overline{\Omega})} \leq R_0, |w_0^m|_Y \leq R_0,$$
$$\text{and } |G(w_0^m)|_{C(\overline{\Omega})} \leq R_0, \text{ for } m = 1, 2, 3, \dots. \tag{3.6.1}$$

Also, by using (A2), it is estimated that:

$$\begin{aligned}|f^{\varepsilon_m}(\partial_x w_0^m)|_{L^1(\Omega)} &\leq \int_\Omega \left(\int_0^1 |(f^{\varepsilon_m})'(\varsigma \partial_x w_0^m)| \, d\varsigma \right) dx \\ &\leq C_0 \left(2L + \frac{1}{2} \int_\Omega |\partial_x w_0^m| \, dx \right) \\ &\leq C_0 \left(2L + \sqrt{\frac{L}{2}} R_0 \right), \text{ for } m = 1, 2, 3, \dots. \end{aligned} \tag{3.6.2}$$

On account of (3.3.13), (3.6.1), and (3.6.2), we compute that:

$$\begin{aligned}&\frac{1}{4T}|w_i^m|_X^2 + \mathscr{F}^{(\varepsilon_m,\delta_m)}(w_i^m) \\ &\quad \leq \frac{1}{2T}|w_0^m|_X^2 + \frac{1}{2\tau}\sum_{i=1}^{l} |w_i^m - w_{i-1}|_X^2 + \mathscr{F}^{(\varepsilon_m,\delta_m)}(w_i^m) \\ &\quad \leq \frac{R_0^2}{2T} + \mathscr{F}^{(\varepsilon_m,\delta_m)}(w_0^m) + \tau M_u^2 |u_i^m|_X^2 \\ &\quad \leq \frac{R_0^2}{2T} + C_0 \left(2L + \sqrt{\frac{L}{2}} R_0 \right) + \frac{\nu^2}{2} R_0^2 + \hat{K}_* + 2LR_0 + \tau M_u^2 R_0^2, \\ &\quad \text{for } i = 1, 2, 3, \dots, n, \text{ and } m = 1, 2, 3, \dots. \end{aligned} \tag{3.6.3}$$

As a consequence of (3.3.12), (3.3.13) and (3.6.3), one can observe that:

(♦ 1) the sequence $\{w^m\}_{m\in\mathbb{N}} = \{[w_1^m, \dots, w_n^m]\}_{m\in\mathbb{N}}$ is bounded in $\mathbb{Y}$.

Furthermore, by virtue of (A3), (3.3.11), (3.3.13), (3.5.9), and (♦ 1), it is derived that:

(♦ 2) the sequence $\{\xi^m\}_{m\in\mathbb{N}} = \{[\xi_1^m, \dots, \xi_n^m]\}_{m\in\mathbb{N}}$ is bounded in $\mathbb{X}$,

(♦ 3) the sequence $\{\partial_x(\varpi^{(*,m)} + \nu^2 \partial_x w^m)\}_{m\in\mathbb{N}} = \{[\partial_x(\varpi_1^{(*,m)} + \nu^2 \partial_x w_1^m), \dots, \partial_x(\varpi_n^{(*,m)} + \nu^2 \partial_x w_n^m)]\}_{m\in\mathbb{N}}$ is bounded in $\mathbb{X}$, with a sequence $\{\varpi_i^{(*,m)}\}_{m\in\mathbb{N}} \subset X$ satisfying $\varpi_i^{(*,m)} \in \partial f^{\varepsilon_m}(\partial_x w_i^m)$ in $\mathbb{R}$, a.e. in Ω.

Note that (3.4.13) leads to:

$$\begin{aligned}\partial_x w_i^m &= (\mu_{\varepsilon_m}^\nu)^{-1}\big(\mu_{\varepsilon_m}^\nu(\partial_x w_i^m)\big) \\ &= (\mu_{\varepsilon_m}^\nu)^{-1}(\varpi_i^{(*,m)} + \nu^2 \partial_x w_i^m), \text{ for any } m \in \mathbb{N} \text{ and } i = 1, 2, 3, \dots, n. \end{aligned} \tag{3.6.4}$$

With Remark 3.6 and (3.6.4) in mind, we apply the generalized chain rule in BV-theory [3, Theorem 3.99], and we infer that:

$$|\partial_x^2 w_i^m|_X = \big|\big((\mu_{\varepsilon_m}^\nu)^{-1}\big)'(\varpi_i^{(*,m)} + \nu^2 \partial_x w_i^m)\partial_x(\varpi_i^{(*,m)} + \nu^2 \partial_x w_i^m)\big|_X$$

$$\leq \frac{1}{\nu^2}|\partial_x(\varpi_i^{(*,m)} + \nu^2\partial_x w_i^m)|_X, \text{ for any } m \in \mathbb{N} \text{ and } i = 1, 2, 3, \dots, n. \qquad (3.6.5)$$

From (♦3) and (3.6.5), it is deduced that:

(♦4) the sequence $\{\partial_x^2 w^m\}_{m\in\mathbb{N}} = \{[\partial_x^2 w_1^m, \dots, \partial_x^2 w_n^m]\}_{m\in\mathbb{N}}$ is bounded in $\mathbb{X}$, and hence $\{w^m\}_{m\in\mathbb{N}}$ is bounded in $[H^2(\Omega)]^n$.

As a consequence of the one-dimensional compact embedding $H^2(\Omega) \subset C^1(\overline{\Omega})$ and (♦4), there exist $\{m_k\}_{k\in\mathbb{N}} \subset \{m\}$, $\tilde{w} = [\tilde{w}_1, \dots, \tilde{w}_n] \in [H^2(\Omega)]^n$ such that:

$$\begin{aligned} &w_i^{m_k} \to \tilde{w}_i \text{ in } Y, \text{ in } C^1(\overline{\Omega}), \text{ and weakly in } H^2(\Omega), \\ &\qquad \text{for any } i = 1, 2, 3, \dots, n, \text{ as } k \to \infty. \end{aligned} \qquad (3.6.6)$$

Next, we verify that the limit $\tilde{w}$ is the solution to the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$. Let us fix $k \in \mathbb{N}$. From Remark 3.11, the solution w^{m_k} admits the following variational inequality:

$$\begin{aligned} &\frac{1}{\tau}(w_i^{m_k} - w_{i-1}^{m_k}, w_i^{m_k} - z)_X + (g(w_i^{m_k}) - M_u u_i^{m_k}, w_i^{m_k} - z)_X \\ &\quad + \Psi^{(\varepsilon_{m_k},\delta_{m_k})}(w_i^{m_k}) - \Psi^{(\varepsilon_{m_k},\delta_{m_k})}(z) \leq 0, \\ &\qquad \text{for any } z \in D(\Psi^{(\varepsilon_{m_k},\delta_{m_k})}) \text{ and } i = 1, 2, 3, \dots, n. \end{aligned} \qquad (3.6.7)$$

Here, we compute the limit of the both sides of (3.6.7), as $k \to \infty$. Then, with (A3), Key-Lemma and (3.6.6) in mind, one can see that:

$$\begin{aligned} &\frac{1}{\tau}(\tilde{w}_i - \tilde{w}_{i-1}, \tilde{w}_i - z)_X + (g(\tilde{w}_i), \tilde{w}_i - z)_X + \Psi^{(\varepsilon,\delta)}(\tilde{w}_i) \\ &\leq \frac{1}{\tau}\lim_{k\to\infty}(w_i^{m_k} - w_{i-1}^{m_k}, w_i^{m_k} - z)_X + \lim_{k\to\infty}(g(w_i^{m_k}), w_i^{m_k} - z)_X + \varliminf_{k\to\infty}\Psi^{(\varepsilon_{m_k},\delta_{m_k})}(w_i^{m_k}) \\ &\leq \Psi^{(\varepsilon,\delta)}(z) + (M_u u_i, \tilde{w}_i - z)_X, \text{ for any } z \in D(\Psi^{(\varepsilon,\delta)}) \text{ and } i = 1, 2, 3, \dots, n. \end{aligned}$$

This implies that $\tilde{w}$ coincides with the solution w to $(\mathrm{AC})^{(\varepsilon,\delta)}$.

Finally, we prove the convergence (3.3.15). Let us fix $i \in \{1, 2, 3, \dots, n\}$. Then, one can see from (3.6.6) that:

$$\partial_x w_i^{m_k} \to \partial_x w_i \text{ in } C(\overline{\Omega}), \text{ as } k \to \infty. \qquad (3.6.8)$$

In the light of (A2) and (3.6.8), we compute that:

$$\begin{aligned} &|f^{\varepsilon_{m_k}}(\partial_x w_i^{m_k}) - f^\varepsilon(\partial_x w_i)| \\ &\quad \leq |f^{\varepsilon_{m_k}}(\partial_x w_i^{m_k}) - f^{\varepsilon_{m_k}}(\partial_x w_i)| + |f^{\varepsilon_{m_k}}(\partial_x w_i) - f^\varepsilon(\partial_x w_i)| \\ &\quad \leq \left|\left(\int_0^1 (f^{\varepsilon_{m_k}})'(\partial_x w_i + \varsigma(\partial_x w_i^{m_k} - \partial_x w_i))\,d\varsigma\right)(\partial_x w_i^{m_k} - \partial_x w_i)\right| \\ &\qquad + |f^{\varepsilon_{m_k}}(\partial_x w_i) - f^\varepsilon(\partial_x w_i)| \\ &\quad \leq \left(\int_0^1 C_0(1 + |\partial_x w_i + \varsigma(\partial_x w_i^{m_k} - \partial_x w_i)|)\,d\varsigma\right)|\partial_x w_i^{m_k} - \partial_x w_i| \\ &\qquad + |f^{\varepsilon_{m_k}}(\partial_x w_i) - f^\varepsilon(\partial_x w_i)| \end{aligned}$$

$$\leq C_0|\partial_x w_i^{m_k} - \partial_x w_i|_{C(\overline{\Omega})} \int_0^1 (1 + \varsigma|\partial_x w_i| + (1-\varsigma)|\partial_x w_i^{m_k}|)\, d\varsigma$$
$$+ |f^{\varepsilon_{m_k}}(\partial_x w_i) - f^{\varepsilon}(\partial_x w_i)|$$
$$= C_0|\partial_x w_i^{m_k} - \partial_x w_i|_{C(\overline{\Omega})} \left(1 + \frac{1}{2}|\partial_x w_i| + \frac{1}{2}|\partial_x w_i^{m_k}|\right)$$
$$+ |f^{\varepsilon_{m_k}}(\partial_x w_i) - f^{\varepsilon}(\partial_x w_i)|$$
$$\to 0 \text{ in the pointwise sense on } \overline{\Omega}, \text{ as } k \to \infty. \tag{3.6.9}$$

On account of (A2) and (3.6.8), we can say that:

$$|f^{\varepsilon_{m_k}}(\partial_x w_i^{m_k})| \leq \left| \int_0^1 (f^{\varepsilon_{m_k}})(\varsigma \partial_x w_i^{m_k})\, d\varsigma \right| \leq \int_0^1 C_0(1 + \varsigma|\partial_x w_i^{m_k}|)\, d\varsigma$$
$$\leq C_0 \left(1 + \frac{1}{2} \sup_{k \in \mathbb{N}} |\partial_x w_i^{m_k}|\right) \text{ on } \overline{\Omega}, \text{ for any } k \in \mathbb{N}. \tag{3.6.10}$$

By (3.6.9), (3.6.10), and Lebesgue's dominated convergence theorem, we can infer that:

$$f^{\varepsilon_{m_k}}(\partial_x w_i^{m_k}) \to f^{\varepsilon}(\partial_x w_i) \text{ in } X, \text{ as } k \to \infty. \tag{3.6.11}$$

Now, taking into account (3.6.6), (3.6.9), (3.6.11), and the uniqueness of the solution w, we conclude the convergences (3.3.14) and (3.3.15), with no use of subsequence. □

3.7 Proof of Main Theorem 3.3

In this section, we prove the third Main Theorem 3.3. Let us fix the constants $\varepsilon, \delta \in [0,1]$, and fix the initial data $w_0 \in Y$ satisfying $\hat{K}^{\delta}(w_0) \in L^1(\Omega)$. Let us fix any forcing term $\bar{u} = [\bar{u}_1, \dots, \bar{u}_n] \in \mathbb{X}$. Then, invoking (1.5.10), it is estimated that:

$$0 \leq \underline{J}^{(\varepsilon,\delta)} := \inf \mathcal{J}^{(\varepsilon,\delta)}(\mathbb{X}) \leq \overline{J}^{(\varepsilon,\delta)} := \mathcal{J}^{(\varepsilon,\delta)}(\bar{u}) < \infty,$$
$$\text{for all } \varepsilon \in [0,1] \text{ and } \delta \in [0,1]. \tag{3.7.1}$$

Also, for any $\varepsilon \in [0,1]$ and $\delta \in [0,1]$, we denote by $\bar{w} = [\bar{w}_1, \dots, \bar{w}_n]$ the solution to $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the forcing term $\bar{u}$ and initial data w_0.

Based on these, the Main Theorem 3.3 is proved as follows.

Proof of Main Theorem 3.3 (III-A). From the estimate (3.7.1), we immediately find a sequence of forcing functions $\{u^m\}_{m \in \mathbb{N}} = \{[u_1^m, \dots, u_n^m]\}_{m \in \mathbb{N}} \subset \mathbb{X}$, such that:

$$\mathcal{J}^{(\varepsilon,\delta)}(u^m) \downarrow \underline{J}^{(\varepsilon,\delta)}, \text{ as } m \to \infty, \tag{3.7.2a}$$

and

$$\sup_{m \in \mathbb{N}} \left| \sqrt{\frac{M_u}{2}} u^m \right|_{\mathbb{X}}^2 \leq \mathcal{J}^{(\varepsilon,\delta)}(\bar{u}) < \infty. \tag{3.7.2b}$$

Also, the estimate (3.7.2b) enables us to take a subsequence of $\{u^m\}_{m\in\mathbb{N}} \subset \mathbb{X}$ (not relabeled), and to find a function $u^* = [u_1^*, \dots, u_n^*] \in \mathbb{X}$, such that:

$$\sqrt{M_u}u^m \to \sqrt{M_u}u^* \text{ weakly in } \mathbb{X}, \text{ as } m \to \infty. \tag{3.7.3}$$

Let $w^* = [w_1^*, \dots, w_n^*] \in \mathbb{X}$ be the solution to $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the forcing term u^* and initial data w_0. Also, for any $m \in \mathbb{N}$, let $w^m = [w_1^m, \dots, w_n^m] \in \mathbb{X}$ be the solution to $(\mathrm{AC})^{(\varepsilon_m,\delta_m)}$ for the forcing term u^m and the initial data w_0. Then, having in mind (3.3.13), (3.7.3), we can apply Main Theorem 3.2, to see that:

$$w^m \to w^* \text{ in } \mathbb{Y}, \text{ in } [C^1(\overline{\Omega})]^n, \text{ and weakly in } [H^2(\Omega)]^n, \text{ as } m \to \infty. \tag{3.7.4}$$

On account of (3.7.2a), (3.7.3), and (3.7.4), it is computed that:

$$\begin{aligned}
\mathcal{J}^{(\varepsilon,\delta)}(u^*) &= \frac{1}{2}\big|[\sqrt{M_w}(w^* - w^{\mathrm{ad}})\big|_{\mathbb{X}}^2 + \frac{1}{2}\big|\sqrt{M_u}u^*\big|_{\mathbb{X}}^2 \\
&\leq \frac{1}{2}\varliminf_{m\to\infty}\big|[\sqrt{M_w}(w^m - w^{\mathrm{ad}})]\big|_{\mathbb{X}}^2 + \frac{1}{2}\varliminf_{m\to\infty}\big|\sqrt{M_u}u^m\big|_{\mathbb{X}}^2 \\
&= \lim_{m\to\infty}\mathcal{J}^{(\varepsilon,\delta)}(u^m) = \underline{J}^{(\varepsilon,\delta)} \ (\leq \mathcal{J}^{(\varepsilon,\delta)}(u^*)),
\end{aligned}$$

and this leads to:

$$\mathcal{J}^{(\varepsilon,\delta)}(u^*) = \min_{u\in\mathbb{X}}\mathcal{J}^{(\varepsilon,\delta)}(u).$$

Thus, we conclude the item (III-A). □

Proof of Main Theorem 3.3 (III-B). Let $\varepsilon, \delta \in [0,1]$, $\{\varepsilon_m\}_{m\in\mathbb{N}} \subset (0,1]$, $\{\delta_m\}_{m\in\mathbb{N}} \subset (0,1]$, and $\{w_0^m\}_{m\in\mathbb{N}} \subset Y$ be the sequences as in (3.3.13). Let $\bar{w} = [\bar{w}_1, \dots, \bar{w}_n] \in \mathbb{X}$ be the solution to $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the forcing term $\bar{u}$ and initial data w_0, and for any $m \in \mathbb{N}$, let $\bar{w}^m = [\bar{w}_1^m, \dots, \bar{w}_n^m] \in \mathbb{X}$ be the solution to $(\mathrm{AC})^{(\varepsilon_m,\delta_m)}$ for the forcing term $\bar{u}$ and initial data w_0^m. Then, applying (Fact 2) and Main Theorem 3.2 to these solutions, it is observed that:

$$\bar{w}^m \to \bar{w} \text{ in } \mathbb{Y}, \text{ in } [C^1(\overline{\Omega})]^n, \text{ and weakly in } [H^2(\Omega)]^n, \tag{3.7.5a}$$

and

$$\sup_{m\in\mathbb{N}}|w_0^m|_Y < \infty, \ \hat{K}^\delta(w_0) \leq \varliminf_{m\to\infty}\int_\Omega \hat{K}^{\delta_m}(w_0^m) \leq \hat{K}_* < \infty. \tag{3.7.5b}$$

and hence,

$$\overline{J}_{\sup} := \sup_{m\in\mathbb{N}} J^{(\varepsilon_m,\delta_m)}(\bar{u}) < \infty. \tag{3.7.6}$$

Next, for any $m \in \mathbb{N}$, let us denote by $w^{(*,m)} = [w_1^{(*,m)}, \dots, w_n^{(*,m)}] \in \mathbb{X}$ the solution to $(\mathrm{AC})^{(\varepsilon_m,\delta_m)}$ for the forcing term $u^{(*,m)} = [u_1^{(*,m)}, \dots, u_n^{(*,m)}]$ of the optimal control of $(\mathrm{OP})^{(\varepsilon_m,\delta_m)}$ and initial data w_0^m. Then, in the light of (3.7.1) and (3.7.6), it is observed that:

$$0 \leq \frac{1}{2}\big|\sqrt{M_u}u^{(*,m)}\big|_{\mathbb{X}}^2 \leq \underline{J}^{(\varepsilon_m,\delta_m)} \leq \overline{J}_{\sup} < \infty, \text{ for any } m \in \mathbb{N}.$$

Therefore, one can find a subsequence $\{m_k\}_{k\in\mathbb{N}} \subset \{m\}$, together with a limiting function $u^{**} = [u_1^{**}, \dots, u_n^{**}] \in \mathbb{X}$, such that:

$$\begin{gathered}\sqrt{M_u}u^{(*,m_k)} \to \sqrt{M_u}u^{**} \text{ weakly in } \mathbb{X}, \text{ as } k \to \infty, \\ \text{and as well as } M_u u^{(*,m_k)} \to M_u u^{**} \text{ weakly in } \mathbb{X}, \text{ as } k \to \infty.\end{gathered} \tag{3.7.7}$$

Now, let us denote by $w^{**} = [w_1^{**}, \dots, w_n^{**}] \in \mathbb{X}$ the solution to $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the forcing term u^{**} and initial data w_0. Then, applying Main Theorem 3.2, again, to the solutions w^{**} and $w^{(*,m_k)}$, $k = 1, 2, 3, \dots$, one can observe that:

$$w^{(*,m_k)} \to w^{**} \text{ in } \mathbb{Y}, \text{ in } [C^1(\overline{\Omega})]^n, \text{ and weakly in } [H^2(\Omega)]^n, \text{ as } k \to \infty. \tag{3.7.8}$$

As a consequence of (3.7.5), (3.7.7), and (3.7.8), it is verified that:

$$\begin{aligned}\mathcal{J}^{(\varepsilon,\delta)}(u^{**}) &= \frac{1}{2}\big|\sqrt{M_w}(w^{**} - w^{\mathrm{ad}})\big|_{\mathbb{X}}^2 + \frac{1}{2}\big|\sqrt{M_u}u^{**}]\big|_{\mathbb{X}}^2 \\ &\leq \frac{1}{2}\varliminf_{k\to\infty}\big|\sqrt{M_w}(w^{(*,m_k)} - w^{\mathrm{ad}})\big|_{\mathbb{X}}^2 + \frac{1}{2}\varliminf_{k\to\infty}\big|\sqrt{M_u}u^{(*,m_k)}\big|_{\mathbb{X}}^2 \\ &= \varliminf_{k\to\infty}\mathcal{J}^{(\varepsilon_{m_k},\delta_{m_k})}(u^{(*,m_k)}) \leq \lim_{k\to\infty}\mathcal{J}^{(\varepsilon_{m_k},\delta_{m_k})}(\bar{u}) \\ &= \frac{1}{2}\lim_{k\to\infty}\big|\sqrt{M_w}(\bar{w}^{m_k} - w^{\mathrm{ad}}))\big|_{\mathbb{X}}^2 + \frac{1}{2}\big|\sqrt{M_u}\bar{u}\big|_{\mathbb{X}}^2 \\ &= \mathcal{J}^{(\varepsilon,\delta)}(\bar{u}).\end{aligned}$$

Since the choice of $\bar{u} \in \mathbb{X}$ is arbitrary, we conclude that:

$$\mathcal{J}^{(\varepsilon,\delta)}(u^{**}) = \min_{u\in\mathbb{X}}\mathcal{J}^{(\varepsilon,\delta)}(u),$$

and complete the proof of Main Theorem 3.3 (III-B). □

3.8 Proof of Main Theorem 3.4

Let $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$ be fixed constants, and let $w_0 \in Y$ be the initial data, satisfying $\hat{K}^\delta(w_0) \in L^1(\Omega)$. Let us take any forcing term $u = [u_1, \dots, u_n] \in \mathbb{X}$, and take the unique solution $w = [w_1, \dots, w_n] \in \mathbb{X}$ to the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$. Also, let us take any constant $\lambda \in (0, 1)$ and any function $h = [h_1, \dots, h_n] \in \mathbb{X}$, and consider another solution $w^\lambda = [w_1^\lambda, \dots, w_n^\lambda] \in \mathbb{X}$ to the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the perturbed forcing term $u + \lambda h = [u_1 + \lambda h_1, \dots, u_n + \lambda h_n]$ and initial data w_0. On this basis, we consider a sequence of function $\{\chi^\lambda\}_{\lambda\in(0,1)} = \{[\chi_1^\lambda, \dots, \chi_n^\lambda]\}_{\lambda\in(0,1)} \subset \mathbb{X}$, defined as:

$$\chi^\lambda = [\chi_1^\lambda, \dots, \chi_n^\lambda] := \frac{w^\lambda - w}{\lambda} = \left[\frac{w_1^\lambda - w_1}{\lambda}, \dots, \frac{w_n^\lambda - w_n}{\lambda}\right] \in \mathbb{X}, \text{ for } \lambda \in (0, 1). \tag{3.8.1}$$

This sequence acts a key-role in the computation of Gâteaux differential of the cost function $\mathcal{J}^{(\varepsilon,\delta)}$, for $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$.

Remark 3.15. Note that for any $\lambda \in (0,1)$, the function $w^\lambda = [w_1^\lambda, \dots, w_n^\lambda] \in \mathbb{X}$ fulfills the following variational forms:

$$
\begin{aligned}
&\frac{1}{\tau}(\chi_i^\lambda - \chi_{i-1}^\lambda, \varphi)_X + \nu^2(\partial_x \chi_i^\lambda, \partial_x \varphi)_X + \int_\Omega \left(\int_0^1 (f^\varepsilon)''(\partial_x w_i + \varsigma\lambda\partial_x\chi_i^\lambda)\, d\varsigma \right) \partial_x\chi_i^\lambda \partial_x\varphi \, dx \\
&\quad + \int_\Omega \left(\int_0^1 g'(w_i + \varsigma\lambda\chi_i^\lambda)\, d\varsigma \right) \chi_i^\lambda \varphi \, dx + \int_\Omega \left(\int_0^1 (K^\delta)'(w_i + \varsigma\lambda\chi_i^\lambda)\, d\varsigma \right) \chi_i^\lambda \varphi \, dx \\
&= (M_u h_i, \varphi)_X, \text{ for any } \varphi \in Y \text{ and } i = 1, 2, 3, \dots, n, \text{ subject to } \chi_0^\lambda = 0 \text{ in } X.
\end{aligned}
$$

In fact, this variational form is obtained by taking the difference between respective two variational forms for $w^\lambda = [w_1^\lambda, \dots, w_n^\lambda]$ and $w = [w_1, \dots, w_n]$, as in Main Theorem 3.1, and by using the following linearization formulas:

$$
\frac{1}{\lambda}\big((f^\varepsilon)'(\partial_x w_i^\lambda) - (f^\varepsilon)'(\partial_x w_i)\big) = \left(\int_0^1 (f^\varepsilon)''(\partial_x w_i + \varsigma\lambda\partial_x\chi_i^\lambda)\, d\varsigma \right) \partial_x\chi_i^\lambda \text{ in } X,
$$

$$
\frac{1}{\lambda}\big(g(w_i^\lambda) - g(w_i)\big) = \left(\int_0^1 g'(w_i + \varsigma\lambda\chi_i^\lambda)\, d\varsigma \right) \chi_i^\lambda \text{ in } X,
$$

and

$$
\frac{1}{\lambda}\big(K^\delta(w_i^\lambda) - K^\delta(w_i)\big) = \left(\int_0^1 (K^\delta)'(w_i + \varsigma\lambda\chi_i^\lambda)\, d\varsigma \right) \chi_i^\lambda \text{ in } X, \text{ for any } i = 1, 2, 3, \dots, n.
$$

Incidentally, the above linearization formulas can be verified as consequences of the assumptions (A1)–(A6) and the mean-value theorem (cf. [54, Theorem 5 in p. 313]).

Now, we prepare the following three Lemmas, for the proof of Main Theorem 3.4.

Lemma 3.1. Let $\tau \in (0, \tau^*)$ be as in the assumption (A5), and let $c \geq 0$ be a fixed constant such that:

$$
0 < c\tau < \frac{1}{2}. \tag{3.8.2}
$$

Let $\{A_i\}_{i=0}^n \subset [0, \infty)$, $\{B_i\}_{i=0}^n \subset [0, \infty)$, and $\{C_i\}_{i=1}^n \subset [0, \infty)$ be sequences such that:

$$
\frac{1}{\tau}(A_i - A_{i-1} + \tau B_i) \leq cA_i + C_i, \ i = 1, 2, 3, \dots, n. \tag{3.8.3}
$$

Then, it is estimated that:

$$
A_i + \tau B_i \leq 2^n \left(A_0 + \tau B_0 + \tau \sum_{j=1}^n C_j \right), \ i = 1, 2, 3, \dots, n. \tag{3.8.4}
$$

Proof. From the assumptions (3.8.2) and (3.8.3), it is easily derived that:

$$\begin{aligned}\frac{1}{2}(A_i+\tau B_i) &\le A_{i-1}+\tau C_i \\ &\le (A_{i-1}+\tau B_{i-1})+\tau C_i,\ \ i=1,2,3,\dots,n.\end{aligned}$$

Next, we put:

$$P_i := (A_i+\tau B_i), i=0,1,2,\dots,n.$$

On this basis, we observe that:

$$\begin{aligned}P_1 &\le 2P_0+2\tau C_1,\\ P_2 &\le 2^2P_0+2^2\tau C_1+2\tau C_2,\\ P_3 &\le 2^3P_0+2^3\tau C_1+2^2\tau C_2+2\tau C_3,\end{aligned}$$

and in general,

$$\begin{aligned}P_i &\le 2^iP_0+\tau(2^iC_1+2^{i-1}C_2+\cdots+2^2C_{i-1}+2C_i)\\ &\le 2^i(P_0+\tau(C_1+\cdots+C_i)),\ \text{ for } i=1,2,3,\dots,n.\end{aligned} \tag{3.8.5}$$

The estimate (3.8.4) is obtained as a straightforward consequence of (3.8.5). □

Lemma 3.2. Under the assumptions (A1)–(A6), let us fix $\varepsilon\in(0,1]$ and $\delta\in(0,1]$, and fix the initial data $w_0\in Y$ satisfying $\hat{K}^\delta(w_0)\in L^1(\Omega)$. Then, for any $u=[u_1,\dots,u_n]\in\mathbb{X}$, the cost function $\mathcal{J}^{(\varepsilon,\delta)}$ admits the Gâteaux derivative $\left(\mathcal{J}^{(\varepsilon,\delta)}\right)'(u)\in\mathbb{X}(=\mathbb{X}^*)$, such that:

$$\begin{gathered}\left((\mathcal{J}^{(\varepsilon,\delta)})'(u),h\right)_{\mathbb{X}}=\left(M_w(w-w^{\mathrm{ad}}),\chi\right)_{\mathbb{X}}+\left(M_uu,h\right)_{\mathbb{X}},\\ \text{for any } h=[h_1,\dots,h_n]\in\mathbb{X}.\end{gathered}$$

In the context, $w=[w_1,\dots,w_n]\in\mathbb{X}$ is the solution to the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the forcing term u and initial data w_0, and $\chi=[\chi_1,\dots,\chi_n]\in\mathbb{X}$ is a unique solution to the following linearization system:

$$\begin{cases}\dfrac{1}{\tau}(\chi_i-\chi_{i-1})-\partial_x((f^\varepsilon)''(\partial_x w_i)\partial_x\chi_i+\nu^2\partial_x\chi_i)+g'(w_i)\chi_i+(K^\delta)'(w_i)\chi_i\\ \qquad =M_uh_i,\ \text{ in }\Omega,\\ \partial_x\chi_i(\pm L)=0,\ \text{ for any } i=1,2,3,\dots,n,\\ \chi_0=0 \text{ in } X.\end{cases} \tag{3.8.6}$$

Proof. We prove this Lemma in accordance with the following two Steps.

First Step: The linearization system (3.8.6) admits a unique solution.

Second Step: The cost function $\mathcal{J}^{(\varepsilon,\delta)}$ admits the Gâteaux derivative $\left(\mathcal{J}^{(\varepsilon,\delta)}\right)'(u)\in\mathbb{X}$, for any $u=[u_1,\dots,u_n]\in\mathbb{X}$.

Verification of First Step. At first, we verify that the linearization system (3.8.6) admits a solution $\chi = [\chi_1, \ldots, \chi_n]$. Let us fix $i \in \{1, 2, 3, \ldots, n\}$ and let us define a functional $\mathcal{G} : X \longrightarrow (-\infty, \infty]$, by letting:

$$z \in X \mapsto \mathcal{G}(z)$$
$$:= \begin{cases} \dfrac{1}{2\tau}\displaystyle\int_\Omega |z - \chi_{i-1}|^2\, dx + \int_\Omega (f^\varepsilon)''(\partial_x w_i)|\partial_x z|^2\, dx + \frac{\nu^2}{2}\int_\Omega |\partial_x z|^2\, dx \\ +\dfrac{1}{2}\displaystyle\int_\Omega g'(w_i)|z|^2\, dx + \frac{1}{2}\int_\Omega (K^\delta)'(w_i)|z|^2\, dx - (M_u h_i, z)_X, \\ \qquad \text{if } z \in Y, \\ \infty, \quad \text{otherwise.} \end{cases}$$

From (A5) and Remark 3.7, it is observed that:

$$\frac{1}{\tau} - g'(w_i) \geq \frac{1}{\tau} - [g']^-(w_i) \geq \frac{1}{\tau} - C_g > 0,$$

and this enables us to say that the functional

$$z \in X \mapsto \frac{1}{2\tau}\int_\Omega |z - \chi_{i-1}|^2\, dx + \frac{1}{2}\int_\Omega g'(w_i)|z|^2\, dx \tag{3.8.7}$$

is strictly convex on X. From the assumptions (A1)–(A5), Remark 3.7, and (3.8.7), it is easily checked that $\mathcal{G}$ is a proper, l.s.c., and strictly convex function on X, such that:

$$\begin{aligned} \mathcal{G}(z) &\geq \frac{1}{4\tau}|z|_X^2 - \frac{1}{2\tau}|\chi_{i-1}|_X^2 + \frac{\nu^2}{2}|\partial_x z|_X^2 \\ &\quad - \frac{C_g}{2}|z|_X^2 - M_u|h_i|_X|z|_X \\ &\geq \frac{1}{4\tau}|z|_X^2 - \frac{1}{2\tau}|\chi_{i-1}|_X^2 + \frac{\nu^2}{2}|\partial_x z|_X^2 \\ &\quad - \frac{C_g}{2}|z|_X^2 - \frac{1}{16\tau}|z|_X^2 - 4\tau M_u^2|h_i|_X^2 \\ &\geq \frac{1}{8\tau}|z|_X^2 - \frac{1}{2\tau}|\chi_{i-1}|_X^2 + \frac{\nu^2}{2}|\partial_x z|_X^2 - 4\tau M_u^2|h_i|_X^2, \\ &\qquad \text{for any } z \in Y. \end{aligned} \tag{3.8.8}$$

(3.8.8) implies that $\mathcal{G}$ is coercive.

Now, applying [24, Proposition 1.2, Chapter II], we find a unique minimizer $\tilde{z} \in Y$ of $\mathcal{G}$, and hence, we can see that:

$$\begin{aligned} 0 &\leq \frac{1}{\lambda}\big(\mathcal{G}(\tilde{z} + \lambda\varphi) - \mathcal{G}(\tilde{z})\big) \\ &= \frac{1}{\tau}\int_\Omega \varphi(\tilde{z} - \chi_{i-1})\, dx + \frac{\lambda}{2\tau}\int_\Omega |\varphi|^2\, dx \\ &\quad + \int_\Omega (f^\varepsilon)''(\partial_x w_i)\partial_x \tilde{z}\partial_x \varphi\, dx + \frac{\lambda}{2}\int_\Omega (f^\varepsilon)''(\partial_x w_i)|\partial_x \varphi|^2\, dx \end{aligned}$$

$$
\begin{aligned}
&+ \nu^2 \int_\Omega \partial_x \tilde{z} \partial_x \varphi \, dx + \frac{\nu^2 \lambda}{2} \int_\Omega |\partial_x \varphi|^2 \, dx + \int_\Omega g'(w_i) \tilde{z} \varphi \, dx + \frac{\lambda}{2} \int_\Omega g'(w_i) |\varphi|^2 \, dx \\
&+ \int_\Omega (K^\delta)'(w_i) \tilde{z} \varphi \, dx + \frac{\lambda}{2} \int_\Omega (K^\delta)'(w_i) |\varphi|^2 \, dx - (M_u h_i, \varphi)_X, \\
&\quad \text{for any } \varphi \in Y, \text{ and } \lambda \in (0,1).
\end{aligned} \tag{3.8.9}
$$

Taking $\lambda \downarrow 0$ in (3.8.9), it is inferred that:

$$
\begin{aligned}
0 \leq \frac{1}{\tau}(\tilde{z} - \chi_{i-1}, \varphi)_X &+ \big((f^\varepsilon)''(\partial_x w_i)\partial_x \tilde{z} + \nu^2 \partial_x \tilde{z}, \partial_x \varphi\big)_X \\
&+ \big(g'(w_i)\tilde{z}, \varphi\big)_X + \big((K^\delta)'(w_i)\tilde{z}, \varphi\big)_X - (M_u h_i, \varphi)_X, \text{ for any } \varphi \in Y,
\end{aligned} \tag{3.8.10}
$$

i.e.

$$
\begin{aligned}
\frac{1}{\tau}(\tilde{z} - \chi_{i-1}, \varphi)_X &+ \big((f^\varepsilon)''(\partial_x w_i)\partial_x \tilde{z} + \nu^2 \partial_x \tilde{z}, \partial_x \varphi\big)_X \\
&+ \big(g'(w_i)\tilde{z}, \varphi\big)_X + \big((K^\delta)'(w_i)\tilde{z}, \varphi\big)_X = (M_u h_i, \varphi)_X, \text{ for any } \varphi \in Y.
\end{aligned} \tag{3.8.11}
$$

In particular, taking any $\varphi_0 \in H_0^1(\Omega)$ and putting $\varphi = \varphi_0$ in (3.8.11),

$$
\begin{aligned}
&\left(M_u h_i - \frac{1}{\tau}(\tilde{z} - \chi_{i-1}) - g'(w_i)\tilde{z} - (K^\delta)'(w_i)\tilde{z}, \varphi_0 \right)_X \\
&= \int_\Omega \big((f^\varepsilon)''(\partial_x w_i)\partial_x \tilde{z} + \nu^2 \partial_x \tilde{z}\big) \partial_x \varphi_0 \, dx, \text{ for any } \varphi_0 \in H_0^1(\Omega),
\end{aligned}
$$

which implies:

$$
\begin{aligned}
&-\partial_x \big((f^\varepsilon)''(\partial_x w_i)\partial_x \tilde{z} + \nu^2 \partial_x \tilde{z}\big) \\
&\quad = M_u h_i - \frac{1}{\tau}(\tilde{z} - \chi_{i-1}) - g'(w_i)\tilde{z} - (K^\delta)'(w_i)\tilde{z} \in X, \text{ in } \mathfrak{D}'(\Omega).
\end{aligned} \tag{3.8.12}
$$

Additionally, having in mind Remark 3.6, (A2), (3.8.12), and $w_i \in H^2(\Omega) \subset C^1(\overline{\Omega})$, we infer that

$$
\begin{aligned}
&(f^\varepsilon)''(\partial_x w_i)\partial_x \tilde{z} + \nu^2 \partial_x \tilde{z} \in Y, \text{ and} \\
&\qquad |\partial_x \tilde{z}(\pm L)| \leq \frac{1}{\nu^2} \big| \big((f^\varepsilon)''(\partial_x w_i)\partial_x \tilde{z} + \nu^2 \partial_x \tilde{z}\big)(\pm L) \big| = 0,
\end{aligned} \tag{3.8.13a}
$$

i.e.

$$
(f^\varepsilon)''(\partial_x w_i)\partial_x \tilde{z} + \nu^2 \partial_x \tilde{z} \in H_0^1(\Omega). \tag{3.8.13b}
$$

As a consequence of (3.8.11)–(3.8.13), we obtain the existence and uniqueness of the solution to the linearization system (3.8.6).

Verification of Second Step. Let us fix any $u = [u_1, \dots, u_n] \in \mathbb{X}$, and take any $\lambda \in (0,1)$ and any $h = [h_1, \dots, h_n] \in \mathbb{X}$. Then, it is easily seen that:

$$
M_u(u + \lambda h) \to M_u u \text{ in } \mathbb{X}, \text{ as } \lambda \downarrow 0,
$$

and also, as a consequence of Main Theorem 3.2, it is observed that:

$$
\begin{aligned}
w^\lambda &= [w_1^\lambda, \dots, w_n^\lambda] \to w = [w_1, \dots, w_n] \\
&\quad \text{in } \mathbb{Y}, \text{ in } [C^1(\overline{\Omega})]^n, \text{ and weakly in } [H^2(\Omega)]^n, \text{ as } \lambda \downarrow 0.
\end{aligned} \tag{3.8.14}
$$

Here, in the light of (A2)–(A4), (3.8.1), and (3.8.14), we can find a constant $R_1 > 0$, independent of $\lambda \in (0,1)$, such that:

$$
|w|_{[C^1(\overline{\Omega})]^n \cap [H^2(\Omega)]^n} \vee \sup_{\lambda \in (0,1)} |w^\lambda|_{[C^1(\overline{\Omega})]^n \cap [H^2(\Omega)]^n} \leq R_1, \tag{3.8.15a}
$$

and

$$
\begin{aligned}
\max_{\substack{1 \leq i \leq n \\ \varsigma \in [0,1]}} \Bigl\{ & |(f^\varepsilon)''(\partial_x w_i + \varsigma \lambda \partial_x \chi_i^\lambda)|_{C(\overline{\Omega})}, |g'(w_i + \varsigma \lambda \chi_i^\lambda)|_{C(\overline{\Omega})}, \\
& |(K^\delta)'(w_i + \varsigma \lambda \chi_i^\lambda)|_{L^\infty(\Omega)} \Bigr\} \leq R_1, \\
& \text{for all } 0 < \lambda < 1.
\end{aligned} \tag{3.8.15b}
$$

Also, taking a subsequence if necessary, we see from the assumptions (A2)–(A4) that:

$$
\begin{cases}
\overline{f}_\lambda^\varepsilon := \left(\displaystyle\int_0^1 (f^\varepsilon)''(\partial_x w_i + \varsigma \lambda \partial_x \chi_i^\lambda) \, d\varsigma \right) \to (f^\varepsilon)''(\partial_x w_i), \\
\overline{g}_\lambda := \left(\displaystyle\int_0^1 g'(w_i + \varsigma \lambda \chi_i^\lambda) \, d\varsigma \right) \to g'(w_i), \\
\overline{K}_\lambda^\delta := \left(\displaystyle\int_0^1 (K^\delta)'(w_i + \varsigma \lambda \chi_i^\lambda) \, d\varsigma \right) \to (K^\delta)'(w_i),
\end{cases}
\\ \text{in the pointwise sense a.e. in } \Omega, \text{ as } \lambda \to 0. \tag{3.8.16}
$$

In the meantime, it is easily seen that:

$$
\begin{aligned}
& \frac{1}{\lambda} \bigl(\mathcal{J}^{(\varepsilon,\delta)}(u + \lambda h) - \mathcal{J}^{(\varepsilon,\delta)}(u) \bigr) \\
& \quad = \left(\frac{M_w}{2}(w^\lambda + w - 2w^{\mathrm{ad}}), \chi^\lambda \right)_{\mathbb{X}} + \left(\frac{M_u}{2}(2u + \lambda h), h \right)_{\mathbb{X}}.
\end{aligned} \tag{3.8.17}
$$

Now, we fix $i \in \{1, 2, 3, \dots, n\}$. By using assumptions (A1)–(A4), and Remark 3.7, and by choosing $\varphi = \chi_i^\lambda$ in Remark 3.15, we can deduce that:

$$
\frac{1}{2\tau}(|\chi_i^\lambda|_X^2 - |\chi_{i-1}^\lambda|_X^2) + I_A + I_B \leq I_C + I_D, \tag{3.8.18a}
$$

via

$$
\begin{aligned}
\frac{1}{\tau}(\chi_i^\lambda - \chi_{i-1}^\lambda, \chi_i^\lambda)_X &\geq \frac{1}{\tau}|\chi_i^\lambda|_X^2 - \frac{1}{\tau}(\chi_{i-1}^\lambda, \chi_i^\lambda)_X \\
&\geq \frac{1}{2\tau}(|\chi_i^\lambda|_X^2 - |\chi_{i-1}^\lambda|_X^2),
\end{aligned} \tag{3.8.18b}
$$

$$
\begin{aligned}
I_A &:= \left(-\partial_x \left(\left(\left(\int_0^1 (f^\varepsilon)''(\partial_x w_i + \varsigma\lambda\partial_x\chi_i^\lambda)\, d\varsigma \right) \partial_x \chi_i^\lambda + \nu^2 \partial_x \chi_i^\lambda \right), \chi_i^\lambda \right)_X \\
&= \int_\Omega \left(\int_0^1 (f^\varepsilon)''(\partial_x w_i + \varsigma\lambda\partial_x\chi_i^\lambda)\, d\varsigma \right) |\partial_x \chi_i^\lambda|^2 dx + (\nu^2 \partial_x \chi_i^\lambda, \partial_x \chi_i^\lambda)_X \\
&\geq \nu^2 |\partial_x \chi_i^\lambda|_X^2,
\end{aligned} \tag{3.8.18c}
$$

$$
\begin{aligned}
I_B &:= \left(\left(\int_0^1 (K^\delta)'(w_i + \varsigma\lambda\chi_i^\lambda)\, d\varsigma \right) \chi_i^\lambda, \chi_i^\lambda \right)_X \\
&= \int_\Omega \left(\int_0^1 (K^\delta)'(w_i + \varsigma\lambda\chi_i^\lambda)\, d\varsigma \right) |\chi_i^\lambda|^2 dx \\
&\geq 0,
\end{aligned} \tag{3.8.18d}
$$

$$
\begin{aligned}
I_C &:= -\left(\left(\int_0^1 g'(w_i + \varsigma\lambda\chi_i^\lambda)\, d\varsigma \right) \chi_i^\lambda, \chi_i^\lambda \right)_X \\
&\leq -\int_\Omega \left(\int_0^1 -[g']^-(w_i + \varsigma\lambda\chi_i^\lambda)\, d\varsigma \right) |\chi_i^\lambda|^2 dx \\
&\leq C_g |\chi_i^\lambda|_X^2,
\end{aligned} \tag{3.8.18e}
$$

and

$$
I_D := (M_u h_i, \chi_i^\lambda)_X \leq \frac{M_u^2}{2}|h_i|^2 + \frac{1}{2}|\chi_i^\lambda|_X^2. \tag{3.8.18f}
$$

Based on the (3.8.18), we compute that:

$$
\frac{1}{\tau}(|\chi_i^\lambda|_X^2 - |\chi_{i-1}^\lambda|_X^2 + 2\tau\nu^2|\partial_x\chi_i^\lambda|_X^2) \leq (1 + 2C_g)|\chi_i^\lambda|_X^2 + M_u^2|h_i|_X^2.
$$

Now, let us set:

$$
\begin{cases}
A_i := |\chi_i^\lambda|_X^2, \quad B_i := 2\nu^2|\partial_x\chi_i^\lambda|_X^2 \text{ with } B_0 := B_1, \\
c := 1 + 2C_g, \quad C_i := M_u^2|h_i|_X^2.
\end{cases}
$$

Then, in the light of Lemma 3.1, one can say that:

($\star$1) the sequence $\{\chi^\lambda\}_{\lambda\in(0,1)} = \{[\chi_1^\lambda, \dots, \chi_n^\lambda\}_{\lambda\in(0,1)}$ is bounded in $\mathbb{Y}$, and compact in $\mathbb{X}$.

As consequences of (A2)–(A4), ($\star$1), (3.8.15), (3.8.16), and Lebesgue's dominated convergence theorem, one can find a sequence $\{\lambda_m\}_{m\in\mathbb{N}} \subset \{\lambda\}_{\lambda\in(0,1)}$ and the function $\chi = [\chi_1, \dots, \chi_n] \in \mathbb{X}$ such that:

$$
0 < |\lambda_m| < 1, \text{ and } \lambda_m \to 0, \text{ as } m \to \infty, \tag{3.8.19a}
$$

$$
\chi_i^{\lambda_m} \to \chi_i \text{ in } X, \text{ and weakly in } Y, \text{ as } m \to \infty, \text{ for any } i = 1, 2, 3, \dots, n, \tag{3.8.19b}
$$

$$\overline{f}^{\varepsilon}_{\lambda_m}\partial_x\varphi \to (f^{\varepsilon})''(\partial_x w_i)\partial_x\varphi \text{ in } X,$$
$$\text{for any } \varphi \in Y, \text{ as } m \to \infty, \text{ for any } i = 1, 2, 3, \dots, n, \tag{3.8.19c}$$

$$\overline{g}_{\lambda_m}\varphi \to g'(w_i)\varphi \text{ in } X, \text{ for any } \varphi \in Y, \text{ as } m \to \infty, \text{ for any } i = 1, 2, 3, \dots, n, \tag{3.8.19d}$$

and

$$\overline{K}^{\delta}_{\lambda_m}\varphi \to (K^{\delta})'(w_i)\varphi \text{ in } X,$$
$$\text{for any } \varphi \in Y, \text{ as } m \to \infty, \text{ for any } i = 1, 2, 3, \dots, n. \tag{3.8.19e}$$

On account of (3.8.19), Remark 3.15, and First Step, we will verify that χ is the unique solution to the linearization system (3.8.6).

Now, taking into account (3.8.14), (3.8.17), and (3.8.19), and the uniqueness of the limit χ, we can compute the directional derivative $D_h\mathcal{J}^{(\varepsilon,\delta)}(u) \in \mathbb{R}$, as follows:

$$D_h\mathcal{J}^{(\varepsilon,\delta)}(u) := \lim_{\lambda\to 0}\frac{1}{\lambda}\bigl(\mathcal{J}^{(\varepsilon,\delta)}(u+\lambda h) - \mathcal{J}^{(\varepsilon,\delta)}(u)\bigr) = (M_w(w - w^{\mathrm{ad}}), \chi)_{\mathbb{X}} + (M_u u, h)_{\mathbb{X}},$$
$$\text{for any } u = [u_1, \dots, u_n] \in \mathbb{X}, \text{ and any direction } h = [h_1, \dots, h_n] \in \mathbb{X}. \tag{3.8.20}$$

Moreover, with Riesz's theorem in mind, we deduce the existence of the Gâteaux derivative $\left(\mathcal{J}^{(\varepsilon,\delta)}\right)'(u) \in \mathbb{X}^*(= \mathbb{X})$ at $u = [u_1, \dots, u_n] \in \mathbb{X}$, i.e.

$$\left(\left(\mathcal{J}^{(\varepsilon,\delta)}\right)'(u), h\right)_{\mathbb{X}} = D_h\mathcal{J}^{(\varepsilon,\delta)}(u),$$
$$\text{for every } u = [u_1, \dots, u_n], h = [h_1, \dots, h_n] \in \mathbb{X}. \tag{3.8.21}$$

On account of (3.8.20) and (3.8.21), we conclude this Lemma. □

Lemma 3.3. Under the assumptions (A1)–(A6), let us fix $\varepsilon \in (0, 1]$ and $\delta \in (0, 1]$, and fix the initial data $w_0 \in Y$ satisfying $\hat{K}^{\delta}(w_0) \in L^1(\Omega)$. Also, let $u^* = [u_1^*, \dots, u_n^*] \in \mathbb{X}$ be an optimal control of the problem $(\mathrm{OP})^{(\varepsilon,\delta)}$, and let $w^* = [w_1^*, \dots, w_n^*]$ be the solution to the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the forcing term u^* and initial data w_0. Then, we can get the following equation:

$$\bigl(p, h\bigr)_{\mathbb{X}} = \bigl(v, \chi\bigr)_{\mathbb{X}}, \text{ for all } h = [h_1, \dots, h_n], v = [v_1, \dots, v_n] \in \mathbb{X}.$$

In the context, $\chi = [\chi_1, \dots, \chi_n] \in \mathbb{X}$ and $p = [p_1, \dots, p_n] \in \mathbb{X}$ are unique solutions for the following linearization system:

$$\begin{cases} \dfrac{1}{\tau}(\chi_i - \chi_{i-1}) - \partial_x((f^{\varepsilon})''(\partial_x w_i^*)\partial_x\chi_i + \nu^2\partial_x\chi_i) + g'(w_i^*)\chi_i + (K^{\delta})'(w_i^*)\chi_i \\ \qquad = h_i, \text{ in } \Omega, \\ \partial_x\chi_i(\pm L) = 0, \text{ for any } i = 1, 2, 3, \dots, n, \\ \chi_0 = 0 \text{ in } X; \end{cases} \tag{3.8.22}$$

and the adjoint system:

$$\begin{cases} \dfrac{1}{\tau}(p_i - p_{i+1}) - \partial_x((f^{\varepsilon})''(\partial_x w_i^*)\partial_x p_i + \nu^2\partial_x p_i) + g'(w_i^*)p_i + (K^{\delta})'(w_i^*)p_i \\ \qquad = v_i, \text{ in } \Omega, \\ \partial_x p_i(\pm L) = 0, \text{ for any } i = n, \dots, 3, 2, 1, \\ p_{n+1} = 0 \text{ in } X; \end{cases} \tag{3.8.23}$$

respectively.

Proof. Let us fix arbitrary functions $h = [h_1, \dots, h_n], v = [v_1, \dots, v_n] \in \mathbb{X}$.

First, we verify (3.8.22) and (3.8.23) admit unique solutions $\chi = [\chi_1, \dots, \chi_n]$ and $p = [p_1, \dots, p_n]$, respectively. The existence and uniqueness of the solution χ to (3.8.22) will be verified, immediately, by applying the same argument as in the First Step in the proof of Lemma 3.2, to the special case when $M_u = 1$.

Now, we prove the existence and uniqueness for the system (3.8.23). Let us fix $i \in \{1, 2, 3, \dots, n\}$ and we define a functional $\mathcal{L} : X \longrightarrow (-\infty, \infty]$, by letting:

$$
\begin{aligned}
&z \in X \mapsto \mathcal{L}(z) \\
&:= \begin{cases}
\dfrac{1}{2\tau}\displaystyle\int_\Omega |z - p_{i+1}|^2 \, dx + \int_\Omega (f^\varepsilon)''(\partial_x w_i^*)|\partial_x z|^2 \, dx + \frac{\nu^2}{2}\int_\Omega |\partial_x z|^2 \, dx \\
+\dfrac{1}{2}\displaystyle\int_\Omega g'(w_i^*)|z|^2 \, dx + \frac{1}{2}\int_\Omega (K^\delta)'(w_i^*)|z|^2 \, dx - (u_i, z)_X, \\
\qquad \text{if } z \in Y, \\
\infty, \quad \text{otherwise.}
\end{cases}
\end{aligned}
$$

Then, by using the assumptions (A1)–(A5), and Remark 3.7, and by applying similar arguments as in (3.8.7)–(3.8.13), we can see the existence of the solution to the adjoint system (3.8.23).

Next, having in mind (A2)–(A4), (3.8.22), (3.8.23), and Main Theorem 3.1, it is deduced that:

$$
\begin{aligned}
&\frac{1}{\tau}\sum_{i=1}^{n}(\chi_i - \chi_{i-1}, p_i)_X \\
&\quad = \frac{1}{\tau}\big((\chi_1 - \chi_0, p_1)_X + \cdots + (\chi_n - \chi_{n-1}, p_n)_X - (\chi_n, p_{n+1})_X\big) \\
&\quad = \frac{1}{\tau}\big((p_1 - p_2, \chi_1)_X + \cdots + (p_{n-1} - p_n, \chi_{n-1})_X + (p_n - p_{n+1}, \chi_n)_X\big) \\
&\quad = \frac{1}{\tau}\sum_{i=1}^{n}(p_i - p_{i+1}, \chi_i)_X,
\end{aligned}
\tag{3.8.24a}
$$

$$
\big((f^\varepsilon)''(\partial_x w_i^*)\partial_x \chi_i, \partial_x p_i\big)_X = \big((f^\varepsilon)''(\partial_x w_i^*)\partial_x p_i, \partial_x \chi_i\big)_X, \tag{3.8.24b}
$$

$$
(\nu^2 \partial_x \chi_i, \partial_x p_i)_X = (\nu^2 \partial_x p_i, \partial_x \chi_i)_X, \tag{3.8.24c}
$$

$$
(g'(w_i^*)\chi_i, p_i)_X = (g'(w_i^*)p_i, \chi_i)_X, \tag{3.8.24d}
$$

and

$$
((K^\delta)'(w_i^*)\chi_i, p_i)_X = ((K^\delta)'(w_i^*)p_i, \chi_i)_X, \text{ for any } i = 1, 2, 3, \dots, n. \tag{3.8.24e}
$$

Here, invoking (3.8.22), (3.8.23), and (3.8.24), we compute that:

$$
\big(p, h\big)_{\mathbb{X}} = \sum_{i=1}^{n}(p_i, h_i)_X = \sum_{i=1}^{n}\langle h_i, p_i\rangle_Y
$$

$$
\begin{aligned}
&= \frac{1}{\tau}\sum_{i=1}^{n}(\chi_i - \chi_{i-1}, p_i)_X + \sum_{i=1}^{n}\big((f^\varepsilon)''(\partial_x w_i^*)\partial_x\chi_i, \partial_x p_i\big)_X \\
&\quad + \sum_{i=1}^{n}(\nu^2\partial_x\chi_i, \partial_x p_i)_X + \sum_{i=1}^{n}(g'(w_i^*)\chi_i, p_i)_X + \sum_{i=1}^{n}((K^\delta)'(w_i^*)\chi_i, p_i)_X \\
&= \frac{1}{\tau}\sum_{i=1}^{n}(p_i - p_{i+1}, \chi_i)_X + \sum_{i=1}^{n}\big((f^\varepsilon)''(\partial_x w_i^*)\partial_x p_i, \partial_x \chi_i\big)_X \\
&\quad + \sum_{i=1}^{n}(\nu^2\partial_x p_i, \partial_x \chi_i)_X + \sum_{i=1}^{n}(g'(w_i^*)p_i, \chi_i)_X + \sum_{i=1}^{n}((K^\delta)'(w_i^*)p_i, \chi_i)_X \\
&= \sum_{i=1}^{n}(v_i, \chi_i)_X = (v, \chi)_{\mathbb{X}}.
\end{aligned}
$$

Thus, we conclude this Lemma. □

Now, we are on the stage to prove the Main Theorem 3.4.

Proof of Main Theorem 3.4. Let $u^* = [u_1^*, \dots, u_n^*] \in \mathbb{X}$ be the optimal control of $(\mathrm{OP})^{(\varepsilon,\delta)}$, with the solution $w^* = [w_1^*, \dots, w_n^*] \in \mathbb{X}$ to the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the forcing term u^* and initial data $w_0 \in Y$ satisfying $\hat{K}^\delta(w_0) \in L^1(\Omega)$. Also, let us fix $h = [h_1, \dots, h_n] \in \mathbb{X}$ and let $\chi^* = [\chi_1^*, \dots, \chi_n^*] \in \mathbb{X}$ be the solution to the linearization system (3.8.22) for the forcing term $M_u h$, and let $p^* = [p_1^*, \dots, p_n^*] \in \mathbb{X}$ be the solution to the adjoint system (3.8.23) for the forcing term $M_w(w^* - w^{\mathrm{ad}})$. Then, by applying Lemma 3.2, and by applying Lemma 3.3 to the case when $h = M_u h$ and $v = M_w(w^* - w^{\mathrm{ad}})$, we compute that:

$$
\begin{aligned}
0 = &\ \big((\mathcal{J}^{(\varepsilon,\delta)})'(u^*), h\big)_{\mathbb{X}} \\
= &\ \lim_{\lambda \to 0}\frac{1}{\lambda}\big(\mathcal{J}^{(\varepsilon,\delta)}(u^* + \lambda h) - \mathcal{J}^{(\varepsilon,\delta)}(u^*)\big) \\
= &\ \big(M_w(w^* - w^{\mathrm{ad}}), \chi\big)_{\mathbb{X}} + \big(M_u u^*, h\big)_{\mathbb{X}} \\
= &\ \big(p^*, M_u h\big)_{\mathbb{X}} + \big(M_u u^*, h\big)_{\mathbb{X}} \\
= &\ \big(M_u(p^* + u^*), h\big)_{\mathbb{X}}, \text{ for any } h = [h_1, \dots, h_n] \in \mathbb{X}.
\end{aligned}
$$

Thus, we conclude Main Theorem 3.4. □

3.9 Proof of Main Theorem 3.5

At first, we prepare the notations used throughout this Section. For every $\varepsilon \in (0,1]$ and $\delta \in (0,1]$, we denote by $u^{(*,(\varepsilon,\delta))} = [u_1^{(*,(\varepsilon,\delta))}, \dots, u_n^{(*,(\varepsilon,\delta))}] \in \mathbb{X}$ the optimal control of the approximating problem $(\mathrm{OP})^{(\varepsilon,\delta)}$, together with the solution $w^{(*,(\varepsilon,\delta))} = [w_1^{(*,(\varepsilon,\delta))}, \dots, w_n^{(*,(\varepsilon,\delta))}] \in \mathbb{X}$ to the state-system $(\mathrm{AC})^{(\varepsilon,\delta)}$ for the forcing term $u^{(*,(\varepsilon,\delta))}$ and initial data $w_0 \in Y$, satisfying $|w_0| \leq 1$, a.e. in Ω. Note that:

$$
\hat{K}^{\tilde{\delta}}(w_0) \equiv 0 \text{ on } \overline{\Omega}, \text{ and hence}
$$

$$|\hat{K}^{\tilde{\delta}}(w_0)|_{L^1(\Omega)} = 0, \text{ for all } \tilde{\delta} \in [0,1].$$

In addition, for every $\varepsilon \in (0,1]$ and $\delta \in (0,1]$, we denote by $p^{(*,(\varepsilon,\delta))} = [p_1^{(*,(\varepsilon,\delta))}, \dots, p_n^{(*,(\varepsilon,\delta))}] \in \mathbb{X}$ the solution to the adjoint system (3.8.23) for the forcing term $M_w(w^{(*,(\varepsilon,\delta))} - w^{\mathrm{ad}})$.

Now, by Main Theorem 3.3 (III-B), we find an optimal control $u^\circ = [u_1^\circ, \dots, u_n^\circ] \in \mathbb{X}$ of $(\mathrm{OP})^{(0,0)}$, with a zero-convergent sequences $\{\varepsilon_m\}_{m\in\mathbb{N}} \subset (0,1]$ and $\{\delta_m\}_{m\in\mathbb{N}} \subset (0,1]$, such that:

$$u^{(*,m)} := u^{(*,(\varepsilon_m,\delta_m))} \to u^\circ \text{ weakly in } \mathbb{X}, \text{ as } m \to \infty. \tag{3.9.1a}$$

Let $w^\circ = [w_1^\circ, \dots, w_n^\circ] \in \mathbb{X}$ be the solution to $(\mathrm{AC})^{(0,0)}$ for the forcing term u° and initial data w_0. Then, having in mind (A3) and Main Theorem 3.2, we can find subsequences of $\{\varepsilon_m\}_{m\in\mathbb{N}}$ and $\{\delta_m\}_{m\in\mathbb{N}}$ (not relabeled) such that:

$$\begin{aligned} w^{(*,m)} := w^{(*,(\varepsilon_m,\delta_m))} &\to w^\circ \text{ in } \mathbb{Y}, \text{ in } [C^1(\overline{\Omega})]^n, \\ &\text{and weakly in } [H^2(\Omega)]^n, \text{ as } m \to \infty, \end{aligned} \tag{3.9.1b}$$

$$\partial_x w^{(*,m)} := \partial_x w^{(*,(\varepsilon_m,\delta_m))} \to \partial_x w^\circ \text{ in } [C(\overline{\Omega})]^n, \text{ as } m \to \infty, \tag{3.9.1c}$$

and

$$\begin{aligned} g'(w_i^{(*,m)}) := g'(w_i^{(*,(\varepsilon_m,\delta_m))}) &\to g'(w_i^\circ) \text{ in } C(\overline{\Omega}), \\ &\text{for any } i = 1,2,3,\dots,n, \text{ as } m \to \infty. \end{aligned} \tag{3.9.1d}$$

Here, in the light of (A3) and (3.9.1), we can find a constant $R_2 > 0$, independent of $m \in \mathbb{N}$, such that:

$$\max_{1\le i\le n} |g'(w_i^{(*,m)})|_{C(\overline{\Omega})} \le R_2, \text{ for all } m \in \mathbb{N}. \tag{3.9.2}$$

Next, let us fix $m \in \mathbb{N}$. Then, by (3.8.23), $p^{(*,m)} := p^{(*,(\varepsilon_m,\delta_m))}$ satisfies the following equation:

$$\begin{aligned} &\frac{1}{\tau}(p_i^{(*,m)} - p_{i+1}^{(*,m)}, \varphi)_X + \big((f^{\varepsilon_m})''(\partial_x w_i^{(*,m)})\partial_x p_i^{(*,m)}, \partial_x \varphi\big)_X \\ &\quad + \nu^2(\partial_x p_i^{(*,m)}, \partial_x \varphi)_X + \big(g'(w_i^{(*,m)})p_i^{(*,m)}, \varphi\big)_X \\ &\quad + \big((K^\delta)'(w_i^{(*,m)})p_i^{(*,m)}, \varphi\big)_X = \big(M_w(w_i^{(*,m)} - w_i^{\mathrm{ad}}), \varphi\big)_X, \\ &\quad \text{for any } \varphi \in Y, \text{ and } i = n, \dots, 3, 2, 1. \end{aligned} \tag{3.9.3}$$

Let us fix $i \in \{1,2,3,\dots,n\}$. Then, by applying the assumptions (A1)–(A4), and Remark 3.7, and by choosing $\varphi = p_i^{(*,m)}$ in (3.9.3), we can deduce that:

$$\frac{1}{2\tau}(|p_i^{(*,m)}|_X^2 - |p_{i+1}^{(*,m)}|_X^2) + \nu^2|\partial_x p_i^{(*,m)}|_X^2 + L_A + L_B \le L_C + L_D, \tag{3.9.4a}$$

with

$$L_A := \big((f^{\varepsilon_m})''(\partial_x w_i^{(*,m)})\partial_x p_i^{(*,m)}, \partial_x p_i^{(*,m)}\big)_X$$

$$= \int_\Omega (f^{\varepsilon_m})''(\partial_x w_i^{(*,m)})|\partial_x p_i^{(*,m)}|^2\,dx \geq 0, \tag{3.9.4b}$$

$$\begin{aligned} L_B &:= \big((K^{\delta_m})'(w_i^{(*,m)})p_i^{(*,m)}, p_i^{(*,m)}\big)_X \\ &= \int_\Omega (K^{\delta_m})'(w_i^{(*,m)})|p_i^{(*,m)}|^2\,dx \geq 0, \end{aligned} \tag{3.9.4c}$$

$$\begin{aligned} L_C &:= -\,\big(g'(w_i^{(*,m)})p_i^{(*,m)}, p_i^{(*,m)}\big)_X \\ &\leq \int_\Omega |[g']^-|_{L^\infty(\mathbb{R})}|p_i^{(*,m)}|^2\,dx \\ &\leq C_g|p_i^{(*,m)}|_X^2, \end{aligned} \tag{3.9.4d}$$

and

$$\begin{aligned} L_D &:= \big(M_w(w_i^{(*,m)} - w_i^{\mathrm{ad}}), p_i^{(*,m)}\big)_X \\ &\leq \frac{1}{2}|p_i^{(*,m)}|_X^2 + \frac{M_w^2}{2}|w_i^{(*,m)} - w_i^{\mathrm{ad}}|_X^2. \end{aligned} \tag{3.9.4e}$$

Based on the (3.9.4), we compute that:

$$\begin{aligned} &\frac{1}{\tau}\big((|p_i^{(*,m)}|_X^2 - |p_{i+1}^{(*,m)}|_X^2) + 2\tau\nu^2|\partial_x p_i^{(*,m)}|_X^2\big) \\ &\qquad \leq (1 + 2C_g)|p_i^{(*,m)}|_X^2 + M_w^2|w_i^{(*,m)} - w_i^{\mathrm{ad}}|_X^2. \end{aligned}$$

Here, let us apply Lemma 3.1 to the case when:

$$\begin{cases} A_i := |p_{n-i+1}^{(*,m)}|_X^2, \quad B_i := 2\nu^2|\partial_x p_{n-i+1}^{(*,m)}|_X^2 \text{ with } B_0 := B_1, \\[1ex] c := 1 + 2C_g, \quad C_i := M_w^2|w_{n-i+1}^{(*,m)} - w_{n-i+1}^{\mathrm{ad}}|_X^2. \end{cases}$$

Then, in the light of Lemma 3.1 and (3.9.1b), one can say that:

($\star$2) the sequence $\{p^{(*,m)}\}_{m\in\mathbb{N}} = \{[p_1^{(*,m)}, \dots, p_n^{(*,m)}]\}_{m\in\mathbb{N}}$ is bounded in $\mathbb{Y}$, and compact in $\mathbb{X}$.

Now, let us fix $m \in \mathbb{N}$ and $i \in \{1, 2, 3, \dots, n\}$, and define a bounded and linear functional $\zeta_i^m \in Y^*$ on Y, by putting:

$$\begin{aligned} \langle \zeta_i^m, \varphi\rangle_Y &:= \big((f^{\varepsilon_m})''(\partial_x w_i^{(*,m)})\partial_x p_i^{(*,m)}, \partial_x\varphi\big)_X \\ &\qquad + \big((K^{\delta_m})'(w_i^{(*,m)})p_i^{(*,m)}, \varphi\big)_X, \text{ for any } \varphi \in Y. \end{aligned} \tag{3.9.5}$$

Then, on account of (3.9.2), (3.9.3), and (3.9.5), we can estimate that:

$$\begin{aligned} |\langle \zeta_i^m, \varphi\rangle_Y| &\leq \frac{1}{\tau}|p_i^{(*,m)} - p_{i+1}^{(*,m)}|_X|\varphi|_X + \nu^2|\partial_x p_i^{(*,m)}|_X|\partial_x\varphi|_X \\ &\quad + \int_\Omega |g'(w_i^{(*,m)})p_i^{(*,m)}\varphi|\,dx + M_w|w_i^{(*,m)} - w_i^{\mathrm{ad}}|_X|\varphi|_X \end{aligned}$$

$$\leq \sup_{m\in\mathbb{N}} \Bigl(\frac{1}{\tau} |p_i^{(*,m)} - p_{i+1}^{(*,m)}|_X| + \nu^2 |\partial_x p_i^{(*,m)}|_X$$

$$+ R_2 |p_i^{(*,m)}|_X + M_w |w_i^{(*,m)} - w_i^{\mathrm{ad}}|_X \Bigr) |\varphi|_Y,$$

$$\text{for any } \varphi \in Y. \tag{3.9.6}$$

From (3.9.1a), (3.9.6), and ($\star$ 2), we can see that:

($\star$ 3) the sequence $\{\zeta^m\}_{m\in\mathbb{N}} = \{[\zeta_1^m, \dots, \zeta_n^m]\}_{m\in\mathbb{N}}$ is bounded in $\mathbb{Y}^*$.

Having in mind (3.9.1), ($\star$ 2), and ($\star$ 3), we can find subsequences of $\{w^{(*,m)}\}_{m\in\mathbb{N}} \subset \mathbb{X}$, $\{p^{(*,m)}\}_{m\in\mathbb{N}} \subset \mathbb{X}$, and $\{\zeta^m\}_{m\in\mathbb{N}} = \{[\zeta_i^m, \dots \zeta_n^m]\}_{m\in\mathbb{N}} \subset \mathbb{Y}^*$ (not relabeled), together with $w^\circ \in \mathbb{X}$, $p^\circ = [p_1^\circ, \dots, p_n^\circ] \in \mathbb{X}$, and $\zeta^\circ = [\zeta_1^\circ, \dots, \zeta_n^\circ] \in \mathbb{Y}^*$, such that:

$$p^{(*,m)} \to p^\circ \text{ in } \mathbb{X}, \text{ weakly in } \mathbb{Y}, \text{ as } m \to \infty, \tag{3.9.7a}$$

and

$$\zeta^m \to \zeta^\circ \text{ weakly in } \mathbb{Y}^*, \text{ as } m \to \infty. \tag{3.9.7b}$$

Now, the properties (3.3.16)–(3.3.18) will be verified through the limiting observations for (3.9.3), as $m \to \infty$, with use of (3.9.1) and (3.9.7).

Next, let us fix $\rho > 0$ and define γ_ρ, as follows:

$$\gamma_\rho(r) := \begin{cases} \gamma_0(r-\rho), & \text{if } r \geq \rho, \\ \gamma_0(r+\rho), & \text{if } r \leq -\rho, \\ 0, & \text{if } |r| < \rho. \end{cases} \tag{3.9.8}$$

Also, we fix $i \in \{1, 2, 3, \dots, n\}$, and we define

$$M_\rho := \{|\partial_x w_i^\circ| \geq \rho\}. \tag{3.9.9}$$

In the light of Main Theorem 3.1, (3.9.1), and (3.9.9), there exists $m_\rho \in \mathbb{N}$ such that:

$$|\partial_x w_i^{(*,m)}| \geq \frac{\rho}{2} \text{ uniformly on } M_\rho, \text{ for all } m \geq m_\rho, \tag{3.9.10}$$

and

$$w_i^\circ(M_\rho) \cup w_i^{(*,m)}(M_\rho) \subset (-1, 1) \text{ for all } m \geq m_\rho,$$

i.e.

$$(K^{\delta_m})'(w_i^{(*,m)}) \equiv 0 \text{ on } M_\rho, \text{ for all } m \geq m_\rho. \tag{3.9.11}$$

By (3.3.1) as in the assumption (A2), (3.9.1), and (3.9.7)–(3.9.10), we can obtain that:

$$\int_{M_\rho} (f^{\varepsilon_m})''(\partial_x w_i^{(*,m)}) \partial_x p_i^{(*,m)} \cdot$$

$$\cdot \bigl(\gamma_\rho'(\partial_x w_i^{(*,m)}) \partial_x^2 w_i^{(*,m)} \varphi + \gamma_\rho(\partial_x w_i^{(*,m)}) \partial_x \varphi \bigr) \, dx$$

$$\to 0, \text{ for any } \varphi \in Y, \text{ as } m \to \infty. \tag{3.9.12}$$

Moreover, from Main Theorem 3.1,

$$\partial_x w_i^{(*,m)}(\pm L) = 0, \text{ for all } m \geq m_\rho. \tag{3.9.13}$$

Hence, as a consequence of $\gamma_\rho(0) = \gamma_\rho'(0) = 0$, (3.9.1), (3.9.12), and (3.9.13), we can compute that:

$$\begin{aligned}
&\langle -\gamma_\rho(\partial_x w_i^{(*,m)})\partial_x\big((f^{\varepsilon_m})''(\partial_x w_i^{(*,m)})\partial_x p_i^{(*,m)} + \nu^2 \partial_x p_i^{(*,m)}\big), \varphi\rangle_Y \\
&= \int_\Omega \big((f^{\varepsilon_m})''(\partial_x w_i^{(*,m)})\partial_x p_i^{(*,m)} + \nu^2 \partial_x p_i^{(*,m)}\big)\partial_x\big(\gamma_\rho(\partial_x w_i^{(*,m)})\varphi\big)\, dx \\
&= \int_{M_\rho} (f^{\varepsilon_m})''(\partial_x w_i^{(*,m)})\partial_x p_i^{(*,m)}\big(\gamma_\rho'(\partial_x w_i^{(*,m)})\partial_x^2 w_i^{(*,m)}\varphi + \gamma_\rho(\partial_x w_i^{(*,m)})\partial_x \varphi\big)\, dx \\
&\quad + \int_\Omega \nu^2 \partial_x p_i^{(*,m)}\big(\gamma_\rho'(\partial_x w_i^{(*,m)})\partial_x^2 w_i^{(*,m)}\varphi + \gamma_\rho(\partial_x w_i^{(*,m)})\partial_x \varphi\big)\, dx \\
&\to \int_\Omega \nu^2 \partial_x p_i^\circ\big(\gamma_\rho'(\partial_x w_i^\circ)\partial_x^2 w_i^\circ \varphi + \gamma_\rho(\partial_x w_i^\circ)\partial_x\varphi\big)\, dx, \text{ for any } \varphi \in Y, \text{ as } m \to \infty.
\end{aligned} \tag{3.9.14}$$

Meanwhile, on account of (3.9.8), (3.9.9), and (3.9.11), we can easily check that:

$$\gamma_\rho(\partial_x w_i^{(*,m)})(K^{\delta_m})'(w_i^{(*,m)}) \equiv 0, \text{ for all } m \geq m_\rho. \tag{3.9.15}$$

Here, we invoke that $p^{(*,m)}$ is the solution to the adjoint system (3.8.23) for the forcing term $M_w(w^{(*,m)} - w^{\mathrm{ad}})$, i.e.

$$\begin{aligned}
&\frac{1}{\tau}(p_i^{(*,m)} - p_{i+1}^{(*,m)}) - \partial_x((f^{\varepsilon_m})''(\partial_x w_i^{(*,m)})\partial_x p_i^{(*,m)} + \nu^2 \partial_x p_i^{(*,m)}) \\
&\quad + g'(w_i^{(*,m)})p_i^{(*,m)} + (K^{\delta_m})'(w_i^{(*,m)})p_i^{(*,m)} \\
&= M_w(w_i^{(*,m)} - w_i^{\mathrm{ad}}) \text{ in } \Omega, \text{ for any } i = n, \dots, 3, 2, 1, \text{ and any } m \in \mathbb{N}.
\end{aligned} \tag{3.9.16}$$

On this basis, we multiply $\gamma_\rho(\partial_x w_i^{(*,m)})$ and passing the limit of the both sides of (3.9.16), as $m \to \infty$. Then, from (3.9.1), (3.9.7), (3.9.14), and (3.9.15), one can see that:

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega \gamma_\rho(\partial_x w_i^\circ)(p_i^\circ - p_{i+1}^\circ)\varphi\, dx + \int_\Omega \nu^2 \partial_x p_i^\circ\big(\gamma_\rho'(\partial_x w_i^\circ)\partial_x^2 w_i^\circ\varphi + \gamma_\rho(\partial_x w_i^\circ)\partial_x\varphi\big)\, dx, \\
&\quad + \int_\Omega \gamma_\rho(\partial_x w_i^\circ) g'(w_i^\circ) p_i^\circ \varphi\, dx = \int_\Omega M_w \gamma_\rho(\partial_x w_i^\circ)(w_i^\circ - w_i^{\mathrm{ad}})\varphi\, dx, \\
&\quad \text{for all } \varphi \in Y, \text{ and } i = n, \dots, 3, 2, 1.
\end{aligned} \tag{3.9.17}$$

Then, taking $\rho \to 0$ in (3.9.17), we can deduce that:

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega \gamma_0(\partial_x w_i^\circ)(p_i^\circ - p_{i+1}^\circ)\varphi\, dx + \int_\Omega \nu^2 \partial_x p_i^\circ\big(\gamma_0'(\partial_x w_i^\circ)\partial_x^2 w_i^\circ\varphi + \gamma_0(\partial_x w_i^\circ)\partial_x\varphi\big)\, dx \\
&\quad + \int_\Omega \gamma_0(\partial_x w_i^\circ) g'(w_i^\circ) p_i^\circ \varphi\, dx = \int_\Omega M_w \gamma_0(\partial_x w_i^\circ)(w_i^\circ - w_i^{\mathrm{ad}})\varphi\, dx, \\
&\quad \text{for all } \varphi \in Y, \text{ and } i = n, \dots, 3, 2, 1.
\end{aligned} \tag{3.9.18}$$

Here, in the light of Main Theorem 3.1 and $\gamma_0(0) = \gamma_0'(0) = 0$, we compute that:

$$\int_\Omega \nu^2 \partial_x p_i^\circ \bigl(\gamma_0'(\partial_x w_i^\circ)\partial_x^2 w_i^\circ \varphi + \gamma_0(\partial_x w_i^\circ)\partial_x \varphi\bigr)\, dx \\ = \int_\Omega \nu^2 \partial_x^2 p_i^\circ \gamma_0(\partial_x w_i^\circ)\varphi \, dx, \text{ for any } \varphi \in Y. \tag{3.9.19}$$

From (3.9.18) and (3.9.19), we deduce that:

$$\left(\frac{1}{\tau}(p_i^\circ - p_{i+1}^\circ) - \nu^2 \partial_x^2 p_i^\circ + g'(w_i^\circ) - M_w(w_i^\circ - w_i^{\mathrm{ad}})\right)\gamma_0(\partial_x w_i^\circ) = 0 \text{ in } Y^*, \tag{3.9.20}$$

and furthermore,

$$\langle \nu^2 \partial_x^2 p_i^\circ \gamma_0(\partial_x w_i^\circ), \varphi\rangle_Y \\ = \left(\left(\frac{1}{\tau}(p_i^\circ - p_{i+1}^\circ) + g'(w_i^\circ) - M_w(w_i^\circ - w_i^{\mathrm{ad}})\right)\gamma_0(\partial_x w_i^\circ), \varphi\right)_X \\ \text{for any } \varphi \in Y. \tag{3.9.21}$$

(3.9.20) and (3.9.21) immediately leads to the required equation (3.3.19).

Thus, we complete the proof of Main Theorem 3.5. □

Remark 3.16. Especially, if we assume that:

$$\varepsilon_0 \in (0, 1] \text{ and } (f^\varepsilon)'' \to (f^{\varepsilon_0})'' \text{ in } C_{\mathrm{loc}}(\mathbb{R}), \text{ as } \varepsilon \to \varepsilon_0; \tag{3.9.22}$$

then we can obtain the characterization (3.3.19), more precisely. In fact, applying (3.9.1) under (3.9.22) yields that:

$$(f^{\varepsilon_m})''(\partial_x w_i^{(*,m)}) \to (f^{\varepsilon_0})''(\partial_x w_i^\circ) \\ \text{in } C(\overline{\Omega}), \text{ for any } i = 1, 2, 3, \dots, n, \text{ as } m \to \infty,$$

and this uniform convergence enables to improve, slightly, the characterization (3.3.19) as follows:

$$\gamma_0(\partial_x w_i^\circ)\left(\frac{1}{\tau}(p_i^\circ - p_{i+1}^\circ) - \partial_x\bigl((f^{\varepsilon_0})''(\partial_x w_i^\circ)\partial_x p_i^\circ + \nu^2 \partial_x p_i^\circ\bigr) \\ + g'(w_i^\circ)p_i^\circ - M_w(w_i^\circ - w_i^{\mathrm{ad}})\right) = 0 \text{ in } X, \text{ for any } i = n, \dots, 3, 2, 1.$$

Chapter 4

A class of approximate optimal control problems for 1-D phase-field system with singularity and its numerical algorithm

In Chapter 4, We study an optimal control problem for a one dimensional phase-filed system associated with the total variation energy, from the view-point of numerical analysis. Our state system consists of two parabolic PDEs: a heat equation and a singular diffusion equation of an order parameter. In this paper, we give a class of approximate optimal control problems for our original phase-filed system with singularity. Then, we show the necessary condition of the optimal pair by using the control problem of the approximate state system. In addition, by means of necessary conditions for the approximate control problem, we propose the numerical scheme to find the stationary point of the cost functional to the approximate control problem, and show the convergence of our numerical algorithm. Furthermore, we perform the simple numerical experiments.

4.1 Notations and basic assumptions

First, we mention the notations that are used throughout this Chapter.

For each dimension $n \in \mathbb{N}$, we denote by $\mathscr{L}^n$ the n-dimensional Lebesgue measure, and we use this measure unless otherwise specified.

For any reflexive Banach space B, we denote by $|\cdot|_B$ the norm of B, and denote by B' the dual space of B. Additionally, we denote by $\langle \cdot, \cdot \rangle_{B',B}$ the duality pairing between B' and B.

In particular, we put $H := L^2(0, L)$ with the usual real Hilbert structure, and denote by $(\cdot, \cdot)_H$ the inner product in H, for simplicity.

Also, let X be the Sobolev space $H^1(0, L)$ with the norm

$$|z|_X := \left\{ |z_x|_H^2 + n_0 \left(|z(0)|^2 + |z(L)|^2 \right) \right\}^{1/2} \qquad \text{for any } z \in X,$$

which is equivalent to the standard norm of $H^1(0, L)$. We denote by X' the dual space of X. Also, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X' and X. By identifying Hilbert spaces with their duals, we suppose that

$$X \subset H = H' \subset X' \tag{4.1.1}$$

with dense and compact embeddings, and then we have $\langle v, z\rangle = (v, z)_H$ for $v \in H$ and $z \in X$. Furthermore, let $F : X \to X'$ be the duality mapping defined by

$$\langle Fv, z\rangle := (v_x, z_x)_H + n_0 \left(v(0)z(0) + v(L)z(L)\right) \quad \text{for all } v, z \in X. \tag{4.1.2}$$

Also, for given $f \in H$, $h \in \mathbb{R}$, $\ell \in \mathbb{R}$, $a_0 \in \mathbb{R}$, $a_1 \in \mathbb{R}$, $a_2 \in \mathbb{R}$, $b_1 \in \mathbb{R}$, $b_2 \in \mathbb{R}$, and $n_0 \in \mathbb{R}$, an element $\widetilde{f} \in X'$ is uniquely determined by

$$\langle \widetilde{f}, z\rangle := (a_0 f, z)_H + (a_1 h + n_0 b_1)z(0) + (a_2 \ell + n_0 b_2)z(L) \quad \text{for all } z \in X.$$

For this $\widetilde{f}$, it is easy to check that $Fv = \widetilde{f}$ is formally equivalent to

$$\begin{cases} -v_{xx} = a_0 f \quad \text{in } (0, L), \\ -v_x(0) + n_0(v(0) - b_1) = a_1 h, \quad v_x(L) + n_0(v(L) - b_2) = a_2 \ell. \end{cases} \tag{4.1.3}$$

Note that X' becomes a Hilbert space with inner product $(\cdot, \cdot)_{X'}$ given by

$$(v, z)_{X'} := \langle v, F^{-1} z\rangle \quad \text{for all } v, z \in X'.$$

We next list some notation and definitions of subdifferentials of convex functions. For a proper (i.e., not identically equal to infinity), l.s.c. (lower semi-continuous), and convex function $\psi : H \to \mathbb{R} \cup \{\infty\}$, the effective domain $D(\psi)$ of ψ is defined by $D(\psi) := \{z \in H;\ \psi(z) < \infty\}$. We denote by $\partial\psi$ the subdifferential of ψ in the topology of H. In general, the subdifferential is a possibly multi-valued operator from H into itself, and for any $z \in H$, the value $\partial\psi(z)$ is defined as:

$$\partial\psi(z) := \{z^* \in H\ ;\ (z^*, y - z)_H \le \psi(y) - \psi(z) \quad \text{for all } y \in H\}.$$

Then, a set $D(\partial\psi) := \{z \in H\ ;\ \partial\psi(z) \neq \emptyset\}$ is called the domain of $\partial\psi$. For various properties and related notions of a proper, l.s.c., convex function ψ and its subdifferential $\partial\psi$, we refer to the monograph by Brézis [18]. In particular, for those in Banach spaces, we quote the books by Barbu [13, 14].

We also recall a notion of convergence for convex functions, developed by Mosco [59].

Definition 4.1 (cf. [59])**.** Let ψ, ψ_n ($n \in \mathbb{N}$) be proper, l.s.c., and convex functions on H. Then, we say that ψ_n converges to ψ on H in the sense of Mosco [59] as $n \to \infty$ if the following two conditions are satisfied:

(i) for any subsequence $\{\psi_{n_k}\}_{k\in\mathbb{N}} \subset \{\psi_n\}_{n\in\mathbb{N}}$, if $z_k \to z$ weakly in H as $k \to \infty$, then

$$\liminf_{k\to\infty} \psi_{n_k}(z_k) \ge \psi(z);$$

(ii) for any $z \in D(\psi)$, there is a sequence $\{z_n\}_{n\in\mathbb{N}}$ in H such that

$$z_n \to z \text{ in } H \text{ as } n \to \infty \quad \text{and} \quad \lim_{n\to\infty} \psi_n(z_n) = \psi(z).$$

As well as, if the sequence of convex functions $\{\psi_\varepsilon\}_{\varepsilon\in\Xi}$ is labeled by a continuous argument $\varepsilon \in \Xi$ with a infinite set $\Xi \subset \mathbb{R}$, then for any $\varepsilon_0 \in \Xi$, the Mosco-convergence of $\{\psi_\varepsilon\}_{\varepsilon\in\Xi}$, as $\varepsilon \to \varepsilon_0$, is defined by those of subsequences $\{\psi_{\varepsilon_n}\}_{n\in\mathbb{N}}$, for all sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \Xi$, satisfying $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$.

Finally, throughout this paper, N_i, $i = 1, 2, 3, \cdots$, denotes positive (or nonnegative) constants depending only on their argument(s).

4.2 Preliminaries

In this section, we recall the fundamentals concerned with the total variation and functions of bounded variation. These notions are rigorously defined as follows.

Definition 4.2. **(I)** Let $z \in L^1(0, L)$. Then, z is called a function of bounded variation, or simply a BV-function, on $(0, L)$, if and only if:

$$V_0(z) := \sup \left\{ \int_0^L z \varpi_x dx; \; \begin{array}{l} \varpi \in C^1[0, L] \text{ with a compact support on } (0, L), \\ |\varpi| \leq 1 \text{ on } [0, L] \end{array} \right\} < \infty.$$

Here, we call $V_0(z)$ the total variation of z.
(II) We denote by $BV(0, L)$ the space of all BV-functions on $(0, L)$.

Here are listed usual properties of BV-functions and the space $BV(0, L)$, in forms of some propositions and remarks.

Proposition 4.1 (cf. [26, Chapter 5])**.** Let $z \in BV(0, L)$. Then, there exists a Radon measure $|Dz|$ on $(0, L)$, and $|Dz|$-measurable function $\sigma_z : (0, L) \to \mathbb{R}$ such that

(i) $V_0(z) = \int_0^L |Dz|$, and $|\sigma_z| = 1$, $|Dz|$-a.e. on $(0, L)$;

(ii) $\int_0^L z \varpi_x dx = - \int_0^L \varpi \, \sigma_z |Dz|$ for any $\varpi \in C^1[0, L]$ with a compact support on $(0, L)$.

Remark 4.1. If z belongs to the Sobolev space $W^{1,1}(0, L)$, then $|Dz|$ is absolutely continuous with respect to the Lebesgue measure, and it follows that:

$$\int_U |Dz| = \int_U |z_x(x)| dx \quad \text{for all Borel subsets } U \subset (0, L)$$

and

$$\sigma_z(x) = \begin{cases} \dfrac{z_x(x)}{|z_x(x)|}, & \text{if } z_x(x) \neq 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{a.a. } x \in (0, L).$$

Proposition 4.2 (cf. [11, Chapter 10], [26, Chapter 5])**.** **(I)** The functional $z \in L^1(0, L) \mapsto V_0(z)$ forms a proper, l.s.c., and convex function on $L^1(0, L)$.
(II) The space $BV(0, L)$ is a Banach space with the norm:

$$|z|_{BV(0,L)} := |z|_{L^1(0,L)} + V_0(z) \quad \text{for all } z \in BV(0, L).$$

Proposition 4.3 (cf. [3, Corollary 3.49], [11, Chapter 10])**.** $BV(0, L)$ is continuously embedded in $L^\infty(0, L)$, and compactly embedded in $L^p(0, L)$ for any $1 \leq p < \infty$.

Next, let us set a proper, l.s.c., and convex functional $\mathcal{I}_{[-1,1]}$ on H, by putting:

$$\mathcal{I}_{[-1,1]}(z) := \int_0^L I_{[-1,1]}(z(x))dx \quad \text{for all } z \in H;$$

to define the following total variation functional V^0 with a constraint by the indicator function $I_{[-1,1]}$:

$$V^0(z) = V_0(z) + \frac{1}{\kappa}\mathcal{I}_{[-1,1]}(z) \quad \text{for all } z \in H. \tag{4.2.1}$$

Clearly, V^0 is proper, l.s.c., and convex on H, and its effective domain is formulated by:

$$D(V^0) = \{z \in BV(0,L)\ ;\ |z| \le 1, \text{ a.e. on } (0,L)\}.$$

Finally, we recall the decomposition result of the subdifferential ∂V^0 of V^0. For the detailed proof, we refer to [77, Theorem 3.1].

Proposition 4.4 (cf. [77, Theorem 3.1])**.** The subdifferential ∂V^0 of V^0 is decomposed into the following form:

$$\partial V^0(z) = \partial\left(V_0|_H\right)(z) + \frac{1}{\kappa}\partial\mathcal{I}_{[-1,1]}(z) \text{ in } H \quad \text{for all } z \in H,$$

where $V_0|_H$ denotes the restriction of V_0 onto H.

4.3 Solvability of (P;f, h, ℓ)$^\varepsilon$

In this section, we discuss the existence-uniqueness of solutions to (P;f,h,ℓ)$^\varepsilon$ for any $\varepsilon \ge 0$.

We begin with giving some assumptions on data. Throughout this paper, we assume the following conditions (A1)–(A4).

(A1) $\widehat{a}^0$ is an absolute value function on $\mathbb{R}$, i.e., $\widehat{a}^0(r) := |r|$ for all $r \in \mathbb{R}$. In addition, let $\{\widehat{a}^\varepsilon\}_{\varepsilon\in(0,1]} \subset C^1(\mathbb{R})$ be a sequence of convex functions and C^1-regularizations for $\widehat{a}^0(\cdot) := |\cdot|$, such that:

$$\widehat{a}^\varepsilon(r) \ge 0 \ \text{ for any } r \in \mathbb{R} \text{ and any } \varepsilon \in (0,1],$$

$$\left\{\begin{array}{ll} \widehat{a}^\varepsilon(r) \to \widehat{a}^0(r) \ \text{ for any } r \in \mathbb{R}, \\ \widehat{a}^\varepsilon(\cdot) \to \widehat{a}^0(\cdot) \ \text{ on } \mathbb{R}, \ \text{ in the sense of Mosco,} \end{array}\right. \text{as } \varepsilon \downarrow 0,$$

and there exists a constant $\delta_0 > 0$, independent of $\varepsilon \in (0,1]$, satisfying:

$$|a^\varepsilon(r)| \le \delta_0(|r|+1) \ \text{ for any } r \in \mathbb{R} \text{ and any } \varepsilon \in (0,1],$$

where $a^\varepsilon := (\widehat{a}^\varepsilon)'$ is the derivative of $\widehat{a}^\varepsilon$. Furthermore, there exist bounded functions $\delta_1 : (0,1] \to (0,1]$ and $\delta_2 : (0,1] \to [0,\infty)$ such that

$$\delta_1(\varepsilon) \to 1, \quad \delta_2(\varepsilon) \to 0 \ \text{ as } \varepsilon \downarrow 0,$$

and

$$\widehat{a}^\varepsilon(r) \ge \delta_1(\varepsilon)\widehat{a}^0(r) - \delta_2(\varepsilon) \ \text{ for any } r \in \mathbb{R} \text{ and any } \varepsilon \in (0,1].$$

(A2) g is a continuous and semi-monotone function on $\mathbb{R}$, i.e., there is a constant $C_g > 0$ such that $g(r) + C_g r$ is monotone in $r \in \mathbb{R}$. In addition, the function $g(r)$ has a non-negative potential function $\widehat{g}(r)$, that is,

$$\widehat{g}(r) \geq 0 \ \text{ and } \ (\widehat{g})'(r) = g(r) \ \text{ for any } r \in \mathbb{R}.$$

(A3) $T > 0$, $L > 0$, $\kappa > 0$, $n_0 > 0$, $c_0 \geq 0$, $c_1 \geq 0$, $m_0 \geq 0$, $m_1 \geq 0$, $m_2 \geq 0$ are fixed constants. Also, a_0, a_1, a_2, b_1, b_2 are fixed real numbers.

(A4) u_d and w_d are the given desired target profiles in $L^2(0,T;H)$.

Remark 4.2 (cf. [22,58])**.** The assumption (A1) was introduced in [22, (A4)]. The similar assumption was found in [58, Definition 3.1]. In addition, the typical examples of $\widehat{a}^\varepsilon$ are the followings:

- (Hyperbola type) $\widehat{a}^\varepsilon(r) = \sqrt{r^2 + \varepsilon^2}$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0,1]$.
- (Hyperbolic-tangent type) $\widehat{a}^\varepsilon(r) = \varepsilon \log\left(\cosh\left(\frac{r}{\varepsilon}\right)\right)$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0,1]$.
- (Arctangent type) $\widehat{a}^\varepsilon(r) = \frac{2\varepsilon}{\pi}\left[\frac{r}{\varepsilon}\tan^{-1}\left(\frac{r}{\varepsilon}\right) - \frac{1}{2}\log\left(1 + \left(\frac{r}{\varepsilon}\right)^2\right)\right]$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0,1]$.
- (p-growth type) $\widehat{a}^\varepsilon(r) = \frac{1}{1+\varepsilon^2}|r^2 + \varepsilon^2|^{\frac{1+\varepsilon^2}{2}}$ for any $r \in \mathbb{R}$ and any $\varepsilon \in (0,1]$.

Clearly, such functions satisfy (A1).

We now give the notion of solutions to (P;f, h, ℓ)$^\varepsilon$. To this end, for given $f \in L^2(0,T;H)$, $h \in L^2(0,T)$, and $\ell \in L^2(0,T)$, we define $\widetilde{f} \in L^2(0,T;X')$ by putting

$$\begin{gathered}\langle \widetilde{f}(t), z \rangle := (a_0 f(t), z)_H + (a_1 h(t) + n_0 b_1) z(0) + (a_2 \ell(t) + n_0 b_2) z(L) \\ \text{for all } z \in X \text{ and a.a. } t \in (0,T).\end{gathered} \tag{4.3.1}$$

In addition, let K^ε be a function on $\mathbb{R}$ defined by (1.5.19). Clearly, K^ε is a C^1-function with derivative $(K^\varepsilon)' \in W^{1,\infty}(\mathbb{R})$. We fix a primitive $\widehat{K}^\varepsilon \in C^2(\mathbb{R}) \cap W^{3,\infty}_{loc}(\mathbb{R})$ of K^ε such that

$$\widehat{K}^\varepsilon(0) = 0 \quad \text{and} \quad \widehat{K}^\varepsilon(r) \geq 0 \quad \text{for all } r \in \mathbb{R}. \tag{4.3.2}$$

Then, for any $\varepsilon \in (0,1]$, let us set:

$$V^\varepsilon(z) := \begin{cases} \displaystyle\int_0^L \widehat{a}^\varepsilon(z_x(x))dx + \frac{\varepsilon}{2}\int_0^L |z_x(x)|^2 dx + \frac{1}{\kappa}\int_0^L \widehat{K}^\varepsilon(z(x))\,dx, & \text{if } z \in X, \\ \infty, & \text{otherwise.} \end{cases} \tag{4.3.3}$$

Clearly, each functional V^ε ($\varepsilon \in (0,1]$) forms a proper, l.s.c., and convex functional on H.

Based on functionals V^ε ($\varepsilon \in (0,1]$) and V^0 (cf. (4.2.1)), the solutions to (P;f, h, ℓ)$^\varepsilon$, for $\varepsilon \geq 0$, are defined as follows.

Definition 4.3. Let $\varepsilon \in [0,1]$, $u_0 \in X'$, and $w_0 \in H$. Then, a couple of functions $[u^\varepsilon, w^\varepsilon]$ is called a solution to (P;$f,h,\ell)^\varepsilon$, or (P;$u_0, w_0, f, h, \ell)^\varepsilon$ when the initial data are specified, on $[0,T]$, if the following conditions are satisfied:

(S1) $u^\varepsilon \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H)$.

(S2) $w^\varepsilon \in W^{1,2}(0,T;H)$ with $V^\varepsilon(w^\varepsilon) \in L^\infty(0,T)$.

(S3) For all $z \in X$ and a.a. $t \in (0,T)$,

$$\langle (u^\varepsilon)'(t), z\rangle + ((w^\varepsilon)'(t), z)_H + \langle Fu^\varepsilon(t), z\rangle = \langle \widetilde{f}(t), z\rangle.$$

(S4) There is a function $(w^\varepsilon)^* \in L^2(0,T;H)$ such that $(w^\varepsilon)^*(t) \in \partial V^\varepsilon(w^\varepsilon(t))$ in H and

$$(w^\varepsilon)'(t) + \kappa (w^\varepsilon)^*(t) + g(w^\varepsilon(t)) = u^\varepsilon(t) \text{ in } H, \text{ a.a. } t \in (0,T).$$

(S5) $u^\varepsilon(0) = u_0$ in X' and $w^\varepsilon(0) = w_0$ in H.

Remark 4.3. By the definition of subdifferentials, we observe that the evolution equation in (S4) of Definition 4.3 is equivalent to the following variational inequality:

$$\begin{gathered} ((w^\varepsilon)'(t) + g(w^\varepsilon(t)) - u^\varepsilon(t), w^\varepsilon(t) - z)_H + \kappa V^\varepsilon(w^\varepsilon(t)) - \kappa V^\varepsilon(z) \le 0 \\ \text{for any } z \in D(V^\varepsilon) \text{ and a.a. } t \in (0,T). \end{gathered} \tag{4.3.4}$$

Note that (4.3.4) corresponds to a weak formulation of the second equation of (P;$f,h,\ell)^\varepsilon$, for any $\varepsilon \ge 0$.

Remark 4.4. Let $\varepsilon \in (0,1]$. Then, note that the subdifferential operator ∂V^ε is single-valued. In addition, we observe from the definition of subdifferential that $w^* = \partial V^\varepsilon(w^\varepsilon)$ if and only if

$$(w^*, z)_H = (a^\varepsilon(w^\varepsilon_x) + \varepsilon w^\varepsilon_x, z_x)_H + \frac{1}{\kappa}\left(K^\varepsilon(w^\varepsilon), z\right)_H, \quad \forall z \in D(V^\varepsilon).$$

The expression of ∂V^ε is obtained by computing the first variations of the convex function V^ε, and the variational inequality (4.3.4) implicitly includes the homogeneous Neumann type boundary condition.

Remark 4.5 (cf. [81, Remarks 3.1, 3.2, and 3.3])**.** Let $\varepsilon = 0$. By Proposition 4.4, the condition (S4) of Definition 4.3 is equivalent to the following condition (S4)$'$:

(S4)$'$ There is a function $(w^\varepsilon_0)^* \in L^2(0,T;H)$ and a function $\xi^\varepsilon \in L^2(0,T;H)$ such that

$$(w^\varepsilon_0)^*(t) \in \partial\left(V_0|_H\right)(w^\varepsilon(t)) \text{ in } H, \quad \xi^\varepsilon(t) \in \partial \mathcal{I}_{[-1,1]}(w^\varepsilon(t)) \text{ in } H,$$

$$(w^\varepsilon)'(t) + \kappa (w^\varepsilon_0)^*(t) + \xi^\varepsilon(t) + g(w^\varepsilon(t)) = u^\varepsilon(t) \text{ in } H$$

for a.a. $t \in (0,T)$.

Note that the function $(w_0^\varepsilon)^* \in L^2(0,T;H)$ as in (S4)' somehow links to the first variation of the total variation functional $V_0|_H$. In addition, as is well-known (cf. [18, Proposition 2.16]),

$$\partial\mathcal{I}_{[-1,1]}(z) = \left\{\xi \in H;\ \xi \in \partial I_{[-1,1]}(z),\ \text{a.e. on } (0,L)\right\} \text{ for any } z \in D(\mathcal{I}_{[-1,1]}).$$

Hence, the subdifferential ∂V^0 corresponds to the rigorous expression of the singular term $-\left(\frac{w_x}{|w_x|}\right)_x + \frac{1}{\kappa}\partial I_{[-1,1]}(w)$ as in (1.5.12), and the variational inequality (4.3.4) implicitly includes the homogeneous Neumann type boundary condition.

We now mention the first main result in this paper, which is concerned with the existence-uniqueness of solutions to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ for each $\varepsilon \in [0,1]$.

Theorem 4.1 (cf. [81, Propositions 3.1 and 3.2])**.** Assume (A1), (A2), (A3), and $\varepsilon \in [0,1]$. Let $[f,h,\ell]$ be arbitrary triplet of functions in $\mathcal{U}$. Then, for each $u_0 \in H$ and $w_0 \in D(V^\varepsilon)$, there is a unique solution $[u^\varepsilon, w^\varepsilon]$ to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ on $[0,T]$. In addition, there is a positive constant N_1, dependent only on T and n_0, and independent of $\varepsilon \in [0,1]$, such that the following bounded estimate holds:

$$\begin{aligned}
&|(u^\varepsilon)'|^2_{L^2(0,T;X')} + |u^\varepsilon|^2_{L^\infty(0,T;H)} + |u^\varepsilon|^2_{L^2(0,T;X)} + |(w^\varepsilon)'|^2_{L^2(0,T;H)} + |w^\varepsilon|^2_{L^\infty(0,T;H)} \\
&\qquad +\kappa \sup_{0\le t\le T} V^\varepsilon(w^\varepsilon(t)) + \sup_{0\le t\le T}\int_0^L \widehat{g}(w^\varepsilon(t,x))\,dx \\
\le\ & N_1\left(|u_0|^2_H + |w_0|^2_H + \kappa V^\varepsilon(w_0) + \int_0^L \widehat{g}(w_0(x))\,dx + a_0^2|f|^2_{L^2(0,T;H)}\right. \\
&\qquad \left. +a_1^2|h|^2_{L^2(0,T)} + a_2^2|\ell|^2_{L^2(0,T)} + b_1^2 + b_2^2\right).
\end{aligned} \tag{4.3.5}$$

Proof. By a similar argument to [40, Theorem 2.1], we get the unique solution $[u^\varepsilon, w^\varepsilon]$ to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ on $[0,T]$. In fact, let $[u_i^\varepsilon, w_i^\varepsilon]$ $(i=1,2)$ be two solutions to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ on $[0,T]$. Then, note that the following variational identity holds:

$$\begin{gathered}
\langle((u_1^\varepsilon)' - (u_2^\varepsilon)')(\tau), z\rangle + (((w_1^\varepsilon)' - (w_2^\varepsilon)')(\tau), z)_H + \langle(Fu_1^\varepsilon - Fu_2^\varepsilon)(\tau), z\rangle = 0 \\
\text{for all } z \in X \text{ and a.a. } \tau \in (0,T).
\end{gathered} \tag{4.3.6}$$

By integrating (4.3.6) in time, we obtain that

$$\begin{gathered}
((u_1^\varepsilon - u_2^\varepsilon)(t), z)_H + ((w_1^\varepsilon - w_2^\varepsilon)(t), z)_H + \left((\textstyle\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau)d\tau)_x, z_x\right)_H \\
+n_0\left\{\left(\textstyle\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,0)d\tau\right) z(0) + \left(\textstyle\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,L)d\tau\right) z(L)\right\} = 0 \\
\text{for all } z \in X \text{ and all } t \in [0,T].
\end{gathered} \tag{4.3.7}$$

Taking $z = (u_1^\varepsilon - u_2^\varepsilon)(t)$ in (4.3.7), we get that

$$\begin{gathered}
|(u_1^\varepsilon - u_2^\varepsilon)(t)|^2_H + ((w_1^\varepsilon - w_2^\varepsilon)(t), (u_1^\varepsilon - u_2^\varepsilon)(t))_H + \frac{1}{2}\frac{d}{dt}\left|\left(\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau)d\tau\right)_x\right|^2_H \\
+\frac{n_0}{2}\frac{d}{dt}\left\{\left|\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,0)d\tau\right|^2 + \left|\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,L)d\tau\right|^2\right\} = 0 \\
\text{for all } t \in [0,T].
\end{gathered} \tag{4.3.8}$$

By using the Schwarz inequality in (4.3.8), and integrating in time, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\int_0^t |(u_1^\varepsilon - u_2^\varepsilon)(\tau)|_H^2 d\tau + \frac{1}{2}\left|\left(\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau)d\tau\right)_x\right|_H^2 \\
&+\frac{n_0}{2}\left\{\left|\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,0)d\tau\right|^2 + \left|\int_0^t (u_1^\varepsilon - u_2^\varepsilon)(\tau,L)d\tau\right|^2\right\} \\
\leq\ &\frac{1}{2}\int_0^t |(w_1^\varepsilon - w_2^\varepsilon)(\tau)|_H^2 d\tau \quad \text{for all } t \in [0,T].
\end{aligned}
\tag{4.3.9}
$$

Note from the monotonicity of ∂V^ε that

$$
(((w_1^\varepsilon)^* - (w_2^\varepsilon)^*)(\tau), (w_1^\varepsilon - w_2^\varepsilon)(\tau))_H \geq 0, \qquad \text{a.a. } \tau \in (0,T),
$$

where $(w_i^\varepsilon)^*(\tau) \in \partial V^\varepsilon(w_i^\varepsilon(\tau))$ in H for a.a. $\tau \in (0,T)$ $(i = 1,2)$. Therefore, it follows from (S4) of Definition 4.3 and (A2), i.e., the monotonicity of $g(r) + C_g r$, that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}|(w_1^\varepsilon - w_2^\varepsilon)(\tau)|_H^2 \leq ((u_1^\varepsilon - u_2^\varepsilon)(\tau), (w_1^\varepsilon - w_2^\varepsilon)(\tau))_H + C_g|(w_1^\varepsilon - w_2^\varepsilon)(\tau)|_H^2 \\
\text{for a.a. } \tau \in (0,T).
\end{aligned}
\tag{4.3.10}
$$

By using the Schwarz inequality in (4.3.10), and integrating in time, we obtain:

$$
\begin{aligned}
\frac{1}{2}|(w_1^\varepsilon - w_2^\varepsilon)(t)|_H^2 \leq \left(\frac{1}{2} + C_g\right)\int_0^t |(w_1^\varepsilon - w_2^\varepsilon)(\tau)|_H^2 d\tau + \frac{1}{2}\int_0^t |(u_1^\varepsilon - u_2^\varepsilon)(\tau)|_H^2 d\tau \\
\text{for all } t \in [0,T].
\end{aligned}
\tag{4.3.11}
$$

Hence, we infer from (4.3.9) and (4.3.11) that

$$
\frac{1}{2}|(w_1^\varepsilon - w_2^\varepsilon)(t)|_H^2 \leq (1 + C_g)\int_0^t |(w_1^\varepsilon - w_2^\varepsilon)(\tau)|_H^2 d\tau \quad \text{for all } t \in [0,T]. \tag{4.3.12}
$$

Thus, applying the Gronwall inequality to (4.3.12), we observe that

$$
w_1^\varepsilon(t) = w_2^\varepsilon(t) \text{ in } H \text{ for all } t \in [0,T]. \tag{4.3.13}
$$

By the quite standard arguments, we conclude from (4.3.6) with (4.3.13) that

$$
u_1^\varepsilon(t) = u_2^\varepsilon(t) \text{ in } H \text{ for all } t \in [0,T]. \tag{4.3.14}
$$

Thus, the solutions to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ on $[0,T]$ is unique.

Now, we show the existence of solutions to (P;$u_0, w_0, f, h, \ell)^\varepsilon$. Note that (P;$u_0, w_0, f, h, \ell)^\varepsilon$ can be reformulated to abstract evolution equations of the form:

$$
(u^\varepsilon)'(t) + (w^\varepsilon)'(t) + \partial\varphi(u^\varepsilon(t)) \ni \widetilde{f}(t) \ \text{ in } X', \quad \text{for } t \in (0,T), \tag{4.3.15}
$$

$$
(w^\varepsilon)'(t) + \kappa\partial V^\varepsilon(w^\varepsilon(t)) + g(w^\varepsilon(t)) \ni u^\varepsilon(t) \ \text{ in } H, \quad \text{for } t \in (0,T), \tag{4.3.16}
$$

$$
u^\varepsilon(0) = u_0 \text{ in } X' \text{ and } \ w^\varepsilon(0) = w_0 \text{ in } H, \tag{4.3.17}
$$

where $\partial\varphi(\cdot)$ is the subdifferential of a convex function $\varphi(\cdot)$ on X' defined by

$$\varphi(z) := \begin{cases} \dfrac{1}{2}|z|_H^2, & \text{if } z \in H, \\ \infty, & \text{otherwise.} \end{cases} \tag{4.3.18}$$

Also, $\partial V^\varepsilon(\cdot)$ is the subdifferential of the convex functional V^ε on H defined in (4.3.3).

We first show the existence of a local (in time) solution to (4.3.15)–(4.3.17) by employing the fixed point argument for continuous operators in compact convex sets. To this end, for $T > 0$ and $M > 0$, we define a (non-empty) compact convex subset $E(T, M)$ of $L^2(0, T; H)$ by

$$E(T,M) := \left\{ u \in L^2(0,T;H) \;\middle|\; \begin{array}{c} u \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H), \\ |u'|^2_{L^2(0,T;X')} + |u|^2_{L^2(0,T;X)} + \sup\limits_{0\le t\le T} |u(t)|_H^2 \le M \end{array} \right\}.$$

Now, for each $\overline{u} \in E(T, M)$ we consider the following problem, denoted by $(\mathrm{P2})_{\overline{u}}$, with a given function $\overline{u} \in E(T, M)$. For a moment, we often omit the superscript $\varepsilon \in [0, 1]$.

Problem $(\mathrm{P2})_{\overline{u}}$. Find a function $w : [0, T] \to H$ which fulfills the following equation:

$$w'(t) + \kappa\partial V^\varepsilon(w(t)) + g(w(t)) \ni \overline{u}(t) \ \text{ in } H, \quad \text{for } t \in (0,T), \tag{4.3.19}$$

$$w(0) = w_0 \ \text{ in } H. \tag{4.3.20}$$

Taking account of (4.2.1) with Propositions 4.2–4.3 and (4.3.3) with (A1), we observe that for each $\varepsilon \in [0, 1]$, V^ε is proper, l.s.c., and convex on H such that the level set of V^ε is compact in H, i.e.,

$$\{z \in H \;;\; V^\varepsilon(z) \le r\} \ \text{ is compact in } H \text{ for any } r > 0. \tag{4.3.21}$$

Therefore, by using the abstract theory established by Brézis [18] and the perturbation theory (cf. [17, 38]), we observe that $(\mathrm{P2})_{\overline{u}}$ has a unique solution $w \in W^{1,2}(0, T; H)$ with $V^\varepsilon(w) \in L^\infty(0, T)$ for each $w_0 \in D(V^\varepsilon)$ and $\overline{u} \in E(T, M)$. Indeed, (4.3.19) is equivalent to the following equation:

$$w'(t) + \kappa\partial V^\varepsilon(w(t)) + g(w(t)) + C_g w(t) - C_g w(t) \ni \overline{u}(t) \ \text{ in } H, \quad \text{for } t \in (0,T),$$

where C_g is the positive constant in (A2). Therefore, by applying the general theory of evolution equations with monotone and Lipschitz linear perturbations, we can get the unique solution to $(\mathrm{P2})_{\overline{u}}$ on $[0, T]$.

Moreover, by the standard calculation (cf. (4.3.29) below), we can obtain the following inequality:

$$\begin{aligned} &\int_0^t |w'(\tau)|_H^2 d\tau + 2\kappa V^\varepsilon(w(t)) + 2\int_0^L \widehat{g}(w(t,x))dx \\ \le\ & 2\kappa V^\varepsilon(w_0) + 2\int_0^L \widehat{g}(w_0(x))dx + \int_0^T |\overline{u}(\tau)|_H^2 d\tau, \quad \forall t \in (0,T). \end{aligned} \tag{4.3.22}$$

Next, for the function w constructed above, we consider the following problem, denoted by $(\mathrm{P1})_w$.

Problem (P1)$_w$. Find a function $u : [0,T] \to X'$ which fulfills the following equation:

$$u'(t) + \partial\varphi(u(t)) \ni \widetilde{f}(t) - w'(t) \ \text{ in } X', \quad \text{for } t \in (0,T), \tag{4.3.23}$$

$$u(0) = u_0 \ \text{ in } X'. \tag{4.3.24}$$

We observe from (4.1.1) and (4.3.18) that φ is proper, l.s.c., and convex on X' such that the level set of φ is compact in X'. Since $\widetilde{f} - w' \in L^2(0,T;X')$, we can apply the abstract theory established by Brézis [18]. Thus, we observe that (P1)$_w$ has a unique solution $u \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H)$ for each $u_0 \in H$ and solution w to (P2)$_{\overline{u}}$. Moreover, by the standard calculation (cf. (4.3.28) below), we can obtain the following inequality:

$$\begin{aligned} & |u(t)|_H^2 + 2\int_0^t |u_x(\tau)|_H^2 d\tau + n_0 \left(\int_0^t |u(\tau,0)|^2 d\tau + \int_0^t |u(\tau,L)|^2 d\tau \right) \\ \le \ & e^{2T} \left(|u_0|_H^2 + a_0^2 \int_0^T |f(\tau)|_H^2 d\tau + \frac{2a_1^2}{n_0} \int_0^T |h(\tau)|^2 d\tau + \frac{2a_2^2}{n_0} \int_0^T |\ell(\tau)|^2 d\tau \right) \\ & + 2n_0 T e^{2T}(b_1^2 + b_2^2) + e^{2T} \int_0^T |w'(\tau)|_H^2 d\tau, \quad \forall t \in (0,T). \end{aligned} \tag{4.3.25}$$

Note that the solution u to (P1)$_w$ satisfies the following identity (cf. (S3) in Definition 4.3):

$$\begin{gathered} \int_0^T \langle u'(t), \zeta(t)\rangle dt + \int_0^T \langle Fu(t), \zeta(t)\rangle dt = \int_0^T \langle \widetilde{f}(t), \zeta(t)\rangle dt - \int_0^T (w'(t), \zeta(t))_H dt \\ \text{for all } \zeta \in L^2(0,T;X). \end{gathered} \tag{4.3.26}$$

From (4.1.2), (4.3.25), and (4.3.26), we infer that

$$\begin{aligned} |u'|_{L^2(0,T;X')}^2 \ \le N_2 \Big(& |u_x|_{L^2(0,T;H)}^2 + n_0 |u(\cdot,0)|_{L^2(0,T)}^2 + n_0 |u(\cdot,L)|_{L^2(0,T)}^2 \\ & + a_0^2 |f|_{L^2(0,T;H)}^2 + \frac{a_1^2}{n_0} |h|_{L^2(0,T)}^2 + \frac{a_2^2}{n_0} |\ell|_{L^2(0,T)}^2 \\ & + |w'|_{L^2(0,T;H)}^2 + n_0 T (b_1^2 + b_2^2) \Big) \end{aligned} \tag{4.3.27}$$

for some constant $N_2 > 0$ independent of the given function $\overline{u} \in E(T,M)$.

Here, we define an operator $S : E(T,M) \longrightarrow L^2(0,T;H)$ as follows. For each $\overline{u} \in E(T,M)$, we denote by w a unique solution to (P2)$_{\overline{u}}$, and subsequently, we denote by u a unique solution to (P1)$_w$. On that basis, for any given $\overline{u} \in E(T,M)$, we put $S\overline{u} = u$ via the solution w.

Now, we show that S is a self-mapping on $E(T_0, M_0)$ for some positive constants T_0 and M_0, i.e., $S\overline{u} (= u) \in E(T_0, M_0)$ for any $\overline{u} \in E(T_0, M_0)$.

Here, we take $M_0 > 0$ so large such that

$$\begin{aligned} & (4N_2 + 8) \left(|u_0|_H^2 + a_0^2 |f|_{L^2(0,T;H)}^2 + \frac{a_1^2}{n_0} |h|_{L^2(0,T)}^2 + \frac{a_2^2}{n_0} |\ell|_{L^2(0,T)}^2 \right) \\ & + N_2 \left(a_0^2 |f|_{L^2(0,T;H)}^2 + \frac{a_1^2}{n_0} |h|_{L^2(0,T)}^2 + \frac{a_2^2}{n_0} |\ell|_{L^2(0,T)}^2 \right) \end{aligned}$$

$$+n_0(b_1^2+b_2^2)+(6N_2+8)\left(\kappa V^\varepsilon(w_0)+\int_0^L \widehat{g}(w_0(x))dx+1\right)\le M_0,$$

and then, choose $T_0 \in (0,T]$ so small such that

$$e^{2T_0}\le 2, \quad M_0T_0\le 1, \quad N_2T_0+4T_0e^{2T_0}+2N_2T_0e^{2T_0}\le 1.$$

Then, estimates (4.3.22), (4.3.25), (4.3.27) implies that $S\overline{u}(=u)$ belongs to the set $E(T_0,M_0)$ for $\overline{u}\in E(T_0,M_0)$. Thus, the mapping S maps the set $E(T_0,M_0)$ into itself for T_0 and M_0 chosen as above.

Moreover, on account of the convergence theory as in [10], we observe that S is continuous in $E(T_0,M_0)$ with respect to the topology of $L^2(0,T;H)$ (cf. Corollary 4.4 below). Therefore, the Schauder fixed point theorem guarantees that S has at least one fixed point u in $E(T_0,M_0)$. The pair of functions $[u,w]$, consisting of the fixed point u of S and the solution w to $(\mathrm{P})_{\overline{u}}$ when $\overline{u}=u$, is a solution to $(\mathrm{P};u_0,w_0,f,h,\ell)^\varepsilon$ on the time interval $[0,T_0]$. Thus, we have shown that the problem $(\mathrm{P};u_0,w_0,f,h,\ell)^\varepsilon$ has a local (in time) solution $[u,w]$ on $[0,T_0]$.

We now give the energy estimate of the local (in time) solution $[u,w]$ to $(\mathrm{P};u_0,w_0,f,h,\ell)^\varepsilon$ on $[0,T_0]$. To this end, take $z=u(t)$ in (S3) of Definition 4.3. Then, by using the Schwarz inequality, we have:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u(t)|_H^2+(w'(t),u(t))_H+|u_x(t)|_H^2+\frac{n_0}{2}|u(t,0)|^2+\frac{n_0}{2}|u(t,L)|^2\\ \le\ &\frac{1}{2}|u(t)|_H^2+\frac{a_0^2}{2}|f(t)|_H^2+\frac{a_1^2}{n_0}|h(t)|^2+\frac{a_2^2}{n_0}|\ell(t)|^2+n_0(b_1^2+b_2^2)\\ &\qquad\text{for a.a. } t\in(0,T_0).\end{aligned} \tag{4.3.28}$$

Next, multiplying (4.3.16) by $w'(t)$ (cf. (S4) of Definition 4.3), we get:

$$\begin{aligned}&|w'(t)|_H^2+\kappa\frac{d}{dt}V^\varepsilon(w(t))+\frac{d}{dt}\int_0^L\widehat{g}(w(t,x))dx=(u(t),w'(t))_H\\ &\qquad\text{for a.a. } t\in(0,T_0).\end{aligned} \tag{4.3.29}$$

Adding (4.3.29) to (4.3.28), and applying the Gronwall inequality to the resultant, we have:

$$\begin{aligned}&\frac{1}{2}|u(t)|_H^2+\kappa V^\varepsilon(w(t))+\int_0^L\widehat{g}(w(t,x))dx+\\ &\quad+\int_0^t\left\{|u_x(\tau)|_H^2+\frac{n_0}{2}|u(\tau,0)|^2+\frac{n_0}{2}|u(\tau,L)|^2+|w'(\tau)|_H^2\right\}d\tau\\ \le\ & e^T\left(\frac{1}{2}|u_0|_H^2+\kappa V^\varepsilon(w_0)+\int_0^L\widehat{g}(w_0(x))dx+\frac{a_0^2}{2}|f|_{L^2(0,T;H)}^2\right.\\ &\quad\left.+\frac{a_1^2}{n_0}|h|_{L^2(0,T)}^2+\frac{a_2^2}{n_0}|\ell|_{L^2(0,T)}^2+n_0T(b_1^2+b_2^2)\right),\quad \forall t\in[0,T_0].\end{aligned} \tag{4.3.30}$$

In addition, it follows from (4.3.26) and (4.3.30) that

$$\begin{aligned} |u'|^2_{L^2(0,T_0;X')} &\le N_3 \left(|u_0|^2_H + \kappa V^\varepsilon(w_0) + \int_0^L \widehat{g}(w_0(x))dx + a_0^2|f|^2_{L^2(0,T;H)} \right. \\ &\qquad \left. + \frac{a_1^2}{n_0}|h|^2_{L^2(0,T)} + \frac{a_2^2}{n_0}|\ell|^2_{L^2(0,T)} + n_0(b_1^2+b_2^2) \right), \end{aligned} \tag{4.3.31}$$

where N_3 is a positive constant independent of T_0 and given data u_0, w_0, f, h, and ℓ.

Multiplying (4.3.16) by $w(t)$ (cf. (S4) of Definition 4.3), we observe from (A2) that:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|w(t)|^2_H + \kappa V^\varepsilon(w(t)) \le C_g|w(t)|^2_H + (u(t), w(t))_H \\ \text{for a.a. } t \in (0, T_0). \end{aligned} \tag{4.3.32}$$

By using the Schwarz inequality in (4.3.32), applying the Gronwall inequality to the resultant, we have:

$$|w(t)|^2_H + 2\kappa \int_0^t V^\varepsilon(w(\tau))d\tau \le e^{(1+2C_g)T}\left(|w_0|^2_H + |u|^2_{L^2(0,T_0;H)}\right), \quad \forall t \in [0, T_0]. \tag{4.3.33}$$

Therefore, by (4.3.30), (4.3.31), and (4.3.33), we can find a positive constant N_4, independent of T_0, such that the following bounded estimate holds:

$$\begin{aligned} &|u'|^2_{L^2(0,T_0;X')} + |u|^2_{L^\infty(0,T_0;H)} + |u|^2_{L^2(0,T_0;X)} + |w'|^2_{L^2(0,T_0;H)} + |w|^2_{L^\infty(0,T_0;H)} \\ &\qquad + \kappa \sup_{0\le t\le T_0} V^\varepsilon(w(t)) + \sup_{0\le t\le T_0}\int_0^L \widehat{g}(w(t,x))\,dx \\ &\le N_4\left(|u_0|^2_H + |w_0|^2_H + \kappa V^\varepsilon(w_0) + \int_0^L \widehat{g}(w_0(x))\,dx + a_0^2|f|^2_{L^2(0,T;H)} \right. \\ &\qquad \left. + a_1^2|h|^2_{L^2(0,T)} + a_2^2|\ell|^2_{L^2(0,T)} + b_1^2 + b_2^2 \right). \end{aligned} \tag{4.3.34}$$

Hence, by (4.3.34), we can extend the solution to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ beyond the time interval $[0, T_0]$. Namely, we get the existence of a solution to (P;$u_0, w_0, f, h, \ell)^\varepsilon$on $[0, T]$.

The a priori estimate (4.3.5) can be obtained by calculations similar to (4.3.34).

Thus, the proof of Theorem 4.1 has been completed. □

4.4 Continuous dependence of solutions to (P;$f, h, \ell)^\varepsilon$

In this section, we discuss the continuous dependence of solutions to systems (P;$u_0, w_0, f, h, \ell)^\varepsilon$ with respect to $\varepsilon \to 0$.

We begin with proving the Mosco convergence of V^ε on H as $\varepsilon \to 0$.

Lemma 4.1 (cf. [58, Theorem 4.1], [73, Lemma 3.1])**.** Let V^0 and V^ε ($\varepsilon \in (0,1]$) be convex functions given in (4.2.1) and (4.3.3), respectively. Then:

$$V^\varepsilon(\cdot) \longrightarrow V^0(\cdot) \text{ on } H \text{ in the sense of Mosco [59] as } \varepsilon \to 0. \tag{4.4.1}$$

Proof. The proof of this lemma is merely a slight modification of that as in [58, Theorem 4.1] and [73, Lemma 3.1].

Indeed, note that:

$$\int_0^L \widehat{K}^\varepsilon(\cdot)dx \longrightarrow \int_0^L I_{[-1,1]}(\cdot)dx \text{ on } H \text{ in the sense of Mosco [59] as } \varepsilon \to 0; \qquad (4.4.2)$$

we easily show (4.4.2), therefore, we omit the detailed proof of (4.4.2).

Now, we show (i) of Definition 4.1 by using (A1), Proposition 4.2(I), and (4.4.2). To this end, assume $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,1]$, $\{z_k\}_{k\in\mathbb{N}} \subset H$, and $z \in H$ so that

$$\varepsilon_k \to 0 \text{ and } z_k \to z \text{ weakly in } H \text{ as } k \to \infty.$$

Note that we may suppose $\liminf_{k\to\infty} V^{\varepsilon_k}(z_k) < \infty$, because the other case is trivial. Then, from (A1), Proposition 4.2(I), and (4.4.2), we infer that:

$$\begin{aligned}
\infty > {} & \liminf_{k\to\infty} V^{\varepsilon_k}(z_k) \\
= {} & \liminf_{k\to\infty} \left[\int_0^L \widehat{a}^{\varepsilon_k}((z_k)_x(x))dx + \frac{\varepsilon_k}{2}\int_0^L |(z_k)_x(x)|^2 dx + \frac{1}{\kappa}\int_0^L \widehat{K}^{\varepsilon_k}(z_k(x))\,dx \right] \\
\geq {} & \liminf_{k\to\infty} \left[\int_0^L \left\{ \delta_1(\varepsilon_k)\widehat{a}^0((z_k)_x(x)) - \delta_2(\varepsilon_k) \right\} dx + \frac{1}{\kappa}\int_0^L \widehat{K}^{\varepsilon_k}(z_k(x))\,dx \right] \\
\geq {} & \lim_{k\to\infty} \delta_1(\varepsilon_k) \liminf_{k\to\infty} \int_0^L \widehat{a}^0((z_k)_x(x))dx - \lim_{k\to\infty} \delta_2(\varepsilon_k)L \\
& \qquad + \frac{1}{\kappa} \liminf_{k\to\infty} \int_0^L \widehat{K}^{\varepsilon_k}(z_k(x))\,dx \\
\geq {} & \liminf_{k\to\infty} V_0(z_k) + \frac{1}{\kappa}\int_0^L I_{[-1,1]}(z(x))dx \\
\geq {} & V_0(z) + \frac{1}{\kappa}\int_0^L I_{[-1,1]}(z(x))dx = V^0(z),
\end{aligned}$$

which implies that (i) of Definition 4.1 holds.

Next, we show (ii) of Definition 4.1. To this end, Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1]$ be any sequence such that $\varepsilon_n \to 0$ as $n \to \infty$, and let z be any element of $D(V^0)$. According to the result in [3, Theorem 3.9] and [26, Chapter 5], there is a sequence $\{\tilde{z}_i\}_{i\in\mathbb{N}\cup\{0\}} \subset C^\infty(0,L)\cap D(V^0)$ such that

$$|\tilde{z}_i - z|_H < \frac{1}{2^{i+1}} \text{ and } |V_0(\tilde{z}_i) - V_0(z)| < \frac{1}{2^{i+1}} \text{ for all } i \in \mathbb{N}\cup\{0\}. \qquad (4.4.3)$$

By (A1), we can find a sequence $\{n_i\}_{i\in\mathbb{N}}$ such that

$$n_0 = 1,\ n_i \geq i,\ n_{i+1} > n_i,$$

and for any $i \in \mathbb{N}\cup\{0\}$,

$$\begin{cases} \displaystyle \sup_{n\geq n_i} \left| \int_0^L \widehat{a}^{\varepsilon_n}((\tilde{z}_i)_x(x))dx - \int_0^L \widehat{a}^0((\tilde{z}_i)_x(x))dx \right| < \frac{1}{2^{i+1}}, \\ \displaystyle \sup_{n\geq n_i} \frac{\varepsilon_n}{2}|(\tilde{z}_i)_x|_H^2 < \frac{1}{2^{i+1}}. \end{cases} \qquad (4.4.4)$$

Based on these, let us define:

$$z_n := \tilde{z}_i \ \text{ if } n_i \le n < n_{i+1} \ \text{ for some } i \in \mathbb{N} \cup \{0\}.$$

Then, we infer from (4.4.3) and (4.4.4) that

$$|z_n - z|_H = |\tilde{z}_i - z|_H < \frac{1}{2^{i+1}}$$

and

$$\begin{aligned} & |V^{\varepsilon_n}(z_n) - V^0(z)| \\ \le\ & \left| \int_0^L \widehat{a}^{\varepsilon_n}((z_n)_x(x))dx - \int_0^L \widehat{a}^0((z_n)_x(x))dx \right| \\ & \quad + \left| \int_0^L \widehat{a}^0((z_n)_x(x))dx - V_0(z) \right| + \frac{\varepsilon_n}{2}|(z_n)_x|_H^2 \\ \le\ & \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}} < \frac{1}{2^{i-1}} \\ & \qquad \text{for any } i \in \mathbb{N} \cup \{0\} \ \text{ and } \ n \ge n_i, \end{aligned}$$

which implies that (ii) of Definition 4.1 holds.

Thus, the proof of Lemma 4.1 is complete. □

Taking account of (4.4.1), we get the following Corollary 4.2. For the detailed proof, we refer to [10] or [30, Appendix], for instance.

Corollary 4.2 (cf. [10], [30, Appendix])**.** Let V^0 and V^ε ($\varepsilon \in (0,1]$) be convex functions given in (4.2.1) and (4.3.3), respectively. Define

$$\widehat{V}^0(z) := \int_0^T V^0(z(t))dt \ \text{ and } \ \widehat{V}^\varepsilon(z) := \int_0^T V^\varepsilon(z(t))dt, \quad \forall z \in L^2(0,T;H).$$

Then:

$$\widehat{V}^\varepsilon(\cdot) \longrightarrow \widehat{V}^0(\cdot) \ \text{ on } L^2(0,T;H) \text{ in the sense of Mosco [59] as } \varepsilon \to 0.$$

Now, we mention the main theorem concerning the continuous dependence of solutions to systems (P;$u_0, w_0, f, h, \ell)^\varepsilon$ with respect to $\varepsilon \to 0$.

Theorem 4.3. Assume (A1), (A2), and (A3). Let $[f,h,\ell] \in \mathcal{U}$, $u_0 \in H$, and $w_0 \in D(V^0)$. Also, let $\varepsilon \in (0,1]$, $\{[f^\varepsilon, h^\varepsilon, \ell^\varepsilon]\}_{\varepsilon\in(0,1]} \subset \mathcal{U}$, $\{u_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset H$, and $\{w_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset D(V^\varepsilon)$. Furthermore, suppose that

$$f^\varepsilon \to f \text{ weakly in } L^2(0,T;H), \tag{4.4.5}$$

$$h^\varepsilon \to h \text{ weakly in } L^2(0,T), \tag{4.4.6}$$

$$\ell^\varepsilon \to \ell \text{ weakly in } L^2(0,T), \tag{4.4.7}$$

$$u_0^\varepsilon \to u_0 \text{ in } X', \quad w_0^\varepsilon \to w_0 \text{ in } H, \quad \text{and} \quad V^\varepsilon(w_0^\varepsilon) \to V^0(w_0) \tag{4.4.8}$$

as $\varepsilon \to 0$. Then, the unique solution $[u^\varepsilon, w^\varepsilon]$ to (P;$u_0^\varepsilon, w_0^\varepsilon, f^\varepsilon, h^\varepsilon, \ell^\varepsilon)^\varepsilon$ converges to the solution $[u,w]$ to (P;$u_0, w_0, f, h, \ell)^0$ in the following sense:

$$[u^\varepsilon, w^\varepsilon] \longrightarrow [u,w] \quad \text{in } L^2(0,T;H) \times C([0,T];H) \text{ as } \varepsilon \to 0. \tag{4.4.9}$$

Proof. Note from the Mosco convergence (4.4.1) in Lemma 4.1 that for each $w_0 \in D(V^0)$, we can always find a sequence $\{w_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset D(V^\varepsilon)$ satisfying (4.4.8).

From (4.4.8), we infer that

$$V^\varepsilon(w_0^\varepsilon) \text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.4.10}$$

Therefore, we observe from (A1), Proposition 4.1(i), Remark 4.1, and the definitions of $V_0(\cdot)$ and $V^\varepsilon(\cdot)$ that

$$V_0(w_0^\varepsilon) \;\text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.4.11}$$

Therefore, it follows from Propositions 4.2–4.3, (4.4.8), and (4.4.11) that $|w_0^\varepsilon|_{BV(0,L)}$ is bounded uniformly in $\varepsilon \in (0,1]$, hence,

$$|w_0^\varepsilon|_{L^\infty(0,L)} \;\text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.4.12}$$

Thus, we observe from (A2) and (4.4.12) that

$$\int_0^L \widehat{g}(w_0^\varepsilon(x))\,dx \;\text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.4.13}$$

Now, let $[u^\varepsilon, w^\varepsilon]$ be the unique solution to (P;$u_0^\varepsilon, w_0^\varepsilon, f^\varepsilon, h^\varepsilon, \ell^\varepsilon)^\varepsilon$ on $[0,T]$. Then, from (4.4.5)–(4.4.8) and the stability estimate (4.3.5) with (4.4.10)–(4.4.13), we observe that

$$u^\varepsilon \text{ is bounded in } W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H), \tag{4.4.14}$$

$$w^\varepsilon \text{ is bounded in } W^{1,2}(0,T;H), \tag{4.4.15}$$

and

$$\sup_{0\le t\le T} V^\varepsilon(w^\varepsilon(t)) \text{ is bounded} \tag{4.4.16}$$

uniformly in $\varepsilon \in (0,1]$.

Additionally, from similar arguments as above (cf. (4.4.10)–(4.4.12)), we infer that

$$\sup_{0\le t\le T} V_0(w^\varepsilon(t)) \;\text{ is bounded uniformly in } \varepsilon \in (0,1], \tag{4.4.17}$$

therefore,

$$\sup_{0\le t\le T} |w^\varepsilon(t)|_{BV(0,L)} \;\text{ is bounded uniformly in } \varepsilon \in (0,1].$$

Hence, we have

$$\sup_{0\le t\le T} |w^\varepsilon(t)|_{L^\infty(0,L)} \;\text{ is bounded uniformly in } \varepsilon \in (0,1] \tag{4.4.18}$$

and

$$\sup_{0\le t\le T} \int_0^L \widehat{g}(w^\varepsilon(t,x))dx \;\text{ is bounded uniformly in } \varepsilon \in (0,1]. \tag{4.4.19}$$

Thus, by (4.4.14)–(4.4.19), there is a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of $\{\varepsilon\}_{\varepsilon\in(0,1]}$ and functions $u \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H)$ and $w \in W^{1,2}(0,T;H) \cap L^\infty(Q)$ with $V_0(w) \in L^\infty(0,T)$ such that $\varepsilon_k \to 0$,

$$\left.\begin{array}{ll} u^{\varepsilon_k} \to u & \text{in } L^2(0,T;H), \\ & \text{in } C([0,T];X'), \\ & \text{weakly in } W^{1,2}(0,T;X'), \\ & \text{weakly in } L^2(0,T;X), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;H), \end{array}\right\} \tag{4.4.20}$$

$$\left.\begin{array}{ll} w^{\varepsilon_k} \to w & \text{in } C([0,T];H), \\ & \text{weakly in } W^{1,2}(0,T;H), \\ & \text{weakly-}* \text{ in } L^\infty(Q), \end{array}\right\} \tag{4.4.21}$$

and

$$w^{\varepsilon_k}(t) \to w(t) \ \text{ weakly-}* \text{ in } BV(0,L), \quad \text{for any } t \in [0,T]$$

as $k \to \infty$.

We now show that the pair of functions $[u,w]$ is the solution to (P;$u_0, w_0, f, h, \ell)^0$ on $[0,T]$. To this end, we recall Corollary 4.2. Let z be any element in $D(\widehat{V}^0)$. Then, by the Mosco convergence of $\hat{V}^\varepsilon(\cdot)$, we can find a sequence $\{z_k\}_{k\in\mathbb{N}} \subset L^2(0,T;H)$ such that

$$z_k \to z \ \text{ in } L^2(0,T;H) \ \text{ and } \ \widehat{V}^{\varepsilon_k}(z_k) \to \widehat{V}^0(z) \tag{4.4.22}$$

as $k \to \infty$.

Since $[u^{\varepsilon_k}, w^{\varepsilon_k}]$ is the unique solution to (P;$u_0^{\varepsilon_k}, w_0^{\varepsilon_k}, f^{\varepsilon_k}, h^{\varepsilon_k}, \ell^{\varepsilon_k})^{\varepsilon_k}$ on $[0,T]$, we easily observe that:

$$\begin{aligned} &\int_0^T \langle (u^{\varepsilon_k})'(t), \varpi(t)\rangle dt + \int_0^T ((w^{\varepsilon_k})'(t), \varpi(t))_H dt + \int_0^T \langle Fu^{\varepsilon_k}(t), \varpi(t)\rangle dt \\ = &\int_0^T (a_0 f^{\varepsilon_k}(t), \varpi(t))_H dt + \int_0^T (a_1 h^{\varepsilon_k}(t) + n_0 b_1)\varpi(t,0) dt \\ &\quad + \int_0^T (a_2 \ell^{\varepsilon_k}(t) + n_0 b_2)\varpi(t,L) dt \quad \text{ for any } \varpi \in L^2(0,T;X), \end{aligned} \tag{4.4.23}$$

$$\begin{aligned} &\int_0^T ((w^{\varepsilon_k})'(t) + g(w^{\varepsilon_k}(t)) - u^{\varepsilon_k}(t), w^{\varepsilon_k}(t) - z_k(t))_H dt \\ &\quad + \kappa \int_0^T V^{\varepsilon_k}(w^{\varepsilon_k}(t)) dt - \kappa \int_0^T V^{\varepsilon_k}(z_k(t)) dt \le 0, \end{aligned} \tag{4.4.24}$$

and

$$u^{\varepsilon_k}(0) = u_0^{\varepsilon_k} \text{ in } X' \text{ and } \ w^{\varepsilon_k}(0) = w_0^{\varepsilon_k} \text{ in } H. \tag{4.4.25}$$

Therefore, from (4.4.5)–(4.4.8), (4.4.20)–(4.4.25), the Mosco convergence of $\hat{V}^\varepsilon(\cdot)$, and the Lebesgue dominated convergence theorem, we observe that:

$$\begin{aligned} &\int_0^T \langle u'(t), \varpi(t)\rangle dt + \int_0^T (w'(t), \varpi(t))_H dt + \int_0^T \langle Fu(t), \varpi(t)\rangle dt \\ = &\int_0^T (a_0 f(t), \varpi(t))_H dt + \int_0^T (a_1 h(t) + n_0 b_1)\varpi(t,0) dt \\ &\quad + \int_0^T (a_2 \ell(t) + n_0 b_2)\varpi(t,L) dt \quad \text{ for any } \varpi \in L^2(0,T;X), \end{aligned} \tag{4.4.26}$$

$$\begin{aligned}&\int_0^T (w'(t)+g(w(t))-u(t), w(t)-z(t))_H dt\\&+\kappa\int_0^T V^0(w(t))dt-\kappa\int_0^T V^0(z(t))dt\le 0\\&\text{for any } z\in L^2(0,T;H) \text{ with } V^0(z)\in L^1(0,T),\end{aligned}\tag{4.4.27}$$

and

$$u(0)=u_0 \text{ in } X' \text{ and } \ w(0)=w_0 \text{ in } H. \tag{4.4.28}$$

Thus, we conclude from (4.4.20), (4.4.21), and (4.4.26)–(4.4.28) that $[u,w]$ is a unique solution to (P;$u_0, w_0, f, h, \ell)^0$ on $[0,T]$, whence (4.4.9) holds without extracting any subsequence from $\{\varepsilon\}_{\varepsilon\in(0,1]}$. Thus, the proof of Theorem 4.3 has been completed. □

By the slight modification of the proof of Theorem 4.3, we have the following convergence result of solutions to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ on $[0,T]$ for the fixed parameter $\varepsilon\in[0,1]$.

Corollary 4.4 (cf. [10], [30, Appendix])**.** Assume (A1), (A2), and (A3). Let $\varepsilon\in[0,1]$ be a fixed parameter, and let $[f,h,\ell]\in\mathcal{U}$, $u_0\in H$, and $w_0\in D(V^\varepsilon)$. Also, let $\{[f_n,h_n,\ell_n]\}_{n\in\mathbb{N}}\subset\mathcal{U}$, $\{u_{0n}\}_{n\in\mathbb{N}}\subset H$, and $\{w_{0n}\}_{n\in\mathbb{N}}\subset D(V^\varepsilon)$. Furthermore, suppose that

$$f_n\to f \text{ weakly in } L^2(0,T;H),$$

$$h_n\to h \text{ weakly in } L^2(0,T),$$

$$\ell_n\to \ell \text{ weakly in } L^2(0,T),$$

$$u_{0n}\to u_0 \text{ in } X', \quad w_{0n}\to w_0 \text{ in } H, \quad \text{and} \quad V^\varepsilon(w_{0n})\to V^\varepsilon(w_0)$$

as $n\to\infty$. Then, the sequence of solutions $[u_n,w_n]$ to (P;$u_{0n}, w_{0n}, f_n, h_n, \ell_n)^\varepsilon$ converges to the solution $[u,w]$ to (P;$u_0, w_0, f, h, \ell)^\varepsilon$ in the following sense:

$$[u_n,w_n]\longrightarrow[u,w] \quad \text{in } L^2(0,T;H)\times C([0,T];H) \text{ as } n\to\infty.$$

4.5 Optimal control to (OP)$^\varepsilon$

In this section, we consider a class of approximate optimal control problems (OP)$^\varepsilon$. Indeed, we prove the following main result, which is concerned with the existence of an optimal control to (OP)$^\varepsilon$ for each $\varepsilon\in[0,1]$ and the relationship between the limits (ω-limit points) of sequences of approximate optimal pairs and the optimal pairs of the limiting problem (OP)^0.

Theorem 4.5. Suppose (A1)–(A4). Then, the following two statements hold.

(I) Let $\varepsilon\in[0,1]$, $u_0^\varepsilon\in H$, and $w_0^\varepsilon\in D(V^\varepsilon)$. Then, the problem (OP)$^\varepsilon$ has at least one optimal control $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]\in\mathcal{U}$, so that:

$$J^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)=\inf_{[f,h,\ell]\in\mathcal{U}} J^\varepsilon(f,h,\ell).$$

(II) Assume $u_0 \in H$, $\{u_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset H$, $w_0 \in D(V^0)$, $\{w_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset D(V^\varepsilon)$,

$$u_0^\varepsilon \to u_0 \text{ in } X', \;\; w_0^\varepsilon \to w_0 \text{ in } H, \;\; \text{and} \;\; V^\varepsilon(w_0^\varepsilon) \longrightarrow V^0(w_0) \text{ as } \varepsilon \to 0. \tag{4.5.1}$$

Let $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] \in \mathcal{U}$ be the optimal control of $(\mathrm{OP})^\varepsilon$ obtained in (I). In addition, let $[u_*^\varepsilon, w_*^\varepsilon]$ be the unique solution to $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)^\varepsilon$ on $[0,T]$. Then, there exist a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$, the triplet of functions $[f_{**}, h_{**}, \ell_{**}] \in \mathcal{U}$, and the unique solution $[u_{**}, w_{**}]$ to $(\mathrm{P};u_0, w_0, f_{**}, h_{**}, \ell_{**})^0$ on $[0,T]$ such that $[f_{**}, h_{**}, \ell_{**}]$ is the optimal control of $(\mathrm{OP})^0$, $\varepsilon_k \to 0$,

$$f_*^{\varepsilon_k} \to f_{**} \quad \text{weakly in } L^2(0,T;H), \tag{4.5.2}$$

$$h_*^{\varepsilon_k} \to h_{**} \quad \text{weakly in } L^2(0,T), \tag{4.5.3}$$

$$\ell_*^{\varepsilon_k} \to \ell_{**} \quad \text{weakly in } L^2(0,T), \tag{4.5.4}$$

and

$$[u_*^{\varepsilon_k}, w_*^{\varepsilon_k}] \longrightarrow [u_{**}, w_{**}] \quad \text{in } L^2(0,T;H) \times C([0,T];H) \tag{4.5.5}$$

as $k \to \infty$.

Proof. By Corollary 4.4 and taking a minimizing sequence $\{[f_n, h_n, \ell_n]\}_{n\in\mathbb{N}} \subset \mathcal{U}$ so that

$$\lim_{n\to\infty} J^\varepsilon(f_n, h_n, \ell_n) = \inf_{[f,h,\ell]\in\mathcal{U}} J^\varepsilon(f,h,\ell),$$

we can prove (I). Such an argument is quite standard, thus, we omit the detailed proof of (I).

Next, let us prove (II), which is concerned with the relationship between the optimal control problems $(\mathrm{OP})^\varepsilon$ and $(\mathrm{OP})^0$. To this end, let us fix any sequence $\{[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]\}_{\varepsilon\in(0,1]} \subset \mathcal{U}$ of the optimal controls $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ to $(\mathrm{OP})^\varepsilon$ for $\varepsilon \in (0,1]$. Let $[f,h,\ell]$ be any function in $\mathcal{U}$. In addition, let $[u^\varepsilon, w^\varepsilon]$ be a unique solution to $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$ on $[0,T]$, and let $[u,w]$ be a unique solution to $(\mathrm{P};u_0, w_0, f, h, \ell)^0$ on $[0,T]$. Then, we observe from Theorem 4.3 with (4.5.1) that

$$[u^\varepsilon, w^\varepsilon] \longrightarrow [u,w] \;\; \text{in } L^2(0,T;H) \times C([0,T];H) \text{ as } \varepsilon \to 0. \tag{4.5.6}$$

Since $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ is the optimal control to $(\mathrm{OP})^\varepsilon$, we observe that

$$\begin{aligned} J^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon) &\leq J^\varepsilon(f,h,\ell) \\ &= \frac{c_0}{2}\int_0^T |(u^\varepsilon - u_d)(t)|_H^2\,dt + \frac{c_1}{2}\int_0^T |(w^\varepsilon - w_d)(t)|_H^2\,dt \\ &\quad + \frac{m_0}{2}\int_0^T a_0^2 |f(t)|_H^2\,dt \\ &\quad + \frac{m_1}{2}\int_0^T a_1^2 |h(t)|^2\,dt + \frac{m_2}{2}\int_0^T a_2^2|\ell(t)|^2\,dt. \end{aligned} \tag{4.5.7}$$

Clearly, it follows from (1.5.26), (4.5.6), and (4.5.7) that $\{[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]\}_{\varepsilon\in(0,1]}$ is bounded in $\mathcal{U}$ with respect to $\varepsilon \in (0,1]$. Therefore, there are a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$ and the triplet of functions $[f_{**}, h_{**}, \ell_{**}] \in \mathcal{U}$ such that $\varepsilon_k \to 0$,

$$f_*^{\varepsilon_k} \to f_{**} \quad \text{weakly in } L^2(0,T;H), \tag{4.5.8}$$

$$h_*^{\varepsilon_k} \to h_{**} \quad \text{weakly in } L^2(0,T), \tag{4.5.9}$$

$$\ell_*^{\varepsilon_k} \to \ell_{**} \quad \text{weakly in } L^2(0,T) \tag{4.5.10}$$

as $k \to \infty$.

Let $[u_*^{\varepsilon_k}, w_*^{\varepsilon_k}]$ be a unique solution to $(\mathrm{P}; u_0^{\varepsilon_k}, w_0^{\varepsilon_k}, f_*^{\varepsilon_k}, h_*^{\varepsilon_k}, \ell_*^{\varepsilon_k})^{\varepsilon_k}$ on $[0,T]$. Then, from Theorem 4.3 with (4.5.1) and (4.5.8)–(4.5.10), we infer that $[u_*^{\varepsilon_k}, w_*^{\varepsilon_k}]$ converges to the unique solution $[u_{**}, w_{**}]$ to $(\mathrm{P}; u_0, w_0, f_{**}, h_{**}, \ell_{**})^0$ on $[0,T]$ in the sense that

$$[u_*^{\varepsilon_k}, w_*^{\varepsilon_k}] \longrightarrow [u_{**}, w_{**}] \ \text{ in } L^2(0,T;H) \times C([0,T];H) \text{ as } k \to \infty, \tag{4.5.11}$$

hence, the convergence (4.5.5) holds.

Now, by using (4.5.6)–(4.5.11) and the weak lower semicontinuity of L^2-norm, we see that

$$J^0(f_{**}, h_{**}, \ell_{**}) \le \liminf_{k\to\infty} J^{\varepsilon_k}(f_*^{\varepsilon_k}, h_*^{\varepsilon_k}, \ell_*^{\varepsilon_k}) \le \lim_{k\to\infty} J^{\varepsilon_k}(f,h,\ell) = J^0(f,h,\ell).$$

Since $[f,h,\ell]$ is any function in $\mathcal{U}$, we infer from the above inequality that $[f_{**}, h_{**}, \ell_{**}]$ is the optimal control to $(\mathrm{OP})^0$. Hence, the assertion (II) of Theorem 4.5 holds. Thus, the proof of Theorem 4.5 has been completed. □

Remark 4.6. Unfortunately, Theorem 4.5 does not cover the uniqueness of optimal controls. Although Hoffmann–Jiang [33] reported the uniqueness of optimal controls for a regular Fix–Caginalp system, their technique is not applicable to our problem $(\mathrm{OP})^\varepsilon$ because of the nonlinear terms $a^\varepsilon(w_x)$ and $K^\varepsilon(w)$. Therefore, the uniqueness question of optimal controls to $(\mathrm{OP})^\varepsilon$ is still open.

Remark 4.7. Theorem 4.5(II) shows that the weak limit function of optimal control of $(\mathrm{OP})^\varepsilon$ is an optimal control for $(\mathrm{OP})^0$. Note that we can approximate any optimal control of $(\mathrm{OP})^0$ by considering the following approximate control problems:

($\star$) Let $\alpha > 0$ be a fixed constant. In addition, let $[f_*, h_*, \ell_*] \in \mathcal{U}$ be any optimal control of $(\mathrm{OP})^0$ obtained in Theorem 4.5(I). Then, for each $\varepsilon \in (0,1]$, we consider the following approximate optimal control problem:

Problem $(\mathrm{OP})_\alpha^\varepsilon$ **.** Find a triplet of control functions $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] \in \mathcal{U}$, called optimal control, such that

$$J_\alpha^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon) = \inf_{[f,h,\ell]\in\mathcal{U}} J_\alpha^\varepsilon(f,h,\ell).$$

Here, $J_\alpha^\varepsilon(f,h,\ell)$ is the cost functional defined by

$$\begin{aligned} J_\alpha^\varepsilon(f,h,\ell) \ &:= \frac{c_0}{2}\int_0^T |(u^\varepsilon - u_d)(t)|_H^2\,dt + \frac{c_1}{2}\int_0^T |(w^\varepsilon - w_d)(t)|_H^2\,dt \\ &\quad + \frac{m_0}{2}\int_0^T a_0^2 |f(t)|_H^2\,dt + \frac{m_1}{2}\int_0^T a_1^2 |h(t)|^2\,dt \\ &\quad + \frac{m_2}{2}\int_0^T a_2^2 |\ell(t)|^2\,dt + \frac{\alpha}{2}\int_0^T |(f-f_*)(t)|_H^2\,dt \\ &\quad + \frac{\alpha}{2}\int_0^T |(h-h_*)(t)|^2\,dt + \frac{\alpha}{2}\int_0^T |(\ell-\ell_*)(t)|^2\,dt, \end{aligned} \tag{4.5.12}$$

where $[f, h, \ell] \in \mathcal{U}$ is the control, and a couple of functions $[u^\varepsilon, w^\varepsilon]$ is a unique solution to the state problem (P;$u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$.

Then, by arguments similar to those in [81, Theorem 3.3(II)], we can prove that there is a subsequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \{\varepsilon\}_{\varepsilon\in(0,1]}$ such that $\varepsilon_k \to 0$,

$$f_*^{\varepsilon_k} \to f_* \text{ in } L^2(0,T;H), \quad h_*^{\varepsilon_k} \to h_* \text{ in } L^2(0,T), \quad \ell_*^{\varepsilon_k} \to \ell_* \text{ in } L^2(0,T),$$

and

$$[u_*^{\varepsilon_k}, w_*^{\varepsilon_k}] \longrightarrow [u_*, w_*] \text{ in } L^2(0,T;H) \times C([0,T];H)$$

as $k \to \infty$, where $[u_*^{\varepsilon_k}, w_*^{\varepsilon_k}]$ is a unique solution to (P;$u_0^{\varepsilon_k}, w_0^{\varepsilon_k}, f_*^{\varepsilon_k}, h_*^{\varepsilon_k}, \ell_*^{\varepsilon_k})^{\varepsilon_k}$ and $[u_*, w_*]$ is a unique solution to (P;$u_0, w_0, f_*, h_*, \ell_*)^0$ on $[0,T]$.

4.6 Optimality condition for (OP)$^\varepsilon$ with $\varepsilon > 0$

In this section we show the necessary condition of an optimal pair $[u_*^\varepsilon, w_*^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ to (OP)$^\varepsilon$ with $\varepsilon > 0$, where $[u_*^\varepsilon, w_*^\varepsilon]$ is the unique solution to (P;$u_0^\varepsilon, w_0^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)^\varepsilon$, and $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] \in \mathcal{U}$ is the optimal control to (OP)$^\varepsilon$ obtained in Theorem 4.5(I).

Theorem 4.6. Suppose the same conditions as in Theorem 4.5. Additionally, assume

(A5) $\{\widehat{a}^\varepsilon\}_{\varepsilon\in(0,1]} \subset C^2(\mathbb{R})$ is a sequence of convex functions and C^2-regularizations for $\widehat{a}^0(\cdot) := |\cdot|$. Moreover, there exists a positive constant δ_3, independent of $\varepsilon \in (0,1]$, such that

$$0 \le (\widehat{a}^\varepsilon)''(r) \le \frac{\delta_3}{\varepsilon} \quad \text{for any } r \in \mathbb{R}.$$

(A6) g is a C^1 function on $\mathbb{R}$.

For the fixed number $\varepsilon \in (0,1]$, let $u_0^\varepsilon \in H$, $w_0^\varepsilon \in D(V^\varepsilon)$, and let $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] \in \mathcal{U}$ be the optimal control to (OP)$^\varepsilon$ obtained in Theorem 4.5(I). In addition, let $[u_*^\varepsilon, w_*^\varepsilon]$ be the unique solution to (P;$u_0^\varepsilon, w_0^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)^\varepsilon$ on $[0,T]$. Then, there exists a unique solution $[p^\varepsilon, q^\varepsilon]$ to the adjoint equation on $[0,T]$ as follows:

$$p^\varepsilon \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X), \tag{4.6.1}$$

$$q^\varepsilon \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H), \tag{4.6.2}$$

$$-(p^\varepsilon)' - p_{xx}^\varepsilon - q^\varepsilon = c_0(u_*^\varepsilon - u_d) \quad \text{in } Q, \tag{4.6.3}$$

$$\begin{aligned} &\int_0^T (-(p^\varepsilon)'(\tau), \zeta(\tau))_H d\tau + \int_0^T \langle -(q^\varepsilon)'(\tau), \zeta(\tau)\rangle d\tau \\ &\quad + \kappa \int_0^T (((a^\varepsilon)'((w_*^\varepsilon)_x(\tau)) + \varepsilon) q_x^\varepsilon(\tau), \zeta_x(\tau))_H \, d\tau \\ &\quad + \int_0^T ((K^\varepsilon)'(w_*^\varepsilon(\tau)) q^\varepsilon(\tau), \zeta(\tau))_H \, d\tau + \int_0^T (g'(w_*^\varepsilon(\tau)) q^\varepsilon(\tau), \zeta(\tau))_H \, d\tau \\ = \;& c_1 \int_0^T ((w_*^\varepsilon - w_d)(\tau), \zeta(\tau))_H \, d\tau \quad \text{for all } \zeta \in L^2(0,T;X), \end{aligned} \tag{4.6.4}$$

$$-p_x^\varepsilon(t,0) + n_0 p^\varepsilon(t,0) = p_x^\varepsilon(t,L) + n_0 p^\varepsilon(t,L) = 0, \quad t \in (0,T), \tag{4.6.5}$$

$$p^\varepsilon(T,x) = q^\varepsilon(T,x) = 0, \quad x \in (0,L), \tag{4.6.6}$$

where $(a^\varepsilon)'(\cdot)$ and $g'(\cdot)$ are the derivatives of $a^\varepsilon(\cdot)$ and $g(\cdot)$, respectively. Moreover, p^ε satisfies the following equations:

$$a_0(p^\varepsilon + m_0 a_0 f_*^\varepsilon) = 0 \ \text{ in } L^2(0,T;H), \tag{4.6.7}$$

$$a_1(p^\varepsilon(\cdot,0) + m_1 a_1 h_*^\varepsilon) = 0 \ \text{ in } L^2(0,T), \tag{4.6.8}$$

$$a_2(p^\varepsilon(\cdot,L) + m_2 a_2 \ell_*^\varepsilon) = 0 \ \text{ in } L^2(0,T). \tag{4.6.9}$$

We prove Theorem 4.6 by showing the result of Gâteaux differentiability of the cost functional $J^\varepsilon(\cdot,\cdot,\cdot)$. To this end, we fix $\varepsilon \in (0,1]$ and the initial data $[u_0^\varepsilon, w_0^\varepsilon] \in H \times D(V^\varepsilon)$. Then, we define the solution operator Λ^ε to (P;$u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$ as follows.

Definition 4.4. (I) We denote by $\Lambda^\varepsilon : \mathcal{U} \to L^2(0,T;H) \times L^2(0,T;H)$ a solution operator to (P;$u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$ that assigns to any control $[f,h,\ell] \in \mathcal{U}$ the unique solution $[u^\varepsilon, w^\varepsilon] := \Lambda^\varepsilon(f,h,\ell)$ to the state system (P;$u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$.
(II) Let $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] \in \mathcal{U}$ be the optimal control to (OP)$^\varepsilon$. Then, $[u_*^\varepsilon, w_*^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] = [\Lambda^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon), f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ is called the optimal pair to the optimal control problem (OP)$^\varepsilon$.

For a moment, we often omit the superscript $\varepsilon \in (0,1]$.

At first, we show the Gâteaux differentiability of Λ^ε and J^ε. For any $\lambda \in [-1,1] \setminus \{0\}$, any $[f,h,\ell] \in \mathcal{U}$, and any $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$, we put $[u_\lambda, w_\lambda] := \Lambda^\varepsilon(f + \lambda\check{f}, h + \lambda\check{h}, \ell + \lambda\check{\ell})$, $[u,w] := \Lambda^\varepsilon(f,h,\ell)$, $\theta_\lambda := \frac{u_\lambda - u}{\lambda}$, and $\chi_\lambda := \frac{w_\lambda - w}{\lambda}$.

Note that the pair of functions $[\theta_\lambda, \chi_\lambda]$ satisfies the following system:

$$\begin{aligned} &\langle \theta_\lambda'(t), z\rangle + \langle \chi_\lambda'(t), z\rangle + ((\theta_\lambda)_x(t), z_x)_H + n_0\theta_\lambda(t,0)z(0) + n_0\theta_\lambda(t,L)z(L) \\ &\qquad = (a_0\check{f}(t), z)_H + a_1\check{h}(t)z(0) + a_2\check{\ell}(t)z(L), \\ &\qquad\qquad \text{a.a. } t \in (0,T), \ \text{ for all } z \in X; \end{aligned} \tag{4.6.10}$$

$$\begin{aligned} &\langle \chi_\lambda'(t), z\rangle + \kappa(\overline{a}_\lambda^\varepsilon(t)(\chi_\lambda)_x(t), z_x)_H + \left(\overline{K}_\lambda^\varepsilon(t)\chi_\lambda(t), z\right)_H + (\overline{g}_\lambda(t)\chi_\lambda(t), z)_H = (\theta_\lambda(t), z)_H, \\ &\qquad \text{a.a. } t \in (0,T), \ \text{ for all } z \in X; \end{aligned} \tag{4.6.11}$$

$$\theta_\lambda(0,x) = \chi_\lambda(0,x) = 0, \quad \text{ a.a. } x \in (0,L), \tag{4.6.12}$$

where notations $\overline{a}_\lambda^\varepsilon$, $\overline{K}_\lambda^\varepsilon$, and $\overline{g}_\lambda$ are functions on Q, given as:

$$\overline{a}_\lambda^\varepsilon(t,x) = \int_0^1 (a^\varepsilon)'(w_x(t,x) + s((w_\lambda)_x(t,x) - w_x(t,x)))ds + \varepsilon;$$

$$\overline{K}_\lambda^\varepsilon(t,x) = \int_0^1 (K^\varepsilon)'(w(t,x) + s(w_\lambda(t,x) - w(t,x)))ds;$$

$$\overline{g}_\lambda(t,x) = \int_0^1 g'\left(w(t,x) + s(w_\lambda(t,x) - w(t,x))\right)ds;$$

for $(t,x) \in Q$, with use of the derivatives $(a^\varepsilon)'$, $(K^\varepsilon)'$, and g' of the single-valued functions.

Now, we give the uniform estimate of solutions $[\theta_\lambda, \chi_\lambda]$ to (4.6.10)–(4.6.12) with respect to $\lambda \in [-1,1] \setminus \{0\}$.

Lemma 4.2. Suppose all the same conditions in Theorem 4.6. Then, there is a positive number $N_5 > 0$, dependent on $\varepsilon, T, \kappa, n_0$ and independent of λ, such that

$$\begin{aligned}
&\sup_{0\le t\le T} |\theta_\lambda(t)|_H^2 + \int_0^T |\theta'_\lambda(t)|_{X'}^2 dt + \int_0^T |\theta_\lambda(t)|_X^2 dt \\
&\quad + \sup_{0\le t\le T} |\chi_\lambda(t)|_H^2 + \int_0^T |\chi'_\lambda(t)|_{X'}^2 dt + \int_0^T |\chi_\lambda(t)|_X^2 dt \\
&\le N_5 \left(a_0^2 |\check{f}|_{L^2(0,T;H)}^2 + a_1^2 |\check{h}|_{L^2(0,T)}^2 + a_2^2 |\check{\ell}|_{L^2(0,T)}^2 \right)
\end{aligned} \tag{4.6.13}$$

for any $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$.

Proof. Clearly, we observe from (A1) and (A5) that $(\widehat{a^\varepsilon})'(\cdot) = a^\varepsilon(\cdot) \in C^1(\mathbb{R})$ and

$$0 \le (a^\varepsilon)'(r) \le \frac{\delta_3}{\varepsilon} \quad \text{for any } r \in \mathbb{R}. \tag{4.6.14}$$

In addition, from the definitions of K^ε in (1.5.19) we infer that

$$0 \le \overline{K}_\lambda^\varepsilon(t,x) \le \frac{1}{\varepsilon}, \quad \text{a.a. } (t,x) \in Q. \tag{4.6.15}$$

Here, from the boundedness (4.3.5) of solutions to (P;f, h, ℓ)$^\varepsilon$, we note that

$$\begin{aligned}
&\sup_{0\le t\le T} |w(t)|_H^2 + \sup_{0\le t\le T} |w_\lambda(t)|_H^2 + \kappa \sup_{0\le t\le T} V^\varepsilon(w(t)) + \kappa \sup_{0\le t\le T} V^\varepsilon(w_\lambda(t)) \\
\le\ & N_6 \Bigg(|u_0^\varepsilon|_H^2 + |w_0^\varepsilon|_H^2 + \kappa V^\varepsilon(w_0^\varepsilon) + \int_0^L \widehat{g}(w_0^\varepsilon(x))dx \\
&\quad + a_0^2 |f|_{L^2(0,T;H)}^2 + a_1^2 |h|_{L^2(0,T)}^2 + a_2^2 |\ell|_{L^2(0,T)}^2 \\
&\quad + a_0^2 |\check{f}|_{L^2(0,T;H)}^2 + a_1^2 |\check{h}|_{L^2(0,T)}^2 + a_2^2 |\check{\ell}|_{L^2(0,T)}^2 + b_1^2 + b_2^2 \Bigg),
\end{aligned} \tag{4.6.16}$$

where $N_6 > 0$ is a positive constant independent of $\lambda \in [-1,1] \setminus \{0\}$. Since the embedding $BV(0,L) \hookrightarrow L^\infty(0,L)$ is continuous (cf. Proposition 4.3), we infer from (4.6.16) that

$$\begin{aligned}
&\sup_{0\le t\le T} |w(t)|_{L^\infty(0,L)}^2 + \sup_{0\le t\le T} |w_\lambda(t)|_{L^\infty(0,L)}^2 \\
\le\ & N'_6 \Bigg(|u_0^\varepsilon|_H^2 + |w_0^\varepsilon|_H^2 + \kappa V^\varepsilon(w_0^\varepsilon) + \int_0^L \widehat{g}(w_0^\varepsilon(x))dx \\
&\quad + a_0^2 |f|_{L^2(0,T;H)}^2 + a_1^2 |h|_{L^2(0,T)}^2 + a_2^2 |\ell|_{L^2(0,T)}^2 \\
&\quad + a_0^2 |\check{f}|_{L^2(0,T;H)}^2 + a_1^2 |\check{h}|_{L^2(0,T)}^2 + a_2^2 |\check{\ell}|_{L^2(0,T)}^2 + b_1^2 + b_2^2 \Bigg),
\end{aligned} \tag{4.6.17}$$

for some positive constant N'_6 independent of $\lambda \in [-1,1] \setminus \{0\}$ (cf. (4.4.18)). Thus, by (4.6.17), we find a positive constant N_7, independent of $\lambda \in [-1,1] \setminus \{0\}$, such that

$$\sup_{0\le t\le T} |\overline{g}_\lambda(t)|_{L^\infty(0,L)} \le N_7, \quad \text{for all } \lambda \in [-1,1] \setminus \{0\}. \tag{4.6.18}$$

Now, we show a priori estimate (4.6.13). Taking account of (4.6.14)–(4.6.18), we can get the following estimate:

$$\begin{aligned}
&\sup_{0\le t\le T} |\theta_\lambda(t)|_H^2 + \int_0^T |\theta_\lambda(t)|_X^2 dt + \sup_{0\le t\le T} |\chi_\lambda(t)|_H^2 + \int_0^T |\chi_\lambda(t)|_X^2 dt \\
\le\ & N_8 \left(a_0^2 |\check{f}|_{L^2(0,T;H)}^2 + a_1^2 |\check{h}|_{L^2(0,T)}^2 + a_2^2 |\check{\ell}|_{L^2(0,T)}^2 \right),
\end{aligned} \tag{4.6.19}$$

where $N_8 > 0$ is some positive constant, dependent on $\varepsilon, T, \kappa, n_0$ and independent of $\lambda \in [-1,1] \setminus \{0\}$. In fact, taking the sum of (4.6.10) with $z = \theta_\lambda$, (4.6.11) with $z = \theta_\lambda$, and (4.6.11) with $z = \frac{\kappa}{\varepsilon}(\frac{\delta_3}{\varepsilon} + \varepsilon)^2 \chi_\lambda$, and applying the Gronwall-type inequality (e.g., [39, Proposition 0.4.1]), we get (4.6.19). Such calculations are standard one, so we omit the detailed arguments (cf. (4.8.34) in Lemma 4.3).

By using (4.6.14), (4.6.15), and (4.6.18), we infer from (4.6.11) that

$$\left| \int_0^T \langle \chi_\lambda'(t), \zeta(t) \rangle dt \right| \leq N_9 \left(|\chi_\lambda|_{L^2(0,T;X)} + |\theta_\lambda|_{L^2(0,T;H)} \right) |\zeta|_{L^2(0,T;X)} \quad \text{for any } \zeta \in L^2(0,T;X), \tag{4.6.20}$$

where $N_9 > 0$ is some positive constant, dependent on ε, κ and independent of $\lambda \in [-1,1] \setminus \{0\}$. Hence, we infer from (4.6.19) and (4.6.20) that

$$|\chi_\lambda'|_{L^2(0,T;X')} \leq N_9' \left(|a_0| |\check{f}|_{L^2(0,T;H)} + |a_1| |\check{h}|_{L^2(0,T)} + |a_2| |\check{\ell}|_{L^2(0,T)} \right) \tag{4.6.21}$$

for some positive constant $N_9' > 0$, dependent on $\varepsilon, T, \kappa, n_0$ and independent of $\lambda \in [-1,1] \setminus \{0\}$.

Similarly, we infer from (4.6.10), (4.6.19), and (4.6.21) that

$$|\theta_\lambda'|_{L^2(0,T;X')} \leq N_{10} \left(|a_0| |\check{f}|_{L^2(0,T;H)} + |a_1| |\check{h}|_{L^2(0,T)} + |a_2| |\check{\ell}|_{L^2(0,T)} \right) \tag{4.6.22}$$

for some positive constant $N_{10} > 0$, dependent on $\varepsilon, T, \kappa, n_0$ and independent of $\lambda \in [-1,1] \setminus \{0\}$.

By (4.6.19), (4.6.21), and (4.6.22), we get the boundedness (4.6.13). Thus, the proof of Lemma 4.2 has been completed. □

Now, let us mention the result of the Gâteaux differentiability of Λ^ε and J^ε.

Proposition 4.5. Assume the same conditions in Theorem 4.6. Then, the following two statements hold.

(I) The solution operator Λ^ε admits the Gâteaux derivative at any $[f,h,\ell] \in \mathcal{U}$. More precisely, for arbitrary $[f,h,\ell] \in \mathcal{U}$, there exists a pair of functions $[\theta, \chi] \in L^2(0,T;H) \times L^2(0,T;H)$ such that:

$$D_{[\check{f},\check{h},\check{\ell}]} \Lambda^\varepsilon(f,h,\ell) := \lim_{\lambda \to 0} \frac{\Lambda^\varepsilon(f + \lambda \check{f}, h + \lambda \check{h}, \ell + \lambda \check{\ell}) - \Lambda^\varepsilon(f,h,\ell)}{\lambda} = [\theta, \chi] \quad \text{for all direction } [\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}, \tag{4.6.23}$$

$$\theta \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H), \tag{4.6.24}$$

$$\chi \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H), \tag{4.6.25}$$

and $[\theta, \chi]$ solves the following linear system:

$$\begin{aligned} &\langle \theta'(t), z \rangle + \langle \chi'(t), z \rangle + (\theta_x(t), z_x)_H + n_0 (\theta(t,0) z(0) + \theta(t,L) z(L)) \\ &\qquad = (a_0 \check{f}(t), z)_H + a_1 \check{h}(t) z(0) + a_2 \check{\ell}(t) z(L), \\ &\qquad \text{a.a. } t \in (0,T), \text{ for all } z \in X; \end{aligned} \tag{4.6.26}$$

$$\langle \chi'(t), z\rangle + \kappa\left(((a^\varepsilon)'(w_x(t)) + \varepsilon)\chi_x(t), z_x\right)_H + ((K^\varepsilon)'(w(t))\chi(t), z)_H$$
$$+(g'(w(t))\chi(t), z)_H = (\theta(t), z)_H, \qquad (4.6.27)$$
$$\text{a.a. } t \in (0,T), \text{ for all } z \in X;$$
$$\theta(0,x) = \chi(0,x) = 0, \qquad \text{a.a. } x \in (0,L). \qquad (4.6.28)$$

(II) The cost function J^ε admits the Gâteaux derivative at any $[f,h,\ell] \in \mathcal{U}$. More precisely,

$$\begin{aligned} D_{[\check{f},\check{h},\check{\ell}]}J^\varepsilon(f,h,\ell) &:= \lim_{\lambda\to 0}\frac{J^\varepsilon(f+\lambda\check{f}, h+\lambda\check{h}, \ell+\lambda\check{\ell}) - J^\varepsilon(f,h,\ell)}{\lambda} \\ &= c_0\int_0^T ((u-u_d)(t), \theta(t))_H dt + c_1\int_0^T ((w-w_d)(t),\chi(t))_H dt \\ &\quad + m_0 a_0^2\int_0^T (f(t),\check{f}(t))_H dt \\ &\quad + m_1 a_1^2 \int_0^T h(t)\check{h}(t)dt + m_2 a_2^2\int_0^T \ell(t)\check{\ell}(t)dt \end{aligned} \qquad (4.6.29)$$

for any $[f,h,\ell] \in \mathcal{U}$ and any direction $[\check{f},\check{h},\check{\ell}] \in \mathcal{U}$, where $[u,w] = \Lambda^\varepsilon(f,h,\ell)$ is the solution to (P;$u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$, u_d and w_d are the given target profiles in $L^2(0,T;H)$, and $[\theta,\chi](= D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f,h,\ell))$ is the pair of functions obtained in the assertion (I).

Proof. At first, we show (I). To this end, we put $[u_\lambda, w_\lambda] := \Lambda^\varepsilon(f+\lambda\check{f}, h+\lambda\check{h}, \ell+\lambda\check{\ell})$, $[u,w] := \Lambda^\varepsilon(f,h,\ell)$, $\theta_\lambda := \frac{u_\lambda - u}{\lambda}$, and $\chi_\lambda := \frac{w_\lambda - w}{\lambda}$ for all $[f,h,\ell]\in\mathcal{U}$, $[\check{f},\check{h},\check{\ell}]\in\mathcal{U}$, and $\lambda \in [-1,1]\setminus\{0\}$. Then, by the uniform estimate (4.6.13) of $[\theta_\lambda, \chi_\lambda]$, there is a subsequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \{\lambda\}_{\lambda\in[-1,1]\setminus\{0\}}$ and the functions $\theta, \chi \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \subset C([0,T];H)$ such that $\lambda_n \to 0$,

$$\left.\begin{array}{ll}\theta_{\lambda_n} \to \theta & \text{in } C([0,T];X'), \\ & \text{in } L^2(0,T;H), \\ & \text{weakly in } W^{1,2}(0,T;X'), \\ & \text{weakly in } L^2(0,T;X), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;H),\end{array}\right\} \qquad (4.6.30)$$

$$\left.\begin{array}{ll}\chi_{\lambda_n} \to \chi & \text{in } C([0,T];X'), \\ & \text{in } L^2(0,T;H), \\ & \text{weakly in } W^{1,2}(0,T;X'), \\ & \text{weakly in } L^2(0,T;X), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;H),\end{array}\right\} \qquad (4.6.31)$$

as $n \to \infty$, and

$$\begin{aligned} &\sup_{0\le t\le T}|\theta(t)|_H^2 + \int_0^T |\theta'(t)|_{X'}^2 dt + \int_0^T |\theta(t)|_X^2 dt \\ &\quad + \sup_{0\le t\le T}|\chi(t)|_H^2 + \int_0^T|\chi'(t)|_{X'}^2 dt + \int_0^T |\chi(t)|_X^2 dt \\ &\le N_5\left(a_0^2|\check{f}|_{L^2(0,T;H)}^2 + a_1^2|\check{h}|_{L^2(0,T)}^2 + a_2^2|\check{\ell}|_{L^2(0,T)}^2\right), \end{aligned} \qquad (4.6.32)$$

where N_5 is the same constant as in Lemma 4.2.

Now, let us show a pair of the limit functions $[\theta, \chi]$ of $[\theta_{\lambda_n}, \chi_{\lambda_n}]$ satisfies (4.6.26)–(4.6.28). To this end, note from (4.6.13) that

$$\begin{aligned} |w_\lambda - w|_{L^2(0,T;X)} &= \lambda |\chi_\lambda|_{L^2(0,T;X)} \\ &\le \lambda N_5^{\frac{1}{2}} \left(|a_0||\check{f}|_{L^2(0,T;H)} + |a_1||\check{h}|_{L^2(0,T)} + |a_2||\check{\ell}|_{L^2(0,T)} \right) \\ &\to 0 \ \text{ as } \lambda \to 0. \end{aligned} \tag{4.6.33}$$

So, taking a subsequence if necessary, we see from the definition of functions $\overline{a}^\varepsilon_\lambda$, $\overline{K}^\varepsilon_\lambda$, $\overline{g}_\lambda$ ($\lambda \in [-1,1] \setminus \{0\}$) and continuity of functions $(a^\varepsilon)'$, $(K^\varepsilon)'$, and $g'(\cdot)$ that

$$\begin{cases} \overline{a}^\varepsilon_{\lambda_n}(t,x) \to (a^\varepsilon)'(w_x(t,x)) + \varepsilon, \\ \overline{K}^\varepsilon_{\lambda_n}(t,x) \to (K^\varepsilon)'(w(t,x)), \qquad \text{a.a. } (t,x) \in Q, \ \text{in the pointwise sense, as } n \to \infty. \\ \overline{g}_{\lambda_n}(t,x) \to g'(w(t,x)), \end{cases}$$

Here, let us fix arbitrary $0 \le t_0 < t_1 \le T$. Since functions $\overline{a}^\varepsilon_\lambda$, $\overline{K}^\varepsilon_\lambda$ and $\overline{g}_\lambda$ ($\lambda \in [-1,1] \setminus \{0\}$) are respectively bounded in senses of (4.6.14), (4.6.15), and (4.6.18), we can apply the Lebesgue dominated convergence theorem to show that

$$\begin{cases} \overline{a}^\varepsilon_{\lambda_n} \to (a^\varepsilon)'(w_x) + \varepsilon, \\ \overline{K}^\varepsilon_{\lambda_n} \to (K^\varepsilon)'(w), \qquad \text{in } L^2(t_0,t_1;H), \ \text{as } n \to \infty. \\ \overline{g}_{\lambda_n} \to g'(w), \end{cases} \tag{4.6.34}$$

Combining (4.6.30), (4.6.31), (4.6.32), and (4.6.34), it is deduced that:

$$\theta_{\lambda_n} \to \theta \ \text{ weakly in } L^2(t_0,t_1;X), \tag{4.6.35}$$

$$\theta'_{\lambda_n} \to \theta' \ \text{ weakly in } L^2(t_0,t_1;X'), \tag{4.6.36}$$

$$\chi_{\lambda_n} \to \chi \ \text{ weakly in } L^2(t_0,t_1;X), \tag{4.6.37}$$

$$\chi'_{\lambda_n} \to \chi' \ \text{ weakly in } L^2(t_0,t_1;X'), \tag{4.6.38}$$

and

$$\begin{cases} \overline{a}^\varepsilon_{\lambda_n}(\chi_{\lambda_n})_x \to ((a^\varepsilon)'(w_x) + \varepsilon)\chi_x, \\ \overline{K}^\varepsilon_{\lambda_n}\chi_{\lambda_n} \to (K^\varepsilon)'(w)\chi, \qquad \text{weakly in } L^2(t_0,t_1;H) \\ \overline{g}_{\lambda_n}\chi_{\lambda_n} \to g'(w)\chi, \end{cases} \tag{4.6.39}$$

as $n \to \infty$.

Here, note from (4.6.10) and (4.6.11) that

$$\begin{aligned} &\int_{t_0}^{t_1} \langle \theta'_{\lambda_n}(t), z \rangle dt + \int_{t_0}^{t_1} \langle \chi'_{\lambda_n}(t), z \rangle dt + \int_{t_0}^{t_1} ((\theta_{\lambda_n})_x(t), z_x)_H \, dt \\ &\quad + n_0 \int_{t_0}^{t_1} \theta_{\lambda_n}(t,0) z(0) dt + n_0 \int_{t_0}^{t_1} \theta_{\lambda_n}(t,L) z(L) dt \\ &= \int_{t_0}^{t_1} (a_0 \check{f}(t), z)_H dt + \int_{t_0}^{t_1} a_1 \check{h}(t) z(0) dt + \int_{t_0}^{t_1} a_2 \check{\ell}(t) z(L) dt \\ &\qquad \text{for all } z \in X \text{ and } n = 1,2,3,\cdots \end{aligned} \tag{4.6.40}$$

and

$$\int_{t_0}^{t_1} \langle \chi'_{\lambda_n}(t), z\rangle dt + \kappa \int_{t_0}^{t_1} (\overline{a}^{\varepsilon}_{\lambda_n}(t)(\chi_{\lambda_n})_x(t), z_x)_H dt + \int_{t_0}^{t_1} (\overline{K}^{\varepsilon}_{\lambda_n}(t)\chi_{\lambda_n}(t), z)_H dt$$
$$+ \int_{t_0}^{t_1} (\overline{g}_{\lambda_n}(t)\chi_{\lambda_n}(t), z)_H dt = \int_{t_0}^{t_1} (\theta_{\lambda_n}(t), z)_H dt$$
$$\text{for all } z \in X \text{ and } n = 1, 2, 3, \cdots. \tag{4.6.41}$$

On account of (4.6.35)–(4.6.39), we obtain the variational form (4.6.26) (resp. (4.6.27)) by taking the limits in (4.6.40) (resp. (4.6.41)) as $n \to \infty$.

On the other hand, by (4.6.30) and (4.6.31),

$$\theta(0,\cdot) = \lim_{n\to\infty} \theta_{\lambda_n}(0,\cdot) = 0 \ (\in H) \ \text{ in } X',$$
$$\chi(0,\cdot) = \lim_{n\to\infty} \chi_{\lambda_n}(0,\cdot) = 0 \ (\in H) \ \text{ in } X',$$

which implies (4.6.28).

Furthermore, by the usual method with helps from the facts that $(a^\varepsilon)' \geq 0$ (on $\mathbb{R}$), $(K^\varepsilon)' \geq 0$ (on $\mathbb{R}$), and $g'(w) + C_g \geq 0$, a.e. in Q, we easily prove that the solutions to the Cauchy problem $\{(4.6.26)\text{–}(4.6.28)\}$ are uniquely determined within (4.6.24)–(4.6.25). Hence, the uniqueness of solution to $\{(4.6.26)\text{–}(4.6.28)\}$ guarantees that of cluster points of the sequence $[\theta_\lambda, \chi_\lambda]$ as $\lambda \to 0$. Namely:

- $(*)$ $[\theta_\lambda, \chi_\lambda]$ originally converges to the unique solution $[\theta, \chi]$ to $\{(4.6.26)\text{–}(4.6.28)\}$ in the senses as in (4.6.30)–(4.6.31), as $\lambda \to 0$, and hence the operator $\mathcal{X}_{[f,h,\ell]} : \mathcal{U} \to L^2(0,T;H) \times L^2(0,T;H)$, defined by $\mathcal{X}_{[f,h,\ell]}(\check{f}, \check{h}, \check{\ell}) := D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f,h,\ell)$ for all direction $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$, is well-defined.

Now, on account of the linearity inherent in (4.6.26)–(4.6.27) and the estimate (4.6.32), we observe that each operator $\mathcal{X}_{[f,h,\ell]}$ ($[f,h,\ell] \in \mathcal{U}$) is a bounded and linear operator from $\mathcal{U}$ into $L^2(0,T;H) \times L^2(0,T;H)$, and hence, the solution operator Λ^ε admits the Gâteaux derivative at any $[f,h,\ell] \in \mathcal{U}$. Thus, we conclude the assertion (I) of this proposition.

Next, we show (II). The Gâteaux differentiability of the cost function J^ε will be a direct consequence of the assertion (I). In fact, note from (4.6.13) that

$$\begin{aligned} |u_\lambda - u|_{L^2(0,T;X)} &= \lambda |\theta_\lambda|_{L^2(0,T;X)} \\ &\leq \lambda N_5^{\frac{1}{2}} \left(|a_0||\check{f}|_{L^2(0,T;H)} + |a_1||\check{h}|_{L^2(0,T)} + |a_2||\check{\ell}|_{L^2(0,T)} \right) \\ &\quad \to 0 \ \text{ as } \lambda \to 0. \end{aligned} \tag{4.6.42}$$

Then, by virtue of (4.6.33), (4.6.42), and $(*)$, it is computed that

$$\begin{aligned} &D_{[\check{f},\check{h},\check{\ell}]} J^\varepsilon(f,h,\ell) \\ := &\lim_{\lambda\to 0} \frac{J^\varepsilon(f + \lambda\check{f}, h + \lambda\check{h}, \ell + \lambda\check{\ell}) - J^\varepsilon(f,h,\ell)}{\lambda} \\ = &\lim_{\lambda\to 0} \left\{ \frac{c_0}{2} \int_0^T ((u_\lambda + u - 2u_d)(t), \theta_\lambda(t))_H dt + \frac{c_1}{2} \int_0^T ((w_\lambda + w - 2w_d)(t), \chi_\lambda(t))_H dt \right. \end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{m_0 a_0^2}{2} \int_0^T ((2f + \lambda \check{f})(t), \check{f}(t))_H dt \\
&\quad + \frac{m_1 a_1^2}{2} \int_0^T (2h + \lambda \check{h})(t) \check{h}(t) dt + \frac{m_2 a_2^2}{2} \int_0^T (2\ell + \lambda \check{\ell})(t) \check{\ell}(t) dt \Bigg\} \\
= \; & c_0 \int_0^T ((u - u_d)(t), \theta(t))_H dt + c_1 \int_0^T ((w - w_d)(t), \chi(t))_H dt \\
&+ m_0 a_0^2 \int_0^T (f(t), \check{f}(t))_H dt + m_1 a_1^2 \int_0^T h(t) \check{h}(t) dt + m_2 a_2^2 \int_0^T \ell(t) \check{\ell}(t) dt
\end{aligned}
$$

for any $[f, h, \ell] \in \mathcal{U}$ and any direction $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$.

Clearly, we infer from (4.6.32) and $(*)$ that for any $[f, h, \ell] \in \mathcal{U}$, the functional:

$$
[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U} \mapsto D_{[\check{f}, \check{h}, \check{\ell}]} J^\varepsilon(f, h, \ell)
$$

will form a bounded linear functional on $\mathcal{U}$. Hence, the cost functional J^ε admits the Gâteaux derivative at any $[f, h, \ell] \in \mathcal{U}$ with the directional derivative as in (4.6.29).

Thus, the proof of Proposition 4.5 has been completed. □

By taking account of Proposition 4.5, we can prove Theorem 4.6 concerning the necessary condition of an optimal pair $[u_*^\varepsilon, w_*^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] = [\Lambda^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon), f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ to $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$.

Proof of Theorem 4.6. By using the Schauder fixed point theorem and the general results by Ladyženskaja–Solonnikov–Ural'ceva [53, Chapter 3], we can get the unique solution $[p^\varepsilon, q^\varepsilon]$ to the adjoint equations (4.6.1)–(4.6.6).

Now, let $[u_*^\varepsilon, w_*^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon] = [\Lambda^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon), f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ be the optimal pair to the problem $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$. Let $[\theta_*^\varepsilon, \chi_*^\varepsilon]$ be the limit of $\frac{\Lambda^\varepsilon(f_*^\varepsilon + \lambda \check{f}, h_*^\varepsilon + \lambda \check{h}, \ell_*^\varepsilon + \lambda \check{\ell}) - \Lambda^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)}{\lambda}$ as $\lambda \to 0$ in the sense of (4.6.23).

Since $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ is a minimizer for $J^\varepsilon(\cdot, \cdot, \cdot)$, we have

$$
\begin{aligned}
0 \; \leq \; & \lim_{\lambda \to 0} \frac{J^\varepsilon(f_*^\varepsilon + \lambda \check{f}, h_*^\varepsilon + \lambda \check{h}, \ell_*^\varepsilon + \lambda \check{\ell}) - J^\varepsilon(f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)}{\lambda} \\
= \; & c_0 \int_0^T ((u_*^\varepsilon - u_d)(t), \theta_*^\varepsilon(t))_H dt + c_1 \int_0^T ((w_*^\varepsilon - w_d)(t), \chi_*^\varepsilon(t))_H dt \\
&+ m_0 a_0^2 \int_0^T (f_*^\varepsilon(t), \check{f}(t))_H dt + m_1 a_1^2 \int_0^T h_*^\varepsilon(t) \check{h}(t) dt + m_2 a_2^2 \int_0^T \ell_*^\varepsilon(t) \check{\ell}(t) dt \\
= & \int_0^T \langle -(p^\varepsilon)'(t), \theta_*^\varepsilon(t) \rangle dt + \int_0^T (p_x^\varepsilon(t), (\theta_*^\varepsilon)_x(t))_H dt + n_0 \int_0^T p^\varepsilon(t, 0) \theta_*^\varepsilon(t, 0) dt \\
&\quad + n_0 \int_0^T p^\varepsilon(t, L) \theta_*^\varepsilon(t, L) dt - \int_0^T (q^\varepsilon(t), \theta_*^\varepsilon(t))_H dt \\
&\quad + \int_0^T (-(p^\varepsilon)'(t), \chi_*^\varepsilon(t))_H dt + \int_0^T \langle -(q^\varepsilon)'(t), \chi_*^\varepsilon(t) \rangle dt \\
&+ \kappa \int_0^T (((a^\varepsilon)'((w_*^\varepsilon)_x(t)) + \varepsilon) q_x^\varepsilon(t), (\chi_*^\varepsilon)_x(t))_H \, dt + \int_0^T ((K^\varepsilon)'(w_*^\varepsilon(t)) q^\varepsilon(t), \chi_*^\varepsilon(t))_H \, dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T (g'(w_*^\varepsilon(t))q^\varepsilon(t), \chi_*^\varepsilon(t))_H\, dt \\
&+m_0a_0^2\int_0^T (f_*^\varepsilon(t), \check{f}(t))_H dt + m_1a_1^2\int_0^T h_*^\varepsilon(t)\check{h}(t)dt + m_2a_2^2\int_0^T \ell_*^\varepsilon(t)\check{\ell}(t)dt \\
&= \int_0^T \langle(\theta_*^\varepsilon)'(t), p^\varepsilon(t)\rangle dt + \int_0^T ((\theta_*^\varepsilon)_x(t), p_x^\varepsilon(t))_H dt + n_0\int_0^T \theta_*^\varepsilon(t,0)p^\varepsilon(t,0)dt \\
&\quad +n_0\int_0^T \theta_*^\varepsilon(t,L)p^\varepsilon(t,L)dt - \int_0^T (q^\varepsilon(t), \theta_*^\varepsilon(t))_H dt \\
&\quad +\int_0^T \langle(\chi_*^\varepsilon)'(t), p^\varepsilon(t)\rangle dt + \int_0^T \langle(\chi_*^\varepsilon)'(t), q^\varepsilon(t)\rangle dt \\
&+\kappa\int_0^T (((a^\varepsilon)'((w_*^\varepsilon)_x(t)) + \varepsilon)(\chi_*^\varepsilon)_x(t), q_x^\varepsilon(t))_H\, dt + \int_0^T ((K^\varepsilon)'(w_*^\varepsilon(t))\chi_*^\varepsilon(t), q^\varepsilon(t))_H\, dt \\
&\quad +\int_0^T (g'(w_*^\varepsilon(t))\chi_*^\varepsilon(t), q^\varepsilon(t))_H\, dt \\
&+m_0a_0^2\int_0^T (f_*^\varepsilon(t), \check{f}(t))_H dt + m_1a_1^2\int_0^T h_*^\varepsilon(t)\check{h}(t)dt + m_2a_2^2\int_0^T \ell_*^\varepsilon(t)\check{\ell}(t)dt \\
&= \int_0^T (a_0p^\varepsilon(t) + m_0a_0^2 f_*^\varepsilon(t), \check{f}(t))_H dt + \int_0^T (a_1p^\varepsilon(t,0) + m_1a_1^2h_*^\varepsilon(t))\check{h}(t)dt \\
&\quad +\int_0^T (a_2p^\varepsilon(t,L) + m_2a_2^2\ell_*^\varepsilon(t))\check{\ell}(t)dt
\end{aligned}
$$

for any $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$. Here, we use the equations (4.6.3)–(4.6.6) and (4.6.26)–(4.6.28) for $[p^\varepsilon, q^\varepsilon]$ and $[\theta_*^\varepsilon, \chi_*^\varepsilon]$, respectively. Since $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$ is arbitrary, we infer from the inequality as above that the equations in (4.6.7)–(4.6.9) hold. Thus, the proof of Theorem 4.6 has been completed. □

4.7 Optimality condition for $(\mathrm{OP})^0$

In previous Section 6, we proved Theorem 4.6, which is concerned with the optimality condition to the approximate problem $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$. But, in general, it is difficult to show the necessary condition of the optimal control to $(\mathrm{OP})^0$, i.e., $\varepsilon = 0$, since (1.5.12) is the singular diffusion equation with constraint $\partial I_{[-1,1]}(\cdot)$. Therefore, by using Theorem 4.6, more precisely, by the limiting observation of $(\mathrm{OP})^\varepsilon$ as $\varepsilon \to 0$, we derive the optimality condition to $(\mathrm{OP})^0$.

Now, we mention the main result in this paper, which is concerned with the necessary condition of the optimal control to $(\mathrm{OP})^0$

Theorem 4.7. Suppose that all the assumptions of Theorem 4.6 are fulfilled. Let $u_0 \in H$, $w_0 \in D(V^0)$, and let $[f_{**}, h_{**}, \ell_{**}]$ be the optimal control to $(\mathrm{OP})^0$ obtained in Theorem 4.5(II). Let $[u_{**}, w_{**}]$ be the unique solution to $(\mathrm{P};u_0, w_0, f_{**}, h_{**}, \ell_{**})^0$ on $[0, T]$. Additionally, let us set:

$$W := \{z \in H^1(Q)\ ;\ z(0,x) = 0,\ \text{a.a. } x \in (0, L)\}.$$

Then, there are the functions $p \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$, $q \in L^\infty(0,T;H)$, and an element $\mu \in W'$ satisfying the following:

$$- p' - p_{xx} - q = c_0(u_{**} - u_d) \ \text{ in } Q, \tag{4.7.1}$$

$$\int_0^T (-p'(\tau), z(\tau))_H d\tau + \int_0^T (q(\tau), z'(\tau))_H d\tau + \langle \mu, z\rangle_{W',W} + \int_0^T (g'(w_{**}(\tau))q(\tau), z(\tau))_H d\tau$$

$$= c_1 \int_0^T ((w_{**} - w_d)(\tau), z(\tau))_H d\tau \quad \text{ for all } z \in W. \tag{4.7.2}$$

$$- p_x(t,0) + n_0 p(t,0) = p_x(t,L) + n_0 p(t,L) = 0, \quad t \in (0,T), \tag{4.7.3}$$

$$p(T,x) = 0, \quad x \in (0,L). \tag{4.7.4}$$

Moreover, p satisfies the following equations:

$$a_0(p + m_0 a_0 f_{**}) = 0 \ \text{ in } L^2(0,T;H), \tag{4.7.5}$$

$$a_1(p(\cdot,0) + m_1 a_1 h_{**}) = 0 \ \text{ in } L^2(0,T), \tag{4.7.6}$$

$$a_2(p(\cdot,L) + m_2 a_2 \ell_{**}) = 0 \ \text{ in } L^2(0,T). \tag{4.7.7}$$

Proof. Let $u_0 \in H$ and $w_0 \in D(V^0)$. Then, note from (4.1.1) and Lemma 4.1 that we find sequences $\{u_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset H$ and $\{w_0^\varepsilon\}_{\varepsilon\in(0,1]} \subset D(V^\varepsilon)$ satisfying

$$u_0^\varepsilon \to u_0 \text{ in } X', \ w_0^\varepsilon \to w_0 \text{ in } H, \ \text{ and } \ V^\varepsilon(w_0^\varepsilon) \longrightarrow V^0(w_0) \text{ as } \varepsilon \to 0. \tag{4.7.8}$$

Now, let $[f_{**}, h_{**}, \ell_{**}]$ be the optimal control to $(\mathrm{OP})^0$ obtained in Theorem 4.5(II). Namely, there exists a subsequence of $\varepsilon \in (0,1]$ (which we also denote ε for simplicity) such that $[f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon]$ is the optimal control to $(\mathrm{OP})^\varepsilon$ and

$$f_*^\varepsilon \to f_{**} \quad \text{weakly in } L^2(0,T;H), \tag{4.7.9}$$

$$h_*^\varepsilon \to h_{**} \quad \text{weakly in } L^2(0,T), \tag{4.7.10}$$

$$\ell_*^\varepsilon \to \ell_{**} \quad \text{weakly in } L^2(0,T), \tag{4.7.11}$$

and

$$[u_*^\varepsilon, w_*^\varepsilon] \longrightarrow [u_{**}, w_{**}] \ \text{ in } L^2(0,T;H) \times C([0,T];H) \tag{4.7.12}$$

as $\varepsilon \to 0$, where $[u_*^\varepsilon, w_*^\varepsilon]$ is the unique solution to $(\mathrm{P}; u_0^\varepsilon, w_0^\varepsilon, f_*^\varepsilon, h_*^\varepsilon, \ell_*^\varepsilon)^\varepsilon$ on $[0,T]$, and $[u_{**}, w_{**}]$ is the unique solution to $(\mathrm{P}; u_0, w_0, f_{**}, h_{**}, \ell_{**})^0$ on $[0,T]$.

Now, by taking the limit with respect to ε, we prove Theorem 4.7. To this end, we give a priori estimate of the solution $[p^\varepsilon, q^\varepsilon]$ to the adjoint equations (4.6.3)–(4.6.6).

Now, we multiply (4.6.3) by p^ε. Then, by applying the Schwarz inequality, we have

$$\begin{aligned} & -\frac{1}{2}\frac{d}{d\tau}|p^\varepsilon(\tau)|_H^2 + |p_x^\varepsilon(\tau)|_H^2 + n_0|p^\varepsilon(\tau,0)|^2 + n_0|p^\varepsilon(\tau,L)|^2 \\ \le \ & |p^\varepsilon(\tau)|_H^2 + \frac{1}{2}|q^\varepsilon(\tau)|_H^2 + \frac{c_0^2}{2}|(u_*^\varepsilon - u_d)(\tau)|_H^2, \quad \text{a.a. } \tau \in (0,T). \end{aligned} \tag{4.7.13}$$

By integrating (4.7.13) in τ over $[T-t,T]$ ($t \in [0,T]$), we have

$$\begin{array}{rl} & \frac{1}{2}|p^\varepsilon(T-t)|_H^2 + \int_{T-t}^T |p_x^\varepsilon(\tau)|_H^2 d\tau \\ & \quad + n_0 \int_{T-t}^T |p^\varepsilon(\tau,0)|^2 d\tau + n_0 \int_{T-t}^T |p^\varepsilon(\tau,L)|^2 d\tau \\ \leq & \int_{T-t}^T |p^\varepsilon(\tau)|_H^2 d\tau + \frac{1}{2}\int_{T-t}^T |q^\varepsilon(\tau)|_H^2 d\tau + \frac{c_0^2}{2}\int_{T-t}^T |(u_*^\varepsilon - u_d)(\tau)|_H^2 d\tau \end{array} \tag{4.7.14}$$

for all $t \in [0,T]$.

Next, multiply (4.6.3) by $-(p^\varepsilon)'$. Then, by applying the Schwarz inequality, we get

$$\begin{array}{rl} & \frac{1}{2}|(p^\varepsilon)'(\tau)|_H^2 - \frac{1}{2}\frac{d}{d\tau}\left\{|p_x^\varepsilon(\tau)|_H^2 + n_0|p^\varepsilon(\tau,0)|^2 + n_0|p^\varepsilon(\tau,L)|^2\right\} \\ \leq & |q^\varepsilon(\tau)|_H^2 + c_0^2|(u_*^\varepsilon - u_d)(\tau)|_H^2, \quad \text{a.a. } \tau \in (0,T). \end{array} \tag{4.7.15}$$

By integrating (4.7.15) in τ over $[T-t,T]$ ($t \in [0,T]$), we have

$$\begin{array}{l} \frac{1}{2}\int_{T-t}^T |(p^\varepsilon)'(\tau)|_H^2 d\tau + \frac{1}{2}\left\{|p_x^\varepsilon(T-t)|_H^2 + n_0|p^\varepsilon(T-t,0)|^2 + n_0|p^\varepsilon(T-t,L)|^2\right\} \\ \leq \int_{T-t}^T |q^\varepsilon(\tau)|_H^2 d\tau + c_0^2\int_{T-t}^T |(u_*^\varepsilon - u_d)(\tau)|_H^2 d\tau, \quad \forall t \in [0,T]. \end{array} \tag{4.7.16}$$

Here, note that the pair of functions $[p^\varepsilon, q^\varepsilon]$ satisfies the following variational identity (cf. (4.6.4)):

$$\begin{array}{l} \int_{T-t}^T (-(p^\varepsilon)'(\tau), \zeta(\tau))_H d\tau + \int_{T-t}^T \langle -(q^\varepsilon)'(\tau), \zeta(\tau)\rangle d\tau \\ \quad + \kappa\int_{T-t}^T \left(((a^\varepsilon)'((w_*^\varepsilon)_x(\tau)) + \varepsilon)q_x^\varepsilon(\tau), \zeta_x(\tau)\right)_H d\tau + \int_{T-t}^T \left((K^\varepsilon)'(w_*^\varepsilon(\tau))q^\varepsilon(\tau), \zeta(\tau)\right)_H d\tau \\ \quad + \int_{T-t}^T \left(g'(w_*^\varepsilon(\tau))q^\varepsilon(\tau), \zeta(\tau)\right)_H d\tau \\ = c_1\int_{T-t}^T \left((w_*^\varepsilon - w_d)(\tau), \zeta(\tau)\right)_H d\tau \quad \text{for all } t \in [0,T] \text{ and all } \zeta \in L^2(T-t,T;X). \end{array} \tag{4.7.17}$$

Therefore, let us assign q^ε to the test function ζ as in (4.7.17). Then, by applying the Schwarz inequality, we see that

$$\begin{array}{rl} \frac{1}{2}|q^\varepsilon(T-t)|_H^2 & \leq (2+C_g)\int_{T-t}^T |q^\varepsilon(\tau)|_H^2 d\tau + \frac{1}{4}\int_{T-t}^T |(p^\varepsilon)'(\tau)|_H^2 d\tau \\ & \quad + \frac{c_1^2}{4}\int_{T-t}^T |(w_*^\varepsilon - w_d)(\tau)|_H^2 d\tau, \quad \forall t \in [0,T], \end{array} \tag{4.7.18}$$

since a^ε and K^ε are nondecreasing on $\mathbb{R}$ (cf. (4.6.14), (4.6.15)), and $g'(w_*^\varepsilon) + C_g \geq 0$, a.e. in Q. Adding (4.7.14), (4.7.16), and (4.7.18), we have

$$\frac{1}{2}\left\{|p^\varepsilon(T-t)|_H^2 + |q^\varepsilon(T-t)|_H^2 + |p_x^\varepsilon(T-t)|_H^2 + n_0|p^\varepsilon(T-t,0)|^2 + n_0|p^\varepsilon(T-t,L)|^2\right\}$$

$$+\frac{1}{4}\int_{T-t}^{T}|(p^\varepsilon)'(\tau)|_H^2d\tau+\int_{T-t}^{T}|p_x^\varepsilon(\tau)|_H^2d\tau+n_0\int_{T-t}^{T}|p^\varepsilon(\tau,0)|^2d\tau+n_0\int_{T-t}^{T}|p^\varepsilon(\tau,L)|^2d\tau$$

$$\le\int_{T-t}^{T}|p^\varepsilon(\tau)|_H^2d\tau+\left(\frac{7}{2}+C_g\right)\int_{T-t}^{T}|q^\varepsilon(\tau)|_H^2d\tau+\frac{3c_0^2}{2}\int_{T-t}^{T}|(u_*^\varepsilon-u_d)(\tau)|_H^2d\tau \qquad (4.7.19)$$

$$+\frac{c_1^2}{4}\int_{T-t}^{T}|(w_*^\varepsilon-w_d)(\tau)|_H^2d\tau,\quad \forall t\in[0,T].$$

Thus, by (4.7.12) and applying the Gronwall-type inequality (e.g., [39, Proposition 0.4.1]) to (4.7.19), we have

$$\int_0^T\left\{|p^\varepsilon(t)|_H^2+|q^\varepsilon(t)|_H^2+|p_x^\varepsilon(t)|_H^2+n_0|p^\varepsilon(t,0)|^2+n_0|p^\varepsilon(t,L)|^2\right\}dt$$

$$\le N_{11}\left(\int_0^T|(u_{**}-u_d)(t)|_H^2dt+\int_0^T|(w_{**}-w_d)(t)|_H^2dt+1\right) \qquad (4.7.20)$$

for some constant $N_{11}>0$, independent of $\varepsilon\in(0,1]$ and dependent on T. Hence, it follows from (4.7.19) and (4.7.20) that

$$\sup_{0\le t\le T}\left\{|p^\varepsilon(t)|_H^2+|q^\varepsilon(t)|_H^2+|p_x^\varepsilon(t)|_H^2+n_0|p^\varepsilon(t,0)|^2+n_0|p^\varepsilon(t,L)|^2\right\}$$

$$+\int_0^T|(p^\varepsilon)'(t)|_H^2dt+\int_0^T|p_x^\varepsilon(t)|_H^2dt+n_0\int_0^T|p^\varepsilon(t,0)|^2dt+n_0\int_0^T|p^\varepsilon(t,L)|^2dt$$

$$\le N_{12}\left(\int_0^T|(u_{**}-u_d)(t)|_H^2dt+\int_0^T|(w_{**}-w_d)(t)|_H^2dt+1\right) \qquad (4.7.21)$$

for some constant $N_{12}>0$, independent of $\varepsilon\in(0,1]$ and dependent on T.

Now, for any $\varepsilon\in(0,1]$, let us define a bounded and linear functional $\mu^\varepsilon\in W'$ on W, by putting, for all $\zeta\in W$,

$$\langle\mu^\varepsilon,\zeta\rangle_{W',W}:=\int_0^T\{(\kappa((a^\varepsilon)'((w_*^\varepsilon)_x(t))+\varepsilon)q_x^\varepsilon(t),\zeta_x(t))_H+((K^\varepsilon)'(w_*^\varepsilon(t))q^\varepsilon(t),\zeta(t))_H\}\,dt.$$

Here, note from (4.3.5) and (4.7.8) that (cf. (4.4.18), (4.6.17)):

$$\{w_*^\varepsilon\}\text{ is bounded in }L^\infty(Q)\text{ uniformly in }\varepsilon\in(0,1]. \qquad (4.7.22)$$

In addition, we infer from (A6) snd (4.7.22) that (cf. (4.4.19), (4.6.18)):

$$\{g'(w_*^\varepsilon)\}\text{ is bounded in }L^\infty(Q)\text{ uniformly in }\varepsilon\in(0,1]. \qquad (4.7.23)$$

Then, on account of (4.6.6), (4.7.12), (4.7.17), and (4.7.21)–(4.7.23), there exists a positive constant N_{13}, independent of $\varepsilon\in(0,1]$, such that

$$|\langle\mu^\varepsilon,\zeta\rangle_{W',W}|\le\left|\int_0^T((p^\varepsilon)'(t),\zeta(t))_H\,dt\right|+\left|\int_0^T\langle(q^\varepsilon)'(t),\zeta(t)\rangle dt\right|$$

$$+\left|\int_0^T(g'(w_*^\varepsilon(t))q^\varepsilon(t),\zeta(t))_H\,dt\right|+\left|c_1\int_0^T((w_*^\varepsilon-w_d)(t),\zeta(t))_H\,dt\right|$$

$$
\begin{aligned}
&= \left|\int_0^T ((p^\varepsilon)'(t),\zeta(t))_H\,dt\right| + \left|\int_0^T (-q^\varepsilon(t),\zeta'(t))_H\,dt\right| \\
&\quad + \left|\int_0^T (g'(w_*^\varepsilon(t))q^\varepsilon(t),\zeta(t))_H\,dt\right| + \left|c_1\int_0^T ((w_*^\varepsilon - w_d)(t),\zeta(t))_H\,dt\right| \\
&\le N_{13}\left(|u_{**}-u_d|_{L^2(0,T;H)} + |w_{**}-w_d|_{L^2(0,T;H)} + 1\right)|\zeta|_W \\
&\qquad \text{for any } \zeta \in W := \{z \in H^1(Q)\,;\, z(0,x) = 0, \ \text{a.a. } x \in (0,L)\}.
\end{aligned}
$$

Therefore, we get

$$
|\mu^\varepsilon|_{W'} \le N_{13}\left(|u_{**}-u_d|_{L^2(0,T;H)} + |w_{**}-w_d|_{L^2(0,T;H)} + 1\right) \tag{4.7.24}
$$

for all $\varepsilon \in (0,1]$.

By the boundedness estimates (4.7.21) and (4.7.24), there are the functions $p \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$, $q \in L^\infty(0,T;H)$ and an element $\mu \in W'$ such that

$$
\left.\begin{aligned}
p^\varepsilon \to p \quad & \text{in } C([0,T];H), \\
& \text{weakly in } W^{1,2}(0,T;H), \\
& \text{weakly-}* \text{ in } L^\infty(0,T;X),
\end{aligned}\right\} \tag{4.7.25}
$$

$$
q^\varepsilon \to q \quad \text{weakly-}* \text{ in } L^\infty(0,T;H), \tag{4.7.26}
$$

$$
\mu^\varepsilon \to \mu \quad \text{weakly in } W' \tag{4.7.27}
$$

as $\varepsilon \to 0$, by taking a subsequence if necessary.

Taking account of the convergence (4.7.9)–(4.7.12) and (4.7.25)–(4.7.27), we observe that the equations (4.7.1)–(4.7.7) hold. In fact, the system $\{(4.6.3)\text{–}(4.6.6)\}$ is equivalent to the following variational identities:

$$
\begin{aligned}
&\int_0^T (-(p^\varepsilon)'(t),\zeta(t))_H dt + \int_0^T (p_x^\varepsilon(t),\zeta_x(t))_H dt + n_0\int_0^T p^\varepsilon(t,0)\zeta(t,0)dt \\
&\quad + n_0\int_0^T p^\varepsilon(t,L)\zeta(t,L)dt - \int_0^T (q^\varepsilon(t),\zeta(t))_H dt \\
= &\int_0^T c_0((u_*^\varepsilon - u_d)(t),\zeta(t))_H dt \qquad \text{for all } \zeta \in L^2(0,T;X)
\end{aligned} \tag{4.7.28}
$$

and

$$
\begin{aligned}
&\int_0^T (-(p^\varepsilon)'(t),z(t))_H dt + \int_0^T (q^\varepsilon(t),z'(t))_H dt + \langle \mu^\varepsilon, z\rangle_{W',W} \\
&\quad + \int_0^T (g'(w_*^\varepsilon(t))q^\varepsilon(t),z(t))_H\,dt \\
= \;& c_1\int_0^T ((w_*^\varepsilon - w_d)(t),z(t))_H\,dt \qquad \text{for all } z \in W.
\end{aligned} \tag{4.7.29}
$$

Thus, we easily see from (4.7.9)–(4.7.12), (A6) with (4.7.23) (cf. (4.6.39)), and (4.7.25)–(4.7.29) that the equations (4.7.1)–(4.7.4) hold. Moreover, we easily see from (4.6.7)–(4.6.9), (4.7.9)–(4.7.11), and (4.7.25) that (4.7.5)–(4.7.7) hold. Thus, the proof of Theorem 4.7 has been completed. □

Remark 4.8. Theorem 4.7 is to be proved through the limiting observation of the approximate situations shown in Theorem 4.6. In addition, the identities (4.6.4) and (4.7.2) can be regarded as some variational forms of the equations:

$$-p_t^\varepsilon - q_t^\varepsilon - \kappa(((a^\varepsilon)'((w_*^\varepsilon)_x) + \varepsilon)q_x^\varepsilon)_x + (K^\varepsilon)'(w_*^\varepsilon)q^\varepsilon + g'(w_*^\varepsilon)q^\varepsilon = c_1(w_*^\varepsilon - w_d)$$

and

$$-p_t - q_t + \mu + g'(w_{**})q = c_1(w_{**} - w_d)$$

in the distribution sense, respectively.

Remark 4.9. In Remark 4.7 we mentioned that any optimal control of $(\mathrm{OP})^0$ can be approximated by the control problem $(\mathrm{OP})_\alpha^\varepsilon$ with $\alpha > 0$. Then, by arguments similar to those in Theorem 4.7 and [81, Theorem 3.5], we can show the necessary conditions for any optimal control of $(\mathrm{OP})^0$, which are the same ones (4.7.5)–(4.7.7) as in Theorem 4.7.

Remark 4.10. In Remark 4.7 we mentioned that for each optimal control $[f_*, h_*, \ell_*]$ of $(\mathrm{OP})^0$, we can find the sequence of optimal controls of $(\mathrm{OP})_\alpha^\varepsilon$ which converges to $[f_*, h_*, \ell_*]$ strongly in $\mathcal{U}$. However, it is very difficult to give the numerical experiments of $(\mathrm{OP})_\alpha^\varepsilon$, since the cost function J_α^ε defined by (4.5.12) depends on the unknown optimal control $[f_*, h_*, \ell_*]$ of $(\mathrm{OP})^0$. Therefore, in the numerical analysis, we are forced to adopt $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$ as the approximate problem of $(\mathrm{OP})^0$, since the cost function J^ε defined by (1.5.26) is independent of optimal controls of $(\mathrm{OP})^0$. Thus, from the viewpoint of applications, the main results for $(\mathrm{OP})^\varepsilon$ would be more useful than those for $(\mathrm{OP})_\alpha^\varepsilon$ with $\alpha > 0$.

4.8 Numerical Scheme for $(\mathrm{OP})^\varepsilon$

Note from the singularity and nonlinearity in (1.5.12) that the numerical consideration of $(\mathrm{OP})^0$ is very difficult (cf. Theorem 4.7 and Remark 4.8). In Section 5, we proved the relationship between the limits (ω-limit points) of sequences of approximate optimal pairs of $(\mathrm{OP})^\varepsilon$ as $\varepsilon \to 0$ and the optimal pairs of the limiting problem $(\mathrm{OP})^0$ (cf. Theorem 4.5(II)). Therefore, it is worth considering the approximate optimal control problem $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$ from the viewpoint of numerical analysis.

In this section, we propose the numerical scheme to find the stationary point of the cost functional J^ε to $(\mathrm{OP})^\varepsilon$ with $\varepsilon > 0$, and show the convergence of our numerical algorithm. To this end, we fix the small parameter $\varepsilon \in (0, 1]$ and the pair of initial data $[u_0^\varepsilon, w_0^\varepsilon] \in H \times D(V^\varepsilon)$. Then, we define the solution operator Λ_{ad}^ε of the adjoint system $\{(4.6.3)–(4.6.6)\}$:

Definition 4.5. We denote by $\Lambda_{ad}^\varepsilon : \mathcal{U} \to L^2(0, T; H) \times L^2(0, T; H)$ the solution operator that assigns to any control $[f, h, \ell] \in \mathcal{U}$ the unique solution $[p^\varepsilon, q^\varepsilon] := \Lambda_{ad}^\varepsilon(f, h, \ell)$ to the adjoint system $\{(4.6.3)–(4.6.6)\}$ on $[0, T]$.

For a moment, we often omit the superscript $\varepsilon \in (0, 1]$.

Now, by the similar idea used in [1, 64, 66, 68, 69, 82], namely, by using the necessary conditions (4.6.7), (4.6.8), and (4.6.9) of $(\mathrm{OP})^\varepsilon$ obtained in Theorem 4.6, we propose the following numerical algorithm, denoted by (NA), to find the stationary point of the cost functional J^ε with $\varepsilon > 0$.

Numerical Algorithm (NA) of $(\mathbf{OP})^\varepsilon$ with $\varepsilon > 0$

(Step 0) Give the stop parameter μ;

(Step 1) Choose the triplet of initial functions $[f, h, \ell] \in \mathcal{U}$, and put $[f_n, h_n, \ell_n] := [f, h, \ell]$;

(Step 2) Solve the approximate state system $(\mathrm{P}; u_0^\varepsilon, w_0^\varepsilon, f_n, h_n, \ell_n)^\varepsilon$ for n, and let $[u_n, w_n] := \Lambda^\varepsilon(f_n, h_n, \ell_n)$, where Λ^ε is the solution operator to $(\mathrm{P}; u_0^\varepsilon, w_0^\varepsilon, f_n, h_n, \ell_n)^\varepsilon$ defined in Definition 4.4(I);

(Step 3) Solve the adjoint system $\{(4.6.3)–(4.6.6)\}$ for n, and let $[p_n, q_n] := \Lambda_{ad}^\varepsilon(f_n, h_n, \ell_n)$;

(Step 4) Put

$$d_{0n} := a_0(p_n + m_0 a_0 f_n),\ d_{1n} := a_1(p_n(\cdot, 0) + m_1 a_1 h_n),\ \text{and}\ d_{2n} := a_2(p_n(\cdot, L) + m_2 a_2 \ell_n).$$

Test: If

$$|[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}} < \mu,$$

then, STOP; Otherwise, go to (Step 5); note here that $\mathcal{U}$ is the product Hilbert space endowed with the usual norm

$$|[f, h, \ell]|_{\mathcal{U}}^2 := |f|_{L^2(0,T;H)}^2 + |h|_{L^2(0,T)}^2 + |\ell|_{L^2(0,T)}^2, \quad \forall [f, h, \ell] \in \mathcal{U}; \tag{4.8.1}$$

(Step 5) Put

$$f_{n+1} := f_n - \rho_n d_{0n},\ h_{n+1} := h_n - \rho_n d_{1n},\ \text{and}\ \ell_{n+1} := \ell_n - \rho_n d_{2n},$$

where ρ_n is some appropriate constant found by using a line search. More precisely, let $\beta \in (0, 1)$. Then, find the minimal constant $\varsigma_n \in \mathbb{N} \cup \{0\}$ such that

$$\begin{aligned} & J^\varepsilon\left(f_n - \beta^{\varsigma_n} d_{0n}, h_n - \beta^{\varsigma_n} d_{1n}, \ell_n - \beta^{\varsigma_n} d_{2n}\right) - J^\varepsilon(f_n, h_n, \ell_n) \\ \leq\ & -\mu \beta^{\varsigma_n} \left|[d_{0n}, d_{1n}, d_{2n}]\right|_{\mathcal{U}}, \end{aligned}$$

and put the constant $\rho_n := \beta^{\varsigma_n}$;

(Step 6) Set $n = n + 1$, and go to (Step 2).

Remark 4.11. In (Step 5), we need to find the constant ρ_n (cf. the so-called "the learning rate" in neural networks) for each step n, because of the nonlinear term $\left(a^\varepsilon(w_x^\varepsilon)\right)_x$ in (1.5.21) (cf. Remark 4.2). If the main diffusion term in (1.5.21) is just only linear (i.e., w_{xx}^ε), we can take the constant $\rho \equiv \rho_n$ independent of n. Indeed, Aiki et al. [1] considered the optimal control problem for phase-field equations of a regular Fix–Caginalp type with dynamic boundary conditions, and proved the existence of a constant ρ, independent of n, in the descend method. For the detailed statement, we refer to [1, Section 4].

Now, we mention our final theoretical result in this paper, which is concerned with the convergence of the numerical algorithm (NA).

Theorem 4.8. Suppose that all the assumptions of Theorem 4.6 are fulfilled. Let $\varepsilon \in (0,1]$ and $[u_0^\varepsilon, w_0^\varepsilon] \in H \times D(V^\varepsilon)$. Let $\{[f_n, h_n, \ell_n]\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{U}$ defined by the numerical algorithm (NA). In addition, let $[p_n, q_n] = \Lambda_{ad}^\varepsilon(f_n, h_n, \ell_n)$. Then:

(I) $\lim_{n\to\infty} J^\varepsilon(f_n, h_n, \ell_n)$ exists.

(II)

$$\lim_{n\to\infty} a_0(p_n + m_0 a_0 f_n) = 0 \ \text{ in } L^2(0,T;H), \tag{4.8.2}$$

$$\lim_{n\to\infty} a_1(p_n(\cdot,0) + m_1 a_1 h_n) = 0 \ \text{ in } L^2(0,T), \tag{4.8.3}$$

$$\lim_{n\to\infty} a_2(p_n(\cdot,L) + m_2 a_2 \ell_n) = 0 \ \text{ in } L^2(0,T). \tag{4.8.4}$$

(III) There are the triplet of functions $[f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon] \in \mathcal{U}$, the pair of functions $[p_{**}^\varepsilon, q_{**}^\varepsilon] \in L^2(0,T;H) \times L^2(0,T;H)$, and a subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \{n\}_{n\in\mathbb{N}}$ such that $p_{**}^\varepsilon \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$, $q_{**}^\varepsilon \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H)$, $[p_{**}^\varepsilon, q_{**}^\varepsilon]$ is a unique solution of the adjoint system $\{(4.6.3)–(4.6.6)\}$ for (P;$u_0^\varepsilon, w_0^\varepsilon, f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon)^\varepsilon$, i.e., $[p_{**}^\varepsilon, q_{**}^\varepsilon] = \Lambda_{ad}^\varepsilon(f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon)$,

$$f_{n_k} \longrightarrow f_{**}^\varepsilon \ \text{ in } L^2(0,T;H), \tag{4.8.5}$$

$$h_{n_k} \longrightarrow h_{**}^\varepsilon \ \text{ in } L^2(0,T), \tag{4.8.6}$$

$$\ell_{n_k} \longrightarrow \ell_{**}^\varepsilon \ \text{ in } L^2(0,T), \tag{4.8.7}$$

$$p_{n_k} \longrightarrow p_{**}^\varepsilon \ \left.\begin{array}{l} \text{in } C([0,T];H), \\ \text{in } L^2(0,T;X), \end{array}\right\} \tag{4.8.8}$$

$$q_{n_k} \longrightarrow q_{**}^\varepsilon \ \text{ in } L^2(0,T;H) \tag{4.8.9}$$

as $k \to \infty$, and

$$a_0(p_{**}^\varepsilon + m_0 a_0 f_{**}^\varepsilon) = 0 \ \text{ in } L^2(0,T;H), \tag{4.8.10}$$

$$a_1(p_{**}^\varepsilon(\cdot,0) + m_1 a_1 h_{**}^\varepsilon) = 0 \ \text{ in } L^2(0,T), \tag{4.8.11}$$

$$a_2(p_{**}^\varepsilon(\cdot,L) + m_2 a_2 \ell_{**}^\varepsilon) = 0 \ \text{ in } L^2(0,T). \tag{4.8.12}$$

Hence,

$$\begin{aligned} &D_{[\check{f},\check{h},\check{\ell}]} J^\varepsilon(f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon) \\ := &\lim_{\lambda\to 0} \frac{J^\varepsilon(f_{**}^\varepsilon + \lambda\check{f}, h_{**}^\varepsilon + \lambda\check{h}, \ell_{**}^\varepsilon + \lambda\check{\ell}) - J^\varepsilon(f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon)}{\lambda} = 0 \end{aligned} \tag{4.8.13}$$

for all direction $[\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}$;

thus, $[f_{**}^\varepsilon, h_{**}^\varepsilon, \ell_{**}^\varepsilon] \in \mathcal{U}$ is the stationary point of the cost functional J^ε with $\varepsilon \in (0,1]$.

To prove Theorem 4.8, we need some lemmas.

Note from Corollary 4.4 that we have the result of continuous dependence of solutions to the approximate state system $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$. In addition, note from Proposition 4.5(I) that the solution operator Λ^ε admits the Gâteaux derivative at any $[f, h, l] \in \mathcal{U}$.

Now we show the continuity of Gâteaux derivative of Λ^ε, which is the key to proving Theorem 4.8.

Lemma 4.3. Assume the same conditions as in Theorem 4.8. Let $\varepsilon \in (0,1]$, $\xi \in [-1,1] \setminus \{0\}$, and fix the pair of initial data $[u_0^\varepsilon, w_0^\varepsilon] \in H \times D(V^\varepsilon)$. Then, the Gâteaux derivative of the control-to-state mapping Λ^ε is continuous in the following sense:

$$\begin{aligned} [\theta_\xi, \chi_\xi] \quad &:= D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3) \\ \longrightarrow \quad [\theta, \chi] \quad &= D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f, h, \ell) \quad \text{ in } L^2(0,T;H) \times L^2(0,T;H) \qquad (4.8.14) \\ &\text{for all } [f,h,\ell] \in \mathcal{U},\ [\varpi_1, \varpi_2, \varpi_3] \in \mathcal{U}, \text{ and all direction } [\check{f}, \check{h}, \check{\ell}] \in \mathcal{U} \end{aligned}$$

as $\xi \to 0$.

Proof. For any $[f,h,\ell] \in \mathcal{U}$, $[\varpi_1, \varpi_2, \varpi_3] \in \mathcal{U}$, and $\xi \in [-1,1] \setminus \{0\}$, we put $[u_\xi, w_\xi] := \Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ and $[u, w] := \Lambda^\varepsilon(f,h,\ell)$. Then, we observe from Corollary 4.4 that

$$[u_\xi, w_\xi] \longrightarrow [u, w] \text{ in } L^2(0,T;H) \times C([0,T];H) \text{ as } \xi \to 0. \qquad (4.8.15)$$

In addition, we have:

$$w_\xi \longrightarrow w \text{ in } L^2(0,T;X) \text{ as } \xi \to 0. \qquad (4.8.16)$$

Indeed, subtract (1.5.21) for $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)^\varepsilon$ from the one for $(\mathrm{P};u_0^\varepsilon, w_0^\varepsilon, f, h, \ell)^\varepsilon$, and multiply it by $w_\xi - w$. Then, from the monotonicity of $a^\varepsilon(w_x^\varepsilon)$, $K^\varepsilon(w^\varepsilon)$, and $g(w^\varepsilon) + C_g w^\varepsilon$ (cf. (A1), (1.5.19), and (A2)), and the Schwarz inequality, we observe that

$$\frac{1}{2}\frac{d}{dt}|(w_\xi - w)(t)|_H^2 + \varepsilon\kappa|(w_\xi - w)_x(t)|_H^2 \le \left(\frac{1}{2} + C_g\right)|(w_\xi - w)(t)|_H^2 + \frac{1}{2}|(u_\xi - u)(t)|_H^2 \qquad (4.8.17)$$

for a.a. $t \in (0,T)$. Hence, applying the Gronwall inequality to (4.8.17), we conclude that

$$\frac{1}{2}\sup_{t\in[0,T]}|(w_\xi - w)(t)|_H^2 + \varepsilon\kappa\int_0^T |(w_\xi - w)_x(t)|_H^2 dt \le \frac{1}{2}e^{(1+2C_g)T}|u_\xi - u|_{L^2(0,T;H)}^2. \qquad (4.8.18)$$

Thus, we infer from (4.8.15) and (4.8.18) that the convergence (4.8.16) holds.

Now, we show (4.8.14) by using the convergences (4.8.15) and (4.8.16). Note from Proposition 4.5(I) that $[\theta_\xi, \chi_\xi] = D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ satisfies the following variational identities:

$$\begin{aligned} &\langle \theta_\xi'(t), z\rangle + \langle \chi_\xi'(t), z\rangle + ((\theta_\xi)_x(t), z_x)_H + n_0\left(\theta_\xi(t,0)z(0) + \theta_\xi(t,L)z(L)\right) \\ &\qquad = (a_0\check{f}(t), z)_H + a_1\check{h}(t)z(0) + a_2\check{\ell}(t)z(L), \qquad (4.8.19) \\ &\qquad\qquad \text{a.a. } t \in (0,T), \text{ for all } z \in X; \end{aligned}$$

$$\begin{aligned} &\langle \chi_\xi'(t), z\rangle + \kappa\left(((a^\varepsilon)'((w_\xi)_x(t)) + \varepsilon)(\chi_\xi)_x(t), z_x\right)_H + ((K^\varepsilon)'(w_\xi(t))\chi_\xi(t), z)_H \\ &\qquad + (g'(w_\xi(t))\chi_\xi(t), z)_H = (\theta_\xi(t), z)_H, \qquad (4.8.20) \\ &\qquad\qquad \text{a.a. } t \in (0,T), \text{ for all } z \in X; \end{aligned}$$

$$\theta_\xi(0,x) = \chi_\xi(0,x) = 0, \quad \text{a.a. } x \in (0,L). \tag{4.8.21}$$

Then, by arguments similar to Lemma 4.2, we can obtain the uniform estimate of functions θ_ξ and χ_ξ with respect to $\xi \in [-1,1] \setminus \{0\}$. Indeed, taking $z = \theta_\xi$ in (4.8.19), using the Schwarz inequality, and integrating in time, we obtain:

$$\begin{aligned} &\frac{1}{2}|\theta_\xi(t)|_H^2 + \int_0^t \langle \chi_\xi'(s), \theta_\xi(s)\rangle ds + \int_0^t |(\theta_\xi)_x(s)|_H^2 ds \\ &\quad + \frac{n_0}{2}\int_0^t |\theta_\xi(s,0)|^2 ds + \frac{n_0}{2}\int_0^t |\theta_\xi(s,L)|^2 ds \\ &\le \frac{1}{2}\int_0^t |\theta_\xi(s)|_H^2 ds + \frac{a_0^2}{2}\int_0^t |\check{f}(s)|_H^2 ds + \frac{a_1^2}{2n_0}\int_0^t |\check{h}(s)|^2 ds + \frac{a_2^2}{2n_0}\int_0^t |\check{\ell}(s)|^2 ds \\ &\qquad \text{for all } t \in [0,T]. \end{aligned} \tag{4.8.22}$$

Next, we note from (A1) and (A5) that $(\widehat{a}^\varepsilon)'(\cdot) = a^\varepsilon(\cdot) \in C^1(\mathbb{R})$ and

$$0 \le (a^\varepsilon)'(r) \le \frac{\delta_3}{\varepsilon} \quad \text{for any } r \in \mathbb{R}. \tag{4.8.23}$$

In addition, note from (1.5.19) that $K^\varepsilon(\cdot) \in C^1(\mathbb{R})$ and

$$0 \le (K^\varepsilon)'(r) \le \frac{1}{\varepsilon} \quad \text{for any } r \in \mathbb{R}. \tag{4.8.24}$$

Furthermore, note from (A2) and (A6) that $g(\cdot) \in C^1(\mathbb{R})$ and

$$g'(r) + C_g \ge 0 \ \text{ for any } r \in \mathbb{R}. \tag{4.8.25}$$

Then, taking $z = \chi_\xi$ in (4.8.20), using (4.8.23)–(4.8.25) and the Schwarz inequality, and integrating in time, we obtain:

$$\begin{aligned} &\frac{1}{2}|\chi_\xi(t)|_H^2 + \varepsilon\kappa\int_0^t |(\chi_\xi)_x(s)|_H^2 ds \le \left(\frac{1}{2} + C_g\right)\int_0^t |\chi_\xi(s)|_H^2 ds + \frac{1}{2}\int_0^t |\theta_\xi(s)|_H^2 ds \\ &\qquad \text{for all } t \in [0,T]. \end{aligned} \tag{4.8.26}$$

Similarly, taking $z = \theta_\xi$ in (4.8.20), using (4.8.23), (4.8.24), and the Höder inequality, and integrating in time, we obtain:

$$\begin{aligned} \left|\int_0^t \langle \chi_\xi'(s), \theta_\xi(s)\rangle ds\right| &\le \kappa\left(\frac{\delta_3}{\varepsilon} + \varepsilon\right)\int_0^t |(\chi_\xi)_x(s)|_H |(\theta_\xi)_x(s)|_H ds \\ &\quad + \frac{1}{\varepsilon}\int_0^t |\chi_\xi(s)|_H |\theta_\xi(s)|_H ds \\ &\quad + \int_0^t |g'(w_\xi(s))\chi_\xi(s)|_H |\theta_\xi(s)|_H ds + \int_0^t |\theta_\xi(s)|_H^2 ds \\ &\quad \text{for all } t \in [0,T]. \end{aligned} \tag{4.8.27}$$

Note from (4.3.5) that we get the the following uniform estimate of solutions $[u_\xi, w_\xi] :=$

$\Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ with respect to $\xi \in [-1,1]\setminus\{0\}$:

$$
\begin{aligned}
&|u'_\xi|^2_{L^2(0,T;X')} + |u_\xi|^2_{L^\infty(0,T;H)} + |u_\xi|^2_{L^2(0,T;X)} + |w'_\xi|^2_{L^2(0,T;H)} \\
&\quad + |w_\xi|^2_{L^\infty(0,T;H)} + \kappa \sup_{0\le t\le T} V^\varepsilon(w_\xi(t)) + \sup_{0\le t\le T} \int_0^L \widehat{g}(w_\xi(t,x))\,dx \\
\le\ & N_1 \left(|u_0^\varepsilon|^2_H + |w_0^\varepsilon|^2_H + \kappa V^\varepsilon(w_0^\varepsilon) + \int_0^L \widehat{g}(w_0^\varepsilon(x))\,dx + a_0^2 |f+\xi\varpi_1|^2_{L^2(0,T;H)} \right. \\
&\quad \left. + a_1^2 |h+\xi\varpi_2|^2_{L^2(0,T)} + a_2^2|\ell+\xi\varpi_3|^2_{L^2(0,T)} + b_1^2 + b_2^2 \right),
\end{aligned}
\tag{4.8.28}
$$

where N_1 is the same positive constant in (4.3.5).

Taking account of (4.8.28), $\xi \in [-1,1]\setminus\{0\}$, and the continuous embedding $BV(0,L) \hookrightarrow L^\infty(0,L)$ (cf. Proposition 4.3), we see that

$$
\begin{aligned}
\sup_{t\in[0,T]} |w_\xi(t)|_{L^\infty(0,L)} \le N_{14} & \left(|u_0^\varepsilon|^2_H + |w_0^\varepsilon|^2_H + \kappa V^\varepsilon(w_0^\varepsilon) + \int_0^L \widehat{g}(w_0^\varepsilon(x))\,dx \right. \\
& + a_0^2|f|^2_{L^2(0,T;H)} + a_0^2|\varpi_1|^2_{L^2(0,T;H)} \\
& + a_1^2|h|^2_{L^2(0,T)} + a_1^2|\varpi_2|^2_{L^2(0,T)} \\
& \left. + a_2^2|\ell|^2_{L^2(0,T)} + a_2^2|\varpi_3|^2_{L^2(0,T)} + b_1^2 + b_2^2 \right),
\end{aligned}
\tag{4.8.29}
$$

hence, we observe from (A6) that

$$
\sup_{t\in[0,T]} |g'(w_\xi(t))|_{L^\infty(0,L)} \le N_{14,\varepsilon}, \tag{4.8.30}
$$

where N_{14} and $N_{14,\varepsilon}$ are positive constants independent of $\xi \in [-1,1]\setminus\{0\}$. Therefore, it follows from (4.8.27), (4.8.30), and the Schwarz inequality that

$$
\begin{aligned}
\left| \int_0^t \langle \chi'_\xi(s), \theta_\xi(s)\rangle ds \right| \le\ & \frac{1}{2}\int_0^t |(\theta_\xi)_x(s)|^2_H ds + \frac{1}{2}\kappa^2 \left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2 \int_0^t |(\chi_\xi)_x(s)|^2_H ds \\
& + \left(\frac{1}{2\varepsilon^2} + \frac{N_{14,\varepsilon}^2}{2}\right) \int_0^t |\chi_\xi(s)|^2_H ds + 2\int_0^t |\theta_\xi(s)|^2_H ds \\
& \text{for all } t \in [0,T].
\end{aligned}
\tag{4.8.31}
$$

Hence, we infer from (4.8.22) and (4.8.31) that

$$
\begin{aligned}
& \frac{1}{2}|\theta_\xi(t)|^2_H + \frac{1}{2}\int_0^t |(\theta_\xi)_x(s)|^2_H ds + \frac{n_0}{2}\int_0^t |\theta_\xi(s,0)|^2 ds + \frac{n_0}{2}\int_0^t |\theta_\xi(s,L)|^2 ds \\
\le\ & \frac{1}{2}\kappa^2 \left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2 \int_0^t |(\chi_\xi)_x(s)|^2_H ds + \frac{5}{2}\int_0^t |\theta_\xi(s)|^2_H ds \\
& + \left(\frac{1}{2\varepsilon^2} + \frac{N_{14,\varepsilon}^2}{2}\right) \int_0^t |\chi_\xi(s)|^2_H ds \\
& + \frac{a_0^2}{2}\int_0^t |\check{f}(s)|^2_H ds + \frac{a_1^2}{2n_0}\int_0^t |\check{h}(s)|^2 ds + \frac{a_2^2}{2n_0}\int_0^t |\check{\ell}(s)|^2 ds \\
& \text{for all } t \in [0,T].
\end{aligned}
\tag{4.8.32}
$$

Now, by adding (4.8.32) and (4.8.26)$\times\frac{\kappa}{\varepsilon}\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2$, we have

$$
\begin{aligned}
&\frac{1}{2}|\theta_\xi(t)|_H^2+\frac{\kappa}{2\varepsilon}\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2|\chi_\xi(t)|_H^2+\frac{1}{2}\int_0^t|(\theta_\xi)_x(s)|_H^2ds\\
&\quad+\frac{1}{2}\kappa^2\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2\int_0^t|(\chi_\xi)_x(s)|_H^2ds\\
&\quad+\frac{n_0}{2}\int_0^t|\theta_\xi(s,0)|^2ds+\frac{n_0}{2}\int_0^t|\theta_\xi(s,L)|^2ds\\
\leq\ &\left(\frac{5}{2}+\frac{\kappa}{2\varepsilon}\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2\right)\int_0^t|\theta_\xi(s)|_H^2ds \qquad (4.8.33)\\
&\quad+\left(\frac{1}{2\varepsilon^2}+\frac{N_{14,\varepsilon}^2}{2}+\frac{\kappa}{\varepsilon}\left(\frac{1}{2}+C_g\right)\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)^2\right)\int_0^t|\chi_\xi(s)|_H^2ds\\
&\quad+\frac{a_0^2}{2}\int_0^t|\check{f}(s)|_H^2ds+\frac{a_1^2}{2n_0}\int_0^t|\check{h}(s)|^2ds+\frac{a_2^2}{2n_0}\int_0^t|\check{\ell}(s)|^2ds\\
&\qquad\qquad\text{for all } t\in[0,T].
\end{aligned}
$$

Thus, by applying the Gronwall inequality to (4.8.33) and the standard calculations, we have

$$
\begin{aligned}
&|\theta_\xi(t)|_H^2+C_1(\varepsilon)|\chi_\xi(t)|_H^2+\int_0^t|(\theta_\xi)_x(s)|_H^2ds+C_2(\varepsilon)\int_0^t|(\chi_\xi)_x(s)|_H^2ds\\
&\quad+n_0\int_0^t|\theta_\xi(s,0)|^2ds+n_0\int_0^t|\theta_\xi(s,L)|^2ds \qquad (4.8.34)\\
\leq\ &N_{15,\varepsilon}\left(a_0^2|\check{f}|_{L^2(0,T;H)}^2+\frac{a_1^2}{n_0}|\check{h}|_{L^2(0,T)}^2+\frac{a_2^2}{n_0}|\check{\ell}|_{L^2(0,T)}^2\right)\quad\text{for all } t\in[0,T],
\end{aligned}
$$

where $C_1(\varepsilon)$, $C_2(\varepsilon)$, and $N_{15,\varepsilon}$ are positive constants dependent on ε and are independent of $\xi\in[-1,1]\setminus\{0\}$. In addition, by (4.8.20), (4.8.23), and (4.8.24), we have (cf. (4.8.27)):

$$
\begin{aligned}
\left|\int_0^T\langle\chi_\xi'(s),z(s)\rangle ds\right|\ &\leq\kappa\left(\frac{\delta_3}{\varepsilon}+\varepsilon\right)|(\chi_\xi)_x|_{L^2(0,T;H)}|z_x|_{L^2(0,T;H)}\\
&\quad+\frac{1}{\varepsilon}|\chi_\xi|_{L^2(0,T;H)}|z|_{L^2(0,T;H)} \qquad (4.8.35)\\
&\quad+|g'(w_\xi)\chi_\xi|_{L^2(0,T;H)}|z|_{L^2(0,T;H)}\\
&\quad+|\theta_\xi|_{L^2(0,T;H)}|z|_{L^2(0,T;H)},\qquad\forall z\in L^2(0,T;X).
\end{aligned}
$$

Hence, we infer from (4.8.30), (4.8.34), and (4.8.35) that

$$
|\chi_\xi'|_{L^2(0,T;X')}\leq N_{16,\varepsilon}\left(a_0|\check{f}|_{L^2(0,T;H)}+\frac{a_1}{\sqrt{n_0}}|\check{h}|_{L^2(0,T)}+\frac{a_2}{\sqrt{n_0}}|\check{\ell}|_{L^2(0,T)}\right), \qquad (4.8.36)
$$

where $N_{16,\varepsilon}$ is a positive constant dependent on ε and is independent of $\xi\in[-1,1]\setminus\{0\}$.

Similarly, we observe from (4.8.19), (4.8.34), and (4.8.36) that

$$
|\theta_\xi'|_{L^2(0,T;X')}\leq\tilde{N}_{16,\varepsilon}\left(a_0|\check{f}|_{L^2(0,T;H)}+\frac{a_1}{\sqrt{n_0}}|\check{h}|_{L^2(0,T)}+\frac{a_2}{\sqrt{n_0}}|\check{\ell}|_{L^2(0,T)}\right), \qquad (4.8.37)
$$

where $\tilde{N}_{16,\varepsilon}$ is a positive constant dependent on ε and is independent of $\xi \in [-1,1] \setminus \{0\}$.

By the uniform estimates (4.8.34), (4.8.36), and (4.8.37) of $[\theta_\xi, \chi_\xi]$, there is a subsequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \{\xi\}_{\xi\in[-1,1]\setminus\{0\}}$ and the functions $\overline{\theta}$, $\overline{\chi} \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H)$ such that $\xi_n \to 0$,

$$\left.\begin{array}{ll} \theta_{\xi_n} \to \overline{\theta} & \text{in } C([0,T];X'), \\ & \text{in } L^2(0,T;H), \\ & \text{weakly in } W^{1,2}(0,T;X'), \\ & \text{weakly in } L^2(0,T;X), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;H), \end{array}\right\} \tag{4.8.38}$$

$$\theta_{\xi_n}(\cdot,0) \to \overline{\theta}(\cdot,0) \ \text{ weakly in } L^2(0,T), \tag{4.8.39}$$

$$\theta_{\xi_n}(\cdot,L) \to \overline{\theta}(\cdot,L) \ \text{ weakly in } L^2(0,T), \tag{4.8.40}$$

and

$$\left.\begin{array}{ll} \chi_{\xi_n} \to \overline{\chi} & \text{in } C([0,T];X'), \\ & \text{in } L^2(0,T;H), \\ & \text{weakly in } W^{1,2}(0,T;X'), \\ & \text{weakly in } L^2(0,T;X), \\ & \text{weakly-}* \text{ in } L^\infty(0,T;H), \end{array}\right\} \tag{4.8.41}$$

as $n \to \infty$.

Here, from (4.8.16), (4.8.23), (4.8.24), (4.8.30), continuity of functions $(a^\varepsilon)'$, $(K^\varepsilon)'$, and g', and the Lebesgue dominated convergence theorem, we note that:

$$\left\{\begin{array}{l} (a^\varepsilon)'((w_{\xi_n})_x) \longrightarrow (a^\varepsilon)'(w_x), \\ (K^\varepsilon)'(w_{\xi_n}) \longrightarrow (K^\varepsilon)'(w), \\ g'(w_{\xi_n}) \longrightarrow g'(w), \end{array}\right. \quad \begin{array}{l} \text{in } L^2(0,T;H), \\ \text{weakly-}* \text{ in } L^\infty(Q), \end{array} \quad \text{as } n \to \infty. \tag{4.8.42}$$

Thus, taking a subsequence if necessary, we observe from (4.8.23), (4.8.24), (4.8.30), and (4.8.38)–(4.8.42) that:

$$\left\{\begin{array}{l} (a^\varepsilon)'((w_{\xi_n})_x)(\chi_{\xi_n})_x \rightharpoonup (a^\varepsilon)'(w_x)\overline{\chi}_x, \\ (K^\varepsilon)'(w_{\xi_n})\chi_{\xi_n} \rightharpoonup (K^\varepsilon)'(w)\overline{\chi}, \\ g'(w_{\xi_n})\chi_{\xi_n} \rightharpoonup g'(w)\overline{\chi}, \end{array}\right. \quad \text{weakly in } L^2(0,T;H), \ \text{ as } n \to \infty. \tag{4.8.43}$$

Note from Proposition 4.5(I) that $[\theta_{\xi_n}, \chi_{\xi_n}] = D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f + \xi_n\varpi_1, h + \xi_n\varpi_2, \ell + \xi_n\varpi_3)$ satisfies the following variational identities:

$$\begin{aligned} &\int_0^T \langle \theta'_{\xi_n}(t), z(t)\rangle dt + \int_0^T \langle \chi'_{\xi_n}(t), z(t)\rangle dt + \int_0^T ((\theta_{\xi_n})_x(t), z_x(t))_H \, dt \\ &\qquad + n_0 \int_0^T \theta_{\xi_n}(t,0) z(t,0) dt + n_0 \int_0^T \theta_{\xi_n}(t,L) z(t,L) dt \\ &= \int_0^T (a_0 \check{f}(t), z(t))_H dt + a_1 \int_0^T \check{h}(t) z(t,0) dt + a_2 \int_0^T \check{\ell}(t) z(t,L) dt \\ &\qquad \text{for all } z \in L^2(0,T;X) \text{ and all direction } [\check{f},\check{h},\check{\ell}] \in \mathcal{U}; \end{aligned} \tag{4.8.44}$$

$$
\begin{aligned}
&\int_0^T \langle \chi'_{\xi_n}(t), z(t)\rangle dt + \kappa \int_0^T (((a^\varepsilon)'((w_{\xi_n})_x(t)) + \varepsilon)(\chi_{\xi_n})_x(t), z_x(t))_H \, dt \\
&+ \int_0^T ((K^\varepsilon)'(w_{\xi_n}(t))\chi_{\xi_n}(t), z(t))_H dt + \int_0^T (g'(w_{\xi_n}(t))\chi_{\xi_n}(t), z(t))_H dt \\
&\qquad = \int_0^T (\theta_{\xi_n}(t), z(t))_H dt \quad \text{for all } z \in L^2(0,T;X);
\end{aligned}
\tag{4.8.45}
$$

$$
\theta_{\xi_n}(0,x) = \chi_{\xi_n}(0,x) = 0 \ (\in H) \ \text{ in } X'. \tag{4.8.46}
$$

By (4.8.38)–(4.8.43), and by taking the limits in (4.8.44)–(4.8.46) as $n \to \infty$, we observe that $[\overline{\theta}, \overline{\chi}]$ satisfies the following system:

$$
\begin{aligned}
&\int_0^T \langle \overline{\theta}'(t), z(t)\rangle dt + \int_0^T \langle \overline{\chi}'(t), z(t)\rangle dt + \int_0^T \left(\overline{\theta}_x(t), z_x(t)\right)_H dt \\
&\qquad + n_0 \int_0^T \overline{\theta}(t,0) z(t,0) dt + n_0 \int_0^T \overline{\theta}(t,L) z(t,L) dt \\
&= \int_0^T (a_0 \check{f}(t), z(t))_H dt + a_1 \int_0^T \check{h}(t) z(t,0) dt + a_2 \int_0^T \check{\ell}(t) z(t,L) dt \\
&\qquad \text{for all } z \in L^2(0,T;X) \text{ and all direction } [\check{f}, \check{h}, \check{\ell}] \in \mathcal{U};
\end{aligned}
\tag{4.8.47}
$$

$$
\begin{aligned}
&\int_0^T \langle \overline{\chi}'(t), z(t)\rangle dt + \kappa \int_0^T (((a^\varepsilon)'(w_x(t)) + \varepsilon)\overline{\chi}_x(t), z_x(t))_H \, dt \\
&\qquad + \int_0^T ((K^\varepsilon)'(w(t))\overline{\chi}(t), z(t))_H dt + \int_0^T (g'(w(t))\overline{\chi}(t), z(t))_H dt \\
&= \int_0^T (\overline{\theta}(t), z(t))_H dt \quad \text{for all } z \in L^2(0,T;X);
\end{aligned}
\tag{4.8.48}
$$

$$
\overline{\theta}(0,x) = \overline{\chi}(0,x) = 0 \ (\in H) \ \text{ in } X'. \tag{4.8.49}
$$

Since the solutions of the Cauchy problem $\{(4.8.47)–(4.8.49)\}$ are uniquely determined, we observe that $[\overline{\theta}, \overline{\chi}] = [\theta, \chi]$ and the convergence (4.8.14) holds without extracting any subsequence from $\{\xi\}_{\xi \in [-1,1] \setminus \{0\}}$, i.e.,

$$
\begin{aligned}
&[\theta_\xi, \chi_\xi] = D_{[\check{f}, \check{h}, \check{\ell}]} \Lambda^\varepsilon (f + \xi \varpi_1, h + \xi \varpi_2, \ell + \xi \varpi_3) \\
\longrightarrow \quad &[\theta, \chi] = D_{[\check{f}, \check{h}, \check{\ell}]} \Lambda^\varepsilon (f, h, \ell) \quad \text{in } L^2(0,T;H) \times L^2(0,T;H) \\
&\text{for all } [f,h,\ell] \in \mathcal{U}, \ [\varpi_1, \varpi_2, \varpi_3] \in \mathcal{U}, \text{ and all direction } [\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}
\end{aligned}
$$

as $\xi \to 0$.

Thus, the proof of this lemma has been completed. □

Note from Proposition 4.5(II) that the cost functional J^ε admits the Gâteaux derivative at any $[f,h,l] \in \mathcal{U}$. Moreover, by Lemma 4.3 we can prove the continuity of Gâteaux derivative of J^ε as follows:

Corollary 4.9. Assume the same conditions as in Theorem 4.8. Let $\varepsilon \in (0,1]$ and $\xi \in [-1,1] \setminus \{0\}$. Then, the Gâteaux derivative of the cost functional J^ε is continuous in the following sense:

$$
\begin{aligned}
&D_{[\check{f}, \check{h}, \check{\ell}]} J^\varepsilon (f + \xi \varpi_1, h + \xi \varpi_2, \ell + \xi \varpi_3) \longrightarrow D_{[\check{f}, \check{h}, \check{\ell}]} J^\varepsilon (f, h, \ell) \\
&\text{for all } [f,h,\ell] \in \mathcal{U}, \ [\varpi_1, \varpi_2, \varpi_3] \in \mathcal{U}, \text{ and all direction } [\check{f}, \check{h}, \check{\ell}] \in \mathcal{U}
\end{aligned}
\tag{4.8.50}
$$

as $\xi \to 0$.

Proof. Note from (4.6.29) that

$$\begin{aligned}
& D_{[\check{f},\check{h},\check{\ell}]}J^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3) \\
= \ & c_0\int_0^T ((u_\xi - u_d)(t), \theta_\xi(t))_H dt + c_1\int_0^T ((w_\xi - w_d)(t), \chi_\xi(t))_H dt \\
& + m_0 a_0^2 \int_0^T ((f+\xi\varpi_1)(t), \check{f}(t))_H dt \\
& + m_1 a_1^2 \int_0^T (h+\xi\varpi_2)(t)\check{h}(t) dt + m_2 a_2^2 \int_0^T (\ell+\xi\varpi_3)(t)\check{\ell}(t) dt
\end{aligned}$$

for any $[f,h,\ell] \in \mathcal{U}$, $[\varpi_1,\varpi_2,\varpi_3] \in \mathcal{U}$, and any direction $[\check{f},\check{h},\check{\ell}] \in \mathcal{U}$, where $[u_\xi, w_\xi] = \Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ and $[\theta_\xi,\chi_\xi] = D_{[\check{f},\check{h},\check{\ell}]}\Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$. Thus, taking account of (4.8.14) and (4.8.15), we easily observe that the convergence (4.8.50) holds. □

Lemma 4.4. Suppose the same conditions as in Theorem 4.8. Fix $\varepsilon \in (0,1]$ and the pair of initial data $[u_0^\varepsilon, w_0^\varepsilon] \in H \times D(V^\varepsilon)$. In addition, for any $\xi \in [-1,1] \setminus \{0\}$, $[f,h,\ell] \in \mathcal{U}$, and $[\varpi_1,\varpi_2,\varpi_3] \in \mathcal{U}$, let $[p_\xi, q_\xi] = \Lambda^\varepsilon_{ad}(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$. Then, $[p_\xi,q_\xi] = \Lambda^\varepsilon_{ad}(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ converges to $[p,q] = \Lambda^\varepsilon_{ad}(f,h,\ell)$ in $L^2(0,T;H) \times L^2(0,T;H)$ as $\xi \to 0$. Furthermore,

$$p_\xi \to p \ \text{ in } L^2(0,T;X) \ \text{ as } \xi \to 0. \tag{4.8.51}$$

Proof. For any $\xi \in [-1,1] \setminus \{0\}$, $[f,h,\ell] \in \mathcal{U}$, and $[\varpi_1,\varpi_2,\varpi_3] \in \mathcal{U}$, let $[u_\xi, w_\xi] = \Lambda^\varepsilon(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$. Then, note from Theorem 4.6 that $[p_\xi,q_\xi] = \Lambda^\varepsilon_{ad}(f+\xi\varpi_1, h+\xi\varpi_2, \ell+\xi\varpi_3)$ satisfies the following:

$$-p_\xi' - (p_\xi)_{xx} - q_\xi = c_0(u_\xi - u_d) \quad \text{in } Q; \tag{4.8.52}$$

$$\begin{aligned}
& \int_t^T (-p_\xi'(\tau), \zeta(\tau))_H d\tau + \int_t^T \langle -q_\xi'(\tau), \zeta(\tau)\rangle d\tau \\
& \quad + \kappa\int_t^T \left(((a^\varepsilon)'((w_\xi)_x(\tau)) + \varepsilon)(q_\xi)_x(\tau), \zeta_x(\tau)\right)_H d\tau \\
& \quad + \int_t^T \left((K^\varepsilon)'(w_\xi(\tau))q_\xi(\tau), \zeta(\tau)\right)_H d\tau + \int_t^T \left(g'(w_\xi(\tau))q_\xi(\tau), \zeta(\tau)\right)_H d\tau \\
= \ & c_1\int_t^T \left((w_\xi - w_d)(\tau), \zeta(\tau)\right)_H d\tau \\
& \qquad \text{for all } t \in [0,T] \text{ and } \zeta \in L^2(0,T;X);
\end{aligned} \tag{4.8.53}$$

$$-(p_\xi)_x(t,0) + n_0 p_\xi(t,0) = (p_\xi)_x(t,L) + n_0 p_\xi(t,L) = 0, \quad t \in (0,T), \tag{4.8.54}$$

$$p_\xi(T,x) = q_\xi(T,x) = 0, \quad x \in (0,L). \tag{4.8.55}$$

Now, we give the uniform estimate of functions p_ξ and q_ξ with respect to $\xi \in [-1,1] \setminus \{0\}$.

Multiplying (4.8.52) by p_ξ and using the Schwarz inequality, we obtain:

$$\begin{aligned} & -\frac{1}{2}\frac{d}{dt}|p_\xi(t)|_H^2 + |(p_\xi)_x(t)|_H^2 + n_0|p_\xi(t,0)|^2 + n_0|p_\xi(t,L)|^2 \\ \leq \; & |p_\xi(t)|_H^2 + \frac{1}{2}|q_\xi(t)|_H^2 + \frac{c_0^2}{2}|(u_\xi - u_d)(t)|_H^2, \quad \text{a.a. } t \in (0,T). \end{aligned} \tag{4.8.56}$$

Then, by (4.8.56) and the standard calculation (e.g., Gronwall inequality), we have

$$\begin{aligned} & |p_\xi(t)|_H^2 + \int_t^T |(p_\xi)_x(s)|_H^2 ds + n_0 \int_t^T |p_\xi(s,0)|^2 ds + n_0 \int_t^T |p_\xi(s,L)|^2 ds \\ \leq \; & N_{17} \left(\int_t^T |q_\xi(s)|_H^2 ds + c_0^2 \int_t^T |(u_\xi - u_d)(s)|_H^2 ds \right) \quad \text{for all } t \in [0,T], \end{aligned} \tag{4.8.57}$$

where N_{17} is a positive constant independent of $\xi \in [-1,1] \setminus \{0\}$.

Next, multiplying (4.8.52) by $-p'_\xi$, using the Schwarz inequality, and integrating in time, we obtain:

$$\begin{aligned} & \frac{1}{2}\int_t^T |p'_\xi(s)|_H^2 ds + \frac{1}{2}|(p_\xi)_x(t)|_H^2 + \frac{n_0}{2}|p_\xi(t,0)| + \frac{n_0}{2}|p_\xi(t,L)| \\ \leq \; & \int_t^T |q_\xi(s)|_H^2 ds + c_0^2 \int_t^T |(u_\xi - u_d)(s)|_H^2 ds \quad \text{for all } t \in [0,T]. \end{aligned} \tag{4.8.58}$$

In addition, taking $\zeta = q_\xi$ in (4.8.53), using (4.8.23)–(4.8.25), (4.8.58), and the Schwarz inequality, and integrating in time, we obtain:

$$\begin{aligned} & \frac{1}{2}|q_\xi(t)|_H^2 + \varepsilon\kappa \int_t^T |(q_\xi)_x(s)|_H^2 ds \\ \leq \; & (2 + C_g) \int_t^T |q_\xi(s)|_H^2 ds + c_0^2 \int_t^T |(u_\xi - u_d)(s)|_H^2 ds \\ & + \frac{c_1^2}{2} \int_t^T |(w_\xi - w_d)(s)|_H^2 ds \quad \text{for all } t \in [0,T]. \end{aligned} \tag{4.8.59}$$

On account of (4.8.15), (4.8.59), and the Gronwall inequality, we can get the following estimate:

$$\begin{aligned} & |q_\xi(t)|_H^2 + \varepsilon\kappa \int_t^T |(q_\xi)_x(s)|_H^2 ds \\ \leq \; & N_{18} \left(c_0^2 |u - u_d|_{L^2(0,T;H)}^2 + c_1^2 |w - w_d|_{L^2(0,T;H)}^2 + 1 \right) \quad \text{for all } t \in [0,T], \end{aligned} \tag{4.8.60}$$

where N_{18} is a positive constant independent of $\xi \in [-1,1] \setminus \{0\}$. Consequently, we infer from (4.8.57), (4.8.58), and (4.8.60) that

$$\begin{aligned} & |p_\xi(t)|_H^2 + \int_t^T |(p_\xi)_x(s)|_H^2 ds + n_0 \int_t^T |p_\xi(s,0)|^2 ds + n_0 \int_t^T |p_\xi(s,L)|^2 ds \\ & + \int_t^T |p'_\xi(s)|_H^2 ds + |(p_\xi)_x(t)|_H^2 + n_0|p_\xi(t,0)| + n_0|p_\xi(t,L)| \\ \leq \; & N_{19} \left(c_0^2 |u - u_d|_{L^2(0,T;H)}^2 + c_1^2 |w - w_d|_{L^2(0,T;H)}^2 + 1 \right) \quad \text{for all } t \in [0,T], \end{aligned} \tag{4.8.61}$$

where N_{19} is a positive constant independent of $\xi \in [-1,1] \setminus \{0\}$.

Additionally, we infer from (4.8.23), (4.8.24), and (4.8.53) that (cf. (4.8.27)):

$$\begin{aligned}
&\left| \int_0^T \langle -q'_\xi(\tau), \zeta(\tau) \rangle d\tau \right| \\
\leq\ & |p'_\xi|_{L^2(0,T;H)} |\zeta|_{L^2(0,T;H)} + \kappa \left(\frac{\delta_3}{\varepsilon} + \varepsilon \right) |(q_\xi)_x|_{L^2(0,T;H)} |\zeta_x|_{L^2(0,T;H)} \\
&+ \frac{1}{\varepsilon} |q_\xi|_{L^2(0,T;H)} |\zeta|_{L^2(0,T;H)} + |g'(w_\xi) q_\xi|_{L^2(0,T;H)} |\zeta|_{L^2(0,T;H)} \\
&+ c_1 |w_\xi - w_d|_{L^2(0,T;H)} |\zeta|_{L^2(0,T;H)} \qquad \text{for all } \zeta \in L^2(0,T;X),
\end{aligned}$$

which implies from (4.8.30), (4.8.60), and (4.8.61) that

$$|q'_\xi|_{L^2(0,T;X')} \leq \tilde{N}_{19,\varepsilon} \left(c_0^2 |u - u_d|^2_{L^2(0,T;H)} + c_1^2 |w - w_d|^2_{L^2(0,T;H)} + 1 \right), \tag{4.8.62}$$

where $\tilde{N}_{19,\varepsilon}$ is a positive constant dependent on ε and is independent of $\xi \in [-1,1] \setminus \{0\}$.

By the uniform estimates (4.8.60)–(4.8.62) of $[p_\xi, q_\xi]$, there are a subsequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \{\xi\}_{\xi \in [-1,1]\setminus\{0\}}$, and the functions $\overline{p} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$, $\overline{q} \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H)$ such that $\xi_n \to 0$,

$$\left.\begin{aligned}
p_{\xi_n} \to \overline{p} \qquad & \text{in } C([0,T];H), \\
& \text{weakly in } W^{1,2}(0,T;H), \\
& \text{weakly-}* \text{ in } L^\infty(0,T;X),
\end{aligned}\right\} \tag{4.8.63}$$

$$p_{\xi_n}(\cdot,0) \to \overline{p}(\cdot,0) \ \text{ weakly in } L^2(0,T), \tag{4.8.64}$$

$$p_{\xi_n}(\cdot,L) \to \overline{p}(\cdot,L) \ \text{ weakly in } L^2(0,T), \tag{4.8.65}$$

and

$$\left.\begin{aligned}
q_{\xi_n} \to \overline{q} \qquad & \text{in } C([0,T];X'), \\
& \text{in } L^2(0,T;H), \\
& \text{weakly in } W^{1,2}(0,T;X'), \\
& \text{weakly in } L^2(0,T;X), \\
& \text{weakly-}* \text{ in } L^\infty(0,T;H),
\end{aligned}\right\} \tag{4.8.66}$$

as $n \to \infty$.

Then, by (4.8.42), the uniqueness of the adjoint system $\{(4.6.3)–(4.6.6)\}$, and the similar argument in Lemma 4.3, we can show that

$$\begin{aligned}
& [p_\xi, q_\xi] = \Lambda^\varepsilon_{ad}(f + \xi\varpi_1, h + \xi\varpi_2, \ell + \xi\varpi_3) \\
\longrightarrow\ & [\overline{p}, \overline{q}] = [p, q] = \Lambda^\varepsilon_{ad}(f, h, \ell) \quad \text{ in } L^2(0,T;H) \times L^2(0,T;H) \text{ as } \xi \to 0.
\end{aligned}$$

Finally, we show (4.8.51). To this end, subtract (4.6.3) for $[p_\xi, q_\xi] = \Lambda^\varepsilon_{ad}(f + \xi\varpi_1, h + \xi\varpi_2, \ell + \xi\varpi_3)$ from the one for $[p, q] = \Lambda^\varepsilon_{ad}(f, h, \ell)$, and multiply it by $p_\xi - p$. Then, using the Schwarz inequality, we obtain:

$$\begin{aligned}
& -\frac{1}{2}\frac{d}{dt} |(p_\xi - p)(t)|^2_H + |(p_\xi - p)_x(t)|^2_H \\
& \qquad + n_0 |(p_\xi - p)(t,0)|^2 + n_0 |(p_\xi - p)(t,L)|^2 \\
\leq\ & |(p_\xi - p)(t)|^2_H + \frac{1}{2} |(q_\xi - q)(t)|^2_H + \frac{c_0^2}{2} |(u_\xi - u)(t)|^2_H, \\
& \qquad \text{for a.a. } t \in (0,T).
\end{aligned} \tag{4.8.67}$$

Then, by (4.8.67) and the standard calculation (e.g., Gronwall inequality), we have

$$
\begin{aligned}
&|(p_\xi - p)(t)|_H^2 + \int_t^T |(p_\xi - p)_x(s)|_H^2 ds \\
&\qquad +n_0 \int_t^T |(p_\xi - p)(s,0)|^2 ds + n_0 \int_t^T |(p_\xi - p)(s,L)|^2 ds \\
\le\ & N_{20} \left(\int_t^T |(q_\xi - q)(s)|_H^2 ds + c_0^2 \int_t^T |(u_\xi - u)(s)|_H^2 ds \right) \quad \text{for all } t \in [0,T],
\end{aligned}
\tag{4.8.68}
$$

where N_{20} is a positive constant independent of $\xi \in [-1,1] \setminus \{0\}$. Hence, we conclude from (4.8.15), (4.8.66), and (4.8.68) that (4.8.51) holds.

Thus, the proof of this lemma has been completed. □

Definition 4.6. We define the function $\gamma : [0,\infty) \to [0,\infty)$ by

$$
\gamma(t) := \inf \left\{ |[\xi\varpi_1, \xi\varpi_2, \xi\varpi_3]|_{\mathcal{U}} \ ; \ \left| {}^{\mathrm{t}}\begin{bmatrix} \xi\varpi_1 + a_0(p_\xi - p) \\ \xi\varpi_2 + a_1(p_\xi - p)(\cdot,0) \\ \xi\varpi_3 + a_2(p_\xi - p)(\cdot,L) \end{bmatrix} \right|_{\mathcal{U}} \ge t \right\}, \quad \text{for } t \ge 0,
\tag{4.8.69}
$$

where $[\varpi_1, \varpi_2, \varpi_3] \in \mathcal{U}$, $\xi \in \mathbb{R}$, the symbol ${}^{\mathrm{t}}\begin{bmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{bmatrix}$ means the transposed matrix of $\begin{bmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{bmatrix}$, namely, ${}^{\mathrm{t}}\begin{bmatrix} \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{bmatrix} = [\varpi_1, \varpi_2, \varpi_3]$, $|\cdot|_{\mathcal{U}}$ is the norm of $\mathcal{U}$ defined in (4.8.1), $[p_\xi, q_\xi] = \Lambda_{ad}^\varepsilon(f + \xi\varpi_1, h + \xi\varpi_2, \ell + \xi\varpi_3)$, and $[p,q] = \Lambda_{ad}^\varepsilon(f,h,\ell)$. Clearly, $\gamma(\cdot)$ is a well-defined increasing function with $\gamma(0) = 0$, because of the continuity of Λ_{ad}^ε and (4.8.51) (cf. Lemma 4.4).

Lemma 4.5. Assume the same conditions as in Theorem 4.8. Let $n \in \mathbb{N}$ be a fixed number, and let $\{[f_k, h_k, \ell_k]; k = 1, 2, \cdots, n\}$ be a sequence in $\mathcal{U}$ defined by the numerical algorithm (NA). Let $[p_n, q_n] = \Lambda_{ad}^\varepsilon(f_n, h_n, \ell_n)$, $\beta \in (0,1)$, and $\mu \in (0,1)$. Put

$$
d_{0n} := a_0(p_n + m_0 a_0 f_n), \quad d_{1n} := a_1(p_n(\cdot,0) + m_1 a_1 h_n), \quad d_{2n} := a_2(p_n(\cdot,L) + m_2 a_2 \ell_n).
$$

Assume that at least one of the following conditions is satisfied:

$$
d_{0n} \ne 0 \text{ in } L^2(0,T;H),\ d_{1n} \ne 0 \text{ in } L^2(0,T), \text{ or } d_{2n} \ne 0 \text{ in } L^2(0,T). \tag{4.8.70}
$$

Then, there is a minimal constant $\varsigma_n \in \mathbb{N} \cup \{0\}$ such that

$$
\begin{aligned}
& J^\varepsilon\left(f_n - \beta^{\varsigma_n} d_{0n}, h_n - \beta^{\varsigma_n} d_{1n}, \ell_n - \beta^{\varsigma_n} d_{2n}\right) - J^\varepsilon(f_n, h_n, \ell_n) \\
\le\ & -\mu \beta^{\varsigma_n} \left|[d_{0n}, d_{1n}, d_{2n}]\right|_{\mathcal{U}}^2 .
\end{aligned}
\tag{4.8.71}
$$

Proof. By assumption (4.8.70) and by the definition of the Gâteaux derivative of $J^\varepsilon(\cdot,\cdot,\cdot)$, there is a constant $\delta_{\mu,n} > 0$ such that

$$\left| \frac{J^\varepsilon(f_n - \lambda d_{0n}, h_n - \lambda d_{1n}, \ell_n - \lambda d_{2n}) - J^\varepsilon(f_n, h_n, \ell_n)}{\lambda} - D_{[-d_{0n},-d_{1n},-d_{2n}]} J^\varepsilon(f_n, h_n, \ell_n) \right|$$

$$< (1-\mu)|[d_{0n}, d_{1n}, d_{2n}]|^2_{\mathcal{U}} \quad \text{for any } \lambda \in (-\delta_{\mu,n}, \delta_{\mu,n}) \setminus \{0\}. \tag{4.8.72}$$

Put $[u_n, w_n] = \Lambda^\varepsilon[f_n, h_n, \ell_n]$ and $[\theta_n, \chi_n] = D_{[-d_{0n},-d_{1n},-d_{2n}]}\Lambda^\varepsilon(f_n, h_n, \ell_n)$. Then, by the proof of Theorem 4.6, we observe that

$$\begin{aligned}
& D_{[-d_{0n},-d_{1n},-d_{2n}]} J^\varepsilon(f_n, h_n, \ell_n) \\
= \;& c_0 \int_0^T ((u_n - u_d)(t), \theta_n(t))_H dt + c_1 \int_0^T ((w_n - w_d)(t), \chi_n(t))_H dt \\
& + m_0 a_0^2 \int_0^T (f_n(t), -d_{0n}(t))_H dt + m_1 a_1^2 \int_0^T h_n(t)(-d_{1n}(t)) dt \\
& + m_2 a_2^2 \int_0^T \ell_n(t)(-d_{2n}(t)) dt \\
= \;& \int_0^T (a_0 p_n(t) + m_0 a_0^2 f_n(t), -d_{0n}(t))_H dt \\
& \quad + \int_0^T (a_1 p_n(t,0) + m_1 a_1^2 h_n(t))(-d_{1n}(t)) dt \\
& \quad + \int_0^T (a_2 p_n(t,L) + m_2 a_2^2 \ell_n(t))(-d_{2n}(t)) dt \\
= \;& -|[d_{0n}, d_{1n}, d_{2n}]|^2_{\mathcal{U}}.
\end{aligned} \tag{4.8.73}$$

Therefore, we observe from (4.8.72) that

$$J^\varepsilon(f_n - \lambda d_{0n}, h_n - \lambda d_{1n}, \ell_n - \lambda d_{2n}) - J^\varepsilon(f_n, h_n, \ell_n) \le -\lambda\mu|[d_{0n}, d_{1n}, d_{2n}]|^2_{\mathcal{U}}$$

for any $\lambda \in (0, \delta_{\mu,n})$. Therefore, we have only to take a minimal constant $\varsigma_n \in \mathbb{N} \cup \{0\}$ such that

$$0 < \beta^{\varsigma_n} < \delta_{\mu,n}.$$

Thus, the proof of this lemma has been completed. □

Lemma 4.6. Assume the same conditions as in Theorem 4.8. Let $n \in \mathbb{N}$ be a fixed number, and let $\{[f_k, h_k, \ell_k]; k = 1, 2, \cdots, n\}$ be a sequence in $\mathcal{U}$ defined by the numerical algorithm (NA). Let $[p_n, q_n] = \Lambda^\varepsilon_{ad}(f_n, h_n, \ell_n)$, $\beta \in (0,1)$, and $\mu \in (0,1)$. Put

$$d_{0n} := a_0(p_n + m_0 a_0 f_n), d_{1n} := a_1(p_n(\cdot, 0) + m_1 a_1 h_n), d_{2n} := a_2(p_n(\cdot, L) + m_2 a_2 \ell_n). \tag{4.8.74}$$

Assume that at least one of the following conditions is satisfied:

$$d_{0n} \neq 0 \text{ in } L^2(0,T;H),\ d_{1n} \neq 0 \text{ in } L^2(0,T), \text{ or } d_{2n} \neq 0 \text{ in } L^2(0,T).$$

Let ς_n be the constant obtained in Lemma 4.5, and put

$$M_{max} := \max\{m_0 a_0^2,\ m_1 a_1^2,\ m_2 a_2^2\}. \tag{4.8.75}$$

Then, we have

$$\beta\gamma\left((1-\mu)|[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}}\right) \leq \beta^{\varsigma_n} M_{max}|[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}}, \tag{4.8.76}$$

where $\gamma(\cdot)$ is the function defined by (4.8.69) in Definition 4.6.

Proof. From the definition of ς_n obtained in Lemma 4.5, we observe that

$$\begin{aligned} & J^\varepsilon\left(f_n - \frac{\beta^{\varsigma_n}}{\beta}d_{0n}, h_n - \frac{\beta^{\varsigma_n}}{\beta}d_{1n}, \ell_n - \frac{\beta^{\varsigma_n}}{\beta}d_{2n}\right) - J^\varepsilon(f_n, h_n, \ell_n) \\ & > -\mu\frac{\beta^{\varsigma_n}}{\beta}\left|[d_{0n}, d_{1n}, d_{2n}]\right|_{\mathcal{U}}^2. \end{aligned} \tag{4.8.77}$$

Here, by (4.8.73), the mean-valued theorem, and the continuity of $D_{[-d_{0n},-d_{1n},-d_{2n}]}J^\varepsilon(f_n+\xi\varpi_1, h_n+\xi\varpi_2, \ell_n+\xi\varpi_3)$ with respect to ξ, there is a constant $\vartheta \in (0,1)$ satisfying

$$\begin{aligned} & J^\varepsilon\left(f_n - \frac{\beta^{\varsigma_n}}{\beta}d_{0n}, h_n - \frac{\beta^{\varsigma_n}}{\beta}d_{1n}, \ell_n - \frac{\beta^{\varsigma_n}}{\beta}d_{2n}\right) - J^\varepsilon(f_n, h_n, \ell_n) \\ = \ & \int_0^{\frac{\beta^{\varsigma_n}}{\beta}} \frac{d}{d\xi}J^\varepsilon\left(f_n - \xi d_{0n}, h_n - \xi d_{1n}, \ell_n - \xi d_{2n}\right)d\xi \\ = \ & \frac{\beta^{\varsigma_n}}{\beta}D_{[-d_{0n},-d_{1n},-d_{2n}]}J^\varepsilon\left(f_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{0n}, h_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{1n}, \ell_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{2n}\right) \\ = \ & \frac{\beta^{\varsigma_n}}{\beta}\left[\int_0^T \left(a_0 p_{n,\vartheta}(t) + m_0 a_0^2\big(f_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{0n}(t)\big), -d_{0n}(t)\right)_H dt \right. \\ & \quad + \int_0^T \Big(a_1 p_{n,\vartheta}(t,0) + m_1 a_1^2(h_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{1n}(t))\Big)(-d_{1n}(t))dt \\ & \quad \left. + \int_0^T \Big(a_2 p_{n,\vartheta}(t,L) + m_2 a_2^2(\ell_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{2n}(t))\Big)(-d_{2n}(t))dt\right], \end{aligned} \tag{4.8.78}$$

where $[p_{n,\vartheta}, q_{n,\vartheta}] = \Lambda_{ad}^\varepsilon\left(f_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{0n}, h_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{1n}, \ell_n - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{2n}\right)$.

It follows from (4.8.77) and (4.8.78) that

$$\begin{aligned} & (1-\mu)\left|[d_{0n}, d_{1n}, d_{2n}]\right|_{\mathcal{U}}^2 \\ \leq \ & |[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}}^2 + \int_0^T \left(a_0 p_{n,\vartheta}(t) + m_0 a_0^2\big(f_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{0n}(t)\big), -d_{0n}(t)\right)_H dt \\ & + \int_0^T \Big(a_1 p_{n,\vartheta}(t,0) + m_1 a_1^2(h_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{1n}(t))\Big)(-d_{1n}(t))dt \\ & + \int_0^T \Big(a_2 p_{n,\vartheta}(t,L) + m_2 a_2^2(\ell_n(t) - \vartheta\frac{\beta^{\varsigma_n}}{\beta}d_{2n}(t))\Big)(-d_{2n}(t))dt \end{aligned}$$

$$= \int_0^T \left(a_0 p_{n,\vartheta}(t) + m_0 a_0^2\big(f_n(t) - \vartheta \frac{\beta^{\varsigma_n}}{\beta} d_{0n}(t)\big) - d_{0n}(t), -d_{0n}(t) \right)_H dt$$
$$+ \int_0^T \Big(a_1 p_{n,\vartheta}(t,0) + m_1 a_1^2 (h_n(t) - \vartheta \frac{\beta^{\varsigma_n}}{\beta} d_{1n}(t)) - d_{1n}(t) \Big)(-d_{1n}(t))dt$$
$$+ \int_0^T \Big(a_2 p_{n,\vartheta}(t,L) + m_2 a_2^2 (\ell_n(t) - \vartheta \frac{\beta^{\varsigma_n}}{\beta} d_{2n}(t)) - d_{2n}(t) \Big)(-d_{2n}(t))dt.$$

Hence, we infer from the Hölder inequality and (4.8.74) that

$$\begin{aligned}(1-\mu)\big|[d_{0n}, d_{1n}, d_{2n}]\big|_{\mathcal{U}} &\le \left| {}^{\mathrm{t}}\begin{bmatrix} a_0 p_{n,\vartheta} + m_0 a_0^2\big(f_n - \vartheta \frac{\beta^{\varsigma_n}}{\beta} d_{0n}\big) - d_{0n} \\ a_1 p_{n,\vartheta}(\cdot,0) + m_1 a_1^2\big(h_n - \vartheta \frac{\beta^{\varsigma_n}}{\beta} d_{1n}\big) - d_{1n} \\ a_2 p_{n,\vartheta}(\cdot,L) + m_2 a_2^2\big(\ell_n - \vartheta \frac{\beta^{\varsigma_n}}{\beta} d_{2n}\big) - d_{2n} \end{bmatrix} \right|_{\mathcal{U}} \\ &= \left| {}^{\mathrm{t}}\begin{bmatrix} -\vartheta \frac{\beta^{\varsigma_n}}{\beta} m_0 a_0^2 d_{0n} + a_0(p_{n,\vartheta} - p_n) \\ -\vartheta \frac{\beta^{\varsigma_n}}{\beta} m_1 a_1^2 d_{1n} + a_1(p_{n,\vartheta}(\cdot,0) - p_n(\cdot,0)) \\ -\vartheta \frac{\beta^{\varsigma_n}}{\beta} m_2 a_2^2 d_{2n} + a_2(p_{n,\vartheta}(\cdot,L) - p_n(\cdot,L)) \end{bmatrix} \right|_{\mathcal{U}}.\end{aligned} \tag{4.8.79}$$

By the definition of the function γ, we observe from (4.8.79) and $\theta \in (0,1)$ that

$$\begin{aligned}\gamma\left((1-\mu)\big|[d_{0n}, d_{1n}, d_{2n}]\big|_{\mathcal{U}}\right) &\le \vartheta \frac{\beta^{\varsigma_n}}{\beta} \big|\big[m_0 a_0^2 d_{0n}, m_1 a_1^2 d_{1n}, m_2 a_2^2 d_{2n}\big]\big|_{\mathcal{U}}, \\ &\le \frac{\beta^{\varsigma_n}}{\beta} M_{max} |[d_{0n}, d_{1n}, d_{2n}]|_{\mathcal{U}},\end{aligned}$$

which implies that the inequality (4.8.76) holds. □

Now, we show our main Theorem 4.8 in this paper, which is concerned with the convergence for numerical algorithm (NA).

Proof of Theorem 4.8. We show (I). By (Step 5) in the numerical algorithm (NA) (cf. (4.8.71) or (4.8.82) below), we easily observe that $J^\varepsilon(f_n, h_n, \ell_n)$ is the non-increasing sequence with respect to n. Thus, from the non-negativity of $J^\varepsilon(\cdot,\cdot,\cdot)$ (cf. (1.5.26)), we infer that $\lim_{n\to\infty} J^\varepsilon(f_n, h_n, \ell_n)$ exists.

We next show (II). Indeed, we first prove (4.8.2) by contradiction. To this end, we assume that (4.8.2) does not hold. Then, there exist a constant $\delta > 0$ and a sequence $\{k\}_{k\in\mathbb{N}}$ such that

$$|a_0(p_k + m_0 a_0 f_k)|_{L^2(0,T;H)} \ge \delta \quad \text{for any } k. \tag{4.8.80}$$

Since $\gamma(\cdot)$ is the increasing function (cf. Definition 4.6), it follows from (4.8.74) and (4.8.80) that

$$\gamma\left((1-\mu)\delta\right) \le \gamma\left((1-\mu)\big|[d_{0k}, d_{1k}, d_{2k}]\big|_{\mathcal{U}}\right) \quad \text{for any } k. \tag{4.8.81}$$

Then, we observe from (4.8.71), (4.8.76), (4.8.80), and (Step 5) that

$$\begin{aligned}
& J^\varepsilon(f_{k+1},h_{k+1},\ell_{k+1}) - J^\varepsilon(f_k,h_k,\ell_k) \\
= \ & J^\varepsilon\left(f_k - \beta^{\varsigma_k} d_{0k}, h_k - \beta^{\varsigma_k} d_{1k}, \ell_k - \beta^{\varsigma_k} d_{2k}\right) - J^\varepsilon(f_k,h_k,\ell_k) \\
\leq \ & -\mu\beta^{\varsigma_k}\left|[d_{0k},d_{1k},d_{2k}]\right|_{\mathcal{U}}^2 \\
\leq \ & -\frac{\mu\beta}{M_{max}}\gamma\left((1-\mu)\left|[d_{0k},d_{1k},d_{2k}]\right|_{\mathcal{U}}\right)\left|[d_{0k},d_{1k},d_{2k}]\right|_{\mathcal{U}} \\
\leq \ & -\frac{\mu\beta}{M_{max}}\gamma\left((1-\mu)\delta\right)\delta \\
< \ & 0 \qquad\qquad \text{for any } k \in \mathbb{N}.
\end{aligned} \tag{4.8.82}$$

By repeating this procedure, we observe from (4.8.82) that

$$\begin{aligned}
J^\varepsilon(f_{k+1},h_{k+1},\ell_{k+1}) & \leq J^\varepsilon(f_k,h_k,\ell_k) - \frac{\mu\beta}{M_{max}}\gamma\left((1-\mu)\delta\right)\delta \\
& \leq J^\varepsilon(f_{k-1},h_{k-1},\ell_{k-1}) - \frac{2\mu\beta}{M_{max}}\gamma\left((1-\mu)\delta\right)\delta \\
& \leq \cdots \\
& \leq J^\varepsilon(f_1,h_1,\ell_1) - \frac{k\mu\beta}{M_{max}}\gamma\left((1-\mu)\delta\right)\delta \quad \text{ for any } k \in \mathbb{N}.
\end{aligned}$$

Therefore, the above inequality implies that

$$J^\varepsilon(f_{k+1},h_{k+1},\ell_{k+1}) \longrightarrow -\infty \quad \text{as } \ k \to \infty. \tag{4.8.83}$$

This contradicts the non-negativity of $J^\varepsilon(\cdot,\cdot,\cdot)$ (cf. (1.5.26)). Hence, (4.8.2) holds.

Similarly, we can show (4.8.3) and (4.8.4), thus, (II) holds.

Now, we show (III). By Theorem 4.8(I) and the definition of $J^\varepsilon(\cdot,\cdot,\cdot)$ (cf. (1.5.26)), we observe that $\{[f_n,h_n,\ell_n]\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{U}$. Therefore, there exist a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\{n\}_{n\in\mathbb{N}}$ and a triplet of functions $[f^\varepsilon_{**},h^\varepsilon_{**},\ell^\varepsilon_{**}] \in \mathcal{U}$ such that $n_k \to \infty$ and

$$\left.\begin{aligned}
f_{n_k} \longrightarrow f^\varepsilon_{**} \ \text{ weakly in } L^2(0,T;H), \\
h_{n_k} \longrightarrow h^\varepsilon_{**} \ \text{ weakly in } L^2(0,T), \\
\ell_{n_k} \longrightarrow \ell^\varepsilon_{**} \ \text{ weakly in } L^2(0,T)
\end{aligned}\right\} \tag{4.8.84}$$

as $k \to \infty$. Then, from Corollary 4.4 concerning the convergence result of solutions to $(\mathrm{P};u^\varepsilon_0,w^\varepsilon_0,f_{n_k},h_{n_k},\ell_{n_k})^\varepsilon$, we observe that

$$\begin{aligned}
[u_{n_k},w_{n_k}] = \Lambda^\varepsilon(f_{n_k},h_{n_k},\ell_{n_k}) \longrightarrow [u^\varepsilon_{**},w^\varepsilon_{**}] = \Lambda^\varepsilon(f^\varepsilon_{**},h^\varepsilon_{**},\ell^\varepsilon_{**}) \\
\text{in } L^2(0,T;H) \times C([0,T];H) \ \text{ as } k \to \infty.
\end{aligned} \tag{4.8.85}$$

In addition, by (4.8.85) and the slight modification of the proof of Lemma 4.4, there are functions $p^\varepsilon_{**} \in W^{1,2}(0,T;H) \cap L^\infty(0,T;X)$, $q^\varepsilon_{**} \in W^{1,2}(0,T;X') \cap L^2(0,T;X) \cap L^\infty(0,T;H)$, and a subsequence of $\{n_k\}_{k\in\mathbb{N}}$ (which we also denote $\{n_k\}_{k\in\mathbb{N}}$ for simplicity) such that $[p_{n_k},q_{n_k}] = \Lambda^\varepsilon_{ad}(f_{n_k},h_{n_k},\ell_{n_k})$ converges to $[p^\varepsilon_{**},q^\varepsilon_{**}] = \Lambda^\varepsilon_{ad}(f^\varepsilon_{**},h^\varepsilon_{**},\ell^\varepsilon_{**})$ in the following sense:

$$\left.\begin{aligned}
p_{n_k} \to p^\varepsilon_{**} \qquad & \text{in } C([0,T];H), \\
& \text{weakly in } W^{1,2}(0,T;H), \\
& \text{weakly-}* \text{ in } L^\infty(0,T;X),
\end{aligned}\right\} \tag{4.8.86}$$

$$p_{n_k}(\cdot,0) \to p^\varepsilon_{**}(\cdot,0) \ \text{ weakly in } L^2(0,T), \tag{4.8.87}$$

$$p_{n_k}(\cdot,L) \to p^\varepsilon_{**}(\cdot,L) \ \text{ weakly in } L^2(0,T), \tag{4.8.88}$$

and

$$\left. q_{n_k} \to q^\varepsilon_{**} \quad \begin{array}{l} \text{in } C([0,T];X'), \\ \text{in } L^2(0,T;H), \\ \text{weakly in } W^{1,2}(0,T;X'), \\ \text{weakly in } L^2(0,T;X), \\ \text{weakly-}* \text{ in } L^\infty(0,T;H), \end{array} \right\} \tag{4.8.89}$$

as $k \to \infty$. In addition, from arguments similar to (4.8.51), we observe that

$$p_{n_k} \to p^\varepsilon_{**} \ \text{ in } L^2(0,T;X) \ \text{ as } k \to \infty, \tag{4.8.90}$$

hence, in particular,

$$p_{n_k}(\cdot,0) \to p^\varepsilon_{**}(\cdot,0) \ \text{ in } L^2(0,T), \quad p_{n_k}(\cdot,L) \to p^\varepsilon_{**}(\cdot,L) \ \text{ in } L^2(0,T) \ \text{ as } k \to \infty. \tag{4.8.91}$$

Therefore, we infer from (4.8.2), (4.8.3), (4.8.4), (4.8.84), (4.8.86), (4.8.90), and (4.8.91) that the assertions (4.8.5)–(4.8.12) hold. In addition, we conclude from Theorem 4.6 and (4.8.10)–(4.8.12) (cf. (4.8.73)) that (4.8.13) holds, hence, $[f^\varepsilon_{**}, h^\varepsilon_{**}, \ell^\varepsilon_{**}] \in \mathcal{U}$ is the stationary point of the cost functional J^ε with $\varepsilon > 0$. Thus, the proof of Theorem 4.8 has been completed. □

4.9 Numerical experiments

In this section, by similar approach as in [65, 66, 68] we perform the simple numerical experiments to $(\mathrm{OP})^\varepsilon$ with some small $\varepsilon > 0$.

4.9.1 State system and its optimal control problem

For the stability of numerics and the propagation speed of the interfaces, we now rescale (t,x) by the small parameter $\sigma > 0$. Indeed, we change the pair of variables (t,x) into $(s,y) := (\sigma t, \sigma x)$. Then, from the formal calculations we observe that (1.5.11) and (1.5.12) are reformulated as follows, respectively:

$$[u+w]_s - \sigma u_{yy} = \frac{a_0 \widetilde{f}(s,y)}{\sigma} \quad \text{in } (s,y) \in Q_\sigma := (0,\sigma T) \times (0,\sigma L), \tag{4.9.1}$$

$$w_s - \kappa \left(\frac{w_y}{|w_y|} \right)_y + \frac{\partial I_{[-1,1]}(w)}{\sigma} + \frac{g(w)}{\sigma} \ni \frac{u}{\sigma} \quad \text{in } Q_\sigma, \tag{4.9.2}$$

where we put $\widetilde{f}(s,y) := f(s/\sigma, y/\sigma)$ for $(s,y) \in Q_\sigma$, for simplicity.

Ohtsuka [65] and Ohtsuka–Shirakawa–Yamazaki [66,68] gave numerical experiments of optimal control problem for the approximate Allen–Cahn type equation associated with total variation energy, in which the singular diffusion term $\left(\frac{w_y}{|w_y|}\right)_y$ was approximated by $\left(\frac{w^\varepsilon_y}{\sqrt{|w^\varepsilon_y|^2+\varepsilon^2}}\right)_y$ for $\varepsilon > 0$.

In this section, by similar approach as in [65, 66, 68] we perform the simple numerical experiments to $(\mathrm{OP})^\varepsilon$ with some small $\varepsilon > 0$. Indeed, we consider a distributed control problem with the heat source as control, more precisely, $(\mathrm{OP})^\varepsilon$ in the case when $a^\varepsilon(r) = \frac{r}{\sqrt{|r|^2+\varepsilon^2}}$, $g(r) = r^3 - r$, and $a_1 = a_2 = b_1 = b_2 = c_0 = 0$.

Now, for the fixed rescale parameter $\sigma \in (0, 1]$, we take $T = \widetilde{T}/\sigma$ and $L = \widetilde{L}/\sigma$ for some positive constants $\widetilde{T}$ and $\widetilde{L}$. Then, we give numerical experiments of the optimal control problem for the following state system that is the approximate problem of (4.9.1) and (4.9.2):

Problem (P;$\widetilde{f}, 0, 0)^\varepsilon$.

$$[u^\varepsilon + w^\varepsilon]_s - \sigma u^\varepsilon_{yy} = \frac{a_0 \widetilde{f}(s, y)}{\sigma} \quad \text{in } (s, y) \in \widetilde{Q}_\sigma := (0, \widetilde{T}) \times (0, \widetilde{L}), \tag{4.9.3}$$

$$w^\varepsilon_s - \kappa \left(\frac{w^\varepsilon_y}{\sqrt{|w^\varepsilon_y|^2 + \varepsilon^2}} + \varepsilon w^\varepsilon_y \right)_y + \frac{K^\varepsilon(w^\varepsilon)}{\sigma} + \frac{(w^\varepsilon)^3 - w^\varepsilon}{\sigma} = \frac{u^\varepsilon}{\sigma} \quad \text{in } \widetilde{Q}_\sigma, \tag{4.9.4}$$

$$-u^\varepsilon_y(s, 0) + u^\varepsilon(s, 0) = u^\varepsilon_y(s, \widetilde{L}) + u^\varepsilon(s, \widetilde{L}) = 0, \quad s \in (0, \widetilde{T}), \tag{4.9.5}$$

$$w^\varepsilon_y(s, 0) = w^\varepsilon_y(s, \widetilde{L}) = 0, \quad s \in (0, \widetilde{T}), \tag{4.9.6}$$

$$u^\varepsilon(0, y) = u^\varepsilon_0(y), \quad w^\varepsilon(0, y) = w^\varepsilon_0(y), \quad y \in (0, \widetilde{L}). \tag{4.9.7}$$

In addition, for simplicity, we consider the following distributed control problem with the heat source as control:

Problem $(\mathrm{OP})^\varepsilon$: Find a control function $\widetilde{f}^\varepsilon_* \in L^2(0, \widetilde{T}; H)$, call *optimal control*, such that

$$J^\varepsilon(\widetilde{f}^\varepsilon_*) = \inf_{\widetilde{f} \in L^2(0, \widetilde{T}; H)} J^\varepsilon(\widetilde{f}). \tag{4.9.8}$$

Here, $J^\varepsilon(\widetilde{f})$ is the cost functional defined by

$$J^\varepsilon(\widetilde{f}) := \frac{c_1}{2} \int_0^{\widetilde{T}} |(w^\varepsilon - w_d)(s)|^2_H ds + \frac{m_0}{2} \int_0^{\widetilde{T}} |\widetilde{f}(s)|^2_H ds, \tag{4.9.9}$$

where c_1, m_0 are nonnegative constants, w_d is the given desired target profile in $L^2(0, \widetilde{T}; H)$, and a couple of functions $[u^\varepsilon, w^\varepsilon]$ is a unique solution to the initial–boundary value state problem (P;$\widetilde{f}, 0, 0)^\varepsilon$ for the control parameter $\widetilde{f} \in L^2(0, \widetilde{T}; H)$.

Note that the rescale parameter σ appears in (4.9.3) and (4.9.4). However, σ is a fixed positive constant. Hence, by the slight modification of the proof of Theorems 4.1, 4.3, 4.5, 4.6, 4.7, and 4.8, we can prove the solvability of state systems $\{(4.9.3)–(4.9.7)\}$ for any $\varepsilon \in (0, 1]$, the existence of optimal controls to (4.9.8), and so on.

4.9.2 Discretization

We perform the numerical experiments of (P;$\widetilde{f}, 0, 0)^\varepsilon$ and $(\mathrm{OP})^\varepsilon$ via the standard explicit finite difference scheme. Indeed, let Δt and Δh be the mesh size of time and space, respectively, and set $w_{n,j} := w^\varepsilon(n\Delta t, j\Delta h)$ and $D_y^\pm w_{n,j} := \pm(w_{n,j\pm 1} - w_{n,j})/\Delta h$. Then, the diffusion term in (4.9.4) is discretized by the following:

$$\mathcal{D}w_{n,j} := \frac{1}{\Delta h}\left[\kappa\left(\frac{D_y^+ w_{n,j}}{\sqrt{|D_y^+ w_{n,j}|^2 + \varepsilon^2}} - \frac{D_y^- w_{n,j}}{\sqrt{|D_y^- w_{n,j}|^2 + \varepsilon^2}} + \varepsilon(D_y^+ w_{n,j} - D_y^- w_{n,j})\right)\right].$$

Other terms are discretized by the standard forms. For instance, we refer to the explicit finite difference scheme used in [65].

4.9.3 Numerical experiments

In this subsection, we give three numerical experiments of $(\mathrm{OP})^\varepsilon$ with sufficient small parameter ε under the following numerical data:

<u>**Numerical data**</u>

- $\sigma = 0.001$, the domain $\widetilde{Q}_\sigma = (0, \widetilde{T}) \times (0, \widetilde{L})$ with $\widetilde{T} = 0.0025$ and $\widetilde{L} = 1.0$, the space mesh size $\Delta h = 0.005$, the time mesh size $\Delta t = 0.1 \times \Delta h^2 = 0.0000025$, $\kappa = 0.001$, $c_1 = 10.0$, $m_0 = 1.0$, $\varepsilon = 0.001$, the stop parameter $\mu = 0.0001$ for (NA), and the given initial data $[u_0^\varepsilon, w_0^\varepsilon] \equiv [0.0, 0.0]$. In addition, we take $f_0 \equiv 0.0$ as the initial control function for (NA).

(Numerical experiment 1)

In the first experiment, we consider a simple target desired profile w_d such that

$$w_d(s,y) := \begin{cases} 1, & \text{if } y \in [0.30, 0.70], \\ -1, & \text{otherwise}, \end{cases} \qquad \forall s \in [0, \widetilde{T}], \tag{4.9.10}$$

whose graph is the dotted line in Figure 4.1.

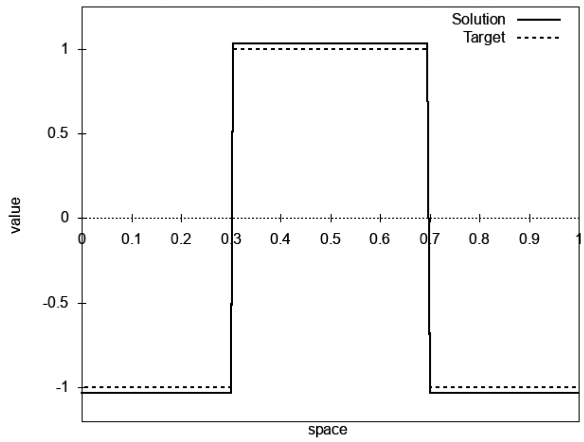

Figure 4.1: Target profile $w_d(\widetilde{T}, y)$ and solution $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 7$.

We perform a numerical experiment of $(\mathrm{OP})^\varepsilon$ by using the numerical algorithm (NA) proposed in Section 8. Then, (NA) is finished when the iteration number is $n = 7$ as in Figure 4.2.

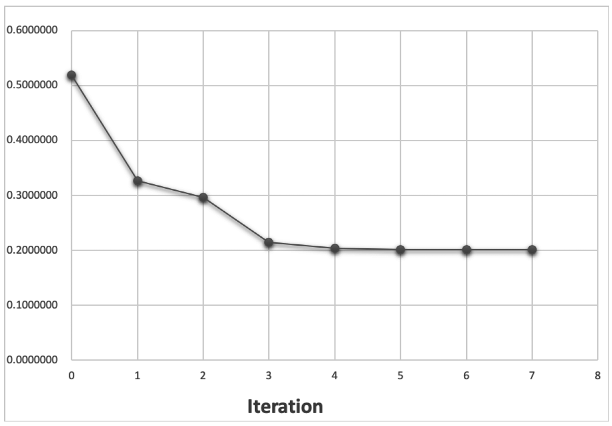

Figure 4.2: The value of the cost functional J^ε for $(\mathrm{OP})^\varepsilon$.

Figure 4.3 is the graph of the control function $\widetilde{f}$ found by (NA) in the case of the iteration number $n = 7$.

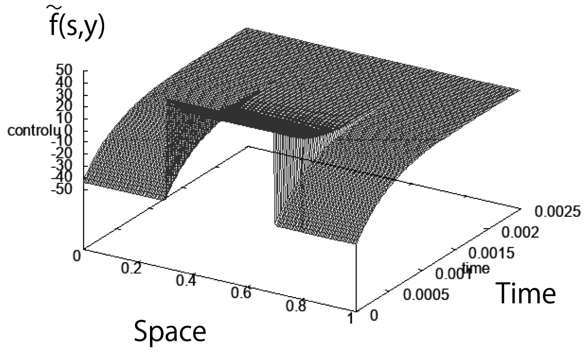

Figure 4.3: The graph of the control function $\widetilde{f}$ found by (NA) at the iteration number $n = 7$.

Figure 4.4 is the picture of the solutions u^ε and w^ε for $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ with initial data $[u_0^\varepsilon, w_0^\varepsilon] \equiv [0.0, 0.0]$ in the case of the iteration number $n = 7$.

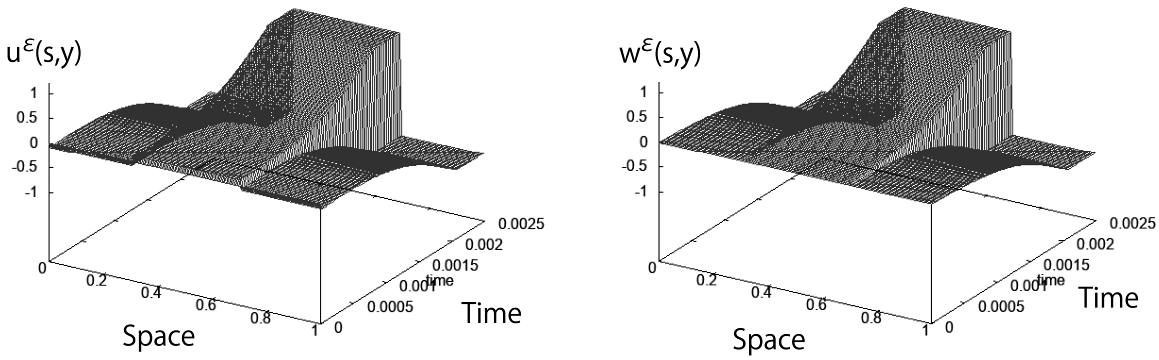

Figure 4.4: The graph of solution $[u^\varepsilon, w^\varepsilon]$ to $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ at the iteration number $n = 7$: (left) $u^\varepsilon(s,y)$; (right) $w^\varepsilon(s,y)$.

In addition, the real line in Figure 4.1 means the graph of $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 7$. We observe from Figures 4.1–4.4 that the solution $w^\varepsilon(\widetilde{T}, y)$ to $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ has the similar profile to the desired one $w_d(\widetilde{T}, y)$ and the data sequence of cost functional J^ε almost reaches a stationary point.

(Numerical experiment 2)

In the second experiment, we consider a target desired profile w_d such that

$$w_d(s,y) := \begin{cases} 1, & \text{if } y \in [0.00, 0.35], \\ 0, & \text{if } y \in (0.35, 0.70], \\ -1, & \text{if } y \in (0.70, 1.00], \end{cases} \quad \forall s \in [0, \widetilde{T}], \tag{4.9.11}$$

whose graph is the dotted line in Figure 4.5.

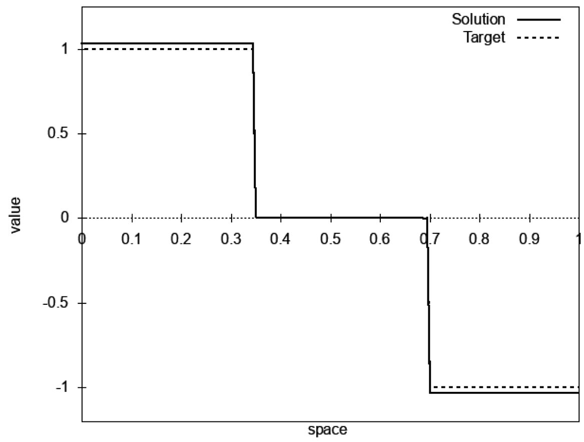

Figure 4.5: Target profile $w_d(\widetilde{T}, y)$ and solution $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 15$.

We perform a numerical experiment of $(\mathrm{OP})^\varepsilon$ by using the numerical algorithm (NA) proposed in Section 8. Then, (NA) is finished when the iteration number is $n = 15$ as in Figure 4.6.

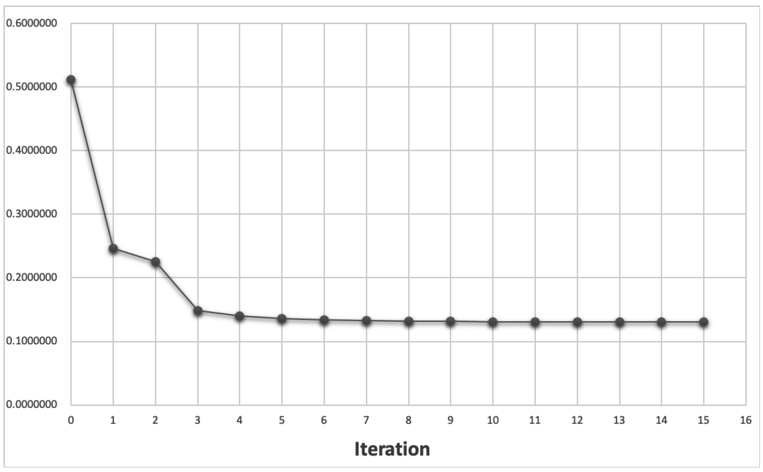

Figure 4.6: The value of the cost functional J^ε for $(\mathrm{OP})^\varepsilon$.

Figure 4.7 is the graph of the control function $\widetilde{f}$ found by (NA) in the case of the iteration number $n = 15$.

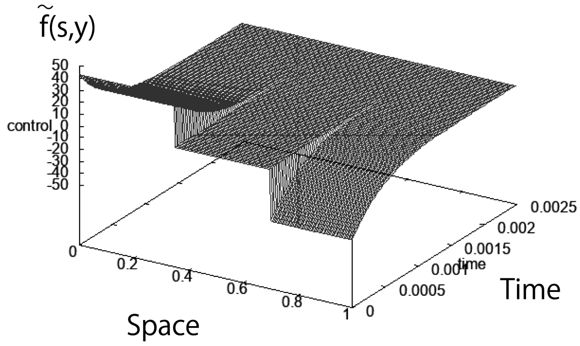

Figure 4.7: The graph of the control function $\widetilde{f}$ found by (NA) at the iteration number $n = 15$.

Figure 4.8 is the picture of the solutions u^ε and w^ε for $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ with initial data $[u_0^\varepsilon, w_0^\varepsilon] \equiv [0.0, 0.0]$ in the case of the iteration number $n = 15$.

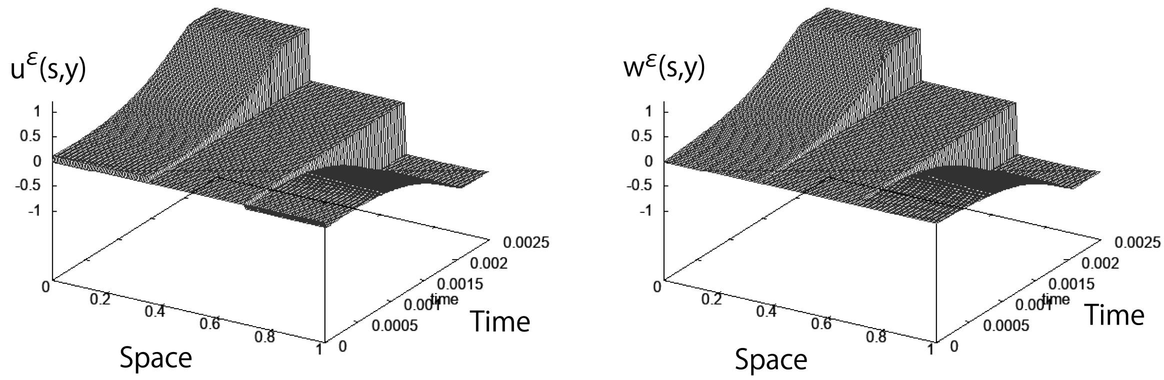

Figure 4.8: The graph of solution $[u^\varepsilon, w^\varepsilon]$ to $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ at the iteration number $n = 15$: (left) $u^\varepsilon(s,y)$; (right) $w^\varepsilon(s,y)$.

In addition, the real line in Figure 4.5 means the graph of $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 15$. We observe from Figures 4.5–4.8 that the solution $w^\varepsilon(\widetilde{T}, y)$ to $(\mathrm{P};\widetilde{f},0,0)^\varepsilon$ has the similar profile to the desired one $w_d(\widetilde{T}, y)$ and the data sequence of cost functional J^ε almost reaches a stationary point.

(Numerical experiment 3)

In the final experiment, we consider a target desired profile w_d such that

$$w_d(s, y) := \cos(2\pi y), \quad y \in \overline{\Omega} = [0.0, 1.0], \ \forall s \in [0, \widetilde{T}], \tag{4.9.12}$$

whose graph is the dotted line in Figure 4.9.

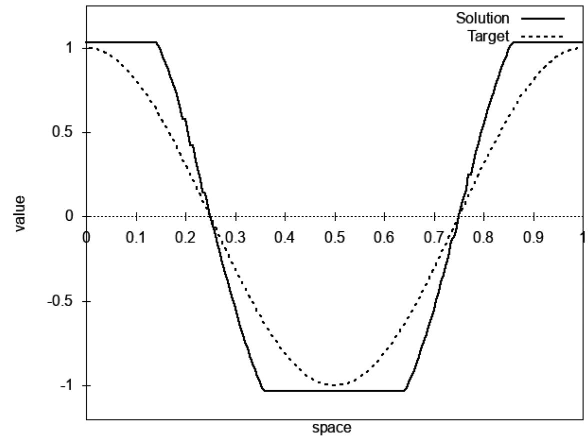

Figure 4.9: Target profile $w_d(\widetilde{T}, y)$ and solution $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 17$.

Here we take the stop parameter $\mu = 0.00025$ for (NA). Then, we perform a numerical experiment of $(\mathrm{OP})^\varepsilon$ by using the numerical algorithm (NA) proposed in Section 8. Then, (NA) is finished when the iteration number is $n = 17$ as in Figure 4.10.

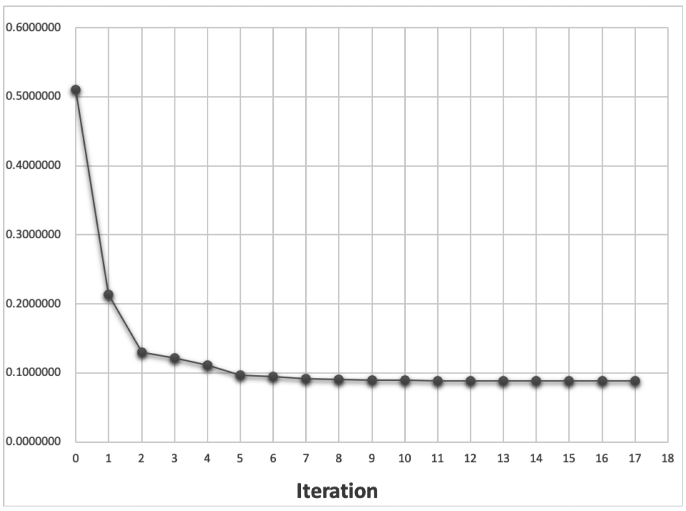

Figure 4.10: The value of the cost functional J^ε for $(\mathrm{OP})^\varepsilon$.

Figure 4.11 is the graph of the control function $\widetilde{f}$ found by (NA) in the case of the iteration number $n = 17$.

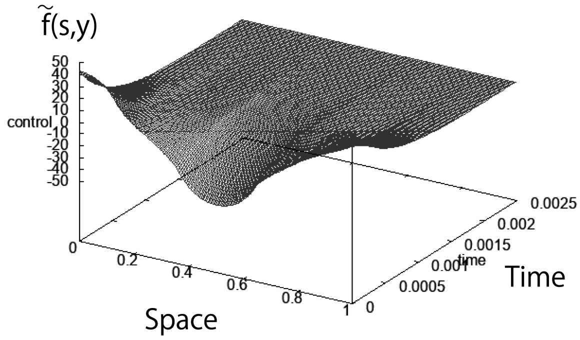

Figure 4.11: The graph of the control function $\widetilde{f}$ found by (NA) at the iteration number $n = 17$.

Figure 4.12 is the picture of the solutions u^ε and w^ε for (P;$\widetilde{f}, 0, 0)^\varepsilon$ with initial data $[u_0^\varepsilon, w_0^\varepsilon] \equiv [0.0, 0.0]$ in the case of the iteration number $n = 17$.

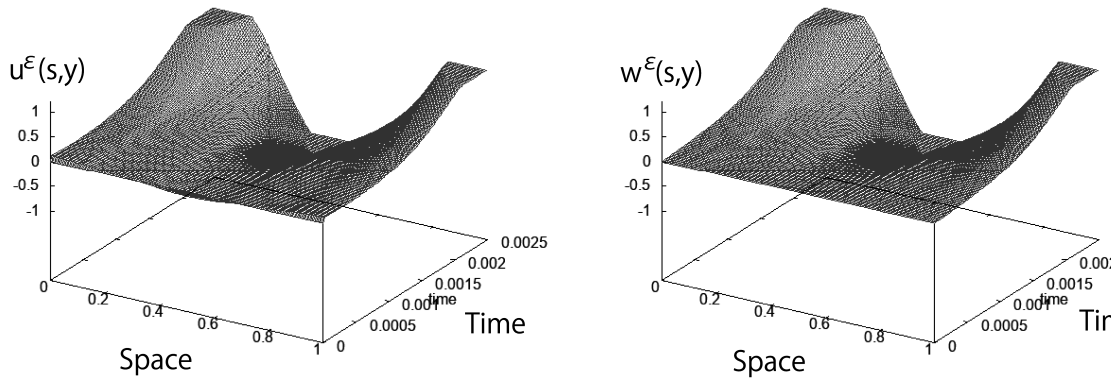

Figure 4.12: The graph of solution $[u^\varepsilon, w^\varepsilon]$ to (P;$\widetilde{f}, 0, 0)^\varepsilon$ at the iteration number $n = 17$: (left) $u^\varepsilon(s, y)$; (right) $w^\varepsilon(s, y)$.

In addition, the real line in Figure 4.9 means the graph of $w^\varepsilon(\widetilde{T}, y)$ at $\widetilde{T} = 0.0025$ and the iteration number $n = 17$. We observe from Figures 4.9–4.12 that the data sequence of cost functional J^ε almost reaches a stationary point, however, there is the slight gap between the solution $w^\varepsilon(\widetilde{T}, y)$ to (P;$\widetilde{f}, 0, 0)^\varepsilon$ and the desired profile $w_d(\widetilde{T}, y)$ (see Figure 4.9). We guess the reason is that the target profile $w_d(\widetilde{T}, y)$ defined by (4.9.12) is not the stable equilibria for (P;$\widetilde{f}, 0, 0)^\varepsilon$, and there is no desired profile u_d of the temperature in $(\mathrm{OP})^\varepsilon$ (cf. (4.9.9)).

In the forthcoming paper we will perform numerical experiments for (P;$f, h, \ell)^\varepsilon$ and $(\mathrm{OP})^\varepsilon$ under various situations (cf. Remark 4.2).

Chapter 5

Optimal control problems governed by 1-D Kobayashi–Warren–Carter type systems

In this Chapter, we recall the class of optimal control problems governed by 1-D K.W.C. models. The class consists of an optimal control problem for a physically realistic state-system of K.W.C. model, and its regularized approximating problems. The main results of this Chapter are stated in three Main Theorems 5.1–5.3. The first Main Theorem 5.1 is concerned with the solvability and continuous dependence for the state-systems. Meanwhile, the second Main Theorem 5.2 is concerned with the solvability of optimal control problems, and some semi-continuous association in the class of our optimal control problems. Finally, in the third Main Theorem 5.3, we derive the first order necessary optimality conditions for optimal controls of the regularized approximating problems. By taking the approximating limit, we also derive the optimality conditions for the optimal controls for the physically realistic problem.

5.1 Preliminaries

We begin by prescribing the notations used throughout this Chapter.

Abstract notations. For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X, and denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between X and its dual X^*. In particular, when X is a Hilbert space, we denote by $(\cdot, \cdot)_X$ the inner product of X. For any subset A of a Banach space X, let $\chi_A : X \longrightarrow \{0, 1\}$ be the characteristic function of A, i.e.:

$$\chi_A : w \in X \mapsto \chi_A(w) := \begin{cases} 1, & \text{if } w \in A, \\ 0, & \text{otherwise.} \end{cases}$$

For two Banach spaces X and Y, we denote by $\mathscr{L}(X; Y)$ the Banach space of bounded linear operators from X into Y, and in particular, we let $\mathscr{L}(X) := \mathscr{L}(X; X)$.

For Banach spaces $X_1, \dots, X_N$, with $1 < N \in \mathbb{N}$, let $X_1 \times \cdots \times X_N$ be the product Banach space endowed with the norm $|\cdot|_{X_1 \times \cdots \times X_N} := |\cdot|_{X_1} + \cdots + |\cdot|_{X_N}$. However, when all $X_1, \dots, X_N$ are Hilbert spaces, $X_1 \times \cdots \times X_N$ denotes the product Hilbert space endowed with the inner product $(\cdot,\cdot)_{X_1\times\cdots\times X_N} := (\cdot,\cdot)_{X_1} + \cdots + (\cdot,\cdot)_{X_N}$ and the norm $|\cdot|_{X_1\times\cdots\times X_N} := \left(|\cdot|_{X_1}^2 + \cdots + |\cdot|_{X_N}^2\right)^{\frac{1}{2}}$. In particular, when all $X_1,\ldots,X_N$ coincide with a Banach space Y, we write:

$$[Y]^N := \overbrace{Y \times \cdots \times Y}^{N \text{ times}}.$$

Additionally, for any transform (operator) $\mathcal{T} : X \longrightarrow Y$, we let:

$$\mathcal{T}[w_1,\ldots,w_N] := \left[\mathcal{T}w_1,\ldots,\mathcal{T}w_N\right] \text{ in } [Y]^N, \quad \text{for any } [w_1,\ldots,w_N] \in [X]^N.$$

Specific notations of this Chapter. As is mentioned in the previous section, let $(0,T) \subset \mathbb{R}$ be a bounded time-interval with a finite constant $T > 0$, and let $\Omega := (0,1) \subset \mathbb{R}$ be a one-dimensional bounded spatial domain. We denote by Γ the boundary $\partial\Omega = \{0,1\}$ of Ω, and we let $Q := (0,T)\times\Omega$ and $\Sigma := (0,T)\times\Gamma$. Especially, we denote by ∂_t and ∂_x the distributional time-derivative and the distributional spatial-derivative, respectively. Also, the measure theoretical phrases, such as "a.e.", "dt", "dx", and so on, are all with respect to the Lebesgue measure in each corresponding dimension.

On this basis, we define

$$\begin{cases} H := L^2(\Omega) \text{ and } \mathscr{H} := L^2(0,T;H), \\ V := H^1(\Omega) \text{ and } \mathscr{V} := L^2(0,T;V), \\ V_0 := H_0^1(\Omega) \text{ and } \mathscr{V}_0 := L^2(0,T;V_0). \end{cases}$$

Also, we identify the Hilbert spaces H and $\mathscr{H}$ with their dual spaces. Based on the identifications, we have the following relationships of continuous embeddings:

$$\begin{cases} V \subset H = H^* \subset V^* \text{ and } \mathscr{V} \subset \mathscr{H} = \mathscr{H}^* \subset \mathscr{V}^*, \\ V_0 \subset H = H^* \subset V_0^* \text{ and } \mathscr{V}_0 \subset \mathscr{H} = \mathscr{H}^* \subset \mathscr{V}_0^*, \end{cases}$$

among the Hilbert spaces H, V, V_0, $\mathscr{H}$, $\mathscr{V}$, and $\mathscr{V}_0$, and the respective dual spaces H^*, V^*, V_0^*, $\mathscr{H}^*$, $\mathscr{V}^*$, and $\mathscr{V}_0^*$. Additionally, in this paper, we define the topology of the Hilbert space V_0 by using the following inner product:

$$(w,\tilde{w})_{V_0} := (\partial_x w, \partial_x \tilde{w})_H, \text{ for all } w, \tilde{w} \in V_0.$$

Remark 5.1. Due to the one-dimensional embeddings $V \subset C(\overline{\Omega})$ and $V_0 \subset C(\overline{\Omega})$, it is easily checked that:

$$\begin{cases} \bullet \text{ if } \check{\mu} \in H \text{ and } \check{p} \in V, \text{ then } \check{\mu}\check{p} \in H, \text{ and } |\check{\mu}\check{p}|_H \leq \sqrt{2}|\check{\mu}|_H|\check{p}|_V, \\ \bullet \text{ if } \hat{\mu} \in L^\infty(0,T;H) \text{ and } \hat{p} \in \mathscr{V}, \text{ then } \hat{\mu}\hat{p} \in \mathscr{H}, \text{ and } |\hat{\mu}\hat{p}|_{\mathscr{H}} \leq \sqrt{2}|\hat{\mu}|_{L^\infty(0,T;H)}|\hat{p}|_{\mathscr{V}}. \end{cases} \tag{5.1.1}$$

Here, we note that the constant $\sqrt{2}$ corresponds to the constant of embedding $V \subset C(\overline{\Omega})$. Moreover, under the setting $\Omega := (0,1)$, this $\sqrt{2}$ can be used as a upper bound of the constants of embeddings $V \subset L^q(\Omega)$ and $V_0 \subset L^q(\Omega)$, for all $1 \leq q \leq \infty$.

Notations in convex analysis. (cf. [18, Chapter II]) For a proper, lower semi-continuous (l.s.c.), and convex function $\Psi : X \to (-\infty, \infty]$ on a Hilbert space X, we denote by $D(\Psi)$ the effective domain of Ψ. Also, we denote by $\partial\Psi$ the subdifferential of Ψ. The subdifferential $\partial\Psi$ corresponds to a generalized derivative of Ψ, and it is known as a maximal monotone graph in the product space $X \times X$. The set $D(\partial\Psi) := \{z \in X \mid \partial\Psi(z) \neq \emptyset\}$ is called the domain of $\partial\Psi$. We often use the notation "$[w_0, w_0^*] \in \partial\Psi$ in $X \times X$", to mean that "$w_0^* \in \partial\Psi(w_0)$ in X for $w_0 \in D(\partial\Psi)$ ", by identifying the operator $\partial\Psi$ with its graph in $X \times X$.

For Hilbert spaces $X_1, \cdots, X_N$, with $1 < N \in \mathbb{N}$, let us consider a proper, l.s.c., and convex function on the product space $X_1 \times \cdots \times X_N$:

$$\tilde{\Psi} : w = [w_1, \cdots, w_N] \in X_1 \times \cdots \times X_N \mapsto \tilde{\Psi}(w) = \tilde{\Psi}(w_1, \cdots, w_N) \in (-\infty, \infty].$$

On this basis, for any $i \in \{1, \ldots, N\}$, we denote by $\partial_{w_i}\tilde{\Psi} : X_1 \times \cdots \times X_N \to X_i$ a set-valued operator, which maps any $w = [w_1, \ldots, w_i, \ldots, w_N] \in X_1 \times \cdots \times X_i \times \cdots \times X_N$ to a subset $\partial_{w_i}\tilde{\Psi}(w) \subset X_i$, prescribed as follows:

$$\begin{aligned}\partial_{w_i}\tilde{\Psi}(w) &= \partial_{w_i}\tilde{\Psi}(w_1, \cdots, w_i, \cdots, w_N) \\ &:= \left\{ \tilde{w}^* \in X_i \;\middle|\; \begin{array}{l} (\tilde{w}^*, \tilde{w} - w_i)_{X_i} \leq \tilde{\Psi}(w_1, \cdots, \tilde{w}, \cdots, w_N) \\ \qquad -\tilde{\Psi}(w_1, \cdots, w_i, \cdots, w_N), \text{ for any } \tilde{w} \in X_i \end{array} \right\}.\end{aligned}$$

As is easily checked,

$$\begin{aligned}&\partial\tilde{\Psi}(w) \subset \partial_{w_1}\tilde{\Psi}(w) \times \cdots \times \partial_{w_N}\tilde{\Psi}(w), \\ &\text{for any } w = [w_1, \ldots, w_N] \in X_1 \times \cdots \times X_N.\end{aligned} \tag{5.1.2}$$

But, it should be noted that the converse inclusion of (5.1.2) is not true, in general.

Remark 5.2 (Examples of the subdifferential)**.** As one of the representatives of the subdifferentials, we exemplify the following set-valued function $\mathrm{Sgn}^N : \mathbb{R}^N \to 2^{\mathbb{R}^N}$, with $N \in \mathbb{N}$, which is defined as:

$$\begin{aligned}\xi = [\xi_1, \ldots, \xi_N] \in \mathbb{R}^N \mapsto \mathrm{Sgn}^N(\xi) &= \mathrm{Sgn}^N(\xi_1, \ldots, \xi_N) \\ &:= \begin{cases} \dfrac{\xi}{|\xi|} = \dfrac{[\xi_1, \ldots, \xi_N]}{\sqrt{\xi_1^2 + \cdots + \xi_N^2}}, & \text{if } \xi \neq 0, \\ \mathbb{D}^N, \quad \text{otherwise,} \end{cases}\end{aligned}$$

where $\mathbb{D}^N$ denotes the closed unit ball in $\mathbb{R}^N$ centered at the origin. Indeed, the set-valued function Sgn^N coincides with the subdifferential of the Euclidean norm $|\cdot| : \xi \in \mathbb{R}^N \mapsto |\xi| = \sqrt{\xi_1^2 + \cdots + \xi_N^2} \in [0, \infty)$, i.e.:

$$\partial|\cdot|(\xi) = \mathrm{Sgn}^N(\xi), \text{ for any } \xi \in D(\partial|\cdot|) = \mathbb{R}^N,$$

and furthermore, it is observed that:

$$\partial|\cdot|(0) = \mathbb{D}^N \subsetneq [-1, 1]^N = \partial_{\xi_1}|\cdot|(0) \times \cdots \times \partial_{\xi_N}|\cdot|(0).$$

Finally, we mention about a notion of functional convergence, known as "Mosco-convergence".

Definition 5.1 (Mosco-convergence: cf. [59])**.** Let X be an abstract Hilbert space. Let $\Psi : X \to (-\infty, \infty]$ be a proper, l.s.c., and convex function, and let $\{\Psi_n\}_{n=1}^{\infty}$ be a sequence of proper, l.s.c., and convex functions $\Psi_n : X \to (-\infty, \infty]$, $n = 1, 2, 3, \dots$. Then, it is said that $\Psi_n \to \Psi$ on X, in the sense of Mosco, as $n \to \infty$, iff. the following two conditions are fulfilled:

(M1) The condition of lower-bound: $\varliminf_{n\to\infty} \Psi_n(\check{w}_n) \geq \Psi(\check{w})$, if $\check{w} \in X$, $\{\check{w}_n\}_{n=1}^{\infty} \subset X$, and $\check{w}_n \to \check{w}$ weakly in X, as $n \to \infty$.

(M2) The condition of optimality: for any $\hat{w} \in D(\Psi)$, there exists a sequence $\{\hat{w}_n\}_{n=1}^{\infty} \subset X$ such that $\hat{w}_n \to \hat{w}$ in X and $\Psi_n(\hat{w}_n) \to \Psi(\hat{w})$, as $n \to \infty$.

As well as, if the sequence of convex functions $\{\tilde{\Psi}_\varepsilon\}_{\varepsilon\in\Xi}$ is labeled by a continuous argument $\varepsilon \in \Xi$ with a infinite set $\Xi \subset \mathbb{R}$, then for any $\varepsilon_0 \in \Xi$, the Mosco-convergence of $\{\tilde{\Psi}_\varepsilon\}_{\varepsilon\in\Xi}$, as $\varepsilon \to \varepsilon_0$, is defined by those of subsequences $\{\tilde{\Psi}_{\varepsilon_n}\}_{n=1}^{\infty}$, for all sequences $\{\varepsilon_n\}_{n=1}^{\infty} \subset \Xi$, satisfying $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$.

Remark 5.3. Let X, Ψ, and $\{\Psi_n\}_{n=1}^{\infty}$ be as in Definition 5.1. Then, the following hold:

(Fact 1) (cf. [10, Theorem 3.66], [39, Chapter 2]) Let us assume that

$$\Psi_n \to \Psi \text{ on } X, \text{ in the sense of Mosco, as } n \to \infty,$$

and

$$\begin{cases} [w, w^*] \in X \times X, \quad [w_n, w_n^*] \in \partial\Psi_n \text{ in } X \times X, \\ n \in \mathbb{N}, \\ w_n \to w \text{ in } X \text{ and } w_n^* \to w^* \text{ weakly in } X, \text{ as} \\ n \to \infty. \end{cases}$$

Then, it holds that:

$$[w, w^*] \in \partial\Psi \text{ in } X \times X, \text{ and } \Psi_n(w_n) \to \Psi(w), \text{ as } n \to \infty.$$

(Fact 2) (cf. [22, Lemma 4.1], [30, Appendix]) Let $N \in \mathbb{N}$ denote dimension constant, and let $S \subset \mathbb{R}^N$ be a bounded open set. Then, a sequence $\{\widehat{\Psi}_n^S\}_{n=1}^{\infty}$ of proper, l.s.c., and convex functions on $L^2(S; X)$, defined as:

$$w \in L^2(S; X) \mapsto \widehat{\Psi}_n^S(w) := \begin{cases} \displaystyle\int_S \Psi_n(w(t))\, dt, \\ \qquad \text{if } \Psi_n(w) \in L^1(S), \ \text{for } n = 1, 2, 3, \dots; \\ \infty, \quad \text{otherwise,} \end{cases}$$

converges to a proper, l.s.c., and convex function $\widehat{\Psi}^S$ on $L^2(S; X)$, defined as:

$$z \in L^2(S; X) \mapsto \widehat{\Psi}^S(z) := \begin{cases} \displaystyle\int_S \Psi(z(t))\, dt, \text{ if } \Psi(z) \in L^1(S), \\ \infty, \quad \text{otherwise;} \end{cases}$$

on $L^2(S; X)$, in the sense of Mosco, as $n \to \infty$.

Remark 5.4 (Example of Mosco-convergence). For any $\varepsilon \geq 0$, let $f_\varepsilon : \mathbb{R} \longrightarrow [0, \infty)$ be a continuous and convex function, defined as:

$$f_\varepsilon : \xi \in \mathbb{R} \mapsto f_\varepsilon(\xi) := \sqrt{\varepsilon^2 + |\xi|^2} \in [0, \infty). \tag{5.1.3}$$

Then, due to the uniform estimate:

$$\big| f_\varepsilon(\xi) - |\xi| \big| \leq \varepsilon, \text{ for all } \xi \in \mathbb{R},$$

we easily see that:

$$f_\varepsilon \to f_0 \ (= |\cdot|) \text{ on } \mathbb{R}, \text{ in the sense of Mosco, as } \varepsilon \downarrow 0.$$

In addition, for any $\varepsilon > 0$, it can be said that the subdifferential ∂f_ε coincides with the usual differential:

$$f'_\varepsilon : \xi \in \mathbb{R} \mapsto f'_\varepsilon(\xi) = \frac{\xi}{\sqrt{\varepsilon^2 + |\xi|^2}} \in \mathbb{R}.$$

5.2 Auxiliary lemmas

In this section, we recall the previous work [9], and set up some auxiliary results. In what follows, we let $\mathscr{Y} := \mathscr{V} \times \mathscr{V}_0$, with the dual $\mathscr{Y}^* := \mathscr{V}^* \times \mathscr{V}_0^*$. Note that $\mathscr{Y}$ is a Hilbert space which is endowed with a uniform convex topology, based on the inner product for product space, as in the Preliminaries (see the paragraph of Abstruct notations).

Besides, we define:

$$\mathscr{Z} := \big(W^{1,2}(0,T;V^*) \cap \mathscr{V}\big) \times \big(W^{1,2}(0,T;V_0^*) \cap \mathscr{V}_0\big),$$

as a Banach space, endowed with the norm:

$$|[\tilde{p}, \tilde{z}]|_{\mathscr{Z}} := |[\tilde{p}, \tilde{z}]|_{[C([0,T];H)]^2} + \big(|[\tilde{p}, \tilde{z}]|_{\mathscr{Y}}^2 + |[\partial_t \tilde{p}, \partial_t \tilde{z}]|_{\mathscr{Y}^*}^2\big)^{\frac{1}{2}}, \text{ for } [\tilde{p}, \tilde{z}] \in \mathscr{Z}.$$

Based on this, let us consider the following linear system of parabolic initial-boundary value problem, denoted by (P):

(P)

$$\begin{cases} \partial_t p - \partial_x^2 p + \mu(t,x)p + \lambda(t,x)p + \omega(t,x)\partial_x z = h(t,x), \\ (t,x) \in Q, \\ \partial_x p(t,x) = 0, \ (t,x) \in \Sigma, \\ p(0,x) = p_0(x), \ x \in \Omega; \end{cases}$$

$$\begin{cases} a(t,x)\partial_t z + b(t,x)z - \partial_x\big(A(t,x)\partial_x z + \nu^2 \partial_x z + \omega(t,x)p\big) \\ \qquad = k(t,x), \ (t,x) \in Q, \\ z(t,x) = 0, \ (t,x) \in \Sigma, \\ z(0,x) = z_0(x), \ x \in \Omega. \end{cases}$$

This system is studied in [9] as a key-problem for the Gâteaux differential of the cost $\mathcal{J}_\varepsilon$. In the context, $[a, b, \mu, \lambda, \omega, A] \in [\mathscr{H}]^6$ is a given sextuplet of functions which belongs to a subclass $\mathscr{S} \subset [\mathscr{H}]^6$, defined as:

$$\mathscr{S} := \left\{ [\tilde{a}, \tilde{b}, \tilde{\mu}, \tilde{\lambda}, \tilde{\omega}, \tilde{A}] \in [\mathscr{H}]^6 \ \middle| \ \begin{array}{l} \bullet\ \tilde{a} \in W^{1,\infty}(Q) \text{ and } \log \tilde{a} \in L^\infty(Q), \\ \bullet\ [\tilde{b}, \tilde{\lambda}, \tilde{\omega}] \in [L^\infty(Q)]^3, \\ \bullet\ \tilde{\mu} \in L^\infty(0,T;H) \text{ with } \tilde{\mu} \geq 0 \text{ a.e. in } Q, \\ \bullet\ \tilde{A} \in L^\infty(Q) \text{ with } \log \tilde{A} \in L^\infty(Q) \end{array} \right\}. \tag{5.2.1}$$

Also, $[p_0, z_0] \in [H]^2$ and $[h, k] \in \mathscr{Y}^*$ are, respectively, an initial pair and forcing pair, in the system (P).

Now, we refer to the previous work [9], to recall the key-properties of the system (P), in forms of Propositions.

Proposition 5.1 (cf. [9, Main Theorem 1 (I-A)])**.** For any sextuplet $[a, b, \mu, \lambda, \omega, A] \in \mathscr{S}$, any initial pair $[p_0, z_0] \in [H]^2$, and any forcing pair $[h, k] \in \mathscr{Y}^*$, the system (P) admits a unique solution, in the sense that:

$$\begin{cases} p \in W^{1,2}(0,T;V^*) \cap L^2(0,T;V) \subset C([0,T];H), \\ z \in W^{1,2}(0,T;V_0^*) \cap L^2(0,T;V_0) \subset C([0,T];H); \end{cases} \tag{5.2.2}$$

$$\begin{aligned} &\langle \partial_t p(t), \varphi \rangle_V + (\partial_x p(t), \partial_x \varphi)_H + (\mu(t) p(t), \varphi)_H \\ &\quad + (\lambda(t) p(t) + \omega(t) \partial_x z(t), \varphi)_H = \langle h(t), \varphi \rangle_V, \\ &\quad \text{for any } \varphi \in V, \text{ a.e. } t \in (0,T), \text{ subject to } p(0) = p_0 \text{ in } H; \end{aligned} \tag{5.2.3}$$

and

$$\begin{aligned} &\langle \partial_t z(t), a(t) \psi \rangle_{V_0} + (b(t) z(t), \psi)_H \\ &\quad + \big(A(t) \partial_x z(t) + \nu^2 \partial_x z(t) + p(t) \omega(t), \partial_x \psi \big)_H = \langle k(t), \psi \rangle_{V_0}, \\ &\quad \text{for any } \psi \in V_0, \text{ a.e. } t \in (0,T), \text{ subject to } z(0) = z_0 \text{ in } H. \end{aligned} \tag{5.2.4}$$

Proposition 5.2 (cf. [9, Main Theorem 1 (I-B)])**.** For every $\ell = 1, 2$, let us take arbitrary $[a^\ell, b^\ell, \mu^\ell, \lambda^\ell, \omega^\ell, A^\ell] \in \mathscr{S}$, $[p_0^\ell, z_0^\ell] \in [H]^2$, and $[h^\ell, k^\ell] \in \mathscr{Y}^*$, and let us denote by $[p^\ell, z^\ell] \in [\mathscr{H}]^2$ the solution to (P), corresponding to the sextuplet $[a^\ell, b^\ell, \mu^\ell, \lambda^\ell, \omega^\ell, A^\ell]$, initial pair $[p_0^\ell, z_0^\ell]$, and forcing pair $[h^\ell, k^\ell]$. Besides, let $C_0^* = C_0^*(a^1, b^1, \lambda^1, \omega^1)$ be a positive constant, depending on a^1, b^1, λ^1, and ω^1, which is defined as:

$$C_0^* := \frac{81(1+\nu^2)}{\min\{1, \nu^2, \inf a^1(Q)\}} \big(1 + |a^1|_{W^{1,\infty}(Q)} + |b^1|_{L^\infty(Q)} + |\lambda^1|_{L^\infty(Q)} + |\omega^1|_{L^\infty(Q)}^2 \big). \tag{5.2.5}$$

Then, it is estimated that:

$$\begin{aligned} &\frac{d}{dt} \big(|(p^1 - p^2)(t)|_H^2 + |\sqrt{a^1(t)} (z^1 - z^2)(t)|_H^2 \big) \\ &\qquad + \big(|(p^1 - p^2)(t)|_V^2 + \nu^2 |(z^1 - z^2)(t)|_{V_0}^2 \big) \\ &\leq 3C_0^* \big(|(p^1 - p^2)(t)|_H^2 + |\sqrt{a^1(t)} (z^1 - z^2)(t)|_H^2 \big) \\ &\qquad + 2C_0^* \big(|(h^1 - h^2)(t)|_{V^*}^2 + |(k^1 - k^2)(t)|_{V_0^*}^2 + R_0^*(t) \big), \end{aligned} \tag{5.2.6}$$

for a.e. $t \in (0,T)$;

where

$$R_0^*(t) := |\partial_t z^2(t)|_{V_0^*}^2 \big(|a^1 - a^2|_{C(\overline{Q})}^2 + |\partial_x (a^1 - a^2)(t)|_{L^4(\Omega)}^2 \big) \\ + |p^2(t)|_V^2 \big(|(\mu^1 - \mu^2)(t)|_H^2 + |(\omega^1 - \omega^2)(t)|_{L^4(\Omega)}^2 \big) \\ + |z^2(t)|_{V_0}^2 \big(|(b^1 - b^2)(t)|_{L^4(\Omega)}^2 + |p^2(t)(\lambda^1 - \lambda^2)(t)|_H^2 \big) \\ + |\partial_x z^2(t)(\omega^1 - \omega^2)(t)|_H^2 + |(A^1 - A^2)(t)\partial_x z^2(t)|_H^2,$$
for a.e. $t \in (0,T)$.

Remark 5.5. In the previous work [9], the constant C_0^* for the estimate (5.2.6) is provided as:

$$C_0^* := \frac{9(1+\nu^2)}{\min\{1, \nu^2, \inf a^1(Q)\}} \cdot \big(1 + (C_V^{L^4})^2 + (C_V^{L^4})^4 + (C_{V_0}^{L^4})^2 \big) \\ \cdot \big(1 + |a^1|_{W^{1,\infty}(Q)} + |b^1|_{L^\infty(Q)} + |\lambda^1|_{L^\infty(Q)} + |\omega^1|_{L^\infty(Q)}^2 \big), \tag{5.2.7}$$

with use of the constants $C_V^{L^4} > 0$ and $C_{V_0}^{L^4} > 0$ of the respective embeddings $V \subset L^4(\Omega)$ and $V_0 \subset L^4(\Omega)$. Note that the setting (5.2.5) corresponds to the special case of the original one (5.2.7), under the one-dimensional situation, as in Remark 5.1.

Proposition 5.3 (cf. [9, Corollary 1])**.** For any $[a,b,\mu,\lambda,\omega,A] \in \mathscr{S}$, let us denote by $\mathcal{P} = \mathcal{P}(a,b,\mu,\lambda,\omega,A) : [H]^2 \times \mathscr{Y}^* \longrightarrow \mathscr{Z}$ a linear operator, which maps any $\big[[p_0,z_0],[h,k]\big] \in [H]^2 \times \mathscr{Y}^*$ to the solution $[p,z] \in \mathscr{Z}$ to the linear system (P), for the sextuplet $[a,b,\mu,\lambda,\omega,A]$, initial pair $[p_0,z_0]$, and forcing pair $[h,k]$. Then, for any sextuplet $[a,b,\mu,\lambda,\omega,A] \in \mathscr{S}$, there exist positive constants $M_0^* = M_0^*(a,b,\mu,\lambda,\omega,A)$ and $M_1^* = M_1^*(a,b,\mu,\lambda,\omega,A)$, depending on a, b, μ, λ, ω, and A, such that:

$$M_0^* \big| \big[[p_0,z_0],[h,k]\big] \big|_{[H]^2 \times \mathscr{Y}^*} \leq |[p,z]|_{\mathscr{Z}} \leq M_1^* \big| \big[[p_0,z_0],[h,k]\big] \big|_{[H]^2 \times \mathscr{Y}^*}, \\ \text{for all } [p_0,z_0] \in [H]^2,\ [h,k] \in \mathscr{Y}^*, \\ \text{and } [p,z] = \mathcal{P}(a,b,\mu,\lambda,\omega,A)\big[[p_0,z_0],[h,k]\big] \in \mathscr{Z},$$

i.e. the operator $\mathcal{P} = \mathcal{P}(a,b,\mu,\lambda,\omega,A)$ is an isomorphism between the Hilbert space $[H]^2 \times \mathscr{Y}^*$ and the Banach space $\mathscr{Z}$.

Proposition 5.4 (cf. [9, Corollary 2])**.** Let us assume:

$$[a,b,\mu,\lambda,\omega,A] \in \mathscr{S}, \ \ \{[a^n,b^n,\mu^n,\lambda^n,\omega^n,A^n]\}_{n=1}^\infty \subset \mathscr{S},$$

$$[a^n, \partial_t a^n, \partial_x a^n, b^n, \lambda^n, \omega^n, A^n] \to [a, \partial_t a, \partial_x a, b, \lambda, \omega, A] \\ \text{weakly-}* \text{ in } [L^\infty(Q)]^7, \text{ and in the pointwise sense a.e. in } Q, \\ \text{as } n \to \infty, \tag{5.2.8}$$

and

$$\begin{cases} \mu^n \to \mu \text{ weakly-}* \text{ in } L^\infty(0,T;H), \\ \mu^n \to \mu \text{ in } H, \text{ for a.e. } t \in (0,T), \end{cases} \quad \text{as } n \to \infty.$$

Let us assume $[p_0, z_0] \in [H]^2$, $[h, k] \in \mathscr{Y}^*$, and let us denote by $[p, z] \in [\mathscr{H}]^2$ the solution to (P), for the initial pair $[p_0, z_0]$ and forcing pair $[h, k]$. Also, let us assume $\{[p_0^n, z_0^n]\}_{n=1}^\infty \subset [H]^2$, $\{[h^n, k^n]\}_{n=1}^\infty \subset \mathscr{Y}^*$, and for any $n \in \mathbb{N}$, let us denote by $[p^n, z^n] \in [\mathscr{H}]^2$ the solution to (P), for the initial pair $[p_0^n, z_0^n]$ and forcing pair $[h^n, k^n]$. Then, the following two items hold.

(A) The convergence:

$$\begin{cases} [p_0^n, z_0^n] \to [p_0, z_0] \text{ in } [H]^2, \\ [h^n, k^n] \to [h, k] \text{ in } \mathscr{Y}^*, \end{cases} \quad \text{as } n \to \infty,$$

implies the convergence:

$$[p^n, z^n] \to [p, z] \text{ in } [C([0, T]; H)]^2, \text{ and in } \mathscr{Y}, \text{ as } n \to \infty.$$

(B) The following two convergences:

$$\begin{cases} [p_0^n, z_0^n] \to [p_0, z_0] \text{ weakly in } [H]^2, \\ [h^n, k^n] \to [h, k] \text{ weakly in } \mathscr{Y}^*, \end{cases} \quad \text{as } n \to \infty,$$

and

$$[p^n, z^n] \to [p, z] \text{ in } [\mathscr{H}]^2, \text{ weakly in } \mathscr{Y}, \\ \text{and weakly in } W^{1,2}(0, T; V^*) \times W^{1,2}(0, T; V_0^*), \text{ as } n \to \infty,$$

are equivalent each other.

5.3 Main Theorems

We begin by setting up some assumptions needed in our Main Theorems.

(A1) $\nu > 0$ is a fixed constant. Let $[\eta_0, \theta_0] \in V \times V_0$ be a fixed initial pair. Let $[\eta_{\mathrm{ad}}, \theta_{\mathrm{ad}}] \in [\mathscr{H}]^2$ be a fixed pair of functions, called the *admissible target profile.*

(A2) $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1-function, which is a Lipschitz continuous on $\mathbb{R}$. Also g has a nonnegative primitive $0 \leq G \in C^2(\mathbb{R})$, i.e. the derivative $G' = \frac{dG}{d\eta}$ coincides with g on $\mathbb{R}$.

(A3) $\alpha : \mathbb{R} \longrightarrow (0, \infty)$ and $\alpha_0 : Q \longrightarrow (0, \infty)$ are Lipschitz continuous functions, such that:

- $\alpha \in C^2(\mathbb{R})$, with the first derivative $\alpha' = \frac{d\alpha}{d\eta}$ and the second one $\alpha'' = \frac{d^2\alpha}{d\eta^2}$;
- $\alpha'(0) = 0$, $\alpha'' \geq 0$ on $\mathbb{R}$, and $\alpha\alpha'$ is a Lipschitz continuous function on $\mathbb{R}$;
- $\alpha \geq \delta_*$ on $\mathbb{R}$, and $\alpha_0 \geq \delta_*$ on $\overline{Q}$, for some constant $\delta_* \in (0, 1)$.

Additionally, for any $\varepsilon \geq 0$, let $f_\varepsilon : \mathbb{R} \longrightarrow [0, \infty)$ be the convex function, defined in (5.1.3).

Now, the Main Theorems of this paper are stated as follows:

Main Theorem 5.1. Let us assume (A1)–(A3). Let us fix a constant $\varepsilon \geq 0$, an initial pair $[\eta_0, \theta_0] \in V \times V_0$, and a forcing pair $[u, v] \in [\mathscr{H}]^2$. Then, the following hold:

(I-A) The state-system $(\mathrm{S})_\varepsilon$ admits a unique solution $[\eta, \theta] \in [\mathscr{H}]^2$, in the sense that:

$$\begin{cases} \eta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \subset C(\overline{Q}), \\ \theta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V_0) \subset C(\overline{Q}); \end{cases} \tag{5.3.1}$$

$$\begin{gathered} \big(\partial_t \eta(t), \varphi\big)_H + \big(\partial_x \eta(t), \partial_x \varphi\big)_H + \big(g(\eta(t)), \varphi\big)_H \\ + \big(\alpha'(\eta(t)) f_\varepsilon(\partial_x \theta(t)), \varphi\big)_H = \big(M_u u(t), \varphi\big)_H, \\ \text{for any } \varphi \in V, \text{ a.e. } t \in (0,T), \text{ subject to } \eta(0) = \eta_0 \text{ in } H; \end{gathered} \tag{5.3.2}$$

and

$$\begin{gathered} \big(\alpha_0(t)\partial_t \theta(t), \theta(t) - \psi\big)_H + \nu^2 \big(\partial_x \theta(t), \partial_x(\theta(t) - \psi)\big)_H \\ + \int_\Omega \alpha(\eta(t)) f_\varepsilon(\partial_x \theta(t)) dx \leq \int_\Omega \alpha(\eta(t)) f_\varepsilon(\partial_x \psi) dx \\ + \big(M_v v(t), \theta(t) - \psi\big)_H, \text{ for any } \psi \in V_0, \\ \text{a.e. } t \in (0,T), \text{ subject to } \theta(0) = \theta_0 \text{ in } H. \end{gathered} \tag{5.3.3}$$

(I-B) Let $\{\varepsilon_n\}_{n=1}^\infty \subset [0,1]$, $\{[\eta_{0,n}, \theta_{0,n}]\}_{n=1}^\infty \subset V \times V_0$, and $\{[u_n, v_n]\}_{n=1}^\infty \subset [\mathscr{H}]^2$ be given sequences such that:

$$\begin{gathered} \varepsilon_n \to \varepsilon, \ [\eta_{0,n}, \theta_{0,n}] \to [\eta_0, \theta_0] \text{ weakly in } V \times V_0, \\ \text{and } [M_u u_n, M_v v_n] \to [M_u u, M_v v] \text{ weakly in } [\mathscr{H}]^2, \text{ as } n \to \infty. \end{gathered} \tag{5.3.4}$$

In addition, let $[\eta, \theta]$ be the unique solution to $(\mathrm{S})_\varepsilon$, for the forcing pair $[u, v]$, and for any $n \in \mathbb{N}$, let $[\eta_n, \theta_n]$ be the unique solution to $(\mathrm{S})_{\varepsilon_n}$, for the initial pair $[\eta_{0,n}, \theta_{0,n}]$ and forcing pair $[u_n, v_n]$. Then, it holds that:

$$\begin{gathered} [\eta_n, \theta_n] \to [\eta, \theta] \text{ in } [C(\overline{Q})]^2, \text{ in } \mathscr{Y}, \text{ weakly in } [W^{1,2}(0,T;H)]^2, \\ \text{and weakly-}* \text{ in } L^\infty(0,T;V) \times L^\infty(0,T;V_0), \text{ as } n \to \infty, \end{gathered} \tag{5.3.5}$$

and in particular,

$$\begin{gathered} \alpha''(\eta_n) f_{\varepsilon_n}(\partial_x \theta_n) \to \alpha''(\eta) f_\varepsilon(\partial_x \theta) \text{ in } \mathscr{H}, \\ \text{and weakly-}* \text{ in } L^\infty(0,T;H), \text{ as } n \to \infty. \end{gathered} \tag{5.3.6}$$

Remark 5.6. As a consequence of (5.3.5) and (5.3.6), we further find a subsequence $\{n_i\}_{i=1}^\infty \subset \{n\}$, such that:

$$\begin{gathered} [\eta_{n_i}, \theta_{n_i}] \to [\eta, \theta], \ [\partial_x \eta_{n_i}, \partial_x \theta_{n_i}] \to [\partial_x \eta, \partial_x \theta], \\ \text{and } \alpha''(\eta_{n_i}) f_{\varepsilon_{n_i}}(\partial_x \theta_{n_i}) \to \alpha''(\eta) f_\varepsilon(\partial_x \theta), \\ \text{in the pointwise sense a.e. in } Q, \text{ as } i \to \infty, \end{gathered} \tag{5.3.7}$$

and

$$\begin{gathered} [\eta_{n_i}(t), \theta_{n_i}(t)] \to [\eta(t), \theta(t)] \text{ in } V \times V_0, \\ \text{and } \alpha''(\eta_{n_i}(t)) f_{\varepsilon_{n_i}}(\partial_x \theta_{n_i}(t)) \to \alpha''(\eta(t)) f_\varepsilon(\partial_x \theta(t)) \text{ in } H, \\ \text{in the pointwise sense for a.e. } t \in (0,T), \text{ as } i \to \infty. \end{gathered} \tag{5.3.8}$$

Main Theorem 5.2. Let us assume (A1)–(A3), and fix any constant $\varepsilon \geq 0$. Then, the following two items hold.

(II-A) The problem $(\mathrm{OP})_\varepsilon$ has at least one optimal control $[u^*, v^*] \in [\mathscr{H}]^2$, so that:

$$\mathcal{J}_\varepsilon(u^*, v^*) = \min_{[u,v]\in[\mathscr{H}]^2} \mathcal{J}_\varepsilon(u, v).$$

(II-B) Let $\{\varepsilon_n\}_{n=1}^\infty \subset [0,1]$ and $\{[\eta_{0,n}, \theta_{0,n}]\}_{n=1}^\infty \subset V \times V_0$ be given sequences such that:

$$\varepsilon_n \to \varepsilon, \text{and } [\eta_{0,n}, \theta_{0,n}] \to [\eta_0, \theta_0] \text{ weakly in } V \times V_0, \text{ as } n \to \infty. \tag{5.3.9}$$

In addition, for any $n \in \mathbb{N}$, let $[u_n^*, v_n^*] \in [\mathscr{H}]^2$ be the optimal control of $(\mathrm{OP})_{\varepsilon_n}$. Then, there exist a subsequence $\{n_i\}_{i=1}^\infty \subset \{n\}$ and a pair of functions $[u^{**}, v^{**}] \in [\mathscr{H}]^2$, such that:

$$\varepsilon_{n_i} \to \varepsilon, \text{ and } [M_u u_{n_i}^*, M_v v_{n_i}^*] \to [M_u u^{**}, M_v v^{**}] \\ \text{weakly in } [\mathscr{H}]^2, \text{ as } i \to \infty,$$

and

$$[u^{**}, v^{**}] \text{ is an optimal control of } (\mathrm{OP})_\varepsilon.$$

Main Theorem 5.3. Under the assumptions (A1)–(A3), the following two items hold.

(III-A) (Necessary condition for $(\mathrm{OP})_\varepsilon$ when $\varepsilon > 0$) For any $\varepsilon > 0$, let $[u_\varepsilon^*, v_\varepsilon^*] \in [\mathscr{H}]^2$ be an optimal control of $(\mathrm{OP})_\varepsilon$, and let $[\eta_\varepsilon^*, \theta_\varepsilon^*]$ be the solution to $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u_\varepsilon^*, v_\varepsilon^*] \in [\mathscr{H}]^2$. Then, it holds that:

$$[M_u(u_\varepsilon^* + p_\varepsilon^*), M_v(v_\varepsilon^* + z_\varepsilon^*)] = [0, 0] \text{ in } [\mathscr{H}]^2, \tag{5.3.10}$$

where $[p_\varepsilon^*, z_\varepsilon^*]$ is a unique solution to the following variational system:

$$\begin{aligned} -\langle \partial_t p_\varepsilon^*(t), \varphi \rangle_V &+ \big(\partial_x p_\varepsilon^*(t), \partial_x \varphi\big)_H + \big([\alpha''(\eta_\varepsilon^*) f_\varepsilon(\partial_x \theta_\varepsilon^*)](t) p_\varepsilon^*(t), \varphi\big)_H \\ &+ \big(g'(\eta_\varepsilon^*(t)) p_\varepsilon^*(t), \varphi\big)_H + \big([\alpha'(\eta_\varepsilon^*) f_\varepsilon'(\partial_x \theta_\varepsilon^*)](t) \partial_x z_\varepsilon^*(t), \varphi\big)_H \\ =&\big(M_\eta(\eta_\varepsilon^* - \eta_{\mathrm{ad}})(t), \varphi\big)_H, \text{ for any } \varphi \in V, \text{ and a.e. } t \in (0, T); \end{aligned} \tag{5.3.11}$$

and

$$\begin{aligned} -\big\langle \partial_t\big(\alpha_0 z_\varepsilon^*\big)(t), \psi \big\rangle_{V_0} &+ \big([\alpha(\eta_\varepsilon^*) f_\varepsilon''(\partial_x \theta_\varepsilon^*)](t) \partial_x z_\varepsilon^*(t) + \nu^2 \partial_x z_\varepsilon^*(t), \partial_x \psi\big)_H \\ &+ \big([\alpha'(\eta_\varepsilon^*) f_\varepsilon'(\partial_x \theta_\varepsilon^*)](t) p_\varepsilon^*(t), \partial_x \psi\big)_H = \big(M_\theta(\theta_\varepsilon^* - \theta_{\mathrm{ad}})(t), \psi\big)_H, \\ &\text{for any } \psi \in V_0, \text{ and a.e. } t \in (0, T); \end{aligned} \tag{5.3.12}$$

subject to the terminal condition:

$$[p_\varepsilon^*(T), z_\varepsilon^*(T)] = [0, 0] \text{ in } [H]^2. \tag{5.3.13}$$

(III-B) Let us define a Hilbert space $\mathscr{W}_0$ as:

$$\mathscr{W}_0 := \left\{ \ \psi \in W^{1,2}(0, T; H) \cap \mathscr{V}_0 \ \middle| \ \psi(0) = 0 \text{ in } H \ \right\}.$$

Then, there exists an optimal control $[u^\circ, v^\circ] \in [\mathscr{H}]^2$ of the problem $(\mathrm{OP})_0$, together with the solution $[\eta^\circ, \theta^\circ]$ to the system $(\mathrm{S})_0$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u^\circ, v^\circ]$, and there exist pairs of functions $[p^\circ, z^\circ] \in \mathscr{Y}$, $[\xi^\circ, \nu^\circ] \in \mathscr{H} \times L^\infty(Q)$, and a distribution $\zeta^\circ \in \mathscr{W}_0^*$, such that:

$$[M_u(u^\circ + p^\circ), M_v(v^\circ + z^\circ)] = [0, 0] \text{ in } [\mathscr{H}]^2; \tag{5.3.14}$$

$$\begin{cases} p^\circ \in W^{1,2}(0, T; V^*) \ (\cap \mathscr{V}), \text{ i.e. } p^\circ \in C([0, T]; H), \\ \nu^\circ \in \mathrm{Sgn}^1(\partial_x \theta^\circ), \text{ a.e. in } Q; \end{cases} \tag{5.3.15}$$

$$\begin{aligned} &\bigl\langle -\partial_t p^\circ, \varphi \bigr\rangle_{\mathscr{V}} + \bigl(\partial_x p^\circ, \partial_x \varphi\bigr)_{\mathscr{H}} + \bigl(\alpha''(\eta^\circ)|\partial_x \theta^\circ| p^\circ, \varphi\bigr)_{\mathscr{H}} \\ &\quad + \bigl(g'(\eta^\circ) p^\circ + \alpha'(\eta^\circ)\xi^\circ, \varphi\bigr)_{\mathscr{H}} = \bigl(M_\eta(\eta^\circ - \eta_{ad}), \varphi\bigr)_{\mathscr{H}}, \\ &\quad \text{for any } \varphi \in \mathscr{V}, \text{ subject to } p^\circ(T) = 0 \text{ in } H; \end{aligned} \tag{5.3.16}$$

and

$$\begin{aligned} &\bigl(\alpha_0 z^\circ, \partial_t \psi\bigr)_{\mathscr{H}} + \bigl\langle \zeta^\circ, \psi \bigr\rangle_{\mathscr{W}_0} + \bigl(\nu^2 \partial_x z^\circ + \alpha'(\eta^\circ)\nu^\circ p^\circ, \partial_x \psi\bigr)_{\mathscr{H}} \\ &\quad = \bigl(M_\theta(\theta^\circ - \theta_{ad}), \psi\bigr)_{\mathscr{H}}, \text{ for any } \psi \in \mathscr{W}_0. \end{aligned} \tag{5.3.17}$$

Remark 5.7. Let $\mathcal{R}_T \in \mathcal{L}(\mathscr{H})$ be an isomorphism, defined as:

$$\bigl(\mathcal{R}_T \varphi\bigr)(t) := \varphi(T - t) \text{ in } H, \text{ for a.e. } t \in (0, T).$$

Also, let us fix $\varepsilon > 0$, and define a bounded linear operator $\mathcal{Q}_\varepsilon^* : [\mathscr{H}]^2 \longrightarrow \mathscr{Z}$ as the restriction $\mathcal{P}|_{\{[0,0]\} \times \mathscr{Y}^*}$ of the linear isomorphism $\mathcal{P} = \mathcal{P}(a, b, \mu, \lambda, \omega, A) : [H]^2 \times \mathscr{Y}^* \longrightarrow \mathscr{Z}$, as in Proposition 5.3, in the case when:

$$\begin{cases} [a, b] = \mathcal{R}_T[\alpha_0, -\partial_t \alpha_0] \text{ in } W^{1,\infty}(Q) \times L^\infty(Q), \\ \mu = \mathcal{R}_T\bigl[\alpha''(\eta_\varepsilon^*) f_\varepsilon(\partial_x \theta_\varepsilon^*)\bigr] \text{ in } L^\infty(0, T; H), \\ [\lambda, \omega, A] = \mathcal{R}_T\bigl[g'(\eta_\varepsilon^*), \alpha'(\eta_\varepsilon^*) f_\varepsilon'(\partial_x \theta_\varepsilon^*), \alpha(\eta_\varepsilon^*) f_\varepsilon''(\partial_x \theta_\varepsilon^*)\bigr] \text{ in } [L^\infty(Q)]^3. \end{cases} \tag{5.3.18}$$

On this basis, let us define:

$$\mathcal{P}_\varepsilon^* := \mathcal{R}_T \circ \mathcal{Q}_\varepsilon^* \circ \mathcal{R}_T \text{ in } \mathscr{L}([\mathscr{H}]^2; \mathscr{Z}).$$

Then, having in mind:

$$\partial_t(\alpha_0 \tilde{z}) = \alpha_0 \partial_t \tilde{z} + \tilde{z} \partial_t \alpha_0 \text{ in } \mathscr{V}_0^*, \text{ for any } \tilde{z} \in W^{1,2}(0, T; V_0^*), \tag{5.3.19}$$

we can obtain the unique solution $[p_\varepsilon^*, z_\varepsilon^*] \in [\mathscr{H}]^2$ to the variational system (5.3.11)–(5.3.13) as follows:

$$[p_\varepsilon^*, z_\varepsilon^*] = \mathcal{P}_\varepsilon^*\bigl[M_\eta(\eta_\varepsilon^* - \eta_{\mathrm{ad}}), M_\theta(\theta_\varepsilon^* - \theta_{\mathrm{ad}})\bigr] \text{ in } \mathscr{Z}.$$

5.4 Proof of main Theorem 5.1

In this section, we give the proof of the first Main Theorem 5.1. Before the proof, we refer to the reformulation method as in [62], and consider to reduce the state-system $(\mathrm{S})_\varepsilon$ to an evolution equation in the Hilbert space $[H]^2$.

Let us fix any $\varepsilon \geq 0$. Besides, let us define time-dependent operators $\mathcal{A}(t) \in \mathscr{L}([H]^2)$, for $t \in [0,T]$, a nonlinear operator $\mathcal{G} : [H]^2 \longrightarrow [H]^2$, and a proper functional $\Phi_\varepsilon : [H]^2 \longrightarrow [0,\infty]$, by setting:

$$\mathcal{A}(t) : w = [\eta, \theta] \in [H]^2 \mapsto \mathcal{A}(t)w := [\eta,\ \alpha_0(t)\theta] \in [H]^2, \text{ for } t \in [0,T], \tag{5.4.1}$$

$$\mathcal{G} : w = [\eta, \theta] \in [H]^2 \mapsto \mathcal{G}(w) := \big[g(\eta) - \eta - \nu^{-2}\alpha(\eta)\alpha'(\eta),\ 0\big] \in [H]^2, \tag{5.4.2}$$

and

$$\Phi_\varepsilon : w = [\eta,\theta] \in [H]^2 \mapsto \Phi_\varepsilon(w) = \Phi_\varepsilon(\eta,\theta)$$
$$:= \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\partial_x \eta|^2 dx + \frac{1}{2}\int_\Omega |\eta|^2\, dx + \frac{1}{2}\int_\Omega \left(\nu f_\varepsilon(\partial_x\theta) + \frac{1}{\nu}\alpha(\eta)\right)^2 dx, \\ \qquad \text{if } [\eta,\theta] \in V \times V_0, \\ \infty, \quad \text{otherwise}, \end{cases} \tag{5.4.3}$$

respectively. Note that the definition of f_ε, as in Remark 5.4, and the assumption (A3) guarantee the lower semi-continuity and convexity of Φ_ε on $[H]^2$.

Remark 5.8. When $\varepsilon > 0$, we can easily check from Remark 5.4 and (A3) that the subdifferential $\partial\Phi_\varepsilon \subset [H]^2 \times [H]^2$ is single-valued, and

$$[w, w^*] \in \partial\Phi_\varepsilon \text{ in } [H]^2 \times [H]^2 \text{ for } w = [\eta,\theta] \in [H]^2 \text{ and } w^* = [\eta^*, \theta^*] \in [H]^2,$$

iff.

$$\begin{cases} \bullet\ w = [\eta,\theta] \in H^2(\Omega) \times V_0 \text{ with } \partial_x \eta(\ell) = 0, \text{ for } \ell \in \Gamma = \{0,1\}, \\ \quad \text{and } \alpha(\eta) f'_\varepsilon(\partial_x\theta) + \nu^2 \partial_x \theta \in V_0, \\ \bullet\ w^* = {}^{\mathrm{t}}\begin{bmatrix} \eta^* \\ \theta^* \end{bmatrix} = {}^{\mathrm{t}}\begin{bmatrix} -\partial_x^2 \eta + \eta + \alpha'(\eta) f_\varepsilon(\partial_x\theta) + \nu^{-2}\alpha(\eta)\alpha'(\eta) \\ -\partial_x\big(\alpha(\eta) f'_\varepsilon(\partial_x\theta) + \nu^2\partial_x\theta\big) \end{bmatrix} \\ \quad \text{in } [H]^2. \end{cases}$$

Therefore, in the case of $\varepsilon > 0$, the state-system $(\mathrm{S})_\varepsilon$ will be equivalent to the following Cauchy problem $(\mathrm{E})_\varepsilon$ of an evolution equation:

$$(\mathrm{E})_\varepsilon \quad \begin{cases} \mathcal{A}(t)w'(t) + \partial\Phi_\varepsilon(w(t)) + \mathcal{G}(w(t)) \ni \mathfrak{f}(t) \text{ in } [H]^2,\ t \in (0,T), \\ w(0) = w_0 \text{ in } [H]^2. \end{cases}$$

In the context, "$\,'\,$" denotes the time-derivative, $w_0 := [\eta_0, \theta_0] \in V \times V_0$ and $\mathfrak{f} := [M_u u, M_v v] \in [\mathscr{H}]^2$ are the initial pair and forcing pair, as in $(\mathrm{S})_\varepsilon$, respectively.

Remark 5.9. In the case of $\varepsilon = 0$, the equivalence between the corresponding state-system $(\mathrm{S})_0$ and Cauchy problem $(\mathrm{E})_0$ is not so obvious. However, we can show a partial relation, such that:

($\star$0) if $w = [\eta, \theta]$ is a solution to $(\mathrm{E})_0$, then it is also a solution to $(\mathrm{S})_0$.

In fact, as is easily seen, the operator $\partial_\eta \Phi_0 : [H]^2 \longrightarrow H$ is single-valued. Besides, for any $\tilde{\theta} \in V_0$, it follows that $[\eta, \eta^*] \in \partial_\eta \Phi_0(\cdot, \tilde{\theta})$ in $H \times H$, iff.:

$$\begin{aligned}\big(\eta^*, \varphi\big)_H = \big(\partial_x \eta, \partial_x \varphi\big)_H + \big(\eta, \varphi\big)_H \\ + \big(\alpha'(\eta)|\partial_x \tilde{\theta}| + \nu^{-2}\alpha(\eta)\alpha'(\eta), \varphi\big)_H, \text{ for any } \varphi \in V.\end{aligned}$$

Similarly, for any $\tilde{\eta} \in V$, one can see that $[\theta, \theta^*] \in \partial_\theta \Phi_0(\tilde{\eta}, \cdot)$ in $H \times H$, iff.:

$$\begin{aligned}\big(-\theta^*, \theta - \psi\big)_H + \nu^2\big(\partial_x \theta, \partial_x(\theta - \psi)\big)_H + \int_\Omega \alpha(\tilde{\eta})|\partial_x \theta|\, dx \\ \leq \int_\Omega \alpha(\tilde{\eta})|\partial_x \psi|\, dx, \text{ for any } \psi \in V_0.\end{aligned} \tag{5.4.4}$$

Taking into account (5.4.1)–(5.4.4), we deduce that the variational problem as in (5.3.1)–(5.3.3) is equivalently reformulated to the following Cauchy problem:

$$(\widetilde{\mathrm{E}}) \quad \begin{cases}\mathcal{A}(t)w'(t) + \big[\partial_\eta \Phi_0(w(t)) \times \partial_\theta \Phi_0(w(t))\big] + \mathcal{G}(w(t)) \ni \mathfrak{f}(t) \text{ in } [H]^2,\ t \in (0, T), \\ w(0) = w_0 \text{ in } [H]^2.\end{cases}$$

The item ($\star$0) is a straightforward consequence of this reformulation and the inclusion $\partial \Phi_0 \subset \big[\partial_\eta \Phi_0 \times \partial_\theta \Phi_0\big]$ in $[H]^2 \times [H]^2$, mentioned in (5.1.2).

Now, we are ready to prove the Main Theorem 5.1.

Proof of Main Theorem 5.1 (I-A) First, we verify the existence part. Under the setting (5.4.1)–(5.4.3), we immediately check that:

(ev.0) for any $t \in [0, T]$, $\mathcal{A}(t) \in \mathscr{L}([H]^2)$ is positive and selfadjoint, and

$$(\mathcal{A}(t)w, w)_{[H]^2} \geq \delta_* |w|^2_{[H]^2}, \text{ for any } w \in [H]^2,$$

with the constant $\delta_* \in (0, 1)$ as in (A3);

(ev.1) $\mathcal{A} \in W^{1,\infty}(0, T; \mathscr{L}([H]^2))$, and

$$A^* := \operatorname*{ess\,sup}_{t \in (0,T)} \big\{\max\{|\mathcal{A}(t)|_{\mathscr{L}([H]^2)}, |\mathcal{A}'(t)|_{\mathscr{L}([H]^2)}\}\big\} \leq 1 + |\alpha_0|_{W^{1,\infty}(Q)} < \infty;$$

(ev.2) $\mathcal{G} : [H]^2 \longrightarrow [H]^2$ is a Lipschitz continuous operator with a Lipschitz constant:

$$L_* := 1 + |g'|_{L^\infty(\mathbb{R})} + \nu^{-2}\big|\tfrac{d}{d\eta}(\alpha\alpha')\big|_{L^\infty(\mathbb{R})},$$

and $\mathcal{G}$ has a C^1-potential functional

$$\widehat{\mathcal{G}} : w = [\eta, \theta] \in [H]^2 \mapsto \widehat{\mathcal{G}}(w) := \int_\Omega \left(G(\eta) - \frac{\eta^2}{2} - \frac{\alpha(\eta)^2}{2\nu^2}\right) dx \in \mathbb{R};$$

(ev.3) $\Phi_\varepsilon \geq 0$ on $[H]^2$, and the sublevel set $\{\tilde{w} \in [H]^2 \,|\, \Phi_\varepsilon(\tilde{w}) \leq r\}$ is contained in a compact set $K_\nu(r)$ in $[H]^2$, defined as

$$K_\nu(r) := \left\{ \ \tilde{w} = [\tilde{\eta}, \tilde{\theta}] \in V \times V_0 \ \middle| \ |\tilde{\eta}|_V^2 \leq 2r \text{ and } |\tilde{\theta}|_{V_0}^2 \leq 2\nu^{-2} r \ \right\},$$

for any $r \geq 0$.

On account of (5.4.1)–(5.4.3) and (ev.0)–(ev.3), we can apply Lemma 5.3 in Appendix, as the case when:

$$X = [H]^2, \ \mathcal{A}_0 = \mathcal{A} \text{ in } W^{1,\infty}(0,T;\mathscr{L}([H]^2)), \ \mathcal{G}_0 = \mathcal{G} \text{ on } [H]^2, \text{ and } \Psi_0 = \Phi_\varepsilon \text{ on } [H]^2,$$

and we can find a solution $w = [\eta, \theta] \in [\mathscr{H}]^2$ to the Cauchy problem (E)$_\varepsilon$. In the light of Remarks 5.8 and 5.9, finding this $w = [\eta, \theta]$ directly leads to the existence of solution to the state-system (S)$_\varepsilon$.

Next, for the verification of the uniqueness part, we suppose that the both pairs of functions $[\eta^\ell, \theta^\ell] \in [\mathscr{H}]^2$, $\ell = 1, 2$, solve the state-system (S)$_\varepsilon$ for the common initial pair $[\eta_0, \theta_0]$ and forcing pair $[u, v] \in [\mathscr{H}]^2$. Besides, let us take the difference between two variational forms (5.3.2) for η^ℓ, $\ell = 1, 2$, and put $\varphi = \eta^1 - \eta^2$. Then, by using the assumptions (A1)–(A3), and Hölder's and Young's inequalities, we have:

$$\frac{1}{2}\frac{d}{dt}|(\eta^1 - \eta^2)(t)|_H^2 + |\partial_x(\eta^1 - \eta^2)(t)|_H^2 = I_A^1 + I_A^2, \tag{5.4.5a}$$

with

$$I_A^1 := -\big(g(\eta^1(t)) - g(\eta^2(t)), (\eta^1 - \eta^2)(t)\big)_H \leq L_*|(\eta^1 - \eta^2)(t)|_H^2, \tag{5.4.5b}$$

and

$$\begin{aligned}
I_A^2 :=& - \big(\alpha'(\eta^1(t)) f_\varepsilon(\partial_x\theta^1(t)) - \alpha'(\eta^2(t)) f_\varepsilon(\partial_x\theta^2(t)), (\eta^1 - \eta^2)(t)\big)_H \\
=& \int_\Omega f_\varepsilon(\partial_x\theta^1(t))\big(\alpha'(\eta^1(t))(\eta^2 - \eta^1)(t)\big)\,dx \\
&\qquad + \int_\Omega f_\varepsilon(\partial_x\theta^2(t))\big(\alpha'(\eta^2(t))(\eta^1 - \eta^2)(t)\big)\,dx \\
\leq& - \int_\Omega \big(f_\varepsilon(\partial_x\theta^1(t)) - f_\varepsilon(\partial_x\theta^2(t))\big)\big(\alpha(\eta^1(t)) - \alpha(\eta^2(t))\big)\,dx \\
\leq& |\alpha'|_{L^\infty(\mathbb{R})}|\partial_x(\theta^1 - \theta^2)(t)|_H|(\eta^1 - \eta^2)(t)|_H \\
\leq& \frac{\nu^2}{4}|\partial_x(\theta^1 - \theta^2)(t)|_H^2 + \frac{|\alpha'|_{L^\infty(\mathbb{R})}^2}{\nu^2}|(\eta^1 - \eta^2)(t)|_H^2, \text{for a.e. } t \in (0,T).
\end{aligned} \tag{5.4.5c}$$

Meanwhile, for any $\ell \in \{1, 2\}$, let us take $\ell^\perp \in \{1, 2\} \setminus \{\ell\}$, and put $\psi = \theta^{\ell^\perp}$ in the variational inequality (5.3.3) for θ^ℓ. Then, adding those two variational inequalities, and using Hölder's and Young's inequalities, one can observe that:

$$\frac{1}{2}\frac{d}{dt}|\sqrt{\alpha_0(t)}(\theta^1 - \theta^2)(t)|_H^2 + \nu^2|\partial_x(\theta^1 - \theta^2)(t)|_H^2 \leq I_A^3 + I_A^4, \tag{5.4.6a}$$

with

$$I_A^3 := \frac{1}{2}\int_\Omega \partial_t\alpha_0(t)|(\theta^1 - \theta^2)(t)|^2\,dx \leq \frac{|\partial_t\alpha_0|_{L^\infty(Q)}}{2}|(\theta^1 - \theta^2)(t)|_H^2, \tag{5.4.6b}$$

and

$$
\begin{aligned}
I_A^4 :=& -\int_\Omega \alpha(\eta^2(t)) f_\varepsilon(\partial_x \theta^2(t))\, dx + \int_\Omega \alpha(\eta^2(t)) f_\varepsilon(\partial_x \theta^1(t)) \\
&\quad - \int_\Omega \alpha(\eta^1(t)) f_\varepsilon(\partial_x \theta^1(t))\, dx + \int_\Omega \alpha(\eta^1(t)) f_\varepsilon(\partial_x \theta^2(t)) \\
=& -\int_\Omega \bigl(f_\varepsilon(\partial_x\theta^1(t)) - f_\varepsilon(\partial_x \theta^2(t))\bigr)\bigl(\alpha(\eta^1(t)) - \alpha(\eta^2(t))\bigr)\, dx \\
\leq& \frac{\nu^2}{4}|\partial_x(\theta^1 - \theta^2)(t)|_H^2 + \frac{|\alpha'|_{L^\infty(\mathbb{R})}^2}{\nu^2}|(\eta^1 - \eta^2)(t)|_H^2, \text{for a.e. } t \in (0, T).
\end{aligned} \tag{5.4.6c}
$$

As the summation of (5.4.5) and (5.4.6), we obtain that:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigl(|(\eta^1 - \eta^2)(t)|_H^2 + |\sqrt{\alpha_0(t)}(\theta^1 - \theta^2)(t)|_H^2\bigr) \\
&\quad \leq\ C_A^1\bigl(|(\eta^1 - \eta^2)(t)|_H^2 + |\sqrt{\alpha_0(t)}(\theta^1 - \theta^2)(t)|_H^2\bigr),
\end{aligned} \tag{5.4.7}
$$

$$
\text{for a.e. } t \in (0, T), \text{ with } C_A^1 := L_* + \frac{2|\alpha'|_{L^\infty(\mathbb{R})}^2}{\nu^2} + \frac{|\partial_t \alpha_0|_{L^\infty(Q)}}{2\delta_*}.
$$

Now, with (A3) in mind, we can verify the uniqueness part of (I-A), just by applying Gronwall's lemma to the estimate (5.4.7). $\square$

Remark 5.10. As a consequence of the uniqueness result in (I-A), we can say that the converse of $(\star 0)$ in Remark 5.9 is also true, i.e. the three problems $(\mathrm{S})_0$, $(\mathrm{E})_0$, and $(\widetilde{\mathrm{E}})$ are equivalent each other.

Proof of Main Theorem 5.1 (I-B) By Remarks 5.8–5.10, the solution $w := [\eta, \theta] \in [\mathscr{H}]^2$ to the state-system $(\mathrm{S})_\varepsilon$ coincides with that to the Cauchy problem $(\mathrm{E})_\varepsilon$ for the initial data $w_0 := [\eta_0, \theta_0] \in V \times V_0$ and forcing term $\mathfrak{f} := [M_u u, M_v v] \in [\mathscr{H}]^2$. Also, putting:

$$
\begin{gathered}
w_n := [\eta_n, \theta_n] \text{ in } [\mathscr{H}]^2,\ w_{0,n} := [\eta_{0,n}, \theta_{0,n}] \text{ in } [H]^2, \\
\text{and } \mathfrak{f}_n := [M_u u_n, M_v v_n] \text{ in } [\mathscr{H}]^2, \text{ for } n = 1, 2, 3, \dots,
\end{gathered}
$$

we can suppose that the sequence $\{w_n\}_{n=1}^\infty = \{[\eta_n, \theta_n]\}_{n=1}^\infty$ of solutions to systems $(\mathrm{S})_{\varepsilon_n}$, $n = 1, 2, 3, \dots$, coincides with that of solutions to the problems $(\mathrm{E})_{\varepsilon_n}$, for the initial data $w_{0,n}$ and forcing terms $\mathfrak{f}_n$, $n = 1, 2, 3, \dots$. In addition:

(ev.4) $\Phi_{\varepsilon_n} \geq 0$ on $[H]^2$, for $n = 1, 2, 3, \dots$, and the union $\bigcup_{n=1}^\infty \bigl\{\tilde{w} \in [H]^2 \bigm| \Phi_{\varepsilon_n}(\tilde{w}) \leq r\bigr\}$ of sublevel sets is contained in the compact set $K_\nu(r) \subset [H]^2$, as in (ev.3), for any $r > 0$;

(ev.5) $\Phi_{\varepsilon_n} \to \Phi_\varepsilon$ on $[H]^2$, in the sense of Mosco, as $n \to \infty$, more precisely, the following estimate

$$
\begin{aligned}
&|\Phi_{\varepsilon_n}(w) - \Phi_\varepsilon(w)| \\
&\quad = \frac{1}{2}\left|\int_\Omega \left(\bigl(\nu f_{\varepsilon_n}(\partial_x \theta) + \nu^{-1}\alpha(\eta)\bigr)^2 - \bigl(\nu f_\varepsilon(\partial_x \theta) + \nu^{-1}\alpha(\eta)\bigr)^2\right) dx\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\nu}{2}\int_\Omega \bigl|\nu\bigl(f_{\varepsilon_n}(\partial_x\theta)+f_\varepsilon(\partial_x\theta)\bigr)+2\nu^{-1}\alpha(\eta)\bigr|\bigl|f_{\varepsilon_n}(\partial_x\theta)-f_\varepsilon(\partial_x\theta)\bigr|\,dx\\
&\leq \frac{\nu^2}{2}|\varepsilon_n-\varepsilon|\int_\Omega \left((\varepsilon_n+\varepsilon)+2|\partial_x\theta|+\frac{2}{\nu^2}\alpha(\eta)\right)dx\\
&\leq \nu^2|\varepsilon_n-\varepsilon|\int_\Omega \left(1+|\partial_x\theta|+\frac{1}{\nu^2}\alpha(\eta)\right)dx,\\
&\qquad \text{for any } w=[\eta,\theta]\in V\times V_0,\ n=1,2,3,\dots, \qquad (5.4.8)
\end{aligned}
$$

where we use the following inequality:

$$
\begin{aligned}
|f_\varepsilon(\omega)-f_\delta(\omega)| &= \left|\frac{\varepsilon^2-\delta^2}{\sqrt{\varepsilon^2+|\omega|^2}+\sqrt{\delta^2+|\omega|^2}}\right|\\
&= \frac{|\varepsilon+\delta|}{\sqrt{\varepsilon^2+|\omega|^2}+\sqrt{\delta^2+|\omega|^2}}|\varepsilon-\delta|\\
&\leq |\varepsilon-\delta|,\ \text{for any } \varepsilon,\delta\in[0,1],\ \text{and } \omega\in\mathbb{R}.
\end{aligned}
$$

Immediately leads to the corresponding lower bound condition and optimality condition, in the Mosco-convergence of $\{\Phi_{\varepsilon_n}\}_{n=1}^\infty$;

(ev.6) $\sup_{n\in\mathbb{N}}\Phi_{\varepsilon_n}(w_{0,n})<\infty$, and

$$
w_{0,n}\to w_0 \text{ in } [H]^2,\ \text{as } n\to\infty,
$$

more precisely, it follows from (5.3.4) and (A3) that

$$
\sup_{n\in\mathbb{N}}\Phi_{\varepsilon_n}(w_{0,n})\leq \sup_{n\in\mathbb{N}}\left(\frac{1}{2}|\eta_{0,n}|_V^2+\nu^2(1+|\theta_{0,n}|_{V_0}^2)+\frac{1}{\nu^2}|\alpha(\eta_{0,n})|_H^2\right)<\infty,
$$

and moreover, the weak convergence of $\{w_{0,n}\}_{n=1}^\infty$ in $V\times V_0$ and the compactness of embedding $V\times V_0\subset[H]^2$ imply the strong convergence of $\{w_{0,n}\}_{n=1}^\infty$ in $[H]^2$.

On account of (5.3.4) and (ev.0)–(ev.6), we can apply Lemma 5.4, to show that:

$$
\begin{cases}
w_n\to w \text{ in } C([0,T];[H]^2) \text{ (i.e. in } [C([0,T];H)]^2),\\
\quad \text{weakly in } W^{1,2}(0,T;[H]^2) \text{ (i.e. weakly in } [W^{1,2}(0,T;H)]^2),\\
\displaystyle\int_0^T \Phi_{\varepsilon_n}(w_n(t))\,dt\to\int_0^T\Phi_\varepsilon(w(t))\,dt,
\end{cases}
\text{as } n\to\infty, \quad (5.4.9a)
$$

$$
\begin{aligned}
\sup_{n\in\mathbb{N}}|w_n|^2_{L^\infty(0,T;V)\times L^\infty(0,T;V_0)} &\leq 4\sup_{n\in\mathbb{N}}|w_n|^2_{L^\infty(0,T;V\times V_0)}\\
&\leq \frac{8}{\min\{1,\nu^2\}}\sup_{n\in\mathbb{N}}|\Phi_{\varepsilon_n}(w_n)|_{L^\infty(0,T)}<\infty,
\end{aligned}
$$

and hence,

$$
w_n\to w \text{ weakly-}* \text{ in } L^\infty(0,T;V)\times L^\infty(0,T;V_0),\ \text{as } n\to\infty. \qquad (5.4.9b)
$$

Also, as a consequence of the one-dimensional compact embeddings $V \subset C(\overline{\Omega})$ and $V_0 \subset C(\overline{\Omega})$, the uniqueness of solution w to $(\mathrm{E})_\varepsilon$, and Ascoli's theorem (cf. [83, Corollary 4]), we can derive from (5.4.9a) that

$$w_n \to w \text{ in } [C(\overline{Q})]^2, \text{ as } n \to \infty. \tag{5.4.10}$$

Furthermore, from (5.4.9), (5.4.10), and the assumptions (A1) and (A3), one can observe that:

$$\begin{cases} \varliminf_{n\to\infty} \frac{1}{2}|\eta_n|_{\mathscr{V}}^2 \geq \frac{1}{2}|\eta|_{\mathscr{V}}^2, \quad \varliminf_{n\to\infty} \frac{\nu^2}{2}|\theta_n|_{\mathscr{V}_0}^2 \geq \frac{\nu^2}{2}|\theta|_{\mathscr{V}_0}^2, \\ \lim_{n\to\infty} \frac{1}{2\nu^2}|\alpha(\eta_n)|_{\mathscr{H}}^2 = \frac{1}{2\nu^2}|\alpha(\eta)|_{\mathscr{H}}^2, \end{cases} \tag{5.4.11a}$$

and

$$\begin{aligned} \varliminf_{n\to\infty} \bigl|\alpha(\eta_n)f_{\varepsilon_n}(\partial_x\theta_n)\bigr|_{L^1(Q)} &= \varliminf_{n\to\infty} \int_0^T \int_\Omega \alpha(\eta_n(t))f_{\varepsilon_n}(\partial_x\theta_n(t))\,dxdt \\ &\geq \varliminf_{n\to\infty} \int_0^T \int_\Omega \alpha(\eta(t))f_{\varepsilon_n}(\partial_x\theta_n(t))\,dxdt \\ &\qquad - \lim_{n\to\infty} |\alpha(\eta_n) - \alpha(\eta)|_{C(\overline{Q})} \cdot \sup_{n\in\mathbb{N}}\bigl(T\varepsilon_n + |\partial_x\theta_n|_{L^1(0,T;L^1(\Omega))}\bigr) \\ &\geq \varliminf_{n\to\infty} \int_0^T \int_\Omega \alpha(\eta(t))f_\varepsilon(\partial_x\theta_n(t))\,dxdt - |\alpha(\eta)|_{C(\overline{Q})} \cdot \lim_{n\to\infty}\bigl(T|\varepsilon_n - \varepsilon|\bigr) \\ &\geq \int_0^T \int_\Omega \alpha(\eta(t))f_\varepsilon(\partial_x\theta(t))\,dxdt = \bigl|\alpha(\eta)f_\varepsilon(\partial_x\theta)\bigr|_{L^1(Q)}. \end{aligned} \tag{5.4.11b}$$

Here, from (5.4.3), it is seen that:

$$\begin{aligned} \int_0^T \Phi_{\tilde{\varepsilon}}(\tilde{w}(t))\,dt &= \int_0^T \Phi_{\tilde{\varepsilon}}(\tilde{\eta}(t), \tilde{\theta}(t))\,dt \\ &= \frac{1}{2}|\tilde{\eta}|_{\mathscr{V}}^2 + \frac{\nu^2}{2}|\tilde{\theta}|_{\mathscr{V}_0}^2 + \bigl|\alpha(\tilde{\eta})f_{\tilde{\varepsilon}}(\partial_x\tilde{\theta})\bigr|_{L^1(Q)} + \frac{1}{2\nu^2}|\alpha(\tilde{\eta})|_{\mathscr{H}}^2 + \frac{\nu^2\tilde{\varepsilon}^2}{2}T \\ &\qquad \text{for all } \tilde{\varepsilon} > 0 \text{ and } \tilde{w} = [\tilde{\eta}, \tilde{\theta}] \in D(\Phi_{\tilde{\varepsilon}}) = \mathscr{Y}. \end{aligned} \tag{5.4.12}$$

Taking into account (5.4.9a), (5.4.11), and (5.4.12), we deduce that:

$$|\eta_n|_{\mathscr{V}}^2 + \nu^2|\theta_n|_{\mathscr{V}_0}^2 \to |\eta|_{\mathscr{V}}^2 + \nu^2|\theta|_{\mathscr{V}_0}^2, \text{and hence, } |w_n|_{\mathscr{Y}} \to |w|_{\mathscr{Y}}, \text{ as } n \to \infty. \tag{5.4.13}$$

Since the norm of Hilbert space $\mathscr{Y}$ is uniformly convex, the convergences (5.4.9b) and (5.4.13) imply the strong convergences:

$$w_n \to w \text{ in } \mathscr{Y}, \text{ as } n \to \infty, \tag{5.4.14a}$$

and

$$\begin{aligned} &|f_{\varepsilon_n}(\partial_x\theta_n) - f_\varepsilon(\partial_x\theta)|_{\mathscr{H}} \\ &\quad \leq |f_{\varepsilon_n}(\partial_x\theta_n) - f_{\varepsilon_n}(\partial_x\theta)|_{\mathscr{H}} + |f_{\varepsilon_n}(\partial_x\theta) - f_\varepsilon(\partial_x\theta)|_{\mathscr{H}} \\ &\quad \leq |\theta_n - \theta|_{\mathscr{V}_0} + \sqrt{T}|\varepsilon_n - \varepsilon| \to 0, \text{ as } n \to \infty. \end{aligned} \tag{5.4.14b}$$

The convergences (5.4.9) and (5.4.14) are sufficient to verify the conclusions (5.3.5) and (5.3.6) of Main Theorem 5.1 (I-B). □

5.5 Proof of main Theorem 5.2

In this section, we prove the second Main Theorem 5.2. Let $[\eta_0, \theta_0] \in V \times V_0$ be the initial pair, fixed in (A1). Also, let us fix arbitrary forcing pair $[\bar{u}, \bar{v}] \in [\mathscr{H}]^2$, and let us invoke the definition of the cost function (1.5.29), to estimate that:

$$0 \leq \underline{J}_\varepsilon := \inf \mathcal{J}_\varepsilon([\mathscr{H}]^2) \leq \overline{J}_\varepsilon := \mathcal{J}_\varepsilon(\bar{u}, \bar{v}) < \infty, \text{ for all } \varepsilon \geq 0. \tag{5.5.1}$$

Also, for any $\varepsilon \geq 0$, we denote by $[\bar{\eta}_\varepsilon, \bar{\theta}_\varepsilon]$ the solution to $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[\bar{u}, \bar{v}]$.

Based on these, the proof of Main Theorem 5.2 is demonstrated as follows.

Proof of Main Theorem 5.2 (II-A) Let us fix any $\varepsilon \geq 0$. Then, from the estimate (5.5.1), we immediately find a sequence of forcing pairs $\{[u_n, v_n]\}_{n=1}^\infty \subset [\mathscr{H}]^2$, such that:

$$\mathcal{J}_\varepsilon(u_n, v_n) \downarrow \underline{J}_\varepsilon, \text{ as } n \to \infty, \tag{5.5.2a}$$

and

$$\sup_{n \in \mathbb{N}} \bigl|[\sqrt{M_u} u_n, \sqrt{M_v} v_n]\bigr|_{[\mathscr{H}]^2}^2 \leq \mathcal{J}_\varepsilon(\bar{u}, \bar{v}) < \infty. \tag{5.5.2b}$$

Also, the estimate (5.5.2b) enables us to take a subsequence of $\{[u_n, v_n]\}_{n=1}^\infty \subset [\mathscr{H}]^2$ (not relabeled), and to find a pair of functions $[u^*, v^*] \in [\mathscr{H}]^2$, such that:

$$[\sqrt{M_u} u_n, \sqrt{M_v} v_n] \to [\sqrt{M_u} u^*, \sqrt{M_v} v^*] \text{ weakly in } [\mathscr{H}]^2, \text{as } n \to \infty,$$

and as well as,

$$[M_u u_n, M_v v_n] \to [M_u u^*, M_v v^*] \text{ weakly in } [\mathscr{H}]^2, \text{ as } n \to \infty. \tag{5.5.3}$$

Let $[\eta^*, \theta^*] \in [\mathscr{H}]^2$ be the solution to $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u^*, v^*]$. As well as, for any $n \in \mathbb{N}$, let $[\eta_n, \theta_n] \in [\mathscr{H}]^2$ be the solution to $(\mathrm{S})_\varepsilon$, for the forcing pair $[u_n, v_n]$. Then, having in mind (5.5.3) and the initial condition:

$$[\eta_n(0), \theta_n(0)] = [\eta^*(0), \theta^*(0)] = [\eta_0, \theta_0] \text{ in } [H]^2, \text{ for } n = 1, 2, 3, \dots,$$

we can apply Main Theorem 5.1 (I-B), to see that:

$$[\eta_n, \theta_n] \to [\eta^*, \theta^*] \text{ in } [C(\overline{Q})]^2, \text{ as } n \to \infty. \tag{5.5.4}$$

On account of (5.5.2a), (5.5.3), and (5.5.4), it is computed that:

$$\begin{aligned}
\mathcal{J}_\varepsilon(u^*, v^*) &= \frac{1}{2} \bigl|[\sqrt{M_\eta}(\eta^* - \eta_{\mathrm{ad}}), \sqrt{M_\theta}(\theta^* - \theta_{\mathrm{ad}})]\bigr|_{[\mathscr{H}]^2}^2 \\
&\qquad + \frac{1}{2} \bigl|[\sqrt{M_u} u^*, \sqrt{M_v} v^*]\bigr|_{[\mathscr{H}]^2}^2 \\
&\leq \frac{1}{2} \lim_{n \to \infty} \bigl|[\sqrt{M_\eta}(\eta_n - \eta_{\mathrm{ad}}), \sqrt{M_\theta}(\theta_n - \theta_{\mathrm{ad}})]\bigr|_{[\mathscr{H}]^2}^2 \\
&\qquad + \frac{1}{2} \varliminf_{n \to \infty} \bigl|[\sqrt{M_u} u_n, \sqrt{M_v} v_n]\bigr|_{[\mathscr{H}]^2}^2 \\
&= \lim_{n \to \infty} \mathcal{J}_\varepsilon(u_n, v_n) = \underline{J}_\varepsilon \ (\leq \mathcal{J}_\varepsilon(u^*, v^*)),
\end{aligned}$$

and it implies that

$$\mathcal{J}_\varepsilon(u^*, v^*) = \min_{[u,v] \in [\mathscr{H}]^2} \mathcal{J}_\varepsilon(u, v).$$

Thus, we conclude the item (II-A). □

Proof of Main Theorem 5.2 (II-B) Let $\varepsilon \in [0,1]$ and $\{\varepsilon_n\}_{n=1}^{\infty} \subset [0,1]$ be as in (5.3.9). Let $[\bar{\eta}_\varepsilon, \bar{\theta}_\varepsilon] \in [\mathscr{H}]^2$ be the solution to the system $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[\bar{u}, \bar{v}]$, and let $[\bar{\eta}_{\varepsilon_n}, \bar{\theta}_{\varepsilon_n}] \in [\mathscr{H}]^2$, $n = 1, 2, 3, \ldots$, be the solutions to $(\mathrm{S})_{\varepsilon_n}$, for the respective initial pairs $[\eta_{0,n}, \theta_{0,n}]$, $n = 1, 2, 3, \ldots$, and the fixed forcing pair $[\bar{u}, \bar{v}]$. On this basis, let us first apply Main Theorem 5.1 (I-B) to the solutions $[\bar{\eta}_\varepsilon, \bar{\theta}_\varepsilon] \in [\mathscr{H}]^2$ and $[\bar{\eta}_{\varepsilon_n}, \bar{\theta}_{\varepsilon_n}] \in [\mathscr{H}]^2$, $n = 1, 2, 3, \ldots$. Then, we have

$$\begin{cases} [\bar{\eta}_{\varepsilon_n}, \bar{\theta}_{\varepsilon_n}] \to [\bar{\eta}_\varepsilon, \bar{\theta}_\varepsilon] \text{ in } [C(\overline{Q})]^2, \\ [\bar{\eta}_n(0), \bar{\theta}_n(0)] = [\eta_{0,n}, \theta_{0,n}] \\ \qquad \to [\eta_0, \theta_0] = [\bar{\eta}_\varepsilon(0), \bar{\theta}_\varepsilon(0)] \text{ in } [C(\overline{\Omega})]^2, \end{cases} \quad \text{as } n \to \infty, \tag{5.5.5}$$

and hence,

$$\overline{J}_{\sup} := \sup_{n \in \mathbb{N}} \mathcal{J}_{\varepsilon_n}(\bar{u}, \bar{v}) < \infty. \tag{5.5.6}$$

Next, for any $n \in \mathbb{N}$, let us denote by $[\eta_n^*, \theta_n^*] \in [\mathscr{H}]^2$ the solution to $(\mathrm{S})_{\varepsilon_n}$, for the initial pair $[\eta_{0,n}, \theta_{0,n}]$ and forcing pair $[u_n^*, v_n^*]$. Then, in the light of (5.5.1) and (5.5.6), we can see that:

$$0 \leq \frac{1}{2} |[\sqrt{M_u} u_n^*, \sqrt{M_v} v_n^*]|_{[\mathscr{H}]^2}^2 \leq \underline{J}_{\varepsilon_n} \leq \overline{J}_{\sup} < \infty, \text{ for } n = 1, 2, 3, \ldots.$$

Therefore, we can find a subsequence $\{n_i\}_{i=1}^{\infty} \subset \{n\}$, together with a pair of functions $[u^{**}, v^{**}] \in [\mathscr{H}]^2$, such that:

$$[\sqrt{M_u} u_{n_i}^*, \sqrt{M_v} v_{n_i}^*] \to [\sqrt{M_u} u^{**}, \sqrt{M_v} v^{**}] \text{ weakly in } [\mathscr{H}]^2, \text{ as } i \to \infty,$$

and as well as,

$$[M_u u_{n_i}^*, M_v v_{n_i}^*] \to [M_u u^{**}, M_v v^{**}] \text{ weakly in } [\mathscr{H}]^2, \text{ as } i \to \infty. \tag{5.5.7}$$

Here, let us denote by $[\eta^{**}, \theta^{**}] \in [\mathscr{H}]^2$ the solution to $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u^{**}, v^{**}]$. Then, applying Main Theorem 5.1 (I-B), again, to the solutions $[\eta^{**}, \theta^{**}]$ and $[\eta_{n_i}^*, \theta_{n_i}^*]$, $i = 1, 2, 3, \ldots$, we can observe that:

$$[\eta_{n_i}^*, \theta_{n_i}^*] \to [\eta^{**}, \theta^{**}] \text{ in } [C(\overline{Q})]^2, \text{ as } i \to \infty. \tag{5.5.8}$$

Now, as a consequence of (5.5.5), (5.5.7), and (5.5.8), it is verified that:

$$\begin{aligned} \mathcal{J}_\varepsilon(u^{**}, v^{**}) &= \frac{1}{2} \big| [\sqrt{M_\eta}(\eta^{**} - \eta_{\mathrm{ad}}), \sqrt{M_\theta}(\theta^{**} - \theta_{\mathrm{ad}})] \big|_{[\mathscr{H}]^2}^2 \\ &\qquad + \frac{1}{2} \big| [\sqrt{M_u} u^{**}, \sqrt{M_v} v^{**}] \big|_{[\mathscr{H}]^2}^2 \\ &\leq \frac{1}{2} \lim_{i \to \infty} \big| [\sqrt{M_\eta}(\eta_{n_i}^* - \eta_{\mathrm{ad}}), \sqrt{M_\theta}(\theta_{n_i}^* - \theta_{\mathrm{ad}})] \big|_{[\mathscr{H}]^2}^2 \\ &\qquad + \frac{1}{2} \varliminf_{i \to \infty} \big| [\sqrt{M_u} u_{n_i}^*, \sqrt{M_v} v_{n_i}^*] \big|_{[\mathscr{H}]^2}^2 \\ &\leq \varliminf_{i \to \infty} \mathcal{J}_{\varepsilon_{n_i}}(u_{n_i}^*, v_{n_i}^*) \leq \lim_{i \to \infty} \mathcal{J}_{\varepsilon_{n_i}}(\bar{u}, \bar{v}) \\ &= \frac{1}{2} \lim_{i \to \infty} \big| [\sqrt{M_\eta}(\bar{\eta}_{\varepsilon_{n_i}} - \eta_{\mathrm{ad}}), \sqrt{M_\theta}(\bar{\theta}_{\varepsilon_{n_i}} - \theta_{\mathrm{ad}})] \big|_{[\mathscr{H}]^2}^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\big|[\sqrt{M_u}\bar{u},\sqrt{M_v}\bar{v}]\big|^2_{[\mathscr{H}]^2}\\
&=\mathcal{J}_\varepsilon(\bar{u},\bar{v}).
\end{aligned}
$$

Since the choice of $[\bar{u},\bar{v}] \in [\mathscr{H}]^2$ is arbitrary, we conclude that:

$$
\mathcal{J}_\varepsilon(u^{**},v^{**}) = \min_{[u,v]\in[\mathscr{H}]^2} \mathcal{J}_\varepsilon(u,v),
$$

and complete the proof of the item (II-B). □

5.6 Proof of main Theorem 5.3

This section is devoted to the proof of Main Theorem 5.3. To this end, we need to start with the case of $\varepsilon > 0$, and prepare some Lemmas, associated with the Gâteaux differential of the regular cost function $\mathcal{J}_\varepsilon$.

Let $\varepsilon > 0$ be a fixed constant, and let $[\eta_0,\theta_0] \in V \times V_0$ be the initial pair, fixed in (A1). Let us take any forcing pair $[u,v] \in [\mathscr{H}]^2$, and consider the unique solution $[\eta,\theta] \in [\mathscr{H}]^2$ to the state-system $(\mathrm{S})_\varepsilon$. Also, let us take any constant $\delta \in (-1,1) \setminus \{0\}$ and any pair of functions $[h,k] \in [\mathscr{H}]^2$, and consider another solution $[\eta^\delta,\theta^\delta] \in [\mathscr{H}]^2$ to the system $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0,\theta_0]$ and a perturbed forcing pair $[u+\delta h, v+\delta k]$. On this basis, we consider a sequence of pairs of functions $\{[\chi^\delta,\gamma^\delta]\}_{\delta\in(-1,1)\setminus\{0\}} \subset [\mathscr{H}]^2$, defined as:

$$
[\chi^\delta,\gamma^\delta] := \left[\frac{\eta^\delta-\eta}{\delta},\ \frac{\theta^\delta-\theta}{\delta}\right] \in [\mathscr{H}]^2, \text{ for } \delta \in (-1,1)\setminus\{0\}. \tag{5.6.1}
$$

This sequence acts a key-role in the computation of Gâteaux differential of the cost function $\mathcal{J}_\varepsilon$, for $\varepsilon > 0$.

Remark 5.11. Note that for any $\delta \in (-1,1)\setminus\{0\}$, the pair of functions $[\chi^\delta,\gamma^\delta] \in [\mathscr{H}]^2$ fulfills the following variational forms:

$$
\begin{aligned}
&(\partial_t\chi^\delta(t),\varphi)_H + (\partial_x\chi^\delta(t),\partial_x\varphi)_H\\
&\quad+\int_\Omega\left(\int_0^1 g'(\eta(t)+\varsigma\delta\chi^\delta(t))\,d\varsigma\right)\chi^\delta(t)\varphi\,dx\\
&\quad+\int_\Omega\left(f_\varepsilon(\partial_x\theta(t))\int_0^1 \alpha''(\eta(t)+\varsigma\delta\chi^\delta(t))\,d\varsigma\right)\chi^\delta(t)\varphi\,dx\\
&\quad+\int_\Omega\left(\alpha'(\eta^\delta(t))\int_0^1 f'_\varepsilon(\partial_x\theta(t)+\varsigma\delta\partial_x\gamma^\delta(t))\,d\varsigma\right)\partial_x\gamma^\delta(t)\varphi\,dx\\
&=(M_u h(t),\varphi)_H,\ \text{for any } \varphi\in V,\text{ a.e. } t\in(0,T),\text{ subject to } \chi^\delta(0)=0 \text{ in } H,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\alpha_0(t)\partial_t\gamma^\delta(t),\psi)_H + \nu^2(\partial_x\gamma^\delta(t),\partial_x\psi)_H\\
&\quad+\int_\Omega\left(\alpha(\eta^\delta(t))\int_0^1 f''_\varepsilon(\partial_x\theta(t)+\varsigma\delta\partial_x\gamma^\delta(t))\,d\varsigma\right)\partial_x\gamma^\delta(t)\partial_x\psi\,dx\\
&\quad+\int_\Omega\left(f'_\varepsilon(\partial_x\theta(t))\int_0^1 \alpha'(\eta(t)+\varsigma\delta\chi^\delta(t))\,d\varsigma\right)\chi^\delta(t)\partial_x\psi\,dx
\end{aligned}
$$

$$=(M_v k(t), \psi)_H, \text{ for any } \psi \in V_0, \text{ a.e. } t \in (0,T), \text{ subject to } \gamma^\delta(0) = 0 \text{ in } H.$$

In fact, these variational forms are obtained by taking the difference between respective two variational forms for $[\eta^\delta, \theta^\delta]$ and $[\eta, \theta]$, as in Main Theorem 5.1 (I-A), and by using the following linearization formulas:

$$\frac{1}{\delta}\big(g(\eta^\delta) - g(\eta)\big) = \left(\int_0^1 g'(\eta + \varsigma\delta\chi^\delta)\, d\varsigma\right) \chi^\delta \text{ in } \mathscr{H},$$

$$\begin{aligned}
&\frac{1}{\delta}\big(\alpha'(\eta^\delta) f_\varepsilon(\partial_x \theta^\delta) - \alpha'(\eta) f_\varepsilon(\partial_x \theta)\big) \\
&\quad = \frac{1}{\delta}\big((\alpha'(\eta^\delta) - \alpha'(\eta)) f_\varepsilon(\partial_x \theta)\big) + \frac{1}{\delta}\big(\alpha'(\eta^\delta)(f_\varepsilon(\partial_x \theta^\delta) - f_\varepsilon(\partial_x \theta))\big) \\
&\quad = \left(f_\varepsilon(\partial_x \theta) \int_0^1 \alpha''(\eta + \varsigma\delta\chi^\delta)\, d\varsigma\right) \chi^\delta \\
&\qquad + \left(\alpha'(\eta^\delta) \int_0^1 f_\varepsilon'(\partial_x \theta + \varsigma\delta\partial_x \gamma^\delta)\, d\varsigma\right) \partial_x \gamma^\delta \text{ in } \mathscr{H},
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\delta}\big(\alpha(\eta^\delta) f_\varepsilon'(\partial_x \theta^\delta) - \alpha(\eta) f_\varepsilon'(\partial_x \theta)\big) \\
&\quad = \frac{1}{\delta}\big(\alpha(\eta^\delta)(f_\varepsilon'(\partial_x \theta^\delta) - f_\varepsilon'(\partial_x \theta))\big) + \frac{1}{\delta}\big((\alpha(\eta^\delta) - \alpha(\eta)) f_\varepsilon'(\partial_x \theta)\big) \\
&\quad = \left(\alpha(\eta^\delta) \int_0^1 f_\varepsilon''(\partial_x \theta + \varsigma\delta\partial_x \gamma^\delta)\, d\varsigma\right) \partial_x \gamma^\delta \\
&\qquad + \left(f_\varepsilon'(\partial_x \theta) \int_0^1 \alpha'(\eta + \varsigma\delta\chi^\delta)\, d\varsigma\right) \chi^\delta \text{ in } \mathscr{H}.
\end{aligned}$$

Incidentally, the above linearization formulas can be verified as consequences of the assumptions (A1)–(A3) and the mean-value theorem (cf. [54, Theorem 5 in p. 313]).

Now, we verify the following two Lemmas.

Lemma 5.1. Let us fix $\varepsilon > 0$, and assume (A1)–(A3). Then, for any $[u, v] \in [\mathscr{H}]^2$, the cost function $\mathcal{J}_\varepsilon$ admits the Gâteaux derivative $\mathcal{J}_\varepsilon'(u, v) \in [\mathscr{H}]^2$ $(= ([\mathscr{H}]^2)^*)$, such that:

$$\begin{aligned}
\big(\mathcal{J}_\varepsilon'(u, v), [h, k]\big)_{[\mathscr{H}]^2} &= \big([M_\eta(\eta - \eta_{\mathrm{ad}}), M_\theta(\theta - \theta_{\mathrm{ad}})], \bar{\mathcal{P}}_\varepsilon[M_u h, M_v k]\big)_{[\mathscr{H}]^2} \\
&\quad + \big([M_u u, M_v v], [h, k]\big)_{[\mathscr{H}]^2}, \text{ for any } [h, k] \in [\mathscr{H}]^2. \qquad (5.6.2)
\end{aligned}$$

In the context, $[\eta, \theta]$ is the solution to the state-system $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u, v]$, and $\bar{\mathcal{P}}_\varepsilon : [\mathscr{H}]^2 \longrightarrow \mathscr{Z}$ is a bounded linear operator, which is given as a restriction $\mathcal{P}|_{\{[0,0]\} \times [\mathscr{H}]^2}$ of the (linear) isomorphism $\mathcal{P} = \mathcal{P}(a, b, \mu, \lambda, \omega, A) : [H]^2 \times \mathscr{Y}^* \longrightarrow \mathscr{Z}$, as in Proposition 5.3, in the case when:

$$\begin{cases}
[a, b] = [\alpha_0, 0] \text{ in } W^{1,\infty}(Q) \times L^\infty(Q), \\
\mu = \bar{\mu}_\varepsilon := \alpha''(\eta) f_\varepsilon(\partial_x \theta) \text{ in } L^\infty(0, T; H), \\
[\lambda, \omega, A] = [\bar{\lambda}_\varepsilon, \bar{\omega}_\varepsilon, \bar{A}_\varepsilon] := \big[g'(\eta), \alpha'(\eta) f_\varepsilon'(\partial_x \theta), \alpha(\eta) f_\varepsilon''(\partial_x \theta)\big] \\
\qquad \text{in } [L^\infty(Q)]^3.
\end{cases} \qquad (5.6.3)$$

Proof. Let us fix any $[u,v] \in [\mathscr{H}]^2$, and take any $\delta \in (-1,1)\backslash\{0\}$ and any $[h,k] \in [\mathscr{H}]^2$. Then, it is easily seen that:

$$
\begin{aligned}
&\frac{1}{\delta}\big(\mathcal{J}_\varepsilon(u+\delta h, v+\delta k) - \mathcal{J}_\varepsilon(u,v)\big) \\
&\quad = \left(\frac{M_\eta}{2}(\eta^\delta + \eta - 2\eta_{\rm ad}), \chi^\delta\right)_{\mathscr{H}} + \left(\frac{M_\theta}{2}(\theta^\delta + \theta - 2\theta_{\rm ad}), \gamma^\delta\right)_{\mathscr{H}} \\
&\qquad + \left(\frac{M_u}{2}(2u+\delta h), h\right)_{\mathscr{H}} + \left(\frac{M_v}{2}(2v+\delta k), k\right)_{\mathscr{H}}.
\end{aligned} \tag{5.6.4}
$$

Here, let us set:

$$
\begin{cases}
\bar{\mu}_\varepsilon^\delta := f_\varepsilon(\partial_x\theta)\displaystyle\int_0^1 \alpha''(\eta + \varsigma\delta\chi^\delta)\,d\varsigma \text{ in } L^\infty(0,T;H), \\
\bar{\lambda}_\varepsilon^\delta := \displaystyle\int_0^1 g'(\eta + \varsigma\delta\chi^\delta)\,d\varsigma \text{ in } L^\infty(Q), \\
\bar{\omega}_\varepsilon^\delta := \alpha'(\eta^\delta)\displaystyle\int_0^1 f_\varepsilon'(\partial_x\theta + \varsigma\delta\partial_x\gamma^\delta)\,d\varsigma \text{ in } L^\infty(Q), \\
\bar{A}_\varepsilon^\delta := \alpha(\eta^\delta)\displaystyle\int_0^1 f_\varepsilon''(\partial_x\theta + \varsigma\delta\partial_x\gamma^\delta)\,d\varsigma \text{ in } L^\infty(Q),
\end{cases} \tag{5.6.5a}
$$

and

$$
\begin{aligned}
\bar{k}_\varepsilon^\delta := M_v k + \partial_x \Bigg[&\chi^\delta f_\varepsilon'(\partial_x\theta)\int_0^1 \alpha'(\eta + \varsigma\delta\chi^\delta)\,d\varsigma \\
&- \chi^\delta\alpha'(\eta^\delta)\int_0^1 f_\varepsilon'(\partial_x\theta + \varsigma\delta\partial_x\gamma^\delta)\,d\varsigma \Bigg] \text{ in } \mathscr{V}_0^*, \\
&\text{for all } \delta \in (-1,1)\setminus\{0\}.
\end{aligned} \tag{5.6.5b}
$$

Then, in the light of Remark 5.11, one can say that:

$$
[\chi^\delta, \gamma^\delta] = \bar{\mathcal{P}}_\varepsilon^\delta[M_u h, \bar{k}_\varepsilon^\delta] \text{ in } \mathscr{Z}, \text{ for } \delta \in (-1,1)\setminus\{0\},
$$

by using the restriction $\bar{\mathcal{P}}_\varepsilon^\delta := \mathcal{P}|_{\{[0,0]\}\times\mathscr{Y}^*} : \mathscr{Y}^* \longrightarrow \mathscr{Z}$ of the (linear) isomorphism $\mathcal{P} = \mathcal{P}(a,b,\mu,\lambda,\omega,A) : [H]^2 \times \mathscr{Y}^* \longrightarrow \mathscr{Z}$, as in Proposition 5.3, in the case when:

$$
\begin{cases}
[a,b,\lambda,\omega,A] = [\alpha_0, 0, \bar{\lambda}_\varepsilon^\delta, \bar{\omega}_\varepsilon^\delta, \bar{A}_\varepsilon^\delta] \text{ in } W^{1,\infty}(Q) \times [L^\infty(Q)]^4, \\
\mu = \bar{\mu}_\varepsilon^\delta \text{ in } L^\infty(0,T;H), \text{ for } \delta \in (-1,1)\setminus\{0\}.
\end{cases}
$$

Besides, taking into account (5.1.3), (5.6.5), (A2), (A3), and Remarks 5.1 and 5.5, we have:

$$
\begin{aligned}
\bar{C}_0^* :=& \frac{81(1+\nu^2)}{\min\{1,\nu^2,\inf\alpha_0(Q)\}}\big(1 + |\alpha_0|_{W^{1,\infty}(Q)} + |g'|_{L^\infty(\mathbb{R})} + |\alpha'|_{L^\infty(\mathbb{R})}\big) \\
\geq& \frac{81(1+\nu^2)}{\min\{1,\nu^2,\inf\alpha_0(Q)\}} \sup_{0<|\delta|<1}\left\{1 + |\alpha_0|_{W^{1,\infty}(Q)} + |\bar{\lambda}_\varepsilon^\delta|_{L^\infty(Q)} + |\bar{\omega}_\varepsilon^\delta|_{L^\infty(Q)}\right\},
\end{aligned} \tag{5.6.6a}
$$

and

$$
\big|\big\langle [M_u h(t), \bar{k}_\varepsilon^\delta(t)], [\varphi,\psi] \big\rangle_{V\times V_0}\big| \leq |\langle M_u h(t), \varphi\rangle_V| + |\langle \bar{k}_\varepsilon^\delta(t), \psi\rangle_{V_0}|
$$

$$\leq |M_u h(t)|_H |\varphi|_H + |M_v k(t)|_H |\psi|_H + 2|\alpha'|_{L^\infty(\mathbb{R})} |\chi^\delta(t)|_H |\partial_x \psi|_H$$
$$\leq M_u |h(t)|_H |\varphi|_V + \bigl(\sqrt{2} M_v |k(t)|_H + 2|\alpha'|_{L^\infty(\mathbb{R})} |\chi^\delta(t)|_H\bigr) |\psi|_{V_0}, \tag{5.6.6b}$$
$$\text{for a.e. } t \in (0,T), \text{ any } [\varphi, \psi] \in V \times V_0, \text{ and any } \delta \in (-1,1) \setminus \{0\},$$

so that

$$\bigl|[M_u h(t), \bar{k}_\varepsilon^\delta(t)]\bigr|^2_{V^* \times V_0^*} \leq \bar{C}_1^* \bigl(\bigl|[h(t), k(t)]\bigr|^2_{[H]^2} + |\chi^\delta(t)|^2_H\bigr),$$
$$\text{for a.e. } t \in (0,T), \text{ and any } \delta \in (-1,1) \setminus \{0\}, \tag{5.6.6c}$$

with a positive constant $\bar{C}_1^* := 4\bigl(M_u^2 + M_v^2 + |\alpha'|^2_{L^\infty(\mathbb{R})}\bigr)$.

Now, having in mind (5.6.6), let us apply Proposition 5.2 to the case when:

$$\begin{cases} [a^1, b^1, \mu^1, \lambda^1, \omega^1, A^1] = [a^2, b^2, \mu^2, \lambda^2, \omega^2, A^2] = [\alpha_0, 0, \overline{\mu}_\varepsilon^\delta, \bar{\lambda}_\varepsilon^\delta, \bar{\omega}_\varepsilon^\delta, \bar{A}_\varepsilon^\delta], \\ [p_0^1, z_0^1] = [p_0^2, z_0^2] = [0,0], \ [h^1, k^1] = [M_u h, \bar{k}_\varepsilon^\delta], \ [h^2, k^2] = [0,0], \\ [p^1, z^1] = [\chi^\delta, \gamma^\delta] = \bar{\mathcal{P}}_\varepsilon^\delta [M_u h, \bar{k}_\varepsilon^\delta], \ [p^2, z^2] = [0,0] = \bar{\mathcal{P}}_\varepsilon^\delta [0,0], \end{cases}$$
$$\text{for } \delta \in (-1,1) \setminus \{0\}.$$

Then, we estimate that:

$$\frac{d}{dt}\bigl(|\chi^\delta(t)|^2_H + |\sqrt{\alpha_0(t)}\gamma^\delta(t)|^2_H\bigr) + \bigl(|\chi^\delta(t)|^2_V + \nu^2 |\gamma^\delta(t)|^2_{V_0}\bigr)$$
$$\leq 3\bar{C}_0^* \bigl(|\chi^\delta(t)|^2_H + |\sqrt{\alpha_0(t)}\gamma^\delta(t)|^2_H\bigr) + 2\bar{C}_0^* \bigl(|M_u h(t)|^2_{V^*} + |\bar{k}_\varepsilon^\delta(t)|^2_{V_0^*}\bigr)$$
$$\leq 3\bar{C}_0^*(1 + \bar{C}_1^*)\bigl(|\chi^\delta(t)|^2_H + |\sqrt{\alpha_0(t)}\gamma^\delta(t)|^2_H\bigr) + 2\bar{C}_0^* \bar{C}_1^* \bigl(|h(t)|^2_H + |k(t)|^2_H\bigr),$$
$$\text{for a.e. } t \in (0,T),$$

and subsequently, by using (A3) and Gronwall's lemma, we observe that:

($\star$1) the sequence $\{[\chi^\delta, \gamma^\delta]\}_{\delta \in (-1,1)\setminus\{0\}}$ is bounded in $[C([0,T];H)]^2 \cap \mathscr{Y}$.

Meanwhile, as consequences of (5.6.1), (5.6.3)–(5.6.6), ($\star$1), (A1)–(A3), Main Theorem 5.1, Remark 5.6, and Lebesgue's dominated convergence theorem, one can find a sequence $\{\delta_n\}_{n=1}^\infty \subset \mathbb{R}$, such that:

$$0 < |\delta_n| < 1, \text{ and } \delta_n \to 0, \text{ as } n \to \infty, \tag{5.6.7a}$$

$$\begin{cases} [\delta_n \chi^{\delta_n}, \delta_n \gamma^{\delta_n}] = [\eta^{\delta_n} - \eta, \theta^{\delta_n} - \theta] \to [0,0] \\ \quad \text{in } [C(\overline{Q})]^2, \text{ and in } \mathscr{Y}, \\ [\delta_n \partial_x \chi^{\delta_n}, \delta_n \partial_x \gamma^{\delta_n}] = [\partial_x(\eta^{\delta_n} - \eta), \partial_x(\theta^{\delta_n} - \theta)] \to [0,0] \\ \quad \text{in } [\mathscr{H}]^2, \text{ and in the pointwise sense a.e. in } Q, \end{cases} \text{as } n \to \infty, \tag{5.6.7b}$$

$$[\bar{\lambda}_\varepsilon^{\delta_n}, \bar{\omega}_\varepsilon^{\delta_n}, \bar{A}_\varepsilon^{\delta_n}] \to [\bar{\lambda}_\varepsilon, \bar{\omega}_\varepsilon, \bar{A}_\varepsilon] \text{ weakly-}* \text{ in } [L^\infty(Q)]^3,$$
$$\text{and in the pointwise sense a.e. in } Q, \quad \text{as } n \to \infty, \tag{5.6.7c}$$

$$\begin{cases} \bar{\mu}_\varepsilon^{\delta_n} \to \bar{\mu}_\varepsilon \text{ weakly-}* \text{ in } L^\infty(0,T;H), \\ \bar{\mu}_\varepsilon^{\delta_n}(t) \to \bar{\mu}_\varepsilon(t) \text{ in } H, \text{ for a.e. } t \in (0,T), \end{cases} \quad \text{as } n \to \infty, \tag{5.6.7d}$$

and

$$\begin{aligned} \langle \bar{k}_\varepsilon^{\delta_n} - M_v k, \psi \rangle_{\mathscr{V}_0} = & - \left(\chi^{\delta_n},\ f_\varepsilon'(\partial_x \theta) \left(\int_0^1 \alpha'(\eta + \varsigma \delta_n \chi^{\delta_n})\, d\varsigma \right) \partial_x \psi \right)_{\mathscr{H}} \\ & + \left(\chi^{\delta_n},\ \alpha'(\eta^{\delta_n}) \left(\int_0^1 f_\varepsilon'(\partial_x \theta + \varsigma \delta_n \partial_x \gamma^{\delta_n})\, d\varsigma \right) \partial_x \psi \right)_{\mathscr{H}} \\ & \to 0, \text{ as } n \to \infty. \end{aligned} \tag{5.6.7e}$$

On account of (5.6.1) and (5.6.3)–(5.6.7), we can apply Proposition 5.4 (B), and can see that:

$$\begin{aligned} [\chi^{\delta_n}, \gamma^{\delta_n}] = \bar{\mathcal{P}}_\varepsilon^{\delta_n}[M_u h, \bar{k}_\varepsilon^{\delta_n}] \to [\chi, \gamma] := \bar{\mathcal{P}}_\varepsilon[M_u h, M_v k] \text{ in } [\mathscr{H}]^2, \text{ weakly in } \mathscr{Y}, \\ \text{and weakly in } W^{1,2}(0,T;V^*) \times W^{1,2}(0,T;V_0^*), \text{ as } n \to \infty. \end{aligned} \tag{5.6.8}$$

Since the uniqueness of the solution $[\chi, \gamma] = \bar{\mathcal{P}}_\varepsilon[M_u h, M_v k]$ is guaranteed by Proposition 5.1, the observations (5.6.4), (5.6.7), and (5.6.8) enable us to compute the directional derivative $D_{[h,k]}\mathcal{J}_\varepsilon(u,v) \in \mathbb{R}$, as follows:

$$\begin{aligned} D_{[h,k]}\mathcal{J}_\varepsilon(u,v) := & \lim_{\delta \to 0} \frac{1}{\delta} \big(\mathcal{J}_\varepsilon(u + \delta h, v + \delta k) - \mathcal{J}_\varepsilon(u,v) \big) \\ = & \big([M_\eta(\eta - \eta_{\mathrm{ad}}), M_\theta(\theta - \theta_{\mathrm{ad}})], \bar{\mathcal{P}}_\varepsilon[M_u h, M_v k] \big)_{[\mathscr{H}]^2} + \big([M_u u, M_v v], [h,k] \big)_{[\mathscr{H}]^2}, \\ & \text{for any } [u,v] \in [\mathscr{H}]^2, \text{ and any direction } [h,k] \in [\mathscr{H}]^2. \end{aligned}$$

Moreover, with Proposition 5.3 and Riesz's theorem in mind, we deduce the existence of the Gâteaux derivative $\mathcal{J}_\varepsilon'(u,v) \in ([\mathscr{H}]^2)^*$ $(= [\mathscr{H}]^2)$ at $[u,v] \in [\mathscr{H}]^2$, i.e.:

$$\big(\mathcal{J}_\varepsilon'(u,v), [h,k] \big)_{[\mathscr{H}]^2} = D_{[h,k]}\mathcal{J}_\varepsilon(u,v), \text{ for every } [u,v], [h,k] \in [\mathscr{H}]^2.$$

Thus, we conclude this lemma with the required property (5.6.2). □

Lemma 5.2. Under the assumptions (A1)–(A3), let $[u_\varepsilon^*, v_\varepsilon^*] \in [\mathscr{H}]^2$ be an optimal control of the problem $(\mathrm{OP})_\varepsilon$, and let $[\eta_\varepsilon^*, \theta_\varepsilon^*]$ be the solution to the system $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u_\varepsilon^*, v_\varepsilon^*]$. Also, let $\mathcal{P}_\varepsilon^* : [\mathscr{H}]^2 \longrightarrow \mathscr{Z}$ be the bounded linear operator, defined in Remark 5.7, with the use of the solution $[\eta_\varepsilon^*, \theta_\varepsilon^*]$. Let $\mathcal{P}_\varepsilon : [\mathscr{H}]^2 \longrightarrow \mathscr{Z}$ be a bounded linear operator, which is defined as a restriction $\mathcal{P}|_{\{[0,0]\} \times [\mathscr{H}]^2}$ of the linear isomorphism $\mathcal{P} = \mathcal{P}(a, b, \mu, \lambda, \omega, A) : [H]^2 \times \mathscr{Y}^* \longrightarrow \mathscr{Z}$, as in Proposition 5.3, in the case when:

$$\begin{cases} [a,b] = [\alpha_0, 0] \text{ in } W^{1,\infty}(Q) \times L^\infty(Q), \\ \mu = \alpha''(\eta_\varepsilon^*) f_\varepsilon(\partial_x \theta_\varepsilon^*) \text{ in } L^\infty(0,T;H), \\ [\lambda, \omega, A] = \big[g'(\eta_\varepsilon^*), \alpha'(\eta_\varepsilon^*) f_\varepsilon'(\partial_x \theta_\varepsilon^*), \alpha(\eta_\varepsilon^*) f_\varepsilon''(\partial_x \theta_\varepsilon^*) \big] \text{ in } [L^\infty(Q)]^3. \end{cases} \tag{5.6.9}$$

Then, the operators $\mathcal{P}_\varepsilon^*$ and $\mathcal{P}_\varepsilon$ have a conjugate relationship, in the following sense:

$$\begin{aligned} \big(\mathcal{P}_\varepsilon^*[u,v], [h,k] \big)_{[\mathscr{H}]^2} = \big([u,v], \mathcal{P}_\varepsilon[h,k] \big)_{[\mathscr{H}]^2}, \\ \text{for all } [h,k], [u,v] \in [\mathscr{H}]^2. \end{aligned}$$

Proof. Let us fix arbitrary pairs of functions $[h, k], [u, v] \in [\mathscr{H}]^2$, and let us put:

$$[\chi_\varepsilon, \gamma_\varepsilon] := \mathcal{P}_\varepsilon[h, k] \quad \text{and} \quad [p_\varepsilon, z_\varepsilon] := \mathcal{P}_\varepsilon^*[u, v], \text{ in } [\mathscr{H}]^2.$$

Then, invoking Proposition 5.1, and the settings as in (5.3.18) and (5.6.9), we compute that:

$$\begin{aligned}
&\bigl(\mathcal{P}_\varepsilon^*[u, v], [h, k]\bigr)_{[\mathscr{H}]^2} = \int_0^T \bigl(p_\varepsilon(t), h(t)\bigr)_H \, dt + \int_0^T \bigl(z_\varepsilon(t), k(t)\bigr)_H \, dt \\
&= \int_0^T \langle h(t), p_\varepsilon(t)\rangle_V \, dt + \int_0^T \langle k(t), z_\varepsilon(t)\rangle_{V_0} \, dt \\
&= \int_0^T \Bigl[\bigl\langle \partial_t \chi_\varepsilon(t), p_\varepsilon(t)\bigr\rangle_V + \bigl(\partial_x \chi_\varepsilon(t), \partial_x p_\varepsilon(t)\bigr)_H \\
&\qquad + \bigl(\alpha''(\eta_\varepsilon^*(t)) f_\varepsilon(\partial_x \theta_\varepsilon^*(t)) \chi_\varepsilon(t), p_\varepsilon(t)\bigr)_H \\
&\qquad + \bigl(g'(\eta_\varepsilon^*(t)) \chi_\varepsilon(t), p_\varepsilon(t)\bigr)_H + \bigl(\alpha'(\eta_\varepsilon^*(t)) f_\varepsilon'(\partial_x \theta_\varepsilon^*(t)) \partial_x \gamma_\varepsilon(t), p_\varepsilon(t)\bigr)_H \Bigr] \, dt \\
&\quad + \int_0^T \Bigl[\bigl\langle \alpha_0(t) \partial_t \gamma_\varepsilon(t), z_\varepsilon(t)\bigr\rangle_{V_0} + \bigl(\alpha'(\eta_\varepsilon^*(t)) f_\varepsilon'(\partial_x \theta_\varepsilon^*(t)) \chi_\varepsilon(t), \partial_x z_\varepsilon(t)\bigr)_H \\
&\qquad + \bigl(\alpha(\eta_\varepsilon^*(t)) f_\varepsilon''(\partial_x \theta_\varepsilon^*(t)) \partial_x \gamma_\varepsilon(t), \partial_x z_\varepsilon(t)\bigr)_H + \nu^2 \bigl(\partial_x \gamma_\varepsilon(t), \partial_x z_\varepsilon(t)\bigr)_H \Bigr] \, dt \\
&= \bigl(p_\varepsilon(T), \chi_\varepsilon(T)\bigr)_H - \bigl(p_\varepsilon(0), \chi_\varepsilon(0)\bigr)_H + \int_0^T \Bigl[\bigl\langle -\partial_t p_\varepsilon(t), \chi_\varepsilon(t)\bigr\rangle_V \\
&\qquad + \bigl(\partial_x p_\varepsilon(t), \partial_x \chi_\varepsilon(t)\bigr)_H + \bigl(\alpha''(\eta_\varepsilon^*(t)) f_\varepsilon(\partial_x \theta_\varepsilon^*(t)) p_\varepsilon(t), \chi_\varepsilon(t)\bigr)_H \\
&\qquad + \bigl(g'(\eta_\varepsilon^*(t)) p_\varepsilon(t), \chi_\varepsilon(t)\bigr)_H + \bigl(\alpha'(\eta_\varepsilon^*(t)) f_\varepsilon'(\partial_x \theta_\varepsilon^*(t)) \partial_x z_\varepsilon(t), \chi_\varepsilon(t)\bigr)_H \Bigr] \, dt \\
&\quad + \bigl(\alpha_0(T) z_\varepsilon(T), \gamma_\varepsilon(T)\bigr)_H - \bigl(\alpha_0(0) z_\varepsilon(0), \gamma_\varepsilon(0)\bigr)_H \\
&\qquad + \int_0^T \Bigl[\bigl\langle -\partial_t \bigl(\alpha_0 z_\varepsilon\bigr)(t), \gamma_\varepsilon(t)\bigr\rangle_{V_0} + \bigl(\alpha'(\eta_\varepsilon^*(t)) f_\varepsilon'(\partial_x \theta_\varepsilon^*(t)) p_\varepsilon(t), \partial_x \gamma_\varepsilon(t)\bigr)_H \\
&\qquad + \bigl(\alpha(\eta_\varepsilon^*(t)) f_\varepsilon''(\partial_x \theta_\varepsilon^*(t)) \partial_x z_\varepsilon(t), \partial_x \gamma_\varepsilon(t)\bigr)_H + \nu^2 \bigl(\partial_x z_\varepsilon(t), \partial_x \gamma_\varepsilon(t)\bigr)_H \Bigr] \, dt \\
&= (u, \chi_\varepsilon)_{\mathscr{H}} + (v, \gamma_\varepsilon)_{\mathscr{H}} = \bigl([u, v], \mathcal{P}_\varepsilon[h, k]\bigr)_{[\mathscr{H}]^2}.
\end{aligned}$$

□

Remark 5.12. Note that the operator $\mathcal{P}_\varepsilon \in \mathscr{L}([\mathscr{H}]^2; \mathscr{Z})$, as in Lemma 5.2, corresponds to the operator $\bar{\mathcal{P}}_\varepsilon \in \mathscr{L}([\mathscr{H}]^2; \mathscr{Z})$, as in the previous Lemma 5.1, under the special setting (5.6.9).

Now, we are ready to prove the Main Theorem 5.3.

Proof of (III-A) of Main Theorem 5.3. Let $[u_\varepsilon^*, v_\varepsilon^*] \in [\mathscr{H}]^2$ be the optimal control of $(\mathrm{OP})_\varepsilon$, with the solution $[\eta_\varepsilon^*, \theta_\varepsilon^*] \in [\mathscr{H}]^2$ to the system $(\mathrm{S})_\varepsilon$ for the initial pair $[\eta_0, \theta_0]$, as in (A1), and forcing pair $[u_\varepsilon^*, v_\varepsilon^*]$, and let $\mathcal{P}_\varepsilon, \mathcal{P}_\varepsilon^* \in \mathscr{L}([\mathscr{H}]^2; \mathscr{Z})$ be the two operators as in Lemma 5.2. Then, on the basis of the previous Lemmas 5.1 and 5.2, Main Theorem 5.3 (III-A) will be demonstrated as follows:

$$
\begin{aligned}
0 = \ & (\mathcal{J}_\varepsilon'(u_\varepsilon^*, v_\varepsilon^*), [h,k])_{[\mathscr{H}]^2} = \lim_{\delta \to 0} \frac{1}{\delta} \bigl(\mathcal{J}_\varepsilon(u_\varepsilon^* + \delta h, v_\varepsilon^* + \delta k) - \mathcal{J}_\varepsilon(u_\varepsilon^*, v_\varepsilon^*) \bigr) \\
= & \bigl([M_\eta(\eta_\varepsilon^* - \eta_{\mathrm{ad}}), M_\theta(\theta_\varepsilon^* - \theta_{\mathrm{ad}})], \mathcal{P}_\varepsilon [M_u h, M_v k] \bigr)_{[\mathscr{H}]^2} + \bigl([M_u u_\varepsilon^*, M_v v_\varepsilon^*], [h,k] \bigr)_{[\mathscr{H}]^2} \\
= & \bigl(\mathcal{P}_\varepsilon^* [M_\eta(\eta_\varepsilon^* - \eta_{\mathrm{ad}}), M_\theta(\theta_\varepsilon^* - \theta_{\mathrm{ad}})], [M_u h, M_v k] \bigr)_{[\mathscr{H}]^2} + \bigl([M_u u_\varepsilon^*, M_v v_\varepsilon^*], [h,k] \bigr)_{[\mathscr{H}]^2} \\
= & \bigl([M_u p_\varepsilon^*, M_v z_\varepsilon^*], [h,k] \bigr)_{[\mathscr{H}]^2} + \bigl([M_u u_\varepsilon^*, M_v v_\varepsilon^*], [h,k] \bigr)_{[\mathscr{H}]^2} \\
= & \bigl([M_u(p_\varepsilon^* + u_\varepsilon^*), M_v(z_\varepsilon^* + v_\varepsilon^*)], [h,k] \bigr)_{[\mathscr{H}]^2}, \text{ for any } [h,k] \in [\mathscr{H}]^2.
\end{aligned}
$$

□

Proof of (III-B) of Main Theorem 5.3. Let $[\eta_0, \theta_0] \in V \times V_0$ be the fixed initial pair as in (A1). For any $\varepsilon > 0$, let $[u_\varepsilon^*, v_\varepsilon^*] \in [\mathscr{H}]^2$, $[\eta_\varepsilon^*, \theta_\varepsilon^*] \in [\mathscr{H}]^2$, and $[p_\varepsilon^*, z_\varepsilon^*] \in \mathscr{Z}$ be as in Main Theorem 5.3 (III-A). Then, by Main Theorem 5.2 (II-B), we find an optimal control $[u^\circ, v^\circ] \in [\mathscr{H}]^2$ of $(\mathrm{OP})_0$, with a zero-convergent sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$, such that:

$$
[u_n^*, v_n^*] := [u_{\varepsilon_n}^*, v_{\varepsilon_n}^*] \to [u^\circ, v^\circ] \text{ weakly in } [\mathscr{H}]^2, \text{ as } n \to \infty. \tag{5.6.10a}
$$

Let $[\eta^\circ, \theta^\circ] \in [\mathscr{H}]^2$ be the solution to $(\mathrm{S})_0$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u^\circ, v^\circ]$. Then, having in mind Main Theorem 5.1 (I-B) and Remark 5.6, we can find a subsequence of $\{\varepsilon_n\}_{n=1}^\infty$ (not relabeled) and a function $\nu^\circ \in L^\infty(Q)$, such that:

$$
\begin{aligned}
[\eta_n^*, \theta_n^*] \ & := [\eta_{\varepsilon_n}^*, \theta_{\varepsilon_n}^*] \to [\eta^\circ, \theta^\circ] \text{ in } [C(\overline{Q})]^2, \text{ in } \mathscr{Y}, \\
& \text{and weakly-}* \text{ in } L^\infty(0,T;V) \times L^\infty(0,T;V_0), \text{ as } n \to \infty,
\end{aligned} \tag{5.6.10b}
$$

$$
\begin{aligned}
& [\partial_x \eta_n, \partial_x \theta_n] \to [\partial_x \eta^\circ, \partial_x \theta^\circ] \text{ in } [\mathscr{H}]^2, \\
& \qquad \text{and in the pointwise sense a.e. in } Q, \text{ as } n \to \infty,
\end{aligned} \tag{5.6.10c}
$$

$$
\begin{cases}
\mu_n^* := \alpha''(\eta_n^*) f_{\varepsilon_n}(\partial_x \theta_n^*) \to \mu^\circ := \alpha''(\eta^\circ)|\partial_x \theta^\circ| \\
\quad \text{weakly-}* \text{ in } L^\infty(0,T;H), \\
\quad \text{and in the pointwise sense a.e. in } Q, \\
\mu_n^*(t) \to \mu^\circ(t) \text{ in } H, \\
\quad \text{and in the pointwise sense for a.e. } t \in (0,T),
\end{cases} \quad \text{as } n \to \infty, \tag{5.6.10d}
$$

$$
\lambda_n^* := g'(\eta_n^*) \to \lambda^\circ := g'(\eta^\circ) \text{ in } C(\overline{Q}), \text{ as } n \to \infty, \tag{5.6.10e}
$$

$$
\begin{cases}
f_{\varepsilon_n}'(\partial_x \theta_n^*) \to \nu^\circ \text{ weakly-}* \text{ in } L^\infty(Q), \text{ as } n \to \infty, \\
|\nu^\circ| \le 1 \text{ a.e. in } Q,
\end{cases} \tag{5.6.10f}
$$

and

$$\omega_n^* := \alpha'(\eta_n^*) f'_{\varepsilon_n}(\partial_x \theta_n^*) \to \alpha'(\eta^\circ)\nu^\circ \text{ weakly-}* \text{ in } L^\infty(Q), \text{ as } n \to \infty. \tag{5.6.10g}$$

Besides, from (5.6.10c), (5.6.10f), Remark 5.3 (Fact 1) and (Fact 2), and [18, Proposition 2.16], one can see that:

$$\nu^\circ \in \partial f_0(\partial_x \theta^\circ) = \mathrm{Sgn}^1(\partial_x \theta^\circ) \text{ a.e. in } Q. \tag{5.6.11}$$

Next, let us put:

$$\begin{cases} [p_n^*, z_n^*] := [p_{\varepsilon_n}^*, z_{\varepsilon_n}^*] \text{ in } [\mathscr{H}]^2, \\ A_n^* := \alpha(\eta_n^*) f''_{\varepsilon_n}(\partial_x \theta_n^*) \text{ in } L^\infty(Q), \end{cases} \quad n = 1, 2, 3, \dots.$$

Then, from (5.3.10)–(5.3.13), and (5.3.19), it follows that:

$$[M_u(u_n^* + p_n^*), M_v(v_n^* + z_n^*)] = [0, 0] \text{ in } [\mathscr{H}]^2, \ n = 1, 2, 3, \dots, \tag{5.6.12a}$$

$$\begin{aligned} &\langle -\partial_t p_n^*, \varphi \rangle_{\mathscr{V}} + \big(\partial_x p_n^*, \partial_x \varphi\big)_{\mathscr{H}} + \big(\mu_n^* p_n^*, \varphi\big)_{\mathscr{H}} + \big(\lambda_n^* p_n^* + \omega_n^* \partial_x z_n^*, \varphi\big)_{\mathscr{H}} \\ &\quad = \big(M_\eta(\eta_n^* - \eta_{\mathrm{ad}}), \varphi\big)_{\mathscr{H}}, \text{for any } \varphi \in \mathscr{V}, \ n = 1, 2, 3, \dots, \end{aligned} \tag{5.6.12b}$$

$$\begin{aligned} &\langle -\alpha_0 \partial_t z_n^*, \psi \rangle_{\mathscr{V}_0} + \big((-\partial_t \alpha_0) z_n^*, \psi\big)_{\mathscr{H}} + \big(A_n^* \partial_x z_n^* + \nu^2 \partial_x z_n^* + \omega_n^* p_n^*, \partial_x \psi\big)_{\mathscr{H}} \\ &\quad = \big(M_\theta(\theta_n^* - \theta_{\mathrm{ad}}), \psi\big)_{\mathscr{H}}, \text{ for any } \psi \in \mathscr{V}_0, \ n = 1, 2, 3, \dots, \end{aligned} \tag{5.6.12c}$$

and

$$[p_n^*(T), z_n^*(T)] = [0, 0] \text{ in } [H]^2, \ n = 1, 2, 3, \dots. \tag{5.6.12d}$$

Here, invoking the operators $\mathcal{Q}_\varepsilon^* \in \mathscr{L}([\mathscr{H}]^2; \mathscr{Z})$ and $\mathcal{R}_T \in \mathscr{L}([\mathscr{H}]^2)$ as in Remark 5.7, we apply Proposition 5.2 to the case when:

$$\begin{cases} [a^1, b^1, \mu^1, \lambda^1, \omega^1, A^1] = [a^2, b^2, \mu^2, \lambda^2, \omega^2, A^2] \\ \qquad = \mathcal{R}_T[\alpha_0, -\partial_t \alpha_0, \mu_n^*, \lambda_n^*, \omega_n^*, A_n^*], \\ [p_0^1, z_0^1] = [p_0^2, z_0^2] = [0, 0], \\ [h^1, k^1] = [\mathcal{R}_T\big(M_\eta(\eta_n^* - \eta_{\mathrm{ad}})\big), \mathcal{R}_T\big(M_\theta(\theta_n^* - \theta_{\mathrm{ad}})\big)], \ [h^2, k^2] = [0, 0], \\ [p^1, z^1] = \mathcal{Q}_{\varepsilon_n}^*\big[\mathcal{R}_T[M_\eta(\eta_n^* - \eta_{\mathrm{ad}}), M_\theta(\theta_n^* - \theta_{\mathrm{ad}})]\big], \\ [p^2, z^2] = [0, 0] = \mathcal{Q}_{\varepsilon_n}^*\big[\mathcal{R}_T[0, 0]\big], \end{cases} \quad \text{for } n \in \mathbb{N}.$$

Then, with use of the constant $\bar{C}_0^*$ as in (5.6.6a), we deduced that:

$$\begin{aligned} &\frac{d}{dt}\big(\big|(\mathcal{R}_T p_n^*)(t)\big|_H^2 + \big|\mathcal{R}_T\big(\sqrt{\alpha_0} z_n^*\big)(t)\big|_H^2\big) \\ &\qquad + \big(\big|(\mathcal{R}_T p_n^*)(t)\big|_V^2 + \nu^2 \big|(\mathcal{R}_T z_n^*)(t)\big|_{V_0}^2\big) \\ &\quad \le 3\bar{C}_0^* \big(\big|(\mathcal{R}_T p_n^*)(t)\big|_H^2 + \big|\mathcal{R}_T\big(\sqrt{\alpha_0} z_n^*\big)(t)\big|_H^2\big) \\ &\qquad + 2\bar{C}_0^* \big(\big|\mathcal{R}_T\big(M_\eta(\eta_n^* - \eta_{\mathrm{ad}})\big)(t)\big|_{V^*}^2 + \big|\mathcal{R}_T\big(M_\theta(\theta_n^* - \theta_{\mathrm{ad}})\big)(t)\big|_{V_0^*}^2\big), \\ &\quad \text{for a.e. } t \in (0, T), \ n = 1, 2, 3, \dots. \end{aligned} \tag{5.6.13}$$

As a consequence of (5.6.6a), (5.6.10b), (5.6.13), (A3), and Gronwall's lemma, it is observed that:

($\star$2) the sequence $\{[p_n^*, z_n^*]\}_{n=1}^\infty$ is bounded in $[C([0,T];H)]^2 \cap \mathscr{Y}$.

Furthermore, from (5.1.1), (5.1.3), (5.6.10d), (5.6.10e), (5.6.10g), (5.6.12b), (5.6.12c), and (A3), we can derive the following estimates:

$$\begin{aligned} \left| \langle \partial_t p_n^*, \varphi \rangle_{\mathscr{V}} \right| &\leq \left| \left(\mu_n^* p_n^*, \varphi \right)_{\mathscr{H}} \right| + \left| \left(\partial_x p_n^*, \partial_x \varphi \right)_{\mathscr{H}} \right| \\ &\quad + \left| \left(\lambda_n^* p_n^* + \omega_n^* \partial_x z_n^*, \varphi \right)_{\mathscr{H}} \right| + \left| \left(M_\eta(\eta_n^* - \eta_{\mathrm{ad}}), \varphi \right)_{\mathscr{H}} \right| \\ &\leq C_1^* |\varphi|_{\mathscr{V}}, \text{ for any } \varphi \in \mathscr{V},\ n = 1, 2, 3, \ldots, \end{aligned} \tag{5.6.14}$$

and

$$\begin{aligned} &\left| \langle -\partial_x (A_n^* \partial_x z_n^*), \psi \rangle_{\mathscr{W}_0} \right| = \left| \left(A_n^* \partial_x z_n^*, \partial_x \psi \right)_{\mathscr{H}} \right| \\ &\quad \leq \left| \left(\alpha_0 z_n^*, \partial_t \psi \right)_{\mathscr{H}} \right| + \left| \left(\nu^2 \partial_x z_n^* + \omega_n^* p_n^*, \partial_x \psi \right)_{\mathscr{H}} \right| + \left| \left(M_\theta(\theta_n^* - \theta_{\mathrm{ad}}), \psi \right)_{\mathscr{H}} \right| \\ &\quad \leq C_2^* |\psi|_{\mathscr{W}_0}, \text{ for any } \psi \in C_{\mathrm{c}}^\infty(Q),\ n = 1, 2, 3, \ldots, \end{aligned} \tag{5.6.15}$$

with n-independent positive constants:

$$C_1^* := 2 \sup_{n \in \mathbb{N}} \left\{ \begin{array}{c} (1 + |\mu_n^*|_{L^\infty(0,T;H)} + |\lambda_n^*|_{L^\infty(Q)} + |\omega_n^*|_{L^\infty(Q)}) \\ \cdot \left(\left| [p_n^*, z_n^*] \right|_{\mathscr{Y}} + \left| M_\eta(\eta_n^* - \eta_{\mathrm{ad}}) \right|_{\mathscr{H}} \right) \end{array} \right\} (< \infty),$$

and

$$C_2^* := 2 \sup_{n \in \mathbb{N}} \left\{ \begin{array}{c} (1 + \nu^2 + |\alpha_0|_{L^\infty(Q)} + |\omega_n^*|_{L^\infty(Q)}) \\ \cdot \left(\left| [p_n^*, z_n^*] \right|_{\mathscr{Y}} + \left| M_\theta(\theta_n^* - \theta_{\mathrm{ad}}) \right|_{\mathscr{H}} \right) \end{array} \right\} (< \infty),$$

respectively.

Due to (5.6.10e)–(5.6.10g), (5.6.14), (5.6.15), ($\star$2), and the compactness theory of Aubin's type (cf. [83, Corollary 4]), we can find subsequences of $\{[p_n^*, z_n^*]\}_{n=1}^\infty \subset \mathscr{Y}$, $\{\omega_n^* \partial_x z_n^*\}_{n=1}^\infty \subset \mathscr{H}$, and $\{-\partial_x (A_n^* \partial_x z_n^*)\}_{n=1}^\infty \subset \mathscr{W}_0^*$ (not relabeled), together with the respective limits $[p^\circ, z^\circ] \in \mathscr{Y}$, $\xi^\circ \in \mathscr{H}$, and $\zeta^\circ \in \mathscr{W}_0^*$, such that:

$$\begin{cases} [p_n^*, z_n^*] \to [p^\circ, z^\circ] \text{ weakly in } \mathscr{Y}, \\ p_n^* \to p^\circ \text{ in } \mathscr{H}, \text{ weakly in } W^{1,2}(0,T;V^*), \\ \quad \text{and in the pointwise sense a.e. in } Q, \end{cases} \quad \text{as } n \to \infty, \tag{5.6.16a}$$

$$\begin{cases} \lambda_n^* p_n^* \to \lambda^\circ p^\circ \text{ in } \mathscr{H}, \\ \omega_n^* p_n^* \to \alpha'(\eta^\circ) \nu^\circ p^\circ \text{ weakly in } \mathscr{H}, \end{cases} \quad \text{as } n \to \infty, \tag{5.6.16b}$$

$$\omega_n^* \partial_x z_n^* \to \xi^\circ \text{ weakly in } \mathscr{H}, \text{ as } n \to \infty, \tag{5.6.16c}$$

and

$$-\partial_x (A_n^* \partial_x z_n^*) \to \zeta^\circ \text{ weakly in } \mathscr{W}_0^*, \text{ as } n \to \infty. \tag{5.6.16d}$$

Now, the properties (5.3.14)–(5.3.17) will be verified through the limiting observations for (5.6.12a)–(5.6.12d), as $n \to \infty$, with use of (5.6.10) and (5.6.16).

Thus, we complete the proof. □

5.7 Appendix

The objective of the Appendix is to reorganize the general theory of nonlinear evolution equation, which enables us to handle the state-systems $(\mathrm{S})_\varepsilon$, for all $\varepsilon \geq 0$ in a unified fashion.

In what follows, let X be an abstract Hilbert space. On this basis, the general theory will be stated by considering two Lemmas, and the proofs will be modified (mixed and reduced) versions of the existing theories, such as [14, 18, 39].

Lemma 5.3. Let $\{\mathcal{A}_0(t) \,|\, t \in [0,T]\} \subset \mathscr{L}(X)$ be a class of time-dependent bounded linear operators, let $\mathcal{G}_0 : X \longrightarrow X$ be a given nonlinear operator, and let $\Psi_0 : X \longrightarrow [0, \infty]$ be a non-negative, proper, l.s.c., and convex function, fulfilling the following conditions:

(cp.0) $\mathcal{A}_0(t) \in \mathscr{L}(X)$ is positive and selfadjoint, for any $t \in [0,T]$, and it holds that

$$(\mathcal{A}_0(t)w, w)_X \geq \kappa_0 |w|_X^2, \text{ for any } w \in X,$$

with some constant $\kappa_0 \in (0,1)$, independent of $t \in [0,T]$ and $w \in X$.

(cp.1) $\mathcal{A}_0 : [0,T] \longrightarrow \mathscr{L}(X)$ is Lipschitz continuous, so that $\mathcal{A}_0$ admits the (strong) time-derivative $\mathcal{A}_0'(t) \in \mathscr{L}(X)$ a.e. in $(0,T)$, and

$$A_T^* := \operatorname*{ess\,sup}_{t \in (0,T)} \bigl\{ \max\{ |\mathcal{A}_0(t)|_{\mathscr{L}(X)}, |\mathcal{A}_0'(t)|_{\mathscr{L}(X)} \} \bigr\} < \infty;$$

(cp.2) $\mathcal{G}_0 : X \longrightarrow X$ is a Lipschitz continuous operator with a Lipschitz constant L_0, and $\mathcal{G}_0$ has a C^1-potential functional $\widehat{\mathcal{G}}_0 : X \longrightarrow \mathbb{R}$, so that the Gâteaux derivative $\widehat{\mathcal{G}}_0'(w) \in X^* \ (= X)$ at any $w \in X$ coincides with $\mathcal{G}_0(w) \in X$;

(cp.3) $\Psi_0 \geq 0$ on X, and the sublevel set $\bigl\{ w \in X \,\big|\, \Psi_0(w) \leq r \bigr\}$ is compact in X, for any $r \geq 0$.

Then, for any initial data $w_0 \in D(\Psi_0)$ and a forcing term $\mathfrak{f}_0 \in L^2(0,T;X)$, the following Cauchy problem of evolution equation:

$$\text{(CP)} \quad \begin{cases} \mathcal{A}_0(t)w'(t) + \partial\Psi_0(w(t)) + \mathcal{G}_0(w(t)) \ni \mathfrak{f}_0(t) \text{ in } X, \quad t \in (0,T), \\ w(0) = w_0 \text{ in } X; \end{cases}$$

admits a unique solution $w \in L^2(0,T;X)$, in the sense that:

$$w \in W^{1,2}(0,T;X), \ \Psi_0(w) \in L^\infty(0,T), \tag{5.7.1}$$

and

$$\begin{gathered} \bigl(\mathcal{A}_0(t)w'(t) + \mathcal{G}_0(w(t)) - \mathfrak{f}_0(t), w(t) - \varpi \bigr)_X + \Psi_0(w(t)) \leq \Psi_0(\varpi), \\ \text{for any } \varpi \in D(\Psi_0), \text{ a.e. } t \in (0,T). \end{gathered} \tag{5.7.2}$$

Moreover, both $t \in [0,T] \mapsto \Psi_0(w(t)) \in [0,\infty)$ and $t \in [0,T] \mapsto \widehat{\mathcal{G}}_0(w(t)) \in \mathbb{R}$ are absolutely continuous functions in time, and

$$\begin{gathered} |\mathcal{A}_0(t)^{\frac{1}{2}} w'(t)|_X^2 + \frac{d}{dt} \Bigl(\Psi_0(w(t)) + \widehat{\mathcal{G}}_0(w(t)) \Bigr) = (\mathfrak{f}_0(t), w'(t))_X, \\ \text{for a.e. } t \in (0,T). \end{gathered} \tag{5.7.3}$$

Remark 5.13. Under the assumptions (cp.0) and (cp.1), it is easily verified that:

$$\begin{aligned}\frac{d}{dt}\big(\mathcal{A}_0(t)w(t),\varpi(t)\big)_X &= \big(\mathcal{A}_0(t)w(t),\varpi'(t)\big)_X \\ &\quad + \big(\mathcal{A}_0'(t)w(t),\varpi(t)\big)_X + \big(\mathcal{A}_0(t)w'(t),\varpi(t)\big)_X, \\ &\qquad \text{for a.e. } t\in(0,T), \text{ and all } w,\varpi\in W^{1,2}(0,T;X).\end{aligned}$$

Additionally, we can identify $\mathcal{A}_0 \in \mathscr{L}(L^2(0,T;X))$, and for arbitrary functions $w, \varpi \in L^2(0,T;X)$ and arbitrary sequences $\{w_n\}_{n=1}^{\infty}, \{\varpi_n\}_{n=1}^{\infty} \subset L^2(0,T;X)$, we can compute that:

$$\begin{aligned}&\big(\mathcal{A}_0 w_n,\varpi_n\big)_{L^2(0,T;X)} = \big(w_n,\mathcal{A}_0\varpi_n\big)_{L^2(0,T;X)} \\ &\quad \to \big(w,\mathcal{A}_0\varpi\big)_{L^2(0,T;X)} = \big(\mathcal{A}_0 w,\varpi\big)_{L^2(0,T;X)}, \text{ as } n\to\infty, \\ &\text{if } \varpi_n \to \varpi \text{ in } L^2(0,T;X), \text{ and } w_n \to w \text{ weakly in } L^2(0,T;X), \\ &\text{as } n\to\infty.\end{aligned}$$

Remark 5.14. Note that the assumptions (cp.2) and (cp.3) imply that the potential $\widehat{\mathcal{G}}_0$ is the so-called λ-convex functional. More precisely, for every $L > L_0$, the functional:

$$\begin{aligned}&\widehat{\mathcal{F}}_L : w \in X \mapsto \widehat{\mathcal{F}}_L(w) := \widehat{\mathcal{G}}_0(w) + L|w|_X^2 + \widehat{C}_0 \in \mathbb{R}, \\ &\qquad \text{with a constant } \widehat{C}_0 := |\widehat{\mathcal{G}}_0(0)| + \frac{|\mathcal{G}_0(0)|_X^2}{2L_0};\end{aligned} \tag{5.7.4}$$

is nonnegative, strictly convex, and coercive on X. Indeed, from the assumption (cp.2), we immediately see the strictly monotonicity property of the Gâteaux differential $\widehat{\mathcal{F}}_L' \in \mathscr{L}(X)$, as follows:

$$\begin{aligned}(\widehat{\mathcal{F}}_L'(w^1) - \widehat{\mathcal{F}}_L'(w^2), w^1 - w^2)_X &= (\mathcal{G}_0(w^1) - \mathcal{G}_0(w^2), w^1 - w^2)_X + 2L|w^1 - w^2|_X^2 \\ &\geq (2L - L_0)|w^1 - w^2|_X^2 > 0, \text{ if } w^\ell \in X,\ \ell = 1,2,\ w^1 \neq w^2, \text{ and } L > L_0.\end{aligned}$$

Hence, for every $L > L_0$, $\widehat{\mathcal{F}}_L$ is strictly convex on X (cf. [70, Theorem B in p. 99]). Moreover, with use of the mean-value theorem (cf. [54, Theorem 5 in p. 313]), one can verify the non-negativity and coercivity of $\widehat{\mathcal{F}}_L$ as follows:

$$\begin{aligned}\widehat{\mathcal{F}}_L(w) = &\ \widehat{\mathcal{G}}_0(0) + \left(\int_0^1 \mathcal{G}_0(\varsigma w)\,d\varsigma, w\right)_X + \big(L|w|_X^2 + \widehat{C}_0\big) \\ \geq &\ -|\widehat{\mathcal{G}}_0(0)| - L_0|w|_X^2 \int_0^1 \varsigma\,d\varsigma + \big(\mathcal{G}_0(0), w\big)_X + \big(L|w|_X^2 + \widehat{C}_0\big) \\ \geq &\ (L - L_0)|w|_X^2 \geq 0, \text{ for all } w \in X.\end{aligned}$$

Proof of Lemma 5.3. The existence result for the problem (CP) can be proved by means of standard time-discretization method, applied to the following iteration scheme:

$$\begin{aligned}&\frac{1}{\tau_n}\mathcal{A}_{0,i}(w_i - w_{i-1}) + 2L(w_i - w_{i-1}) + \partial\Psi_0(w_i) + \mathcal{G}_0(w_i) \ni \mathfrak{f}_{0,i} \text{ in } X, \\ &\qquad \text{for } i = 1,\dots,n, \text{ starting from the initial data } w_0 \in D(\Psi_0).\end{aligned} \tag{5.7.5}$$

In the context, $n \in \mathbb{N}$ is a given (large) number, $\tau_n := T/n$ is the time-step-size, $\{t_i\}_{i=0}^n := \{i\tau_n\}_{i=0}^n$ is the partition of the time-interval $[0, T]$, and

$$\begin{cases} \mathcal{A}_{0,i} := \mathcal{A}_0(t_i) \text{ in } \mathscr{L}(X), \ i = 0, 1, \dots, n, \\ \mathfrak{f}_{0,i} := \dfrac{1}{\tau_n} \displaystyle\int_{t_{i-1}}^{t_i} \mathfrak{f}_0(\tau)\, d\tau \text{ in } X, \ i = 1, \dots, n. \end{cases} \tag{5.7.6}$$

Here, let us set:

$$\begin{cases} [\widehat{w}]_n(t) := \chi_{(-\infty,0]}(t) w_0 + \displaystyle\sum_{i=1}^n \chi_{(t_{i-1},t_i]}(t) \left(w_i + \frac{t - t_i}{\tau_n}(w_i - w_{i-1}) \right) \text{ in } X, \\ [\overline{w}]_n(t) := \chi_{(-\infty,0]}(t) w_0 + \displaystyle\sum_{i=1}^n \chi_{(t_{i-1},t_i]}(t) w_i \text{ in } X, \end{cases}$$
$$\text{for all } t \in [0, \infty), \text{ and } n = 1, 2, 3, \dots,$$

and

$$\begin{cases} [\overline{\mathcal{A}_0}]_n := \chi_{(-\infty,0]}(t) \mathcal{A}_{0,0} + \displaystyle\sum_{i=1}^n \chi_{(t_{i-1},t_i]}(t) \mathcal{A}_{0,i} \text{ in } \mathscr{L}(X), \\ [\overline{\mathfrak{f}_0}]_n(t) := \displaystyle\sum_{i=1}^n \chi_{(t_{i-1},t_i]}(t)\, \mathfrak{f}_{0,i} \text{ in } X, \end{cases}$$
$$\text{for all } t \in [0, \infty), \text{ and } n = 1, 2, 3, \dots.$$

Then, it is easily checked from (5.7.6), (cp.1), and $\mathfrak{f}_0 \in L^2(0, T; X)$ that

$$\begin{cases} [\overline{\mathcal{A}_0}]_n \to \mathcal{A}_0 \text{ in } C([0, T]; \mathscr{L}(X)), \\ [\overline{\mathfrak{f}_0}]_n \to \mathfrak{f}_0 \text{ in } L^2(0, T; X), \end{cases} \quad \text{as } n \to \infty. \tag{5.7.7}$$

Now, let us fix a constant $L > L_0$, and take $n \in \mathbb{N}$ so large to satisfy $(5L + A_T^*)\tau_n < \kappa_0$ (< 1). Then, the existence and uniqueness of the scheme (5.7.5) will be reduced to those of the minimization problems for the following proper, l.s.c., strictly convex, and coercive functions:

$$\begin{aligned} \varpi \in X \mapsto & \frac{1}{2\tau_n} |\mathcal{A}_{0,i}^{\frac{1}{2}}(\varpi - w_{i-1})|_X^2 + \Psi_0(\varpi) + \widehat{\mathcal{F}}_L(\varpi) \\ & + L|\varpi - w_{i-1}|_X^2 - L|\varpi|_X^2 - \widehat{C}_0 - (\mathfrak{f}_{0,i}, \varpi)_X \in (-\infty, \infty], \ i = 1, \dots, n. \end{aligned}$$

On this basis, let us multiply the both sides of the scheme (5.7.5) by $w_i - w_0$. Then, as a consequence of (cp.0)–(cp.3), Remark 5.14, and Young's inequality, we infer that:

$$\begin{aligned} & \frac{1}{2\tau_n} \left(\left|\mathcal{A}_{0,i}^{\frac{1}{2}}(w_i - w_0)\right|_X^2 - \left|\mathcal{A}_{0,i-1}^{\frac{1}{2}}(w_{i-1} - w_0)\right|_X^2 \right) \\ & \quad \leq \frac{5L + A_T^*}{\kappa_0} \left(\frac{\left|\mathcal{A}_{0,i}^{\frac{1}{2}}(w_i - w_0)\right|_X^2 + \left|\mathcal{A}_{0,i-1}^{\frac{1}{2}}(w_{i-1} - w_0)\right|_X^2}{2} \right) \\ & \qquad + \frac{1 + 2L^2}{2L} \left(|\mathfrak{f}_{0,i}|_X^2 + |w_0|_X^2 + \Psi_0(w_0) + \widehat{\mathcal{F}}_L(w_0) \right), \ \text{for } i = 1, \dots, n; \end{aligned} \tag{5.7.8}$$

via the following calculations:

$$\left(\frac{1}{\tau_n} \mathcal{A}_{0,i}(w_i - w_{i-1}), w_i - w_0 \right)_X \geq \frac{1}{2\tau_n} \left(\left| \mathcal{A}_{0,i}^{\frac{1}{2}}(w_i - w_0) \right|_X^2 - \left| \mathcal{A}_{0,i}^{\frac{1}{2}}(w_{i-1} - w_0) \right|_X^2 \right)$$
$$= \frac{1}{2\tau_n} \left(\left| \mathcal{A}_{0,i}^{\frac{1}{2}}(w_i - w_0) \right|_X^2 - \left| \mathcal{A}_{0,i-1}^{\frac{1}{2}}(w_{i-1} - w_0) \right|_X^2 \right)$$
$$- \frac{1}{2} \left(\frac{1}{\tau_n} (\mathcal{A}_{0,i} - \mathcal{A}_{0,i-1})(w_{i-1} - w_0), w_{i-1} - w_0 \right)_X$$
$$\geq \frac{1}{2\tau_n} \left(\left| \mathcal{A}_{0,i}^{\frac{1}{2}}(w_i - w_0) \right|_X^2 - \left| \mathcal{A}_{0,i-1}^{\frac{1}{2}}(w_{i-1} - w_0) \right|_X^2 \right) - \frac{A_T^*}{2} |w_{i-1} - w_0|_X^2,$$

$$\left(w_i^*, w_i - w_0 \right)_X \geq \Psi_0(w_i) - \Psi_0(w_0),$$
$$\text{with } w_i^* := \mathfrak{f}_{0,i} - \frac{1}{\tau_n} \mathcal{A}_{0,i}(w_i - w_{i-1}) - 2L(w_i - w_{i-1}) - \mathcal{G}_0(w_i) \in \partial \Psi_0(w_i), \tag{5.7.9}$$

$$\left(2L(w_i - w_{i-1}), w_i - w_0 \right)_X + \left(\mathcal{G}_0(w_i), w_i - w_0 \right)_X$$
$$= \left(\widehat{\mathcal{F}}_L'(w_i), w_i - w_0 \right)_X - 2L(w_{i-1}, w_i - w_0)_X$$
$$\geq \widehat{\mathcal{F}}_L(w_i) - \widehat{\mathcal{F}}_L(w_0) - 2L|w_i - w_0|_X |w_{i-1} - w_0|_X - 2L|w_0|_X |w_i - w_0|_X$$
$$\geq \widehat{\mathcal{F}}_L(w_i) - \widehat{\mathcal{F}}_L(w_0) - 2L|w_i - w_0|_X^2 - L|w_{i-1} - w_0|_X^2 - L|w_0|_X^2,$$

$$(\mathfrak{f}_{0,i}, w_i - w_0)_X \leq \frac{L}{2} |w_i - w_0|_X^2 + \frac{1}{2L} |\mathfrak{f}_{0,i}|_X^2,$$

and

$$|w_i - w_0|_X^2 \leq \frac{1}{\kappa_0} \left(\mathcal{A}_{0,i}(w_i - w_0), w_i - w_0 \right)_X$$
$$= \frac{1}{\kappa_0} \left| \mathcal{A}_{0,i}^{\frac{1}{2}}(w_i - w_0) \right|_X^2, \text{ for } i = 1, \ldots, n.$$

So, applying the discrete version of Gronwall's lemma (cf. [25, Section 3.1]) to (5.7.8), and having in mind (5.7.7), it is observed that:

$$\left| \mathcal{A}_{0,i}^{\frac{1}{2}}(w_i - w_0) \right|_X^2$$
$$\leq \frac{1 + 2L^2}{L} e^{\frac{4T(A_T^* + 5L)}{\kappa_0}} \left(\sup_{n \in \mathbb{N}} \left| [\overline{\mathfrak{f}}_0]_n \right|_{L^2(0,T;X)}^2 + T(|w_0|_X^2 + \Psi_0(w_0) + \widehat{\mathcal{F}}_L(w_0)) \right)$$
$$=: r_0^* < \infty, \text{ for } i = 1, \ldots, n,$$

and

$$|w_i|_X^2 \leq 2 \left(|w_0|_X^2 + \frac{1}{\kappa_0} |\mathcal{A}_{0,i}^{\frac{1}{2}}(w_i - w_0)|_X^2 \right)$$
$$\leq 2 \left(|w_0|_X^2 + \frac{r_0^*}{\kappa_0} \right) =: r_1^* < \infty, \text{ for } i = 1, \ldots, n. \tag{5.7.10}$$

Additionally, multiplying the both sides of (5.7.5) by $w_i - w_{i-1}$, and using (cp.0)–(cp.3) and (5.7.10), we infer that:

$$\frac{\kappa_0}{2\tau_n}|w_i - w_{i-1}|_X^2 + \big(\Psi_0(w_i) + \widehat{\mathcal{F}}_L(w_i)\big) - \big(\Psi_0(w_{i-1}) + \widehat{\mathcal{F}}_L(w_{i-1})\big)$$
$$\leq \frac{1+4L^2}{\kappa_0} \cdot \tau_n\big(r_1^* + |\mathfrak{f}_{0,i}|_X^2\big), \text{ for } i = 1, \dots, n, \tag{5.7.11}$$

via the following calculations:

$$\begin{aligned}
&\big(w_i^*, w_i - w_{i-1}\big)_X + 2L|w_i - w_{i-1}|_X^2 + \big(\mathcal{G}_0(w_i), w_i - w_{i-1}\big)_X \\
&\quad \geq \Psi_0(w_i) - \Psi_0(w_{i-1}) + \big(\widehat{\mathcal{F}}_L'(w_i), w_i - w_{i-1}\big)_X - 2L(w_{i-1}, w_i - w_{i-1})_X \\
&\quad \geq \big(\Psi_0(w_i) + \widehat{\mathcal{F}}_L(w_i)\big) - \big(\Psi_0(w_{i-1}) + \widehat{\mathcal{F}}_L(w_{i-1})\big) \\
&\qquad\qquad - \frac{\kappa_0}{4\tau_n}|w_i - w_{i-1}|_X^2 - \frac{4L^2}{\kappa_0} \cdot \tau_n r_1^*,
\end{aligned}$$

with the element $w_i^* \in \partial\Psi_0(w_i)$, as in (5.7.9),

and

$$(\mathfrak{f}_{0,i}, w_i - w_{i-1})_X \leq \frac{\kappa_0}{4\tau_n}|w_i - w_{i-1}|_X^2 + \frac{1}{\kappa_0} \cdot \tau_n|\mathfrak{f}_{0,i}|_X^2, \text{ for } i = 1, \dots, n.$$

So, summing up (5.7.11), for $i = 1, \dots, n$, and invoking (5.7.7), we can derive the following estimate:

$$\begin{aligned}
&\frac{\kappa_0}{2}\int_0^t \big|[\widehat{w}]_n'(\varsigma)\big|_X^2 \, d\varsigma + \Psi_0([\overline{w}]_n(t)) + \widehat{\mathcal{F}}_L([\overline{w}]_n(t)) \\
&\qquad \leq \Psi_0(w_0) + \widehat{\mathcal{F}}_L(w_0) + \frac{1+4L^2}{\kappa_0}\left(Tr_1^* + \sup_{n\in\mathbb{N}}\big|[\overline{\mathfrak{f}_0}]_n\big|_{L^2(0,T;X)}^2\right) \\
&\qquad =: r_2^* < \infty, \text{ for all } t \in [0,T], \text{ and } n = 1, 2, 3, \dots.
\end{aligned}$$

This estimate enable us to say that:

($\star$3) $\{[\widehat{w}]_n\}_{n=1}^\infty$ is bounded in $W^{1,2}(0,T;X)$, and $\{[\overline{w}]_n\}_{n=1}^\infty$ is bounded in $L^\infty(0, T;X)$;

($\star$4) $\big\{[\overline{w}]_n(t), [\widehat{w}]_n(t) \,\big|\, t \in [0,T],\ n = 1, 2, 3, \dots\big\}$ is contained in a compact sublevel set $\big\{\varpi \in X \,\big|\, \Psi_0(\varpi) \leq r_2^*\big\}$.

By virtue of ($\star$3) and ($\star$4), we can apply the general theories of compactness, such as Ascoli's and Alaoglu's theorems (cf. [83, Corollary 4], [86, Section 1.2], and so on), and we can find a limit function $w \in W^{1,2}(0,T;X)$ for some subsequences of $\{[\widehat{w}]_n\}_{n=1}^\infty$ and $\{[\overline{w}]_n\}_{n=1}^\infty$ (not relabeled), such that:

$$\begin{aligned}
&[\widehat{w}]_n \to w \text{ in } C([0,T;X]), \\
&\qquad\qquad \text{and weakly in } W^{1,2}(0,T;X), \text{ as } n \to \infty.
\end{aligned} \tag{5.7.12a}$$

Here, having in mind:

$$|[\widehat{w}]_n - [\overline{w}]_n|_{L^\infty(0,T;X)} \leq \tau_n^{\frac{1}{2}}|[\widehat{w}]_n'|_{L^2(0,T;X)} \to 0, \text{ as } n \to \infty,$$

we can also see that

$$[\overline{w}]_n \to w \text{ in } L^\infty(0,T;X), \text{ as } n \to \infty. \tag{5.7.12b}$$

Taking into account (5.7.5), (5.7.7), (5.7.12), and (cp.0)–(cp.3), we deduce that:

$$\begin{aligned}
&\int_I \bigl(\mathcal{A}_0(t)w'(t), w(t)-\varpi\bigr)_X\,dt + \int_I \bigl(\mathcal{G}_0(w(t)) - \mathfrak{f}_0(t), w(t)-\varpi\bigr)_X\,dt \\
&\qquad + \int_I \Psi_0(w(t))\,dt - \int_I \Psi_0(\varpi)\,dt \\
&\quad \leq \lim_{n\to\infty} \int_I \bigl([\widehat{w}]_n'(t), [\overline{\mathcal{A}}_0]_n(t)([\overline{w}]_n(t)-\varpi)\bigr)_X\,dt \\
&\qquad + \lim_{n\to\infty} \tau_n \int_I \bigl(2L[\widehat{w}]_n'(t), [\overline{w}]_n(t)-\varpi\bigr)_X\,dt \\
&\qquad + \lim_{n\to\infty} \int_I \bigl(\mathcal{G}_0([\overline{w}]_n(t)) - [\overline{\mathfrak{f}_0}]_n(t), [\overline{w}]_n(t)-\varpi\bigr)_X\,dt \\
&\qquad + \varliminf_{n\to\infty} \int_I \Psi_0([\overline{w}]_n(t))\,dt - \int_I \Psi_0(\varpi)\,dt \leq 0, \\
&\qquad \text{for any } \varpi \in D(\Psi_0), \text{ and any open interval } I \subset (0,T).
\end{aligned}$$

This implies that w is a solution to the problem (CP).

Next, for the proof of uniqueness, we suppose that the both $w^\ell \in L^2(0,T;X)$, $\ell = 1,2$, are solutions to (CP). Then, by virtue of (cp.0)–(cp.3), it is immediately verified that:

$$\begin{aligned}
&\bigl(\mathfrak{f}_0 - \mathcal{A}_0(w^\ell)' - \mathcal{G}_0(w^\ell)\bigr)(t) \in \partial\Psi_0(w^\ell(t)) \text{ in } X, \\
&\qquad\qquad \text{for a.e. } t \in (0,T),\ \ell = 1,2,
\end{aligned} \tag{5.7.13a}$$

$$\begin{aligned}
&\bigl(\mathcal{A}_0(t)(w^1-w^2)'(t), (w^1-w^2)(t)\bigr)_X \\
&\quad = \frac{1}{2}\bigl([\mathcal{A}_0(w^1-w^2)]'(t), (w^1-w^2)(t)\bigr)_X \\
&\qquad - \frac{1}{2}\bigl(\mathcal{A}_0'(t)(w^1-w^2)(t), (w^1-w^2)(t)\bigr)_X \\
&\qquad + \frac{1}{2}\bigl(\mathcal{A}_0(t)(w^1-w^2)(t), (w^1-w^2)'(t)\bigr)_X \\
&\quad \geq \frac{1}{2}\frac{d}{dt}|\mathcal{A}_0(t)^{\frac{1}{2}}(w^1-w^2)(t)|_X^2 \\
&\qquad - \frac{A_T^*}{2\kappa_0}|\mathcal{A}_0(t)^{\frac{1}{2}}(w^1-w^2)(t)|_X^2, \text{ for a.e. } t \in (0,T),
\end{aligned} \tag{5.7.13b}$$

and

$$\begin{aligned}
&\bigl(\mathcal{G}_0(w^1(t)) - \mathcal{G}_0(w^2(t)), (w^1-w^2)(t)\bigr)_X \\
&\qquad \geq -\frac{L_0}{\kappa_0}|\mathcal{A}_0(t)^{\frac{1}{2}}(w^1-w^2)(t)|_X^2, \text{ for a.e. } t \in (0,T).
\end{aligned} \tag{5.7.13c}$$

Hence, the uniqueness for the problem (CP) will be verified via the following Gronwall type estimate:

$$\begin{aligned}
\frac{d}{dt}|\mathcal{A}_0(t)^{\frac{1}{2}}(w^1-w^2)(t)|_X^2 &\leq \frac{A_T^*+2L_0}{\kappa_0}|\mathcal{A}_0(t)^{\frac{1}{2}}(w^1-w^2)(t)|_X^2 \\
&\text{for a.e. } t \in (0,T),
\end{aligned}$$

that will be obtained by referring to the standard method, i.e.: by taking the difference between two equations, as in (5.7.13a); by multiplying the both sides by $(w^1 - w^2)(t)$; and by applying (5.7.13b) and (5.7.13c), the monotonicity of $\partial\Psi_0$ in $X \times X$, and the initial condition $w^1(0) = w^2(0) = w_0$ in X.

Finally, we verify (5.7.3). Owing to (cp.2) and [18, Lemma 3.3], one can say that the both functions $t \in [0,T] \mapsto \Psi_0(w(t)) \in [0,\infty)$ and $t \in [0,T] \mapsto \widehat{\mathcal{G}}_0(w(t)) \in \mathbb{R}$ are absolutely continuous, and:

$$\frac{d}{dt}\Bigl(\Psi_0(w(t)) + \widehat{\mathcal{G}}_0(w(t))\Bigr) = \bigl(\mathfrak{f}_0(t) - \mathcal{A}_0(t)w'(t), w'(t)\bigr)_X, \text{ for a.e. } t \in (0,T). \tag{5.7.14}$$

The equality (5.7.3) will be obtained as a consequence of (5.7.14) and (cp.0). □

Lemma 5.4. Under the notations $\mathcal{A}_0$, $\mathcal{G}_0$, and Ψ_0, and assumptions (cp.0)–(cp.3) as in the previous Lemma 5.3, let us fix $w_0 \in D(\Psi_0)$ and $\mathfrak{f}_0 \in L^2(0,T;X)$, and take the unique solution $w \in L^2(0,T;X)$ to the Cauchy problem (CP). Let $\{\Psi_n\}_{n=1}^\infty$, $\{w_{0,n}\}_{n=1}^\infty \subset X$, and $\{\mathfrak{f}_n\}_{n=1}^\infty$ be, respectively, a sequence of proper, l.s.c., and convex functions on X, a sequence of initial data in X, and a sequence of forcing terms in $L^2(0,T;X)$, such that:

(cp.4) $\Psi_n \geq 0$ on X, for $n = 1,2,3,\dots$, and the union $\bigcup_{n=1}^\infty \bigl\{w \in X \,\big|\, \Psi_n(w) \leq r\bigr\}$ of sublevel sets is relatively compact in X, for any $r \geq 0$;

(cp.5) Ψ_n converges to Ψ_0 on X, in the sense of Mosco, as $n \to \infty$;

(cp.6) $\sup_{n\in\mathbb{N}} \Psi_n(w_{0,n}) < \infty$, and $w_{0,n} \to w_0$ in X, as $n \to \infty$;

(cp.7) $\mathfrak{f}_n \to \mathfrak{f}_0$ weakly in $L^2(0,T;X)$, as $n \to \infty$.

Let $w_n \in W^{1,2}(0,T;X)$ be the solution to the Cauchy problem (CP), for the initial data $w_{0,n} \in D(\Psi_n)$ and forcing term $\mathfrak{f}_n \in L^2(0,T;X)$. Then,

$$w_n \to w \text{ in } C([0,T];X), \text{ weakly in } W^{1,2}(0,T;X),$$
$$\int_0^T \Psi_n(w_n(t))\,dt \to \int_0^T \Psi_0(w(t))\,dt, \text{ as } n \to \infty,$$

and

$$\bigl|\Psi_0(w)\bigr|_{C([0,T])} \leq \sup_{n\in\mathbb{N}} \bigl|\Psi_n(w_n)\bigr|_{C([0,T])} < \infty.$$

Proof. This Lemma is proved by referring to the method of proof as in [39, Theorem 2.7.1] (also see [22, Main Theorem 2]).

First, let us apply (5.7.3) to the solutions w_n, for $n = 1,2,3,\dots$. Then, we have:

$$|\mathcal{A}_0(t)^{\frac{1}{2}} w_n'(t)|_X^2 + \frac{d}{dt}\Bigl(\Psi_n(w_n(t)) + \widehat{\mathcal{G}}_0(w_n(t))\Bigr) = \bigl(\mathfrak{f}_n(t), w_n'(t)\bigr)_X, \text{ for a.e. } t \in (0,T),\ n = 1,2,3,\dots. \tag{5.7.15}$$

Besides, for simplicity of description, we define:

$$\widehat{\Psi}_0(\varpi) := \int_0^T \Psi_0(\varpi(t))\,dt \text{ and } \widehat{\Psi}_n(\varpi) := \int_0^T \Psi_n(\varpi(t))\,dt,\ n = 1,2,3,\dots, \text{ for any } \varpi \in L^2(0,T;X).$$

By (cp.5), Remark 5.3 (Fact 2), and [18, Proposition 2.16], the above $\widehat{\Psi}_0$ and $\widehat{\Psi}_n$, $n = 1, 2, 3, \dots$, form proper, l.s.c., and convex functions on $L^2(0, T; X)$, such that:

$$\begin{cases} \bigl[w, \mathfrak{f}_0 - \mathcal{A}_0 w' - \mathcal{G}_0(w)\bigr] \in \partial \widehat{\Psi}_0 \text{ in } L^2(0, T; X) \times L^2(0, T; X), \\ \bigl[w_n, \mathfrak{f}_{0,n} - \mathcal{A}_0 w_n' - \mathcal{G}_0(w_n)\bigr] \in \partial \widehat{\Psi}_n \text{ in } L^2(0, T; X) \times L^2(0, T; X), \\ \qquad \text{for } n = 1, 2, 3, \dots, \end{cases} \tag{5.7.16a}$$

and

$$\widehat{\Psi}_n \to \widehat{\Psi}_0 \text{ on } L^2(0, T; X), \text{ in the sense of Mosco, as } n \to \infty. \tag{5.7.16b}$$

Next, let us take arbitrary $t \in [0, T]$, and integrate the both sides of (5.7.15) over $[0, t]$. Then, by using Hölder's and Young's inequalities, and by applying (cp.0), (cp.2), (cp.6), (cp.7), and the mean-value theorem (cf. [54, Theorem 5 in p. 313]), we deduce that:

$$\begin{aligned} \frac{\kappa_0}{2} \int_0^t |w_n'(\tau)|_X^2 \, d\tau + \Bigl(\Psi_n(w_n(t)) + \widehat{\mathcal{G}}_0(w_n(t)) \Bigr) \\ \leq \Bigl(\Psi_n(w_{0,n}) + \widehat{\mathcal{G}}_0(w_{0,n}) \Bigr) + \frac{1}{2\kappa_0} \int_0^T |\mathfrak{f}_n(t)|_X^2 \, dt \\ \leq \sup_{n \in \mathbb{N}} \Biggl(\Psi_n(w_{0,n}) + \frac{1}{2\kappa_0} |\mathfrak{f}_n|_{L^2(0,T;X)}^2 \\ + |\widehat{\mathcal{G}}_0(0)| + |w_{0,n}|_X \bigl(|\mathcal{G}_0(0)|_X + L_0 |w_{0,n}|_X \bigr) \Biggr) \\ =: r_3^* < \infty, \text{ for all } t \in [0, T], \text{ and } n = 1, 2, 3, \dots. \end{aligned} \tag{5.7.17}$$

From the above estimate, one can say that:

- $\{w_n\}_{n=1}^\infty$ is bounded in $W^{1,2}(0, T; X)$, and is also bounded in $C([0, T]; X)$,
- $\bigl\{ w_n(t) \bigm| t \in [0, T],\ n = 1, 2, 3, \dots \bigr\}$ is contained in a relatively compact set $\bigcup_{n=1}^\infty \bigl\{ \varpi \in X \bigm| \Psi_n(\varpi) \leq r_3^* \bigr\}$.

Therefore, applying (cp.1)–(cp.7), and the general theories of compactness, such as Ascoli's and Alaoglu's theorems (cf. [83, Corollary 4], [86, Section 1.2], and so on), we find a limit function $\bar{w} \in W^{1,2}(0, T; X)$, with a subsequence of $\{w_n\}_{n=1}^\infty$ (not relabeled), such that:

$$\begin{aligned} w_n \to \bar{w} \text{ in } C([0, T]; X), \text{ weakly in } W^{1,2}(0, T; X), \\ \text{and in particular, } w_{0,n} = w_n(0) \to w_0 = \bar{w}(0), \text{ as } n \to \infty, \end{aligned} \tag{5.7.18a}$$

$$\begin{aligned} \mathfrak{f}_n - \mathcal{A}_0 w_n' - \mathcal{G}_0(w_n) \to \mathfrak{f}_0 - \mathcal{A}_0 \bar{w}' - \mathcal{G}_0(\bar{w}) \\ \text{weakly in } L^2(0, T; X), \text{ as } n \to \infty, \end{aligned} \tag{5.7.18b}$$

and

$$0 \leq \Psi_0(\bar{w}(t)) \leq \liminf_{n \to \infty} \Psi_n(w_n(t)) \leq \sup_{n \in \mathbb{N}} \bigl| \Psi_n(w_n) \bigr|_{C([0,T])}$$

$$\leq r_3^* < \infty, \text{ for any } t \in [0, T]. \tag{5.7.18c}$$

On account of (5.7.16), (5.7.18), and Remark 5.3 (Fact 1), we can observe that $\bar{w}$ coincides with the unique solution w to the problem (CP), and we can conclude this Lemma. □

Chapter 6

Optimal control problems for 1D parabolic state-systems of KWC types with dynamic boundary conditions

Throughout Chapter 6, we recall the class of optimal control problems governed by 1D parabolic state-systems of K.W.C. models with dynamic boundary conditions. In the context, the dynamic boundary conditions are supposed to reproduce the transmitted heat exchanges between interior and boundary of a polycrystal body. Our optimal control problems are labeled by using a constant $\varepsilon \geq 0$, and roughly summarized, the case when $\varepsilon = 0$ and the cases when $\varepsilon > 0$ correspond to the physically realistic setting, and its regularized approximating ones, respectively. Under suitable assumptions, the mathematical results concerned with: the solvability and continuous dependence for the state-systems; the solvability and ε-dependence of optimal control problems; and the first order necessary optimality conditions in the problems when $\varepsilon > 0$ and the limiting optimality condition as $\varepsilon \downarrow 0$; will be obtained in forms of three Main Theorems of this Chapter.

6.1 Preliminaries

We begin by prescribing the notations used throughout this Chapter.

<u>Abstract notations.</u> For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X, and denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between X and its dual X^*. In particular, when X is a Hilbert space, we denote by $(\cdot, \cdot)_X$ the inner product of X.

For any subset A of a Banach space X, let $\chi_A : X \longrightarrow \{0, 1\}$ be the characteristic function of A, i.e.:

$$\chi_A : w \in X \mapsto \chi_A(w) := \begin{cases} 1, & \text{if } w \in A, \\ 0, & \text{otherwise.} \end{cases}$$

For two Banach spaces X and Y, we denote by $\mathscr{L}(X; Y)$ the Banach space of bounded linear operators from X into Y, and in particular, we let $\mathscr{L}(X) := \mathscr{L}(X; X)$.

For Banach spaces $X_1, \dots, X_N$, with $1 < N \in \mathbb{N}$, let $X_1 \times \cdots \times X_N$ be the product Banach space endowed with the norm $|\cdot|_{X_1 \times \cdots \times X_N} := |\cdot|_{X_1} + \cdots + |\cdot|_{X_N}$. However, when all $X_1, \dots, X_N$ are Hilbert spaces, $X_1 \times \cdots \times X_N$ denotes the product Hilbert space endowed with the inner product $(\cdot,\cdot)_{X_1 \times \cdots \times X_N} := (\cdot,\cdot)_{X_1} + \cdots + (\cdot,\cdot)_{X_N}$ and the norm $|\cdot|_{X_1 \times \cdots \times X_N} := \bigl(|\cdot|_{X_1}^2 + \cdots + |\cdot|_{X_N}^2\bigr)^{\frac{1}{2}}$. In particular, when all $X_1, \dots, X_N$ coincide with a Banach space Y, we write:

$$[Y]^N := \overbrace{Y \times \cdots \times Y}^{N \text{ times}}.$$

Additionally, for any (possibly nonlinear) transform $\mathcal{T} : X \longrightarrow Y$, we let:

$$\mathcal{T}[w_1, \dots, w_N] := \bigl[\mathcal{T} w_1, \dots, \mathcal{T} w_N\bigr] \text{ in } [Y]^N, \quad \text{for any } [w_1, \dots, w_N] \in [X]^N.$$

Specific notations of this Chapter. As is mentioned in the introduction, let $(0,T) \subset \mathbb{R}$ be a bounded time-interval with a finite constant $T > 0$, and let $\Omega := (0,1) \subset \mathbb{R}$ be a one-dimensional bounded spatial domain. We denote by Γ the boundary $\partial\Omega = \{0,1\}$ of Ω, and we define

$$n_\Gamma(\ell) := (-1)^{\ell-1} \text{ for any } \ell \in \Gamma = \{0,1\}.$$

Besides, we let $Q := (0,T) \times \Omega$ and $\Sigma := (0,T) \times \Gamma$.

Throughout this paper, we denote by ∂_t and ∂_x the distributional time-derivative and the distributional spatial-derivative, respectively. Also, the measure theoretical phrases, such as "a.e.", "dt", "dx", and so on, are all with respect to the Lebesgue measure in each corresponding dimension. Additionally, "$|_\Gamma$" denotes the trace on Γ for a Sobolev function.

On this basis, we define

$$\begin{cases} H := L^2(\Omega), \quad H_\Gamma := \bigl\{ \tilde{w} \bigm| \tilde{w} : \Gamma \longrightarrow \mathbb{R} \bigr\} \ (\sim \mathbb{R}^2), \\ V := H^1(\Omega), \quad V_0 := H_0^1(\Omega), \end{cases}$$

$$\begin{cases} \mathbb{X} := H \times H_\Gamma, \quad \mathbb{V} = V \times H_\Gamma, \\ \mathbb{W} := \bigl\{ [\tilde{w}, \tilde{w}_\Gamma] \in \mathbb{V} \bigm| \tilde{w}_{|_\Gamma}(\ell) = \tilde{w}_\Gamma(\ell), \ \ell \in \Gamma \bigr\}, \end{cases}$$

$$\begin{cases} \mathscr{H} := L^2(0,T;H), \quad \mathscr{H}_\Gamma := L^2(0,T;H_\Gamma), \\ \mathscr{V} := L^2(0,T;V), \quad \mathscr{V}_0 := L^2(0,T;V_0), \end{cases}$$

and

$$\mathfrak{X} := \mathscr{H} \times \mathscr{H}_\Gamma \ (= L^2(0,T;\mathbb{X})), \quad \text{and} \quad \mathfrak{W} := L^2(0,T;\mathbb{W}).$$

Note that $\mathbb{W}$ is a closed linear subspace in the Hilbert space $\mathbb{V}$, so that $\mathbb{W}$ is also a Hilbert space endowed with the inner product of $\mathbb{V}$.

In this paper, we identify the Hilbert spaces H and $\mathscr{H}$ with their dual spaces. On this basis, we have the following relationships of continuous embeddings:

$$\begin{cases} V \subset H = H^* \subset V^*, \ \mathscr{V} \subset \mathscr{H} = \mathscr{H}^* \subset \mathscr{V}^*, \\ \mathbb{W} \subset \mathbb{V} \subset \mathbb{X} = \mathbb{X}^* \subset \mathbb{V}^* \subset \mathbb{W}^*, \text{ and } \mathfrak{W} \subset \mathfrak{X} = \mathfrak{X}^* \subset \mathfrak{W}^*, \end{cases}$$

among the Hilbert spaces H, V, $\mathscr{H}$, $\mathscr{V}$, $\mathbb{X}$, $\mathbb{V}$, $\mathbb{W}$, $\mathfrak{X}$, and $\mathfrak{W}$, and the respective dual spaces H^*, V^*, $\mathscr{H}^*$, $\mathscr{V}^*$, $\mathbb{X}^*$, $\mathbb{V}^*$, $\mathbb{W}^*$, $\mathfrak{X}^*$, and $\mathfrak{W}^*$.

Remark 6.1. Due to the one-dimensional embeddings $V \subset C(\overline{\Omega})$ and $V_0 \subset C(\overline{\Omega})$, it is easily checked that:

$$\begin{cases} \bullet \text{ if } \check{\mu} \in H \text{ and } \check{p} \in V, \text{ then } \check{\mu}\check{p} \in H, \text{ and } \\ \quad |\check{\mu}\check{p}|_H \leq \sqrt{2}|\check{\mu}|_H|\check{p}|_V, \\ \bullet \text{ if } \hat{\mu} \in L^\infty(0,T;H) \text{ and } \hat{p} \in \mathscr{V}, \text{ then } \hat{\mu}\hat{p} \in \mathscr{H}, \\ \quad \text{and } |\hat{\mu}\hat{p}|_{\mathscr{H}} \leq \sqrt{2}|\hat{\mu}|_{L^\infty(0,T;H)}|\hat{p}|_{\mathscr{V}}. \end{cases} \tag{6.1.1}$$

Here, we note that the constant $\sqrt{2}$ corresponds to the constant of embedding $V \subset C(\overline{\Omega})$. Moreover, under the setting $\Omega := (0,1)$, this $\sqrt{2}$ can be used as a upper bound of the constants of embeddings $V \subset L^q(\Omega)$ and $V_0 \subset L^q(\Omega)$, for all $1 \leq q \leq \infty$.

Remark 6.2. Let us take any $\tilde{a} \in W^{1,\infty}(Q) \cup L^\infty(0,T;W^{1,\infty}(\Omega))$ and any $w \in \mathscr{V}_0^*$. Then, we can say that $\tilde{a}w$ $(= w\tilde{a}) \in \mathscr{V}_0^*$, via the following variational form:

$$\langle \tilde{a}w, \psi \rangle_{\mathscr{V}_0} := \langle w, \tilde{a}\psi \rangle_{\mathscr{V}_0}, \text{ for any } \psi \in \mathscr{V}_0,$$

and can estimate that:

$$|\tilde{a}w|_{\mathscr{V}_0^*} \leq (1+\sqrt{2})\big(|\tilde{a}|_{L^\infty(Q)} + |\partial_x \tilde{a}|_{L^\infty(Q)}\big)|w|_{\mathscr{V}_0^*},$$

by using the constant $\sqrt{2}$ of the embedding $V_0 \subset H$. Also, if $\{\tilde{a}_n\}_{n=1}^\infty \subset W^{1,\infty}(Q) \cup L^\infty(0,T;W^{1,\infty}(\Omega))$, $\{w_n\}_{n=1}^\infty \subset \mathscr{V}_0^*$, and

$$\begin{cases} \tilde{a}_n \to \tilde{a} \text{ in } L^\infty(Q), \\ \partial_x \tilde{a}_n \to \partial_x \tilde{a} \text{ in } L^\infty(Q), \end{cases}$$

and

$$w_n \to w \text{ weakly in } \mathscr{V}_0^*, \text{ as } n \to \infty,$$

it holds that:

$$\tilde{a}_n w_n \to \tilde{a}w \text{ weakly in } \mathscr{V}_0^*, \text{ as } n \to \infty,$$

since

$$\tilde{a}\psi \in \mathscr{V}_0,\ \{\tilde{a}_n w_n\}_{n=1}^\infty \subset \mathscr{V}_0, \text{ and } \tilde{a}_n\psi \to \tilde{a}\psi \text{ in } \mathscr{V}_0 \text{ as } n \to \infty,$$
$$\text{for any } \psi \in \mathscr{V}_0\ .$$

In particular, if $\tilde{a} \in W^{1,\infty}(Q)$ and $w \in W^{1,2}(0,T;V_0^*)$, then

$$\tilde{a}w \in W^{1,2}(0,T;V_0^*), \text{ and } \partial_t(\tilde{a}w) = \tilde{a}\partial_t w + w\partial_t \tilde{a} \text{ in } \mathscr{V}_0^*.$$

Moreover, if $\tilde{a} \in W^{1,\infty}(Q) \cup L^\infty(0,T;W^{1,\infty}(\Omega))$, and $\log \tilde{a} \in L^\infty(Q)$, then it is estimated that:

$$|\tilde{a}w|_{\mathscr{V}_0^*} \geq \frac{\inf \tilde{a}(Q)^2}{(1+\sqrt{2})(\inf \tilde{a}(Q) + |\partial_x \tilde{a}|_{L^\infty(Q)})}|w|_{\mathscr{V}_0^*}.$$

Notations for the time-discretization. Let $\tau \in (0,1)$ be a constant that denotes the time-step size, and let $\{t_i\}_{i=0}^\infty \subset [0,\infty)$ be a sequence of time defined as:

$$t_i := i\tau,\ i = 0,1,2,\ldots. \tag{6.1.2}$$

Let X be a Banach space. Then, for any sequence $\{[t_i, \gamma_i]\}_{i=0}^{\infty} \subset [0, \infty) \times X$, we define the *forward time-interpolation* $[\overline{\gamma}]_\tau \in L^\infty_{\mathrm{loc}}([0,\infty); X)$, the *backward time-interpolation* $[\underline{\gamma}]_\tau \in L^\infty_{\mathrm{loc}}([0,\infty); X)$, and the *linear time-interpolation* $[\gamma]_\tau \in W^{1,2}_{\mathrm{loc}}([0,\infty); X)$, by letting:

$$\begin{cases} [\overline{\gamma}]_\tau(t) := \chi_{(-\infty,0]}(t)\gamma_0 + \displaystyle\sum_{i=1}^{\infty} \chi_{(t_{i-1},t_i]}(t)\gamma_i, \\ [\underline{\gamma}]_\tau(t) := \chi_{(-\infty,0]}(t)\gamma_0 + \displaystyle\sum_{i=0}^{\infty} \chi_{(t_i,t_{i+1}]}(t)\gamma_i, \qquad \text{in } X, \text{ for } t \geq 0, \\ [\gamma]_\tau(t) := \displaystyle\sum_{i=1}^{\infty} \chi_{[t_{i-1},t_i)}(t) \left(\frac{t - t_{i-1}}{\tau}\gamma_i + \frac{t_i - t}{\tau}\gamma_{i-1} \right), \end{cases} \tag{6.1.3}$$

respectively.

Remark 6.3. For an interval $I \subset \mathbb{R}$, a Banach space X, and a constant $q \in [1, \infty]$, we say that $L^q(I; X) \subset L^q_{\mathrm{loc}}(\mathbb{R}; X)$ (resp. $L^q_{\mathrm{loc}}(\mathbb{R}; X) \subset L^q(I; X)$) by identifying X-valued functions on I (resp. on $\mathbb{R}$) with the zero-extensions onto $\mathbb{R}$ (resp. the restriction onto I). Besides, under the notations as in (6.1.2) and (6.1.3), the following facts can be verified.

(Fact 0) • If $q \in [1, \infty)$, $\gamma \in L^q(0, T; X)$, and the sequence $\{\gamma_i\}_{i=0}^{\infty} \subset X$ is given by:

$$\gamma_i := \frac{1}{\tau} \int_{t_{i-1}}^{t_i} \gamma(\varsigma)\, d\varsigma \text{ in } X,\ i = 0, 1, 2, \dots, \text{ with } t_{-1} := -\tau, \tag{6.1.4}$$

then

$$[\overline{\gamma}]_\tau \to \gamma,\ [\underline{\gamma}]_\tau \to \gamma, \text{ and } [\gamma]_\tau \to \gamma \text{ in } L^q(\mathbb{R}; X), \\ \text{especially in } L^q(0, T; X), \text{ as } \tau \downarrow 0,$$

and

$$[\overline{\gamma}]_\tau(t) \to \gamma(t),\ [\underline{\gamma}]_\tau(t) \to \gamma(t), \text{ and } [\gamma]_\tau(t) \to \gamma(t) \\ \text{in } X, \text{ a.e. } t \in \mathbb{R}, \text{ as } \tau \downarrow 0.$$

- If X is a reflexive Banach space, and $\gamma \in L^\infty(0, T; X)$, then the sequence $\{\gamma_i\}_{i=0}^{\infty} \subset X$ given by (6.1.4) fulfills that:

$$\sup_{\tau \in (0,1)} \left\{ |[\overline{\gamma}]_\tau|_{L^\infty(0,T;X)}, |[\underline{\gamma}]_\tau|_{L^\infty(0,T;X)}, |[\gamma]_\tau|_{L^\infty(0,T;X)} \right\} \leq |\gamma|_{L^\infty(0,T;X)},$$

$$\begin{cases} [\overline{\gamma}]_\tau \to \gamma,\ [\underline{\gamma}]_\tau \to \gamma, \text{ and } [\gamma]_\tau \to \gamma \\ \qquad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}; X), \text{ for any } q \in [1, \infty), \quad \text{as } \tau \downarrow 0, \\ \qquad \text{weakly-}* \text{ in } L^\infty(\mathbb{R}; X), \end{cases}$$

$$[\overline{\gamma}]_\tau(t) \to \gamma(t),\ [\underline{\gamma}]_\tau(t) \to \gamma(t), \text{ and } [\gamma]_\tau(t) \to \gamma(t) \\ \text{in } X, \text{ a.e. } t \in \mathbb{R}, \text{ as } \tau \downarrow 0.$$

- If $\gamma \in W^{1,\infty}(Q)$, and the sequence $\{\gamma_i\}_{i=0}^{\infty} \subset W^{1,\infty}(\Omega)$ is given as:

$$\gamma_i := \begin{cases} \gamma(t_i) \text{ in } W^{1,\infty}(\Omega), \text{ if } t_i \leq T, \\ \gamma(t_{i-1}) \text{ in } W^{1,\infty}(\Omega), \text{ if } t_{i-1} \leq T < t_i, \qquad i = 0, 1, 2, \dots, \\ 0 \text{ in } W^{1,\infty}(\Omega), \text{ otherwise,} \end{cases}$$

then

$$\begin{cases} \sup_{\tau\in(0,1)} \left\{ |[\overline{\gamma}]_\tau|_{L^\infty(Q)}, |[\underline{\gamma}]_\tau|_{L^\infty(Q)}, |[\gamma]_\tau|_{C(\overline{Q})} \right\} \leq |\gamma|_{C(\overline{Q})}, \\ \sup_{\tau\in(0,1)} \left\{ |\partial_x[\overline{\gamma}]_\tau|_{L^\infty(Q)}, |\partial_x[\underline{\gamma}]_\tau|_{L^\infty(Q)}, |\partial_x[\gamma]_\tau|_{L^\infty(Q)} \right\} \leq |\partial_x \gamma|_{L^\infty(Q)}, \\ \sup_{\tau\in(0,1)} |\partial_t[\gamma]_\tau|_{L^\infty(Q)} \leq |\partial_t \gamma|_{L^\infty(Q)}, \end{cases}$$

$$\begin{cases} [\overline{\gamma}]_\tau \to \gamma \text{ and } [\underline{\gamma}]_\tau \to \gamma, \text{ in } L^\infty(0,T;C(\overline{\Omega})), \\ [\gamma]_\tau \to \gamma \text{ in } C(\overline{Q}), \end{cases}$$

$$\begin{cases} \partial_t[\gamma]_\tau \to \partial_t \gamma \text{ weakly-}* \text{ in } L^\infty(Q), \\ \qquad \text{and in the pointwise sense a.e. in } Q, \end{cases}$$

and

$$\begin{cases} \partial_x[\overline{\gamma}]_\tau \to \partial_x \gamma, \ \partial_x[\underline{\gamma}]_\tau \to \partial_x \gamma, \text{ and } \partial_x[\gamma]_\tau \to \partial_x \gamma \\ \qquad \text{weakly-}* \text{ in } L^\infty(Q), \\ \qquad \text{and in the pointwise sense a.e. in } Q, \end{cases} \qquad \text{as } \tau \downarrow 0.$$

Notations in convex analysis. (cf. [18, Chapter II]) For a proper, lower semi-con- tinuous (l.s.c.), and convex function $\Psi : X \to (-\infty, \infty]$ on a Hilbert space X, we denote by $D(\Psi)$ the effective domain of Ψ. Also, we denote by $\partial\Psi$ the subdifferential of Ψ. The subdifferential $\partial\Psi$ corresponds to a generalized derivative of Ψ, and it is known as a maximal monotone graph in the product space $X \times X$. The set $D(\partial\Psi) := \{z \in X \mid \partial\Psi(z) \neq \emptyset\}$ is called the domain of $\partial\Psi$. We often use the notation "$[w_0, w_0^*] \in \partial\Psi$ in $X \times X$", to mean that "$w_0^* \in \partial\Psi(w_0)$ in X for $w_0 \in D(\partial\Psi)$", by identifying the operator $\partial\Psi$ with its graph in $X \times X$.

For Hilbert spaces $X_1, \cdots, X_N$, with $1 < N \in \mathbb{N}$, let us consider a proper, l.s.c., and convex function on the product space $X_1 \times \cdots \times X_N$:

$$\tilde{\Psi} : w = [w_1, \cdots, w_N] \in X_1 \times \cdots \times X_N \mapsto \tilde{\Psi}(w) = \tilde{\Psi}(w_1, \cdots, w_N) \in (-\infty, \infty].$$

On this basis, for any $i \in \{1, \ldots, N\}$, we denote by $\partial_{w_i}\tilde{\Psi} : X_1 \times \cdots \times X_N \to X_i$ a set-valued operator, which maps any $w = [w_1, \ldots, w_i, \ldots, w_N] \in X_1 \times \cdots \times X_i \times \cdots \times X_N$ to a subset $\partial_{w_i}\tilde{\Psi}(w) \subset X_i$, prescribed as follows:

$$\begin{aligned} \partial_{w_i}\tilde{\Psi}(w) &= \partial_{w_i}\tilde{\Psi}(w_1, \cdots, w_i, \cdots, w_N) \\ &:= \left\{ \tilde{w}^* \in X_i \;\middle|\; \begin{array}{l} (\tilde{w}^*, \tilde{w} - w_i)_{X_i} \leq \tilde{\Psi}(w_1, \cdots, \tilde{w}, \cdots, w_N) \\ \qquad -\tilde{\Psi}(w_1, \cdots, w_i, \cdots, w_N), \text{ for any } \tilde{w} \in X_i \end{array} \right\}. \end{aligned}$$

As is easily checked,

$$\begin{gathered} \partial\tilde{\Psi}(w) \subset \partial_{w_1}\tilde{\Psi}(w) \times \cdots \times \partial_{w_N}\tilde{\Psi}(w), \\ \text{for any } w = [w_1, \ldots, w_N] \in X_1 \times \cdots \times X_N. \end{gathered} \tag{6.1.5}$$

But, it should be noted that the converse inclusion of (6.1.5) is not true, in general.

Remark 6.4 (Examples of the subdifferential)**.** As one of the representatives of the subdifferentials, we exemplify the following set-valued function $\mathrm{Sgn}^N : \mathbb{R}^N \to 2^{\mathbb{R}^N}$, with $N \in \mathbb{N}$, which is defined as:

$$\xi = [\xi_1, \dots, \xi_N] \in \mathbb{R}^N \mapsto \mathrm{Sgn}^N(\xi) = \mathrm{Sgn}^N(\xi_1, \dots, \xi_N) := \begin{cases} \dfrac{\xi}{|\xi|} = \dfrac{[\xi_1, \dots, \xi_N]}{\sqrt{\xi_1^2 + \cdots + \xi_N^2}}, & \text{if } \xi \neq 0, \\ \mathbb{D}^N, & \text{otherwise}, \end{cases}$$

where $\mathbb{D}^N$ denotes the closed unit ball in $\mathbb{R}^N$ centered at the origin. Indeed, the set-valued function Sgn^N coincides with the subdifferential of the Euclidean norm $|\cdot| : \xi \in \mathbb{R}^N \mapsto |\xi| = \sqrt{\xi_1^2 + \cdots + \xi_N^2} \in [0, \infty)$, i.e.:

$$\partial|\cdot|(\xi) = \mathrm{Sgn}^N(\xi), \text{ for any } \xi \in D(\partial|\cdot|) = \mathbb{R}^N,$$

and furthermore, it is observed that:

$$\partial|\cdot|(0) = \mathbb{D}^N \subsetneq [-1, 1]^N = \partial_{\xi_1}|\cdot|(0) \times \cdots \times \partial_{\xi_N}|\cdot|(0).$$

Finally, we mention about a notion of functional convergence, known as "Mosco-convergence".

Definition 6.1 (Mosco-convergence: cf. [59])**.** Let X be an abstract Hilbert space. Let $\Psi : X \to (-\infty, \infty]$ be a proper, l.s.c., and convex function, and let $\{\Psi_n\}_{n=1}^\infty$ be a sequence of proper, l.s.c., and convex functions $\Psi_n : X \to (-\infty, \infty]$, $n = 1, 2, 3, \dots$. Then, it is said that $\Psi_n \to \Psi$ on X, in the sense of Mosco, as $n \to \infty$, iff. the following two conditions are fulfilled:

(M1) The condition of lower-bound: $\varliminf_{n\to\infty} \Psi_n(\check{w}_n) \geq \Psi(\check{w})$, if $\check{w} \in X$, $\{\check{w}_n\}_{n=1}^\infty \subset X$, and $\check{w}_n \to \check{w}$ weakly in X, as $n \to \infty$.

(M2) The condition of optimality: for any $\hat{w} \in D(\Psi)$, there exists a sequence $\{\hat{w}_n\}_{n=1}^\infty \subset X$ such that $\hat{w}_n \to \hat{w}$ in X and $\Psi_n(\hat{w}_n) \to \Psi(\hat{w})$, as $n \to \infty$.

As well as, if the sequence of convex functions $\{\tilde{\Psi}_\varepsilon\}_{\varepsilon\in\Xi}$ is labeled by a continuous argument $\varepsilon \in \Xi$ with a infinite set $\Xi \subset \mathbb{R}$, then for any $\varepsilon_0 \in \Xi$, the Mosco-convergence of $\{\tilde{\Psi}_\varepsilon\}_{\varepsilon\in\Xi}$, as $\varepsilon \to \varepsilon_0$, is defined by those of subsequences $\{\tilde{\Psi}_{\varepsilon_n}\}_{n=1}^\infty$, for all sequences $\{\varepsilon_n\}_{n=1}^\infty \subset \Xi$, satisfying $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$.

Remark 6.5. Let X, Ψ, and $\{\Psi_n\}_{n=1}^\infty$ be as in Definition 6.1. Then, the following facts hold.

(Fact 1) (cf. [10, Theorem 3.66], [39, Chapter 2]) Let us assume that

$$\Psi_n \to \Psi \text{ on } X, \text{ in the sense of Mosco, as } n \to \infty,$$

and

$$\begin{cases} [w, w^*] \in X \times X, \quad [w_n, w_n^*] \in \partial\Psi_n \text{ in } X \times X, \ n \in \mathbb{N}, \\ w_n \to w \text{ in } X \text{ and } w_n^* \to w^* \text{ weakly in } X, \text{ as } n \to \infty. \end{cases}$$

Then, it holds that:

$$[w, w^*] \in \partial\Psi \text{ in } X \times X, \text{ and } \Psi_n(w_n) \to \Psi(w), \text{ as } n \to \infty.$$

(Fact 2) (cf. [22, Lemma 4.1], [30, Appendix]) Let $N \in \mathbb{N}$ denote the dimension constant, and let $S \subset \mathbb{R}^N$ be a bounded open set. Then, a sequence $\{\widehat{\Psi}_n^S\}_{n=1}^\infty$ of proper, l.s.c., and convex functions on $L^2(S;X)$, defined as:

$$w \in L^2(S;X) \mapsto \widehat{\Psi}_n^S(w) := \begin{cases} \displaystyle\int_S \Psi_n(w(t))\,dt, \\ \qquad \text{if } \Psi_n(w) \in L^1(S), \quad \text{for } n = 1,2,3,\ldots; \\ \infty, \quad \text{otherwise,} \end{cases}$$

converges to a proper, l.s.c., and convex function $\widehat{\Psi}^S$ on $L^2(S;X)$, defined as:

$$z \in L^2(S;X) \mapsto \widehat{\Psi}^S(z) := \begin{cases} \displaystyle\int_S \Psi(z(t))\,dt, \text{ if } \Psi(z) \in L^1(S), \\ \infty, \quad \text{otherwise;} \end{cases}$$

on $L^2(S;X)$, in the sense of Mosco, as $n \to \infty$.

Remark 6.6 (Example of Mosco-convergence)**.** For any $\varepsilon \geq 0$, let $f_\varepsilon : \mathbb{R} \longrightarrow [0,\infty)$ be a continuous and convex function, defined as:

$$f_\varepsilon : \xi \in \mathbb{R} \mapsto f_\varepsilon(\xi) := \sqrt{\varepsilon^2 + |\xi|^2} \in [0,\infty). \tag{6.1.6}$$

Then, due to the uniform estimate:

$$\begin{gathered} \bigl|f_\varepsilon(\xi) - f_{\tilde{\varepsilon}}(\xi)\bigr| = \bigl|\sqrt{\varepsilon^2 + |\xi|^2} - \sqrt{\tilde{\varepsilon}^2 + |\xi|^2}\bigr| \leq |\varepsilon - \tilde{\varepsilon}|, \\ \text{for all } \xi \in \mathbb{R}, \text{ and } \varepsilon, \tilde{\varepsilon} \geq 0, \end{gathered} \tag{6.1.7}$$

we easily see that:

$$f_\varepsilon \to f_0 \ (= |\cdot|) \text{ on } \mathbb{R}, \text{ in the sense of Mosco, as } \varepsilon \downarrow 0.$$

In addition, for any $\varepsilon > 0$, it can be said that the subdifferential ∂f_ε coincides with the single-valued function of usual differential:

$$f'_\varepsilon : \xi \in \mathbb{R} \mapsto f'_\varepsilon(\xi) = \frac{\xi}{\sqrt{\varepsilon^2 + |\xi|^2}} \in \mathbb{R}.$$

6.2 Auxiliary results

In this Section, we prepare some auxiliary results for our study. The auxiliary results are discussed through the following two Subsections.

§ 4.1.1 Abstract theory for the state-system $(\mathrm{S})_\varepsilon$;

§ 4.1.2 Mathematical theory for the linearized system of $(\mathrm{S})_\varepsilon$.

6.2.1 Abstract theory for the state-system $(\mathrm{S})_\varepsilon$

In this Subsection, we refer to [7, Appendix] to overview the abstract theory of nonlinear evolution equation, which enables us to handle the state-systems $(\mathrm{S})_\varepsilon$, for all $\varepsilon \geq 0$, in a unified fashion.

The general theory consists of the following two Propositions.

Proposition 6.1 (cf. [7, Lemma 8.1]). Let $\{\mathcal{A}_0(t) \,|\, t \in [0,T]\} \subset \mathscr{L}(X)$ be a class of time-dependent bounded linear operators, let $\mathcal{G}_0 : X \longrightarrow X$ be a given nonlinear operator, and let $\Psi_0 : X \longrightarrow [0,\infty]$ be a non-negative, proper, l.s.c., and convex function, fulfilling the following conditions:

(cp.0) $\mathcal{A}_0(t) \in \mathscr{L}(X)$ is positive and selfadjoint, for any $t \in [0,T]$, and it holds that

$$(\mathcal{A}_0(t)w, w)_X \geq \kappa_0 |w|_X^2, \text{ for any } w \in X,$$

with some constant $\kappa_0 \in (0,1)$, independent of $t \in [0,T]$ and $w \in X$.

(cp.1) $\mathcal{A}_0 : [0,T] \longrightarrow \mathscr{L}(X)$ is Lipschitz continuous, so that $\mathcal{A}_0$ admits the (strong) time-derivative $\mathcal{A}_0'(t) \in \mathscr{L}(X)$ a.e. in $(0,T)$, and

$$A_T^* := \operatorname*{ess\,sup}_{t \in (0,T)} \left\{ \max\{|\mathcal{A}_0(t)|_{\mathscr{L}(X)}, |\mathcal{A}_0'(t)|_{\mathscr{L}(X)}\} \right\} < \infty;$$

(cp.2) $\mathcal{G}_0 : X \longrightarrow X$ is a Lipschitz continuous operator with a Lipschitz constant L_0, and $\mathcal{G}_0$ has a C^1-potential functional $\widehat{\mathcal{G}}_0 : X \longrightarrow \mathbb{R}$, so that the Gâteaux derivative $\widehat{\mathcal{G}}_0'(w) \in X^* \; (= X)$ at any $w \in X$ coincides with $\mathcal{G}_0(w) \in X$;

(cp.3) $\Psi_0 \geq 0$ on X, and the sublevel set $\left\{ w \in X \,\middle|\, \Psi_0(w) \leq r \right\}$ is compact in X, for any $r \geq 0$.

Then, for any initial data $w_0 \in D(\Psi_0)$ and a forcing term $\mathfrak{f}_0 \in L^2(0,T;X)$, the following Cauchy problem of evolution equation:

$$\text{(CP)} \quad \begin{cases} \mathcal{A}_0(t)w'(t) + \partial\Psi_0(w(t)) + \mathcal{G}_0(w(t)) \ni \mathfrak{f}_0(t) \text{ in } X, \quad t \in (0,T), \\ w(0) = w_0 \text{ in } X; \end{cases}$$

admits a unique solution $w \in L^2(0,T;X)$, in the sense that:

$$w \in W^{1,2}(0,T;X), \; \Psi_0(w) \in L^\infty(0,T),$$

and

$$\big(\mathcal{A}_0(t)w'(t) + \mathcal{G}_0(w(t)) - \mathfrak{f}_0(t), w(t) - \varpi\big)_X + \Psi_0(w(t)) \leq \Psi_0(\varpi),$$
$$\text{for any } \varpi \in D(\Psi_0), \text{ a.e. } t \in (0,T).$$

Moreover, both $t \in [0,T] \mapsto \Psi_0(w(t)) \in [0,\infty)$ and $t \in [0,T] \mapsto \widehat{\mathcal{G}}_0(w(t)) \in \mathbb{R}$ are absolutely continuous functions in time, and

$$|\mathcal{A}_0(t)^{\frac{1}{2}} w'(t)|_X^2 + \frac{d}{dt}\Big(\Psi_0(w(t)) + \widehat{\mathcal{G}}_0(w(t))\Big) = (\mathfrak{f}_0(t), w'(t))_X,$$
$$\text{for a.e. } t \in (0,T).$$

Proposition 6.2 (cf. [7, Lemma 8.2]). Under the notations $\mathcal{A}_0$, $\mathcal{G}_0$, Ψ_0, and assumptions (cp.0)–(cp.3), as in the previous Proposition 6.1, let us fix $w_0 \in D(\Psi_0)$ and $\mathfrak{f}_0 \in L^2(0,T;X)$, and take the unique solution $w \in L^2(0,T;X)$ to the Cauchy problem (CP). Let $\{\Psi_n\}_{n=1}^\infty$, $\{w_{0,n}\}_{n=1}^\infty \subset X$, and $\{\mathfrak{f}_n\}_{n=1}^\infty$ be, respectively, a sequence of proper, l.s.c., and convex functions on X, a sequence of initial data in X, and a sequence of forcing terms in $L^2(0,T;X)$, such that:

(cp.4) $\Psi_n \geq 0$ on X, for $n = 1, 2, 3, \dots$, and the union $\bigcup_{n=1}^\infty \{ w \in X \,|\, \Psi_n(w) \leq r \}$ of sublevel sets is relatively compact in X, for any $r \geq 0$;

(cp.5) Ψ_n converges to Ψ_0 on X, in the sense of Mosco, as $n \to \infty$;

(cp.6) $\sup_{n \in \mathbb{N}} \Psi_n(w_{0,n}) < \infty$, and $w_{0,n} \to w_0$ in X, as $n \to \infty$;

(cp.7) $\mathfrak{f}_n \to \mathfrak{f}_0$ weakly in $L^2(0,T;X)$, as $n \to \infty$.

Let $w_n \in L^2(0,T;X)$ be the solution to the Cauchy problem (CP), for the initial data $w_{0,n} \in D(\Psi_n)$ and forcing term $\mathfrak{f}_n \in L^2(0,T;X)$. Then,

$$w_n \to w \text{ in } C([0,T];X), \text{ weakly in } W^{1,2}(0,T;X),$$
$$\int_0^T \Psi_n(w_n(t))\,dt \to \int_0^T \Psi_0(w(t))\,dt, \text{ as } n \to \infty,$$

and

$$\left|\Psi_0(w)\right|_{C([0,T])} \leq \sup_{n \in \mathbb{N}} \left|\Psi_n(w_n)\right|_{C([0,T])} < \infty.$$

In this paper, the readers are recommended to see [7, Appendix] for the detailed proofs of the above Propositions 6.1 and 6.2. Roughly summarized, these Propositions can be obtained by means of modified (mixed and reduced) methods of the existing theories, such as [14, 18, 39].

6.2.2 Mathematical theory for the linearized system of $(\mathrm{S})_\varepsilon$

In this Subsection, we set up auxiliary results for linearized systems of $(\mathrm{S})_\varepsilon$, which are associated with the first necessary optimality conditions in our optimal control problems $(\mathrm{OP})_\varepsilon$, for $\varepsilon \geq 0$. The linearized systems are generally reduced to the following type of parabolic initial-boundary value problem, denoted by (P).

(P):

$$\begin{cases} \partial_t p - \partial_x^2 p + \mu(t,x)p + \omega(t,x)\partial_x z = h(t,x),\ (t,x) \in Q, \\ \partial_t p_\Gamma(t,\ell) + (-1)^{\ell-1}\partial_x p_{|_\Gamma}(t,\ell) = h_\Gamma(t,\ell), \quad (t,\ell) \in \Sigma, \\ p_{|_\Gamma} = p_\Gamma \text{ on } \Sigma, \\ p(0,x) = p_0(x),\ x \in \Omega; \end{cases}$$

$$\begin{cases} a(t,x)\partial_t z + b(t,x)z - \partial_x\big(A(t,x)\partial_x z + \nu^2\partial_x z + \omega(t,x)p\big) \\ \qquad = k(t,x),\ (t,x) \in Q, \\ z(t,x) = 0,\ (t,x) \in \Sigma, \\ z(0,x) = z_0(x),\ x \in \Omega. \end{cases}$$

This system is a key-problem for the Gâteaux differential of the cost functional $\mathcal{J}_\varepsilon$. In the context, $[a, b, \lambda, \omega, A] \in [\mathscr{H}]^5$ is a given quintet of functions which belongs to a class $\mathscr{S} \subset [\mathscr{H}]^5$, defined as:

$$\mathscr{S} := \left\{ [\tilde{a}, \tilde{b}, \tilde{\mu}, \tilde{\omega}, \tilde{A}] \in [\mathscr{H}]^5 \;\middle|\; \begin{array}{l} \tilde{a} \in W^{1,\infty}(Q) \text{ with } \log \tilde{a} \in L^\infty(Q), \\ [\tilde{b}, \tilde{\omega}] \in [L^\infty(Q)]^2, \ \tilde{\mu} \in L^\infty(0,T;H), \\ \text{and } \tilde{A} \in L^\infty(Q) \text{ with } \tilde{A} \geq 0 \text{ a.e. in } Q \end{array} \right\}. \tag{6.2.1}$$

Also, $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0 = [p_0, p_{\Gamma,0}]$ and $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{V}_0^*$ with $\boldsymbol{h} = [h, h_\Gamma]$ are the initial triplet and forcing triplet in the system (P), respectively.

Remark 6.7. If $[a, b, \mu, \omega, A] \in \mathscr{S}$, then the condition:

$$a \in W^{1,\infty}(Q) \text{ with } \log a \in L^\infty(Q),$$

brought by (6.2.1), implies the no degeneration property:

$$\delta_*(a) := \inf a(Q) > 0, \tag{6.2.2}$$

of the coefficient a in the system (P).

Now, as the key-properties of the system (P), we can verify the following three Theorems.

Theorem 6.1. Let us assume $[a, b, \mu, \omega, A] \in \mathscr{S}$, $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0 = [p_0, p_{\Gamma,0}]$, and $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{V}_0^*$ with $\boldsymbol{h} = [h, h_\Gamma]$. Then, the system (P) admits a unique solution $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{p} = [p, p_\Gamma]$, in the sense that:

$$\begin{cases} \boldsymbol{p} = [p, p_\Gamma] \in W^{1,2}(0,T;\mathbb{X}) \cap L^\infty(0,T;\mathbb{W}) \subset C(\overline{Q}) \times C(\overline{\Sigma}), \\ z \in W^{1,2}(0,T;V_0^*) \cap \mathscr{V}_0 \subset C([0,T];H); \end{cases}$$

$$\begin{aligned} &(\partial_t \boldsymbol{p}(t), \boldsymbol{\varphi})_{\mathbb{X}} + (\partial_x p(t), \partial_x \varphi)_H + (\mu(t) p(t), \varphi)_H + (\omega(t) \partial_x z(t), \varphi)_H \\ &\qquad = (\boldsymbol{h}(t), \boldsymbol{\varphi})_{\mathbb{X}}, \text{ for any } \boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] \in \mathbb{W}, \text{ a.e. } t \in (0,T), \\ &\text{subject to } \boldsymbol{p}(0) = [p(0), p_\Gamma(0)] = \boldsymbol{p}_0 = [p_0, p_{\Gamma,0}] \text{ in } \mathbb{X}; \end{aligned}$$

and

$$\begin{aligned} &\langle a(t) \partial_t z(t), \psi \rangle_{V_0} + (b(t) z(t), \psi)_H \\ &\qquad + \big(A(t) \partial_x z(t) + \nu^2 \partial_x z(t) + p(t) \omega(t), \partial_x \psi \big)_H = \langle k(t), \psi \rangle_{V_0}, \\ &\text{for any } \psi \in V_0, \text{ a.e. } t \in (0,T), \text{ subject to } z(0) = z_0 \text{ in } H. \end{aligned}$$

Theorem 6.2. Let us take arbitrary $[a, b, \mu, \omega, A] \in \mathscr{S}$, $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0 = [p_0, p_{\Gamma,0}]$, and $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{V}_0^*$ with $\boldsymbol{h} = [h, h_\Gamma]$, and let us denote by $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{p} = [p, p_\Gamma]$ the solution to (P). Additionally, let $\delta_*(a)$ be the positive constant as in (6.2.2). Then, the following two items hold.

(I) Let C_0^* be a positive constant, defined as:

$$C_0^* := \frac{16\big(1 + |a|_{W^{1,\infty}(Q)} + |b|_{L^\infty(Q)} + |\mu|^2_{L^\infty(0,T;H)} + |\omega|^2_{L^\infty(Q)}\big)}{\min\{1, \nu^2, \delta_*(a)\}}. \tag{6.2.3}$$

Then, it is estimated that:

$$\begin{aligned}&\frac{d}{dt}\big(|\boldsymbol{p}(t)|^2_{\mathbb{X}} + |\sqrt{a(t)}z(t)|^2_H\big) + \big(|\boldsymbol{p}(t)|^2_{\mathbb{W}} + \nu^2|z(t)|^2_{V_0}\big)\\ &\quad \leq C_0^*\big(|\boldsymbol{p}(t)|^2_{\mathbb{X}} + |\sqrt{a(t)}z(t)|^2_H\big) + C_0^*\big(|\boldsymbol{h}(t)|^2_{\mathbb{X}} + |k(t)|^2_{V_0^*}\big), \\ &\qquad \text{for a.e. } t \in (0,T).\end{aligned} \tag{6.2.4}$$

(II) Let C_0^* be the positive constant given in (6.2.3), and let C_ℓ^*, $\ell = 1, 2$, be positive constants, defined as:

$$\begin{cases} C_1^* := 4(C_0^*)^2 e^{\frac{3}{2}C_0^* T}, \\ C_2^* := 4(C_0^*)^6 e^{\frac{3}{2}C_0^* T}(1 + |a|_{W^{1,\infty}(Q)})^2 \cdot \\ \qquad \cdot(1 + \nu + |b|_{L^\infty(Q)} + |\omega|_{L^\infty(Q)} + |A|_{L^\infty(Q)})^2. \end{cases} \tag{6.2.5}$$

Then, it is estimated that:

$$\begin{cases} |\partial_t \boldsymbol{p}|^2_{\mathfrak{X}} + |p|^2_{L^\infty(0,T;V)} \leq C_1^*\big(|\boldsymbol{p}_0|^2_{\mathbb{W}} + |\sqrt{a(0,\cdot)}z_0|^2_H + |\boldsymbol{h}|^2_{\mathfrak{X}} + |k|^2_{\mathscr{V}_0^*}\big), \\ |\partial_t z|^2_{\mathscr{V}_0^*} \leq C_2^*\big(|\boldsymbol{p}_0|^2_{\mathbb{W}} + |\sqrt{a(0,\cdot)}z_0|^2_H + |\boldsymbol{h}|^2_{\mathfrak{X}} + |k|^2_{\mathscr{V}_0^*}\big). \end{cases} \tag{6.2.6}$$

Remark 6.8. By applying Gronwall's lemma to the inequality in Theorem 6.2 (I), we also estimate that:

$$\begin{aligned}&\big(|\boldsymbol{p}|^2_{C([0,T];\mathbb{X})} + |\sqrt{a}z|^2_{C([0,T];H)}\big) + \big(|\boldsymbol{p}|^2_{\mathfrak{W}} + \nu^2|z|^2_{\mathscr{V}_0}\big)\\ &\qquad \leq 2C_0^* e^{C_0^* T}\big(|\boldsymbol{p}_0|^2_{\mathbb{X}} + |\sqrt{a(0,\cdot)}z_0|^2_H + |\boldsymbol{h}|^2_{\mathfrak{X}} + |k|^2_{\mathscr{V}_0^*}\big).\end{aligned}$$

Theorem 6.3. Let us assume:

$$[a, b, \mu, \omega, A] \in \mathscr{S}, \ \ \{[a^n, b^n, \mu^n, \omega^n, A^n]\}_{n=1}^\infty \subset \mathscr{S}, \tag{6.2.7a}$$

$$\begin{aligned}&[a^n, \partial_t a^n, \partial_x a^n, b^n, \omega^n, A^n] \to [a, \partial_t a, \partial_x a, b, \omega, A] \text{ weakly-}* \text{ in } [L^\infty(Q)]^6, \\ &\qquad \text{and in the pointwise sense a.e. in } Q, \text{ as } n \to \infty,\end{aligned} \tag{6.2.7b}$$

and

$$\begin{cases} \mu^n \to \mu \text{ weakly-}* \text{ in } L^\infty(0,T;H), \\ \mu^n(t) \to \mu(t) \text{ in } H, \text{ in the pointwise sense,} \\ \qquad \text{for a.e. } t \in (0,T), \end{cases} \quad \text{as } n \to \infty. \tag{6.2.7c}$$

Let us assume $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0 = [p_0, p_{\Gamma,0}]$, $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{V}_0^*$ with $\boldsymbol{h} = [h, h_\Gamma]$, and let us denote by $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{p} = [p, p_\Gamma]$ the solution to (P), for the initial triplet $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0]$ and forcing triplet

$[\boldsymbol{h}, k] = [h, h_\Gamma, k]$. Also, for any $n \in \mathbb{N}$, let us assume $[\boldsymbol{p}_0^n, z_0^n] = [p_0^n, p_{\Gamma,0}^n, z_0^n] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0^n = [p_0^n, p_{\Gamma,0}^n]$, $[\boldsymbol{h}^n, k^n] = [h^n, h_\Gamma^n, k^n] \in \mathfrak{X} \times \mathscr{V}_0^*$ with $\boldsymbol{h}^n = [h^n, h_\Gamma^n]$, and let us denote by $[\boldsymbol{p}^n, z^n] = [p^n, p_\Gamma^n, z^n] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{p}^n = [p^n, p_\Gamma^n]$ the solution to (P), for the initial triplet $[\boldsymbol{p}_0^n, z_0^n] = [p_0^n, p_{\Gamma,0}^n, z_0^n]$ and forcing triplet $[\boldsymbol{h}^n, k^n] = [h^n, h_\Gamma^n, k^n]$. Then, the convergences of given data:

$$\begin{cases} [\boldsymbol{p}_0^n, z_0^n] \to [\boldsymbol{p}_0, z_0] \text{ weakly in } \mathbb{W} \times H, \\ [\boldsymbol{h}^n, k^n] \to [\boldsymbol{h}, k] \text{ weakly in } \mathfrak{X} \times \mathscr{V}_0^*, \end{cases} \quad \text{as } n \to \infty, \tag{6.2.8}$$

implies the convergence of solutions, in the sense that:

$$\begin{aligned} &[\boldsymbol{p}^n, z^n] \to [\boldsymbol{p}, z] \text{ in } [C(\overline{Q}) \times C(\overline{\Sigma})] \times \mathscr{H}, \text{ weakly in } \mathfrak{W} \times \mathscr{V}_0, \\ &\qquad \text{and weakly in } W^{1,2}(0,T;\mathbb{X}) \times W^{1,2}(0,T;V_0^*), \text{ as } n \to \infty. \end{aligned} \tag{6.2.9}$$

Remark 6.9 (Review of Theorems 6.1–6.3)**.** Let us define:

$$\mathfrak{Y} := [W^{1,2}(0,T;\mathbb{X}) \cap L^\infty(0,T;\mathbb{W})] \times [W^{1,2}(0,T;V_0^*) \cap \mathscr{V}_0].$$

as a Banach space endowed with the following norm

$$\begin{aligned} \bigl|[\tilde{\boldsymbol{p}}, z]\bigr|_{\mathfrak{Y}} := \Bigl(\bigl|[\partial_t \tilde{p}, \partial_t \tilde{p}_\Gamma]\bigr|_{\mathfrak{X}}^2 + \bigl|[\tilde{p}, \tilde{p}_\Gamma]\bigr|_{L^\infty(0,T;\mathbb{W})}^2 + |\partial_t \tilde{z}|_{\mathscr{V}_0^*}^2 + |\tilde{z}|_{\mathscr{V}_0}^2\Bigr)^{\frac{1}{2}}, \\ \text{for any } [\tilde{\boldsymbol{p}}, \tilde{z}] = [\tilde{p}, \tilde{p}_\Gamma, \tilde{z}] \in \mathfrak{Y} \text{ with } \tilde{\boldsymbol{p}} = [\tilde{p}, \tilde{p}_\Gamma]. \end{aligned}$$

The Banach space $\mathfrak{Y}$ is to characterize the regularity of solution to the linearized system of $(\mathrm{S})_\varepsilon$. Due to the compactness theory of Aubin's type (cf. [83, Corollary 4]), this Banach space $\mathfrak{Y}$ is compactly embedded into the Banach space $[C(\overline{Q}) \times C(\overline{\Sigma})] \times \mathscr{H}$.

Now, for any quintet of functions $[a, b, \mu, \omega, A] \in \mathscr{S}$, the first and second Theorems 6.1 and 6.2 will enable us to define a bounded linear operator $\mathcal{P} = P(a, b, \mu, \omega, A) : [\mathbb{W} \times H] \times [\mathfrak{X} \times \mathscr{V}_0^*] \longrightarrow \mathfrak{Y}$, which maps any pair $\bigl[[\boldsymbol{p}_0, z_0], [\boldsymbol{h}, k]\bigr] \in [\mathbb{W} \times H] \times [\mathfrak{X} \times \mathscr{V}_0^*]$ of the initial triplet $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0 = [p_0, p_{\Gamma,0}]$ and the forcing triplet $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{V}_0^*$ with $\boldsymbol{h} = [h, h_\Gamma]$, to the solution $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathfrak{Y}$ with $\boldsymbol{p} = [p, p_\Gamma]$ to the linear system (P). Moreover, the third Theorem 6.3 will be to guarantee the continuous dependence of the solution operator $\mathcal{P} = \mathcal{P}(a, b, \mu, \omega, A)$, in the following sense:

$$\begin{aligned} \mathcal{P}(a^n, b^n, \mu^n, \omega^n, A^n)\bigl[[\boldsymbol{p}_0^n, z_0^n], [\boldsymbol{h}^n, k^n]\bigr] \to \mathcal{P}(a, b, \mu, \omega, A)\bigl[[\boldsymbol{p}_0, z_0], [\boldsymbol{h}, k]\bigr] \\ \text{in the topologies as in (6.2.9), whenever (6.2.7) and (6.2.8) are fulfilled.} \end{aligned}$$

The proofs of the three Theorems 6.1–6.3 will be given in the appendix, that is assigned to the last Section 7 of this paper.

6.3 Main Theorems

We begin by setting up the assumptions needed in our Main Theorems.

(A0) $\nu > 0$ is a fixed constant. Let $[\boldsymbol{\eta}_{\mathrm{ad}}, \theta_{\mathrm{ad}}] = [\eta_{\mathrm{ad}}, \eta_{\Gamma,\mathrm{ad}}, \theta_{\mathrm{ad}}] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta}_{\mathrm{ad}} = [\eta_{\mathrm{ad}}, \eta_{\Gamma,\mathrm{ad}}]$ be a fixed triplet of functions, called the *admissible target profile*.

(A1) $\alpha : \mathbb{R} \longrightarrow (0, \infty)$ and $\alpha_0 : Q \longrightarrow (0, \infty)$ are Lipschitz continuous functions, such that:

- $\alpha \in C^2(\mathbb{R})$, with the first derivative $\alpha' = \frac{d\alpha}{d\eta} \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and the second one $\alpha'' = \frac{d^2\alpha}{d\eta^2} \in C(\mathbb{R})$;
- $\alpha'(0) = 0$, $\alpha'' \geq 0$ on $\mathbb{R}$, and $\alpha\alpha'$ is a Lipschitz continuous function on $\mathbb{R}$;
- $\alpha \geq \delta_*$ on $\mathbb{R}$, and $\alpha_0 \geq \delta_*$ on $\overline{Q}$, for some constant $\delta_* \in (0, 1)$.

(A2) For any $\varepsilon \geq 0$, let $f_\varepsilon : \mathbb{R} \longrightarrow [0, \infty)$ be the convex function, defined in (6.1.6).

(A3) $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1-function, which is a Lipschitz continuous on $\mathbb{R}$. Also g has a nonnegative primitive $0 \leq G \in C^2(\mathbb{R})$, i.e. the derivative $G' = \frac{dG}{d\eta}$ coincides with g on $\mathbb{R}$.

Now, the Main Theorems of this paper are stated as follows:

Main Theorem 6.1. Let us assume (A0)–(A3). Let us fix a constant $\varepsilon \geq 0$, an initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \in \mathbb{W} \times V_0$ with $\boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}]$, and a forcing triplet $[\boldsymbol{u}, v] = [u, u_\Gamma, v] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u} = [u, u_\Gamma]$. Then, the following two items hold.

(I-A) The state-system $(\mathrm{S})_\varepsilon$ admits a unique solution $[\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta} = [\eta, \eta_\Gamma]$, in the sense that:

$$\begin{cases} \boldsymbol{\eta} = [\eta, \eta_\Gamma] \in W^{1,2}(0,T;\mathbb{X}) \cap L^\infty(0,T;\mathbb{W}) \subset C(\overline{Q}) \times C(\overline{\Sigma}), \\ \theta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V_0) \subset C(\overline{Q}); \end{cases}$$

$$\begin{gathered} \big(\partial_t \boldsymbol{\eta}(t), \boldsymbol{\varphi}\big)_{\mathbb{X}} + \big(\partial_x \eta(t), \partial_x \varphi\big)_H + \big(g(\eta(t)), \varphi\big)_H \\ + \big(\alpha'(\eta(t)) f_\varepsilon(\partial_x \theta(t)), \varphi\big)_H = \big(Lu(t), \varphi\big)_H + \big(L_\Gamma u_\Gamma(t), \varphi_\Gamma\big)_{H_\Gamma}, \\ \text{for any } \boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] \in \mathbb{W}, \text{ a.e. } t \in (0,T), \\ \text{subject to } \boldsymbol{\eta}(0) = [\eta(0), \eta_\Gamma(0)] = \boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}] \text{ in } \mathbb{X}; \end{gathered}$$

and

$$\begin{gathered} \big(\alpha_0(t) \partial_t \theta(t), \theta(t) - \psi\big)_H + \nu^2 \big(\partial_x \theta(t), \partial_x(\theta(t) - \psi)\big)_H \\ + \int_\Omega \alpha(\eta(t)) f_\varepsilon(\partial_x \theta(t)) \, dx \leq \int_\Omega \alpha(\eta(t)) f_\varepsilon(\partial_x \psi) \, dx \\ + \big(Mv(t), \theta(t) - \psi\big)_H, \text{ for any } \psi \in V_0, \\ \text{a.e. } t \in (0,T), \text{ subject to } \theta(0) = \theta_0 \text{ in } H. \end{gathered}$$

(I-B) Let $\{\varepsilon_n\}_{n=1}^\infty \subset [0,1]$, $\{[\boldsymbol{\eta}_{0,n}, \theta_{0,n}]\}_{n=1}^\infty = \{[\eta_{0,n}, \eta_{\Gamma,0,n}, \theta_{0,n}]\}_{n=1}^\infty \subset \mathbb{W} \times V_0$ with $\{\boldsymbol{\eta}_{0,n}\}_{n=1}^\infty = \{[\eta_{0,n}, \eta_{\Gamma,0,n}]\}_{n=1}^\infty$, and $\{[\boldsymbol{u}_n, v_n]\}_{n=1}^\infty = \{[u_n, u_{\Gamma,n}, v_n]\}_{n=1}^\infty \subset \mathfrak{X} \times \mathscr{H}$ with $\{\boldsymbol{u}_n\}_{n=1}^\infty = \{[u_n, u_{\Gamma,n}]\}_{n=1}^\infty$, be given sequences such that:

$$\begin{gathered} \varepsilon_n \to \varepsilon, \ [\boldsymbol{\eta}_{0,n}, \theta_{0,n}] \to [\boldsymbol{\eta}_0, \theta_0] \text{ weakly in } \mathbb{W} \times V_0, \\ \text{and } [Lu_n, L_\Gamma u_{\Gamma,n}, Mv_n] \to [Lu, L_\Gamma u_\Gamma, Mv] \text{ weakly in } \mathfrak{X} \times \mathscr{H}, \text{ as } n \to \infty. \end{gathered} \tag{6.3.1}$$

Let $[\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta} = [\eta, \eta_\Gamma]$ be the unique solution to $(\mathrm{S})_\varepsilon$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0]$ and forcing triplet $[\boldsymbol{u}, v] = [u, u_\Gamma, v]$. Additionally, for any $n \in \mathbb{N}$, let $[\boldsymbol{\eta}_n, \theta_n] = [\eta_n, \eta_{\Gamma,n}, \theta_n] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta}_n = [\eta_n, \eta_{\Gamma,n}]$ be the unique solution to $(\mathrm{S})_{\varepsilon_n}$, for the initial triplet $[\boldsymbol{\eta}_{0,n}, \theta_{0,n}] = [\eta_{0,n}, \eta_{\Gamma,0,n}, \theta_{0,n}]$ and forcing triplet $[\boldsymbol{u}_n, v_n] = [u_n, u_{\Gamma,n}, v_n]$. Then, it holds that:

$$
\begin{aligned}
&[\boldsymbol{\eta}_n, \theta_n] \to [\boldsymbol{\eta}, \theta] \text{ in } [C(\overline{Q}) \times C(\overline{\Sigma})] \times C(\overline{Q}), \\
&\qquad \text{in } \mathfrak{W} \times \mathscr{V}_0, \text{ weakly in } W^{1,2}(0,T;\mathbb{X}) \times W^{1,2}(0,T;H), \\
&\qquad \text{and weakly-}* \text{ in } L^\infty(0,T;\mathbb{W}) \times L^\infty(0,T;V_0), \text{ as } n \to \infty,
\end{aligned} \tag{6.3.2}
$$

and in particular,

$$
\begin{aligned}
&\alpha''(\eta_n) f_{\varepsilon_n}(\partial_x \theta_n) \to \alpha''(\eta) f_\varepsilon(\partial_x \theta) \text{ in } \mathscr{H}, \\
&\qquad \text{and weakly-}* \text{ in } L^\infty(0,T;H), \text{ as } n \to \infty.
\end{aligned} \tag{6.3.3}
$$

Remark 6.10. As a consequence of (6.3.2) and (6.3.3), we further find a subsequence $\{n_i\}_{i=1}^\infty \subset \{n\}$, such that:

$$
\begin{aligned}
&[\eta_{n_i}, \theta_{n_i}] \to [\eta, \theta], \ [\partial_x \eta_{n_i}, \partial_x \theta_{n_i}] \to [\partial_x \eta, \partial_x \theta], \\
&\qquad \text{and } \alpha''(\eta_{n_i}) f_{\varepsilon_{n_i}}(\partial_x \theta_{n_i}) \to \alpha''(\eta) f_\varepsilon(\partial_x \theta), \\
&\qquad\qquad \text{in the pointwise sense a.e. in } Q, \text{ as } i \to \infty,
\end{aligned}
$$

and

$$
\begin{aligned}
&[\eta_{n_i}(t), \theta_{n_i}(t)] \to [\eta(t), \theta(t)] \text{ in } V \times V_0, \\
&\qquad \text{and } \alpha''(\eta_{n_i}(t)) f_{\varepsilon_{n_i}}(\partial_x \theta_{n_i}(t)) \to \alpha''(\eta(t)) f_\varepsilon(\partial_x \theta(t)) \text{ in } H, \\
&\qquad\qquad \text{in the pointwise sense for a.e. } t \in (0,T), \text{ as } i \to \infty.
\end{aligned}
$$

Main Theorem 6.2. Under the assumptions (A0)–(A3), let us fix any constant $\varepsilon \geq 0$, and any initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \in \mathbb{W} \times V_0$ with $\boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}]$. Then, the following two items hold.

(II-A) The problem $(\mathrm{OP})_\varepsilon$ has at least one optimal control $[\boldsymbol{u}^*, v^*] = [u^*, u_\Gamma^*, v^*] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u}^* = [u^*, u_\Gamma^*]$, so that:

$$
\mathscr{J}_\varepsilon(\boldsymbol{u}^*, v^*) = \mathscr{J}_\varepsilon(u^*, u_\Gamma^*, v^*) = \min_{[\boldsymbol{u}, v] \in \mathfrak{X} \times \mathscr{H}} \mathscr{J}_\varepsilon(\boldsymbol{u}, v) = \min_{[u, u_\Gamma, v] \in \mathfrak{X} \times \mathscr{H}} \mathscr{J}_\varepsilon(u, u_\Gamma, v).
$$

(II-B) Let $\{\varepsilon_n\}_{n=1}^\infty \subset [0,1]$ and $\{[\boldsymbol{\eta}_{0,n}, \theta_{0,n}]\}_{n=1}^\infty = \{[\eta_{0,n}, \eta_{\Gamma,0,n}, \theta_{0,n}]\}_{n=1}^\infty \subset \mathbb{W} \times V_0$ with $\{\boldsymbol{\eta}_{0,n}\}_{n=1}^\infty = \{[\eta_{0,n}, \eta_{\Gamma,0,n}]\}_{n=1}^\infty$ be given sequences such that:

$$
\varepsilon_n \to \varepsilon, \ \text{ and } \ [\boldsymbol{\eta}_{0,n}, \theta_{0,n}] \to [\boldsymbol{\eta}_0, \theta_0] \text{ weakly in } \mathbb{W} \times V_0, \text{ as } n \to \infty. \tag{6.3.4}
$$

In addition, for any $n \in \mathbb{N}$, let $[\boldsymbol{u}_n^*, v_n^*] = [u_n^*, u_{\Gamma,n}^*, v_n^*] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u}_n^* = [u_n^*, u_{\Gamma,n}^*]$ be the optimal control of $(\mathrm{OP})_{\varepsilon_n}$. Then, there exist a subsequence $\{n_i\}_{i=1}^\infty \subset \{n\}$ and a triplet of functions $[\boldsymbol{u}^{**}, v^{**}] = [u^{**}, u_\Gamma^{**}, v^{**}] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u}^{**} = [u^{**}, u_\Gamma^{**}]$, such that:

$$
\begin{aligned}
&\varepsilon_{n_i} \to \varepsilon, \text{ and } [L u_{n_i}^*, L_\Gamma u_{\Gamma,n_i}^*, M v_{n_i}^*] \to [L u^{**}, L_\Gamma u_\Gamma^{**}, M v^{**}] \\
&\qquad \text{weakly in } \mathfrak{X} \times \mathscr{H}, \text{ as } i \to \infty,
\end{aligned}
$$

and

$$
[\boldsymbol{u}^{**}, v^{**}] = [u^{**}, u_\Gamma^{**}, v^{**}] \text{ is an optimal control of } (\mathrm{OP})_\varepsilon.
$$

Main Theorem 6.3. Under the assumptions (A0)–(A3), let us fix any initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \in \mathbb{W} \times V_0$ with $\boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}]$. Then, the following two items hold.

(III-A) (Necessary condition for $(\mathrm{OP})_\varepsilon$ when $\varepsilon > 0$) For any $\varepsilon > 0$, let $[\boldsymbol{u}_\varepsilon^*, v_\varepsilon^*] = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*, v_\varepsilon^*] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u}_\varepsilon^* = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*]$ be an optimal control of $(\mathrm{OP})_\varepsilon$, and let $[\boldsymbol{\eta}_\varepsilon^*, \theta_\varepsilon^*] = [\eta_\varepsilon^*, \eta_{\Gamma,\varepsilon}^*, \theta_\varepsilon^*] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta}_\varepsilon^* = [\eta_\varepsilon^*, \eta_{\Gamma,\varepsilon}^*]$ be the solution to $(\mathrm{S})_\varepsilon$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0]$ and forcing triplet $[\boldsymbol{u}_\varepsilon^*, v_\varepsilon^*] = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*, v_\varepsilon^*]$. Then, it holds that:

$$[L(u_\varepsilon^* + p_\varepsilon^*), L_\Gamma(u_{\Gamma,\varepsilon}^* + p_{\Gamma,\varepsilon}^*), M(v_\varepsilon^* + z_\varepsilon^*)] = [0,0,0] \text{ in } \mathfrak{X} \times \mathscr{H}, \tag{6.3.5}$$

where $[\boldsymbol{p}_\varepsilon^*, z_\varepsilon^*] \in \mathfrak{Y}$ is a unique solution to the following variational system:

$$\begin{aligned} &\bigl(-\partial_t \boldsymbol{p}_\varepsilon^*(t), \varphi\bigr)_{\mathbb{X}} + \bigl(\partial_x p_\varepsilon^*(t), \partial_x \varphi\bigr)_H + \bigl([\alpha''(\eta_\varepsilon^*) f_\varepsilon(\partial_x \theta_\varepsilon^*)](t) p_\varepsilon^*(t), \varphi\bigr)_H \\ &\quad + \bigl(g'(\eta_\varepsilon^*(t)) p_\varepsilon^*(t), \varphi\bigr)_H + \bigl([\alpha'(\eta_\varepsilon^*) f_\varepsilon'(\partial_x \theta_\varepsilon^*)](t) \partial_x z_\varepsilon^*(t), \varphi\bigr)_H \\ &\quad = \bigl(K(\eta_\varepsilon^* - \eta_{\mathrm{ad}})(t), \varphi\bigr)_H + \bigl(K_\Gamma(\eta_{\Gamma,\varepsilon}^* - \eta_{\Gamma,\mathrm{ad}})(t), \varphi_\Gamma\bigr)_{H_\Gamma}, \\ &\qquad \text{for any } \boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] \in \mathbb{W}, \text{ and a.e. } t \in (0,T); \end{aligned} \tag{6.3.6}$$

and

$$\begin{aligned} &\bigl\langle -\partial_t\bigl(\alpha_0 z_\varepsilon^*\bigr)(t), \psi\bigr\rangle_{V_0} + \bigl([\alpha(\eta_\varepsilon^*) f_\varepsilon''(\partial_x \theta_\varepsilon^*)](t) \partial_x z_\varepsilon^*(t) + \nu^2 \partial_x z_\varepsilon^*(t), \partial_x \psi\bigr)_H \\ &\quad + \bigl([\alpha'(\eta_\varepsilon^*) f_\varepsilon'(\partial_x \theta_\varepsilon^*)](t) p_\varepsilon^*(t), \partial_x \psi\bigr)_H = \bigl(\Lambda(\theta_\varepsilon^* - \theta_{\mathrm{ad}})(t), \psi\bigr)_H, \\ &\qquad \text{for any } \psi \in V_0, \text{ and a.e. } t \in (0,T); \end{aligned} \tag{6.3.7}$$

subject to the terminal condition:

$$[\boldsymbol{p}_\varepsilon^*(T), z_\varepsilon^*(T)] = [p_\varepsilon^*(T), p_{\Gamma,\varepsilon}^*(T), z_\varepsilon^*(T)] = [0,0,0] \text{ in } \mathbb{X} \times H. \tag{6.3.8}$$

(III-B) Let us define a Hilbert space $\mathscr{U}_0$ as follows:

$$\mathscr{U}_0 := \left\{ \ \psi \in W^{1,2}(0,T;H) \cap \mathscr{V}_0 \ \middle| \ \psi(0) = 0 \text{ in } H \ \right\}.$$

Then, there exists an optimal control $[\boldsymbol{u}^\circ, v^\circ] = [u^\circ, u_\Gamma^\circ, v^\circ] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u}^\circ = [u^\circ, u_\Gamma^\circ]$ of the problem $(\mathrm{OP})_0$, together with the solution $[\boldsymbol{\eta}^\circ, \theta^\circ] = [\eta^\circ, \eta_\Gamma^\circ, \theta^\circ] \in \mathfrak{X} \times \mathscr{H}$ to the system $(\mathrm{S})_0$ with $\boldsymbol{\eta}^\circ = [\eta^\circ, \eta_\Gamma^\circ]$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0]$ and forcing triplet $[\boldsymbol{u}^\circ, v^\circ] = [u^\circ, u_\Gamma^\circ, v^\circ]$, and moreover, there exist a triplet of functions $[\boldsymbol{p}^\circ, z^\circ] = [p^\circ, p_\Gamma^\circ, z^\circ] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{p}^\circ = [p^\circ, p_\Gamma^\circ]$, a pair of functions $[\xi^\circ, \nu^\circ] \in \mathscr{H} \times L^\infty(Q)$, and a distribution $\zeta^\circ \in \mathscr{U}_0^*$, such that:

$$[L(u^\circ + p^\circ), L_\Gamma(u_\Gamma^\circ + p_\Gamma^\circ), M(v^\circ + z^\circ)] = [0,0,0] \text{ in } \mathfrak{X} \times \mathscr{H}; \tag{6.3.9}$$

$$\begin{cases} \boldsymbol{p}^\circ = [p^\circ, p_\Gamma^\circ] \in W^{1,2}(0,T;\mathbb{X}) \cap L^\infty(0,T;\mathbb{W}) \subset C(\overline{Q}) \times C(\overline{\Sigma}), \\ z^\circ \in L^\infty(0,T;H) \cap \mathscr{V}_0, \\ \nu^\circ \in \mathrm{Sgn}^1(\partial_x \theta^\circ), \text{ a.e. in } Q; \end{cases} \tag{6.3.10}$$

$$\begin{aligned} &\bigl(-\partial_t \boldsymbol{p}^\circ, \boldsymbol{\varphi}\bigr)_{\mathfrak{X}} + \bigl(\partial_x p^\circ, \partial_x \varphi\bigr)_{\mathscr{H}} + \bigl(\alpha''(\eta^\circ)|\partial_x \theta^\circ| p^\circ, \varphi\bigr)_{\mathscr{H}} \\ &\quad + \bigl(g'(\eta^\circ) p^\circ, \varphi\bigr)_{\mathscr{H}} + \bigl(\alpha'(\eta^\circ) \xi^\circ, \varphi\bigr)_{\mathscr{H}} \end{aligned}$$

$$= \big(K(\eta^\circ - \eta_{ad}), \varphi\big)_{\mathscr{H}} + \big(K_\Gamma(\eta_\Gamma^\circ - \eta_{\Gamma,ad}), \varphi_\Gamma\big)_{\mathscr{H}_\Gamma}, \qquad (6.3.11)$$
$$\text{for any } \boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] \in \mathfrak{W},$$
$$\text{subject to } \boldsymbol{p}^\circ(T) = [p^\circ(T), p_\Gamma^\circ(T)] = [0,0] \text{ in } \mathbb{X};$$

and

$$\begin{aligned} &\big(\alpha_0 z^\circ, \partial_t \psi\big)_{\mathscr{H}} + \big\langle \zeta^\circ, \psi \big\rangle_{\mathscr{U}_0} + \big(\nu^2 \partial_x z^\circ + \alpha'(\eta^\circ)\nu^\circ p^\circ, \partial_x \psi\big)_{\mathscr{H}} \\ &\quad = \big(\Lambda(\theta^\circ - \theta_{ad}), \psi\big)_{\mathscr{H}}, \text{ for any } \psi \in \mathscr{U}_0. \end{aligned} \qquad (6.3.12)$$

Remark 6.11. Let $\mathcal{R}_T \in \mathscr{L}(\mathscr{H})$ be an isomorphism, defined as:

$$\big(\mathcal{R}_T \varphi\big)(t) := \varphi(T-t) \text{ in } H, \text{ for a.e. } t \in (0,T).$$

Also, let us fix $\varepsilon > 0$, and denote by $\mathcal{Q}_\varepsilon^* \in \mathscr{L}(\mathfrak{X} \times \mathscr{H}; \mathfrak{Y})$ the restriction $\mathcal{P}|_{\{[0,0,0]\} \times [\mathfrak{X} \times \mathscr{H}]}$ of the bounded linear operator $\mathcal{P} = \mathcal{P}(a,b,\mu,\omega,A) : [\mathbb{W} \times H] \times [\mathfrak{X} \times \mathscr{V}_0^*] \longrightarrow \mathfrak{Y}$, as in Remark 6.9, in the case when:

$$\begin{cases} [a,b] = \mathcal{R}_T[\alpha_0, -\partial_t \alpha_0] \text{ in } W^{1,\infty}(Q) \times L^\infty(Q), \\ \mu = \mathcal{R}_T\big[g'(\eta_\varepsilon^*) + \alpha''(\eta_\varepsilon^*) f_\varepsilon(\partial_x \theta_\varepsilon^*)\big] \text{ in } L^\infty(0,T;H), \\ [\omega, A] = \mathcal{R}_T\big[\alpha'(\eta_\varepsilon^*) f_\varepsilon'(\partial_x \theta_\varepsilon^*), \alpha(\eta_\varepsilon^*) f_\varepsilon''(\partial_x \theta_\varepsilon^*)\big] \text{ in } [L^\infty(Q)]^2. \end{cases} \qquad (6.3.13)$$

On this basis, let us define:

$$\mathcal{P}_\varepsilon^* := \mathcal{R}_T \circ \mathcal{Q}_\varepsilon^* \circ \mathcal{R}_T \text{ in } \mathscr{L}(\mathfrak{X} \times \mathscr{H}; \mathfrak{Y}).$$

Then, having in mind:

$$\partial_t(\alpha_0 \tilde{z}) = \alpha_0 \partial_t \tilde{z} + \tilde{z} \partial_t \alpha_0 \text{ in } \mathscr{V}_0^*, \text{ for any } \tilde{z} \in W^{1,2}(0,T;V_0^*), \qquad (6.3.14)$$

we can obtain the unique solution $[\boldsymbol{p}_\varepsilon^*, z_\varepsilon^*] = [p_\varepsilon^*, p_{\Gamma,\varepsilon}^*, z_\varepsilon^*] \in \mathfrak{Y}$ with $\boldsymbol{p}_\varepsilon^* = [p_\varepsilon^*, p_{\Gamma,\varepsilon}^*]$ to the variational system (6.3.6)–(6.3.8) as follows:

$$[\boldsymbol{p}_\varepsilon^*, z_\varepsilon^*] = [p_\varepsilon^*, p_{\Gamma,\varepsilon}^*, z_\varepsilon^*] = \mathcal{P}_\varepsilon^*\big[K(\eta_\varepsilon^* - \eta_{\rm ad}), K_\Gamma(\eta_{\Gamma,\varepsilon}^* - \eta_{\Gamma,\rm ad}), \Lambda(\theta_\varepsilon^* - \theta_{\rm ad})\big] \text{ in } \mathfrak{Y}.$$

6.4 Proof of Main Theorem 6.1

In this Section, we give the proof of the first Main Theorem 6.1. Before the proof, we refer to the reformulation method as in [62], and reduce the state-system $(\mathrm{S})_\varepsilon$ to an evolution equation in the Hilbert space $\mathbb{X} \times H$.

Let us fix any $\varepsilon \geq 0$. Besides, for any $R \geq 0$, let us define a proper functional $\Phi_\varepsilon^R : \mathbb{X} \times H \longrightarrow [0, \infty]$, by setting:

$$\Phi_\varepsilon^R : w = [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathbb{X} \times H \mapsto \Phi_\varepsilon^R(w) = \Phi_\varepsilon^R(\boldsymbol{\eta}, \theta) = \Phi_\varepsilon^R(\eta, \eta_\Gamma, \theta)$$
$$:= \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\partial_x \eta|^2\,dx + \dfrac{R}{2}\int_\Omega |\eta|^2\,dx + \dfrac{1}{2}\int_\Omega \left(\nu f_\varepsilon(\partial_x \theta) + \dfrac{1}{\nu}\alpha(\eta)\right)^2 dx, \\ \qquad \text{if } [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathbb{W} \times V_0 \text{ with } \boldsymbol{\eta} = [\eta, \eta_\Gamma], \\ \infty, \quad \text{otherwise.} \end{cases} \qquad (6.4.1)$$

Note that the assumptions (A1) and (A2) guarantee the lower semi-continuity and convexity of Φ_ε^R on $\mathbb{X} \times H$.

Remark 6.12. As consequences of standard variational methods, we easily check the following facts.

(Fact 3) For the operator $\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R : \mathbb{X} \times H \longrightarrow 2^{\mathbb{X}}$,

$$D(\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R) = \left\{ \ [\tilde{\boldsymbol{\eta}}, \tilde{\theta}] = [\tilde{\eta}, \tilde{\eta}_\Gamma, \tilde{\theta}] \in \mathbb{W} \times V_0 \ \middle| \ \tilde{\eta} \in H^2(\Omega) \text{ with } \partial_x \tilde{\eta} \cdot n_\Gamma = 0 \text{ on } \Gamma \ \right\},$$

and $\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R$ is a single-valued operator such that:

$$\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R(w) = {}^{\mathrm{t}}\begin{bmatrix} -\partial_x^2 \eta + R\eta + \alpha'(\eta) f_\varepsilon(\partial_x \theta) + \nu^{-2} \alpha(\eta) \alpha'(\eta) \\ 0 \end{bmatrix} \text{ in } \mathbb{X},$$

for any $w = [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in D(\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R)$.

(Fact 4) $\theta \in D(\partial_\theta \Phi_\varepsilon^R)$, and $\theta^* \in \partial_\theta \Phi_\varepsilon^R(w) = \partial_\theta \Phi_\varepsilon^R(\boldsymbol{\eta}, \theta) = \partial_\theta \Phi_\varepsilon^R(\eta, \eta_\Gamma, \theta)$, iff. $\theta \in V_0$, and

$$(\theta^*, \theta - \psi)_H \geq \nu^2 (\partial_x \theta, \partial_x (\theta - \psi))_H + \int_\Omega \alpha(\eta) f_\varepsilon(\partial_x \theta)\, dx - \int_\Omega \alpha(\eta) f_\varepsilon(\partial_x \psi),$$
$$\text{for any } \psi \in V_0.$$

In addition, let us define time-dependent operators $\mathcal{A}(t) \in \mathscr{L}(\mathbb{X} \times H)$, for $t \in [0, T]$, nonlinear operators $\mathcal{G}^R : \mathbb{X} \times H \longrightarrow \mathbb{X} \times H$, for $R \geq 0$, by setting:

$$\begin{aligned} \mathcal{A}(t) : w &= [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathbb{X} \times H \\ &\mapsto \mathcal{A}(t) w := [\eta,\ \eta_\Gamma,\ \alpha_0(t)\theta] \in \mathbb{X} \times H, \text{ for } t \in [0, T], \end{aligned} \tag{6.4.2}$$

and

$$\begin{aligned} \mathcal{G}^R : w &= [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathbb{X} \times H \\ &\mapsto \mathcal{G}^R(w) := \bigl[g(\eta) - R\eta - \nu^{-2} \alpha(\eta) \alpha'(\eta),\ 0,\ 0\bigr] \in \mathbb{X} \times H, \end{aligned} \tag{6.4.3}$$

respectively. Then, based on the above (Fact 3) and (Fact 4), it is verified that the state-system $(\mathrm{S})_\varepsilon$ is equivalent to the following Cauchy problem:

$$\begin{cases} \mathcal{A}(t) w'(t) + \bigl[\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R \times \partial_\theta \Phi_\varepsilon^R\bigr](w(t)) + \mathcal{G}^R(w(t)) \ni \mathfrak{f}(t) \text{ in } \mathbb{X} \times H, \\ \qquad \text{a.e. } t \in (0, T), \\ w(0) = w_0 \text{ in } \mathbb{X} \times H. \end{cases}$$

In the context, "$'$" is the time-derivative, and

$$\begin{cases} \bullet\ w_0 := [\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \in \mathbb{W} \times V_0 \text{ with } \boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}] \\ \quad \text{is the initial data of } w = [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta], \\ \bullet\ \mathfrak{f} := [Lu, L_\Gamma u_\Gamma, Mv] \in \mathfrak{X} \times \mathscr{H} \text{ is the forcing term of the} \\ \quad \text{Cauchy problem.} \end{cases} \tag{6.4.4}$$

Now, before the proof of Main Theorem 6.1, we prepare the following Key-Lemma and Corollary.

Key-Lemma 1. *Let us assume (A0)–(A3), and let us fix any $\varepsilon \geq 0$. Then, there exists a positive constant $R_0 > 0$ such that:*

$$\partial \Phi_\varepsilon^{R_0} = \bigl[\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^{R_0} \times \partial_\theta \Phi_\varepsilon^{R_0}\bigr] \ \textit{in } [\mathbb{X} \times H] \times [\mathbb{X} \times H].$$

Proof. We set:

$$R_0 := 1 + \frac{2}{\nu^2}|\alpha|^2_{L^\infty(\mathbb{R})}, \tag{6.4.5}$$

and prove this R_0 is the required constant.

In the light of (6.1.5), it is immediately verified that:

$$\partial\Phi_\varepsilon^{R_0} \subset \left[\partial_{\boldsymbol{\eta}}\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right] \text{ in } [\mathbb{X} \times H] \times [\mathbb{X} \times H].$$

Hence, with the maximality of the monotone graph $\partial\Phi_\varepsilon^{R_0}$ in $[\mathbb{X} \times H] \times [\mathbb{X} \times H]$ in mind, we can reduce our task to show the monotonicity of $\left[\partial_{\boldsymbol{\eta}}\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right]$ in $[\mathbb{X}\times H]\times[\mathbb{X}\times H]$.

Let us assume:

$$[w, w^*] \in \left[\partial_{\boldsymbol{\eta}}\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right] \text{ and } [\tilde{w}, \tilde{w}^*] \in \left[\partial_{\boldsymbol{\eta}}\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right] \text{ in } [\mathbb{X} \times H] \times [\mathbb{X} \times H].$$

Then, by using (6.4.2), (6.4.3), (Fact 3), (Fact 4), and Young's inequality, we compute that:

$$(w^* - \tilde{w}^*, w - \tilde{w})_{\mathbb{X}\times H} \geq I_A^1 + I_A^2 + I_A^3, \tag{6.4.6a}$$

with

$$I_A^1 := |\partial_x(\eta - \tilde{\eta})|_H^2 + R_0|\eta - \tilde{\eta}|_H^2 + \nu^2|\partial_x(\theta - \tilde{\theta})|_H^2, \tag{6.4.6b}$$

$$\begin{aligned} I_A^2 &:= (\alpha'(\eta)f_\varepsilon(\partial_x\theta) - \alpha'(\tilde{\eta})f_\varepsilon(\partial_x\tilde{\theta}), \eta - \tilde{\eta})_H \\ &= \int_\Omega f_\varepsilon(\partial_x\theta)(\alpha'(\eta) - \alpha'(\tilde{\eta}))(\eta - \tilde{\eta}))\,dx \\ &\qquad + \int_\Omega \alpha'(\tilde{\eta})(f_\varepsilon(\partial_x\theta) - f_\varepsilon(\partial_x\tilde{\theta}))(\eta - \tilde{\eta})\,dx \\ &\geq -|\alpha'|_{L^\infty(\mathbb{R})}|\eta - \tilde{\eta}|_H|\partial_x(\theta - \tilde{\theta})|_H \\ &\geq -\frac{|\alpha'|^2_{L^\infty(\mathbb{R})}}{\nu^2}|\eta - \tilde{\eta}|_H^2 - \frac{\nu^2}{4}|\partial_x(\theta - \tilde{\theta})|_H^2, \end{aligned} \tag{6.4.6c}$$

and

$$\begin{aligned} I_A^3 &:= \int_\Omega (\alpha(\eta) - \alpha(\tilde{\eta}))(f_\varepsilon(\partial_x\theta) - f_\varepsilon(\partial_x\tilde{\theta}))\,dx \\ &\geq -|\alpha'|_{L^\infty(\mathbb{R})}|\eta - \tilde{\eta}|_H|\partial_x(\theta - \tilde{\theta})|_H \\ &\geq -\frac{|\alpha'|^2_{L^\infty(\mathbb{R})}}{\nu^2}|\eta - \tilde{\eta}|_H^2 - \frac{\nu^2}{4}|\partial_x(\theta - \tilde{\theta})|_H^2. \end{aligned} \tag{6.4.6d}$$

Due to (6.4.5), the inequalities in (6.4.6) lead to:

$$(w^* - \tilde{w}^*, w - \tilde{w})_{\mathbb{X}\times H} \geq |\eta - \tilde{\eta}|_V^2 + \frac{\nu^2}{2}|\theta - \tilde{\theta}|_{V_0}^2 \geq 0,$$

which implies the monotonicity of the operator $\left[\partial_{\boldsymbol{\eta}}\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right]$ in $[\mathbb{X}\times H]\times[\mathbb{X}\times H]$. □

Corollary 6.4. Under the notations and assumptions as in the previous Key-Lemma 1, it holds that

$$\partial\Phi_\varepsilon^R = \left[\partial_{\boldsymbol{\eta}}\Phi_\varepsilon^R \times \partial_\theta\Phi_\varepsilon^R\right] \text{ in } [\mathbb{X} \times H] \times [\mathbb{X} \times H], \text{ for any } R \geq 0.$$

Proof. Let us take arbitrary two constants $0 \leq R, \tilde{R} < \infty$. Then from (Fact 3), we immediately have

$$D(\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R) = D(\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^{\tilde{R}}) \text{ in } \mathbb{W}, \tag{6.4.7a}$$

and

$$\begin{aligned} \partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R(w) &= {}^{\mathrm{t}}\begin{bmatrix} -\partial_x^2 \eta + \tilde{R}\eta + (R - \tilde{R})\eta + \alpha'(\eta) f_\varepsilon(\partial_x \theta) + \nu^{-2}\alpha(\eta)\alpha'(\eta) \\ 0 \end{bmatrix} \\ &= \partial_{\boldsymbol{\eta}} \Phi_\varepsilon^{\tilde{R}}(w) + (R - \tilde{R})[\eta, 0] \text{ in } \mathbb{X}, \end{aligned} \tag{6.4.7b}$$

$$\text{for any } w = [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in D(\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R) = D(\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^{\tilde{R}}).$$

Also, as a straightforward consequence of (Fact 4), it is seen that:

$$\partial_\theta \Phi_\varepsilon^R = \partial_\theta \Phi_\varepsilon^{\tilde{R}} \text{ in } H \times H. \tag{6.4.8}$$

In the meantime, invoking (6.4.1), [14, Theorem 2.10], and [18, Corollary 2.11], we will infer that

$$D(\partial \Phi_\varepsilon^R) = D(\partial \Phi_\varepsilon^{\tilde{R}}) \text{ in } \mathbb{W} \times V_0, \tag{6.4.9a}$$

and

$$\partial \Phi_\varepsilon^R(w) = \partial \Phi_\varepsilon^{\tilde{R}}(w) + (R - \tilde{R})[\eta, 0, 0] \text{ in } \mathbb{X} \times H. \tag{6.4.9b}$$

Now, let us take the constant $R_0 > 0$ obtained in Key-Lemma 1. Then, owing to (6.4.7)–(6.4.9), and Key-Lemma 1, we can compute that

$$\begin{aligned} \big[\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R \times \partial_\theta \Phi_\varepsilon^R\big](w) &= \big[\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^{R_0} \times \partial_\theta \Phi_\varepsilon^{R_0}\big](w) + (R - R_0)[\eta, 0, 0] \\ &= \partial \Phi_\varepsilon^{R_0}(w) + (R - R_0)[\eta, 0, 0] = \partial \Phi_\varepsilon^R(w) \text{ in } \mathbb{X} \times H, \\ &\text{for any } w \in D(\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R \times \partial_\theta \Phi_\varepsilon^R) = D(\partial_{\boldsymbol{\eta}} \Phi_\varepsilon^R) \cap D(\partial_\theta \Phi_\varepsilon^R). \end{aligned} \tag{6.4.10}$$

In the light of (6.1.5), the above (6.4.10) is sufficient to conclude this Corollary. □

Remark 6.13. Let $\varepsilon \geq 0$ be arbitrary constant. Then, as a consequence of (Fact 3), (Fact 4), Key-Lemma 1, and Corollary 6.4, we can say that the state-system $(\mathrm{S})_\varepsilon$ is equivalent to the following Cauchy problem of evolution equation, denoted by $(\mathrm{E})_\varepsilon$.

$(\mathrm{E})_\varepsilon$:

$$\begin{cases} \mathcal{A}(t)w'(t) + \partial \Phi_\varepsilon^R(w(t)) + \mathcal{G}^R(w(t)) \ni \mathfrak{f}(t) \text{ in } \mathbb{X} \times H, \text{ a.e. } t \in (0, T), \\ w(0) = w_0 \text{ in } \mathbb{X} \times H, \end{cases}$$

for any $R \geq 0$.

Now, we are ready to prove the Main Theorem 6.1.

Proof of Main Theorem 6.1 (I-A). Let us fix any $R > 0$. Then, under the setting (6.4.1)–(6.4.4), we immediately check that:

(ev.0) for any $t \in [0,T]$, $\mathcal{A}(t) \in \mathscr{L}(\mathbb{X} \times H)$ is positive and selfadjoint, and

$$(\mathcal{A}(t)w, w)_{\mathbb{X}\times H} \geq \delta_* |w|^2_{\mathbb{X}\times H}, \text{ for any } w \in \mathbb{X} \times H,$$

with the constant $\delta_* \in (0,1)$ as in (A1);

(ev.1) $\mathcal{A} \in W^{1,\infty}(0,T; \mathscr{L}(\mathbb{X} \times H))$, and

$$A^* := \operatorname*{ess\,sup}_{t\in(0,T)} \bigl\{ \max\{ |\mathcal{A}(t)|_{\mathscr{L}(\mathbb{X}\times H)}, |\mathcal{A}'(t)|_{\mathscr{L}(\mathbb{X}\times H)} \} \bigr\} \leq 1 + |\alpha_0|_{W^{1,\infty}(Q)} < \infty;$$

(ev.2) $\mathcal{G}^R : \mathbb{X} \times H \longrightarrow \mathbb{X} \times H$ is a Lipschitz continuous operator with a Lipschitz constant:

$$L_* := R + |g'|_{L^\infty(\mathbb{R})} + \nu^{-2} \bigl| (\alpha\alpha')' \bigr|_{L^\infty(\mathbb{R})},$$

and $\mathcal{G}^R$ has a C^1-potential functional

$$\begin{aligned} \widehat{\mathcal{G}}^R : w &= [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathbb{X} \times H \\ &\mapsto \widehat{\mathcal{G}}^R(w) := \int_\Omega \left(G(\eta) - \frac{R\eta^2}{2} - \frac{\alpha(\eta)^2}{2\nu^2} \right) dx \in \mathbb{R}; \end{aligned}$$

(ev.3) $\Phi^R_\varepsilon \geq 0$ on $\mathbb{X} \times H$, and the sublevel set $\bigl\{ \tilde{w} \in \mathbb{X} \times H \,\big|\, \Phi^R_\varepsilon(\tilde{w}) \leq r \bigr\}$ is contained in a compact set $K^R_\nu(r)$ in $\mathbb{X} \times H$, defined as

$$K^R_\nu(r) := \left\{ \ \tilde{w} = [\tilde{\boldsymbol{\eta}}, \tilde{\theta}] = [\tilde{\eta}, \tilde{\eta}_\Gamma, \tilde{\theta}] \in \mathbb{W} \times V_0 \ \middle| \ |\tilde{\boldsymbol{\eta}}|^2_{\mathbb{W}} + |\tilde{\theta}|^2_{V_0} \leq \tfrac{2r}{\min\{1, R, \nu^2\}} \ \right\},$$

for any $r \geq 0$.

On account of (6.4.1)–(6.4.4) and (ev.0)–(ev.3), we can apply Proposition 6.1, as the case when:

$$\begin{gathered} X = \mathbb{X} \times H, \ \mathcal{A}_0 = \mathcal{A} \text{ in } W^{1,\infty}(0,T; \mathscr{L}(\mathbb{X} \times H)), \\ \mathcal{G}_0 = \mathcal{G}^R \text{ on } \mathbb{X} \times H, \ \Psi_0 = \Phi^R_\varepsilon \text{ on } \mathbb{X} \times H, \text{ and } \mathfrak{f}_0 = \mathfrak{f} \text{ in } \mathfrak{X} \times \mathscr{H}, \end{gathered}$$

and we can find a solution $w = [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta} = [\eta, \eta_\Gamma]$ to the Cauchy problem $(\mathrm{E})_\varepsilon$. In the light of Proposition 6.1 and Remark 6.13, finding this $w = [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta]$ directly leads to the existence and uniqueness of solution to the state-system $(\mathrm{S})_\varepsilon$. □

Proof of Main Theorem 6.1 (I-B). Under the assumptions and notations as in Theorem 1 (I-A), we first fix a constant $R > 0$, and invoke Remark 6.13 to confirm that the solution $w := [\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta} = [\eta, \eta_\Gamma]$ to $(\mathrm{S})_\varepsilon$ coincides with the solution to the Cauchy problem $(\mathrm{E})_\varepsilon$, and as well as, the solutions $w_n := [\boldsymbol{\eta}_n, \theta_n] = [\eta_n, \eta_{\Gamma,n}, \theta_n] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta}_n = [\eta_n, \eta_{\Gamma,n}]$ to $(\mathrm{S})_{\varepsilon_n}$, $n = 1, 2, 3, \ldots$, coincide with the solutions to the Cauchy problems $(\mathrm{E})_{\varepsilon_n}$ for the initial data $w_{0,n} := [\boldsymbol{\eta}_{0,n} \theta_{0,n}] = [\eta_{0,n}, \eta_{\Gamma,0,n}, \theta_{0,n}] \in \mathbb{W} \times V_0$ with $\boldsymbol{\eta}_{0,n} = [\eta_{0,n}, \eta_{\Gamma,0,n}]$, and forcing terms $\mathfrak{f}_\mathfrak{n} = [Lu_n, L_\Gamma u_{\Gamma,n}, Mv_n] \in \mathfrak{X} \times \mathscr{H}$, $n = 1, 2, 3, \ldots$.

On this basis, we next verify:

(ev.4) $\Phi^R_{\varepsilon_n} \geq 0$ on $\mathbb{X}\times H$, for $n = 1, 2, 3, \dots$, and the union $\bigcup_{n=1}^{\infty}\{\tilde{w} \in \mathbb{X}\times H \,|\, \Phi^R_{\varepsilon_n}(\tilde{w}) \leq r\}$ of sublevel sets is contained in the compact set $K^R_\nu(r) \subset \mathbb{X} \times H$, as in (ev.3), for any $r > 0$;

(ev.5) $\Phi^R_{\varepsilon_n} \to \Phi^R_{\varepsilon}$ on $\mathbb{X}\times H$, in the sense of Mosco, as $n \to \infty$, more precisely, the uniform estimate (6.1.7) will lead to the corresponding lower bound condition and optimality condition, in the Mosco-convergence of $\{\Phi^R_{\varepsilon_n}\}_{n=1}^{\infty}$;

(ev.6) $\sup_{n\in\mathbb{N}} \Phi^R_{\varepsilon_n}(w_{0,n}) < \infty$, and

$$w_{0,n} \to w_0 \text{ in } \mathbb{X} \times H, \text{ as } n \to \infty,$$

more precisely, it follows from (6.3.1), (A0), and (A1) that

$$\sup_{n\in\mathbb{N}} \Phi^R_{\varepsilon_n}(w_{0,n}) \leq \sup_{n\in\mathbb{N}} \left(\frac{1+R}{2}|\eta_{0,n}|^2_V + \nu^2(1 + |\theta_{0,n}|^2_{V_0}) + \frac{1}{\nu^2}|\alpha(\eta_{0,n})|^2_H \right) < \infty,$$

and the weak convergence of $\{w_{0,n}\}_{n=1}^{\infty}$ in $\mathbb{W}\times V_0$ and the compactness of embedding $\mathbb{W} \times V_0 \subset \mathbb{X} \times H$ imply the strong convergence of $\{w_{0,n}\}_{n=1}^{\infty}$ in $\mathbb{X} \times H$.

On account of (6.3.1) and (ev.0)–(ev.6), we can apply Proposition 6.2, to show that:

$$\begin{cases} w_n \to w \text{ in } C([0,T]; \mathbb{X} \times H) \\ \quad \text{(i.e. in } C([0,T]; \mathbb{X}) \times C([0,T]; H)), \\ \quad \text{weakly in } W^{1,2}(0,T; \mathbb{X} \times H) \\ \quad \text{(i.e. weakly in } W^{1,2}(0,T; \mathbb{X}) \times W^{1,2}(0,T; H)), \\ \displaystyle\int_0^T \Phi^R_{\varepsilon_n}(w_n(t))\,dt \to \int_0^T \Phi^R_{\varepsilon}(w(t))\,dt, \end{cases} \text{as } n \to \infty, \tag{6.4.11a}$$

$$\begin{aligned} \sup_{n\in\mathbb{N}} |w_n|^2_{L^\infty(0,T;\mathbb{W})\times L^\infty(0,T;V_0)} &\leq 4 \sup_{n\in\mathbb{N}} |w_n|^2_{L^\infty(0,T;\mathbb{W}\times V_0)} \\ &\leq \frac{8}{\min\{1, \nu^2, R\}} \sup_{n\in\mathbb{N}} |\Phi^R_{\varepsilon_n}(w_n)|_{L^\infty(0,T)} < \infty, \end{aligned}$$

and hence,

$$w_n \to w \text{ weakly-}* \text{ in } L^\infty(0,T; \mathbb{W}) \times L^\infty(0,T; V_0), \text{ as } n \to \infty. \tag{6.4.11b}$$

Also, as a consequence of the one-dimensional compact embeddings $V \subset C(\overline{\Omega})$ and $V_0 \subset C(\overline{\Omega})$, the uniqueness of solution w to $(\mathrm{E})_\varepsilon$, and Ascoli's theorem (cf. [83, Corollary 4]), we can derive from (6.4.11a) that

$$w_n \to w \text{ in } [C(\overline{Q}) \times C(\overline{\Sigma})] \times C(\overline{Q}), \text{ as } n \to \infty. \tag{6.4.12}$$

Furthermore, from (6.1.6), (6.1.7), (6.4.11), (6.4.12), and the assumptions (A0)–(A2), one can observe that:

$$\begin{cases} \displaystyle\varliminf_{n\to\infty} \frac{1}{2}|\partial_x \eta_n|^2_{\mathscr{H}} \geq \frac{1}{2}|\partial_x \eta|^2_{\mathscr{H}}, \quad \varliminf_{n\to\infty} \frac{R}{2}|\eta_n|^2_{\mathscr{H}} \geq \frac{R}{2}|\eta|^2_{\mathscr{H}}, \\ \displaystyle\varliminf_{n\to\infty} \frac{\nu^2}{2}|\theta_n|^2_{\mathscr{V}_0} \geq \frac{\nu^2}{2}|\theta|^2_{\mathscr{V}_0}, \quad \lim_{n\to\infty} \frac{1}{2\nu^2}|\alpha(\eta_n)|^2_{\mathscr{H}} = \frac{1}{2\nu^2}|\alpha(\eta)|^2_{\mathscr{H}}, \end{cases} \tag{6.4.13a}$$

and

$$\begin{aligned}
\varliminf_{n\to\infty}\bigl|\alpha(\eta_n)f_{\varepsilon_n}(\partial_x\theta_n)\bigr|_{L^1(Q)} &= \varliminf_{n\to\infty}\int_0^T\int_\Omega \alpha(\eta_n(t))f_{\varepsilon_n}(\partial_x\theta_n(t))\,dxdt \\
&\geq \varliminf_{n\to\infty}\int_0^T\int_\Omega \alpha(\eta(t))f_{\varepsilon_n}(\partial_x\theta_n(t))\,dxdt \\
&\qquad - \lim_{n\to\infty}|\alpha(\eta_n)-\alpha(\eta)|_{C(\overline{Q})}\cdot\sup_{n\in\mathbb{N}}\bigl(T\varepsilon_n+|\partial_x\theta_n|_{L^1(0,T;L^1(\Omega))}\bigr) \\
&\geq \varliminf_{n\to\infty}\int_0^T\int_\Omega \alpha(\eta(t))f_{\varepsilon}(\partial_x\theta_n(t))\,dxdt - |\alpha(\eta)|_{C(\overline{Q})}\cdot\lim_{n\to\infty}\bigl(T|\varepsilon_n-\varepsilon|\bigr) \\
&\geq \int_0^T\int_\Omega \alpha(\eta(t))f_\varepsilon(\partial_x\theta(t))\,dxdt = \bigl|\alpha(\eta)f_\varepsilon(\partial_x\theta)\bigr|_{L^1(Q)}.
\end{aligned} \tag{6.4.13b}$$

Here, from (6.4.1), it is seen that:

$$\begin{aligned}
&\int_0^T \Phi_{\tilde{\varepsilon}}^R(\tilde{w}(t))\,dt = \int_0^T \Phi_{\tilde{\varepsilon}}^R(\tilde{\eta}(t),\tilde{\eta}_\Gamma(t),\tilde{\theta}(t))\,dt \\
&= \frac{1}{2}|\partial_x\tilde{\eta}|_{\mathscr{H}}^2 + \frac{R}{2}|\tilde{\eta}|_{\mathscr{H}}^2 + \frac{\nu^2}{2}|\tilde{\theta}|_{\mathscr{V}_0}^2 + \bigl|\alpha(\tilde{\eta})f_{\tilde{\varepsilon}}(\partial_x\tilde{\theta})\bigr|_{L^1(Q)} + \frac{1}{2\nu^2}|\alpha(\tilde{\eta})|_{\mathscr{H}}^2 + \frac{\nu^2\tilde{\varepsilon}^2}{2}T \\
&\qquad \text{for all } \tilde{\varepsilon}>0 \text{ and } \tilde{w}=[\tilde{\boldsymbol{\eta}},\tilde{\theta}]=[\tilde{\eta},\tilde{\eta}_\Gamma,\tilde{\theta}]\in D(\Phi_{\tilde{\varepsilon}})=\mathfrak{W}\times\mathscr{V}_0.
\end{aligned} \tag{6.4.14}$$

Taking into account (6.4.11a), (6.4.13), and (6.4.14), we deduce that:

$$\begin{aligned}
&|\partial_x\eta_n|_{\mathscr{H}}^2 + R|\eta_n|_{\mathscr{H}}^2 + \nu^2|\theta_n|_{\mathscr{V}_0}^2 \to |\partial_x\eta|_{\mathscr{H}}^2 + R|\eta|_{\mathscr{H}}^2 + \nu^2|\theta|_{\mathscr{V}_0}^2, \\
&\qquad \text{and hence, } |[\eta_n,\theta_n]|_{\mathscr{V}\times\mathscr{V}_0} \to |[\eta,\theta]|_{\mathscr{V}\times\mathscr{V}_0}, \text{ as } n\to\infty.
\end{aligned} \tag{6.4.15}$$

Since the norm of Hilbert space $\mathscr{V}\times\mathscr{V}_0$ is uniformly convex, the convergences (6.4.11b) and (6.4.15) imply the strong convergences:

$$w_n \to w \text{ in } \mathfrak{W}\times\mathscr{V}_0, \text{ as } n\to\infty, \tag{6.4.16a}$$

and furthermore, it follows from (6.1.7) and (6.4.16a) that:

$$\begin{aligned}
|f_{\varepsilon_n}(\partial_x\theta_n) - f_\varepsilon(\partial_x\theta)|_{\mathscr{H}} &\leq |f_{\varepsilon_n}(\partial_x\theta_n) - f_{\varepsilon_n}(\partial_x\theta)|_{\mathscr{H}} + |f_{\varepsilon_n}(\partial_x\theta) - f_\varepsilon(\partial_x\theta)|_{\mathscr{H}} \\
&\leq |\theta_n-\theta|_{\mathscr{V}_0} + \sqrt{T}|\varepsilon_n-\varepsilon| \to 0, \text{ as } n\to\infty.
\end{aligned} \tag{6.4.16b}$$

The convergences (6.4.11), (6.4.12), and (6.4.16) are sufficient to verify the conclusions (6.3.2) and (6.3.3) of Main Theorem 6.1 (I-B). $\square$

6.5 Proof of Main Theorem 6.2

In this Section, we prove the second Main Theorem 6.2. Let $[\boldsymbol{\eta}_0,\theta_0]=[\eta_0,\eta_{\Gamma,0},\theta_0]\in\mathbb{W}\times V_0$ with $\boldsymbol{\eta}_0=[\eta_0,\eta_{\Gamma,0}]$ be the initial triplet. Also, let us fix arbitrary forcing triplet $[\bar{\boldsymbol{u}},\bar{v}]=[\bar{u},\bar{u}_\Gamma,\bar{v}]\in\mathfrak{X}\times\mathscr{H}$ with $\bar{\boldsymbol{u}}=[\bar{u},\bar{u}_\Gamma]$, and let us invoke the definition of the cost functional (1.5.32), to estimate that:

$$0 \leq \underline{J}_\varepsilon := \inf_{[\boldsymbol{u},v]\in\mathfrak{X}\times\mathscr{H}} \mathcal{J}_\varepsilon(\boldsymbol{u},v) \leq \overline{J}_\varepsilon := \mathcal{J}_\varepsilon(\bar{\boldsymbol{u}},\bar{v}) = \mathcal{J}_\varepsilon(\bar{u},\bar{u}_\Gamma,\bar{v}) < \infty, \text{ for all } \varepsilon\geq 0. \tag{6.5.1}$$

Also, for any $\varepsilon \geq 0$, we denote by $[\bar{\boldsymbol{\eta}}_\varepsilon, \bar{\theta}_\varepsilon] = [\bar{\eta}_\varepsilon, \bar{\eta}_{\Gamma,\varepsilon}, \bar{\theta}_\varepsilon] \in \mathfrak{X} \times \mathscr{H}$ with $\bar{\boldsymbol{\eta}}_\varepsilon = [\bar{\eta}_\varepsilon, \bar{\eta}_{\Gamma,\varepsilon}]$ the solution to $(\mathrm{S})_\varepsilon$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0]$ and forcing triplet $[\bar{\boldsymbol{u}}, \bar{v}] = [\bar{u}, \bar{u}_\Gamma, \bar{v}]$.

Based on these, the Main Theorem 6.2 is proved as follows.

Proof of Main Theorem 6.2 (II-A). Let us fix any $\varepsilon \geq 0$. Then, from the estimate (6.5.1), we immediately find a sequence of forcing triplets $\{[\boldsymbol{u}_n, v_n]\}_{n=1}^\infty = \{[u_n, u_{\Gamma,n}, v_n]\}_{n=1}^\infty \subset \mathfrak{X} \times \mathscr{H}$ with $\{\boldsymbol{u}_n\}_{n=1}^\infty = \{[u_n, u_{\Gamma,n}]\}_{n=1}^\infty$, such that:

$$\mathscr{J}_\varepsilon(\boldsymbol{u}_n, v_n) = \mathscr{J}_\varepsilon(u_n, u_{\Gamma,n}, v_n) \downarrow \underline{J}_\varepsilon, \text{ as } n \to \infty, \tag{6.5.2a}$$

and

$$\frac{1}{2} \sup_{n \in \mathbb{N}} \left| [\sqrt{L} u_n, \sqrt{L_\Gamma} u_{\Gamma,n}, \sqrt{M} v_n] \right|^2_{\mathfrak{X} \times \mathscr{H}} \leq \mathscr{J}_\varepsilon(\bar{u}, \bar{u}_\Gamma, \bar{v}) < \infty. \tag{6.5.2b}$$

Also, the estimate (6.5.2b) enables us to take a subsequence of $\{[\boldsymbol{u}_n, v_n]\}_{n=1}^\infty = \{[u_n, u_{\Gamma,n}, v_n]\}_{n=1}^\infty$ (not relabeled), and to find a triplet of functions $[\boldsymbol{u}^*, v^*] = [u^*, u^*_\Gamma, v^*] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u}^* = [u^*, u^*_\Gamma]$, such that:

$$\begin{aligned} &[\sqrt{L} u_n, \sqrt{L_\Gamma} u_\Gamma, \sqrt{M} v_n] \to [\sqrt{L} u^*, \sqrt{L_\Gamma} u^*_\Gamma, \sqrt{M} v^*] \\ &\qquad \text{weakly in } \mathfrak{X} \times \mathscr{H}, \text{ as } n \to \infty, \end{aligned} \tag{6.5.3a}$$

and as well as,

$$[L u_n, L_\Gamma u_{\Gamma,n}, M v_n] \to [L u^*, L_\Gamma u^*_\Gamma, M_v v^*] \text{ weakly in } \mathfrak{X} \times \mathscr{H}, \text{ as } n \to \infty. \tag{6.5.3b}$$

Let $[\boldsymbol{\eta}^*, \theta^*] = [\eta^*, \eta^*_\Gamma, \theta^*] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta}^* = [\eta^*, \eta^*_\Gamma]$ be the solution to $(\mathrm{S})_\varepsilon$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0]$ and forcing triplet $[\boldsymbol{u}^*, v^*] = [u^*, u^*_\Gamma, v^*]$. As well as, for any $n \in \mathbb{N}$, let $[\boldsymbol{\eta}_n, \theta_n] = [\eta_n, \eta_{\Gamma,n}, \theta_n] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta}_n = [\eta_n, \eta_{\Gamma,n}]$ be the solution to $(\mathrm{S})_\varepsilon$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0]$ and the forcing triplet $[\boldsymbol{u}_n, v_n] = [u_n, u_{\Gamma,n}, v_n]$. Then, having in mind (6.5.3) and the initial condition:

$$\begin{aligned} [\boldsymbol{\eta}_n(0), \theta_n(0)] &= [\eta_n(0), \eta_{\Gamma,n}(0), \theta_n(0)] \\ &= [\boldsymbol{\eta}^*(0), \theta^*(0)] = [\eta^*(0), \eta^*_\Gamma(0), \theta^*(0)] \\ &= [\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \text{ in } \mathbb{X} \times H, \text{ for } n = 1, 2, 3, \ldots, \end{aligned}$$

we can apply Main Theorem 6.1 (I-B), to see that:

$$[\boldsymbol{\eta}_n, \theta_n] \to [\boldsymbol{\eta}^*, \theta^*] \text{ in } [C(\overline{Q}) \times C(\overline{\Sigma})] \times C(\overline{Q}), \text{ as } n \to \infty. \tag{6.5.4}$$

On account of (6.5.2a), (6.5.3a), and (6.5.4), it is computed that:

$$\begin{aligned} \mathscr{J}_\varepsilon(\boldsymbol{u}^*, v^*) &= \mathscr{J}_\varepsilon(u^*, u^*_\Gamma, v^*) \\ &= \frac{1}{2} \left| [\sqrt{K}(\eta^* - \eta_{\mathrm{ad}}), \sqrt{K_\Gamma}(\eta^*_\Gamma - \eta_{\Gamma,\mathrm{ad}}), \sqrt{\Lambda}(\theta^* - \theta_{\mathrm{ad}})] \right|^2_{\mathfrak{X} \times \mathscr{H}} \\ &\quad + \frac{1}{2} \left| [\sqrt{L} u^*, \sqrt{L_\Gamma} u^*_\Gamma, \sqrt{M} v^*] \right|^2_{\mathfrak{X} \times \mathscr{H}} \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\lim_{n\to\infty}\bigl|[\sqrt{K}(\eta_n-\eta_{\mathrm{ad}}),\sqrt{K_\Gamma}(\eta_{\Gamma,n}-\eta_{\Gamma,\mathrm{ad}}),\sqrt{\Lambda}(\theta_n-\theta_{\mathrm{ad}})]\bigr|^2_{\mathfrak{X}\times\mathscr{H}}\\
&\qquad+\frac{1}{2}\varliminf_{n\to\infty}\bigl|[\sqrt{L}u_n,\sqrt{L_\Gamma}u_{\Gamma,n},\sqrt{M}v_n]\bigr|^2_{\mathfrak{X}\times\mathscr{H}}\\
&=\lim_{n\to\infty}\mathcal{J}_\varepsilon(\boldsymbol{u}_n,v_n)=\lim_{n\to\infty}\mathcal{J}_\varepsilon(u_n,u_{\Gamma,n},v_n)=\underline{J}_\varepsilon\ (\le\mathcal{J}_\varepsilon(\boldsymbol{u}^*,v^*)),
\end{aligned}
$$

and it implies that

$$
\mathcal{J}_\varepsilon(\boldsymbol{u}^*,v^*)=\mathcal{J}_\varepsilon(u^*,u^*_\Gamma,v^*)=\min_{[\boldsymbol{u},v]\in\mathfrak{X}\times\mathscr{H}}\mathcal{J}_\varepsilon(\boldsymbol{u},v)=\min_{[u,u_\Gamma,v]\in\mathfrak{X}\times\mathscr{H}}\mathcal{J}_\varepsilon(u,u_\Gamma,v).
$$

Thus, we conclude the item (II-A). □

Proof of Main Theorem 6.2 (II-B). Let $\varepsilon\in[0,1]$ and $\{\varepsilon_n\}_{n=1}^\infty\subset[0,1]$ be as in (6.3.4). Let $[\bar{\boldsymbol{\eta}}_\varepsilon,\bar{\theta}_\varepsilon]=[\bar{\eta}_\varepsilon,\bar{\eta}_{\Gamma,\varepsilon},\bar{\theta}_\varepsilon]\in\mathfrak{X}\times\mathscr{H}$ with $\bar{\boldsymbol{\eta}}_\varepsilon=[\bar{\eta}_\varepsilon,\bar{\eta}_{\Gamma,\varepsilon}]$ be the solution to the system $(\mathrm{S})_\varepsilon$, for the initial triplet $[\boldsymbol{\eta}_0,\theta_0]=[\eta_0,\eta_{\Gamma,0},\theta_0]$ and forcing triplet $[\bar{\boldsymbol{u}},\bar{v}]=[\bar{u},\bar{u}_\Gamma,\bar{v}]$, and let $[\bar{\boldsymbol{\eta}}_{\varepsilon_n},\bar{\theta}_{\varepsilon_n}]=[\bar{\eta}_{\varepsilon_n},\bar{\eta}_{\Gamma,\varepsilon_n},\bar{\theta}_{\varepsilon_n}]\in\mathfrak{X}\times\mathscr{H}$ with $\bar{\boldsymbol{\eta}}_{\varepsilon_n}=[\bar{\eta}_{\varepsilon_n},\bar{\eta}_{\Gamma,\varepsilon_n}]$, $n=1,2,3,\ldots$, be the solutions to $(\mathrm{S})_{\varepsilon_n}$, for the respective initial triplets $[\boldsymbol{\eta}_{0,n},\theta_{0,n}]=[\eta_{0,n},\eta_{\Gamma,0,n},\theta_{0,n}]\in\mathbb{W}\times V_0$ with $\boldsymbol{\eta}_{0,n}=[\eta_{0,n},\eta_{\Gamma,0,n}]$, $n=1,2,3,\ldots$, and the fixed forcing triplet $[\bar{\boldsymbol{u}},\bar{v}]=[\bar{u},\bar{u}_\Gamma,\bar{v}]$. On this basis, let us first apply Main Theorem 6.1 (I-B) to the solutions $[\bar{\boldsymbol{\eta}}_\varepsilon,\bar{\theta}_\varepsilon]=[\bar{\eta}_\varepsilon,\bar{\eta}_{\Gamma,\varepsilon},\bar{\theta}_\varepsilon]$ and $[\bar{\boldsymbol{\eta}}_{\varepsilon_n},\bar{\theta}_{\varepsilon_n}]=[\bar{\eta}_{\varepsilon_n},\bar{\eta}_{\Gamma,\varepsilon_n},\bar{\theta}_{\varepsilon_n}]$, $n=1,2,3,\ldots$. Then, we have

$$
\begin{cases}
[\bar{\boldsymbol{\eta}}_{\varepsilon_n},\theta_{\varepsilon_n}]\to[\bar{\boldsymbol{\eta}}_\varepsilon,\theta_\varepsilon]\text{ in }[C(\overline{Q})\times C(\overline{\Sigma})]\times C(\overline{Q}),\\[1ex]
[\bar{\boldsymbol{\eta}}_n(0),\bar{\theta}_n(0)]=[\boldsymbol{\eta}_{0,n},\theta_{0,n}]\to[\bar{\boldsymbol{\eta}}_\varepsilon(0),\bar{\theta}_\varepsilon(0)]=[\boldsymbol{\eta}_0,\theta_0]\\[1ex]
\qquad\text{in }[C(\overline{\Omega})\times C(\Gamma)]\times C(\overline{\Omega}),\text{ as }n\to\infty,
\end{cases}
\tag{6.5.5}
$$

and hence,

$$
\overline{J}_{\sup}:=\sup_{n\in\mathbb{N}}\mathcal{J}_{\varepsilon_n}(\bar{\boldsymbol{u}},\bar{v})=\sup_{n\in\mathbb{N}}\mathcal{J}_{\varepsilon_n}(\bar{u},\bar{u}_\Gamma,\bar{v})<\infty.
\tag{6.5.6}
$$

Next, for any $n\in\mathbb{N}$, let us denote by $[\boldsymbol{\eta}^*_n,\theta^*_n]=[\eta^*_n,\eta^*_{\Gamma,n},\theta^*_n]\in\mathfrak{X}\times\mathscr{H}$ with $\boldsymbol{\eta}^*_n=[\eta^*_n,\eta^*_{\Gamma,n}]$ the solution to $(\mathrm{S})_{\varepsilon_n}$ for the initial triplet $[\boldsymbol{\eta}_{0,n},\theta_{0,n}]=[\eta_{0,n},\eta_{\Gamma,0,n},\theta_{0,n}]$ and forcing triplet $[\boldsymbol{u}^*_n,v^*_n]=[u^*_n,u^*_{\Gamma,n},v^*_n]\in\mathfrak{X}\times\mathscr{H}$ with $\boldsymbol{u}^*_n=[u^*_n,u^*_{\Gamma,n}]$. Then, in the light of (6.5.1) and (6.5.6), we can see that:

$$
0\le\frac{1}{2}|[\sqrt{L}u^*_n,\sqrt{L_\Gamma}u^*_{\Gamma,n},\sqrt{M}v^*_n]|^2_{\mathfrak{X}\times\mathscr{H}}\le\underline{J}_{\varepsilon_n}\le\overline{J}_{\sup}<\infty,\ \text{ for }n=1,2,3,\ldots.
$$

Therefore, we can find a subsequence $\{n_i\}_{i=1}^\infty\subset\{n\}$, together with a triplet of functions $[\boldsymbol{u}^{**},v^{**}]=[u^{**},u^{**}_{\Gamma,n},v^{**}]\in\mathfrak{X}\times\mathscr{H}$ with $\boldsymbol{u}^{**}=[u^{**},u^{**}_{\Gamma,n}]$, such that:

$$
\begin{aligned}
&[\sqrt{L}u^*_{n_i},\sqrt{L_\Gamma}u^*_{\Gamma,n_i},\sqrt{M}v^*_{n_i}]\to[\sqrt{L}u^{**},\sqrt{L_\Gamma}u^{**}_\Gamma,\sqrt{M}v^{**}]\\
&\qquad\text{weakly in }\mathfrak{X}\times\mathscr{H},\ \text{ as }i\to\infty,
\end{aligned}
\tag{6.5.7}
$$

and as well as,

$$
[Lu^*_{n_i},L_\Gamma u^*_{\Gamma,n_i},Mv^*_{n_i}]\to[Lu^{**},L_\Gamma u^{**}_\Gamma,Mv^{**}]\text{ weakly in }\mathfrak{X}\times\mathscr{H},\ \text{ as }i\to\infty.
$$

Here, let us denote by $[\boldsymbol{\eta}^{**}, \theta^{**}] = [\eta^{**}, \eta_\Gamma^{**}, \theta^{**}] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta}^{**} = [\eta^{**}, \eta_\Gamma^{**}]$ the solution to $(\mathrm{S})_\varepsilon$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_{0,n}, \eta_{\Gamma,0,n}, \theta_{0,n}]$ and forcing triplet $[\boldsymbol{u}^{**}, v^{**}] = [u^{**}, u_\Gamma^{**}, v^{**}]$. Then, applying Main Theorem 6.1 (I-B), again, to the solutions $[\boldsymbol{\eta}^{**}, \theta^{**}] = [\eta^{**}, \eta_\Gamma^{**}, \theta^{**}]$ and $[\boldsymbol{\eta}^*_{n_i}, \theta^*_{n_i}] = [\eta^*_{n_i}, \eta^*_{\Gamma,n_i}, \theta^*_{n_i}]$, $i = 1, 2, 3, \dots$, we can observe that:

$$[\boldsymbol{\eta}^*_{n_i}, \theta^*_{n_i}] \to [\boldsymbol{\eta}^{**}, \theta^{**}] \text{ in } [C(\overline{Q}) \times C(\overline{\Sigma})] \times C(\overline{Q}), \text{ as } i \to \infty. \tag{6.5.8}$$

As a consequence of (6.5.5), (6.5.7), and (6.5.8), it is verified that:

$$\begin{aligned}
\mathscr{J}_\varepsilon(\boldsymbol{u}^{**}, v^{**}) &= \mathscr{J}_\varepsilon(u^{**}, u_\Gamma^{**}, v^{**}) \\
&= \frac{1}{2}\big|[\sqrt{K}(\eta^{**} - \eta_{\mathrm{ad}}), \sqrt{K_\Gamma}(\eta_\Gamma^{**} - \eta_{\Gamma,\mathrm{ad}}), \sqrt{\Lambda}(\theta^{**} - \theta_{\mathrm{ad}}))]\big|^2_{\mathfrak{X}\times\mathscr{H}} \\
&\qquad + \frac{1}{2}\big|[\sqrt{L}u^{**}, \sqrt{L_\Gamma}u_\Gamma^{**}, \sqrt{M}v^{**}]\big|^2_{\mathfrak{X}\times\mathscr{H}} \\
&\leq \frac{1}{2}\lim_{i\to\infty}\big|[\sqrt{K}(\eta^*_{n_i} - \eta_{\mathrm{ad}}), \sqrt{K_\Gamma}(\eta^*_{\Gamma,n_i} - \eta_{\Gamma,\mathrm{ad}}), \sqrt{\Lambda}(\theta^*_{n_i} - \theta_{\mathrm{ad}})]\big|^2_{\mathfrak{X}\times\mathscr{H}} \\
&\qquad + \frac{1}{2}\varliminf_{i\to\infty}\big|[\sqrt{L}u^*_{n_i}, \sqrt{L_\Gamma}u^*_{\Gamma,n_i}, \sqrt{M}v^*_{n_i}]\big|^2_{\mathfrak{X}\times\mathscr{H}} \\
&= \varliminf_{i\to\infty}\mathscr{J}_{\varepsilon_{n_i}}(\boldsymbol{u}^*_{n_i}, v^*_{n_i}) = \varliminf_{i\to\infty}\mathscr{J}_{\varepsilon_{n_i}}(u^*_{n_i}, u^*_{\Gamma,n_i}, v^*_{n_i}) \\
&\leq \lim_{i\to\infty}\mathscr{J}_{\varepsilon_{n_i}}(\bar{\boldsymbol{u}}, \bar{v}) = \lim_{i\to\infty}\mathscr{J}_{\varepsilon_{n_i}}(\bar{u}, \bar{u}_\Gamma, \bar{v}) \\
&= \frac{1}{2}\lim_{i\to\infty}\big|[\sqrt{K}(\bar{\eta}_{\varepsilon_{n_i}} - \eta_{\mathrm{ad}}), \sqrt{K_\Gamma}(\bar{\eta}_{\Gamma,\varepsilon_{n_i}} - \eta_{\Gamma,\mathrm{ad}}), \sqrt{\Lambda}(\bar{\theta}_{\varepsilon_{n_i}} - \theta_{\mathrm{ad}})]\big|^2_{\mathfrak{X}\times\mathscr{H}} \\
&\qquad + \frac{1}{2}\big|[\sqrt{L}\bar{u}, \sqrt{L_\Gamma}\bar{u}_\Gamma, \sqrt{M}\bar{v}]\big|^2_{\mathfrak{X}\times\mathscr{H}} \\
&= \mathscr{J}_\varepsilon(\bar{\boldsymbol{u}}, \bar{v}) = \mathscr{J}_\varepsilon(\bar{u}, \bar{u}_\Gamma, \bar{v}).
\end{aligned}$$

Since the choice of $[\bar{\boldsymbol{u}}, \bar{v}] = [\bar{u}, \bar{u}_\Gamma, \bar{v}] \in \mathfrak{X} \times \mathscr{H}$ is arbitrary, we conclude that:

$$\mathscr{J}_\varepsilon(\boldsymbol{u}^{**}, v^{**}) = \mathscr{J}_\varepsilon(u^{**}, u_\Gamma^{**}, v^{**}) = \min_{[\boldsymbol{u}, v] \in \mathfrak{X}\times\mathscr{H}} \mathscr{J}_\varepsilon(\boldsymbol{u}, v) = \min_{[u, u_\Gamma, v] \in \mathfrak{X}\times\mathscr{H}} \mathscr{J}_\varepsilon(u, u_\Gamma, v),$$

and complete the proof of the item (II-B). □

6.6 Proof of Main Theorem 6.3

This Section is devoted to the proof of the third Main Theorem 6.3. To this end, we need to start with the case of $\varepsilon > 0$, and prepare some Lemmas, associated with the Gâteaux differential of the regular cost functional $\mathscr{J}_\varepsilon$.

Let $\varepsilon > 0$ be a fixed constant, and let $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \in \mathbb{W} \times V_0$ with $\boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}]$ be the initial triplet. Let us take any forcing triplet $[\boldsymbol{u}, v] = [u, u_\Gamma, v] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u} = [u, u_\Gamma]$, and consider the unique solution $[\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta} = [\eta, \eta_\Gamma]$ to the state-system $(\mathrm{S})_\varepsilon$. Also, let us take any constant $\delta \in (-1, 1) \setminus \{0\}$ and any triplet of functions $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{h} = [h, h_\Gamma]$, and consider another solution $[\boldsymbol{\eta}^\delta, \theta^\delta] = [\eta^\delta, \eta_\Gamma^\delta, \theta^\delta] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta}^\delta = [\eta^\delta, \eta_\Gamma^\delta]$ to the system $(\mathrm{S})_\varepsilon$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0]$ and a perturbed forcing triplet $[\boldsymbol{u} + \delta\boldsymbol{h}, v + \delta k] = [u + \delta h, u_\Gamma + \delta h_\Gamma, v + \delta k] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u} + \delta\boldsymbol{h} = [u + \delta h, u_\Gamma + \delta h_\Gamma]$. On this basis, we consider a sequence of triplets of functions $\{[\boldsymbol{\chi}^\delta, \gamma^\delta]\}_{\delta \in (-1,1) \setminus \{0\}} = \{[\chi^\delta, \chi_\Gamma^\delta, \gamma^\delta]\}_{\delta \in (-1,1) \setminus \{0\}} \subset \mathfrak{X} \times \mathscr{H}$ with $\{\boldsymbol{\chi}^\delta\}_{\delta \in (-1,1) \setminus \{0\}} = \{[\chi^\delta, \chi_\Gamma^\delta]\}_{\delta \in (-1,1) \setminus \{0\}}$, defined as:

$$[\boldsymbol{\chi}^\delta, \gamma^\delta] = [\chi^\delta, \chi_\Gamma^\delta, \gamma^\delta] := \left[\frac{\boldsymbol{\eta}^\delta - \boldsymbol{\eta}}{\delta}, \ \frac{\theta^\delta - \theta}{\delta}\right] = \left[\frac{\eta^\delta - \eta}{\delta}, \ \frac{\eta_\Gamma^\delta - \eta_\Gamma}{\delta}, \ \frac{\theta^\delta - \theta}{\delta}\right] \in \mathfrak{X} \times \mathscr{H}$$
$$\text{with } \boldsymbol{\chi}^\delta = [\chi^\delta, \chi_\Gamma^\delta] = \left[\frac{\eta^\delta - \eta}{\delta}, \ \frac{\eta_\Gamma^\delta - \eta_\Gamma}{\delta}\right] \text{ for } \delta \in (-1,1) \setminus \{0\}. \qquad (6.6.1)$$

This sequence acts a key-role in the computation of Gâteaux differential of the cost functional $\mathcal{J}_\varepsilon$, for $\varepsilon > 0$.

Remark 6.14. Note that for any $\delta \in (-1,1) \setminus \{0\}$, the triplet of functions $[\boldsymbol{\chi}^\delta, \gamma^\delta] = [\chi^\delta, \chi_\Gamma^\delta, \gamma^\delta] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\chi}^\delta = [\chi^\delta, \chi_\Gamma^\delta]$ fulfills the following variational forms:

$$\begin{aligned}
&(\partial_t \boldsymbol{\chi}^\delta(t), \boldsymbol{\varphi})_{\mathbb{X}} + (\partial_x \chi^\delta(t), \partial_x \varphi)_H \\
&\quad + \int_\Omega \left(\int_0^1 g'(\eta(t) + \varsigma \delta \chi^\delta(t)) \, d\varsigma \right) \chi^\delta(t) \varphi \, dx \\
&\quad + \int_\Omega \left(f_\varepsilon(\partial_x \theta(t)) \int_0^1 \alpha''(\eta(t) + \varsigma \delta \chi^\delta(t)) \, d\varsigma \right) \chi^\delta(t) \varphi \, dx \\
&\quad + \int_\Omega \left(\alpha'(\eta^\delta(t)) \int_0^1 f_\varepsilon'(\partial_x \theta(t) + \varsigma \delta \partial_x \gamma^\delta(t)) \, d\varsigma \right) \partial_x \gamma^\delta(t) \varphi \, dx \\
&= (Lh(t), \varphi)_H + (L_\Gamma h_\Gamma(t), \varphi_\Gamma)_{H_\Gamma}, \text{ for any } \boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] \in \mathbb{W}, \\
&\qquad \text{a.e. } t \in (0,T), \text{ subject to } \boldsymbol{\chi}^\delta(0) = [\chi^\delta(0), \chi_\Gamma^\delta(0)] = [0,0] \text{ in } \mathbb{X},
\end{aligned}$$

and

$$\begin{aligned}
&(\alpha_0(t) \partial_t \gamma^\delta(t), \psi)_H + \nu^2 (\partial_x \gamma^\delta(t), \partial_x \psi)_H \\
&\quad + \int_\Omega \left(\alpha(\eta^\delta(t)) \int_0^1 f_\varepsilon''(\partial_x \theta(t) + \varsigma \delta \partial_x \gamma^\delta(t)) \, d\varsigma \right) \partial_x \gamma^\delta(t) \partial_x \psi \, dx \\
&\quad + \int_\Omega \left(f_\varepsilon'(\partial_x \theta(t)) \int_0^1 \alpha'(\eta(t) + \varsigma \delta \chi^\delta(t)) \, d\varsigma \right) \chi^\delta(t) \partial_x \psi \, dx \\
&= (Mk(t), \psi)_H, \text{ for any } \psi \in V_0, \text{ a.e. } t \in (0,T), \text{ subject to } \gamma^\delta(0) = 0 \text{ in } H.
\end{aligned}$$

In fact, these variational forms are obtained by taking the difference between respective two variational forms for $[\boldsymbol{\eta}^\delta, \theta^\delta] = [\eta^\delta, \eta_\Gamma^\delta, \theta^\delta] \in \mathfrak{X} \times \mathscr{H}$ and $[\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathfrak{X} \times \mathscr{H}$, as in Main Theorem 6.1 (I-A), and by using the following linearization formulas:

$$\frac{1}{\delta}\big(g(\eta^\delta) - g(\eta)\big) = \left(\int_0^1 g'(\eta + \varsigma \delta \chi^\delta) \, d\varsigma \right) \chi^\delta \text{ in } \mathscr{H},$$

$$\begin{aligned}
&\frac{1}{\delta}\big(\alpha'(\eta^\delta) f_\varepsilon(\partial_x \theta^\delta) - \alpha'(\eta) f_\varepsilon(\partial_x \theta)\big) \\
&\qquad = \frac{1}{\delta}\big(\alpha'(\eta^\delta) - \alpha'(\eta)\big) f_\varepsilon(\partial_x \theta) + \frac{1}{\delta} \alpha'(\eta^\delta) \big(f_\varepsilon(\partial_x \theta^\delta) - f_\varepsilon(\partial_x \theta)\big)
\end{aligned}$$

$$= \left(f_\varepsilon(\partial_x \theta) \int_0^1 \alpha''(\eta + \varsigma\delta\chi^\delta)\, d\varsigma \right) \chi^\delta$$
$$+ \left(\alpha'(\eta^\delta) \int_0^1 f_\varepsilon'(\partial_x \theta + \varsigma\delta\partial_x\gamma^\delta)\, d\varsigma \right) \partial_x \gamma^\delta \text{ in } \mathscr{H},$$

and

$$\frac{1}{\delta}\big(\alpha(\eta^\delta) f_\varepsilon'(\partial_x\theta^\delta) - \alpha(\eta) f_\varepsilon'(\partial_x\theta)\big)$$
$$= \frac{1}{\delta}\alpha(\eta^\delta)\big(f_\varepsilon'(\partial_x\theta^\delta) - f_\varepsilon'(\partial_x\theta)\big) + \frac{1}{\delta}\big(\alpha(\eta^\delta) - \alpha(\eta)\big) f_\varepsilon'(\partial_x\theta)$$
$$= \left(\alpha(\eta^\delta) \int_0^1 f_\varepsilon''(\partial_x\theta + \varsigma\delta\partial_x\gamma^\delta)\, d\varsigma \right) \partial_x\gamma^\delta$$
$$+ \left(f_\varepsilon'(\partial_x\theta) \int_0^1 \alpha'(\eta + \varsigma\delta\chi^\delta)\, d\varsigma \right) \chi^\delta \text{ in } \mathscr{H}.$$

Incidentally, the above linearization formulas can be verified as consequences of the assumptions (A0)–(A3) and the mean-value theorem (cf. [54, Theorem 5 in p. 313]).

Now, we verify the following two Lemmas.

Lemma 6.1. Let us fix $\varepsilon > 0$, and assume (A0)–(A3). Then, for any $[\boldsymbol{u}, v] = [u, u_\Gamma, v] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u} = [u, u_\Gamma]$, the cost functional $\mathcal{J}_\varepsilon$ admits the Gâteaux derivative $\mathcal{J}_\varepsilon'(\boldsymbol{u}, v) = \mathcal{J}_\varepsilon'(u, u_\Gamma, v) \in \mathfrak{X} \times \mathscr{H}$ $(= [\mathfrak{X} \times \mathscr{H}]^*)$, such that:

$$\begin{aligned}
\big(\mathcal{J}_\varepsilon'(\boldsymbol{u}, v), [\boldsymbol{h}, k]\big)_{\mathfrak{X}\times\mathscr{H}} &= \big(\mathcal{J}_\varepsilon'(u, u_\Gamma, v), [h, h_\Gamma, k]\big)_{\mathfrak{X}\times\mathscr{H}} \\
&= \big([K(\eta - \eta_{\rm ad}), K_\Gamma(\eta_\Gamma - \eta_{\Gamma,\rm ad}), \Lambda(\theta - \theta_{\rm ad})], \bar{\mathcal{P}}_\varepsilon[Lh, L_\Gamma u_\Gamma, Mk]\big)_{\mathfrak{X}\times\mathscr{H}} \\
&\quad + \big([Lu, L_\Gamma u_\Gamma, Mv], [h, h_\Gamma, k]\big)_{\mathfrak{X}\times\mathscr{H}}, \\
&\text{for any } [\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{H} \text{ with } \boldsymbol{h} = [h, h_\Gamma].
\end{aligned} \tag{6.6.2}$$

In the context, $[\boldsymbol{\eta}, \theta] = [\eta, \eta_\Gamma, \theta] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta} = [\eta, \eta_\Gamma]$ is the solution to the state-system $(\mathrm{S})_\varepsilon$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \in \mathbb{W} \times V_0$ with $\boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}]$ and forcing triplet $[\boldsymbol{u}, v] = [u, u_\Gamma, v]$, and $\bar{\mathcal{P}}_\varepsilon \in \mathscr{L}(\mathfrak{X} \times \mathscr{H}; \mathfrak{Y})$ is the restriction $\mathcal{P}|_{\{[0,0,0]\}\times[\mathfrak{X}\times\mathscr{H}]}$ of the bounded linear operator $\mathcal{P} = \mathcal{P}(a, b, \mu, \omega, A) : [\mathbb{W} \times H] \times [\mathfrak{X} \times \mathscr{V}_0^*] \longrightarrow \mathfrak{Y}$, as in Remark 6.9, in the case when:

$$\begin{cases}
[a, b] = [\alpha_0, 0] \text{ in } W^{1,\infty}(Q) \times L^\infty(Q), \\
\mu = \bar{\mu}_\varepsilon := g'(\eta) + \alpha''(\eta) f_\varepsilon(\partial_x\theta) \text{ in } L^\infty(0, T; H), \\
[\omega, A] = [\bar{\omega}_\varepsilon, \bar{A}_\varepsilon] := \big[\alpha'(\eta) f_\varepsilon'(\partial_x\theta), \alpha(\eta) f_\varepsilon''(\partial_x\theta)\big] \text{ in } [L^\infty(Q)]^2.
\end{cases} \tag{6.6.3}$$

Proof. Let us fix any $[\boldsymbol{u}, v] = [u, u_\Gamma, v] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u} = [u, u_\Gamma]$, and take any $\delta \in (-1, 1) \setminus \{0\}$ and any $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{h} = [h, h_\Gamma]$. Then, it is easily seen that:

$$\frac{1}{\delta}\big(\mathcal{J}_\varepsilon(\boldsymbol{u} + \delta\boldsymbol{h}, v + \delta k) - \mathcal{J}_\varepsilon(\boldsymbol{u}, v)\big)$$
$$= \frac{1}{\delta}\big(\mathcal{J}_\varepsilon(u + \delta h, u_\Gamma + \delta h_\Gamma, v + \delta k) - \mathcal{J}_\varepsilon(u, u_\Gamma, v)\big)$$

$$= \left(\frac{K}{2}(\eta^\delta + \eta - 2\eta_{\rm ad}), \chi^\delta \right)_{\mathscr{H}} + \left(\frac{K_\Gamma}{2}(\eta_\Gamma^\delta + \eta_\Gamma - 2\eta_{\Gamma,\rm ad}), \chi_\Gamma^\delta \right)_{\mathscr{H}_\Gamma}$$
$$+ \left(\frac{\Lambda}{2}(\theta^\delta + \theta - 2\theta_{\rm ad}), \gamma^\delta \right)_{\mathscr{H}} + \left(\frac{L}{2}(2u + \delta h), h \right)_{\mathscr{H}}$$
$$+ \left(\frac{L_\Gamma}{2}(2u_\Gamma + \delta h_\Gamma), h_\Gamma \right)_{\mathscr{H}_\Gamma} + \left(\frac{M}{2}(2v + \delta k), k \right)_{\mathscr{H}}. \tag{6.6.4}$$

Here, let us set:

$$\begin{cases} \bar{\mu}_\varepsilon^\delta := \displaystyle\int_0^1 g'(\eta + \varsigma\delta\chi^\delta)\, d\varsigma + f_\varepsilon(\partial_x \theta) \int_0^1 \alpha''(\eta + \varsigma\delta\chi^\delta)\, d\varsigma \text{ in } L^\infty(0,T;H), \\ \bar{\omega}_\varepsilon^\delta := \alpha'(\eta^\delta) \displaystyle\int_0^1 f_\varepsilon'(\partial_x\theta + \varsigma\delta\partial_x\gamma^\delta)\, d\varsigma \text{ in } L^\infty(Q), \\ \bar{A}_\varepsilon^\delta := \alpha(\eta^\delta) \displaystyle\int_0^1 f_\varepsilon''(\partial_x\theta + \varsigma\delta\partial_x\gamma^\delta)\, d\varsigma \text{ in } L^\infty(Q), \end{cases} \tag{6.6.5a}$$

and

$$\bar{k}_\varepsilon^\delta := Mk + \partial_x \left[\chi^\delta f_\varepsilon'(\partial_x\theta) \int_0^1 \alpha'(\eta + \varsigma\delta\chi^\delta)\, d\varsigma \right.$$
$$\left. - \chi^\delta \alpha'(\eta^\delta) \int_0^1 f_\varepsilon'(\partial_x\theta + \varsigma\delta\partial_x\gamma^\delta)\, d\varsigma \right] \text{ in } \mathscr{V}_0^*, \tag{6.6.5b}$$
$$\text{for all } \delta \in (-1,1) \setminus \{0\}.$$

Then, in the light of Remark 6.14, one can say that:

$$[\boldsymbol{\chi}^\delta, \gamma^\delta] = [\chi^\delta, \chi_\Gamma^\delta, \gamma^\delta] = \bar{\mathcal{P}}_\varepsilon^\delta [Lh, L_\Gamma h_\Gamma, \bar{k}_\varepsilon^\delta] \text{ in } \mathfrak{Y}, \text{ for } \delta \in (-1,1) \setminus \{0\},$$

by using the restriction $\bar{\mathcal{P}}_\varepsilon^\delta := \mathcal{P}|_{\{[0,0,0]\} \times [\mathfrak{X} \times \mathscr{V}_0^*]} : \mathfrak{X} \times \mathscr{V}_0^* \longrightarrow \mathfrak{Y}$ of the bounded linear operator $\mathcal{P} = \mathcal{P}(a,b,\mu,\omega,A) : [\mathbb{W} \times H] \times [\mathfrak{X} \times \mathscr{V}_0^*] \longrightarrow \mathfrak{Y}$, as in Remark 6.9, in the case when:

$$\begin{cases} [a,b,\omega,A] = [\alpha_0, 0, \bar{\omega}_\varepsilon^\delta, \bar{A}_\varepsilon^\delta] \text{ in } W^{1,\infty}(Q) \times [L^\infty(Q)]^3, \\ \mu = \bar{\mu}_\varepsilon^\delta \text{ in } L^\infty(0,T;H), \text{ for } \delta \in (-1,1) \setminus \{0\}. \end{cases}$$

Besides, taking into account (6.1.6), (6.6.5), (A0)–(A3), we have:

$$\bar{C}_0^* := \frac{16}{\min\{1, \nu^2, \delta_*\}} \big(1 + |\alpha_0|_{W^{1,\infty}(Q)} + 2|g'|_{L^\infty(\mathbb{R})}^2$$
$$+ 2|\alpha''|_{L^\infty(\mathbb{R})}^2 |f_\varepsilon(\partial_x\theta)|_{L^\infty(0,T;H)}^2 + |\alpha'|_{L^\infty(\mathbb{R})}^2 \big) \tag{6.6.6a}$$
$$\geq \frac{16}{\min\{1, \nu^2, \delta_*\}} \sup_{0<|\delta|<1} \left\{ 1 + |\alpha_0|_{W^{1,\infty}(Q)} + |\bar{\mu}_\varepsilon^\delta|_{L^\infty(0,T;H)}^2 + |\bar{\omega}_\varepsilon^\delta|_{L^\infty(Q)}^2 \right\},$$

and

$$\left| \left\langle [Lh(t)\,, L_\Gamma h_\Gamma(t), \bar{k}_\varepsilon^\delta(t)], [\varphi, \varphi_\Gamma, \psi] \right\rangle_{\mathbb{X} \times V_0} \right|$$
$$\leq \left| \big(Lh(t), \varphi\big)_H \right| + \left| \big(L_\Gamma h_\Gamma(t), \varphi_\Gamma\big)_{H_\Gamma} \right| + |\langle \bar{k}_\varepsilon^\delta(t), \psi \rangle_{V_0}|$$
$$\leq L|h(t)|_H |\varphi|_H + L_\Gamma |h_\Gamma(t)|_{H_\Gamma} |\varphi_\Gamma|_{H_\Gamma}$$

$$
\begin{aligned}
&\quad + M|k(t)|_H|\psi|_H + 2|\alpha'|_{L^\infty(\mathbb{R})}|\chi^\delta(t)|_H|\partial_x\psi|_H \\
&\leq L|h(t)|_H|\varphi|_V + L_\Gamma|h_\Gamma(t)|_{H_\Gamma}|\varphi_\Gamma|_{H_\Gamma} \\
&\quad + \bigl(\sqrt{2}M|k(t)|_H + 2|\alpha'|_{L^\infty(\mathbb{R})}|\chi^\delta(t)|_H\bigr)|\psi|_{V_0}, \qquad (6.6.6\text{b}) \\
&\text{for a.e. } t \in (0,T), \text{ any } [\boldsymbol{\varphi}, \psi] = [\varphi, \varphi_\Gamma, \psi] \in \mathbb{X} \times V_0 \\
&\qquad \text{with } \boldsymbol{\varphi} = [\varphi, \varphi_\Gamma], \text{ and any } \delta \in (-1,1) \setminus \{0\},
\end{aligned}
$$

so that

$$
\begin{aligned}
&\bigl|[Lh(t), L_\Gamma h_\Gamma(t), \bar{k}_\varepsilon^\delta(t)]\bigr|_{\mathbb{X}\times V_0^*}^2 \leq \bar{B}_0^*\bigl(\bigl|[\boldsymbol{h}(t), k(t)]\bigr|_{\mathbb{X}\times H}^2 + |\chi^\delta(t)|_H^2\bigr), \\
&\qquad \text{for a.e. } t \in (0,T), \text{ and any } \delta \in (-1,1) \setminus \{0\}, \qquad (6.6.6\text{c})
\end{aligned}
$$

with a positive constant $\bar{B}_0^* := 4\bigl(L^2 + L_\Gamma^2 + M^2 + |\alpha'|_{L^\infty(\mathbb{R})}^2\bigr)$.

Now, having in mind (6.6.6), let us apply Theorem 6.2 (I) to the case when:

$$
\begin{cases}
[a, b, \mu, \omega, A] = [\alpha_0, 0, \overline{\mu}_\varepsilon^\delta, \bar{\omega}_\varepsilon^\delta, \bar{A}_\varepsilon^\delta], \\
[\boldsymbol{h}, k] = [h, h_\Gamma, k] = [Lh, L_\Gamma h_\Gamma, \bar{k}_\varepsilon^\delta], \\
[\boldsymbol{p}, z] = [p, p_\Gamma, z] = [\boldsymbol{\chi}^\delta, \gamma^\delta] = [\chi^\delta, \chi_\Gamma^\delta, \gamma^\delta] = \bar{\mathcal{P}}_\varepsilon^\delta[Lh, L_\Gamma h_\Gamma, \bar{k}_\varepsilon^\delta],
\end{cases}
\quad \text{for } \delta \in (-1,1) \setminus \{0\}.
$$

Then, we estimate that:

$$
\begin{aligned}
&\frac{d}{dt}\bigl(|\boldsymbol{\chi}^\delta(t)|_{\mathbb{X}}^2 + |\sqrt{\alpha_0(t)}\gamma^\delta(t)|_H^2\bigr) + \bigl(|\boldsymbol{\chi}^\delta(t)|_{\mathbb{W}}^2 + \nu^2|\gamma^\delta(t)|_{V_0}^2\bigr) \\
&\quad \leq \bar{C}_0^*\bigl(|\boldsymbol{\chi}^\delta(t)|_{\mathbb{X}}^2 + |\sqrt{\alpha_0(t)}\gamma^\delta(t)|_H^2\bigr) + \bar{C}_0^*\bigl(|Lh(t)|_{V^*}^2 + |L_\Gamma h_\Gamma(t)|_{H_\Gamma^*}^2 + |\bar{k}_\varepsilon^\delta(t)|_{V_0^*}^2\bigr) \\
&\quad \leq \bar{C}_0^*(1 + \bar{B}_0^*)\bigl(|\boldsymbol{\chi}^\delta(t)|_{\mathbb{X}}^2 + |\sqrt{\alpha_0(t)}\gamma^\delta(t)|_H^2\bigr) + \bar{C}_0^*\bar{B}_0^*\bigl(|\boldsymbol{h}(t)|_{\mathbb{X}}^2 + |k(t)|_H^2\bigr), \\
&\qquad \text{for a.e. } t \in (0,T),
\end{aligned}
$$

and subsequently, by using (A1) and Gronwall's lemma, we observe that:

($\star$1) the sequence $\{[\boldsymbol{\chi}^\delta, \gamma^\delta]\}_{\delta\in(-1,1)\setminus\{0\}} = \{[\chi^\delta, \chi_\Gamma^\delta, \gamma^\delta]\}_{\delta\in(-1,1)\setminus\{0\}}$ is bounded in $[C([0,T];\mathbb{X}) \times C([0,T];H)] \cap [\mathfrak{W} \times \mathscr{V}_0]$.

Meanwhile, as consequences of (6.6.1), (6.6.3)–(6.6.6), ($\star$1), (A0)–(A3), Main Theorem 6.1 (I-B), Remark 6.10, and Lebesgue's dominated convergence theorem, one can find a sequence $\{\delta_n\}_{n=1}^\infty \subset \mathbb{R}$, such that:

$$
0 < |\delta_n| < 1, \text{ and } \delta_n \to 0, \text{ as } n \to \infty, \qquad (6.6.7\text{a})
$$

$$
\begin{cases}
[\delta_n \boldsymbol{\chi}^{\delta_n}, \delta_n \gamma^{\delta_n}] = [\boldsymbol{\eta}^{\delta_n} - \eta, \theta^{\delta_n} - \theta] \to [0,0,0] \\
\qquad \text{in } [C(\overline{Q}) \times C(\overline{\Sigma})] \times C(\overline{Q}), \text{ and in } \mathfrak{W} \times \mathscr{V}_0, \\
[\delta_n \partial_x \chi^{\delta_n}, \delta_n \partial_x \gamma^{\delta_n}] = [\partial_x(\eta^{\delta_n} - \eta), \partial_x(\theta^{\delta_n} - \theta)] \to [0,0] \\
\qquad \text{in } [\mathscr{H}]^2, \text{ and in the pointwise sense a.e. in } Q,
\end{cases}
\quad \text{as } n \to \infty, \qquad (6.6.7\text{b})
$$

$$[\bar{\omega}_\varepsilon^{\delta_n}, \bar{A}_\varepsilon^{\delta_n}] \to [\bar{\omega}_\varepsilon, \bar{A}_\varepsilon] \text{ weakly-}* \text{ in } [L^\infty(Q)]^2,$$
$$\text{and in the pointwise sense a.e. in } Q, \quad \text{as } n \to \infty, \tag{6.6.7c}$$

$$\begin{cases} \bar{\mu}_\varepsilon^{\delta_n} \to \bar{\mu}_\varepsilon \text{ weakly-}* \text{ in } L^\infty(0,T;H), \\ \bar{\mu}_\varepsilon^{\delta_n}(t) \to \bar{\mu}_\varepsilon(t) \text{ in } H, \text{ for a.e. } t \in (0,T), \end{cases} \quad \text{as } n \to \infty, \tag{6.6.7d}$$

and

$$\begin{aligned} \langle \bar{k}_\varepsilon^{\delta_n} - Mk, \psi \rangle_{\mathscr{V}_0} = & - \left(\chi^{\delta_n},\ f_\varepsilon'(\partial_x \theta) \left(\int_0^1 \alpha'(\eta + \varsigma \delta_n \chi^{\delta_n})\, d\varsigma \right) \partial_x \psi \right)_{\mathscr{H}} \\ & + \left(\chi^{\delta_n},\ \alpha'(\eta^{\delta_n}) \left(\int_0^1 f_\varepsilon'(\partial_x \theta + \varsigma \delta_n \partial_x \gamma^{\delta_n})\, d\varsigma \right) \partial_x \psi \right)_{\mathscr{H}} \to 0, \text{ as } n \to \infty. \end{aligned} \tag{6.6.7e}$$

On account of (6.6.1), (6.6.3)–(6.6.7), and Remark 6.9, we can apply Theorem 6.3, and can see that:

$$\begin{aligned} [\boldsymbol{\chi}^{\delta_n}, \gamma^{\delta_n}] &= [\chi^{\delta_n}, \chi_\Gamma^{\delta_n}, \gamma^{\delta_n}] = \bar{\mathcal{P}}_\varepsilon^{\delta_n}[Lh, L_\Gamma h_\Gamma, \bar{k}_\varepsilon^{\delta_n}] \\ &\to [\boldsymbol{\chi}, \gamma] = [\chi, \chi_\Gamma, \gamma] := \bar{\mathcal{P}}_\varepsilon[Lh, L_\Gamma h_\Gamma, Mk] \\ &\text{in } [C(\overline{Q}) \times C(\overline{\Sigma})] \times \mathscr{H}, \text{ as } n \to \infty. \end{aligned} \tag{6.6.8}$$

Since the uniqueness of the solution $[\boldsymbol{\chi}, \gamma] = [\chi, \chi_\Gamma, \gamma] = \bar{\mathcal{P}}_\varepsilon[Lh, L_\Gamma h_\Gamma, Mk]$ is guaranteed in Theorem 6.1, the observations (6.6.4), (6.6.7), and (6.6.8) enable us to compute the directional derivative $D_{[\boldsymbol{h},k]}\mathcal{J}_\varepsilon(\boldsymbol{u}, v) = D_{[h,h_\Gamma,k]}\mathcal{J}_\varepsilon(u, u_\Gamma, v) \in \mathbb{R}$, as follows:

$$\begin{aligned} D_{[\boldsymbol{h},k]}\mathcal{J}_\varepsilon(\boldsymbol{u}, v) &= D_{[h,h_\Gamma,k]}\mathcal{J}_\varepsilon(u, u_\Gamma, v) := \lim_{\delta \to 0} \frac{1}{\delta} \bigl(\mathcal{J}_\varepsilon(\boldsymbol{u} + \delta \boldsymbol{h}, v + \delta k) - \mathcal{J}_\varepsilon(\boldsymbol{u}, v) \bigr) \\ &= \lim_{\delta \to 0} \frac{1}{\delta} \bigl(\mathcal{J}_\varepsilon(u + \delta h, u_\Gamma + \delta h_\Gamma, v + \delta k) - \mathcal{J}_\varepsilon(u, u_\Gamma, v) \bigr) \\ &= \bigl([K(\eta - \eta_{\mathrm{ad}}), K_\Gamma(\eta_\Gamma - \eta_{\Gamma,\mathrm{ad}}), \Lambda(\theta - \theta_{\mathrm{ad}})], \bar{\mathcal{P}}_\varepsilon[Lh, L_\Gamma h_\Gamma, Mk] \bigr)_{\mathfrak{X} \times \mathscr{H}} \\ &\quad + \bigl([Lu, L_\Gamma u_\Gamma, Mv], [h, h_\Gamma, k] \bigr)_{\mathfrak{X} \times \mathscr{H}}, \\ &\text{for any } [\boldsymbol{u}, v] = [u, u_\Gamma, v] \in \mathfrak{X} \times \mathscr{H} \text{ with } \boldsymbol{u} = [u, u_\Gamma], \\ &\text{and any direction } [\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{H} \text{ with } \boldsymbol{h} = [h, h_\Gamma]. \end{aligned}$$

Moreover, with Remark 6.9 and Riesz's theorem in mind, we deduce the existence of the Gâteaux derivative $\mathcal{J}_\varepsilon'(\boldsymbol{u}, v) = \mathcal{J}_\varepsilon'(u, u_\Gamma, v) \in [\mathfrak{X} \times \mathscr{H}]^*$ $(= \mathfrak{X} \times \mathscr{H})$ at $[\boldsymbol{u}, v] = [u, u_\Gamma, v] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u} = [u, u_\Gamma]$, i.e.:

$$\begin{aligned} &\bigl(\mathcal{J}_\varepsilon'(u, u_\Gamma, v), [h, h_\Gamma, k] \bigr)_{\mathfrak{X} \times \mathscr{H}} = D_{[h,h_\Gamma,k]}\mathcal{J}_\varepsilon(u, u_\Gamma, v), \\ &\text{for every } [\boldsymbol{u}, v] = [u, u_\Gamma, v] \text{ and } [\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{H}, \\ &\text{with } \boldsymbol{u} = [u, u_\Gamma] \text{ and } \boldsymbol{h} = [h, h_\Gamma], \text{ respectively.} \end{aligned}$$

Thus, we conclude this lemma with the required property (6.6.2). □

Lemma 6.2. Under the assumptions (A0)–(A3), let $[\boldsymbol{u}_\varepsilon^*, v_\varepsilon^*] = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*, v_\varepsilon^*] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u}_\varepsilon^* = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*]$ be an optimal control of the problem $(\mathrm{OP})_\varepsilon$, and let $[\boldsymbol{\eta}_\varepsilon^*, \theta_\varepsilon^*] = [\eta_\varepsilon^*, \eta_{\Gamma,\varepsilon}^*, \theta_\varepsilon^*] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta}_\varepsilon^* = [\eta_\varepsilon^*, \eta_{\Gamma,\varepsilon}^*]$ be the solution to the system $(\mathrm{S})_\varepsilon$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \in \mathbb{W} \times V_0$ with $\boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}]$ and forcing triplet $[\boldsymbol{u}_\varepsilon^*, v_\varepsilon^*] = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*, v_\varepsilon^*]$. Also, let $\mathcal{P}_\varepsilon^* : \mathfrak{X} \times \mathscr{H} \longrightarrow \mathfrak{Y}$ be the bounded linear operator, defined in Remark 6.11, with use of the solution $[\boldsymbol{\eta}_\varepsilon^*, \theta_\varepsilon^*] = [\eta_\varepsilon^*, \eta_{\Gamma,\varepsilon}^*, \theta_\varepsilon^*]$. Let $\mathcal{P}_\varepsilon \in \mathscr{L}(\mathfrak{X} \times \mathscr{H}; \mathfrak{Y})$ be the restriction $\mathcal{P}|_{\{[0,0,0]\} \times [\mathfrak{X} \times \mathscr{H}]}$ of the bounded linear operator $\mathcal{P} = \mathcal{P}(a, b, \mu, \omega, A) : [\mathbb{W} \times H] \times [\mathfrak{X} \times \mathscr{V}_0^*] \longrightarrow \mathfrak{Y}$, as in Remark 6.9, in the case when:

$$\begin{cases} [a, b] = [\alpha_0, 0] \text{ in } W^{1,\infty}(Q) \times L^\infty(Q), \\ \mu = g'(\eta_\varepsilon^*) + \alpha''(\eta_\varepsilon^*) f_\varepsilon(\partial_x \theta_\varepsilon^*) \text{ in } L^\infty(0, T; H), \\ [\omega, A] = \bigl[\alpha'(\eta_\varepsilon^*) f_\varepsilon'(\partial_x \theta_\varepsilon^*), \alpha(\eta_\varepsilon^*) f_\varepsilon''(\partial_x \theta_\varepsilon^*)\bigr] \text{ in } [L^\infty(Q)]^2. \end{cases} \tag{6.6.9}$$

Then, the operators $\mathcal{P}_\varepsilon^*$ and $\mathcal{P}_\varepsilon$ have a conjugate relationship, in the following sense:

$$\begin{aligned} &\bigl(\mathcal{P}_\varepsilon^* [\boldsymbol{u}, v][\boldsymbol{h}, k]\bigr)_{\mathfrak{X} \times \mathscr{H}} = \bigl(\mathcal{P}_\varepsilon^*[u, u_\Gamma, v], [h, h_\Gamma, k]\bigr)_{\mathfrak{X} \times \mathscr{H}} \\ &\quad = \bigl([\boldsymbol{u}, v], \mathcal{P}_\varepsilon[\boldsymbol{h}, k]\bigr)_{\mathfrak{X} \times \mathscr{H}} = \bigl([u, u_\Gamma, v], \mathcal{P}_\varepsilon[h, h_\Gamma, k]\bigr)_{\mathfrak{X} \times \mathscr{H}}, \\ &\quad \text{for all } [\boldsymbol{h}, k] = [h, h_\Gamma, k] \text{ and } [\boldsymbol{u}, v] = [u, u_\Gamma, v] \in \mathfrak{X} \times \mathscr{H} \\ &\quad \text{with } \boldsymbol{h} = [h, h_\Gamma] \text{ and } \boldsymbol{u} = [u, u_\Gamma], \text{ respectively.} \end{aligned}$$

Proof. Let us fix arbitrary triplets of functions $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{h} = [h, h_\Gamma]$ and $[\boldsymbol{u}, v] = [u, u_\Gamma, v] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{u} = [u, u_\Gamma]$, and let us put:

$$\begin{aligned} &[\boldsymbol{\chi}_\varepsilon, \gamma_\varepsilon] = [\chi_\varepsilon, \chi_{\Gamma,\varepsilon}, \gamma_\varepsilon] := \mathcal{P}_\varepsilon[\boldsymbol{h}, k] = \mathcal{P}_\varepsilon[h, h_\Gamma, k] \text{ and} \\ &[\boldsymbol{p}_\varepsilon, z_\varepsilon] = [p_\varepsilon, p_{\Gamma,\varepsilon}, z_\varepsilon] := \mathcal{P}_\varepsilon^*[\boldsymbol{u}, v] = \mathcal{P}_\varepsilon^*[u, u_\Gamma, v], \text{ in } \mathfrak{X} \times \mathscr{H}. \end{aligned}$$

Then, invoking Theorem 6.1, and the settings as in (6.3.13) and (6.6.9), we compute that:

$$\begin{aligned} &\bigl(\mathcal{P}_\varepsilon^*[\boldsymbol{u}, v], [\boldsymbol{h}, k]\bigr)_{\mathfrak{X} \times \mathscr{H}} = \int_0^T \bigl(\boldsymbol{p}_\varepsilon(t), \boldsymbol{h}(t)\bigr)_{\mathbb{X}} \, dt + \int_0^T \bigl(z_\varepsilon(t), k(t)\bigr)_H \, dt \\ &= \int_0^T \bigl(\boldsymbol{h}(t), \boldsymbol{p}_\varepsilon(t)\bigr)_{\mathbb{X}} \, dt + \int_0^T \bigl(k(t), z_\varepsilon(t)\bigr)_H \, dt \\ &= \int_0^T \Bigl[\bigl(\partial_t \boldsymbol{\chi}_\varepsilon(t), \boldsymbol{p}_\varepsilon(t)\bigr)_{\mathbb{X}} + \bigl(\partial_x \chi_\varepsilon(t), \partial_x p_\varepsilon(t)\bigr)_H \\ &\qquad + \bigl(\alpha''(\eta_\varepsilon^*(t)) f_\varepsilon(\partial_x \theta_\varepsilon^*(t)) \chi_\varepsilon(t), p_\varepsilon(t)\bigr)_H \\ &\qquad + \bigl(g'(\eta_\varepsilon^*(t)) \chi_\varepsilon(t), p_\varepsilon(t)\bigr)_H + \bigl(\alpha'(\eta_\varepsilon^*(t)) f_\varepsilon'(\partial_x \theta_\varepsilon^*(t)) \partial_x \gamma_\varepsilon(t), p_\varepsilon(t)\bigr)_H \Bigr] \, dt \\ &\quad + \int_0^T \Bigl[\bigl\langle \alpha_0(t) \partial_t \gamma_\varepsilon(t), z_\varepsilon(t) \bigr\rangle_{V_0} + \bigl(\alpha'(\eta_\varepsilon^*(t)) f_\varepsilon'(\partial_x \theta_\varepsilon^*(t)) \chi_\varepsilon(t), \partial_x z_\varepsilon(t)\bigr)_H \\ &\qquad + \bigl(\alpha(\eta_\varepsilon^*(t)) f_\varepsilon''(\partial_x \theta_\varepsilon^*(t)) \partial_x \gamma_\varepsilon(t), \partial_x z_\varepsilon(t)\bigr)_H + \nu^2 \bigl(\partial_x \gamma_\varepsilon(t), \partial_x z_\varepsilon(t)\bigr)_H \Bigr] \, dt \end{aligned}$$

$$\begin{aligned} &= \bigl(\boldsymbol{p}_\varepsilon(T), \boldsymbol{\chi}_\varepsilon(T)\bigr)_{\mathbb{X}} - \bigl(\boldsymbol{p}_\varepsilon(0), \boldsymbol{\chi}_\varepsilon(0)\bigr)_{\mathbb{X}} + \int_0^T \Bigl[\bigl(-\partial_t \boldsymbol{p}_\varepsilon(t), \boldsymbol{\chi}_\varepsilon(t)\bigr)_{\mathbb{X}} \\ &\qquad + \bigl(\partial_x p_\varepsilon(t), \partial_x \chi_\varepsilon(t)\bigr)_H + \bigl(\alpha''(\eta_\varepsilon^*(t)) f_\varepsilon(\partial_x \theta_\varepsilon^*(t)) p_\varepsilon(t), \chi_\varepsilon(t)\bigr)_H \\ &\qquad + \bigl(g'(\eta_\varepsilon^*(t)) p_\varepsilon(t), \chi_\varepsilon(t)\bigr)_H + \bigl(\alpha'(\eta_\varepsilon^*(t)) f_\varepsilon'(\partial_x \theta_\varepsilon^*(t)) \partial_x z_\varepsilon(t), \chi_\varepsilon(t)\bigr)_H \Bigr] \, dt \end{aligned}$$

$$
\begin{aligned}
&+ \big(\alpha_0(T)z_\varepsilon(T), \gamma_\varepsilon(T)\big)_H - \big(\alpha_0(0)z_\varepsilon(0), \gamma_\varepsilon(0)\big)_H \\
&\qquad + \int_0^T \Big[\big\langle -\partial_t\big(\alpha_0 z_\varepsilon\big)(t), \gamma_\varepsilon(t)\big\rangle_{V_0} + \big(\alpha'(\eta_\varepsilon^*(t))f_\varepsilon'(\partial_x\theta_\varepsilon^*(t))p_\varepsilon(t), \partial_x\gamma_\varepsilon(t)\big)_H \\
&\qquad + \big(\alpha(\eta_\varepsilon^*(t))f_\varepsilon''(\partial_x\theta_\varepsilon^*(t))\partial_x z_\varepsilon(t), \partial_x\gamma_\varepsilon(t)\big)_H + \nu^2\big(\partial_x z_\varepsilon(t), \partial_x\gamma_\varepsilon(t)\big)_H \Big]\, dt \\
=&(\boldsymbol{u}, \boldsymbol{\chi}_\varepsilon)_{\mathfrak{X}} + (v, \gamma_\varepsilon)_{\mathscr{H}} = \big([\boldsymbol{u}, v], \mathcal{P}_\varepsilon[\boldsymbol{h}, k]\big)_{\mathfrak{X}\times\mathscr{H}}.
\end{aligned}
$$

□

Remark 6.15. Note that the operator $\mathcal{P}_\varepsilon \in \mathscr{L}(\mathfrak{X}\times\mathscr{H};\mathfrak{Y})$, as in Lemma 6.2, corresponds to the operator $\bar{\mathcal{P}}_\varepsilon \in \mathscr{L}(\mathfrak{X}\times\mathscr{H};\mathfrak{Y})$, as in the previous Lemma 6.1, under the special setting (6.6.9).

Now, we are ready to prove the Main Theorem 6.3.

Proof of (III-A) of Main Theorem 6.3. Let $[\boldsymbol{u}_\varepsilon^*, v_\varepsilon^*] = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*, v_\varepsilon^*] \in \mathfrak{X}\times\mathscr{H}$ with $\boldsymbol{u}_\varepsilon^* = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*]$ be the optimal control of $(\mathrm{OP})_\varepsilon$, let $[\boldsymbol{\eta}_\varepsilon^*, \theta_\varepsilon^*] = [\eta_\varepsilon^*, \eta_{\Gamma,\varepsilon}^*, \theta_\varepsilon^*] \in \mathfrak{X}\times\mathscr{H}$ with $\boldsymbol{\eta}_\varepsilon^* = [\eta_\varepsilon^*, \eta_{\Gamma,\varepsilon}^*]$ be the solution to the system $(\mathrm{S})_\varepsilon$ for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \in \mathbb{W}\times V_0$ with $\boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}]$, and forcing triplet $[\boldsymbol{u}_\varepsilon^*, v_\varepsilon^*] = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*, v_\varepsilon^*]$, and let $\mathcal{P}_\varepsilon \in \mathscr{L}(\mathfrak{X}\times\mathscr{H};\mathfrak{Y})$ and $\mathcal{P}_\varepsilon^* \in \mathscr{L}(\mathfrak{X}\times\mathscr{H};\mathfrak{Y})$ be the two operators as in Lemma 6.2. Then, on the basis of the previous Lemmas 6.1 and 6.2, Main Theorem 6.3 (III-A) will be demonstrated as follows:

$$
\begin{aligned}
0 =\ & (\mathcal{J}_\varepsilon'(\boldsymbol{u}_\varepsilon^*, v_\varepsilon^*), [\boldsymbol{h}, k])_{\mathfrak{X}\times\mathscr{H}} = \big(\mathcal{J}_\varepsilon'(u_\varepsilon^*, u_{\Gamma,\varepsilon}^*, v_\varepsilon^*), [h, h_\Gamma, k]\big)_{\mathfrak{X}\times\mathscr{H}} \\
=\ & \lim_{\delta\to 0}\frac{1}{\delta}\big(\mathcal{J}_\varepsilon(\boldsymbol{u}_\varepsilon^* + \delta\boldsymbol{h}, v_\varepsilon^* + \delta k) - \mathcal{J}_\varepsilon(\boldsymbol{u}_\varepsilon^*, v_\varepsilon^*)\big) \\
=\ & \lim_{\delta\to 0}\frac{1}{\delta}\big(\mathcal{J}_\varepsilon(u_\varepsilon^* + \delta h, u_{\Gamma,\varepsilon}^* + \delta h_\Gamma, v_\varepsilon^* + \delta k) - \mathcal{J}_\varepsilon(u_\varepsilon^*, u_{\Gamma,\varepsilon}, v_\varepsilon^*)\big) \\
=\ & \big([K(\eta_\varepsilon^* - \eta_{\mathrm{ad}}), K_\Gamma(\eta_{\Gamma,\varepsilon}^* - \eta_{\Gamma,\mathrm{ad}}), \Lambda(\theta_\varepsilon^* - \theta_{\mathrm{ad}})], \mathcal{P}_\varepsilon[Lh, L_\Gamma h_\Gamma, Mk]\big)_{\mathfrak{X}\times\mathscr{H}} \\
& + \big([Lu_\varepsilon^*, L_\Gamma u_{\Gamma,\varepsilon}^*, Mv_\varepsilon^*], [h, h_\Gamma, k]\big)_{\mathfrak{X}\times\mathscr{H}} \\
=\ & \big(\mathcal{P}_\varepsilon^*[K(\eta_\varepsilon^* - \eta_{\mathrm{ad}}), K_\Gamma(\eta_{\Gamma,\varepsilon}^* - \eta_{\Gamma,\mathrm{ad}}), \Lambda(\theta_\varepsilon^* - \theta_{\mathrm{ad}})], [Lh, L_\Gamma h_\Gamma, Mk]\big)_{\mathfrak{X}\times\mathscr{H}} \\
& + \big([Lu_\varepsilon^*, L_\Gamma u_{\Gamma,\varepsilon}^*, Mv_\varepsilon^*], [h, h_\Gamma, k]\big)_{\mathfrak{X}\times\mathscr{H}} \\
=\ & \big([Lp_\varepsilon^*, L_\Gamma p_{\Gamma,\varepsilon}^*, Mz_\varepsilon^*], [h, h_\Gamma, k]\big)_{\mathfrak{X}\times\mathscr{H}} + \big([Lu_\varepsilon^*, L_\Gamma u_{\Gamma,\varepsilon}^*, Mv_\varepsilon^*], [h, h_\Gamma, k]\big)_{\mathfrak{X}\times\mathscr{H}} \\
=\ & \big([L(p_\varepsilon^* + u_\varepsilon^*), L_\Gamma(p_{\Gamma,\varepsilon}^* + u_{\Gamma,\varepsilon}^*), M(z_\varepsilon^* + v_\varepsilon^*)], [h, h_\Gamma, k]\big)_{\mathfrak{X}\times\mathscr{H}}, \\
& \text{for any } [\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X}\times\mathscr{H} \text{ with } \boldsymbol{h} = [h, h_\Gamma].
\end{aligned}
$$

□

Proof of (III-B) of Main Theorem 6.3. Let $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0] \in \mathbb{W}\times V_0$ with $\boldsymbol{\eta}_0 = [\eta_0, \eta_{\Gamma,0}]$ be the fixed initial triplet. For any $\varepsilon > 0$, let $[\boldsymbol{u}_\varepsilon^*, v_\varepsilon^*] = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*, v_\varepsilon^*] \in \mathfrak{X}\times\mathscr{H}$ with $\boldsymbol{u}_\varepsilon^* = [u_\varepsilon^*, u_{\Gamma,\varepsilon}^*]$, $[\boldsymbol{\eta}_\varepsilon^*, \theta_\varepsilon^*] = [\eta_\varepsilon^*, \eta_{\Gamma,\varepsilon}^*, \theta_\varepsilon^*] \in \mathfrak{X}\times\mathscr{H}$ with $\boldsymbol{\eta}_\varepsilon^* = [\eta_\varepsilon^*, \eta_{\Gamma,\varepsilon}^*]$, and $[\boldsymbol{p}_\varepsilon^*, z_\varepsilon^*] = [p_\varepsilon^*, p_{\Gamma,\varepsilon}^*, z_\varepsilon^*] \in \mathfrak{Y}$ with $\boldsymbol{p}_\varepsilon^* = [p_\varepsilon^*, p_{\Gamma,\varepsilon}^*]$ be as in Main Theorem 6.3 (III-A). Then, by Main Theorem 6.2 (II-B), we find an optimal control $[\boldsymbol{u}^\circ, v^\circ] = [u^\circ, u_\Gamma^\circ, v^\circ] \in \mathfrak{X}\times\mathscr{H}$

with $\boldsymbol{u}^\circ = [u^\circ, u_\Gamma^\circ]$ of $(\mathrm{OP})_0$, and find a zero-convergent sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$, such that:

$$
\begin{aligned}
[Lu_n^*, L_\Gamma u_{\Gamma,n}^*, Mv_n^*] := [Lu_{\varepsilon_n}^*, L_\Gamma u_{\Gamma,\varepsilon_n}^*, Mv_{\varepsilon_n}^*] \to [Lu^\circ, L_\Gamma u_\Gamma^\circ, Mv^\circ] \\
\text{weakly in } \mathfrak{X} \times \mathscr{H}, \text{ as } n \to \infty.
\end{aligned}
\tag{6.6.10a}
$$

Let $[\boldsymbol{\eta}^\circ, \theta^\circ] \in \mathfrak{X} \times \mathscr{H}$ with $\boldsymbol{\eta}^\circ = [\eta^\circ, \eta_\Gamma^\circ]$ be the solution to $(\mathrm{S})_0$, for the initial triplet $[\boldsymbol{\eta}_0, \theta_0] = [\eta_0, \eta_{\Gamma,0}, \theta_0]$ and forcing triplet $[\boldsymbol{u}^\circ, v^\circ] = [u^\circ, u_\Gamma^\circ, v^\circ]$. Then, having in mind (6.6.10a), Main Theorem 6.1 (I-B), and Remark 6.10, we can find a subsequence of $\{\varepsilon_n\}_{n=1}^\infty$ (not relabeled) and a function $\nu^\circ \in L^\infty(Q)$, such that:

$$
\begin{aligned}
[\boldsymbol{\eta}_n^*, \theta_n^*] &:= [\boldsymbol{\eta}_{\varepsilon_n}^*, \theta_{\varepsilon_n}^*] \to [\boldsymbol{\eta}^\circ, \theta^\circ] \text{ in } [C(\overline{Q}) \times C(\overline{\Sigma})] \times C(\overline{Q}), \text{ in } \mathfrak{W} \times \mathscr{V}_0, \\
&\text{and weakly-}* \text{ in } L^\infty(0,T;\mathbb{V}) \times L^\infty(0,T;V_0),
\end{aligned}
\tag{6.6.10b}
$$

$$
\begin{aligned}
[\partial_x \eta_n, \partial_x \theta_n] \to [\partial_x \eta^\circ, \partial_x \theta^\circ] \text{ in } [\mathscr{H}]^2, \\
\text{and in the pointwise sense a.e. in } Q,
\end{aligned}
\tag{6.6.10c}
$$

$$
\begin{cases}
\mu_n^* := g'(\eta_n^*) + \alpha''(\eta_n^*) f_{\varepsilon_n}(\partial_x \theta_n^*) \to \mu^\circ := g'(\eta^\circ) + \alpha''(\eta^\circ)|\partial_x \theta^\circ| \\
\quad \text{weakly-}* \text{ in } L^\infty(0,T;H), \text{ and} \\
\quad \text{in the pointwise sense a.e. in } Q, \\
\mu_n^*(t) \to \mu^\circ(t) \text{ in } H, \text{ in the pointwise sense for a.e. } t \in (0,T),
\end{cases}
\tag{6.6.10d}
$$

$$
\begin{cases}
f_{\varepsilon_n}'(\partial_x \theta_n^*) \to \nu^\circ \text{ weakly-}* \text{ in } L^\infty(Q), \\
|\nu^\circ| \leq 1 \text{ a.e. in } Q,
\end{cases}
\tag{6.6.10e}
$$

and

$$
\omega_n^* := \alpha'(\eta_n^*) f_{\varepsilon_n}'(\partial_x \theta_n^*) \to \alpha'(\eta^\circ)\nu^\circ \text{ weakly-}* \text{ in } L^\infty(Q), \text{ as } n \to \infty.
\tag{6.6.10f}
$$

Besides, from (6.6.10c), (6.6.10e), Remark 6.5 (Fact 1) and (Fact 2), and [18, Proposition 2.16], one can see that:

$$
\nu^\circ \in \partial f_0(\partial_x \theta^\circ) = \mathrm{Sgn}^1(\partial_x \theta^\circ) \text{ a.e. in } Q.
\tag{6.6.11}
$$

Next, let us put:

$$
\begin{cases}
[\boldsymbol{p}_n^*, z_n^*] = [p_n^*, p_{\Gamma,n}^*, z_n^*] := [\boldsymbol{p}_{\varepsilon_n}^*, z_{\varepsilon_n}^*] = [p_{\varepsilon_n}^*, p_{\Gamma,\varepsilon_n}^*, z_{\varepsilon_n}^*] \text{ in } \mathfrak{X} \times \mathscr{H}, \\
A_n^* := \alpha(\eta_n^*) f_{\varepsilon_n}''(\partial_x \theta_n^*) \text{ in } L^\infty(Q),
\end{cases}
\quad n = 1, 2, 3, \dots.
$$

Then, from (6.3.5)–(6.3.8), and (6.3.14), it follows that:

$$
[L(u_n^* + p_n^*), L_\Gamma(u_{\Gamma,n}^* + p_{\Gamma,n}^*), M(v_n^* + z_n^*)] = [0,0,0] \text{ in } \mathfrak{X} \times \mathscr{H}, \ n = 1, 2, 3, \dots,
\tag{6.6.12a}
$$

$$
\left(-\partial_t \boldsymbol{p}_n^*, \boldsymbol{\varphi}\right)_{\mathfrak{X}} + \left(\partial_x p_n^*, \partial_x \varphi\right)_{\mathscr{H}} + \left(\mu_n^* p_n^*, \varphi\right)_{\mathscr{H}} + \left(\omega_n^* \partial_x z_n^*, \varphi\right)_{\mathscr{H}}
$$

$$= \big(K(\eta_n^* - \eta_{\rm ad}), \varphi\big)_{\mathscr{H}} + \big(K_\Gamma(\eta_{\Gamma,n}^* - \eta_{\Gamma,{\rm ad}}), \varphi_\Gamma\big)_{\mathscr{H}_\Gamma},$$
$$\text{for any } \boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] \in \mathfrak{W},\ n = 1, 2, 3, \dots, \tag{6.6.12b}$$

$$\big\langle -\alpha_0 \partial_t z_n^*, \psi \big\rangle_{\mathscr{V}_0} + \big((-\partial_t \alpha_0) z_n^*, \psi\big)_{\mathscr{H}} + \big(A_n^* \partial_x z_n^* + \nu^2 \partial_x z_n^* + \omega_n^* p_n^*, \partial_x \psi\big)_{\mathscr{H}}$$
$$= \big(\Lambda(\theta_n^* - \theta_{\rm ad}), \psi\big)_{\mathscr{H}}, \text{ for any } \psi \in \mathscr{V}_0,\ n = 1, 2, 3, \dots, \tag{6.6.12c}$$

and

$$[\boldsymbol{p}_n^*(T), z_n^*(T)] = [p_n^*(T), p_{\Gamma,n}^*(T), z_n^*(T)] = [0, 0, 0] \text{ in } \mathbb{X} \times H,\ n = 1, 2, 3, \dots. \tag{6.6.12d}$$

Here, invoking the operators $\mathscr{Q}_\varepsilon^* \in \mathscr{L}(\mathfrak{X} \times \mathscr{H}; \mathfrak{Y})$ and $\mathcal{R}_T \in \mathscr{L}(\mathfrak{X} \times \mathscr{H})$ as in Remark 6.11, we apply Theorem 6.2 to the case when:

$$\begin{cases} [a, b, \mu, \omega, A] = \mathcal{R}_T[\alpha_0, -\partial_t \alpha_0, \mu_n^*, \omega_n^*, A_n^*], \\ [\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] = [0, 0, 0], \\ [\boldsymbol{h}, k] = [h, h_\Gamma, k] \\ \quad = \mathcal{R}_T\big[K(\eta_n^* - \eta_{\rm ad}), K_\Gamma(\eta_{\Gamma,n}^* - \eta_{\Gamma,{\rm ad}}), \Lambda(\theta_n^* - \theta_{\rm ad})\big], \\ [\boldsymbol{p}, z] = [p, p_\Gamma, z] \\ \quad = \mathscr{Q}_{\varepsilon_n}^*\big[\mathcal{R}_T[K(\eta_n^* - \eta_{\rm ad}), K_\Gamma(\eta_{\Gamma,n}^* - \eta_{\Gamma,{\rm ad}}), \Lambda(\theta_n^* - \theta_{\rm ad})]\big], \end{cases} \quad \text{for } n \in \mathbb{N}.$$

Then, with use of the constant $\bar{C}_0^*$ as in (6.6.6a), it is deduced that:

$$\frac{d}{dt}\Big(\big|(\mathcal{R}_T \boldsymbol{p}_n^*)(t)\big|_{\mathbb{X}}^2 + \big|\mathcal{R}_T\big(\sqrt{\alpha_0} z_n^*\big)(t)\big|_H^2\Big)$$
$$+ \Big(\big|(\mathcal{R}_T \boldsymbol{p}_n^*)(t)\big|_{\mathbb{W}}^2 + \nu^2 \big|(\mathcal{R}_T z_n^*)(t)\big|_{V_0}^2\Big)$$
$$\leq \bar{C}_0^*\Big(\big|(\mathcal{R}_T \boldsymbol{p}_n^*)(t)\big|_{\mathbb{X}}^2 + \big|\mathcal{R}_T\big(\sqrt{\alpha_0} z_n^*\big)(t)\big|_H^2\Big)$$
$$+ \bar{C}_0^*\Big(\big|\mathcal{R}_T\big(K(\eta_n^* - \eta_{\rm ad})\big)(t)\big|_{V^*}^2 + \big|\mathcal{R}_T\big(K_\Gamma(\eta_{\Gamma,n}^* - \eta_{\Gamma,{\rm ad}})\big)(t)\big|_{H_\Gamma}^2$$
$$+ \big|\mathcal{R}_T\big(\Lambda(\theta_n^* - \theta_{\rm ad})\big)(t)\big|_{V_0^*}^2\Big), \tag{6.6.13a}$$
$$\text{for a.e. } t \in (0, T),\ n = 1, 2, 3, \dots,$$

and

$$|\partial_t(\mathcal{R}_T \boldsymbol{p}_n^*)|_{\mathfrak{X}}^2 \leq \bar{C}_1^*\Big(\big|\big[\mathcal{R}_T\big(K(\eta_n^* - \eta_{\rm ad})\big), \mathcal{R}_T\big(K_\Gamma(\eta_{\Gamma,n}^* - \eta_{\Gamma,{\rm ad}})\big)\big]\big|_{\mathfrak{X}}^2$$
$$+ \big|\mathcal{R}_T\big(\Lambda(\theta_n^* - \theta_{\rm ad})\big)\big|_{\mathscr{V}_0^*}^2\Big), \tag{6.6.13b}$$

with n-independent positive constant:

$$\bar{C}_1^* := 4(\bar{C}_0^*)^2 e^{\frac{3}{2}\bar{C}_0^* T}.$$

As a consequence of (6.6.6a), (6.6.10b), (6.6.13), (A1), and Gronwall's lemma, we can observe that:

($\star$2) the sequence $\{[\boldsymbol{p}_n^*, z_n^*]\}_{n=1}^\infty = \{[p_n^*, p_{\Gamma,n}^*, z_n^*]\}_{n=1}^\infty$ is bounded in $[C([0,T]; \mathbb{X}) \times C([0,T]; H)] \cap [\mathfrak{X} \times \mathscr{V}_0]$, and $\{\boldsymbol{p}_n^*\}_{n=1}^\infty$ is bounded in $W^{1,2}(0, T; \mathbb{X})$.

Furthermore, from (6.1.1), (6.1.6), (6.6.10b), (6.6.10f), (6.6.12c), ($\star$2), and (A1), we can derive the following estimate:

$$\begin{aligned} &\left|\langle -\partial_x(A_n^*\partial_x z_n^*), \psi\rangle_{\mathscr{U}_0}\right| = \left|\left(A_n^*\partial_x z_n^*, \partial_x\psi\right)_{\mathscr{H}}\right| \\ &\quad \leq \left|\left(\alpha_0 z_n^*, \partial_t\psi\right)_{\mathscr{H}}\right| + \left|\left(\nu^2\partial_x z_n^* + \omega_n^* p_n^*, \partial_x\psi\right)_{\mathscr{H}}\right| + \left|\left(\Lambda(\theta_n^* - \theta_{\mathrm{ad}}), \psi\right)_{\mathscr{H}}\right| \\ &\quad \leq \bar{C}_2^*|\psi|_{\mathscr{U}_0}, \text{ for any } \psi \in C_{\mathrm{c}}^\infty(Q),\ n = 1, 2, 3, \ldots, \end{aligned} \tag{6.6.14}$$

with n-independent positive constant:

$$\bar{C}_2^* := 2\sup_{n\in\mathbb{N}} \left\{ \begin{array}{c} (1 + \nu^2 + |\alpha_0|_{L^\infty(Q)} + |\omega_n^*|_{L^\infty(Q)}) \\ \cdot\left(\left|[\boldsymbol{p}_n^*, z_n^*]\right|_{\mathfrak{W}\times\mathscr{V}_0} + \left|\Lambda(\theta_n^* - \theta_{\mathrm{ad}})\right|_{\mathscr{H}}\right) \end{array} \right\} (< \infty).$$

Due to (6.6.10e), (6.6.10f), (6.6.13b), (6.6.14), ($\star$2), and the compactness theory of Aubin's type (cf. [83, Corollary 4]), we can find subsequences of $\{[\boldsymbol{p}_n^*, z_n^*]\}_{n=1}^\infty = \{[p_n^*, p_{\Gamma,n}^*, z_n^*]\}_{n=1}^\infty \subset \mathfrak{W} \times \mathscr{V}_0$, $\{\omega_n^*\partial_x z_n^*\}_{n=1}^\infty \subset \mathscr{H}$, and $\{-\partial_x(A_n^*\partial_x z_n^*)\}_{n=1}^\infty \subset \mathscr{U}_0^*$ (not relabeled), together with the respective limits $[\boldsymbol{p}^\circ, z^\circ] = [p^\circ, p_\Gamma^\circ, z^\circ] \in \mathfrak{W}\times\mathscr{V}_0$ with $\boldsymbol{p}^\circ = [p^\circ, p_\Gamma^\circ]$, $\xi^\circ \in \mathscr{H}$, and $\zeta^\circ \in \mathscr{U}_0^*$, such that:

$$\begin{cases} [\boldsymbol{p}_n^*, z_n^*] \to [\boldsymbol{p}^\circ, z^\circ] \text{ weakly in } \mathfrak{W} \times \mathscr{V}_0, \\ \boldsymbol{p}_n^* \to \boldsymbol{p}^\circ \text{ in } \mathfrak{X}, \text{ weakly in } W^{1,2}(0,T;\mathbb{V}^*), \\ \quad \text{and in the pointwise sense a.e. in } Q, \end{cases} \tag{6.6.15a}$$

$$\omega_n^* p_n^* \to \alpha'(\eta^\circ)\nu^\circ p^\circ \text{ weakly in } \mathscr{H}, \tag{6.6.15b}$$

$$\omega_n^*\partial_x z_n^* \to \xi^\circ \text{ weakly in } \mathscr{H}, \tag{6.6.15c}$$

and

$$-\partial_x(A_n^*\partial_x z_n^*) \to \zeta^\circ \text{ weakly in } \mathscr{U}_0^*, \text{ as } n \to \infty. \tag{6.6.15d}$$

Now, the properties (6.3.9)–(6.3.12) will be verified through the limiting observations for (6.6.12), as $n \to \infty$, with use of (6.6.10), (6.6.11), and (6.6.15).

Thus, we complete the proof. □

6.7 Appendix

The objective of the appendix is to give the proofs of three Theorems 6.1–6.3, that are stated as a part of auxiliary results in Section 2.

The three Theorems 6.1–6.3 are proved by means of the time-discretization method. In view of this, we divide the rest part in two Subsections, which are concerned with the auxiliary Lemmas in the time-discretization, and the proofs of Theorems 6.1–6.3.

6.7.1 Auxiliary Lemmas in the time-discretization

Let $[a, b, \mu, \omega, A] \in \mathscr{S}$ be a fixed quintet of functions, and let $\delta_*(a)$ be the positive constant as in (6.2.2). Let $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0 = [p_0, p_{\Gamma,0}]$ be a fixed initial triplet, and let $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{V}_0^*$ with $\boldsymbol{h} = [h, h_\Gamma]$ be a fixed forcing triplet.

On this basis, we denote by $\tau \in (0, 1)$ the constant of time-step size, and consider the following time-discretization scheme for (P), denoted by (DP)$_\tau$.

(DP)$_\tau$ Find a sequence $\{[\boldsymbol{p}_i, z_i]\}_{i=1}^{n} = \{[p_i, p_{\Gamma,i} z_i]\}_{i=1}^{\infty}$ of triplets of functions $[\boldsymbol{p}_i, z_i] = [p_i, p_{\Gamma,i}, z_i] \in \mathbb{W} \times V_0$ with $\boldsymbol{p}_i = [p_i, p_{\Gamma,i}]$, $i = 1, 2, 3, \dots$, such that:

$$\frac{1}{\tau}\big(\boldsymbol{p}_i - \boldsymbol{p}_{i-1}, \boldsymbol{\varphi}\big)_{\mathbb{X}} + (\partial_x p_i, \partial_x \varphi)_H + \big(\mu_i p_i + \omega_i \partial_x z_i, \varphi\big)_H = (\boldsymbol{h}_i, \boldsymbol{\varphi})_{\mathbb{X}}, \quad \text{for every } \boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] \in \mathbb{W},\ i = 1, 2, 3, \dots, \tag{6.7.1}$$

$$\frac{1}{\tau}\big(a_i(z_i - z_{i-1}), \psi\big)_H + (b_i z_i, \psi)_H + \big(A_i \partial_x z_i + \nu^2 \partial_x z_i + p_i \omega_i, \partial_x \psi\big)_H = \langle k_i, \psi\rangle_{V_0}, \text{ for every } \psi \in V_0,\ i = 1, 2, 3, \dots, \tag{6.7.2}$$

starting from the initial triplet $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0 = [p_0, p_{\Gamma,0}]$.

In the context, $\{[a_i, b_i, \mu_i, \omega_i, A_i]\}_{i=0}^{\infty}$ is a bounded sequence in $W^{1,\infty}(\Omega) \times L^\infty(\Omega) \times H \times L^\infty(\Omega) \times L^\infty(\Omega)$, such that:

$$\begin{cases} \sup_{i \ge 0} |a_i|_{W^{1,\infty}(\Omega)} \le |a|_{W^{1,\infty}(Q)}, \ \sup_{i \ge 0} |b_i|_{L^\infty(\Omega)} \le |b|_{L^\infty(Q)}, \\ \sup_{i \ge 0} |\mu_i|_H \le |\mu|_{L^\infty(0,T;H)}, \ \sup_{i \ge 0} |\omega_i|_{L^\infty(\Omega)} \le |\omega|_{L^\infty(Q)}, \\ \sup_{i \ge 0} |A_i|_{L^\infty(\Omega)} \le |A|_{L^\infty(Q)}, \ \sup_{i \ge 0} |\log A_i|_{L^\infty(Q)} \le |\log A|_{L^\infty(Q)}, \end{cases} \tag{6.7.3a}$$

$$a_i \ge \delta_*(a), \text{ a.e. in } \Omega, \text{ for } i = 0, 1, 2, \dots, \tag{6.7.3b}$$

$$\begin{cases} [\overline{a}]_\tau \to a \text{ in } L^\infty(0, T; C(\overline{\Omega})), \\ [a]_\tau \to a \text{ in } C(\overline{Q}), \end{cases} \tag{6.7.3c}$$

$$\begin{cases} [\overline{\mu}]_\tau \to \mu \text{ weakly-}* \text{ in } L^\infty(0, T; H), \\ [\overline{\mu}]_\tau(t) \to \mu(t) \text{ in } H, \text{ a.e. } t \in (0, T), \end{cases} \tag{6.7.3d}$$

$$\big[\partial_t [a]_\tau, \partial_x [\overline{a}]_\tau, [\overline{b}]_\tau, [\overline{\omega}]_\tau, [\overline{A}]_\tau\big] \to [\partial_t a, \partial_x a, b, \omega, A] \text{ weakly-}* \text{ in } [L^\infty(Q)]^5, \text{ and in the pointwise sense a.e. in } Q, \text{ as } \tau \downarrow 0, \tag{6.7.3e}$$

and $\{[\boldsymbol{h}_i, k_i]\}_{i=0}^{\infty} = \{[h_i, h_{\Gamma,i}, k_i]\}_{i=0}^{\infty} \subset \mathbb{X} \times V_0^*$ with $\{\boldsymbol{h}_i\}_{i=0}^{\infty} = \{[h_i, h_{\Gamma,i}]\}_{i=0}^{\infty}$ is a bounded sequence, such that:

$$\begin{cases} K^* := \sup_{\tau \in (0,1)} \big| \big[[\overline{\boldsymbol{h}}]_\tau, [\overline{k}]_\tau\big] \big|_{\mathfrak{X} \times \mathscr{H}} < \infty, \\ \big[[\overline{\boldsymbol{h}}]_\tau, [\overline{k}]_\tau\big] \to [\boldsymbol{h}, k] \text{ in } \mathfrak{X} \times \mathscr{H}, \text{ as } \tau \downarrow 0. \end{cases} \tag{6.7.3f}$$

Remark 6.16. Notice that it is straightforward to obtain $\{[a_i, b_i, \mu_i, \omega_i, A_i]\}_{i=0}^{\infty}$ and $\{[\boldsymbol{h}_i, k_i]\}_{i=0}^{\infty} = \{[h_i, h_{\Gamma,i}, k_i]\}_{i=0}^{\infty}$ fulfilling (6.7.3), because the assumptions $[a, b, \mu, \omega, A] \in \mathscr{S}$ and $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{V}_0^*$ allow us to apply the standard method as in Remark 6.3 (Fact 0).

Now, for the solvability of the time-discretization scheme $(\mathrm{DP})_\tau$, we prepare the following lemma.

Lemma 6.3. Let the assume $[a, b, \mu, \omega, A] \in \mathscr{S}$ and $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{V}_0^*$ with $\boldsymbol{h} = [h, h_\Gamma]$. Let $a^\circ \in L^\infty(\Omega)$, $b^\circ \in L^\infty(\Omega)$, $\mu^\circ \in H$, $\omega^\circ \in L^\infty(\Omega)$, and $A^\circ \in L^\infty(\Omega)$ be functions, such that

$$\begin{cases} |a^\circ|_{L^\infty(\Omega)} \leq |a|_{L^\infty(Q)}, \ |b^\circ|_{L^\infty(\Omega)} \leq |b|_{L^\infty(Q)}, \\ |\mu^\circ|_H \leq |\mu|_{L^\infty(0,T;H)}, \ |\omega^\circ|_{L^\infty(\Omega)} \leq |\omega|_{L^\infty(Q)}, \\ |A^\circ|_{L^\infty(\Omega)} \leq |A|_{L^\infty(Q)}, \ |\log A^\circ|_{L^\infty(\Omega)} \leq |\log A|_{L^\infty(Q)}, \end{cases} \tag{6.7.4a}$$

and

$$a^\circ \geq \delta_*(a), \text{ a.e. in } \Omega. \tag{6.7.4b}$$

Additionally, let us assume:

$$0 < \tau < \tau_0 := \frac{\min\{1, \nu^2, \delta_*(a)\}}{16(1 + |b|_{L^\infty(Q)} + |\mu|^2_{L^\infty(0,T;H)} + |\omega|^2_{L^\infty(Q)})}. \tag{6.7.5}$$

Then, for every pairs of functions $[\boldsymbol{h}^\circ, k^\circ] = [h^\circ, h_\Gamma^\circ, k^\circ] \in \mathbb{X} \times V_0^*$ with $\boldsymbol{h}^\circ = [h^\circ, h_\Gamma^\circ]$ and $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0 = [p_0, p_{\Gamma,0}]$, the following variational system admits a unique solution $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathbb{W} \times V_0$ with $\boldsymbol{p} = [p, p_\Gamma]$:

$$\begin{aligned} \frac{1}{\tau}(\boldsymbol{p} - \boldsymbol{p}_0, \boldsymbol{\varphi})_{\mathbb{X}} + (\partial_x p, \partial_x \varphi)_H + \big(\mu^\circ p + \omega^\circ \partial_x z, \varphi\big)_H \\ = (\boldsymbol{h}^\circ, \boldsymbol{\varphi})_{\mathbb{X}}, \text{ for any } \boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] \in \mathbb{W}, \end{aligned} \tag{6.7.6}$$

$$\begin{aligned} \frac{1}{\tau}(a^\circ(z - z_0), \psi)_H + (b^\circ z, \psi)_H + \big(A^\circ \partial_x z + \nu^2 \partial_x z + p\omega^\circ, \partial_x \psi\big)_H \\ = \langle k^\circ, \psi \rangle_{V_0}, \text{ for any } \psi \in V_0. \end{aligned} \tag{6.7.7}$$

Proof. First, for the proof of existence, we define a (non-convex) functional $\mathcal{E} : \mathbb{X} \times H \longrightarrow (-\infty, \infty]$, by letting:

$$\mathcal{E}(\boldsymbol{p}, z) = \mathcal{E}(p, p_\Gamma, z) := \begin{cases} \frac{1}{2\tau}\big(|\boldsymbol{p} - \boldsymbol{p}_0|_{\mathbb{X}}^2 + |\sqrt{a^\circ}(z - z_0)|_H^2\big) \\ \qquad + \frac{1}{2}\int_\Omega \big(|\partial_x p|^2 + |[A^\circ]^{\frac{1}{2}} \partial_x z|^2 + \nu^2 |\partial_x z|^2\big)\, dx \\ \qquad + \frac{1}{2}\int_\Omega \big(\mu^\circ |p|^2 + b^\circ |z|^2\big)\, dx + \int_\Omega p\big(\omega^\circ \cdot \partial_x z\big)\, dx \\ \qquad - (\boldsymbol{h}^\circ, \boldsymbol{p})_{\mathbb{X}} - \langle k^\circ, z \rangle_{V_0}, \\ \qquad \text{if } [\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathbb{W} \times V_0 \text{ with } \boldsymbol{p} = [p, p_\Gamma], \\ \infty, \quad \text{otherwise}, \end{cases}$$

for any $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathbb{X} \times H$ with $\boldsymbol{p} = [p, p_\Gamma]$.

Then, by using the assumption (6.7.5), Remark 6.1, and Young's inequality, one can easily check that $\mathcal{E}$ is a proper lower semi-continuous functional on $\mathbb{X} \times H$, such that:

$$\mathcal{E}(\boldsymbol{p}, z) = \mathcal{E}(p, p_\Gamma, z) \geq \frac{1}{8\tau}\big(|\boldsymbol{p}|_{\mathbb{X}}^2 + \delta_*(a)|z|_H^2\big) + \frac{1}{4}\big(|\partial_x p|_H^2 + \nu^2 |\partial_x z|_H^2\big)$$
$$- \frac{1}{2\tau}\big(|\boldsymbol{p}_0|_{\mathbb{X}}^2 + |a|_{L^\infty(Q)}|z_0|_H^2\big) - \left(\frac{1}{2}|\boldsymbol{h}^\circ|_{\mathbb{X}}^2 + \frac{2}{\nu^2}|k^\circ|_{V_0^*}^2\right),$$
$$\text{for any } [\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathbb{W} \times V_0 \text{ with } \boldsymbol{p} = [p, p_\Gamma],$$

via the following computations:

$$\frac{1}{2\tau}\big(|\boldsymbol{p} - \boldsymbol{p}_0|_{\mathbb{X}}^2 + |\sqrt{a^\circ}(z - z_0)|_H^2\big)$$
$$\geq \frac{1}{4\tau}\big(|\boldsymbol{p}|_{\mathbb{X}}^2 + \delta_*(a)|z|_H^2\big) - \frac{1}{2\tau}\big(|\boldsymbol{p}_0|_{\mathbb{X}}^2 + |a|_{L^\infty(Q)}|z_0|_H^2\big),$$

$$\frac{1}{2}\int_\Omega \mu^\circ |p|^2\,dx \geq -\frac{1}{2}\big|\mu^\circ p\big|_H |p|_H \geq -\frac{1}{\sqrt{2}}|\mu^\circ|_H |p|_H\big(|p|_H + |\partial_x p|_H\big)$$
$$\geq -\frac{1}{4}|\partial_x p|_H^2 - \left(\frac{|\mu^\circ|_H}{\sqrt{2}} + \frac{|\mu^\circ|_H^2}{2}\right)|p|_H^2$$
$$\geq -\frac{1}{4}|\partial_x p|_H^2 - \left(|\mu|_{L^\infty(0,T;H)}^2 + \frac{1}{4}\right)|p|_H^2, \tag{6.7.8a}$$

$$\int_\Omega p(\omega^\circ \cdot \partial_x z)\,dx + \frac{1}{2}\int_\Omega b^\circ |z|^2\,dx$$
$$\geq -\frac{\nu^2}{8}|\partial_x z|_H^2 - \frac{2}{\nu^2}|\omega|_{L^\infty(Q)}^2 |p|_H^2 - \frac{1}{2}|b|_{L^\infty(Q)}|z|_H^2, \tag{6.7.8b}$$

and

$$-(\boldsymbol{h}^\circ, \boldsymbol{p})_{\mathbb{X}} - \langle k^\circ, z\rangle_{V_0} \geq -\frac{1}{2}|\boldsymbol{p}|_{\mathbb{X}}^2 - \frac{\nu^2}{8}|z|_{V_0}^2 - \frac{1}{2}|\boldsymbol{h}^\circ|_{\mathbb{X}}^2 - \frac{2}{\nu^2}|k^\circ|_{V_0^*}^2. \tag{6.7.8c}$$

Additionally, when $\tau \in (0, \tau_0)$, the system {(6.7.6),(6.7.7)} coincides with the stationarity system for $\min \mathcal{E}$, and hence, the solution to {(6.7.6),(6.7.7)} is immediately obtained, by means of the direct method of calculus of variations (cf. [11, Theorem 3.2.1]).

Next, to prove uniqueness, we assume that there are two solutions $[\boldsymbol{p}^\ell, z^\ell] = [p^\ell, p_\Gamma^\ell, z^\ell] \in \mathbb{W} \times V_0$ with $\boldsymbol{p}^\ell = [p^\ell, p_\Gamma^\ell]$, $\ell = 1, 2$, to the system {(6.7.6),(6.7.7)}. Besides, let us take the difference between the equations (6.7.6) (resp. (6.7.7)) corresponding to $\boldsymbol{p}^\ell = [p^\ell, p_\Gamma^\ell]$ (resp. z^ℓ), $\ell = 1, 2$, and put $\boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] = [p^1 - p^2, p_\Gamma^1 - p_\Gamma^2]$ (resp. $\psi = z^1 - z^2$). Then, taking the sum of the results, we arrive at

$$\frac{1}{\tau}\big(|\boldsymbol{p}^1 - \boldsymbol{p}^2|_{\mathbb{X}}^2 + |\sqrt{a^\circ}(z^1 - z^2)|_H^2\big) + |\partial_x(p^1 - p^2)|_H^2$$
$$+ |[A^\circ]^{\frac{1}{2}}\partial_x(z^1 - z^2)|_H^2 + \nu^2|\partial_x(z^1 - z^2)|_H^2 + \int_\Omega \mu^\circ |p^1 - p^2|^2\,dx$$

$$+ 2\int_\Omega (p^1 - p^2)\omega^\circ \cdot \partial_x (z^1 - z^2)\, dx + \int_\Omega b^\circ |z^1 - z^2|^2\, dx = 0.$$

Here, applying (6.7.8a) and (6.7.8b) to the case when:

$$[\boldsymbol{p}, z] = [p, p_\Gamma, z] = [\boldsymbol{p}^1 - \boldsymbol{p}^2, z^1 - z^2] = [p^1 - p^2, p_\Gamma^1 - p_\Gamma^2, z^1 - z^2],$$

and invoking (6.7.4), (6.7.5), and Young's inequality, it is inferred that:

$$\frac{1}{2\tau}\big(|\boldsymbol{p}^1 - \boldsymbol{p}^2|_{\mathbb{X}}^2 + \delta_*(a)|z^1 - z^2|_H^2\big) \leq 0, \text{ whenever } \tau \in (0, \tau_0).$$

Since $\delta_*(a) > 0$ (cf. Remark 6.7), the proof is finished. □

Remark 6.17. The existence and uniqueness of the solution to the time-discretization scheme $(\mathrm{DP})_\tau$ are verified by applying Lemma 6.3, inductively, for every time-steps $i = 1, 2, 3, \ldots$. Here, we note that we can obtain the solution to the scheme $(\mathrm{DP})_\tau$, for any sequence data of forcing $\{[\boldsymbol{h}_i, k_i]\}_{i=0}^\infty = \{[h_i, h_{\Gamma,i}, k_i]\}_{i=0}^\infty \subset \mathbb{X} \times V_0^*$ with $\{\boldsymbol{h}_i\}_{i=0}^\infty = \{[h_i, h_{\Gamma,i}]\}_{i=0}^\infty$, and in particular, we do not need the assumption (6.7.3f) for the solvability of $(\mathrm{DP})_\tau$.

Next, we prepare the following Lemma, for the limiting observation of the time-discretization scheme as $\tau \downarrow 0$.

Lemma 6.4. Let C_0^* be the constant given in (6.2.3). Let us assume $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0 = [p_0, p_{\Gamma,0}]$, and assume that $\{[a_i, b_i, \mu_i, \omega_i, A_i]\}_{i=0}^\infty \subset W^{1,\infty}(\Omega) \times L^\infty(\Omega) \times H \times L^\infty(\Omega) \times L^\infty(\Omega)$ is a given sequence satisfying (6.7.3a) and (6.7.3b). Let $\{[\boldsymbol{h}_i, k_i]\}_{i=0}^\infty = \{[h_i, h_{\Gamma,i}, k_i]\}_{i=0}^\infty \subset \mathbb{X} \times V_0^*$ with $\{\boldsymbol{h}_i\}_{i=0}^\infty = \{[h_i, h_{\Gamma,i}]\}_{i=0}^\infty$ be a given sequence, and let $\{[\boldsymbol{p}_i, z_i]\}_{i=1}^\infty = \{[p_i, p_{\Gamma,i}, z_i]\}_{i=1}^\infty \subset \mathbb{W} \times V_0$ with $\{\boldsymbol{p}_i\}_{i=1}^\infty = \{[p_i, p_{\Gamma,i}]\}_{i=1}^\infty$ be the solution to the scheme $(\mathrm{DP})_\tau$. Then, it holds that:

$$\begin{aligned}
&\frac{1}{\tau}\big(|\boldsymbol{p}_i|^2 - |\boldsymbol{p}_{i-1}|_{\mathbb{X}}^2\big) + \frac{1}{\tau}\big(\big|\sqrt{a_i} z_i\big|_H^2 - \sqrt{a_{i-1}} z_{i-1}\big|_H^2\big) + |\boldsymbol{p}_i|_{\mathbb{W}}^2 + \nu^2 |z_i|_{V_0}^2 \\
&\quad \leq \frac{C_0^*}{2}\Big(\big(|\boldsymbol{p}_i|_{\mathbb{X}}^2 + |\boldsymbol{p}_{i-1}|_{\mathbb{X}}^2\big) + \big(\big|\sqrt{a_i} z_i\big|_H^2 + \sqrt{a_{i-1}} z_{i-1}\big|_H^2\Big) \\
&\qquad + C_0^*\big(|\boldsymbol{h}_i|_{\mathbb{X}}^2 + |k_i|_{V_0^*}^2\big), \ i = 1, 2, 3, \ldots,
\end{aligned} \tag{6.7.9}$$

and

$$\begin{aligned}
&\frac{1}{\tau}|\boldsymbol{p}_i - \boldsymbol{p}_{i-1}|_{\mathbb{X}}^2 + \big(|\partial_x p_i|_H^2 - |\partial_x p_{i-1}|_H^2\big) \\
&\quad \leq C_0^* \tau \big(|p_i|_V^2 + \nu^2 |z_i|_{V_0}^2\big) + 2\tau |\boldsymbol{h}_i|_{\mathbb{X}}^2, \ i = 1, 2, 3, \ldots.
\end{aligned} \tag{6.7.10}$$

Proof. Let us fix any integer $i \in \mathbb{N}$ of the time-step, and let us put $\boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] = \boldsymbol{p}_i = [p_i, p_{\Gamma,i}] \in \mathbb{W}$ in (6.7.1). Then, by using Young's inequality, it is easily seen that:

$$\frac{1}{2\tau}\big(|\boldsymbol{p}_i|_{\mathbb{X}}^2 - |\boldsymbol{p}_{i-1}|_{\mathbb{X}}^2\big) + |\partial_x p_i|_H^2 \leq -\int_\Omega \mu_i |p_i|^2\, dx - \int_\Omega \omega_i p_i \partial_x z_i\, dx + (\boldsymbol{h}_i, \boldsymbol{p}_i)_{\mathbb{X}}. \tag{6.7.11}$$

Also, putting $\psi = z_i \in V_0$ in (6.7.2), we have:

$$\frac{1}{2\tau}\big(|\sqrt{a_i} z_i|_H^2 - |\sqrt{a_{i-1}} z_{i-1}|_H^2\big) + \nu^2 |\partial_x z_i|_H^2$$

$$\leq \frac{|a|_{W^{1,\infty}(Q)}}{2\delta_*(a)}|\sqrt{a_{i-1}}z_{i-1}|_H^2 - \int_\Omega b_i|z_i|^2\,dx - \int_\Omega \omega_i p_i \partial_x z_i\,dx + \langle k_i, z_i\rangle_{V_0}, \qquad (6.7.12)$$

via the computation:

$$\begin{aligned}
&\frac{1}{\tau}\big(\sqrt{a_i}(z_i - z_{i-1}), z_i\big)_H \\
&\quad \geq \frac{1}{2\tau}\int_\Omega \big(a_i|z_i|^2 - a_{i-1}|z_{i-1}|^2\big)\,dx - \frac{1}{2}\int_\Omega \left(\frac{a_i - a_{i-1}}{\tau}\right)|z_{i-1}|^2\,dx \\
&\quad \geq \frac{1}{2\tau}\big(|\sqrt{a_i}z_i|_H^2 - |\sqrt{a_{i-1}}z_{i-1}|_H^2\big) - \frac{|a|_{W^{1,\infty}(Q)}}{2\delta_*(a)}|\sqrt{a_{i-1}}z_{i-1}|_H^2,
\end{aligned}$$

with the use of (6.7.3a), (6.7.3b), and Young's inequality.

Now, the required inequality (6.7.9) will be verified by taking the sum of (6.7.11) and (6.7.12), by invoking (6.2.3), and by applying (6.7.8) to the case when:

$$\begin{cases}
[a^\circ, b^\circ, \mu^\circ, \omega^\circ, A^\circ] = [a_i, b_i, \mu_i, \omega_i, A_i], \\
[\boldsymbol{p}, z] = [p, p_\Gamma, z] = [\boldsymbol{p}_i, z_i] = [p_i, p_{\Gamma,i}, z_i], \\
[\boldsymbol{h}^\circ, k^\circ] = [h^\circ, h_\Gamma^\circ, k^\circ] = [\boldsymbol{h}_i, k_i] = [h_i, h_{\Gamma,i}, k_i].
\end{cases}$$

Next, let us put $\boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] = \boldsymbol{p}_i = [p_i - p_{i-1}, p_{\Gamma,i} - p_{\Gamma,i-1}] \in \mathbb{W}$ in (6.7.1). Then, having in mind (6.2.3), and using (6.1.1) and Young's inequality, we will observe that:

$$\begin{aligned}
&\frac{1}{\tau}|\boldsymbol{p}_i - \boldsymbol{p}_{i-1}|_{\mathbb{X}}^2 + \frac{1}{2}|\partial_x p_i|_H^2 - \frac{1}{2}|\partial_x p_{i-1}|_H^2 \\
&\leq \sqrt{2}|\mu_i|_H |p_i|_V |p_i - p_{i-1}|_H + |\omega_i|_{L^\infty(\Omega)}|\partial_x z_i|_H |p_i - p_{i-1}|_H + |\boldsymbol{h}_i|_{\mathbb{X}}|\boldsymbol{p}_i - \boldsymbol{p}_{i-1}|_{\mathbb{X}} \\
&\leq \frac{1}{2\tau}|\boldsymbol{p}_i - \boldsymbol{p}_{i-1}|_{\mathbb{X}}^2 + \frac{C_0^*\tau}{2}\big(|p_i|_V^2 + \nu^2|z_i|_{V_0}^2\big) + \tau|\boldsymbol{h}_i|_{\mathbb{X}}.
\end{aligned}$$

This inequality directly leads to the required (6.7.10). □

Finally, we prove the following Lemma concerned with a time-discrete version of Gronwall's inequality.

Lemma 6.5. Let $c \geq 0$ be a fixed constant, and let $\tau \in (0,1)$ be a time-step size satisfying:

$$0 < c\tau < 2. \qquad (6.7.13)$$

Let $0 < T < \infty$ be a constant of time, and let $N_{[\frac{T}{\tau}]} \in \mathbb{N}$ be a time-step such that:

$$(N_{[\frac{T}{\tau}]} - 1)\tau < T \leq N_{[\frac{T}{\tau}]}\tau. \qquad (6.7.14)$$

Let $\{P_i\}_{i=0}^\infty \subset [0,\infty)$ and $\{Q_i\}_{i=1}^\infty \subset [0,\infty)$ be sequences such that:

$$\frac{1}{\tau}(P_i - P_{i-1}) \leq \frac{c}{2}(P_i + P_{i-1}) + Q_i, \ \ i = 1, 2, 3, \ldots. \qquad (6.7.15)$$

Then, it is estimated that:

$$P_i \ \leq 2e^{\frac{3}{2}cT}\left(P_0 + \tau\sum_{i=1}^{N_{[\frac{T}{\tau}]}} Q_i\right), \ \ i = 1, \ldots, N_{[\frac{T}{\tau}]}. \qquad (6.7.16)$$

Proof. From the assumptions (6.7.13) and (6.7.15), it is easily derived that:

$$P_i \le \frac{1+\frac{c\tau}{2}}{1-\frac{c\tau}{2}}\, P_{i-1} + \frac{\tau}{1-\frac{c\tau}{2}}\, Q_i, \;\; i = 1, 2, 3, \ldots .$$

On this basis, we observe that:

$$\begin{aligned}
P_1 &\le \frac{1+\frac{c\tau}{2}}{1-\frac{c\tau}{2}} P_0 + \frac{\tau}{1-\frac{c\tau}{2}} Q_1, \\
P_2 &\le \left(\frac{1+\frac{c\tau}{2}}{1-\frac{c\tau}{2}}\right)^2 P_0 + \tau\left(\frac{1+\frac{c\tau}{2}}{(1-\frac{c\tau}{2})^2} Q_1 + \frac{1}{1-\frac{c\tau}{2}} Q_2\right), \\
P_3 &\le \left(\frac{1+\frac{c\tau}{2}}{1-\frac{c\tau}{2}}\right)^3 P_0 + \tau\left(\frac{(1+\frac{c\tau}{2})^2}{(1-\frac{c\tau}{2})^3} Q_1 + \frac{1+\frac{c\tau}{2}}{(1-\frac{c\tau}{2})^2} Q_2 + \frac{1}{1-\frac{c\tau}{2}} Q_3\right),
\end{aligned}$$

and in general,

$$P_i \le \left(\frac{1+\frac{c\tau}{2}}{1-\frac{c\tau}{2}}\right)^i P_0 + \tau \sum_{j=1}^{i} \frac{(1+\frac{c\tau}{2})^{i-j}}{(1-\frac{c\tau}{2})^{i-j+1}}\, Q_j \;\le \left(\frac{1+\frac{c\tau}{2}}{1-\frac{c\tau}{2}}\right)^{N_{[\frac{T}{\tau}]}} \left(P_0 + \tau \sum_{i=1}^{N_{[\frac{T}{\tau}]}} Q_i\right),$$
$$\text{for } i = 1, \ldots, N_{[\frac{T}{\tau}]}. \tag{6.7.17}$$

Here, in view of (6.7.14), it is inferred that:

$$\begin{aligned}
\left(\frac{1+\frac{c\tau}{2}}{1-\frac{c\tau}{2}}\right)^{N_{[\frac{T}{\tau}]}} &\le \left(1 + \frac{1}{\frac{1}{c\tau} - \frac{1}{2}}\right)\left(1 + \frac{1}{\frac{1}{c\tau} - \frac{1}{2}}\right)^{N_{[\frac{T}{\tau}]}-1} \\
&\le 2\left(1 + \frac{1}{\frac{N_{[\frac{T}{\tau}]}-1}{cT} - \frac{1}{2}}\right)^{N_{[\frac{T}{\tau}]}-1} \le 2\left(1+\frac{1}{\tilde{N}}\right)^{\tilde{N}\cdot cT}\left(1+\frac{1}{\tilde{N}}\right)^{\frac{1}{2}cT} \\
&\le 2e^{\frac{3}{2}cT}, \;\text{ with } \tilde{N} := \tfrac{N_{[\frac{T}{\tau}]}-1}{cT} - \tfrac{1}{2}.
\end{aligned} \tag{6.7.18}$$

The estimate (6.7.16) is obtained as a straightforward consequence of (6.7.17) and (6.7.18). □

6.7.2 Proof of Theorems 6.1–6.3

For efficiency of explanation, we prove the three Theorems 6.1–6.3 in accordance with the following Steps.

Step 1: proof of the existence part of Theorem 6.1.

Step 2: proof of Theorem 6.2 (I).

Step 3: proof of the uniqueness part of Theorem 6.1.

Step 4: proof of Theorem 6.2 (II).

Step 5: proof of Theorem 6.3.

<u>**Step 1: proof of the existence part of Theorem 6.1.**</u> Let C_0^* be the positive constant given in (6.2.3), and let $\tau_0 \in (0,1)$ be the constant given in (6.7.5). Besides, we assume that the time-step size $\tau \in (0,1)$ is so small to satisfy that:

$$0 < \tau \leq \tau_1 := \frac{1}{2C_0^*} \left(\leq \frac{\tau_0}{2} \right),$$

and we set

$$T_{[\tau]} := N_{[\frac{T}{\tau}]}\tau \text{ with use of the time-step } N_{[\frac{T}{\tau}]} \in \mathbb{N} \text{ as in (6.7.14).}$$

On this basis, let us apply Lemma 6.5 to the inequality (6.7.9) in Lemma 6.4, under the setting:

$$\begin{cases} c = C_0^* \ (\geq 1), \\ P_i = |\boldsymbol{p}_i|_{\mathbb{X}}^2 + |\sqrt{a_i} z_i|_H^2 + \tau \displaystyle\sum_{j=0}^{i} \big(|\boldsymbol{p}_j|_{\mathbb{W}}^2 + \nu^2 |z_j|_{V_0}^2 \big) \\ \qquad\qquad -\tau \big(|\boldsymbol{p}_0|_{\mathbb{W}}^2 + \nu^2 |z_0|_{V_0}^2 \big), \ i = 0, 1, 2, 3, \dots, \\ Q_i = C_0^* \big(|\boldsymbol{h}_i|_{\mathbb{X}}^2 + |k_i|_{V_0^*}^2 \big), \ i = 1, 2, 3, \dots. \end{cases}$$

Then, we have:

$$\begin{aligned} & |\boldsymbol{p}_i|_{\mathbb{X}}^2 + |\sqrt{a_i} z_i|_H^2 + \tau \sum_{j=1}^{i} \big(|\boldsymbol{p}_j|_{\mathbb{W}}^2 + \nu^2 |z_j|_{V_0}^2 \big) \\ & \qquad \leq 2e^{\frac{3}{2}C_0^* T} \left(\big(|\boldsymbol{p}_0|_{\mathbb{X}}^2 + |\sqrt{a_0} z_0|_H^2 \big) + C_0^* \tau \sum_{i=1}^{N_{[\frac{T}{\tau}]}} \big(|\boldsymbol{h}_i|_{\mathbb{X}}^2 + |k_i|_{V_0^*}^2 \big) \right), \\ & \qquad\qquad \text{for } i = 1, \dots, N_{[\frac{T}{\tau}]}, \end{aligned}$$

which leads to:

$$\begin{aligned} & \max \left\{ \begin{array}{c} \big| [\boldsymbol{p}]_\tau \big|_{C([0,T];\mathbb{X})}^2, \big| [\overline{\boldsymbol{p}}]_\tau \big|_{L^\infty(0,T;\mathbb{X})}^2, \big| [\underline{\boldsymbol{p}}]_\tau \big|_{L^\infty(0,T;\mathbb{X})}^2, \\ \delta_*(a) \big| [z]_\tau \big|_{C([0,T];H)}^2, \delta_*(a) \big| [\overline{z}]_\tau \big|_{L^\infty(0,T;H)}^2, \delta_*(a) \big| [\underline{z}]_\tau \big|_{L^\infty(0,T;H)}^2, \\ \big| [\overline{\boldsymbol{p}}]_\tau \big|_{L^2(0,T_{[\tau]};\mathbb{W})}^2 + \nu^2 \big| [\overline{z}]_\tau \big|_{L^2(0,T_{[\tau]};V_0)}^2 \end{array} \right\} \\ & \leq 2C_0^* e^{\frac{3}{2}C_0^* T} \big(|\boldsymbol{p}_0|_{\mathbb{X}}^2 + |\sqrt{a_0} z_0|_H^2 + \big| [\overline{\boldsymbol{h}}]_\tau \big|_{L^2(0,T_{[\tau]};\mathbb{X})}^2 + \big| [\overline{k}]_\tau \big|_{L^2(0,T_{[\tau]};V_0^*)}^2 \big). \end{aligned} \tag{6.7.19}$$

Also, taking the sum of the inequality in (6.7.10), for $i = 1, \dots, N_{[\frac{T}{\tau}]}$, it is estimated that:

$$\begin{aligned} & \big| \partial_t [\boldsymbol{p}]_\tau \big|_{L^2(0,T_{[\tau]};\mathbb{X})}^2 + \max \big\{ \big| \partial_x [p]_\tau \big|_{L^\infty(0,T;H)}^2, \big| \partial_x [\overline{p}]_\tau \big|_{L^\infty(0,T;H)}^2, \big| \partial_x [\underline{p}]_\tau \big|_{L^\infty(0,T;H)}^2 \big\} \\ & \quad \leq 2|p_0|_V^2 + C_0^* \big(\big| [\overline{p}]_\tau \big|_{L^2(0,T_{[\tau]};V)}^2 + \nu^2 \big| [\overline{z}]_\tau \big|_{L^2(0,T_{[\tau]};V_0)}^2 \big) + 2 \big| [\overline{\boldsymbol{h}}]_\tau \big|_{L^2(0,T_{[\tau]};\mathbb{X})}^2 \\ & \quad \leq 4(C_0^*)^2 e^{\frac{3}{2}C_0^* T} \big(|\boldsymbol{p}_0|_{\mathbb{W}}^2 + |\sqrt{a_0} z_0|_H^2 + \big| [\overline{\boldsymbol{h}}]_\tau \big|_{L^2(0,T_{[\tau]};\mathbb{X})}^2 + \big| [\overline{k}]_\tau \big|_{L^2(0,T_{[\tau]};V_0^*)}^2 \big). \end{aligned} \tag{6.7.20}$$

Meanwhile, since (6.7.2) implies that:

$$\begin{aligned}
&\left|\frac{1}{\tau}\big(a_i(z_i - z_{i-1}), \psi\big)_H\right| \\
&\leq \big|A_i \partial_x z_i + \nu^2 \partial_x z_i + p_i \omega_i\big|_H \, |\partial_x \psi|_H + 2|b_i|_{L^\infty(\Omega)} |z_i|_{V_0} |\psi|_{V_0} + |k_i|_{V_0^*} |\psi|_{V_0} \\
&\leq \left(\left(\frac{|A|_{L^\infty(Q)} + 2|b_i|_{L^\infty(\Omega)}}{\nu} + \nu \right) \nu |z_i|_{V_0} + |\omega_i|_{L^\infty(\Omega)} |p_i|_H + |k_i|_{V_0^*} \right) |\psi|_{V_0} \\
&\leq \frac{2}{\min\{1, \nu\}} \big(1 + \nu + |b|_{L^\infty(Q)} + |\omega|_{L^\infty(Q)} + |A|_{L^\infty(Q)}\big) \big(|p_i|_H^2 + \nu^2 |z_i|_{V_0}^2 + |k_i|_{V_0^*}^2\big)^{\frac{1}{2}} |\psi|_{V_0},
\end{aligned}$$

for all $\psi \in V_0$, and $i = 1, \ldots, N_{[\frac{T}{\tau}]}$,

one can deduce from (6.2.3) that:

$$\begin{aligned}
\big|[\overline{a}]_\tau(t)\, \partial_t [z]_\tau(t)\big|_{V_0^*}^2 &\leq C_0^* (1 + \nu + |b|_{L^\infty(Q)} + |\omega|_{L^\infty(Q)} + |A|_{L^\infty(Q)})^2 \cdot \\
&\cdot \left(|[\overline{p}_\tau(t)]|_H^2 + \nu^2 |[\overline{z}_\tau(t)]|_{V_0}^2 + |[\overline{k}]_\tau(t)|_{V_0^*}^2 \right), \text{ for any } t \in [0, T_{[\tau]}].
\end{aligned} \tag{6.7.21}$$

Additionally, integrating the both sides of (6.7.21) over $[0, T_{[\tau]}]$, and invoking Remark 6.2, we obtain that:

$$\begin{aligned}
\big|\partial_t\, [z]_\tau\big|_{L^2(0, T_{[\tau]}; V_0^*)}^2 &\leq \frac{(1 + \sqrt{2})^2 (|[\overline{a}]_\tau|_{L^\infty(Q)} + |\partial_x [\overline{a}]_\tau|_{L^\infty(Q)})^2}{\delta_*(a)^4} \big|[\overline{a}]_\tau\, \partial_t [z]_\tau\big|_{L^2(0, T_{[\tau]}; V_0^*)}^2 \\
&\leq (C_0^*)^5 (1 + |a|_{W^{1,\infty}(Q)})^2 \big(1 + \nu + |b|_{L^\infty(Q)} + |\omega|_{L^\infty(Q)} + |A|_{L^\infty(Q)}\big)^2 \cdot \\
&\qquad \cdot \left(\big|[\overline{p}]_\tau\big|_{L^2(0, T_{[\tau]}; H)}^2 + \nu^2 \big|[\overline{z}]_\tau\big|_{L^2(0, T_{[\tau]}; V_0)}^2 + \big|[\overline{k}]_\tau\big|_{L^2(0, T_{[\tau]}; V_0^*)}^2 \right) \\
&\leq 4 (C_0^*)^6 e^{\frac{3}{2} C_0^* T} (1 + |a|_{W^{1,\infty}(Q)})^2 (1 + \nu + |b|_{L^\infty(Q)} + |\omega|_{L^\infty(Q)} + |A|_{L^\infty(Q)})^2 \cdot \\
&\qquad \cdot \left(|\boldsymbol{p}_0|_{\mathbb{X}}^2 + |\sqrt{a_0} z_0|_H^2 + \big|[\overline{\boldsymbol{h}}]_\tau\big|_{L^2(0, T_{[\tau]}; H)}^2 + \big|[\overline{k}]_\tau\big|_{L^2(0, T_{[\tau]}; V_0^*)}^2 \right).
\end{aligned} \tag{6.7.22}$$

Now, on account of the estimates (6.7.19)–(6.7.22), we can say that:

($\star$3) $\{[\boldsymbol{p}]_\tau\}_{\tau \in (0, \tau_1]} = \big\{\big[[p]_\tau, [p_\Gamma]_\tau\big]\big\}_{\tau \in (0, \tau_1]}$ is bounded in $W^{1,2}(0, T; \mathbb{X}) \cap L^\infty(0, T; \mathbb{W})$, and $\{[\overline{\boldsymbol{p}}]_\tau\}_{\tau \in (0, \tau_1]} = \big\{\big[[\overline{p}]_\tau, [\overline{p_\Gamma}]_\tau\big]\big\}_{\tau \in (0, \tau_1]}$ and $\{[\underline{\boldsymbol{p}}]_\tau\}_{\tau \in (0, \tau_1]} = \big\{\big[[\underline{p}]_\tau, [\underline{p_\Gamma}]_\tau\big]\big\}_{\tau \in (0, \tau_1]}$ are bounded in $L^\infty(0, T; \mathbb{W})$;

($\star$4) $\{[z]_\tau\}_{\tau \in (0, \tau_1]}$ is bounded in $W^{1,2}(0, T; V_0^*) \cap C([0, T]; H) \cap \mathscr{V}_0$, and $\{[\overline{z}]_\tau\}_{\tau \in (0, \tau_1]}$ and $\{[\underline{z}]_\tau\}_{\tau \in (0, \tau_1]}$ are bounded in $L^\infty(0, T; H) \cap \mathscr{V}_0$.

In this light, we can apply the compactness theory of Aubin's type (cf. [83, Corollary 4]) with the one-dimensional compact embeddings $V \subset C(\overline{\Omega})$ and $V_0 \subset C(\overline{\Omega})$, and we can find a sequence $\{\tau_n\}_{n=2}^\infty \subset (0, \tau_1)$, and a limiting point $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathfrak{Y}$ with $\boldsymbol{p} = [p, p_\Gamma]$ such that:

$$\tau_1 > \tau_2 > \tau_3 > \cdots > \tau_n \downarrow 0, \text{ as } n \to \infty, \tag{6.7.23}$$

$[\boldsymbol{p}]_{\tau_n} \to \boldsymbol{p}$ in $C(\overline{Q}) \times C(\overline{\Sigma})$, in $\mathfrak{W}$,

weakly in $W^{1,2}(0,T;\mathbb{X})$, weakly-$*$ in $L^\infty(0,T;\mathbb{W})$,
and in the pointwise sense a.e. in Q, (6.7.24a)

$$\begin{cases} [\overline{\boldsymbol{p}}]_{\tau_n} \to \boldsymbol{p}, \\ [\underline{\boldsymbol{p}}]_{\tau_n} \to \boldsymbol{p}, \end{cases} \text{ in } L^\infty(Q) \times L^\infty(\Sigma),$$

in $\mathfrak{W}$, weakly-$*$ in $L^\infty(0,T;\mathbb{W})$,
and in the pointwise sense a.e. in Q, (6.7.24b)

$[z]_{\tau_n} \to z$ in $C([0,T];V_0^*)$, in $\mathscr{H}$, weakly in $\mathscr{V}_0$,
weakly in $W^{1,2}(0,T;V_0^*)$, weakly-$*$ in $L^\infty(0,T;H)$,
and in the pointwise sense a.e. in Q, (6.7.25a)

and

$[\overline{z}]_{\tau_n} \to z$ and $[\underline{z}]_{\tau_n} \to z$ in $L^\infty([0,T];V_0^*)$, in $\mathscr{H}$,
weakly in $\mathscr{V}_0$, weakly-$*$ in $L^\infty(0,T;H)$,
and in the pointwise sense a.e. in Q, as $n \to \infty$. (6.7.25b)

Furthermore, with (6.7.1)–(6.7.3), (6.7.24), (6.7.25), and Remark 6.2 in mind, it will be inferred that:

$$\begin{cases} \partial_t [p]_{\tau_n} + [\overline{\mu}]_{\tau_n} [\overline{p}]_{\tau_n} + [\overline{\omega}]_{\tau_n} \partial_x [\overline{z}]_{\tau_n} - [\overline{h}]_{\tau_n} \\ \qquad \to \partial_t p + \mu p + \omega \partial_x z - h \text{ weakly in } \mathscr{H}, \\ \partial_t [p_\Gamma]_{\tau_n} - [\overline{h_\Gamma}]_{\tau_n} \to \partial_t p_\Gamma - h_\Gamma \text{ weakly in } \mathscr{H}_\Gamma, \end{cases} \tag{6.7.26}$$

$$\begin{cases} [\overline{a}]_{\tau_n} \partial_t [z]_{\tau_n} + [\overline{b}]_{\tau_n} [\overline{z}]_{\tau_n} - [\overline{k}]_{\tau_n} \to a \partial_t z + bz - k \\ \qquad \text{weakly in } \mathscr{V}_0^*, \\ \big([\overline{A}]_{\tau_n} + \nu^2\big) \partial_x [\overline{z}]_{\tau_n} + [\overline{p}]_{\tau_n} [\overline{\omega}]_{\tau_n} \to \big(A + \nu^2\big) \partial_x z + p\omega \\ \qquad \text{weakly in } \mathscr{H}, \end{cases} \text{as } n \to \infty, \tag{6.7.27}$$

$$\begin{aligned} &\int_s^t \Big(\big(\partial_t [p]_{\tau_n} + [\overline{\mu}]_{\tau_n} [\overline{p}]_{\tau_n} + [\overline{\omega}]_{\tau_n} \partial_x [\overline{z}]_{\tau_n} - [\overline{h}]_{\tau_n} \big)(\varsigma), \varphi \Big)_H \, d\varsigma \\ &\quad + \int_s^t \Big(\big(\partial_t [p_\Gamma]_{\tau_n} - [\overline{h_\Gamma}]_{\tau_n} \big)(\varsigma), \varphi_\Gamma \Big)_{H_\Gamma} d\varsigma + \int_s^t \big(\partial_x [\overline{p}]_{\tau_n}(\varsigma), \partial_x \varphi \big)_H \, d\varsigma = 0, \end{aligned} \tag{6.7.28a}$$

and

$$\begin{aligned} &\int_s^t \Big\langle \big([\overline{a}]_{\tau_n} \partial_t [z]_{\tau_n} + [\overline{b}]_{\tau_n} [\overline{z}]_{\tau_n} - [\overline{k}]_{\tau_n} \big)(\varsigma), \psi \Big\rangle_{V_0} d\varsigma \\ &\quad + \int_s^t \Big(\big(([\overline{A}]_{\tau_n} + \nu^2) \partial_x [\overline{z}]_{\tau_n} + [\overline{p}]_{\tau_n} [\overline{\omega}]_{\tau_n} \big)(\varsigma), \partial_x \psi \Big)_H \, d\varsigma = 0, \end{aligned} \tag{6.7.28b}$$

for all $\boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] \in \mathbb{W}$, $\psi \in V_0$, $0 \le s \le t \le T$, and $n = 1, 2, 3, \dots$.

On account of (6.7.26)–(6.7.28), and the arbitrary choices of $0 \le s \le t \le T$, we will verify that the limit $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathfrak{Y}$ with $\boldsymbol{p} = [p, p_\Gamma]$ will be a solution to the system (P), by letting $n \to \infty$ in (6.7.28). □

Step 2: proof of Theorem 6.2 (I). Let $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathfrak{Y}$ with $\boldsymbol{p} = [p, p_\Gamma]$ be a solution to the system (P). Besides, let us put $\boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] = \boldsymbol{p}(t) = [p(t), p_\Gamma(t)] \in \mathbb{W}$ in (6.7.1), put $\psi = z(t)$ in (6.7.2), and take the sum of results. Then, it is seen that:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\bigl(\bigl|\boldsymbol{p}(t)\bigr|_{\mathbb{X}}^2 + \bigl|(\sqrt{a}z)(t)\bigr|_H^2\bigr) &+ |\partial_x p(t)|_H^2 + \nu^2 |z(t)|_{V_0}^2 \\ &\le \bigl(\boldsymbol{h}(t), p(t)\bigr)_{\mathbb{X}} + \langle k(t), z(t)\rangle_{V_0} + \frac{1}{2}\int_\Omega \partial_t a(t)|z(t)|^2\,dx - \int_\Omega \mu(t)|p(t)|^2\,dx \\ &\quad - \int_\Omega b(t)z(t)\,dx - 2\int_\Omega p(t)\omega(t)\partial_x z(t)\,dx, \text{ a.e. } t \in (0,T).\end{aligned}$$

Here, referring to the computations as in (6.7.8), and using the positive constant C_0^* as in (6.2.3), we arrive at the conclusion (6.2.4) in Theorem 6.2 (I). □

Step 3: proof of the uniqueness part of Theorem 6.1. For every $\ell = 1, 2$, let $[\boldsymbol{p}^\ell, z^\ell] = [p^\ell, p_\Gamma^\ell, z^\ell] \in \mathfrak{Y}$ with $\boldsymbol{p}^\ell = [p^\ell, p_\Gamma^\ell]$ be the solutions to the system (P) for the same initial triplet $[\boldsymbol{p}_0, z_0] = [p_0, p_{\Gamma,0}, z_0] \in \mathbb{W} \times H$ with $\boldsymbol{p}_0 = [p_0, p_{\Gamma,0}]$, and the same forcing triplet $[\boldsymbol{h}, k] = [h, h_\Gamma, k] \in \mathfrak{X} \times \mathscr{V}_0^*$ with $\boldsymbol{h} = [h, h_\Gamma]$. Then, since (P) is a linear system, the difference of solutions $[\boldsymbol{p}^1 - \boldsymbol{p}^2, z^1 - z^2] = [p^1 - p^2, p_\Gamma^1 - p_\Gamma^2, z^1 - z^2]$ is also a solution to (P) for the homogeneous initial triplet $[0,0,0] \in \mathbb{W} \times H$ and homogeneous forcing triplet $[0,0,0] \in \mathfrak{X} \times \mathscr{V}_0^*$. So, from Theorem 6.2 (I), it immediately follows that:

$$\begin{aligned}\frac{d}{dt}\bigl(\bigl|(\boldsymbol{p}^1 - \boldsymbol{p}^2)(t)\bigr|_{\mathbb{X}}^2 &+ \bigl|\sqrt{a}(z^1 - z^2)(t)\bigr|_H^2\bigr) \\ &\le C_0^*\bigl(\bigl|(\boldsymbol{p}^1 - \boldsymbol{p}^2)(t)\bigr|_{\mathbb{X}}^2 + \bigl|\sqrt{a}(z^1 - z^2)(t)\bigr|_H^2\bigr), \text{ a.e. } t \in (0,T). \qquad (6.7.29)\end{aligned}$$

The uniqueness result will be verified by applying Gronwall's lemma to (6.7.29) with the assumption (6.2.2). □

Remark 6.18. By virtue of the uniqueness result in Theorem 6.1, we can also conclude the convergence result of the time-discretization scheme $(\mathrm{DP})_\tau$, as $\tau \downarrow 0$. More precisely, we can obtain the convergences as in (6.7.24)–(6.7.27) for any sequence $\{\tau_n\}_{n=1}^\infty$ (subsequence) satisfying (6.7.23).

Step 4: proof of Theorem 6.2 (II). As consequences of (6.2.5), (6.7.19), (6.7.20), and (6.7.22), it is inferred that:

$$\begin{cases}\bigl|\partial_t[\boldsymbol{p}]_\tau\bigr|_{\mathfrak{X}}^2 + \bigl|[p]_\tau\bigr|_{L^\infty(0,T;V)}^2 \\ \qquad \le C_1^*\bigl(|\boldsymbol{p}_0|_{\mathbb{W}}^2 + |\sqrt{a}z_0|_H^2 + \bigl|[\overline{\boldsymbol{h}}]_\tau\bigr|_{L^2(0,T_{[\tau]};H)}^2 + \bigl|[\overline{k}]_\tau\bigr|_{L^2(0,T_{[\tau]};V_0^*)}^2\bigr), \\ \bigl|\partial_t[z]_\tau\bigr|_{\mathscr{V}_0^*}^2 \le C_2^*\bigl(|\boldsymbol{p}_0|_{\mathbb{X}}^2 + |\sqrt{a}z_0|_H^2 + \bigl|[\overline{\boldsymbol{h}}]_\tau\bigr|_{L^2(0,T_{[\tau]};H)}^2 + \bigl|[\overline{k}]_\tau\bigr|_{L^2(0,T_{[\tau]};V_0^*)}^2\bigr).\end{cases} \qquad (6.7.30)$$

Hence, having in mind (6.7.3), (6.7.24), (6.7.25), and Remark 6.18, we can verify the estimate (6.2.6) in Theorem 6.2 (II), just by letting $\tau \downarrow 0$ in (6.7.30). □

<u>**Step 5: proof of Theorem 6.3.**</u> Since (6.2.7b) implies that:

$$a^n \to a \text{ in } C(\overline{Q}), \text{ as } n \to \infty,$$

we may suppose:

$$a^n \geq \frac{\delta_*(a)}{2}, \text{ for } n = 1, 2, 3, \dots,$$

without loss of generality. Here, for any $n \in \mathbb{N}$, we define:

$$C_0^n := \frac{16\big(1 + |a^n|_{W^{1,\infty}(Q)} + |b^n|_{L^\infty(Q)} + |\mu^n|^2_{L^\infty(0,T;H)} + |\omega^n|^2_{L^\infty(Q)}\big)}{\min\{1, \nu^2, \frac{\delta_*(a)}{2}\}}, \tag{6.7.31}$$

and

$$\begin{cases} C_1^n := 4(C_0^n)^2 e^{\frac{3}{2}C_0^n T}, \\ C_2^n := 4(C_0^n)^6 e^{\frac{3}{2}C_0^n T}(1 + |a^n|_{W^{1,\infty}(Q)})^2 \cdot \\ \qquad \cdot (1 + \nu + |b^n|_{L^\infty(Q)} + |\omega^n|_{L^\infty(Q)} + |A^n|_{L^\infty(Q)})^2. \end{cases} \tag{6.7.32}$$

Then, in view of (6.2.3), (6.2.5), (6.2.7), (6.2.8), (6.7.3a), (6.7.3b), (6.7.31), (6.7.32), and Remark 6.8, we will infer that:

$$\begin{cases} |\boldsymbol{p}^n|^2_{C([0,T];\mathbb{X})} + |\sqrt{a^n} z^n|^2_{C([0,T];H)} + |\boldsymbol{p}^n|^2_{L^2(0,T;\mathbb{W})} + \nu^2 |z^n|^2_{\mathbb{V}_0} \leq C_3^*, \\ |\partial_t \boldsymbol{p}^n|^2_{\mathfrak{X}} + |p^n|^2_{L^\infty(0,T;V)} + |\partial_t z^n|^2_{\mathscr{V}_0^*} \leq C_3^*, \end{cases}$$
$$\text{for } n = 1, 2, 3, \dots,$$

with use of a uniform positive constant C_3^*:

$$C_3^* := \sup_{n \in \mathbb{N}} \Big\{ \big(C_1^n + C_2^n\big)\big(|\boldsymbol{p}_0^n|^2_{\mathbb{W}} + |\sqrt{a^n} z_0^n|^2_H + |\boldsymbol{h}^n|^2_{\mathfrak{X}} + |k^n|^2_{\mathscr{V}_0^*}\big) \Big\} < \infty.$$

Now, we can say that:

($\star$5) $\{\boldsymbol{p}^n\}_{n=1}^\infty = \{[p^n, p^n_\Gamma]\}_{n=1}^\infty$ is bounded in $W^{1,2}(0,T;\mathbb{X}) \cap L^\infty(0,T;\mathbb{W})$;

($\star$6) $\{z^n\}_{n=1}^\infty$ is bounded in $W^{1,2}(0,T;V_0^*) \cap C([0,T];H) \cap \mathscr{V}_0$.

In this light, we can apply the compactness theory of Aubin's type (cf. [83, Corollary 4]) with the one-dimensional compact embeddings $V \subset C(\overline{\Omega})$ and $V_0 \subset C(\overline{\Omega})$, and can find a subsequence of $\{[\boldsymbol{p}^n, z^n]\}_{n=1}^\infty$ (not relabeled), and a limiting point $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathfrak{Y}$ with $\boldsymbol{p} = [p, p_\Gamma]$ such that:

$$\begin{aligned} \boldsymbol{p}^n \ \to \boldsymbol{p} & \text{ in } C(\overline{Q}) \times C(\overline{\Sigma}), \text{ in } \mathfrak{W}, \\ & \text{weakly in } W^{1,2}(0,T;\mathbb{X}), \text{ weakly-}* \text{ in } L^\infty(0,T;\mathbb{W}), \\ & \text{and in the pointwise sense a.e. in } Q, \end{aligned} \tag{6.7.33}$$

and

$$z^n \to z \text{ in } C([0,T];V_0^*), \text{ in } \mathscr{H}, \text{ weakly in } \mathscr{V}_0,$$

weakly in $W^{1,2}(0,T;V_0^*)$, weakly-$*$ in $L^\infty(0,T;H)$,
and in the pointwise sense a.e. in Q, as $n \to \infty$. (6.7.34)

Therefore, with (6.2.7), (6.2.8), (6.7.33), (6.7.34), and Remark 6.2 in mind, we can see that:

$$\begin{cases} \partial_t p^n + \mu^n p^n + \omega^n \partial_x z^n - h^n \\ \qquad \to \partial_t p + \mu p + \omega \partial_x z - h \text{ weakly in } \mathscr{H}, \\ \partial_t p_\Gamma^n - h_\Gamma^n \to \partial_t p_\Gamma - h_\Gamma \text{ weakly in } \mathscr{H}_\Gamma, \end{cases} \tag{6.7.35}$$

$$\begin{cases} a^n \partial_t z^n + b^n z^n - k^n \to a \partial_t z + bz - k \text{ weakly in } \mathscr{V}_0^*, \\ \bigl(A^n + \nu^2\bigr)\partial_x z^n + p^n \omega^n \\ \qquad \to \bigl(A + \nu^2\bigr)\partial_x z + p\omega \text{ weakly in } \mathscr{H}, \end{cases} \text{as } n \to \infty, \tag{6.7.36}$$

$$[\boldsymbol{p}(0), z(0)] = \lim_{n\to\infty} [\boldsymbol{p}^n(0), z^n(0)] = \lim_{n\to\infty} [\boldsymbol{p}_0^n, z_0^n] = [\boldsymbol{p}_0, z_0] \text{ in } [C(\overline{\Omega}) \times H_\Gamma] \times V_0^*, \tag{6.7.37}$$

$$\begin{aligned} &\int_s^t \Bigl(\bigl(\partial_t p^n + \mu^n p^n + \omega^n \partial_x z^n - h^n\bigr)(\varsigma), \varphi\Bigr)_H \, d\varsigma \\ &\qquad + \int_s^t \Bigl(\bigl(\partial_t p_\Gamma^n - h_\Gamma^n\bigr)(\varsigma), \varphi_\Gamma\Bigr)_{H_\Gamma} \, d\varsigma + \int_s^t \bigl(\partial_x p^n(\varsigma), \partial_x \varphi\bigr)_H \, d\varsigma = 0, \end{aligned} \tag{6.7.38a}$$

and

$$\begin{aligned} &\int_s^t \Bigl\langle \bigl(a^n \partial_t z^n + b^n z^n - k^n\bigr)(\varsigma), \psi \Bigr\rangle_{V_0} \, d\varsigma \\ &\qquad + \int_s^t \Bigl(\bigl((A^n + \nu^2)\partial_x z^n + p^n \omega^n\bigr)(\varsigma), \partial_x \psi\Bigr)_H \, d\varsigma = 0, \end{aligned} \tag{6.7.38b}$$

for all $\boldsymbol{\varphi} = [\varphi, \varphi_\Gamma] \in \mathbb{W}$, $\psi \in V_0$, $0 \le s \le t \le T$, and $n = 1, 2, 3, \dots$.

As a consequence of (6.7.35)–(6.7.38), and the arbitrary choices of $0 \le s \le t \le T$, we will verify that the limit $[\boldsymbol{p}, z] = [p, p_\Gamma, z] \in \mathfrak{Y}$ with $\boldsymbol{p} = [p, p_\Gamma]$ will be a solution to the system (P), by letting $n \to \infty$ in (6.7.38). Furthermore, on account of the convergences as in (6.7.33) and (6.7.34), and the uniqueness result in Theorem 6.1, we will conclude the required convergence (6.2.9).

Thus, the proof of Theorem 6.3 is finished. □

Chapter 7

Constrained Optimization Problems Governed by PDE Models of Grain Boundary Motions

In Chapter 7, we recall the class of optimal control problems governed by state-equations of K.W.C. models. The control is given by physical temperature. The focus is on problems in dimensions less than equal to 4. The main results are divided in four Main Theorems, concerned with: solvability and parameter-dependence of state-equations and optimal control problems; the first order necessary optimality conditions for these regularized optimal control problems. Subsequently, we derive the limiting systems and optimality conditions and study their well-posedness.

7.1 Preliminaries

We begin by prescribing the notations used throughout this Chapter.

Basic notations. For arbitrary r_0, $s_0 \in [-\infty, \infty]$, we define:

$$r_0 \vee s_0 := \max\{r_0, s_0\} \text{ and } r_0 \wedge s_0 := \min\{r_0, s_0\},$$

and in particular, we set:

$$[r]^+ := r \vee 0 \text{ and } [r]^- := -(r \wedge 0), \text{ for any } r \in \mathbb{R}.$$

For any dimension $d \in \mathbb{N}$, we denote by $\mathcal{L}^d$ the d-dimensional Lebesgue measure. The measure theoretical phrases, such as "a.e.", "dt", "dx", and so on, are all with respect to the Lebesgue measure in each corresponding dimension.

Abstract notations. For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X, and denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between X and its dual X^*. In particular, when X is a Hilbert space, we denote by $(\cdot, \cdot)_X$ the inner product of X. Moreover, when there is no possibility of confusion, we uniformly denote by $|\cdot|$ the norm of Euclidean spaces, and for any dimension $d \in \mathbb{N}$, we write the inner product (scalar product) of $\mathbb{R}^d$, as follows:

$$y \cdot \tilde{y} = \sum_{i=1}^{d} y_i \tilde{y}_i, \text{ for all } y = [y_1, \dots, y_d], \ \tilde{y} = [\tilde{y}_1, \dots, \tilde{y}_d] \in \mathbb{R}^d.$$

For any subset A of a Banach space X, let $\chi_A : X \longrightarrow \{0, 1\}$ be the characteristic function of A, i.e.:

$$\chi_A : w \in X \mapsto \chi_A(w) := \begin{cases} 1, & \text{if } w \in A, \\ 0, & \text{otherwise.} \end{cases}$$

For two Banach spaces X and Y, we denote by $\mathscr{L}(X; Y)$ the Banach space of bounded linear operators from X into Y, and in particular, we let $\mathscr{L}(X) := \mathscr{L}(X; X)$.

For Banach spaces $X_1, \dots, X_d$, with $1 < d \in \mathbb{N}$, let $X_1 \times \cdots \times X_d$ be the product Banach space endowed with the norm $|\cdot|_{X_1 \times \cdots \times X_d} := |\cdot|_{X_1} + \cdots + |\cdot|_{X_d}$. However, when all $X_1, \dots, X_d$ are Hilbert spaces, $X_1 \times \cdots \times X_d$ denotes the product Hilbert space endowed with the inner product $(\cdot, \cdot)_{X_1 \times \cdots \times X_d} := (\cdot, \cdot)_{X_1} + \cdots + (\cdot, \cdot)_{X_d}$ and the norm $|\cdot|_{X_1 \times \cdots \times X_d} := \left(|\cdot|_{X_1}^2 + \cdots + |\cdot|_{X_d}^2\right)^{\frac{1}{2}}$. In particular, when all $X_1, \dots, X_d$ coincide with a Banach space Y, we write:

$$[Y]^d := \overbrace{Y \times \cdots \times Y}^{d \text{ times}}.$$

Additionally, for any transform (operator) $\mathcal{T} : X \longrightarrow Y$, we let:

$$\mathcal{T}[w_1, \dots, w_d] := \left[\mathcal{T} w_1, \dots, \mathcal{T} w_d\right] \text{ in } [Y]^d, \quad \text{for any } [w_1, \dots, w_d] \in [X]^d.$$

Specific notations of this Chapter. As is mentioned in the previous section, let $(0, T) \subset \mathbb{R}$ be a bounded time-interval with a finite constant $T > 0$, and let $N \in \{2, 3, 4\}$ be a constant of spatial dimension. Let $\Omega \subset \mathbb{R}^N$ be a fixed spatial bounded domain with a smooth boundary $\Gamma := \partial\Omega$. We denote by n_Γ the unit outward normal vector on Γ. Besides, we set $Q := (0, T) \times \Omega$ and $\Sigma := (0, T) \times \Gamma$. Especially, we denote by ∂_t, ∇, and div the distributional time-derivative, the distributional gradient, and distributional divergence, respectively.

On this basis, we define

$$\begin{cases} H := L^2(\Omega) \text{ and } \mathscr{H} := L^2(0, T; H), \\ V := H^1(\Omega) \text{ and } \mathscr{V} := L^2(0, T; V), \\ V_0 := H_0^1(\Omega) \text{ and } \mathscr{V}_0 := L^2(0, T; V_0), \\ \mathscr{X} := L^\infty(Q) \times \mathscr{H}. \end{cases}$$

Also, we identify the Hilbert spaces H and $\mathscr{H}$ with their dual spaces. Based on the identifications, we have the following relationships of continuous embeddings:

$$\begin{cases} V \subset H = H^* \subset V^* \text{ and } \mathscr{V} \subset \mathscr{H} = \mathscr{H}^* \subset \mathscr{V}^*, \\ V_0 \subset H = H^* \subset V_0^* \text{ and } \mathscr{V}_0 \subset \mathscr{H} = \mathscr{H}^* \subset \mathscr{V}_0^*, \end{cases}$$

among the Hilbert spaces H, V, V_0, $\mathscr{H}$, $\mathscr{V}$, and $\mathscr{V}_0$, and the respective dual spaces H^*, V^*, V_0^*, $\mathscr{H}^*$, $\mathscr{V}^*$, and $\mathscr{V}_0^*$. Additionally, in this paper, we define the topology of the Hilbert space V_0 by using the following inner product:

$$(w, \tilde{w})_{V_0} := (\nabla w, \nabla \tilde{w})_{[H]^N}, \text{ for all } w, \tilde{w} \in V_0.$$

Remark 7.1. (cf. [9, Remark 3]) Due to the restriction $N \in \{2, 3, 4\}$ of spatial dimension, we can suppose the continuous embedding $V \subset L^4(\Omega)$, and we can easily check that:

(i) if $0 \leq \check{\mu} \in H$ and $\check{p} \in V$, then $\sqrt{\check{\mu}}\check{p} \in H$, $\check{\mu}\check{p} \in V^*$, and

$$\begin{cases} (\sqrt{\check{\mu}}\check{p}, \psi)_H \leq C_V^{L^4} |\check{\mu}|_H^{\frac{1}{2}} |\check{p}|_V |\psi|_H, \text{ for any } \psi \in H, \\ \langle \check{\mu}\check{p}, \check{\psi} \rangle_V = (\sqrt{\check{\mu}}\check{p}, \sqrt{\check{\mu}}\check{\psi})_H \leq (C_V^{L^4})^2 |\check{\mu}|_H |\check{p}|_V |\check{\psi}|_V, \text{ for any } \check{\psi} \in V; \end{cases}$$

(ii) if $0 \leq \hat{\mu} \in L^\infty(0, T; H)$ and $\hat{p} \in \mathscr{V}$, then $\sqrt{\hat{\mu}}\hat{p} \in \mathscr{H}$, $\hat{\mu}\hat{p} \in \mathscr{V}^*$, and

$$\begin{cases} (\sqrt{\hat{\mu}}\hat{p}, \varphi)_{\mathscr{H}} \leq C_V^{L^4} |\hat{\mu}|_{L^\infty(0,T;H)}^{\frac{1}{2}} |\hat{p}|_{\mathscr{V}} |\varphi|_{\mathscr{H}}, \text{ for any } \varphi \in \mathscr{H}, \\ \langle \hat{\mu}\hat{p}, \hat{\varphi} \rangle_{\mathscr{V}} = (\sqrt{\hat{\mu}}\hat{p}, \sqrt{\hat{\mu}}\hat{\varphi})_{\mathscr{H}} \leq (C_V^{L^4})^2 |\hat{\mu}|_{L^\infty(0,T;H)} |\hat{p}|_{\mathscr{V}} |\hat{\varphi}|_{\mathscr{V}}, \text{ for any } \hat{\varphi} \in \mathscr{V}; \end{cases}$$

where $C_V^{L^4} > 0$ is the constant of embedding $V \subset L^4(\Omega)$.

Finally, we define:

$$D := V \times V_0, \quad \text{and} \quad D_0 := \bigl(V \cap L^\infty(\Omega)\bigr) \times V_0,$$

as the notations to specify the range of the initial pair $[\eta_0, \theta_0]$ in the state-system.

Notations in convex analysis. (cf. [18, Chapter II]) Let X be an abstract Hilbert space X. Then, any closed and convex set $K \subset X$ defines a single-valued operator $\mathrm{proj}_K : X \longrightarrow K$, which maps any $w \in X$ to a point $\mathrm{proj}_K(w) \in K$, satisfying:

$$\bigl|\mathrm{proj}_K(w) - w\bigr|_X = \min \left\{ \ |\tilde{w} - w|_X \ \middle| \ \tilde{w} \in K \ \right\}.$$

The operator proj_K is called the *orthogonal projection* (or *projection* in short) onto K.

Remark 7.2 (Key-properties of the projection)**.** Let K be a closed and convex set in a Hilbert space X. Then, the following facts hold.

(Fact 1) The projection $\mathrm{proj}_K : X \longrightarrow K$ is a nonexpansive operator from X into itself, i.e.:

$$|\mathrm{proj}_K(w^1) - \mathrm{proj}_K(w^2)|_X \leq |w^1 - w^2|_X, \text{ for all } w^\ell \in X, \ell = 1, 2.$$

(Fact 2) $w_K^\circ = \mathrm{proj}_K(w)$ in X, iff. $(w - w_K^\circ, \tilde{w} - w_K^\circ)_X \leq 0$, for any $\tilde{w} \in K$.

Remark 7.3 (Examples of projections)**.** Based on Remark 7.2, we can also see the following facts.

(Fact 3) If $-\infty < r^\ell \leq s^\ell < \infty$, $\ell = 1, 2$, then the projections $\mathrm{proj}_{[r^\ell, s^\ell]} : \mathbb{R} \longrightarrow [r^\ell, s^\ell]$ onto compact intervals $[r^\ell, s^\ell] \subset \mathbb{R}$ fulfills that:

$$\left|\mathrm{proj}_{[r^1,s^1]}(\xi) - \mathrm{proj}_{[r^2,s^2]}(\xi)\right| \leq |r^1 - r^2| \vee |s^1 - s^2|, \text{ for any } \xi \in \mathbb{R}.$$

(Fact 4) Let $\mathfrak{K}$ be the class of constraints defined in (1.5.33), and let $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$ be the constraint with the obstacles $\kappa^\ell : Q \longrightarrow [-\infty, \infty]$, $\ell = 0, 1$. Then, for the projection $\mathrm{proj}_K : \mathscr{H} \longrightarrow K$, it holds that:

$$\begin{aligned}\big[\mathrm{proj}_K(u)\big](t,x) &= \mathrm{proj}_{[\kappa^0(t,x), \kappa^1(t,x)] \cap \mathbb{R}}(u(t,x)) \\ &= \big(\kappa^0 \vee (\kappa^1 \wedge u)\big)(t,x) = \begin{cases} \kappa^1(t,x), & \text{if } u(t,x) > \kappa^1(t,x), \\ u(t,x), & \text{if } \kappa^0(t,x) \leq u(t,x) \leq \kappa^1(t,x), \\ \kappa^0(t,x), & \text{if } u(t,x) < \kappa^0(t,x), \end{cases}\end{aligned}$$

a.e. $(t,x) \in Q$, for any $u \in \mathscr{H}$.

For a proper, lower semi-continuous (l.s.c.), and convex function $\Psi : X \to (-\infty, \infty]$ on a Hilbert space X, we denote by $D(\Psi)$ the effective domain of Ψ. Also, we denote by $\partial\Psi$ the subdifferential of Ψ. The subdifferential $\partial\Psi$ corresponds to a weak differential of convex function Ψ, and it is known as a maximal monotone graph in the product space $X \times X$. The set $D(\partial\Psi) := \{z \in X \mid \partial\Psi(z) \neq \emptyset\}$ is called the domain of $\partial\Psi$. We often use the notation "$[w_0, w_0^*] \in \partial\Psi$ in $X \times X$", to mean that "$w_0^* \in \partial\Psi(w_0)$ in X for $w_0 \in D(\partial\Psi)$", by identifying the operator $\partial\Psi$ with its graph in $X \times X$.

Next, for Hilbert spaces $X_1, \cdots, X_d$, with $1 < d \in \mathbb{N}$, let us consider a proper, l.s.c., and convex function on the product space $X_1 \times \cdots \times X_d$:

$$\widehat{\Psi} : w = [w_1, \cdots, w_d] \in X_1 \times \cdots \times X_d \mapsto \widehat{\Psi}(w) = \widehat{\Psi}(w_1, \cdots, w_d) \in (-\infty, \infty].$$

Besides, for any $i \in \{1, \dots, d\}$, we denote by $\partial_{w_i}\widehat{\Psi} : X_1 \times \cdots \times X_d \to X_i$ a set-valued operator, which maps any $w = [w_1, \dots, w_i, \dots, w_d] \in X_1 \times \cdots \times X_i \times \cdots \times X_d$ to a subset $\partial_{w_i}\widehat{\Psi}(w) \subset X_i$, prescribed as follows:

$$\begin{aligned}\partial_{w_i}\widehat{\Psi}(w) &= \partial_{w_i}\widehat{\Psi}(w_1, \cdots, w_i, \cdots, w_d) \\ &:= \left\{ \tilde{w}^* \in X_i \;\middle|\; \begin{array}{c} (\tilde{w}^*, \tilde{w} - w_i)_{X_i} \leq \widehat{\Psi}(w_1, \cdots, \tilde{w}, \cdots, w_d) \\ -\widehat{\Psi}(w_1, \cdots, w_i, \cdots, w_d), \text{ for any } \tilde{w} \in X_i \end{array} \right\}.\end{aligned}$$

As is easily checked,

$$\partial\widehat{\Psi} \subset \big[\partial_{w_1}\widehat{\Psi} \times \cdots \times \partial_{w_d}\widehat{\Psi}\big] \text{ in } [X_1 \times \cdots \times X_d]^2, \tag{7.1.1}$$

where $\big[\partial_{w_1}\widehat{\Psi} \times \cdots \times \partial_{w_d}\widehat{\Psi}\big] : X_1 \times \cdots \times X_d \longrightarrow 2^{X_1 \times \cdots \times X_d}$ is a set-valued operator, defined as:

$$\big[\partial_{w_1}\widehat{\Psi} \times \cdots \times \partial_{w_d}\widehat{\Psi}\big](w) := \partial_{w_1}\widehat{\Psi}(w) \times \cdots \times \partial_{w_d}\widehat{\Psi}(w) \text{ in } X_1 \times \cdots \times X_d,$$

for any $w = [w_1, \dots, w_d] \in D\big(\big[\partial_{w_1}\widehat{\Psi} \times \cdots \times \partial_{w_d}\widehat{\Psi}\big]\big) := D(\partial_{w_1}\widehat{\Psi}) \cap \cdots \cap D(\partial_{w_d}\widehat{\Psi})$.

But, it should be noted that the converse inclusion of (7.1.1) is not true, in general.

Example 7.1 (Examples of the subdifferential)**.** As one of the representatives of the subdifferentials, we exemplify the following set-valued signal function $\mathrm{Sgn}^d : \mathbb{R}^d \to 2^{\mathbb{R}^d}$, with $d \in \mathbb{N}$, which is defined as:

$$\xi = [\xi_1, \dots, \xi_d] \in \mathbb{R}^d \mapsto \mathrm{Sgn}^d(\xi) = \mathrm{Sgn}^d(\xi_1, \dots, \xi_d)$$
$$:= \begin{cases} \dfrac{\xi}{|\xi|} = \dfrac{[\xi_1, \dots, \xi_d]}{\sqrt{\xi_1^2 + \cdots + \xi_d^2}}, & \text{if } \xi \neq 0, \\ \mathbb{D}^d, & \text{otherwise}, \end{cases} \tag{7.1.2}$$

where $\mathbb{D}^d$ denotes the closed unit ball in $\mathbb{R}^d$ centered at the origin. Indeed, the set-valued function Sgn^d coincides with the subdifferential of the Euclidean norm $|\cdot| : \xi \in \mathbb{R}^d \mapsto |\xi| = \sqrt{\xi_1^2 + \cdots + \xi_d^2} \in [0, \infty)$, i.e.:

$$\partial|\cdot|(\xi) = \mathrm{Sgn}^d(\xi), \text{ for any } \xi \in D(\partial|\cdot|) = \mathbb{R}^d,$$

and furthermore, it is observed that:

$$\partial|\cdot|(0) = \mathbb{D}^d \subsetneq [-1, 1]^d = \big[\partial_{\xi_1}|\cdot| \times \cdots \times \partial_{\xi_d}|\cdot|\big](0).$$

Example 7.2. Let $d \in \mathbb{N}$ be the constant of dimension. For any $\varepsilon \geq 0$, let $f_\varepsilon : \mathbb{R}^d \longrightarrow [0, \infty)$ be a continuous and convex function, defined as:

$$f_\varepsilon : y \in \mathbb{R}^d \mapsto f_\varepsilon(y) := \sqrt{\varepsilon^2 + |y|^2} \in [0, \infty). \tag{7.1.3}$$

When $\varepsilon = 0$, the convex function f_0 of this case coincides with the d-dimensional Euclidean norm $|\cdot|$, and hence, the subdifferential ∂f_0 coincides with the set valued signal function $\mathrm{Sgn}^d : \mathbb{R}^d \longrightarrow 2^{\mathbb{R}^d}$, defined in (7.1.2).

In the meantime, when $\varepsilon > 0$, the convex function f_ε belongs to C^∞-class, and the subdifferential ∂f_ε is identified with the (single-valued) usual gradient:

$$\nabla f_\varepsilon : y \in \mathbb{R}^d \mapsto \nabla f_\varepsilon(y) = \frac{y}{\sqrt{\varepsilon^2 + |y|^2}} \in \mathbb{R}^d.$$

Moreover, since:

$$f_\varepsilon(y) = \big|[\varepsilon, y]\big|_{\mathbb{R}^{d+1}} = \big|[\varepsilon, y_1, \dots, y_d]\big|_{\mathbb{R}^{d+1}}, \text{ for all } [\varepsilon, y] = [\varepsilon, y_1, \dots, y_d] \in \mathbb{R}^{d+1},$$
$$\text{with } \varepsilon \geq 0 \text{ and } y = [y_1, \dots, y_d] \in \mathbb{R}^d,$$

it will be estimated that:

$$|f_\varepsilon(y) - f_{\tilde{\varepsilon}}(\tilde{y})| \leq \big|[\varepsilon, y] - [\tilde{\varepsilon}, \tilde{y}]\big|_{\mathbb{R}^{d+1}} \leq |\varepsilon - \tilde{\varepsilon}| + |y - \tilde{y}|_{\mathbb{R}^d},$$
$$\text{for all } \varepsilon, \tilde{\varepsilon} \geq 0 \text{ and } y, \tilde{y} \in \mathbb{R}^d, \tag{7.1.4a}$$

$$\begin{cases} \big|\nabla f_\varepsilon(y)\big|_{\mathbb{R}^d} = \left|\dfrac{y}{\big|[\varepsilon, y]\big|_{\mathbb{R}^{d+1}}}\right|_{\mathbb{R}^d} \leq \left|\dfrac{[\varepsilon, y]}{\big|[\varepsilon, y]\big|_{\mathbb{R}^{d+1}}}\right|_{\mathbb{R}^{d+1}} = 1, \\ \big|\nabla f_\varepsilon(y) - \nabla f_{\tilde{\varepsilon}}(\tilde{y})\big|_{\mathbb{R}^d} \leq \left|\dfrac{[\varepsilon, y]}{\big|[\varepsilon, y]\big|_{\mathbb{R}^{d+1}}} - \dfrac{[\tilde{\varepsilon}, \tilde{y}]}{\big|[\tilde{\varepsilon}, \tilde{y}]\big|_{\mathbb{R}^{d+1}}}\right|_{\mathbb{R}^{d+1}} \\ \qquad \leq \dfrac{2}{\varepsilon \wedge \tilde{\varepsilon}}\big(|\varepsilon - \tilde{\varepsilon}| + |y - \tilde{y}|_{\mathbb{R}^d}\big), \text{ for all } \varepsilon, \tilde{\varepsilon} > 0 \text{ and } y, \tilde{y} \in \mathbb{R}^d, \end{cases} \tag{7.1.4b}$$

and

$$
\begin{cases}
|\nabla^2 f_\varepsilon(y)|_{\mathbb{R}^{d\times d}} \leq \left|\nabla^2\big[|\cdot|_{\mathbb{R}^{d+1}}\big]([\varepsilon, y])\right|_{\mathbb{R}^{(d+1)\times(d+1)}} \leq \dfrac{d+1}{\varepsilon}, \\
\left|\nabla^2 f_\varepsilon(y) - \nabla^2 f_{\tilde{\varepsilon}}(\tilde{y})\right|_{\mathbb{R}^{d\times d}} \\
\qquad \leq \left|\nabla^2\big[|\cdot|_{\mathbb{R}^{d+1}}\big]([\varepsilon, y]) - \nabla^2\big[|\cdot|_{\mathbb{R}^{d+1}}\big]([\tilde{\varepsilon}, \tilde{y}])\right|_{\mathbb{R}^{(d+1)\times(d+1)}} \\
\qquad \leq \dfrac{3(d+1)^2}{(\varepsilon \wedge \tilde{\varepsilon})^2}\big(|\varepsilon - \tilde{\varepsilon}| + |y - \tilde{y}|_{\mathbb{R}^d}\big), \text{ for all } \varepsilon, \tilde{\varepsilon} > 0 \text{ and } y, \tilde{y} \in \mathbb{R}^d.
\end{cases}
\tag{7.1.4c}
$$

Finally, we mention about a notion of functional convergence, known as "Mosco-convergence".

Definition 7.1 (Mosco-convergence: cf. [59])**.** Let X be an abstract Hilbert space. Let $\Psi : X \to (-\infty, \infty]$ be a proper, l.s.c., and convex function, and let $\{\Psi_n\}_{n=1}^\infty$ be a sequence of proper, l.s.c., and convex functions $\Psi_n : X \to (-\infty, \infty]$, $n = 1, 2, 3, \dots$. Then, it is said that $\Psi_n \to \Psi$ on X, in the sense of Mosco, as $n \to \infty$, iff. the following two conditions are fulfilled:

(M1) The condition of lower-bound: $\varliminf_{n\to\infty} \Psi_n(\check{w}_n) \geq \Psi(\check{w})$, if $\check{w} \in X$, $\{\check{w}_n\}_{n=1}^\infty \subset X$, and $\check{w}_n \to \check{w}$ weakly in X, as $n \to \infty$.

(M2) The condition of optimality: for any $\hat{w} \in D(\Psi)$, there exists a sequence $\{\hat{w}_n\}_{n=1}^\infty \subset X$ such that $\hat{w}_n \to \hat{w}$ in X and $\Psi_n(\hat{w}_n) \to \Psi(\hat{w})$, as $n \to \infty$.

As well as, if the sequence of convex functions $\{\widehat{\Psi}_\varepsilon\}_{\varepsilon\in\Xi}$ is labeled by a continuous argument $\varepsilon \in \Xi$ with a range $\Xi \subset \mathbb{R}$, then for any $\varepsilon_0 \in \Xi$, the Mosco-convergence of $\{\widehat{\Psi}_\varepsilon\}_{\varepsilon\in\Xi}$, as $\varepsilon \to \varepsilon_0$, is defined by those of subsequences $\{\widehat{\Psi}_{\varepsilon_n}\}_{n=1}^\infty$, for all sequences $\{\varepsilon_n\}_{n=1}^\infty \subset \Xi$, satisfying $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$.

Remark 7.4. Let X, Ψ, and $\{\Psi_n\}_{n=1}^\infty$ be as in Definition 7.1. Then, the following hold.

(Fact 5) (cf. [10, Theorem 3.66] and [39, Chapter 2]) Let us assume that

$$\Psi_n \to \Psi \text{ on } X, \text{ in the sense of Mosco, as } n \to \infty, \tag{7.1.5}$$

and

$$
\begin{cases}
[w, w^*] \in X \times X, \quad [w_n, w_n^*] \in \partial\Psi_n \text{ in } X \times X, \, n \in \mathbb{N}, \\
w_n \to w \text{ in } X \text{ and } w_n^* \to w^* \text{ weakly in } X, \text{ as } n \to \infty.
\end{cases}
$$

Then, it holds that:

$$[w, w^*] \in \partial\Psi \text{ in } X \times X, \text{ and } \Psi_n(w_n) \to \Psi(w), \text{ as } n \to \infty.$$

(Fact 6) (cf. [22, Lemma 4.1] and [30, Appendix]) Let $d \in \mathbb{N}$ denote dimension constant, and let $S \subset \mathbb{R}^d$ be a bounded open set. Then, under the Mosco-convergence as in (7.1.5), a sequence $\{\widehat{\Psi}_n^S\}_{n=1}^\infty$ of proper, l.s.c., and convex functions on $L^2(S; X)$, defined as:

$$
w \in L^2(S; X) \mapsto \widehat{\Psi}_n^S(w) := \begin{cases}
\displaystyle\int_S \Psi_n(w(t))\, dt, \\
\qquad \text{if } \Psi_n(w) \in L^1(S), \quad \text{for } n = 1, 2, 3, \dots; \\
\infty, \quad \text{otherwise},
\end{cases}
$$

converges to a proper, l.s.c., and convex function $\widehat{\Psi}^S$ on $L^2(S;X)$, defined as:

$$z \in L^2(S;X) \mapsto \widehat{\Psi}^S(z) := \begin{cases} \displaystyle\int_S \Psi(z(t))\,dt, & \text{if } \Psi(z) \in L^1(S), \\ \infty, & \text{otherwise}; \end{cases}$$

on $L^2(S;X)$, in the sense of Mosco, as $n \to \infty$.

Example 7.3 (Example of Mosco-convergence)**.** Let $d \in \mathbb{N}$ be the constant of dimension, and let $\{f_\varepsilon\}_{\varepsilon \geq 0} \subset C(\mathbb{R}^d)$ be the sequence of nonexpansive convex functions, as in (7.1.3) and (7.1.4). Then, the uniform estimate (7.1.4a) immediately leads to:

$$f_\varepsilon \to f_{\varepsilon_0} \text{ on } \mathbb{R}^d, \text{ in the sense of Mosco, as } \varepsilon \to \varepsilon_0, \text{ for any } \varepsilon_0 \geq 0.$$

7.2 Auxiliary results

In this Section, we prepare some auxiliary results for our study. The auxiliary results are stated in the following two Subsections.

§ 5.1.1 Abstract theory for the state-system $(\mathrm{S})_\varepsilon$;

§ 5.1.2 Mathematical theory for the linearized system of $(\mathrm{S})_\varepsilon$.

7.2.1 Abstract theory for the state-system $(\mathrm{S})_\varepsilon$

In this Subsection, we refer to [7, Appendix] to overview the abstract theory of nonlinear evolution equation in an abstract Hilbert space X, which enables us to handle the state-systems $(\mathrm{S})_\varepsilon$, for all $\varepsilon \geq 0$, in a unified fashion.

The general theory consists of the following two Propositions.

Proposition 7.1 (cf. [7, Lemma 8.1])**.** Let $\{\mathcal{A}_0(t) \,|\, t \in [0,T]\} \subset \mathscr{L}(X)$ be a class of time-dependent bounded linear operators, let $\mathcal{G}_0 : X \longrightarrow X$ be a given nonlinear operator, and let $\Psi_0 : X \longrightarrow [0,\infty]$ be a non-negative, proper, l.s.c., and convex function, fulfilling the following conditions:

(cp.0) $\mathcal{A}_0(t) \in \mathscr{L}(X)$ is positive and selfadjoint, for any $t \in [0,T]$, and it holds that

$$(\mathcal{A}_0(t)w, w)_X \geq \kappa_0 |w|_X^2, \text{ for any } w \in X,$$

with some constant $\kappa_0 \in (0,1)$, independent of $t \in [0,T]$ and $w \in X$.

(cp.1) $\mathcal{A}_0 : [0,T] \longrightarrow \mathscr{L}(X)$ is Lipschitz continuous, so that $\mathcal{A}_0$ admits the (strong) time-derivative $\mathcal{A}_0'(t) \in \mathscr{L}(X)$ a.e. in $(0,T)$, and

$$A_T^* := \operatorname*{ess\,sup}_{t \in (0,T)} \left\{ \max\{ |\mathcal{A}_0(t)|_{\mathscr{L}(X)}, |\mathcal{A}_0'(t)|_{\mathscr{L}(X)} \} \right\} < \infty;$$

(cp.2) $\mathcal{G}_0 : X \longrightarrow X$ is a Lipschitz continuous operator, and $\mathcal{G}_0$ has a C^1-potential functional $\widehat{\mathcal{G}}_0 : X \longrightarrow \mathbb{R}$, so that the Gâteaux derivative $\widehat{\mathcal{G}}_0'(w) \in X^* \,(= X)$ at any $w \in X$ coincides with $\mathcal{G}_0(w) \in X$;

(cp.3) $\Psi_0 \geq 0$ on X, and the sublevel set $\{w \in X \,|\, \Psi_0(w) \leq r\}$ is compact in X, for any $r \geq 0$.

Then, for any initial data $w_0 \in D(\Psi_0)$ and a forcing term $\mathfrak{f}_0 \in L^2(0,T;X)$, the following Cauchy problem of evolution equation:

$$\text{(CP)} \quad \begin{cases} \mathcal{A}_0(t)w'(t) + \partial\Psi_0(w(t)) + \mathcal{G}_0(w(t)) \ni \mathfrak{f}_0(t) \text{ in } X, \quad t \in (0,T), \\ w(0) = w_0 \text{ in } X; \end{cases}$$

admits a unique solution $w \in L^2(0,T;X)$, in the sense that:

$$w \in W^{1,2}(0,T;X), \;\; \Psi_0(w) \in L^\infty(0,T),$$

and

$$\bigl(\mathcal{A}_0(t)w'(t) + \mathcal{G}_0(w(t)) - \mathfrak{f}_0(t), w(t) - \varpi\bigr)_X + \Psi_0(w(t)) \leq \Psi_0(\varpi),$$
$$\text{for any } \varpi \in D(\Psi_0), \text{ a.e. } t \in (0,T).$$

Moreover, both $t \in [0,T] \mapsto \Psi_0(w(t)) \in [0,\infty)$ and $t \in [0,T] \mapsto \widehat{\mathcal{G}}_0(w(t)) \in \mathbb{R}$ are absolutely continuous functions in time, and

$$|\mathcal{A}_0(t)^{\frac{1}{2}} w'(t)|_X^2 + \frac{d}{dt}\Bigl(\Psi_0(w(t)) + \widehat{\mathcal{G}}_0(w(t))\Bigr) = (\mathfrak{f}_0(t), w'(t))_X,$$
$$\text{for a.e. } t \in (0,T).$$

Proposition 7.2 (cf. [7, Lemma 8.2])**.** Under the notations $\mathcal{A}_0$, $\mathcal{G}_0$, Ψ_0, and assumptions (cp.0)–(cp.3), as in the previous Proposition 7.1, let us fix $w_0 \in D(\Psi_0)$ and $\mathfrak{f}_0 \in L^2(0,T;X)$, and take the unique solution $w \in L^2(0,T;X)$ to the Cauchy problem (CP). Let $\{\Psi_n\}_{n=1}^\infty$, $\{w_{0,n}\}_{n=1}^\infty$, and $\{\mathfrak{f}_n\}_{n=1}^\infty$ be, respectively, a sequence of proper, l.s.c., and convex functions on X, a sequence of initial data in X, and a sequence of forcing terms in $L^2(0,T;X)$, such that:

(cp.4) $\Psi_n \geq 0$ on X, for $n = 1,2,3,\ldots$, and the union $\bigcup_{n=1}^\infty \{w \in X \,|\, \Psi_n(w) \leq r\}$ of sublevel sets is relatively compact in X, for any $r \geq 0$;

(cp.5) Ψ_n converges to Ψ_0 on X, in the sense of Mosco, as $n \to \infty$;

(cp.6) $\sup_{n\in\mathbb{N}} \Psi_n(w_{0,n}) < \infty$, and $w_{0,n} \to w_0$ in X, as $n \to \infty$;

(cp.7) $\mathfrak{f}_n \to \mathfrak{f}_0$ weakly in $L^2(0,T;X)$, as $n \to \infty$.

For any $n \in \mathbb{N}$, let $w_n \in L^2(0,T;X)$ be the solution to the Cauchy problem (CP), for the initial data $w_{0,n} \in D(\Psi_n)$ and forcing term $\mathfrak{f}_n \in L^2(0,T;X)$. Then,

$$w_n \to w \text{ in } C([0,T];X), \text{ weakly in } W^{1,2}(0,T;X),$$
$$\int_0^T \Psi_n(w_n(t))\,dt \to \int_0^T \Psi_0(w(t))\,dt, \text{ as } n \to \infty,$$

and

$$\bigl|\Psi_0(w)\bigr|_{C([0,T])} \leq \sup_{n\in\mathbb{N}} \bigl|\Psi_n(w_n)\bigr|_{C([0,T])} < \infty.$$

In this paper, the readers are recommended to see [7, Appendix] for the detailed proofs of the above Propositions 7.1 and 7.2. Roughly summarized, these Propositions can be obtained by means of modified (mixed and reduced) methods of the existing theories, such as [14, 18, 39].

7.2.2 Mathematical theory for the linearized system of $(\mathrm{S})_\varepsilon$

In this Subsection, we recall the previous work [9], and set up some auxiliary results. In what follows, we let $\mathscr{Y} := \mathscr{V} \times \mathscr{V}_0$, with the dual $\mathscr{Y}^* := \mathscr{V}^* \times \mathscr{V}_0^*$. Note that $\mathscr{Y}$ is a Hilbert space which is endowed with a uniform convex topology, based on the inner product for product space, as in the Preliminaries (see the paragraph of Abstract notations).

Besides, we define:

$$\mathscr{Z} := \bigl(W^{1,2}(0,T;V^*) \cap \mathscr{V}\bigr) \times \bigl(W^{1,2}(0,T;V_0^*) \cap \mathscr{V}_0\bigr),$$

as a Banach space, endowed with the norm:

$$|[\tilde{p},\tilde{z}]|_{\mathscr{Z}} := |[\tilde{p},\tilde{z}]|_{[C([0,T];H)]^2} + \bigl(|[\tilde{p},\tilde{z}]|_{\mathscr{Y}}^2 + |[\partial_t\tilde{p},\partial_t\tilde{z}]|_{\mathscr{Y}^*}^2\bigr)^{\frac{1}{2}}, \text{ for } [\tilde{p},\tilde{z}] \in \mathscr{Z}.$$

Based on this, let us consider the following linear system of parabolic initial-boundary value problem, denoted by (P):

(P)

$$\begin{cases} \partial_t p - \Delta p + \mu(t,x)p + \lambda(t,x)p + \omega(t,x)\cdot\nabla z = h(t,x),\ (t,x) \in Q, \\ \nabla p(t,x)\cdot n_\Gamma = 0,\ (t,x) \in \Sigma, \\ p(0,x) = p_0(x),\ x \in \Omega; \end{cases}$$

$$\begin{cases} a(t,x)\partial_t z + b(t,x)z - \mathrm{div}\bigl(A(t,x)\nabla z + \nu^2\nabla z + \omega(t,x)p\bigr) \\ \qquad = k(t,x),\ (t,x) \in Q, \\ z(t,x) = 0,\ (t,x) \in \Sigma, \\ z(0,x) = z_0(x),\ x \in \Omega. \end{cases}$$

This system is studied in [9] as a key-problem for the Gâteaux differential of the cost $\mathcal{J}_\varepsilon$ for $\varepsilon > 0$. In the context, $[a,b,\mu,\lambda,\omega,A] \in [\mathscr{H}]^6$ is a given sextuplet of functions which belongs to a subclass $\mathscr{S} \subset [\mathscr{H}]^6$, defined as:

$$\mathscr{S} := \left\{ [\tilde{a},\tilde{b},\tilde{\mu},\tilde{\lambda},\tilde{\omega},\tilde{A}] \in [\mathscr{H}]^6 \;\middle|\; \begin{array}{l} \bullet\ \tilde{a} \in W^{1,\infty}(Q) \text{ and } \log\tilde{a} \in L^\infty(Q), \\ \bullet\ [\tilde{b},\tilde{\lambda}] \in [L^\infty(Q)]^2, \\ \bullet\ \tilde{\mu} \in L^\infty(0,T;H) \text{ with } \tilde{\mu} \geq 0 \text{ a.e. in } Q, \\ \bullet\ \tilde{\omega} \in [L^\infty(Q)]^N, \\ \bullet\ \tilde{A} \in [L^\infty(Q)]^{N\times N}, \text{ and the value } \\ \quad \tilde{A}(t,x) \in \mathbb{R}^{N\times N} \text{ is positive and sym-} \\ \quad \text{metric matrix, for a.e. } (t,x) \in Q \end{array} \right\}. \tag{7.2.1}$$

Also, $[p_0,z_0] \in [H]^2$ and $[h,k] \in \mathscr{Y}^*$ are, respectively, an initial pair and forcing pair, in the system (P).

Now, we refer to the previous work [9], to recall the key-properties of the system (P), in forms of Propositions.

Proposition 7.3 (cf. [9, Main Theorem 1 (I-A)]). For any sextuplet $[a, b, \mu, \lambda, \omega, A] \in \mathscr{S}$, any initial pair $[p_0, z_0] \in [H]^2$, and any forcing pair $[h, k] \in \mathscr{Y}^*$, the system (P) admits a unique solution, in the sense that:

$$\begin{cases} p \in W^{1,2}(0, T; V^*) \cap L^2(0, T; V) \subset C([0, T]; H), \\ z \in W^{1,2}(0, T; V_0^*) \cap L^2(0, T; V_0) \subset C([0, T]; H); \end{cases} \tag{7.2.2}$$

$$\begin{aligned} &\langle \partial_t p(t), \varphi \rangle_V + (\nabla p(t), \nabla \varphi)_{[H]^N} + \langle \mu(t) p(t), \varphi \rangle_V \\ &\qquad + (\lambda(t) p(t) + \omega(t) \cdot \nabla z(t), \varphi)_H = \langle h(t), \varphi \rangle_V, \\ &\qquad \text{for any } \varphi \in V, \text{ a.e. } t \in (0, T), \text{ subject to } p(0) = p_0 \text{ in } H; \end{aligned} \tag{7.2.3}$$

and

$$\begin{aligned} &\langle \partial_t z(t), a(t) \psi \rangle_{V_0} + (b(t) z(t), \psi)_H \\ &\qquad + \bigl(A(t) \nabla z(t) + \nu^2 \nabla z(t) + p(t) \omega(t), \nabla \psi \bigr)_{[H]^N} = \langle k(t), \psi \rangle_{V_0}, \\ &\qquad \text{for any } \psi \in V_0, \text{ a.e. } t \in (0, T), \text{ subject to } z(0) = z_0 \text{ in } H. \end{aligned} \tag{7.2.4}$$

Proposition 7.4 (cf. [9, Main Theorem 1 (I-B)]). For each $\ell \in \{1, 2\}$, let us take arbitrary $[a^\ell, b^\ell, \mu^\ell, \lambda^\ell, \omega^\ell, A^\ell] \in \mathscr{S}$, $[p_0^\ell, z_0^\ell] \in [H]^2$, and $[h^\ell, k^\ell] \in \mathscr{Y}^*$, and let us denote by $[p^\ell, z^\ell] \in [\mathscr{H}]^2$ the solution to (P), corresponding to the sextuplet $[a^\ell, b^\ell, \mu^\ell, \lambda^\ell, \omega^\ell, A^\ell]$, initial pair $[p_0^\ell, z_0^\ell]$, and forcing pair $[h^\ell, k^\ell]$. Besides, let $C_0^* = C_0^*(a^1, b^1, \lambda^1, \omega^1)$ be a positive constant, depending on a^1, b^1, λ^1, and ω^1, which is defined as:

$$\begin{aligned} C_0^* :=& \frac{9(1 + \nu^2)}{1 \wedge \nu^2 \wedge \inf a^1(Q)} \cdot \bigl(1 + (C_V^{L^4})^2 + (C_V^{L^4})^4 + (C_{V_0}^{L^4})^2 \bigr) \\ &\cdot \bigl(1 + |a^1|_{W^{1,\infty}(Q)} + |b^1|_{L^\infty(Q)} + |\lambda^1|_{L^\infty(Q)} + |\omega^1|^2_{[L^\infty(Q)]^N} \bigr), \end{aligned} \tag{7.2.5}$$

with use of the constants $C_V^{L^4} > 0$ and $C_{V_0}^{L^4} > 0$ of the respective embeddings $V \subset L^4(\Omega)$ and $V_0 \subset L^4(\Omega)$. Then, it is estimated that:

$$\begin{aligned} &\frac{d}{dt} \bigl(|(p^1 - p^2)(t)|_H^2 + |\sqrt{a^1(t)} (z^1 - z^2)(t)|_H^2 \bigr) \\ &\qquad + \bigl(|(p^1 - p^2)(t)|_V^2 + \nu^2 |(z^1 - z^2)(t)|_{V_0}^2 \bigr) \\ \leq& 3C_0^* \bigl(|(p^1 - p^2)(t)|_H^2 + |\sqrt{a^1(t)} (z^1 - z^2)(t)|_H^2 \bigr) \\ &\qquad + 2C_0^* \bigl(|(h^1 - h^2)(t)|_{V^*}^2 + |(k^1 - k^2)(t)|_{V_0^*}^2 + R_0^*(t) \bigr), \\ &\qquad \text{for a.e. } t \in (0, T); \end{aligned} \tag{7.2.6}$$

where

$$\begin{aligned} R_0^*(t) :=& |\partial_t z^2(t)|_{V_0^*}^2 \bigl(|a^1 - a^2|^2_{C(\overline{Q})} + |\nabla (a^1 - a^2)(t)|^2_{[L^4(\Omega)]^N} \bigr) \\ &+ |p^2(t)|_V^2 \bigl(|(\mu^1 - \mu^2)(t)|_H^2 + |(\omega^1 - \omega^2)(t)|^2_{[L^4(\Omega)]^N} \bigr) \\ &+ |z^2(t)|_{V_0}^2 \bigl(|(b^1 - b^2)(t)|^2_{L^4(\Omega)} + |p^2(t)(\lambda^1 - \lambda^2)(t)|_H^2 \bigr) \\ &+ |\nabla z^2(t) (\omega^1 - \omega^2)(t)|_H^2 + |(A^1 - A^2)(t) \nabla z^2(t)|^2_{[H]^N}, \\ &\text{for a.e. } t \in (0, T). \end{aligned}$$

Proposition 7.5 (cf. [9, Corollary 1])**.** For any $[a, b, \mu, \lambda, \omega, A] \in \mathscr{S}$, let us denote by $\mathcal{P} = \mathcal{P}(a, b, \mu, \lambda, \omega, A) : [H]^2 \times \mathscr{Y}^* \longrightarrow \mathscr{Z}$ a linear operator, which maps any pair of data $\bigl[[p_0, z_0], [h, k]\bigr] \in [H]^2 \times \mathscr{Y}^*$ to the solution $[p, z] \in \mathscr{Z}$ to the corresponding linear system (P), for the sextuplet $[a, b, \mu, \lambda, \omega, A]$, initial pair $[p_0, z_0]$, and forcing pair $[h, k]$. Then, for any sextuplet $[a, b, \mu, \lambda, \omega, A] \in \mathscr{S}$, there exist positive constants $M_0^* = M_0^*(a, b, \mu, \lambda, \omega, A)$ and $M_1^* = M_1^*(a, b, \mu, \lambda, \omega, A)$, depending on a, b, μ, λ, ω, and A, such that:

$$\begin{aligned} M_0^* \bigl|\bigl[[p_0, z_0], [h, k]\bigr]\bigr|_{[H]^2 \times \mathscr{Y}^*} \leq |[p, z]|_{\mathscr{Z}} \leq M_1^* \bigl|\bigl[[p_0, z_0], [h, k]\bigr]\bigr|_{[H]^2 \times \mathscr{Y}^*}, \\ \text{for all } [p_0, z_0] \in [H]^2, \ [h, k] \in \mathscr{Y}^*, \\ \text{and } [p, z] = \mathcal{P}(a, b, \mu, \lambda, \omega, A)\bigl[[p_0, z_0], [h, k]\bigr] \in \mathscr{Z}, \end{aligned}$$

i.e. the operator $\mathcal{P} = \mathcal{P}(a, b, \mu, \lambda, \omega, A)$ is an isomorphism between the Hilbert space $[H]^2 \times \mathscr{Y}^*$ and the Banach space $\mathscr{Z}$.

Proposition 7.6 (cf. [9, Corollary 2])**.** Let us assume:

$$[a, b, \mu, \lambda, \omega, A] \in \mathscr{S}, \ \{[a_n, b_n, \mu_n, \lambda_n, \omega_n, A_n]\}_{n=1}^{\infty} \subset \mathscr{S},$$

$$\begin{aligned} &[a_n, \partial_t a_n, \nabla a_n, b_n, \lambda_n, \omega_n, A_n] \to [a, \partial_t a, \nabla a, b, \lambda, \omega, A] \text{ weakly-}* \text{ in} \\ &\quad L^\infty(Q) \times L^\infty(Q) \times [L^\infty(Q)]^N \times L^\infty(Q) \times L^\infty(Q) \times [L^\infty(Q)]^N \times [L^\infty(Q)]^{N \times N}, \\ &\quad \text{and in the pointwise sense a.e. in } Q, \text{ as } n \to \infty, \end{aligned} \tag{7.2.7}$$

and

$$\begin{cases} \mu_n \to \mu \text{ weakly-}* \text{ in } L^\infty(0, T; H), \\ \mu_n(t) \to \mu(t) \text{ in } H, \text{ for a.e. } t \in (0, T), \end{cases} \quad \text{as } n \to \infty.$$

Let us assume $[p_0, z_0] \in [H]^2$, $[h, k] \in \mathscr{Y}^*$, and let us denote by $[p, z] \in [\mathscr{H}]^2$ the solution to (P), for the initial pair $[p_0, z_0]$ and forcing pair $[h, k]$. Also, let us assume $\{[p_{0,n}, z_{0,n}]\}_{n=1}^{\infty} \subset [H]^2$, $\{[h_n, k_n]\}_{n=1}^{\infty} \subset \mathscr{Y}^*$, and for any $n \in \mathbb{N}$, let us denote by $[p_n, z_n] \in [\mathscr{H}]^2$ the solution to (P), for the sextuplet $[a_n, b_n, \mu_n, \lambda_n, \omega_n, A_n] \in \mathscr{S}$, initial pair $[p_{0,n}, z_{0,n}]$ and forcing pair $[h_n, k_n]$. Then, the following two items hold.

(A) The convergence:

$$\begin{cases} [p_{0,n}, z_{0,n}] \to [p_0, z_0] \text{ in } [H]^2, \\ [h_n, k_n] \to [h, k] \text{ in } \mathscr{Y}^*, \end{cases} \quad \text{as } n \to \infty,$$

implies the convergence:

$$[p_n, z_n] \to [p, z] \text{ in } [C([0, T]; H)]^2, \text{ and in } \mathscr{Y}, \text{ as } n \to \infty.$$

(B) The following two convergences:

$$\begin{cases} [p_{0,n}, z_{0,n}] \to [p_0, z_0] \text{ weakly in } [H]^2, \\ [h_n, k_n] \to [h, k] \text{ weakly in } \mathscr{Y}^*, \end{cases} \quad \text{as } n \to \infty,$$

and

$$[p_n, z_n] \to [p, z] \text{ in } [\mathscr{H}]^2, \text{ weakly in } \mathscr{Y}, \\ \text{and weakly in } W^{1,2}(0,T;V^*) \times W^{1,2}(0,T;V_0^*), \text{ as } n \to \infty,$$

are equivalent each other.

Remark 7.5. In the previous work [9], one of the essential requirements is to use the continuous embedding $V \subset L^4(\Omega)$, as in Remark 7.1, which is satisfied under the restriction $N \leq 4$ of the spatial dimension $N \in \mathbb{N}$. Therefore, under the assumption $N \in \{2,3,4\}$ of this paper, the Propositions 7.3–7.6 will be applicable, although the previous results as in [9] were obtained under strict assumption $N \in \{1,2,3\}$.

Finally, we recall an auxiliary result, which was indirectly obtained in the proof of [9, Key-Lemma 2].

Lemma 7.1. Let us assume that $\hat{\mu} \in L^\infty(0,T;H)$, $\{\hat{\mu}_n\}_{n=1}^\infty \subset L^\infty(0,T;H)$, $\hat{p} \in \mathscr{V}$, $\{\hat{p}_n\}_{n=1}^\infty \subset \mathscr{V}$,

$$\hat{\mu} \geq 0 \text{ and } \hat{\mu}_n \geq 0, \text{ a.e. in } Q,\ n = 1,2,3,\ldots, \tag{7.2.8a}$$

$$\begin{cases} \hat{\mu}_n \to \hat{\mu} \text{ weakly-}* \text{ in } L^\infty(0,T;H), \\ \hat{\mu}_n(t) \to \hat{\mu}(t) \text{ in } H, \text{ for a.e. } t \in (0,T), \end{cases} \tag{7.2.8b}$$

and

$$\hat{p}_n \to \hat{p} \text{ in } \mathscr{H}, \text{ and weakly in } \mathscr{V}, \text{ as } n \to \infty. \tag{7.2.8c}$$

Then, it holds that:

$$\hat{\mu}_n \hat{p}_n \to \hat{\mu}\hat{p} \text{ weakly in } \mathscr{V}^*, \text{ as } n \to \infty. \tag{7.2.9}$$

Proof. From (7.2.8) and Remark 7.1, we can see that:

$$\sup_{n\in\mathbb{N}} |\hat{\mu}_n \hat{p}_n|_{\mathscr{V}^*} \leq (C_V^{L^4})^2 \sup_{n\in\mathbb{N}} |\hat{\mu}_n|_{L^\infty(0,T;H)} |\hat{p}_n|_{\mathscr{V}} < \infty, \tag{7.2.10}$$

with the use of the constant $C_V^{L^4} > 0$ of embedding $V \subset L^4(\Omega)$. This implies that:

($\star$0) the sequence $\{\hat{\mu}_n \hat{p}_n\}_{n=1}^\infty$ is weakly compact in $\mathscr{V}^*$.

Also, with (7.2.8b) and the dominated convergence theorem [56, Theorem 10 on page 36] in mind, we can derive that:

$$\hat{\mu}_n \to \hat{\mu} \text{ in } \mathscr{H}, \text{ as } n \to \infty. \tag{7.2.11}$$

Now, on the basis of ($\star$0), let us take any $\hat{q}^* \in \mathscr{V}^*$, such that $\hat{q}^* \in \mathscr{V}^*$ is a weak limit of a subsequence of $\{\hat{\mu}_n \hat{p}_n\}_{n=1}^\infty$ (not relabeled), i.e.:

$$\hat{\mu}_n \hat{p}_n \to \hat{q}^* \text{ weakly in } \mathscr{V}^*, \text{ as } n \to \infty. \tag{7.2.12}$$

Besides, by taking subsequences if necessary, (7.2.8c) and (7.2.11) enable us to say that:

$$\hat{\mu}_n \to \hat{\mu} \text{ and } \hat{p}_n \to \hat{p} \text{ in the pointwise sense,}$$

$$\text{a.e. in } Q, \text{ as } n \to \infty. \tag{7.2.13}$$

Additionally, by (7.2.8) and Remark 7.1, we can compute that:

$$\begin{aligned} &\left|\sqrt{\hat{\mu}_n(t)}\hat{\varphi}(t) - \sqrt{\hat{\mu}(t)}\hat{\varphi}(t)\right|_H^2 \leq \left|\sqrt{|\hat{\mu}_n - \hat{\mu}|(t)}\,\hat{\varphi}(t)\right|_H^2 \\ &\qquad \leq (C_V^{L^4})^2 |(\hat{\mu}_n - \hat{\mu})(t)|_H |\hat{\varphi}(t)|_V^2 \to 0, \text{ as } n \to \infty, \end{aligned} \tag{7.2.14a}$$

$$\begin{aligned} &\sup_{n\in\mathbb{N}}\left|\sqrt{\hat{\mu}_n(t)}\hat{\varphi}(t) - \sqrt{\hat{\mu}(t)}\hat{\varphi}(t)\right|_H^2 \\ &\qquad \leq (C_V^{L^4})^2 \sup_{n\in\mathbb{N}} |(\hat{\mu}_n - \hat{\mu})(t)|_H |\hat{\varphi}(t)|_V^2 < \infty, \\ &\text{for any } \hat{\varphi} \in \mathscr{V}, \text{ and a.e. } t \in (0,T), \end{aligned} \tag{7.2.14b}$$

and

$$\sup_{n\in\mathbb{N}}\left|\sqrt{\hat{\mu}_n}\hat{p}_n\right|_{\mathscr{H}}^2 \leq (C_V^{L^4})^2 \sup_{n\in\mathbb{N}}\left\{|\hat{\mu}_n|_{L^\infty(0,T;H)}|\hat{p}_n|_{\mathscr{V}}^2\right\} < \infty. \tag{7.2.14c}$$

Taking into account (7.2.14), Remark 7.1, Lions's lemma [55, Lemma 1.3 on page 12], and the dominated convergence theorem [56, Theorem 10 on page 36], one can observe that:

$$\begin{cases} \sqrt{\hat{\mu}_n}\varphi \to \sqrt{\hat{\mu}}\varphi \text{ in } \mathscr{H}, \\ \sqrt{\hat{\mu}_n}\hat{p}_n \to \sqrt{\hat{\mu}}\hat{p} \text{ weakly in } \mathscr{H}, \text{ as } n \to \infty, \end{cases}$$

and therefore,

$$\begin{aligned} \langle \hat{\mu}_n\hat{p}_n, \hat{\varphi}\rangle_{\mathscr{V}} = (\sqrt{\hat{\mu}_n}\hat{p}_n,\, \sqrt{\hat{\mu}_n}\hat{\varphi})_{\mathscr{H}} \to (\sqrt{\hat{\mu}}\hat{p}, \sqrt{\hat{\mu}}\hat{\varphi})_{\mathscr{H}} = \langle \hat{\mu}\hat{p}, \hat{\varphi}\rangle_{\mathscr{V}} \\ \text{as } n \to \infty, \text{ for any } \hat{\varphi} \in \mathscr{V}. \end{aligned} \tag{7.2.15}$$

(7.2.12) and (7.2.15) imply the uniqueness of the weak limit $\hat{q}^* = \hat{\mu}\hat{p}$ of subsequences of $\{\hat{\mu}_n\hat{p}_n\}_{n=1}^\infty$ in $\mathscr{V}^*$. Hence, invoking the separability of the Hilbert space $\mathscr{V}^*$, we conclude the weak convergence (7.2.9) with non-necessity of subsequences. □

7.3 Main Theorems

We begin by setting up the assumptions needed in our Main Theorems. All Main Theorems are discussed under the following assumptions.

(A1) Let $\nu > 0$ be a fixed constant. Let $[\eta_{\mathrm{ad}}, \theta_{\mathrm{ad}}] \in [\mathscr{H}]^2$ be a fixed pair of the *admissible target profile*.

(A2) For any $\varepsilon \geq 0$, let $f_\varepsilon : \mathbb{R}^N \longrightarrow [0,\infty)$ be the convex function, defined in (7.1.3).

(A3) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^1-function, which is Lipschitz continuous on $\mathbb{R}$. Also, g has a nonnegative primitive $0 \leq G \in C^2(\mathbb{R})$, i.e. the derivative $G' = \frac{dG}{d\eta}$ coincides with g on $\mathbb{R}$. Moreover, g satisfies that:

$$\liminf_{\xi\downarrow-\infty} g(\xi) = -\infty \text{ and } \limsup_{\xi\uparrow\infty} g(\xi) = \infty.$$

(A4) Let $\alpha : \mathbb{R} \longrightarrow (0, \infty)$ and $\alpha_0 : Q \longrightarrow (0, \infty)$ be Lipschitz continuous functions, such that:

- $\alpha \in C^2(\mathbb{R})$, with the first derivative $\alpha' = \frac{d\alpha}{d\eta}$ and the second one $\alpha'' = \frac{d^2\alpha}{d\eta^2}$;
- $\alpha'(0) = 0$, $\alpha'' \geq 0$ on $\mathbb{R}$, and $\alpha\alpha'$ is Lipschitz continuous on $\mathbb{R}$;
- $\alpha \geq \delta_*$ on $\mathbb{R}$, and $\alpha_0 \geq \delta_*$ on $\overline{Q}$, for some constant $\delta_* \in (0, 1)$.

(A5) Let $\mathfrak{K}$ and $\mathfrak{K}_0$ be the classes of constraints given in (1.5.33) and (1.5.37), respectively, and for any constraint $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$, with the measurable obstacles $\kappa^\ell : Q \longrightarrow [-\infty, \infty]$, $\ell = 0, 1$, let $\mathscr{U}_{\rm ad}^K \subset [\mathscr{H}]^2$ be a class of admissible controls $[u, v]$, which is defined as:

$$\begin{aligned} \mathscr{U}_{\rm ad}^K &:= \left\{ \ [\tilde{u}, \tilde{v}] \in [\mathscr{H}]^2 \ \middle| \ \tilde{u} \in K \ \right\} \\ &= \left\{ \ [\tilde{u}, \tilde{v}] \in [\mathscr{H}]^2 \ \middle| \ \kappa^0 \leq \tilde{u} \leq \kappa^1 \text{ a.e. in } Q \ \right\}. \end{aligned}$$

Moreover, the following extra assumption will be adopted to verify the dependence of optimal controls with respect to the constraint $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$.

(A6) The constraint $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$ satisfies that:

$$\begin{gathered} \kappa^\ell \in L^1\big(Q \setminus |\kappa^\ell|^{-1}(\infty)\big), \text{ with} \\ |\kappa^\ell|^{-1}(\infty) := \left\{ \ (t, x) \in Q \ \middle| \ |\kappa^\ell|(t, x) = \infty \ \right\}, \text{ for } \ell = 0, 1, \end{gathered}$$

and $\{K_n\}_{n=1}^\infty = \{[\![\kappa_n^0, \kappa_n^1]\!]\}_{n=1}^\infty \subset \mathfrak{K}$ is a sequence of constraints such that:

$$\begin{gathered} \kappa_n^\ell(t, x) \to \kappa^\ell(t, x) \ (\in [-\infty, \infty]) \text{ as } n \to \infty, \\ \text{for a.e. } (t, x) \in Q, \text{ and } \ell = 0, 1, \end{gathered}$$

$$\int_{Q \setminus |\kappa^\ell|^{-1}(\infty)} |\kappa_n^\ell - \kappa^\ell| \, dxdt \to 0 \text{ as } n \to \infty, \text{ for } \ell = 0, 1,$$

and moreover, $\bigcap_{n=1}^\infty K_n \neq \emptyset$, i.e. there exists $\bar{\kappa} \in \mathscr{H}$ satisfying

$$\begin{cases} \kappa_n^0 \leq \bar{\kappa} \leq \kappa_n^1 \text{ a.e. in } Q, \text{ for } n = 1, 2, 3, \dots, \\ \kappa^0 \leq \bar{\kappa} \leq \kappa^1 \text{ a.e. in } Q. \end{cases}$$

Remark 7.6. The assumption (A4) leads to the boundedness of the second derivative α'' of α. In fact, from the Lipschitz continuity of α and $\alpha\alpha'$, one can see that:

$$|\alpha''(\eta)| \leq \frac{1}{\delta_*}\Big(\big|\tfrac{d}{d\eta}(\alpha\alpha')\big|_{L^\infty(\mathbb{R})} + |\alpha'|_{L^\infty(\mathbb{R})}^2\Big) < \infty, \text{ for any } \eta \in \mathbb{R}.$$

Now, the Main Theorems of this paper are stated as follows.

Main Theorem 7.1. Under the assumptions (A1)–(A4), let us fix a constant $\varepsilon \geq 0$, an initial pair $[\eta_0, \theta_0] \in D$, and a forcing pair $[u, v] \in [\mathscr{H}]^2$. Then, the following hold.

(I-A) The state-system $(\mathrm{S})_\varepsilon$ admits a unique solution $[\eta, \theta] \in [\mathscr{H}]^2$, in the sense that:

$$\begin{cases} \eta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)) \subset C([0,T];H), \\ \theta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V_0) \subset C([0,T];H), \end{cases} \tag{7.3.1}$$

$$\begin{gathered} \bigl(\partial_t \eta(t), \varphi\bigr)_H + \bigl(\nabla \eta(t), \nabla \varphi\bigr)_{[H]^N} + \bigl(g(\eta(t)), \varphi\bigr)_H \\ + \bigl(\alpha'(\eta(t)) f_\varepsilon(\nabla \theta(t)), \varphi\bigr)_H = \bigl(M_u u(t), \varphi\bigr)_H, \\ \text{for any } \varphi \in V, \text{ a.e. } t \in (0,T), \text{ subject to } \eta(0) = \eta_0 \text{ in } H; \end{gathered} \tag{7.3.2}$$

and

$$\begin{gathered} \bigl(\alpha_0(t)\partial_t \theta(t), \theta(t) - \psi\bigr)_H + \nu^2 \bigl(\nabla \theta(t), \nabla(\theta(t) - \psi)\bigr)_{[H]^N} \\ + \int_\Omega \alpha(\eta(t)) f_\varepsilon(\nabla \theta(t)) dx \leq \int_\Omega \alpha(\eta(t)) f_\varepsilon(\nabla \psi) dx \\ + \bigl(M_v v(t), \theta(t) - \psi\bigr)_H, \text{ for any } \psi \in V_0, \\ \text{a.e. } t \in (0,T), \text{ subject to } \theta(0) = \theta_0 \text{ in } H. \end{gathered} \tag{7.3.3}$$

In particular, if $\eta_0 \in L^\infty(\Omega)$ and $u \in L^\infty(Q)$, then $\eta \in L^\infty(Q)$.

(I-B) Let $\{\varepsilon_n\}_{n=1}^\infty \subset [0, \infty)$, $\{[\eta_{0,n}, \theta_{0,n}]\}_{n=1}^\infty \subset D$, and $\{[u_n, v_n]\}_{n=1}^\infty \subset [\mathscr{H}]^2$ be given sequences such that:

$$\varepsilon_n \to \varepsilon, \ [\eta_{0,n}, \theta_{0,n}] \to [\eta_0, \theta_0] \text{ weakly in } V \times V_0, \tag{7.3.4}$$

$$\text{and } [M_u u_n, M_v v_n] \to [M_u u, M_v v] \text{ weakly in } [\mathscr{H}]^2, \text{ as } n \to \infty. \tag{7.3.5}$$

In addition, let $[\eta, \theta]$ be the unique solution to $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u, v]$. Also, for any $n \in \mathbb{N}$, let $[\eta_n, \theta_n]$ be the unique solution to $(\mathrm{S})_{\varepsilon_n}$, for the initial pair $[\eta_{0,n}, \theta_{0,n}]$ and forcing pair $[u_n, v_n]$. Then, it holds that:

$$\begin{gathered} [\eta_n, \theta_n] \to [\eta, \theta] \text{ in } [C([0,T];H)]^2, \text{ in } \mathscr{Y}, \text{ weakly in } [W^{1,2}(0,T;H)]^2, \\ \text{and weakly-}* \text{ in } L^\infty(0,T;V) \times L^\infty(0,T;V_0), \text{ as } n \to \infty. \end{gathered} \tag{7.3.6}$$

In particular, if:

$$\begin{cases} \{\eta_{0,n}\}_{n=1}^\infty \subset L^\infty(\Omega), \ \{u_n\}_{n=1}^\infty \subset L^\infty(Q), \\ \sup_{n \in \mathbb{N}} |\eta_{0,n}|_{L^\infty(\Omega)} \vee \sup_{n \in \mathbb{N}} |u_n|_{L^\infty(Q)} < \infty, \end{cases} \tag{7.3.7}$$

then

$$\eta_n \to \eta \text{ weakly-}* \text{ in } L^\infty(Q), \text{ as } n \to \infty. \tag{7.3.8}$$

Remark 7.7. As a consequence of (7.3.6) and Remark 7.6, we further find a subsequence $\{n_i\}_{i=1}^\infty \subset \{n\}$, such that:

$$\begin{gathered} [\eta_{n_i}, \theta_{n_i}] \to [\eta, \theta], \ [\nabla \eta_{n_i}, \nabla \theta_{n_i}] \to [\nabla \eta, \nabla \theta], \\ \text{in the pointwise sense a.e. in } Q, \end{gathered}$$

$$\alpha''(\eta_{n_i}) f_{\varepsilon_{n_i}}(\nabla \theta_{n_i}) \to \alpha''(\eta) f_\varepsilon(\nabla \theta) \text{ weakly-}* \text{ in } L^\infty(0,T;H),$$

and in the pointwise sense a.e. in Q,

and

$$[\eta_{n_i}(t), \theta_{n_i}(t)] \to [\eta(t), \theta(t)] \text{ in } V \times V_0,$$
$$\text{and } \alpha''(\eta_{n_i}(t)) f_{\varepsilon_{n_i}}(\nabla \theta_{n_i}(t)) \to \alpha''(\eta(t)) f_\varepsilon(\nabla \theta(t)) \text{ in } H,$$
$$\text{for a.e. } t \in (0,T), \text{ as } i \to \infty.$$

Main Theorem 7.2. Let us assume (A1)–(A5). Let us fix any constant $\varepsilon \geq 0$, any initial data $[\eta_0, \theta_0] \in D$, and any constraint $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$. Then, the following two items hold.

(II-A) The problem $(\mathrm{OP})_\varepsilon^K$ has at least one optimal control $[u^*, v^*] \in \mathscr{U}_{\mathrm{ad}}^K$, so that:

$$\mathcal{J}_\varepsilon(u^*, v^*) = \min \left\{ \ \mathcal{J}_\varepsilon(u, v) \ \middle| \ [u, v] \in \mathscr{U}_{\mathrm{ad}}^K \ \right\}.$$

(II-B) Let us assume the extra assumption (A6), for the sequence of constraints $\{K_n\}_{n=1}^\infty = \{[\![\kappa_n^0, \kappa_n^1]\!]\}_{n=1}^\infty \subset \mathfrak{K}$, and let us take the sequences $\{\varepsilon_n\}_{n=1}^\infty \subset [0, \infty)$ and $\{[\eta_{0,n}, \theta_{0,n}]\}_{n=1}^\infty \subset D$ as in (7.3.4). In addition, for any $n \in \mathbb{N}$, let $[u_n^*, v_n^*] \in \mathscr{U}_{\mathrm{ad}}^{K_n}$ be the optimal control of $(\mathrm{OP})_{\varepsilon_n}^{K_n}$ in the case when the initial pair of corresponding state system $(\mathrm{S})_{\varepsilon_n}$ is given by $[\eta_{0,n}, \theta_{0,n}]$. Then, there exist a subsequence $\{n_i\}_{i=1}^\infty \subset \{n\}$ and a pair of functions $[u^{**}, v^{**}] \in \mathscr{U}_{\mathrm{ad}}^K$, such that:

$$\begin{cases} \bullet \ [M_u u_{n_i}^*, M_v v_{n_i}^*] \to [M_u u^{**}, M_v v^{**}] \text{ weakly in } [\mathscr{H}]^2, \text{ as } i \to \infty, \\ \bullet \ [u^{**}, v^{**}] \text{ is an optimal control of } (\mathrm{OP})_\varepsilon^K. \end{cases}$$

Main Theorem 7.3. In addition to the assumptions (A1)–(A5), let us suppose the restricted situation (r.s.0) as in the Introduction, i.e.:

(r.s.0) $\varepsilon > 0$, $[\eta_0, \theta_0] \in D_0$, and $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}_0$ $(= \mathfrak{K} \cap 2^{L^\infty(Q)})$.

Let $[u^*, v^*] \in \mathscr{U}_{\mathrm{ad}}^K$ be an optimal control of $(\mathrm{OP})_\varepsilon^K$, and let $[\eta_\varepsilon^*, \theta_\varepsilon^*]$ be the solution to $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u^*, v^*]$. Then, the following two items hold.

(III-A) (Necessary condition for $(\mathrm{OP})_\varepsilon^K$ under $\varepsilon > 0$ and $K \in \mathfrak{K}_0$) For the optimal control $[u^*, v^*] \in \mathscr{U}_{\mathrm{ad}}^K$ of $(\mathrm{OP})_\varepsilon^K$, it holds that:

$$M_u \left(u^* - \mathrm{proj}_K(-p_\varepsilon^*) \right) = 0, \text{in } \mathscr{H}, \tag{7.3.9a}$$

$$M_v (v^* + z_\varepsilon^*) = 0 \text{ in } \mathscr{H}. \tag{7.3.9b}$$

In this context, $[p_\varepsilon^*, z_\varepsilon^*]$ is a unique solution to the following variational system:

$$\begin{aligned} -\langle \partial_t p_\varepsilon^*(t), \varphi \rangle_V &+ \left(\nabla p_\varepsilon^*(t), \nabla \varphi \right)_{[H]^N} + \langle [\alpha''(\eta_\varepsilon^*) f_\varepsilon(\nabla \theta_\varepsilon^*)](t) p_\varepsilon^*(t), \varphi \rangle_V \\ &+ \left(g'(\eta_\varepsilon^*(t)) p_\varepsilon^*(t), \varphi \right)_H + \left([\alpha'(\eta_\varepsilon^*) \nabla f_\varepsilon(\nabla \theta_\varepsilon^*)](t) \cdot \nabla z_\varepsilon^*(t), \varphi \right)_H \\ =& \left(M_\eta (\eta_\varepsilon^* - \eta_{\mathrm{ad}})(t), \varphi \right)_H, \text{ for any } \varphi \in V, \text{ and a.e. } t \in (0,T); \end{aligned} \tag{7.3.10}$$

and

$$-\langle \partial_t \left(\alpha_0 z_\varepsilon^* \right)(t), \psi \rangle_{V_0} + \left([\alpha(\eta_\varepsilon^*) \nabla^2 f_\varepsilon(\nabla \theta_\varepsilon^*)](t) \nabla z_\varepsilon^*(t) + \nu^2 \nabla z_\varepsilon^*(t), \nabla \psi \right)_{[H]^N}$$

$$+\left(p_\varepsilon^*(t)[\alpha'(\eta_\varepsilon^*)\nabla f_\varepsilon(\nabla\theta_\varepsilon^*)](t), \nabla\psi\right)_{[H]^N} = \left(M_\theta(\theta_\varepsilon^* - \theta_{\mathrm{ad}})(t), \psi\right)_H, \qquad (7.3.11)$$

for any $\psi \in V_0$, and a.e. $t \in (0, T)$;

subject to the terminal condition:

$$[p_\varepsilon^*(T), z_\varepsilon^*(T)] = [0, 0] \text{ in } [H]^2. \qquad (7.3.12)$$

(III-B) Let $\{\varepsilon_n\}_{n=1}^\infty \subset [0, \infty)$ and $\{[\eta_{0,n}, \theta_{0,n}]\}_{n=1}^\infty \subset D$ be sequences as in (7.3.4). Also, let $\{K_n\}_{n=1}^\infty = \{[\![\kappa_n^0, \kappa_n^1]\!]\}_{n=1}^\infty \subset \mathfrak{K}$ be a sequence of constraints, fulfilling (A6). In addition, let us assume:

$$\begin{cases} \bullet\ \{K_n\}_{n=1}^\infty = \{[\![\kappa_n^0, \kappa_n^1]\!]\}_{n=1}^\infty \subset \mathfrak{K}_0 \ \left(= \mathfrak{K} \cap 2^{L^\infty(Q)}\right), \\ \bullet\ \{[\eta_{0,n}, \theta_{0,n}]\}_{n=1}^\infty \subset D_0 \ \left(= (V \cap L^\infty(\Omega)) \times V_0\right), \\ \bullet\ \sup_{n \in \mathbb{N}} \left\{ |\eta_{0,n}|_{L^\infty(\Omega)} \vee |\kappa_n^0|_{L^\infty(Q)} \vee |\kappa_n^1|_{L^\infty(Q)} \right\} < \infty. \end{cases} \qquad (7.3.13)$$

Then, the subsequence $\{n_i\}_{i=1}^\infty \subset \{n\}$ and the limiting optimal control $[u^{**}, v^{**}] \in \mathscr{U}_{\mathrm{ad}}^K$ as in Main Theorem 7.2 (II-B) fulfill that:

$$u^{**} \in L^\infty(Q),\ M_v v^{**} \in W^{1,2}(0, T; V_0^*) \cap L^\infty(0, T; V_0) \subset C([0, T]; H), \qquad (7.3.14a)$$

$$\begin{cases} M_u u_{n_i}^* \to M_u u^{**} \text{ in } \mathscr{H}, \\ u_{n_i}^* \to u^{**} \text{ weakly-}* \text{ in } L^\infty(Q), \end{cases} \text{as } i \to \infty, \qquad (7.3.14b)$$

and

$$\begin{aligned} M_v v_{n_i}^* \to M_v v^{**} &\text{ in } C([0, T]; H), \text{ in } \mathscr{V}_0, \\ &\text{weakly in } W^{1,2}(0, T; V_0^*), \text{ as } i \to \infty. \end{aligned} \qquad (7.3.14c)$$

Remark 7.8. Let $\mathcal{R}_T \in \mathscr{L}(\mathscr{H})$ be an isomorphism, defined as:

$$\left(\mathcal{R}_T\varphi\right)(t) := \varphi(T - t) \text{ in } H, \text{ for a.e. } t \in (0, T).$$

Also, let us fix $\varepsilon > 0$, and define a bounded linear operator $\mathcal{Q}_\varepsilon^* : [\mathscr{H}]^2 \longrightarrow \mathscr{Z}$ as the restriction $\mathcal{P}|_{\{[0,0]\} \times \mathscr{Y}^*}$ of the linear isomorphism $\mathcal{P} = \mathcal{P}(a, b, \mu, \lambda, \omega, A) : [H]^2 \times \mathscr{Y}^* \longrightarrow \mathscr{Z}$, as in Proposition 7.5, in the case when:

$$\begin{cases} [a, b] = \mathcal{R}_T[\alpha_0, -\partial_t\alpha_0] \text{ in } W^{1,\infty}(Q) \times L^\infty(Q), \\ \mu = \mathcal{R}_T\left[\alpha''(\eta_\varepsilon^*) f_\varepsilon(\nabla\theta_\varepsilon^*)\right] \text{ in } L^\infty(0, T; H), \\ \lambda = \mathcal{R}_T\left[g'(\eta_\varepsilon^*)\right] \text{ in } L^\infty(Q), \\ \omega = \mathcal{R}_T\left[\alpha'(\eta_\varepsilon^*)\nabla f_\varepsilon(\nabla\theta_\varepsilon^*)\right] \text{ in } [L^\infty(Q)]^N, \\ A = \mathcal{R}_T\left[\alpha(\eta_\varepsilon^*)\nabla^2 f_\varepsilon(\nabla\theta_\varepsilon^*)\right] \text{ in } [L^\infty(Q)]^{N \times N}. \end{cases} \qquad (7.3.15)$$

On this basis, let us define:

$$\mathcal{P}_\varepsilon^* := \mathcal{R}_T \circ \mathcal{Q}_\varepsilon^* \circ \mathcal{R}_T \text{ in } \mathscr{L}([\mathscr{H}]^2; \mathscr{Z}). \qquad (7.3.16)$$

Then, since the embedding $V_0 \subset L^4(\Omega)$ and $\alpha_0 \in W^{1,\infty}(Q)$ guarantee:

$$\partial_t(\alpha_0 \tilde{z}) = \alpha_0 \partial_t \tilde{z} + \tilde{z}\partial_t \alpha_0 \text{ in } \mathscr{V}_0^*, \text{ for any } \tilde{z} \in W^{1,2}(0,T;V_0^*), \tag{7.3.17}$$

we can obtain the unique solution $[p_\varepsilon^*, z_\varepsilon^*] \in [\mathscr{H}]^2$ to the variational system (7.3.10)–(7.3.12) as follows:

$$[p_\varepsilon^*, z_\varepsilon^*] = \mathcal{P}_\varepsilon^* \big[M_\eta(\eta_\varepsilon^* - \eta_{\mathrm{ad}}), M_\theta(\theta_\varepsilon^* - \theta_{\mathrm{ad}})\big] \text{ in } \mathscr{X}.$$

Main Theorem 7.4. Let us assume (A1)–(A5), and let us assume that the situation is not under (r.s.0), i.e. it is under:

¬(r.s.0) either $\varepsilon = 0$, or $[\eta_0, \theta_0] \in D \setminus D_0$, or $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K} \setminus \mathfrak{K}_0$ is satisfied.

Also, let us define a Hilbert space $\mathscr{W}_0$ as follows:

$$\mathscr{W}_0 := \left\{ \ \psi \in W^{1,2}(0,T;H) \cap \mathscr{V}_0 \ \middle| \ \psi(0) = 0 \text{ in } H \ \right\}.$$

Then, there exists an optimal control $[u^\circ, v^\circ] \in \mathscr{U}_{\mathrm{ad}}^K$ of the problem $(\mathrm{OP})_\varepsilon^K$, together with the solution $[\eta^\circ, \theta^\circ]$ to the system $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u^\circ, v^\circ] \in \mathscr{U}_{\mathrm{ad}}^K$, and moreover, there exist pairs of functions $[p^\circ, z^\circ] \in \mathscr{Y}$, $[\xi^\circ, \sigma^\circ] \in \mathscr{H} \times [L^\infty(Q)]^N$, and a distribution $\zeta^\circ \in \mathscr{W}_0^*$, such that:

$$M_u\left(u^\circ - \mathrm{proj}_K(-p^\circ)\right) = 0, \text{ in } \mathscr{H}, \tag{7.3.18a}$$

$$M_v(v^\circ + z^\circ) = 0 \text{ in } \mathscr{H}, \tag{7.3.18b}$$

$$p^\circ \in W^{1,2}(0,T;V^*) \cap \mathscr{V} \subset C([0,T];H), \tag{7.3.19a}$$

$$\sigma^\circ \in \partial f_\varepsilon(\nabla \theta^\circ), \text{ a.e. in } Q; \tag{7.3.19b}$$

$$\begin{aligned} &\big\langle -\partial_t p^\circ, \varphi \big\rangle_{\mathscr{V}} + \big(\nabla p^\circ, \nabla \varphi\big)_{L^2(0,T;[H]^N)} + \big\langle \alpha''(\eta^\circ) f_\varepsilon(\nabla \theta^\circ) p^\circ, \varphi \big\rangle_{\mathscr{V}} \\ &\quad + \big(g'(\eta^\circ)p^\circ + \alpha'(\eta^\circ)\xi^\circ, \varphi\big)_{\mathscr{H}} = \big(M_\eta(\eta^\circ - \eta_{\mathrm{ad}}), \varphi\big)_{\mathscr{H}}, \\ &\quad \text{for any } \varphi \in \mathscr{V}, \text{ subject to } p^\circ(T) = 0 \text{ in } H; \end{aligned} \tag{7.3.20}$$

and

$$\begin{aligned} &\big(\alpha_0 z^\circ, \partial_t \psi\big)_{\mathscr{H}} + \big\langle \zeta^\circ, \psi \big\rangle_{\mathscr{W}_0} + \big(\nu^2 \nabla z^\circ + \alpha'(\eta^\circ)\sigma^\circ p^\circ, \nabla \psi\big)_{L^2(0,T;[H]^N)} \\ &\quad = \big(M_\theta(\theta^\circ - \theta_{\mathrm{ad}}), \psi\big)_{\mathscr{H}}, \text{ for any } \psi \in \mathscr{W}_0. \end{aligned} \tag{7.3.21}$$

In particular, if $\varepsilon > 0$, i.e. the situation is under:

(r.s.1) $\varepsilon > 0$, while either $[\eta_0, \theta_0] \in D \setminus D_0$ or $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K} \setminus \mathfrak{K}_0$ is satisfied;

then:

$$\begin{cases} \sigma^\circ = \nabla f_\varepsilon(\nabla \theta^\circ), \text{ a.e. in } Q, \\ \xi^\circ = \sigma^\circ \cdot \nabla z^\circ = \nabla f_\varepsilon(\nabla \theta^\circ) \cdot \nabla z^\circ \text{ in } \mathscr{H}, \\ \zeta^\circ = -\mathrm{div}\big(\alpha(\eta^\circ) \nabla^2 f_\varepsilon(\nabla \theta^\circ) \nabla z^\circ\big) \text{ in } \mathscr{W}_0^*. \end{cases} \tag{7.3.22}$$

Remark 7.9. When $\varepsilon = 0$, the inclusion (7.3.19b) is equivalent to:

$$\sigma^\circ \in \mathrm{Sgn}^N(\nabla\theta^\circ), \text{ a.e. in } Q.$$

In the meantime, when $\varepsilon > 0$, (7.3.19)–(7.3.22) imply that the pair of functions $[p^\circ, z^\circ]$ solves the following system:

$$\begin{cases} -\partial_t p^\circ - \mathit{\Delta} p^\circ + \alpha''(\eta^\circ) f_\varepsilon(\nabla\theta^\circ) p^\circ + g'(\eta^\circ) p^\circ + \alpha'(\eta^\circ) \nabla f_\varepsilon(\nabla\theta^\circ) \cdot \nabla z^\circ \\ \quad = M_\eta(\eta^\circ - \eta_{\mathrm{ad}}), \\ -\partial_t(\alpha_0 z^\circ) - \mathrm{div}\big(\alpha(\eta^\circ)\nabla^2 f_\varepsilon(\nabla\theta^\circ)\nabla z^\circ + \nu^2 \nabla z^\circ + p^\circ \alpha'(\eta^\circ)\nabla f_\varepsilon(\nabla\theta^\circ)\big) \\ \quad = M_\theta(\theta^\circ - \theta_{\mathrm{ad}}), \end{cases}$$

in the sense of distribution on Q. Note that the above system corresponds to the distributional form of the variational system (7.3.10)–(7.3.12), as in Main Theorem 7.3 (III-A).

Remark 7.10. Moreover, in the light of (7.3.9a), (7.3.18a), and Remark 7.3 (Fact 4), we will observe that:

$$\begin{cases} M_u u^*(t,x) = M_u\big[\mathrm{proj}_K(-p_\varepsilon^*)\big](t,x) = M_u\big(\kappa^0 \vee (\kappa^1 \wedge (-p_\varepsilon^*))\big)(t,x), \\ M_u u^\circ(t,x) = M_u\big[\mathrm{proj}_K(-p^\circ)\big](t,x) = M_u\big(\kappa^0 \vee (\kappa^1 \wedge (-p^\circ))\big)(t,x), \end{cases}$$
$$\text{for a.e. } (t,x) \in Q.$$

7.4 Proof of Main Theorem 7.1

In this Section, we give the proof of the first Main Theorem 7.1. Before the proof, we refer to the reformulation method as in [62], and consider to reduce the state-system $(\mathrm{S})_\varepsilon$ to an evolution equation in the Hilbert space $[H]^2$.

Let us fix any $\varepsilon \geq 0$. Besides, for any $R \geq 0$, let us define a proper functional $\Phi_\varepsilon^R : [H]^2 \longrightarrow [0,\infty]$, by setting:

$$\Phi_\varepsilon^R : w = [\eta,\theta] \in [H]^2 \mapsto \Phi_\varepsilon^R(w) = \Phi_\varepsilon^R(\eta,\theta)$$
$$:= \begin{cases} \displaystyle \frac{1}{2}\int_\Omega |\nabla\eta|^2 dx + \frac{R}{2}\int_\Omega |\eta|^2\,dx + \frac{1}{2}\int_\Omega \left(\nu f_\varepsilon(\nabla\theta) + \frac{1}{\nu}\alpha(\eta)\right)^2 dx, \\ \qquad \text{if } [\eta,\theta] \in V \times V_0, \\ \infty, \quad \text{otherwise.} \end{cases} \tag{7.4.1}$$

Note that the assumptions (A2) and (A4) guarantee the lower semi-continuity and convexity of Φ_ε^R on $[H]^2$.

Remark 7.11. As consequences of standard variational methods, we easily check the following facts.

(Fact 7) For the operator $\partial_\eta \Phi_\varepsilon^R : [H]^2 \longrightarrow 2^H$,

$$D(\partial_\eta \Phi_\varepsilon^R) = \left\{ [\tilde\eta, \tilde\theta] \in D \;\middle|\; \begin{array}{l} \tilde\eta \in H^2(\Omega) \text{ subject to} \\ \nabla\eta \cdot n_\Gamma = 0 \text{ in } H^{\frac{1}{2}}(\Gamma) \end{array} \right\},$$

independent of $R \geq 0$, and $\partial_\eta \Phi_\varepsilon^R$ is a single-valued operator such that:

$$\partial_\eta \Phi_\varepsilon^R(w) = \partial_\eta \Phi_\varepsilon(\eta, \theta) = -\Delta\eta + R\eta + \alpha'(\eta) f_\varepsilon(\nabla\theta) + \frac{1}{\nu^2}\alpha(\eta)\alpha'(\eta) \text{ in } H,$$

for all $w = [\eta, \theta] \in D(\partial_\eta \Phi_\varepsilon^R)$, and $R \geq 0$.

(Fact 8) $\partial_\theta \Phi_\varepsilon^R : [H]^2 \longrightarrow 2^H$ is independent of $R \geq 0$, and $\theta \in D(\partial_\theta \Phi_\varepsilon^R)$, and $\theta^* \in \partial_\theta \Phi_\varepsilon^R(w) = \partial_\theta \Phi_\varepsilon^R(\eta, \theta)$, iff. $\theta \in V_0$, and

$$\begin{aligned}(\theta^*, \theta - \psi)_H \geq \int_\Omega \alpha(\eta) f_\varepsilon(\nabla\theta)\, dx - \int_\Omega \alpha(\eta) f_\varepsilon(\nabla\psi)\, dx \\ +\nu^2\big(\nabla\theta, \nabla(\theta - \psi)\big)_{[H]^N}, \text{ for all } \psi \in V_0, \text{ and } R \geq 0.\end{aligned}$$

In addition, let us define time-dependent operators $\mathcal{A}(t) \in \mathscr{L}([H]^2)$, for $t \in [0, T]$, nonlinear operators $\mathcal{G}^R : [H]^2 \longrightarrow [H]^2$, for $R \geq 0$, by setting:

$$\mathcal{A}(t) : w = [\eta, \theta] \in [H]^2 \mapsto \mathcal{A}(t)w := [\eta,\ \alpha_0(t)\theta] \in [H]^2, \quad \text{for } t \in [0, T], \tag{7.4.2}$$

$$\mathcal{G}^R : w = [\eta, \theta] \in [H]^2 \mapsto \mathcal{G}^R(w) := \big[g(\eta) - R\eta - \nu^{-2}\alpha(\eta)\alpha'(\eta),\ 0\big] \in [H]^2, \quad \text{for } R \geq 0, \tag{7.4.3}$$

respectively. Then, based on the above (Fact 7) and (Fact 8), it is verified that the state-system $(\mathrm{S})_\varepsilon$ is equivalent to the following Cauchy problem.

$$\begin{cases} \mathcal{A}(t)w'(t) + \big[\partial_\eta \Phi_\varepsilon^R \times \partial_\theta \Phi_\varepsilon^R\big](w(t)) + \mathcal{G}^R(w(t)) \ni \mathfrak{f}(t) \text{ in } [H]^2, \\ \qquad \text{a.e. } t \in (0, T), \\ w(0) = w_0 \text{ in } [H]^2. \end{cases}$$

In the context, "$'$" is the time-derivative, and

$$\begin{cases} \bullet\ w_0 := [\eta_0, \theta_0] \in D \text{ is the initial data of } w = [\eta, \theta], \\ \bullet\ \mathfrak{f} := [M_u u, M_v v] \in [\mathscr{H}]^2 \text{ is the forcing term of the Cauchy problem.} \end{cases} \tag{7.4.4}$$

Now, before the proof of Main Theorem 7.1, we prepare the following Key-Lemma and its Corollary.

Key-Lemma 2. *Let us assume (A1)–(A4). Then, there exists a positive constant $R_0 > 0$ such that:*

$$\partial \Phi_\varepsilon^{R_0} = \big[\partial_\eta \Phi_\varepsilon^{R_0} \times \partial_\theta \Phi_\varepsilon^{R_0}\big] \text{ in } [H]^2 \times [H]^2.$$

Proof. We set:

$$R_0 := 1 + \frac{2}{\nu^2}|\alpha|^2_{L^\infty(\mathbb{R})}, \tag{7.4.5}$$

and prove this R_0 is the required constant.

In the light of (7.1.1), it is immediately verified that:

$$\partial\Phi_\varepsilon^{R_0} \subset \left[\partial_\eta\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right] \text{ in } [H]^2 \times [H]^2.$$

Hence, having in mind the maximality of the monotone graph $\partial\Phi_\varepsilon^{R_0}$ in $[H]^2 \times [H]^2$, we can reduce our task to show the monotonicity of $\left[\partial_\eta\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right]$ in $[H]^2 \times [H]^2$. Let us assume:

$$\begin{gathered}
[w, w^*] \in \left[\partial_\eta\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right] \text{ and } [\tilde{w}, \tilde{w}^*] \in \left[\partial_\eta\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right] \text{ in } [H]^2 \times [H]^2, \\
\text{with } w = [\eta, \theta] \in [H]^2, \ w^* = [\eta^*, \theta^*] \in [H]^2, \\
\tilde{w} = [\tilde{\eta}, \tilde{\theta}] \in [H]^2, \text{ and } \tilde{w}^* = [\tilde{\eta}^*, \tilde{\theta}^*] \in [H]^2, \text{ respectively.}
\end{gathered}$$

Then, by using (7.1.4a), (Fact 7), (Fact 8), (A4), and Young's inequality, we compute that:

$$(w^* - \tilde{w}^*, w - \tilde{w})_{[H]^2} = I_1 + I_2 + I_3, \tag{7.4.6a}$$

with

$$I_1 := |\nabla(\eta - \tilde{\eta})|^2_{[H]^N} + R_0|\eta - \tilde{\eta}|^2_H + \nu^2|\nabla(\theta - \tilde{\theta})|^2_{[H]^N}, \tag{7.4.6b}$$

$$\begin{aligned}
I_2 :=& (\alpha'(\eta)f_\varepsilon(\nabla\theta) - \alpha'(\tilde{\eta})f_\varepsilon(\nabla\tilde{\theta}), \eta - \tilde{\eta})_H \\
&\quad + \frac{1}{\nu^2}(\alpha(\eta)\alpha'(\eta) - \alpha(\tilde{\eta})\alpha'(\tilde{\eta}), \eta - \tilde{\eta})_H \\
=& \int_\Omega f_\varepsilon(\nabla\theta)(\alpha'(\eta) - \alpha'(\tilde{\eta}))(\eta - \tilde{\eta})\,dx \qquad (7.4.6c) \\
&\quad + \int_\Omega \alpha'(\tilde{\eta})(f_\varepsilon(\nabla\theta) - f_\varepsilon(\nabla\tilde{\theta}))(\eta - \tilde{\eta})\,dx \\
&\quad + \frac{1}{2\nu^2}\int_\Omega \left(\tfrac{d}{d\eta}[\alpha^2](\eta) - \tfrac{d}{d\eta}[\alpha^2](\tilde{\eta})\right)(\eta - \tilde{\eta})\,dx \\
\geq& - |\alpha'|_{L^\infty(\mathbb{R})}|\eta - \tilde{\eta}|_H|\nabla(\theta - \tilde{\theta})|_{[H]^N} \\
\geq& - \frac{|\alpha'|^2_{L^\infty(\mathbb{R})}}{\nu^2}|\eta - \tilde{\eta}|^2_H - \frac{\nu^2}{4}|\nabla(\theta - \tilde{\theta})|^2_{[H]^N}, \qquad (7.4.6d)
\end{aligned}$$

and

$$\begin{aligned}
I_3 :=& \int_\Omega (\alpha(\eta) - \alpha(\tilde{\eta}))(f_\varepsilon(\nabla\theta) - f_\varepsilon(\nabla\tilde{\theta}))\,dx \\
&\geq - |\alpha'|_{L^\infty(\mathbb{R})}|\eta - \tilde{\eta}|_H|\nabla(\theta - \tilde{\theta})|_{[H]^N} \\
&\geq - \frac{|\alpha'|^2_{L^\infty(\mathbb{R})}}{\nu^2}|\eta - \tilde{\eta}|^2_H - \frac{\nu^2}{4}|\nabla(\theta - \tilde{\theta})|^2_{[H]^N}. \qquad (7.4.6e)
\end{aligned}$$

Due to (7.4.5), the inequalities in (7.4.6) lead to:

$$(w^* - \tilde{w}^*, w - \tilde{w})_{[H]^2} \geq |\eta - \tilde{\eta}|^2_V + \frac{\nu^2}{2}|\theta - \tilde{\theta}|^2_{V_0} \geq 0, \tag{7.4.7}$$

which implies the (strict) monotonicity of the operator $\left[\partial_\eta\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right]$ in $[H]^2 \times [H]^2$. □

Corollary 1. *Under the notations and assumptions as in the previous Key-Lemma 2, it further holds that*

$$\partial\Phi_\varepsilon^R = \left[\partial_\eta\Phi_\varepsilon^R \times \partial_\theta\Phi_\varepsilon^R\right] \text{ in } [H]^2 \times [H]^2, \text{ for any } R \geq 0.$$

Proof. Let us take arbitrary two constants $0 \leq R, \tilde{R} < \infty$. Then, from (Fact 7), [14, Theorem 2.10], and [18, Corollary 2.11], we immediately have

$$D(\partial_\eta\Phi_\varepsilon^R) = D(\partial_\eta\Phi_\varepsilon^{\tilde{R}}) \text{ in } V, \tag{7.4.8a}$$

and for any $w = [\eta, \theta] \in D(\partial_\eta\Phi_\varepsilon^R) = D(\partial_\eta\Phi_\varepsilon^{\tilde{R}})$,

$$\begin{aligned} \partial_\eta\Phi_\varepsilon^R(w) &= -\Delta\eta + \tilde{R}\eta + (R - \tilde{R})\eta + \alpha'(\eta)f_\varepsilon(\nabla\theta) + \nu^{-2}\alpha(\eta)\alpha'(\eta) \\ &= \partial_\eta\Phi_\varepsilon^{\tilde{R}}(w) + (R - \tilde{R})\eta \text{ in } H. \end{aligned} \tag{7.4.8b}$$

Also, as a straightforward consequence of (Fact 8), it is seen that:

$$\partial_\theta\Phi_\varepsilon^R = \partial_\theta\Phi_\varepsilon^{\tilde{R}} \text{ in } H \times H. \tag{7.4.9}$$

In the meantime, invoking (7.4.1), [14, Theorem 2.10], and [18, Corollary 2.11], we will infer that

$$D(\partial\Phi_\varepsilon^R) = D(\partial\Phi_\varepsilon^{\tilde{R}}) \text{ in } D, \tag{7.4.10a}$$

and

$$\partial\Phi_\varepsilon^R(w) = \partial\Phi_\varepsilon^{\tilde{R}}(w) + (R - \tilde{R})[\eta, 0] \text{ in } [H]^2. \tag{7.4.10b}$$

Now, let us take the constant $R_0 > 0$ obtained in Key-Lemma 2. Then, owing to (7.4.8)–(7.4.10), and Key-Lemma 2, we can compute that

$$\begin{aligned} &\left[\partial_\eta\Phi_\varepsilon^R \times \partial_\theta\Phi_\varepsilon^R\right](w) = \left[\partial_\eta\Phi_\varepsilon^{R_0} \times \partial_\theta\Phi_\varepsilon^{R_0}\right](w) + (R - R_0)[\eta, 0] \\ &\quad = \partial\Phi_\varepsilon^{R_0}(w) + (R - R_0)[\eta, 0] = \partial\Phi_\varepsilon^R(w) \text{ in } [H]^2, \\ &\quad \text{for any } w \in D(\partial_\eta\Phi_\varepsilon^R \times \partial_\theta\Phi_\varepsilon^R) = D(\partial_\eta\Phi_\varepsilon^R) \cap D(\partial_\theta\Phi_\varepsilon^R). \end{aligned} \tag{7.4.11}$$

In the light of (7.1.1), the above (7.4.11) is sufficient to conclude this Corollary. □

Lemma 7.2. Let us assume (A1)–(A4), and fix functions $\bar{\theta} \in L^\infty(0, T; V_0)$, $\eta_0 \in V$, and $u \in \mathscr{H}$. Then, the initial-boundary value problem:

$$\begin{cases} \partial_t\eta - \Delta\eta + g(\eta) + \alpha'(\eta)f_\varepsilon(\nabla\bar{\theta}) = M_u u \text{ a.e. in } Q, \\ \nabla\eta \cdot n_\Gamma = 0 \text{ on } \Sigma, \\ \eta(0, x) = \eta_0(x), \ x \in \Omega; \end{cases} \tag{7.4.12}$$

admits a unique solution $\eta \in W^{1,2}(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega))$, and in particular, if:

$$\eta_0 \in L^\infty(\Omega), \text{ and } u \in L^\infty(Q), \tag{7.4.13}$$

then it holds that $\eta \in L^\infty(Q)$.

Proof. Let us fix $\bar{\theta} \in L^\infty(0,T;V_0)$, $\eta_0 \in V$, and $u \in \mathscr{H}$. Then, referring to the general theories of nonlinear evolution equations (e.g. [14,18,39]), we immediately find a solution $\eta \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$, in the variational sense:

$$\begin{aligned}(\partial_t \eta(t), \varphi)_H &+ (\nabla \eta(t), \nabla \varphi)_{[H]^N} + \big(g(\eta(t)) + \alpha'(\eta(t)) f_\varepsilon(\nabla \bar{\theta}(t)), \varphi\big)_H \\ &= (M_u u(t), \varphi)_H, \text{ for any } \varphi \in V, \text{ a.e. } t \in (0,T). \end{aligned} \tag{7.4.14}$$

Next, we assume $\eta_0 \in V \cap L^\infty(\Omega)$ and $u \in L^\infty(Q)$, and verify the L^∞-regularity of the solution η as in (7.4.13). To this end, we invoke the assumption (A3), and take a large constant $L_0 > 0$, such that:

$$L_0 \geq |\eta_0|_{L^\infty(\Omega)}, \; g(L_0) \geq M_u |u|_{L^\infty(Q)}, \text{ and } g(-L_0) \leq -M_u |u|_{L^\infty(Q)}. \tag{7.4.15}$$

On this basis, we set our remaining task to show that:

$$|\eta|_{L^\infty(Q)} \leq L_0, \text{ i.e. } -L_0 \leq \eta \leq L_0 \text{ a.e. in } Q. \tag{7.4.16}$$

Due to (7.4.15) and (A4), the constants L_0 and $-L_0$ fulfill that:

$$\partial_t L_0 - \varDelta L_0 + g(L_0) + \alpha'(L_0) f_\varepsilon(\nabla \bar{\theta}) \geq M_u u(t,x), \text{ a.e. } (t,x) \in Q, \tag{7.4.17a}$$

and

$$\partial_t(-L_0) - \varDelta(-L_0) + g(-L_0) + \alpha'(-L_0) f_\varepsilon(\nabla \bar{\theta}) \leq M_u u(t,x), \text{ a.e. } (t,x) \in Q, \tag{7.4.17b}$$

respectively, together with the initial values L_0 and $-L_0$, and the zero-Neumann boundary conditions.

Now, let us take the difference between PDEs in (7.4.12) and (7.4.17a) (resp. (7.4.17b) and (7.4.12)), and multiply the both sides by $[\eta - L_0]^+$ (resp. $[-L_0 - \eta]^+$). Then, from (A2)–(A4), it is inferred that:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\big(\big|[\eta - L_0]^+(t)\big|_H^2 + \big|[-L_0 - \eta]^+(t)\big|_H^2\big) \\ &\leq |g'|_{L^\infty(\mathbb{R})}\big(\big|[\eta - L_0]^+(t)\big|_H^2 + \big|[-L_0 - \eta]^+(t)\big|_H^2\big), \text{ a.e. } t \in (0,T).\end{aligned}$$

Applying Gronwall's lemma, and invoking (7.4.15), we obtain:

$$\big|[\eta - L_0]^+(t)\big|_H^2 + \big|[-L_0 - \eta]^+(t)\big|_H^2 \leq 0, \text{ a.e. } t \in (0,T),$$

which implies the validity of (7.4.16). □

Remark 7.12. Let $\varepsilon \geq 0$ be arbitrary constant. Then, as a consequence of (Fact 7), (Fact 8), Key-Lemma 2, Corollary 1, and Lemma 7.2, we can say that the state-system $(\mathrm{S})_\varepsilon$ is equivalent to the following Cauchy problem of evolution equation, denoted by $(\mathrm{E})_\varepsilon$.

$(\mathrm{E})_\varepsilon$:
$$\begin{cases}\mathcal{A}(t) w'(t) + \partial \Phi_\varepsilon^R(w(t)) + \mathcal{G}^R(w(t)) \ni \mathfrak{f}(t) \text{ in } [H]^2, \text{ a.e. } t \in (0,T), \\ w(0) = w_0 \text{ in } [H]^2,\end{cases}$$

for any $R \geq 0$.

Now, we are ready to prove the Main Theorem 7.1.

Proof of Main Theorem 7.1 (I-A). Let us fix any $R > 0$. Then, under the setting (7.4.1)–(7.4.4), we immediately check that:

(ev.0) for any $t \in [0, T]$, $\mathcal{A}(t) \in \mathscr{L}([H]^2)$ is positive and selfadjoint, and

$$(\mathcal{A}(t)w, w)_{[H]^2} \geq \delta_* |w|^2_{[H]^2}, \text{ for any } w \in [H]^2,$$

with the constant $\delta_* \in (0, 1)$ as in (A4);

(ev.1) $\mathcal{A} \in W^{1,\infty}(0, T; \mathscr{L}([H]^2))$, and

$$A^* := \operatorname*{ess\,sup}_{t \in (0,T)} \left\{ \max\{ |\mathcal{A}(t)|_{\mathscr{L}([H]^2)}, |\mathcal{A}'(t)|_{\mathscr{L}([H]^2)} \} \right\} \leq 1 + |\alpha_0|_{W^{1,\infty}(Q)} < \infty;$$

(ev.2) $\mathcal{G}^R : [H]^2 \longrightarrow [H]^2$ is a Lipschitz continuous operator with a Lipschitz constant:

$$\mathrm{Lip}(\mathcal{G}) := R + |g'|_{L^\infty(\mathbb{R})} + \nu^{-2} \left| \tfrac{d}{d\eta}(\alpha\alpha') \right|_{L^\infty(\mathbb{R})},$$

and $\mathcal{G}^R$ has a C^1-potential functional

$$\widehat{\mathcal{G}}^R : w = [\eta, \theta] \in [H]^2 \mapsto \widehat{\mathcal{G}}^R(w) := \int_\Omega \left(G(\eta) - \frac{R\eta^2}{2} - \frac{\alpha(\eta)^2}{2\nu^2} \right) dx \in \mathbb{R};$$

(ev.3) $\Phi_\varepsilon^R \geq 0$ on $[H]^2$, and the sublevel set $\left\{ \tilde{w} \in [H]^2 \,\middle|\, \Phi_\varepsilon^R(\tilde{w}) \leq r \right\}$ is contained in a compact set $K_\nu^R(r)$ in $[H]^2$, defined as

$$K_\nu^R(r) := \left\{ \ \tilde{w} = [\tilde{\eta}, \tilde{\theta}] \in D \ \middle| \ |\tilde{\eta}|_V^2 + |\tilde{\theta}|_{V_0}^2 \leq \tfrac{2r}{1 \wedge R \wedge \nu^2} \ \right\},$$

for any $r \geq 0$.

On account of (7.4.1)–(7.4.4) and (ev.0)–(ev.3), we can apply Proposition 7.1, as the case when:

$$X = [H]^2, \ \mathcal{A}_0 = \mathcal{A} \text{ in } W^{1,\infty}(0, T; \mathscr{L}([H]^2)),$$
$$\mathcal{G}_0 = \mathcal{G}^R \text{ on } [H]^2, \ \Psi_0 = \Phi_\varepsilon^R \text{ on } [H]^2, \text{ and } \mathfrak{f}_0 = \mathfrak{f} \text{ in } [\mathscr{H}]^2,$$

and we can find a solution $w = [\eta, \theta] \in [\mathscr{H}]^2$ to the Cauchy problem $(\mathrm{E})_\varepsilon$. In the light of Proposition 7.1 and Remark 7.12, finding this $w = [\eta, \theta]$ directly leads to the existence and uniqueness of solution to the state-system $(\mathrm{S})_\varepsilon$.

Moreover, if $\eta_0 \in L^\infty(\Omega)$ and $u \in L^\infty(Q)$, then the regularity $\eta \in L^\infty(Q)$ will be immediately seen from Lemma 7.2. □

Proof of Main Theorem 7.1 (I-B). Under the assumptions and notations as in Main Theorem 7.1, we first fix a constant $R > 0$, and invoke Remark 7.12 to confirm that the solution $w := [\eta, \theta] \in [\mathscr{H}]^2$ to $(\mathrm{S})_\varepsilon$ coincides with the solution to the Cauchy problem $(\mathrm{E})_\varepsilon$, and as well as, the solutions $w_n := [\eta_n, \theta_n] \in [\mathscr{H}]^2$ to $(\mathrm{S})_{\varepsilon_n}$, $n = 1, 2, 3, \dots$, coincide with the solutions to the Cauchy problems $(\mathrm{E})_{\varepsilon_n}$ for the initial data $w_{0,n} := [\eta_{0,n}, \theta_{0,n}] \in D$, and forcing terms $\mathfrak{f}_n = [M_u u_n, M_v v_n] \in [\mathscr{H}]^2, n = 1, 2, 3, \dots$, respectively.

On this basis, we next verify:

(ev.4) $\Phi^R_{\varepsilon_n} \geq 0$ on $[H]$, for $n = 1, 2, 3, \dots$, and the union $\bigcup_{n=1}^{\infty} \{ \tilde{w} \in [H]^2 \,|\, \Phi^R_{\varepsilon_n}(\tilde{w}) \leq r \}$ of sublevel sets is contained in the compact set $K^R_\nu(r) \subset [H]^2$, as in (ev.3), for any $r > 0$;

(ev.5) $\Phi^R_{\varepsilon_n} \to \Phi^R_\varepsilon$ on $[H]^2$, in the sense of Mosco, as $n \to \infty$, more precisely, the uniform estimate (7.1.4a) will lead to the corresponding lower bound condition and optimality condition, in the Mosco-convergence of $\{\Phi^R_{\varepsilon_n}\}_{n=1}^{\infty}$;

(ev.6) $\sup_{n\in\mathbb{N}} \Phi^R_{\varepsilon_n}(w_{0,n}) < \infty$, and $w_{0,n} \to w_0$ in $[H]^2$, as $n \to \infty$, more precisely, it follows from (7.3.4), (A1), and (A4) that

$$\sup_{n\in\mathbb{N}} \Phi^R_{\varepsilon_n}(w_{0,n}) \leq \sup_{n\in\mathbb{N}} \bigl(\tfrac{1+R}{2} |\eta_{0,n}|^2_V + \nu^2 (\mathcal{L}^N(\Omega) + |\theta_{0,n}|^2_{V_0}) + \tfrac{1}{\nu^2} |\alpha(\eta_{0,n})|^2_H \bigr) < \infty,$$

and the weak convergence of $\{w_{0,n}\}_{n=1}^{\infty}$ in $D = V \times V_0$ and the compactness of embedding $D \subset [H]^2$ imply the strong convergence of $\{w_{0,n}\}_{n=1}^{\infty}$ in $[H]^2$.

On account of (7.3.4) and (ev.0)–(ev.6), we can apply Proposition 7.2, to show that:

$$\begin{cases} w_n \to w \text{ in } C([0,T];[H]^2) \text{ (i.e. in } [C([0,T];H)]^2), \\ \quad \text{weakly in } W^{1,2}(0,T;[H]^2) \text{ (i.e. weakly in } [W^{1,2}(0,T;H)]^2), \\ \displaystyle\int_0^T \Phi^R_{\varepsilon_n}(w_n(t))\,dt \to \int_0^T \Phi^R_\varepsilon(w(t))\,dt, \end{cases} \quad \text{as } n \to \infty, \tag{7.4.18a}$$

$$\begin{aligned} \sup_{n\in\mathbb{N}} |w_n|^2_{L^\infty(0,T;V)\times L^\infty(0,T;V_0)} &\leq 4 \sup_{n\in\mathbb{N}} |w_n|^2_{L^\infty(0,T;V\times V_0)} \\ &\leq \frac{8}{1 \wedge \nu^2 \wedge R} \sup_{n\in\mathbb{N}} |\Phi^R_{\varepsilon_n}(w_n)|_{L^\infty(0,T)} < \infty, \end{aligned}$$

and hence,

$$w_n \to w \text{ weakly-}* \text{ in } L^\infty(0,T;V) \times L^\infty(0,T;V_0), \text{ as } n \to \infty. \tag{7.4.18b}$$

Furthermore, from (7.1.3), (7.1.4a), (7.4.18), and the assumptions (A2) and (A4), one can observe that:

$$\begin{cases} \displaystyle\varliminf_{n\to\infty} \frac{1}{2} |\nabla \eta_n|^2_{[\mathscr{H}]^N} \geq \frac{1}{2} |\nabla \eta|^2_{[\mathscr{H}]^N}, \quad \varliminf_{n\to\infty} \frac{R}{2} |\eta_n|^2_{\mathscr{H}} \geq \frac{R}{2} |\eta|^2_{\mathscr{H}}, \\ \displaystyle\varliminf_{n\to\infty} \frac{\nu^2}{2} |\theta_n|^2_{\mathscr{V}_0} \geq \frac{\nu^2}{2} |\theta|^2_{\mathscr{V}_0}, \quad \lim_{n\to\infty} \frac{1}{2\nu^2} |\alpha(\eta_n)|^2_{\mathscr{H}} = \frac{1}{2\nu^2} |\alpha(\eta)|^2_{\mathscr{H}}, \end{cases} \tag{7.4.19a}$$

and

$$\begin{aligned} \varliminf_{n\to\infty} \bigl| \alpha(\eta_n) f_{\varepsilon_n}(\nabla\theta_n) \bigr|_{L^1(Q)} &= \varliminf_{n\to\infty} \int_0^T \int_\Omega \alpha(\eta_n(t)) f_{\varepsilon_n}(\nabla\theta_n(t))\,dxdt \\ &\geq \varliminf_{n\to\infty} \int_0^T \int_\Omega \alpha(\eta(t)) f_{\varepsilon_n}(\nabla\theta_n(t))\,dxdt \\ &\qquad - \lim_{n\to\infty} |\alpha(\eta_n) - \alpha(\eta)|_{\mathscr{H}} \cdot \sup_{n\in\mathbb{N}} \bigl(\sqrt{\mathcal{L}^{N+1}(Q)}\, \varepsilon_n + |\theta_n|_{\mathscr{V}_0} \bigr) \end{aligned}$$

$$\geq \varliminf_{n\to\infty} \int_0^T \int_\Omega \alpha(\eta(t)) f_\varepsilon(\nabla\theta_n(t))\, dxdt - |\alpha(\eta)|_{L^1(Q)} \cdot \lim_{n\to\infty} |\varepsilon_n - \varepsilon|$$

$$\geq \int_0^T \int_\Omega \alpha(\eta(t)) f_\varepsilon(\nabla\theta(t))\, dxdt = \bigl|\alpha(\eta) f_\varepsilon(\nabla\theta)\bigr|_{L^1(Q)}. \tag{7.4.19b}$$

Here, from (7.4.1), it is seen that:

$$\int_0^T \Phi_{\tilde\varepsilon}^R(\tilde w(t))\, dt = \int_0^T \Phi_{\tilde\varepsilon}^R(\tilde\eta(t), \tilde\theta(t))\, dt$$

$$= \frac{1}{2}|\nabla\tilde\eta|_{[\mathscr{H}]^N}^2 + \frac{R}{2}|\tilde\eta|_{\mathscr{H}}^2 + \frac{\nu^2}{2}|\tilde\theta|_{\mathscr{V}_0}^2 + \bigl|\alpha(\tilde\eta) f_{\tilde\varepsilon}(\nabla\tilde\theta)\bigr|_{L^1(Q)} + \frac{1}{2\nu^2}|\alpha(\tilde\eta)|_{\mathscr{H}}^2$$

$$+ \frac{\nu^2\tilde\varepsilon^2}{2}\mathcal{L}^{N+1}(Q), \text{ for all } \tilde\varepsilon \geq 0 \text{ and } \tilde w = [\tilde\eta, \tilde\theta] \in \mathscr{Y}. \tag{7.4.20}$$

Taking into account (7.4.18a), (7.4.19), and (7.4.20), we deduce that:

$$|\nabla\eta_n|_{[\mathscr{H}]^N}^2 + R|\eta_n|_{\mathscr{H}}^2 + \nu^2|\theta_n|_{\mathscr{V}_0}^2 \to |\nabla\eta|_{[\mathscr{H}]^N}^2 + R|\eta|_{\mathscr{H}}^2 + \nu^2|\theta|_{\mathscr{V}_0}^2,$$

$$\text{and hence, } |[\eta_n, \theta_n]|_{\mathscr{Y}} \to |[\eta, \theta]|_{\mathscr{Y}}, \text{ as } n \to \infty. \tag{7.4.21}$$

Since the norm of Hilbert space $\mathscr{Y} := \mathscr{V} \times \mathscr{V}_0$ is uniformly convex, the convergences (7.4.18b) and (7.4.21) imply the strong convergence:

$$w_n \to w \text{ in } \mathscr{Y}, \text{ as } n \to \infty, \tag{7.4.22a}$$

and furthermore, it follows from (7.1.4a) and (7.4.22a) that:

$$|f_{\varepsilon_n}(\nabla\theta_n) - f_\varepsilon(\nabla\theta)|_{\mathscr{H}} \leq |f_{\varepsilon_n}(\nabla\theta_n) - f_{\varepsilon_n}(\nabla\theta)|_{\mathscr{H}} + |f_{\varepsilon_n}(\nabla\theta) - f_\varepsilon(\nabla\theta)|_{\mathscr{H}}$$

$$\leq |\theta_n - \theta|_{\mathscr{V}_0} + \sqrt{\mathcal{L}^{N+1}(Q)}|\varepsilon_n - \varepsilon| \to 0, \text{ as } n \to \infty. \tag{7.4.22b}$$

The convergences (7.4.18) and (7.4.22) are sufficient to obtain the convergence (7.3.6) as in Main Theorem 7.1 (I-B).

Finally, let us assume (7.3.7) to verify (7.3.8). In the light of (A3), we can take a large constant $L_* > 0$, independent of n, such that:

$$L_* \geq \sup_{n\in\mathbb{N}} |\eta_{0,n}|_{L^\infty(\Omega)}, \ g(L_*) \geq M_u \sup_{n\in\mathbb{N}} |u_n|_{L^\infty(Q)},$$

$$\text{and } g(-L_*) \leq -M_u \sup_{n\in\mathbb{N}} |u_n|_{L^\infty(Q)}. \tag{7.4.23}$$

Then, just as in the derivation of (7.4.16), we can show that:

$$\sup_{n\in\mathbb{N}} |\eta_n|_{L^\infty(Q)} \leq L_*, \text{ i.e. } -L_* \leq \eta_n \leq L_* \text{ a.e. in } Q,\ n = 1, 2, 3, \ldots. \tag{7.4.24}$$

The convergence (7.3.6), and the L^∞-weak-$*$ compactness brought by (7.4.24) lead to the convergence (7.3.8). □

7.5 Proof of Main Theorem 7.2

In this section, we prove the second Main Theorem 7.2. Before the proof, we prepare the following lemma.

Lemma 7.3. Let us assume (A5) and (A6), and let us fix the function $\bar{\kappa} \in \bigcap_{n=1}^{\infty} K_n$ as in (A6). Besides, let us take any function $u \in K$, and define a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathscr{H}$, by setting:

$$u_n := \mathrm{proj}_{K_n}(u) = \kappa_n^0 \vee (\kappa_n^1 \wedge u) \in K_n \text{ in } \mathscr{H}, \text{ for } n = 1, 2, 3, \ldots.$$

Then, it holds that:

$$u_n \to u \text{ in } \mathscr{H} \text{ as } n \to \infty. \tag{7.5.1}$$

Proof. As is easily seen,

$$u_n(t,x) = \begin{cases} \kappa_n^1(t,x), & \text{if } u(t,x) > \kappa_n^1(t,x), \\ u(t,x), & \text{if } \kappa_n^0(t,x) \leq u(t,x) \leq \kappa_n^1(t,x), \\ \kappa_n^0(t,x), & \text{if } u(t,x) < \kappa_n^0(t,x), \end{cases} \tag{7.5.2}$$
$$\text{a.e. } (t,x) \in Q,\ n = 1, 2, 3, \ldots,$$

so that:

$$|u_n - u| \to 0, \text{ in the pointwise sense, a.e. in } Q, \text{ as } n \to \infty. \tag{7.5.3}$$

Also, owing to the presence of $\bar{\kappa} \in \bigcap_{n=1}^{\infty} K_n$, as in (A6),

$$-[u - \bar{\kappa}]^- \leq u_n - \bar{\kappa} \leq [u - \bar{\kappa}]^+, \text{ a.e. in } Q,$$
$$\text{i.e. } |u_n - \bar{\kappa}| \leq [u - \bar{\kappa}]^+ + [u - \bar{\kappa}]^- = |u - \bar{\kappa}|, \text{ a.e. in } Q,$$

which leads to:

$$|u_n - u| \leq |u_n - \bar{\kappa}| + |u - \bar{\kappa}| \leq 2|u - \bar{\kappa}| \text{ a.e. in } Q,$$
$$\text{with } |u - \bar{\kappa}| \in \mathscr{H}. \tag{7.5.4}$$

The convergence (7.5.1) will be deduced as a straightforward consequence of (7.5.3), (7.5.4), and the dominated convergence theorem [56, Theorem 10 on page 36]. □

Now, let $[\eta_0, \theta_0] \in D$ be the initial pair, and any constraint $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$. Also, let us fix arbitrary forcing pair $[\bar{u}, \bar{v}] \in \mathscr{U}_{\mathrm{ad}}^K$, and let us invoke the definition of the cost function $\mathcal{J}_\varepsilon$, defined in (1.5.34), to estimate that:

$$0 \leq \underline{J}_\varepsilon := \inf \mathcal{J}_\varepsilon(\mathscr{U}_{\mathrm{ad}}^K) \leq \overline{J}_\varepsilon := \mathcal{J}_\varepsilon(\bar{u}, \bar{v}) < \infty, \text{ for all } \varepsilon \geq 0. \tag{7.5.5}$$

Also, for any $\varepsilon \geq 0$, we denote by $[\bar{\eta}, \bar{\theta}]$ the solution to $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[\bar{u}, \bar{v}]$.

Based on these, the Main Theorem 7.2 is proved as follows.

Proof of Main Theorem 7.2 (II-A). Let us fix any $\varepsilon \geq 0$. Then, from the estimate (7.5.5), we immediately find a sequence of forcing pairs $\{[u_n, v_n]\}_{n=1}^{\infty} \subset \mathscr{U}_{\rm ad}^K$, such that:

$$\mathcal{J}_\varepsilon(u_n, v_n) \downarrow \underline{J}_\varepsilon, \text{ as } n \to \infty, \tag{7.5.6a}$$

and

$$\frac{1}{2} \sup_{n \in \mathbb{N}} \bigl| [\sqrt{M_u} u_n, \sqrt{M_v} v_n] \bigr|_{[\mathscr{H}]^2}^2 \leq \mathcal{J}_\varepsilon(\bar{u}, \bar{v}) < \infty. \tag{7.5.6b}$$

Also, the estimate (7.5.6b) and the assumption (A5) enable us to take a subsequence of $\{[u_n, v_n]\}_{n=1}^{\infty} \subset \mathscr{U}_{\rm ad}^K$ (not relabeled), and to find a pair of functions $[u^*, v^*] \in \mathscr{U}_{\rm ad}^K$, such that:

$$[\sqrt{M_u} u_n, \sqrt{M_v} v_n] \to [\sqrt{M_u} u^*, \sqrt{M_v} v^*] \text{ weakly in } [\mathscr{H}]^2, \text{as } n \to \infty, \tag{7.5.7}$$

Let $[\eta^*, \theta^*] \in [\mathscr{H}]^2$ be the solution to $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u^*, v^*]$. Also, for any $n \in \mathbb{N}$, let $[\eta_n, \theta_n] \in [\mathscr{H}]^2$ be the solution to $(\mathrm{S})_{\varepsilon_n}$, for the forcing pair $[u_n, v_n]$. Then, having in mind (7.3.4), (7.5.7), and the initial condition:

$$[\eta_n(0), \theta_n(0)] = [\eta^*(0), \theta^*(0)] = [\eta_0, \theta_0] \text{ in } [H]^2, \text{ for } n = 1, 2, 3, \ldots,$$

we can apply Main Theorem 7.1 (I-B), to see that:

$$[\eta_n, \theta_n] \to [\eta^*, \theta^*] \text{ in } [C([0,T]; H)]^2, \text{ as } n \to \infty. \tag{7.5.8}$$

On account of (7.5.6a), (7.5.7), and (7.5.8), it is computed that:

$$\begin{aligned}
\mathcal{J}_\varepsilon(u^*, v^*) &= \frac{1}{2} \bigl| [\sqrt{M_\eta}(\eta^* - \eta_{\rm ad}), \sqrt{M_\theta}(\theta^* - \theta_{\rm ad})] \bigr|_{[\mathscr{H}]^2}^2 \\
&\qquad + \frac{1}{2} \bigl| [\sqrt{M_u} u^*, \sqrt{M_v} v^*] \bigr|_{[\mathscr{H}]^2}^2 \\
&\leq \frac{1}{2} \lim_{n \to \infty} \bigl| [\sqrt{M_\eta}(\eta_n - \eta_{\rm ad}), \sqrt{M_\theta}(\theta_n - \theta_{\rm ad})] \bigr|_{[\mathscr{H}]^2}^2 \\
&\qquad + \frac{1}{2} \varliminf_{n \to \infty} \bigl| [\sqrt{M_u} u_n, \sqrt{M_v} v_n] \bigr|_{[\mathscr{H}]^2}^2 \\
&= \lim_{n \to \infty} \mathcal{J}_\varepsilon(u_n, v_n) = \underline{J}_\varepsilon \ (\leq \mathcal{J}_\varepsilon(u^*, v^*)),
\end{aligned}$$

and this leads to:

$$\mathcal{J}_\varepsilon(u^*, v^*) = \min_{[u,v] \in \mathscr{U}_{\rm ad}^K} \mathcal{J}_\varepsilon(u, v).$$

Thus, we conclude the item (II-A). □

Proof of Main Theorem 7.2 (II-B). Let us take $\varepsilon \geq 0$, $\{\varepsilon_n\}_{n=1}^{\infty} \subset [0, \infty)$, and $\{[\eta_{0,n}, \theta_{0,n}]\}_{n=1}^{\infty} \subset D$ as in (7.3.4). Besides, for the pair of functions $[\bar{u}, \bar{v}] \in \mathcal{U}_{\rm ad}^K$ as in (7.5.5), let us define:

$$\bar{u}_n := \mathrm{proj}_{K_n}(\bar{u}) = \kappa_n^0 \vee (\kappa_n^1 \wedge \bar{u}) \in K_n, \ n = 1, 2, 3, \ldots.$$

Then, from Lemma 7.3, it immediately follows that:

$$\bar{u}_n \to \bar{u} \text{ in } \mathscr{H}, \text{ as } n \to \infty. \tag{7.5.9}$$

Here, let $[\bar{\eta}, \bar{\theta}] \in [\mathscr{H}]^2$ be the solution to $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[\bar{u}, \bar{v}]$, and let $[\bar{\eta}_n, \bar{\theta}_n] \in [\mathscr{H}]^2$, $n = 1, 2, 3, \dots$, be solutions to $(\mathrm{S})_{\varepsilon_n}$, for the initial pairs $[\eta_{0,n}, \theta_{0,n}]$, and forcing pairs $[\bar{u}_n, \bar{v}]$, $n = 1, 2, 3, \dots$, respectively. Then, invoking (7.3.4) and (7.5.9), we can apply Main Theorem 7.1 (I-B) to these solutions, and we can see that:

$$[\bar{\eta}_n, \bar{\theta}_n] \to [\bar{\eta}, \bar{\theta}] \text{ in } [C([0,T]; H)]^2, \tag{7.5.10a}$$

and in particular,

$$\begin{aligned} [\eta_{0,n}, \theta_{0,n}] = [\bar{\eta}_n(0), \bar{\theta}_n(0)] \to [\eta_0, \theta_0] = [\bar{\eta}(0), \bar{\theta}(0)] \\ \text{in } [H]^2, \text{ as } n \to \infty. \end{aligned} \tag{7.5.10b}$$

The convergences (7.5.9) and (7.5.10) enable us to estimate:

$$\overline{J}_{\sup} := \sup_{n \in \mathbb{N}} \mathcal{J}_{\varepsilon_n}(\bar{u}_n, \bar{v}) < \infty. \tag{7.5.11}$$

Next, for any $n \in \mathbb{N}$, let us denote by $[\eta_n^*, \theta_n^*] \in [\mathscr{H}]^2$ the solution to $(\mathrm{S})_{\varepsilon_n}$, for the initial pair $[\eta_{0,n}, \theta_{0,n}]$, and forcing pair $[u_n^*, v_n^*]$ of the optimal control of $(\mathrm{OP})_{\varepsilon_n}^{K_n}$. Then, in the light of (7.5.5) and (7.5.11), it is observed that:

$$0 \leq \frac{1}{2} \bigl| [\sqrt{M_u} u_n^*, \sqrt{M_v} v_n^*] \bigr|_{[\mathscr{H}]^2}^2 \leq \underline{J}_{\varepsilon_n} \leq \overline{J}_{\sup} < \infty, \; n = 1, 2, 3, \dots.$$

Therefore, one can find a subsequence $\{n_i\}_{i=1}^\infty \subset \{n\}$, together with a limiting pair of functions $[u^{**}, v^{**}] \in [\mathscr{H}]^2$, such that:

$$\begin{aligned} [\sqrt{M_u} u_{n_i}^*, \sqrt{M_v} v_{n_i}^*] \to [\sqrt{M_u} u^{**}, \sqrt{M_v} v^{**}] \text{ weakly in } [\mathscr{H}]^2, \text{ as } i \to \infty, \\ \text{and as well as } [M_u u_{n_i}^*, M_v v_{n_i}^*] \to [M_u u^{**}, M_v v^{**}] \text{ weakly in } [\mathscr{H}]^2, \text{ as } i \to \infty. \end{aligned} \tag{7.5.12}$$

Additionally, for every $\ell = 0, 1$, the convex functionals on $L^1(Q \setminus |\kappa^\ell|^{-1}(\infty))$, defined as:

$$\tilde{u} \in L^1(Q \setminus |\kappa^\ell|^{-1}(\infty)) \mapsto \int_{Q \setminus |\kappa^\ell|^{-1}(\infty)} [\tilde{u}]^+ \, dxdt \in [0, \infty), \; \ell = 0, 1,$$

are weakly lower semi-continuous. Therefore, we can observe from (7.5.12) and (A6) that:

$$\begin{cases} [\kappa^0 - u^{**}]^+ \leq [\bar{\kappa} - u^{**}]^+ \in \mathscr{H} \subset L^1(Q), \\ [u^{**} - \kappa^1]^+ \leq [u^{**} - \bar{\kappa}]^+ \in \mathscr{H} \subset L^1(Q), \end{cases}$$

$$\begin{aligned} \bigl| M_u [\kappa^0 - u^{**}]^+ \bigr|_{L^1(Q)} &= \int_{Q \setminus |\kappa^0|^{-1}(\infty)} [M_u(\kappa^0 - u^{**})]^+ \, dxdt \\ &\leq \varliminf_{i \to \infty} \int_{Q \setminus |\kappa^0|^{-1}(\infty)} [M_u(\kappa_{\varepsilon_{n_i}}^0 - u_{n_i}^*)]^+ \, dxdt = 0, \end{aligned} \tag{7.5.13a}$$

and

$$\left|M_u[u^{**}-\kappa^1]^+\right|_{L^1(Q)} = \int_{Q\setminus|\kappa^1|^{-1}(\infty)} [M_u(u^{**}-\kappa^1)]^+ \, dxdt$$
$$\leq \varliminf_{i\to\infty} \int_{Q\setminus|\kappa^1|^{-1}(\infty)} [M_u(u^*_{n_i}-\kappa^1_{\varepsilon_{n_i}})]^+ \, dxdt = 0. \qquad (7.5.13b)$$

Since the limit u^{**}, when $M_u = 0$, can be taken arbitrary, the estimates as in (7.5.13) enable us to suppose that:

$$\kappa^0 \leq u^{**} \leq \kappa^1 \text{ a.e. in } Q, \text{ i.e. } [u^{**}, v^{**}] \in \mathscr{U}_{\mathrm{ad}}^K.$$

Now, let us denote by $[\eta^{**}, \theta^{**}] \in [\mathscr{H}]^2$ the solution to $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u^{**}, v^{**}]$. Then, applying Main Theorem 7.1 (I-B), again, to the solutions $[\eta^{**}, \theta^{**}]$ and $[\eta^*_{n_i}, \theta^*_{n_i}]$, $i = 1, 2, 3, \dots$, one can see that:

$$\begin{aligned}
&[\eta^*_{n_i}, \theta^*_{n_i}] \to [\eta^{**}, \theta^{**}] \text{ in } [C([0,T];H)]^2, \text{ in } \mathscr{Y}, \\
&\qquad \text{weakly in } [W^{1,2}(0,T;H)]^2, \text{ and} \\
&\qquad \text{weakly-}* \text{ in } L^\infty(0,T;V) \times L^\infty(0,T;V_0), \text{ as } i \to \infty.
\end{aligned} \qquad (7.5.14)$$

As a consequence of (7.5.9), (7.5.10), (7.5.12), and (7.5.14), it is verified that:

$$\begin{aligned}
\mathcal{J}_\varepsilon(u^{**}, v^{**}) &= \frac{1}{2}\left|[\sqrt{M_\eta}(\eta^{**}-\eta_{\mathrm{ad}}), \sqrt{M_\theta}(\theta^{**}-\theta_{\mathrm{ad}})]\right|^2_{[\mathscr{H}]^2} \\
&\quad + \frac{1}{2}\left|[\sqrt{M_u}u^{**}, \sqrt{M_v}v^{**}]\right|^2_{[\mathscr{H}]^2} \\
&\leq \frac{1}{2}\lim_{i\to\infty}\left|[\sqrt{M_\eta}(\eta^*_{n_i}-\eta_{\mathrm{ad}}), \sqrt{M_\theta}(\theta^*_{n_i}-\theta_{\mathrm{ad}})]\right|^2_{[\mathscr{H}]^2} \\
&\quad + \frac{1}{2}\varliminf_{i\to\infty}\left|[\sqrt{M_u}u^*_{n_i}, \sqrt{M_v}v^*_{n_i}]\right|^2_{[\mathscr{H}]^2} \\
&= \varliminf_{i\to\infty} \mathcal{J}_{\varepsilon_{n_i}}(u^*_{n_i}, v^*_{n_i}) \leq \lim_{i\to\infty} \mathcal{J}_{\varepsilon_{n_i}}(\bar{u}_{n_i}, \bar{v}) \\
&= \frac{1}{2}\lim_{i\to\infty}\left|[\sqrt{M_\eta}(\bar{\eta}_{n_i}-\eta_{\mathrm{ad}}), \sqrt{M_\theta}(\bar{\theta}_{n_i}-\theta_{\mathrm{ad}})]\right|^2_{[\mathscr{H}]^2} \\
&\quad + \frac{1}{2}\lim_{i\to\infty}\left|[\sqrt{M_u}\bar{u}_{n_i}, \sqrt{M_v}\bar{v}]\right|^2_{[\mathscr{H}]^2} \\
&= \mathcal{J}_\varepsilon(\bar{u}, \bar{v}).
\end{aligned}$$

Since the choice of $[\bar{u}, \bar{v}] \in \mathscr{U}_{\mathrm{ad}}^K$ is arbitrary, we conclude that:

$$\mathcal{J}_\varepsilon(u^{**}, v^{**}) = \min_{[u,v]\in\mathscr{U}_{\mathrm{ad}}^K} \mathcal{J}_\varepsilon(u, v),$$

and complete the proof of Main Theorem 7.2 (II-B). □

7.6 Proof of Main Theorem 7.3

Throughout this Section, we suppose the situation (r.s.0). Let $\varepsilon > 0$ be a fixed constant, and let $[\eta_0, \theta_0] \in D_0$ be the initial pair. Let us take any forcing pair $[u, v] \in \mathscr{X}$ (=

$L^\infty(Q) \times \mathscr{H})$, and consider the unique solution $[\eta, \theta] \in [\mathscr{H}]^2$ to the state-system $(\mathrm{S})_\varepsilon$. Also, let us take any constant $\delta \in (0,1)$ and any pair of functions $[h, k] \in \mathscr{X}$, and consider another solution $[\eta^\delta, \theta^\delta] \in [\mathscr{H}]^2$ to the system $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and a perturbed forcing pair $[u + \delta h, v + \delta k]$. On this basis, we consider a sequence of pairs of functions $\{[\chi^\delta, \gamma^\delta]\}_{\delta \in (0,1)} \subset [\mathscr{H}]^2$, defined as:

$$[\chi^\delta, \gamma^\delta] := \left[\frac{\eta^\delta - \eta}{\delta},\ \frac{\theta^\delta - \theta}{\delta}\right] \in [\mathscr{H}]^2, \text{ for } \delta \in (0,1). \tag{7.6.1}$$

This sequence acts a key-role in the computation of Gâteaux differential of the cost function $\mathcal{J}_\varepsilon$, for $\varepsilon > 0$.

Remark 7.13. Note that for any $\delta \in (0,1)$, the pair of functions $[\chi^\delta, \gamma^\delta] \in [\mathscr{H}]^2$ fulfills the following variational forms:

$$\begin{aligned}
&(\partial_t \chi^\delta(t), \varphi)_H + (\nabla \chi^\delta(t), \nabla \varphi)_{[H]^N} \\
&\quad + \int_\Omega \left(\int_0^1 g'(\eta(t) + \varsigma\delta\chi^\delta(t))\, d\varsigma \right) \chi^\delta(t) \varphi \, dx \\
&\quad + \int_\Omega \left(f_\varepsilon(\nabla\theta(t)) \int_0^1 \alpha''(\eta(t) + \varsigma\delta\chi^\delta(t))\, d\varsigma \right) \chi^\delta(t) \varphi \, dx \\
&\quad + \int_\Omega \left(\alpha'(\eta^\delta(t)) \int_0^1 \nabla f_\varepsilon(\nabla\theta(t) + \varsigma\delta\nabla\gamma^\delta(t))\, d\varsigma \right) \cdot \nabla\gamma^\delta(t) \varphi \, dx \\
&= (M_u h(t), \varphi)_H, \text{ for any } \varphi \in V, \text{ a.e. } t \in (0,T), \text{ subject to } \chi^\delta(0) = 0 \text{ in } H,
\end{aligned}$$

and

$$\begin{aligned}
&(\alpha_0(t)\partial_t \gamma^\delta(t), \psi)_H + \nu^2 (\nabla \gamma^\delta(t), \nabla \psi)_{[H]^N} \\
&\quad + \int_\Omega \left(\alpha(\eta^\delta(t)) \int_0^1 \nabla^2 f_\varepsilon(\nabla\theta(t) + \varsigma\delta\nabla\gamma^\delta(t))\, d\varsigma \right) \nabla\gamma^\delta(t) \cdot \nabla\psi \, dx \\
&\quad + \int_\Omega \left(\left(\int_0^1 \alpha'(\eta(t) + \varsigma\delta\chi^\delta(t))\, d\varsigma \right) \chi^\delta(t) \right) \nabla f_\varepsilon(\nabla\theta(t)) \cdot \nabla\psi \, dx \\
&= (M_v k(t), \psi)_H, \text{ for any } \psi \in V_0, \text{ a.e. } t \in (0,T), \text{ subject to } \gamma^\delta(0) = 0 \text{ in } H.
\end{aligned}$$

In fact, these variational forms are obtained by taking the difference between respective two variational forms for $[\eta^\delta, \theta^\delta]$ and $[\eta, \theta]$, as in Main Theorem 7.1 (I-A), and by using the following linearization formulas:

$$\frac{1}{\delta}\big(g(\eta^\delta) - g(\eta)\big) = \left(\int_0^1 g'(\eta + \varsigma\delta\chi^\delta)\, d\varsigma \right) \chi^\delta \text{ in } \mathscr{H},$$

$$\begin{aligned}
&\frac{1}{\delta}\big(\alpha'(\eta^\delta) f_\varepsilon(\nabla\theta^\delta) - \alpha'(\eta) f_\varepsilon(\nabla\theta)\big) \\
&\quad = \frac{1}{\delta}\big(\alpha'(\eta^\delta) - \alpha'(\eta)\big) f_\varepsilon(\nabla\theta) + \frac{1}{\delta}\alpha'(\eta^\delta)\big(f_\varepsilon(\nabla\theta^\delta) - f_\varepsilon(\nabla\theta)\big) \\
&\quad = \left(f_\varepsilon(\nabla\theta) \int_0^1 \alpha''(\eta + \varsigma\delta\chi^\delta)\, d\varsigma \right) \chi^\delta
\end{aligned}$$

$$+\left(\alpha'(\eta^\delta)\int_0^1 \nabla f_\varepsilon(\nabla\theta+\varsigma\delta\nabla\gamma^\delta)\,d\varsigma\right)\cdot\nabla\gamma^\delta \text{ in } \mathscr{H},$$

and

$$\begin{aligned}
&\frac{1}{\delta}\big(\alpha(\eta^\delta)\nabla f_\varepsilon(\nabla\theta^\delta)-\alpha(\eta)\nabla f_\varepsilon(\nabla\theta)\big)\\
&\quad=\frac{1}{\delta}\alpha(\eta^\delta)\big(\nabla f_\varepsilon(\nabla\theta^\delta)-\nabla f_\varepsilon(\nabla\theta)\big)+\frac{1}{\delta}\big(\alpha(\eta^\delta)-\alpha(\eta)\big)\nabla f_\varepsilon(\nabla\theta)\\
&\quad=\left(\alpha(\eta^\delta)\int_0^1 \nabla^2 f_\varepsilon(\nabla\theta+\varsigma\delta\nabla\gamma^\delta)\,d\varsigma\right)\nabla\gamma^\delta\\
&\qquad+\left(\left(\int_0^1 \alpha'(\eta+\varsigma\delta\chi^\delta)\,d\varsigma\right)\chi^\delta\right)\nabla f_\varepsilon(\nabla\theta) \text{ in } [\mathscr{H}]^N.
\end{aligned}$$

Incidentally, the above linearization formulas can be verified as consequences of the assumptions (A1)–(A4) and the mean-value theorem (cf. [54, Theorem 5 in p. 313]).

Remark 7.14. Note that the situation (r.s.0) implies $\eta_0 \in L^\infty(\Omega)$ and $u \in L^\infty(Q)$. Therefore, under (r.s.0), we can suppose $\eta \in L^\infty(Q)$ for the solution $[\eta,\theta] \in [\mathscr{H}]^2$ to the system $(\mathrm{S})_\varepsilon$.

Now, we prepare the following two Lemmas, for the proof of Main Theorem 7.3.

Lemma 7.4. Under the assumptions (A1)–(A5), let us fix $\varepsilon > 0$, and suppose (r.s.0) as in Main Theorem 7.3. Then, the restriction of the cost $\mathcal{J}_\varepsilon|_{\mathscr{X}} : \mathscr{X} \longrightarrow \mathbb{R}$ is Gâteaux differentiable over $\mathscr{X}$. Moreover, for any $[u,v] \in \mathscr{X}$, the Gâteaux derivative $(\mathcal{J}_\varepsilon|_{\mathscr{X}})'(u,v) \in \mathscr{X}^*$ admits a unique extension $\mathcal{J}_\varepsilon'(u,v) \in ([\mathscr{H}]^2)^* = [\mathscr{H}]^2$, such that:

$$\mathcal{J}_\varepsilon'(u,v) = (\mathcal{J}_\varepsilon|_{\mathscr{X}})'(u,v) \text{ in } \mathscr{X}^*, \tag{7.6.2}$$

and

$$\begin{aligned}
\big(\mathcal{J}_\varepsilon'(u,v),[h,k]\big)_{[\mathscr{H}]^2} &= \big([M_\eta(\eta-\eta_{\mathrm{ad}}), M_\theta(\theta-\theta_{\mathrm{ad}})], \bar{\mathcal{P}}_\varepsilon[M_u h, M_v k]\big)_{[\mathscr{H}]^2}\\
&\quad+\big([M_u u, M_v v],[h,k]\big)_{[\mathscr{H}]^2}, \text{ for any } [h,k] \in \mathscr{X}.
\end{aligned} \tag{7.6.3}$$

In the context, $[\eta,\theta]$ is the solution to the state-system $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0,\theta_0]$ and forcing pair $[u,v]$, and $\bar{\mathcal{P}}_\varepsilon : [\mathscr{H}]^2 \longrightarrow \mathscr{Z}$ is a bounded linear operator, which is given as a restriction $\mathcal{P}|_{\{[0,0]\}\times[\mathscr{H}]^2}$ of the (linear) isomorphism $\mathcal{P} = \mathcal{P}(a,b,\mu,\lambda,\omega,A) : [H]^2 \times \mathscr{Y}^* \longrightarrow \mathscr{Z}$, as in Proposition 7.5, in the case when:

$$\begin{cases}
[a,b] = [\alpha_0, 0] \text{ in } W^{1,\infty}(Q) \times L^\infty(Q),\\
\mu = \bar{\mu}_\varepsilon := \alpha''(\eta) f_\varepsilon(\nabla\theta) \text{ in } L^\infty(0,T;H),\\
\lambda = \bar{\lambda}_\varepsilon := g'(\eta) \text{ in } L^\infty(Q),\\
\omega = \bar{\omega}_\varepsilon := \alpha'(\eta)\nabla f_\varepsilon(\nabla\theta) \text{ in } [L^\infty(Q)]^N,\\
A = \bar{A}_\varepsilon := \alpha(\eta)\nabla^2 f_\varepsilon(\nabla\theta) \text{ in } [L^\infty(Q)]^{N\times N}.
\end{cases} \tag{7.6.4}$$

Proof. Let us fix any $[u, v] \in \mathscr{X}$, and take any $\delta \in (0,1)$ and any $[h, k] \in \mathscr{X}$. Then, due to the assumptions (r.s.0) and $[u, v], [h, k] \in \mathscr{X}$, we can see that:

$$|\eta_0|_{L^\infty(\Omega)} \vee \sup_{\tilde{\delta} \in (0,1)} |u + \tilde{\delta} h|_{L^\infty(Q)} < \infty,$$

and

$$[M_u(u + \delta h), M_v(v + \delta k)] \to [M_u u, M_v v] \text{ in } \mathscr{X}, \text{ as } \delta \downarrow 0.$$

Therefore, as a consequence of Main Theorem 7.1 (I-B), it is observed that:

$$\begin{aligned} [\eta^\delta, \theta^\delta] \to [\eta, \theta] \text{ in } & [C([0,T]; H)]^2, \text{ in } \mathscr{Y}, \\ & \text{weakly in } [W^{1,2}(0,T;H)]^2, \\ & \text{and weakly-}* \text{ in } L^\infty(0,T;V) \times L^\infty(0,T;V_0), \end{aligned} \tag{7.6.5a}$$

and

$$\eta^\delta \to \eta \text{ weakly-}* \text{ in } L^\infty(Q), \text{ as } \delta \downarrow 0. \tag{7.6.5b}$$

In the meantime, it is easily computed that:

$$\begin{aligned} \frac{1}{\delta} & \big(\mathcal{J}_\varepsilon(u + \delta h, v + \delta k) - \mathcal{J}_\varepsilon(u, v) \big) \\ & = \left(\frac{M_\eta}{2} (\eta^\delta + \eta - 2\eta_{\mathrm{ad}}), \chi^\delta \right)_{\mathscr{H}} + \left(\frac{M_\theta}{2} (\theta^\delta + \theta - 2\theta_{\mathrm{ad}}), \gamma^\delta \right)_{\mathscr{H}} \\ & \qquad + \left(\frac{M_u}{2} (2u + \delta h), h \right)_{\mathscr{H}} + \left(\frac{M_v}{2} (2v + \delta k), k \right)_{\mathscr{H}}. \end{aligned} \tag{7.6.6}$$

Here, let us set:

$$\begin{cases} \bar{\mu}_\varepsilon^\delta := f_\varepsilon(\nabla \theta) \displaystyle\int_0^1 \alpha''(\eta + \varsigma \delta \chi^\delta) \, d\varsigma \text{ in } L^\infty(0,T;H), \\ \bar{\lambda}_\varepsilon^\delta := \displaystyle\int_0^1 g'(\eta + \varsigma \delta \chi^\delta) \, d\varsigma \text{ in } L^\infty(Q), \\ \bar{\omega}_\varepsilon^\delta := \alpha'(\eta^\delta) \displaystyle\int_0^1 \nabla f_\varepsilon(\nabla \theta + \varsigma \delta \nabla \gamma^\delta) \, d\varsigma \text{ in } [L^\infty(Q)]^N, \\ \bar{A}_\varepsilon^\delta := \alpha(\eta^\delta) \displaystyle\int_0^1 \nabla^2 f_\varepsilon(\nabla \theta + \varsigma \delta \nabla \gamma^\delta) \, d\varsigma \text{ in } [L^\infty(Q)]^{N \times N}, \end{cases} \tag{7.6.7a}$$

and

$$\begin{aligned} \bar{k}_\varepsilon^\delta := M_v k + \mathrm{div} \Big[& \chi^\delta \nabla f_\varepsilon(\nabla \theta) \int_0^1 \alpha'(\eta + \varsigma \delta \chi^\delta) \, d\varsigma \\ & - \chi^\delta \alpha'(\eta^\delta) \int_0^1 \nabla f_\varepsilon(\nabla \theta + \varsigma \delta \nabla \gamma^\delta) \, d\varsigma \Big] \text{ in } \mathscr{V}_0^*, \\ & \text{for all } \delta \in (0,1). \end{aligned} \tag{7.6.7b}$$

Then, in the light of (7.6.5) and Remark 7.13, one can say that:

$$[\chi^\delta, \gamma^\delta] = \bar{\mathcal{P}}_\varepsilon^\delta [M_u h, \bar{k}_\varepsilon^\delta] \text{ in } \mathscr{Z}, \text{ for } \delta \in (0,1),$$

by using the restriction $\bar{\mathcal{P}}_\varepsilon^\delta := \mathcal{P}|_{\{[0,0]\}\times\mathscr{Y}^*} : \mathscr{Y}^* \longrightarrow \mathscr{X}$ of the (linear) isomorphism $\mathcal{P} = \mathcal{P}(a, b, \mu, \lambda, \omega, A) : [H]^2 \times \mathscr{Y}^* \longrightarrow \mathscr{X}$, as in Proposition 7.5, in the case when:

$$\begin{cases} [a, b, \lambda] = [\alpha_0, 0, \bar{\lambda}_\varepsilon^\delta] \text{ in } W^{1,\infty}(Q) \times [L^\infty(Q)]^2, \\ \omega = \bar{\omega}_\varepsilon^\delta \text{ in } [L^\infty(Q)]^N, \\ A = \bar{A}_\varepsilon^\delta \text{ in } [L^\infty(Q)]^{N\times N}, \\ \mu = \bar{\mu}_\varepsilon^\delta \text{ in } L^\infty(0,T;H), \text{ for } \delta \in (0,1). \end{cases}$$

Besides, taking into account (7.1.3), (7.2.5), (7.6.7), (A3), (A4), and Remark 7.1, we have:

$$\begin{aligned} \bar{C}_0^* :=& \frac{9(1+\nu^2)}{1 \wedge \nu^2 \wedge \delta_*} \cdot \bigl(1 + (C_V^{L^4})^2 + (C_V^{L^4})^4 + (C_{V_0}^{L^4})^2\bigr) \\ & \cdot \bigl(1 + |\alpha_0|_{W^{1,\infty}(Q)} + |g'|_{L^\infty(\mathbb{R})} + |\alpha'|_{L^\infty(\mathbb{R})}^2\bigr) \\ \geq& \frac{9(1+\nu^2)}{1 \wedge \nu^2 \wedge \inf \alpha_0(Q)} \cdot \bigl(1 + (C_V^{L^4})^2 + (C_V^{L^4})^4 + (C_{V_0}^{L^4})^2\bigr) \\ & \cdot \sup_{\delta \in (0,1)} \bigl(1 + |\alpha_0|_{W^{1,\infty}(Q)} + |\bar{\lambda}_\varepsilon^\delta|_{L^\infty(Q)} + |\bar{\omega}_\varepsilon^\delta|_{[L^\infty(Q)]^N}^2\bigr), \end{aligned} \tag{7.6.8a}$$

and

$$\begin{aligned} \bigl|\bigl\langle [M_u\, h(t), \bar{k}_\varepsilon^\delta(t)], [\varphi, \psi] \bigr\rangle_{V \times V_0}\bigr| &\leq |\langle M_u h(t), \varphi \rangle_V| + |\langle \bar{k}_\varepsilon^\delta(t), \psi \rangle_{V_0}| \\ \leq M_u |h(t)|_H |\varphi|_H &+ M_v |k(t)|_H |\psi|_H + 2|\alpha'|_{L^\infty(\mathbb{R})} |\chi^\delta(t)|_H |\nabla \psi|_{[H]^N} \\ \leq M_u |h(t)|_H |\varphi|_V &+ \bigl(M_v C_{V_0}^H |k(t)|_H + 2|\alpha'|_{L^\infty(\mathbb{R})} |\chi^\delta(t)|_H\bigr) |\psi|_{V_0}, \\ &\text{for a.e. } t \in (0,T), \text{ any } [\varphi, \psi] \in V \times V_0, \text{ and any } \delta \in (0,1), \end{aligned} \tag{7.6.8b}$$

with use of the constant $C_{V_0}^H > 0$ of the embedding $V_0 \subset H$, so that

$$\begin{aligned} \bigl|[M_u h(t), \bar{k}_\varepsilon^\delta(t)]\bigr|_{V^* \times V_0^*}^2 \leq \bar{C}_1^* \bigl(\bigl|[h(t), k(t)]\bigr|_{[H]^2}^2 + |\chi^\delta(t)|_H^2\bigr), \\ \text{for a.e. } t \in (0,T), \text{ and any } \delta \in (0,1), \end{aligned} \tag{7.6.8c}$$

with a positive constant $\bar{C}_1^* := 4\bigl(M_u^2 + M_v^2 (C_{V_0}^H)^2 + |\alpha'|_{L^\infty(\mathbb{R})}^2\bigr)$.

Now, having in mind (7.6.8), let us apply Proposition 7.4 to the case when:

$$\begin{cases} [a^1, b^1, \mu^1, \lambda^1, \omega^1, A^1] = [a^2, b^2, \mu^2, \lambda^2, \omega^2, A^2] = [\alpha_0, 0, \bar{\mu}_\varepsilon^\delta, \bar{\lambda}_\varepsilon^\delta, \bar{\omega}_\varepsilon^\delta, \bar{A}_\varepsilon^\delta], \\ [p_0^1, z_0^1] = [p_0^2, z_0^2] = [0,0], \ [h^1, k^1] = [M_u h, \bar{k}_\varepsilon^\delta], \ [h^2, k^2] = [0,0], \\ [p^1, z^1] = [\chi^\delta, \gamma^\delta] = \bar{\mathcal{P}}_\varepsilon^\delta [M_u h, \bar{k}_\varepsilon^\delta], \ [p^2, z^2] = [0,0] = \bar{\mathcal{P}}_\varepsilon^\delta [0,0], \quad \text{for } \delta \in (0,1). \end{cases}$$

Then, we estimate that:

$$\begin{aligned} \frac{d}{dt} \bigl(|\chi^\delta(t)|_H^2 + |\sqrt{\alpha_0(t)}\gamma^\delta(t)|_H^2\bigr) + \bigl(|\chi^\delta(t)|_V^2 + \nu^2 |\gamma^\delta(t)|_{V_0}^2\bigr) \\ \leq 3\bar{C}_0^* \bigl(|\chi^\delta(t)|_H^2 + |\sqrt{\alpha_0(t)}\gamma^\delta(t)|_H^2\bigr) + 2\bar{C}_0^* \bigl(|M_u h(t)|_{V^*}^2 + |\bar{k}_\varepsilon^\delta(t)|_{V_0^*}^2\bigr) \end{aligned}$$

$$\leq 3\bar{C}_0^*(1+\bar{C}_1^*)\bigl(|\chi^\delta(t)|_H^2 + |\sqrt{\alpha_0(t)}\gamma^\delta(t)|_H^2\bigr) + 2\bar{C}_0^*\bar{C}_1^*\bigl(|h(t)|_H^2 + |k(t)|_H^2\bigr), \quad \text{for a.e. } t \in (0,T),$$

and subsequently, by using Gronwall's lemma, we observe that:

($\star$1) the sequence $\{[\chi^\delta, \gamma^\delta]\}_{\delta \in (0,1)}$ is bounded in $[C([0,T];H)]^2 \cap \mathscr{Y}$.

Meanwhile, as consequences of (7.6.1), (7.6.4)–(7.6.8), ($\star$1), (A1)–(A5), Main Theorem 7.1, Remark 7.7, and the dominated convergence theorem [56, Theorem 10 on page 36], one can find a sequence $\{\delta_n\}_{n=1}^\infty \subset \mathbb{R}$, such that:

$$0 < |\delta_n| < 1, \text{ and } \delta_n \to 0, \text{ as } n \to \infty, \tag{7.6.9a}$$

$$\begin{cases} [\delta_n \chi^{\delta_n}, \delta_n \gamma^{\delta_n}] = [\eta^{\delta_n} - \eta, \theta^{\delta_n} - \theta] \to [0,0] \\ \quad \text{in } [C([0,T];H)]^2, \text{ and in } \mathscr{Y}, \\ [\delta_n \nabla \chi^{\delta_n}, \delta_n \nabla \gamma^{\delta_n}] = [\nabla(\eta^{\delta_n} - \eta), \nabla(\theta^{\delta_n} - \theta)] \to [0,0] \\ \quad \text{in } [L^2(0,T;[H]^N)]^2, \text{ and in the pointwise sense a.e. in } Q, \end{cases} \text{as } n \to \infty, \tag{7.6.9b}$$

$$[\bar{\lambda}_\varepsilon^{\delta_n}, \bar{\omega}_\varepsilon^{\delta_n}, \bar{A}_\varepsilon^{\delta_n}] \to [\bar{\lambda}_\varepsilon, \bar{\omega}_\varepsilon, \bar{A}_\varepsilon] \text{ weakly-}* \text{ in } L^\infty(Q) \times [L^\infty(Q)]^N \times [L^\infty(Q)]^{N \times N}, \\ \text{and in the pointwise sense a.e. in } Q, \quad \text{as } n \to \infty, \tag{7.6.9c}$$

$$\begin{cases} \bar{\mu}_\varepsilon^{\delta_n} \to \bar{\mu}_\varepsilon \text{ weakly-}* \text{ in } L^\infty(0,T;H), \\ \bar{\mu}_\varepsilon^{\delta_n}(t) \to \bar{\mu}_\varepsilon(t) \text{ in } H, \text{ for a.e. } t \in (0,T), \end{cases} \text{as } n \to \infty, \tag{7.6.9d}$$

and

$$\begin{aligned} \langle \bar{k}_\varepsilon^{\delta_n} - M_v k, \psi \rangle_{\mathscr{V}_0} = & -\left(\chi^{\delta_n}, \ \nabla f_\varepsilon(\nabla\theta) \left(\int_0^1 \alpha'(\eta + \varsigma\delta_n\chi^{\delta_n})\, d\varsigma \right) \cdot \nabla\psi \right)_{\mathscr{H}} \\ & + \left(\chi^{\delta_n}, \ \alpha'(\eta^{\delta_n}) \left(\int_0^1 \nabla f_\varepsilon(\nabla\theta + \varsigma\delta_n\nabla\gamma^{\delta_n})\, d\varsigma \right) \cdot \nabla\psi \right)_{\mathscr{H}} \\ & \to 0, \text{ as } n \to \infty. \end{aligned} \tag{7.6.9e}$$

On account of (7.6.1) and (7.6.4)–(7.6.9), we can apply Proposition 7.6 (B), and can see that:

$$[\chi^{\delta_n}, \gamma^{\delta_n}] = \bar{\mathcal{P}}_\varepsilon^{\delta_n}[M_u h, \bar{k}_\varepsilon^{\delta_n}] \to [\chi, \gamma] := \bar{\mathcal{P}}_\varepsilon[M_u h, M_v k] \text{ in } [\mathscr{H}]^2, \text{ weakly in } \mathscr{Y}, \\ \text{and weakly in } W^{1,2}(0,T;V^*) \times W^{1,2}(0,T;V_0^*), \text{ as } n \to \infty. \tag{7.6.10}$$

Since the Hilbert space $\mathscr{Y}$ is separable, and the uniqueness of the solution $[\chi, \gamma] = \bar{\mathcal{P}}_\varepsilon[M_u h, M_v k]$ is guaranteed by Proposition 7.3, the observations (7.6.6), (7.6.9), and (7.6.10) enable us to compute the directional derivative $D_{[h,k]}\mathcal{J}_\varepsilon(u,v) \in \mathbb{R}$, as follows:

$$\begin{aligned} D_{[h,k]}\mathcal{J}_\varepsilon(u,v) &:= \lim_{\delta \to 0} \frac{1}{\delta}\bigl(\mathcal{J}_\varepsilon(u+\delta h, v+\delta k) - \mathcal{J}_\varepsilon(u,v)\bigr) \\ &= \bigl([M_\eta(\eta - \eta_{\mathrm{ad}}), M_\theta(\theta - \theta_{\mathrm{ad}})], \bar{\mathcal{P}}_\varepsilon[M_u h, M_v k]\bigr)_{[\mathscr{H}]^2} + \bigl([M_u u, M_v v], [h,k]\bigr)_{[\mathscr{H}]^2}, \\ &\quad \text{for any direction } [h,k] \in \mathscr{X}. \end{aligned} \tag{7.6.11}$$

Moreover, in the light of (7.6.4), (7.6.11), and Proposition 7.5, we can observe that:

($\star$2) the mapping $[h,k] \in \mathscr{X} \mapsto D_{[h,k]}\mathcal{J}_\varepsilon(u,v) \in \mathbb{R}$ is a linear functional;

($\star$3) there exists a constant M_1^{**}, independent of $[h,k] \in \mathscr{X}$, such that

$$|D_{[h,k]}\mathcal{J}_\varepsilon(u,v)| \leq M_1^{**}|[h,k]|_{[\mathscr{H}]^2}, \text{ for any } [h,k] \in \mathscr{X}.$$

As a consequence of ($\star$2), ($\star$3), the continuous and dense embedding $\mathscr{X} \subset [\mathscr{H}]^2$, and Riesz's theorem, we can obtain the required functional $\mathcal{J}_\varepsilon'(u,v) \in \big([\mathscr{H}]^2\big)^*$ $\big(= [\mathscr{H}]^2\big)$, satisfying (7.6.2) and (7.6.3), as the unique extension of the Gâteaux differential $(\mathcal{J}_\varepsilon|_{\mathscr{X}})'(u,v) \in \mathscr{X}^*$ at $[u,v] \in \mathscr{X}$.

Thus, we complete the proof of this lemma. □

Lemma 7.5. Under the assumptions (A1)–(A5) with (r.s.0), let $[u_\varepsilon^*, v_\varepsilon^*] \in \mathscr{U}_{\mathrm{ad}}^K$ be an optimal control of the problem $(\mathrm{OP})_\varepsilon^K$, and let $[\eta_\varepsilon^*, \theta_\varepsilon^*]$ be the solution to the system $(\mathrm{S})_\varepsilon$, for the initial pair $[\eta_0, \theta_0]$ and forcing pair $[u_\varepsilon^*, v_\varepsilon^*]$. Also, let $\mathcal{P}_\varepsilon^* : [\mathscr{H}]^2 \longrightarrow \mathscr{Z}$ be the bounded linear operator, defined in Remark 7.8, with use of the solution $[\eta_\varepsilon^*, \theta_\varepsilon^*]$. Let $\mathcal{P}_\varepsilon : [\mathscr{H}]^2 \longrightarrow \mathscr{Z}$ be a bounded linear operator, which is defined as a restriction $\mathcal{P}|_{\{[0,0]\}\times[\mathscr{H}]^2}$ of the linear isomorphism $\mathcal{P} = \mathcal{P}(a,b,\mu,\lambda,\omega,A) : [H]^2 \times \mathscr{Y}^* \longrightarrow \mathscr{Z}$, as in Proposition 7.5, in the case when:

$$\begin{cases} [a,b] = [\alpha_0, 0] \text{ in } W^{1,\infty}(Q) \times L^\infty(Q), \\ \mu = \alpha''(\eta_\varepsilon^*) f_\varepsilon(\nabla\theta_\varepsilon^*) \text{ in } L^\infty(0,T;H), \\ \lambda = g'(\eta_\varepsilon^*) \text{ in } L^\infty(Q), \\ \omega = \alpha'(\eta_\varepsilon^*)\nabla f_\varepsilon(\nabla\theta_\varepsilon^*) \text{ in } [L^\infty(Q)]^N, \\ A = \alpha(\eta_\varepsilon^*)\nabla^2 f_\varepsilon(\nabla\theta_\varepsilon^*) \text{ in } [L^\infty(Q)]^{N\times N}. \end{cases} \tag{7.6.12}$$

Then, the operators $\mathcal{P}_\varepsilon^*$ and $\mathcal{P}_\varepsilon$ have a conjugate relationship, in the following sense:

$$\big(\mathcal{P}_\varepsilon^*[u,v],[h,k]\big)_{[\mathscr{H}]^2} = \big([u,v],\mathcal{P}_\varepsilon[h,k]\big)_{[\mathscr{H}]^2}, \\ \text{for all } [h,k],[u,v] \in [\mathscr{H}]^2.$$

Proof. Let us fix arbitrary pairs of functions $[h,k],[u,v] \in [\mathscr{H}]^2$, and let us put:

$$[\chi_\varepsilon, \gamma_\varepsilon] := \mathcal{P}_\varepsilon[h,k] \quad \text{and} \quad [p_\varepsilon, z_\varepsilon] := \mathcal{P}_\varepsilon^*[u,v], \text{ in } [\mathscr{H}]^2.$$

Then, invoking Proposition 7.3, and the settings as in (7.3.15) and (7.6.12), we compute that:

$$\big(\mathcal{P}_\varepsilon^*[u,v],[h,k]\big)_{[\mathscr{H}]^2} = \int_0^T \big(p_\varepsilon(t), h(t)\big)_H\,dt + \int_0^T \big(z_\varepsilon(t), k(t)\big)_H\,dt$$
$$= \int_0^T \langle h(t), p_\varepsilon(t)\rangle_V\,dt + \int_0^T \langle k(t), z_\varepsilon(t)\rangle_{V_0}\,dt$$

$$= \int_0^T \Big[\big\langle \partial_t\chi_\varepsilon(t), p_\varepsilon(t)\big\rangle_V + \big(\nabla\chi_\varepsilon(t), \nabla p_\varepsilon(t)\big)_{[H]^N}$$

$$
\begin{aligned}
&\quad + \big(\alpha''(\eta_\varepsilon^*(t)) f_\varepsilon(\nabla\theta_\varepsilon^*(t))\chi_\varepsilon(t), p_\varepsilon(t)\big)_H \\
&\quad + \big(g'(\eta_\varepsilon^*(t))\chi_\varepsilon(t), p_\varepsilon(t)\big)_H + \big(\alpha'(\eta_\varepsilon^*(t))\nabla f_\varepsilon(\nabla\theta_\varepsilon^*(t))\cdot\nabla\gamma_\varepsilon(t), p_\varepsilon(t)\big)_H \Big]\, dt \\
&+ \int_0^T \Big[\big\langle \partial_t\gamma_\varepsilon(t), \alpha_0(t) z_\varepsilon(t)\big\rangle_{V_0} + \big(\alpha'(\eta_\varepsilon^*(t))\chi_\varepsilon(t)\nabla f_\varepsilon(\nabla\theta_\varepsilon^*(t)), \nabla z_\varepsilon(t)\big)_{[H]^N} \\
&\quad + \big(\alpha(\eta_\varepsilon^*(t))\nabla^2 f_\varepsilon(\nabla\theta_\varepsilon^*(t))\nabla\gamma_\varepsilon(t), \nabla z_\varepsilon(t)\big)_{[H]^N} + \nu^2\big(\nabla\gamma_\varepsilon(t), \nabla z_\varepsilon(t)\big)_{[H]^N} \Big]\, dt
\end{aligned}
$$

$$
\begin{aligned}
=&\big(p_\varepsilon(T), \chi_\varepsilon(T)\big)_H - \big(p_\varepsilon(0), \chi_\varepsilon(0)\big)_H + \int_0^T \Big[\big\langle -\partial_t p_\varepsilon(t), \chi_\varepsilon(t)\big\rangle_V \\
&\quad + \big(\nabla p_\varepsilon(t), \nabla\chi_\varepsilon(t)\big)_{[H]^N} + \big(\alpha''(\eta_\varepsilon^*(t)) f_\varepsilon(\nabla\theta_\varepsilon^*(t)) p_\varepsilon(t), \chi_\varepsilon(t)\big)_H \\
&\quad + \big(g'(\eta_\varepsilon^*(t)) p_\varepsilon(t), \chi_\varepsilon(t)\big)_H + \big(\alpha'(\eta_\varepsilon^*(t))\nabla f_\varepsilon(\nabla\theta_\varepsilon^*(t))\cdot\nabla z_\varepsilon(t), \chi_\varepsilon(t)\big)_H \Big]\, dt \\
&+ \big(\alpha_0(T) z_\varepsilon(T), \gamma_\varepsilon(T)\big)_H - \big(\alpha_0(0) z_\varepsilon(0), \gamma_\varepsilon(0)\big)_H \\
&\quad + \int_0^T \Big[\big\langle -\partial_t\big(\alpha_0 z_\varepsilon\big)(t), \gamma_\varepsilon(t)\big\rangle_{V_0} + \big(\alpha'(\eta_\varepsilon^*(t)) p_\varepsilon(t)\nabla f_\varepsilon(\nabla\theta_\varepsilon^*(t)), \nabla\gamma_\varepsilon(t)\big)_{[H]^N} \\
&\quad + \big(\alpha(\eta_\varepsilon^*(t))\nabla^2 f_\varepsilon(\nabla\theta_\varepsilon^*(t))\nabla z_\varepsilon(t), \nabla\gamma_\varepsilon(t)\big)_{[H]^N} + \nu^2\big(\nabla z_\varepsilon(t), \nabla\gamma_\varepsilon(t)\big)_{[H]^N} \Big]\, dt \\
=&(u, \chi_\varepsilon)_{\mathscr{H}} + (v, \gamma_\varepsilon)_{\mathscr{H}} = \big([u, v], \mathcal{P}_\varepsilon[h, k]\big)_{[\mathscr{H}]^2}.
\end{aligned}
$$

This finishes the proof of Lemma 7.5. □

Remark 7.15. Note that the operator $\mathcal{P}_\varepsilon \in \mathscr{L}([\mathscr{H}]^2; \mathscr{Z})$, as in Lemma 7.5, corresponds to the operator $\bar{\mathcal{P}}_\varepsilon \in \mathscr{L}([\mathscr{H}]^2; \mathscr{Z})$, as in the previous Lemma 7.4, under the special setting (7.6.12).

Now, we are ready to prove the Main Theorem 7.3 (III-A).

Proof of Main Theorem 7.3 (III-A). Let $[u_\varepsilon^*, v_\varepsilon^*] \in \mathscr{U}_{\rm ad}^K$ be the optimal control of $({\rm OP})_\varepsilon^K$, with the solution $[\eta_\varepsilon^*, \theta_\varepsilon^*] \in [\mathscr{H}]^2$ to the system $({\rm S})_\varepsilon$ for the initial pair $[\eta_0, \theta_0] \in D_0$, as in (r.s.0), and forcing pair $[u_\varepsilon^*, v_\varepsilon^*]$, and let $\mathcal{P}_\varepsilon, \mathcal{P}_\varepsilon^* \in \mathscr{L}([\mathscr{H}]^2; \mathscr{Z})$ be the two operators as in Lemma 7.5. In addition, let us put $[p_\varepsilon^*, z_\varepsilon^*] := \mathcal{P}_\varepsilon^*[M_\eta(\eta_\varepsilon^* - \eta_{\rm ad}), M_\theta(\theta_\varepsilon^* - \theta_{\rm ad})]$. Then, on the basis of the previous Lemmas 7.4 and 7.5, we compute that:

$$
\begin{aligned}
0 \leq\ & (\mathcal{J}_\varepsilon'(u_\varepsilon^*, v_\varepsilon^*), [h, k])_{[\mathscr{H}]^2} \\
=\ & \lim_{\delta\downarrow 0} \frac{1}{\delta}\big(\mathcal{J}_\varepsilon(u_\varepsilon^* + \delta(h - u_\varepsilon^*), v_\varepsilon^* + \delta k) - \mathcal{J}_\varepsilon(u_\varepsilon^*, v_\varepsilon^*)\big) \\
=\ & \big([M_\eta(\eta_\varepsilon^* - \eta_{\rm ad}), M_\theta(\theta_\varepsilon^* - \theta_{\rm ad})], \mathcal{P}_\varepsilon[M_u(h - u_\varepsilon^*), M_v k]\big)_{[\mathscr{H}]^2} \\
&\quad + \big([M_u u_\varepsilon^*, M_v v_\varepsilon^*], [h - u_\varepsilon^*, k]\big)_{[\mathscr{H}]^2} \\
=\ & \big(\mathcal{P}_\varepsilon^*[M_\eta(\eta_\varepsilon^* - \eta_{\rm ad}), M_\theta(\theta_\varepsilon^* - \theta_{\rm ad})], [M_u(h - u_\varepsilon^*), M_v k]\big)_{[\mathscr{H}]^2}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \big([M_u u_\varepsilon^*, M_v v_\varepsilon^*], [h - u_\varepsilon^*, k]\big)_{[\mathscr{H}]^2} \\
&= \big(M_u(p_\varepsilon^* + u_\varepsilon^*), h - u_\varepsilon^*\big)_{\mathscr{H}} + \big(M_v(z_\varepsilon^* + v_\varepsilon^*), k\big)_{\mathscr{H}}, \qquad (7.6.13)\\
&\quad \text{for any } [h, k] \in \mathscr{U}_{\rm ad}^K.
\end{aligned}
$$

Now, in (7.6.13), let us consider the case when $[h, k] = [h, 0] \in \mathscr{U}_{\rm ad}^K$ with arbitrary $h \in K$. Then, we have:

$$
0 \leq \big(M_u(p_\varepsilon^* + u_\varepsilon^*), h - u_\varepsilon^*\big)_{\mathscr{H}} = -M_u\big(-p_\varepsilon^* - u_\varepsilon^*, h - u_\varepsilon^*\big)_{\mathscr{H}} \quad \text{for any } h \in K. \qquad (7.6.14)
$$

It is equivalent to (7.3.9a). Indeed, if $M_u > 0$, then the equivalence of (7.3.9a) and (7.6.14) is a straightforward consequence of (Fact 2). Also, if $M_u = 0$, then the both (7.3.9a) and (7.6.14) coincides with the tautology "$0 = 0$".

In the meantime, putting $[h, k] = [u_\varepsilon^*, k] \in \mathscr{U}_{\rm ad}^K$ with arbitrary $k \in \mathscr{H}$, one can see that:

$$
\big(M_v(v^* + z_\varepsilon^*), k\big)_{\mathscr{H}} \geq 0 \text{ for any } k \in \mathscr{H}.
$$

This implies the equality (7.3.9b).

Thus, we conclude Main Theorem 7.3 (III-A). □

Next, before the proof of Main Theorem 7.3 (III-B), we prepare the following lemma.

Lemma 7.6. Let us assume (A5) and (A6), and fix a constraint $K = [\![\kappa^0, \kappa^1]\!] \in \mathfrak{K}$. Also, let us assume that:

$$
\tilde{p} \in \mathscr{H}, \ \{\tilde{p}_n\}_{n=1}^\infty \subset \mathscr{H}, \text{ and } \tilde{p}_n \to \tilde{p} \text{ in } \mathscr{H} \text{ as } n \to \infty, \qquad (7.6.15a)
$$

and let us put:

$$
\begin{cases}
\tilde{u} := \mathrm{proj}_K(\tilde{p}) \text{ in } \mathscr{H}, \\
\tilde{u}_n := \mathrm{proj}_{K_n}(\tilde{p}_n) \text{ in } \mathscr{H}, \text{ for } n = 1, 2, 3, \ldots.
\end{cases} \qquad (7.6.15b)
$$

Then, it holds that:

$$
\tilde{u}_n \to \tilde{u} \text{ in } \mathscr{H}, \text{ as } n \to \infty. \qquad (7.6.16)
$$

Proof. By using the assumptions as in (7.6.15), Remark 7.2, and Lemma 7.3, this Lemma is easily verified as follows.

$$
\begin{aligned}
|\tilde{u}_n - \tilde{u}|_{\mathscr{H}} &\leq |\mathrm{proj}_{K_n}(\tilde{p}_n) - \mathrm{proj}_{K_n}(\tilde{p})|_{\mathscr{H}} + |\mathrm{proj}_{K_n}(\tilde{p}) - \mathrm{proj}_K(\tilde{p})|_{\mathscr{H}} \\
&\leq |\tilde{p}_n - \tilde{p}|_{\mathscr{H}} + |\mathrm{proj}_{K_n}(\tilde{p}) - \mathrm{proj}_K(\tilde{p})|_{\mathscr{H}} \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

□

Now, we are on the stage to prove Main Theorem 7.3 (III-B).

Proof of Main Theorem 7.3 (III-B). Let us notice that the assumptions (7.3.4) and (7.3.13) guarantee that:

- the sequence of initial pairs $\{[\eta_{0,n}, \theta_{0,n}]\}_{n=1}^{\infty}$ is bounded in $D_0 = \bigl(V \cap L^\infty(\Omega)\bigr) \times V_0$;
- the sequence $\{u_n^* \in K_n\}_{n=1}^{\infty}$, consisting of the first components of optimal controls $[u_n^*, v_n^*] \in \mathscr{U}_{\mathrm{ad}}^{K_n}$ of $(\mathrm{OP})_{\varepsilon_n}^{K_n}$, $n = 1, 2, 3, \dots$, is bounded in $L^\infty(Q)$.

Hence, with the compact embedding $D = V \times V_0 \subset [H]^2$ and Alaoglu's theorem in mind, we may suppose that:

$$\begin{cases} [\eta_{0,n_i}, \theta_{0,n_i}] \to [\eta_0, \theta_0] \text{ in } [H]^2, \text{ and weakly in } V \times V_0, \\ \eta_{0,n_i} \to \eta_0 \text{ weakly-}* \text{ in } L^\infty(\Omega), \\ u_{n_i}^* \to u^{**} \text{ weakly-}* \text{ in } L^\infty(Q), \text{ as } i \to \infty, \end{cases} \tag{7.6.17}$$

for the subsequence $\{n_i\}_{i=1}^{\infty} \subset \{n\}$ and the limiting optimal control $[u^{**}, v^{**}] \in \mathscr{U}_{\mathrm{ad}}^{K}$, as in Main Theorem 7.2 (II-B).

By (7.3.4) and (7.6.17), we can apply Main Theorem 7.1 (I-B), to the solutions $[\eta^{**}, \theta^{**}] \in [\mathscr{H}]^2$ and $[\eta_{n_i}^*, \theta_{n_i}^*] \in [\mathscr{H}]^2$, $i = 1, 2, 3, \dots$, as in (7.5.14), and can deduce that:

$$\eta_{n_i}^* \to \eta^{**} \text{ weakly-}* \text{ in } L^\infty(Q), \text{ as } i \to \infty. \tag{7.6.18}$$

Meanwhile, by taking more subsequence(s) if necessary, one can see from (7.3.15), (7.5.14), and (7.6.18) that:

$$\lambda_i^* := \mathcal{R}_T\bigl[g'(\eta_{n_i}^*)\bigr] \to \lambda^{**} := \mathcal{R}_T\bigl[g'(\eta^{**})\bigr] \text{ weakly-}* \text{ in } L^\infty(Q), \\ \text{and in the pointwise sense a.e. in } Q, \tag{7.6.19a}$$

$$\omega_i^* := \mathcal{R}_T\bigl[\alpha'(\eta_{n_i}^*)\nabla f_{\varepsilon_{n_i}}(\nabla \theta_{n_i}^*)\bigr] \to \omega^{**} := \mathcal{R}_T\bigl[\alpha'(\eta^{**})\nabla f_\varepsilon(\nabla \theta^{**})\bigr] \\ \text{weakly-}* \text{ in } [L^\infty(Q)]^N, \text{ and in the pointwise sense a.e. in } Q, \tag{7.6.19b}$$

$$A_i^* := \mathcal{R}_T\bigl[\alpha(\eta_{n_i}^*)\nabla^2 f_{\varepsilon_{n_i}}(\nabla \theta_{n_i}^*)\bigr] \to A^{**} := \mathcal{R}_T\bigl[\alpha(\eta^{**})\nabla^2 f_\varepsilon(\nabla \theta^{**})\bigr] \\ \text{weakly-}* \text{ in } [L^\infty(Q)]^{N \times N}, \text{ and in the pointwise sense a.e. in } Q, \tag{7.6.19c}$$

$$\mu_i^* := \mathcal{R}_T\bigl[\alpha''(\eta_{n_i}^*) f_{\varepsilon_{n_i}}(\nabla \theta_{n_i}^*)\bigr] \to \mu^{**} := \mathcal{R}_T\bigl[\alpha''(\eta^{**}) f_\varepsilon(\nabla \theta^{**})\bigr] \\ \text{weakly-}* \text{ in } L^\infty(0, T; H), \tag{7.6.19d}$$

and

$$\mu_i^*(t) \to \mu^{**}(t) \text{ in } H, \text{ for a.e. } t \in (0, T), \text{ as } i \to \infty. \tag{7.6.19e}$$

Now, let us denote by $\mathcal{P}_0^{**}$ and $\mathcal{P}_i^{**}$, $i = 1, 2, 3, \dots$, the operators $\mathcal{P}_\varepsilon^*$, as in Remark 7.8, in the cases when:

$$[\mu, \lambda, \omega, A] = [\mu^{**}, \lambda^{**}, \omega^{**}, A^{**}] \\ \text{in } L^\infty(0, T; H) \times L^\infty(Q) \times [L^\infty(Q)]^N \times [L^\infty(Q)]^{N \times N},$$

$$\varepsilon = \varepsilon_i, \text{ and } [\mu, \lambda, \omega, A] = [\mu_i^*, \lambda_i^*, \omega_i^*, A_i^*]$$
$$\text{in } L^\infty(0,T;H) \times L^\infty(Q) \times [L^\infty(Q)]^N \times [L^\infty(Q)]^{N\times N},\ i = 1,2,3,\dots,$$

respectively. Then, as a consequence of Proposition 7.6, Main Theorem 7.3 (III-A), and Remark 7.8, we can derive from (7.5.14) and (7.6.19) that:

$$\begin{aligned}
&[p_i^*, z_i^*] := \mathcal{P}_i^{**}\big[M_\eta(\eta_{n_i}^* - \eta_{\mathrm{ad}}), M_\theta(\theta_{n_i}^* - \theta_{\mathrm{ad}})\big] \\
&\quad \to [p^{**}, z^{**}] := \mathcal{P}_0^{**}\big[M_\eta(\eta^{**} - \eta_{\mathrm{ad}}), M_\theta(\theta^{**} - \theta_{\mathrm{ad}})\big] \text{ in } [C([0,T];H)]^2, \\
&\quad \text{in } \mathscr{Y}, \text{ and weakly in } W^{1,2}(0,T;V^*) \times W^{1,2}(0,T;V_0^*), \text{ as } i \to \infty.
\end{aligned} \tag{7.6.20}$$

Furthermore, taking into account (7.6.20) and Lemma 7.6, one can infer that:

$$M_u u_{n_i}^* = M_u \mathrm{proj}_{K_{n_i}}(-p_i^*) \to M_u u^{**} = M_u \mathrm{proj}_K(-p^{**}) \text{ in } \mathscr{H}, \text{ as } i \to \infty, \tag{7.6.21a}$$

and

$$\begin{aligned}
M_v v_{n_i}^* &= -M_v z_i^* \to M_v v^{**} = -M_v z^{**} \text{ in } C([0,T];H), \text{ in } \mathscr{Y}, \\
&\text{and weakly in } W^{1,2}(0,T;V_0^*), \text{ as } i \to \infty.
\end{aligned} \tag{7.6.21b}$$

(7.6.17) and (7.6.21) are sufficient to verify the convergences as in (7.3.14), and to conclude Main Theorem 7.3 (III-B). □

7.7 Proof of Main Theorem 7.4

Under the assumptions (A1)–(A5) and the situation ¬(r.s.0), let us set:

$$\begin{cases}
\varepsilon_n := \varepsilon + \frac{1}{n}, \\
\eta_{0,n} := (-n) \vee (n \wedge \eta_0) \text{ a.e. in } \Omega, \\
\theta_{0,n} := \theta_0 \text{ a.e. in } \Omega, \\
\kappa_n^\ell := (-n) \vee (n \wedge \kappa^\ell) \text{ a.e. in } Q,\ \ell = 0, 1,
\end{cases} \quad n = 1, 2, 3, \dots.$$

Then, we immediately see that:

($\star$4) $\{\varepsilon_n\}_{n=1}^\infty \subset (\varepsilon, \infty)$, $\{[\eta_{0,n}, \theta_{0,n}]\}_{n=1}^\infty \subset D_0$, and $\{K_n\}_{n=1}^\infty := \{[\![\kappa_n^0, \kappa_n^1]\!]\} \subset \mathfrak{K}_0$, and these sequences fulfill the assumptions (7.3.4) and (A6), as in Main Theorems 7.1–7.3.

Additionally, we can apply Main Theorem 7.1 (I-A) and Main Theorem 7.2 (II-A), and can take sequences of functional pairs $\{[u_n^\circ, v_n^\circ]\}_{n=1}^\infty$ and $\{[\eta_n^\circ, \theta_n^\circ]\}_{n=1}^\infty$, such that:

- for any $n \in \mathbb{N}$, $[u_n^\circ, v_n^\circ] \in \mathscr{U}_{\mathrm{ad}}^{K_n}$ is an optimal control of $(\mathrm{OP})_{\varepsilon_n}^{K_n}$;
- for any $n \in \mathbb{N}$, $[\eta_n^\circ, \theta_n^\circ] \in [\mathscr{H}]^2$ is the solution to $(\mathrm{S})_{\varepsilon_n}$, for the initial pair $[\eta_{0,n}, \theta_{0,n}]$ and forcing pair $[u_n^\circ, v_n^\circ]$.

Also, applying Main Theorem 7.1 (I-B) and Main Theorem 7.2 (II-B), we can find subsequences of $\{[u_n^\circ, v_n^\circ]\}_{n=1}^\infty$ and $\{[\eta_n^\circ, \theta_n^\circ]\}_{n=1}^\infty$ (not relabeled), together with limiting pairs $[u^\circ, v^\circ] \in [\mathscr{H}]^2$ and $[\eta^\circ, \theta^\circ] \in [\mathscr{H}]^2$, and a limiting function $\sigma^\circ \in [L^\infty(Q)]^N$, such that:

$$[M_u u_n^\circ, M_v v_n^\circ] \to [M_u u^\circ, M_v v^\circ] \text{ weakly in } [\mathscr{H}]^2, \tag{7.7.1a}$$

$$\begin{aligned} [\eta_n^\circ, \theta_n^\circ] \ &\to [\eta^\circ, \theta^\circ] \text{ in } [C([0,T];H)]^2, \text{ in } \mathscr{Y}, \\ &\text{and weakly-}* \text{ in } L^\infty(0,T;V) \times L^\infty(0,T;V_0), \end{aligned} \tag{7.7.1b}$$

$$\begin{aligned} [\nabla \eta_n^\circ, \nabla \theta_n^\circ] &\to [\nabla \eta^\circ, \nabla \theta^\circ] \text{ in } [L^2(0,T;[H]^N)]^2, \\ &\text{and in the pointwise sense a.e. in } Q, \end{aligned} \tag{7.7.1c}$$

$$\begin{cases} \mu_n^\circ := \alpha''(\eta_n^\circ) f_{\varepsilon_n}(\nabla \theta_n^\circ) \to \mu^\circ := \alpha''(\eta^\circ) f_\varepsilon(\nabla \theta^\circ) \\ \quad \text{weakly-}* \text{ in } L^\infty(0,T;H), \\ \quad \text{and in the pointwise sense a.e. in } Q, \\ \mu_n^\circ(t) \to \mu^\circ(t) \text{ in } H, \text{ for a.e. } t \in (0,T), \end{cases} \tag{7.7.1d}$$

$$\begin{aligned} \lambda_n^\circ := \ g'(\eta_n^\circ) \to \lambda^\circ := g'(\eta^\circ) \text{ in } \mathscr{H}, \text{ weakly-}* \text{ in } L^\infty(Q), \\ \text{and in the pointwise sense a.e. in } Q, \end{aligned} \tag{7.7.1e}$$

$$\nabla f_{\varepsilon_n}(\nabla \theta_n^\circ) \to \sigma^\circ \text{ weakly-}* \text{ in } [L^\infty(Q)]^N, \tag{7.7.1f}$$

and

$$\begin{aligned} \omega_n^\circ := \alpha'(\eta_n^\circ) \ \nabla f_{\varepsilon_n}(\nabla \theta_n^\circ) &\to \alpha'(\eta^\circ)\sigma^\circ \\ &\text{weakly-}* \text{ in } [L^\infty(Q)]^N, \text{ as } n \to \infty. \end{aligned} \tag{7.7.1g}$$

Additionally, from (7.7.1c), (7.7.1f), Remark 7.4, and [18, Proposition 2.16], one can observe that:

$$\sigma^\circ \in \partial f_\varepsilon(\nabla \theta^\circ) = \begin{cases} \{\nabla f_\varepsilon(\nabla \theta^\circ)\}, \text{ if } \varepsilon > 0, \\ \mathrm{Sgn}^N(\nabla \theta^\circ), \text{ if } \varepsilon = 0, \end{cases} \quad \text{a.e. in } Q. \tag{7.7.2}$$

Next, for any $n \in \mathbb{N}$, let us put:

$$A_n^\circ := \alpha(\eta_n^\circ) \nabla^2 f_{\varepsilon_n}(\nabla \theta_n^\circ) \text{ in } [L^\infty(Q)]^{N \times N}, \tag{7.7.3}$$

and let us denote by $\mathcal{P}_n^\circ$ the operator $\mathcal{P}_\varepsilon^* \in \mathscr{L}([\mathscr{H}]^2; \mathscr{Z})$, as in Remark 7.8, in the case when the constant $\varepsilon > 0$ (in Remark 7.8) and the sextuplet $[a, b, \mu, \lambda, \omega, A] \in \mathscr{S}$ is replaced by $\varepsilon_n > 0$ and $\mathcal{R}_T[\alpha_0, -\partial_t \alpha_0, \mu_n^\circ, \lambda_n^\circ, \omega_n^\circ, A_n^\circ] \in \mathscr{S}$, respectively.

On this basis, let us set:

$$[p_n^\circ, z_n^\circ] := \mathcal{P}_n^\circ \big[M_\eta(\eta_n^\circ - \eta_{\mathrm{ad}}), M_\theta(\theta_n^\circ - \theta_{\mathrm{ad}})\big] \text{ in } \mathscr{Z}, \text{ for } n = 1, 2, 3, \ldots.$$

Then, from Main Theorem 7.3 (III-A), it is inferred that:

$$(M_u(p_n^\circ + u_n^\circ), h - u_n^\circ)_{\mathscr{H}} \geq 0, \quad \text{for any } h \in K_n = [\![\kappa_n^0, \kappa_n^1]\!], \tag{7.7.4a}$$

$$M_v(z_n^\circ + v_n^\circ) = 0 \ \text{ in } \mathscr{H}, \tag{7.7.4b}$$

$$\begin{aligned} &\big\langle -\partial_t p_n^\circ, \varphi \big\rangle_{\mathscr{V}} + \big(\nabla p_n^\circ, \nabla \varphi\big)_{[\mathscr{H}]^N} + \big\langle \mu_n^\circ p_n^\circ, \varphi \big\rangle_{\mathscr{V}} \\ &\quad + \big(\lambda_n^\circ p_n^\circ + \omega_n^\circ \cdot \nabla z_n^\circ, \varphi\big)_{\mathscr{H}} = \big(M_\eta(\eta_n^\circ - \eta_{\mathrm{ad}}), \varphi\big)_{\mathscr{H}}, \text{for any } \varphi \in \mathscr{V}, \end{aligned} \tag{7.7.4c}$$

$$\begin{aligned} &\big\langle -\alpha_0 \partial_t z_n^\circ, \psi \big\rangle_{\mathscr{V}_0} + \big((-\partial_t \alpha_0) z_n^\circ, \psi\big)_{\mathscr{H}} + \big(A_n^\circ \nabla z_n^\circ + \nu^2 \nabla z_n^\circ + p_n^\circ \omega_n^\circ, \nabla \psi\big)_{[\mathscr{H}]^N} \\ &\qquad = \big(M_\theta(\theta_n^\circ - \theta_{\mathrm{ad}}), \psi\big)_{\mathscr{H}}, \text{ for any } \psi \in \mathscr{V}_0, \end{aligned} \tag{7.7.4d}$$

and

$$[p_n^\circ(T), z_n^\circ(T)] = [0, 0] \text{ in } [H]^2, \ n = 1, 2, 3, \ldots. \tag{7.7.4e}$$

Also, having in mind (7.7.1)–(7.7.3), and applying Proposition 7.4 to the case when:

$$\begin{cases} [a^1, b^1, \mu^1, \lambda^1, \omega^1, A^1] = [a^2, b^2, \mu^2, \lambda^2, \omega^2, A^2] \\ \qquad = \mathcal{R}_T[\alpha_0, -\partial_t \alpha_0, \mu_n^\circ, \lambda_n^\circ, \omega_n^\circ, A_n^\circ], \\ [p_0^1, z_0^1] = [p_0^2, z_0^2] = [0, 0], \\ [h^1, k^1] = \mathcal{R}_T[M_\eta(\eta_n^\circ - \eta_{\mathrm{ad}}), M_\theta(\theta_n^\circ - \theta_{\mathrm{ad}})], \\ [h^2, k^2] = [0, 0], \\ [p^1, z^1] = \mathcal{R}_T[p_n^\circ, z_n^\circ], \ [p^2, z^2] = [0, 0], \end{cases} \quad \text{for } n = 1, 2, 3, \ldots.$$

we deduce that:

$$\begin{aligned} &\frac{d}{dt}\Big(\big|(\mathcal{R}_T p_n^\circ)(t)\big|_H^2 + \big|\mathcal{R}_T\big(\sqrt{\alpha_0} z_n^\circ\big)(t)\big|_H^2\Big) \\ &\qquad + \Big(\big|(\mathcal{R}_T p_n^\circ)(t)\big|_V^2 + \nu^2 \big|(\mathcal{R}_T z_n^\circ)(t)\big|_{V_0}^2\Big) \\ &\quad \leq 3\bar{C}_0^* \Big(\big|(\mathcal{R}_T p_n^\circ)(t)\big|_H^2 + \big|\mathcal{R}_T\big(\sqrt{\alpha_0} z_n^\circ\big)(t)\big|_H^2\Big) \\ &\qquad + 2\bar{C}_0^* \Big(\big|\mathcal{R}_T\big(M_\eta(\eta_n^\circ - \eta_{\mathrm{ad}})\big)(t)\big|_{V^*}^2 + \big|\mathcal{R}_T\big(M_\theta(\theta_n^\circ - \theta_{\mathrm{ad}})\big)(t)\big|_{V_0^*}^2\Big), \\ &\quad \text{for a.e. } t \in (0, T), \ n = 1, 2, 3, \ldots, \end{aligned} \tag{7.7.5}$$

with use of the constant $\bar{C}_0^*$ as in (7.6.8a). As a consequence of (7.7.1b), (7.7.5), (A4), and Gronwall's lemma, it is observed that:

($\star$5) the sequence $\{[p_n^\circ, z_n^\circ]\}_{n=1}^\infty$ is bounded in $[C([0, T]; H)]^2 \cap \mathscr{Y}$.

In the meantime, from (7.1.3), (7.7.1b)–(7.7.1g), (7.7.4c), (7.7.4d), (A3), and Remark 7.1, we can derive the following estimates:

$$\begin{aligned} \big|\big\langle \partial_t p_n^\circ, \varphi \big\rangle_{\mathscr{V}}\big| &\leq \big|\big\langle \mu_n^\circ p_n^\circ, \varphi \big\rangle_{\mathscr{V}}\big| + \big|\big(\nabla p_n^\circ, \nabla \varphi\big)_{[\mathscr{H}]^N}\big| + \big|\big(\lambda_n^\circ p_n^\circ + \omega_n^\circ \cdot \nabla z_n^\circ, \varphi\big)_{\mathscr{H}}\big| \\ &\quad + \big|\big(M_\eta(\eta_n^\circ - \eta_{\mathrm{ad}}), \varphi\big)_{\mathscr{H}}\big| \leq C_1^\circ |\varphi|_{\mathscr{V}}, \text{ for any } \varphi \in \mathscr{V}, \end{aligned} \tag{7.7.6a}$$

and

$$
\begin{aligned}
\left|\langle -\mathrm{div}(A_n^\circ \nabla z_n^\circ), \psi\rangle_{\mathscr{W}_0}\right| &= \left|\left(A_n^\circ \nabla z_n^\circ, \nabla \psi\right)_{[\mathscr{H}]^N}\right| \leq \left|\left(\alpha_0 z_n^\circ, \partial_t \psi\right)_{\mathscr{H}}\right| \\
&\quad + \left|\left(\nu^2 \nabla z_n^\circ + p_n^\circ \omega_n^\circ, \nabla \psi\right)_{[\mathscr{H}]^N}\right| + \left|\left(M_\theta(\theta_n^\circ - \theta_{\mathrm{ad}}), \psi\right)_{\mathscr{H}}\right| \\
&\quad \leq C_2^\circ |\psi|_{\mathscr{W}_0}, \text{ for any } \psi \in C_{\mathrm{c}}^\infty(Q),\ n = 1, 2, 3, \dots,
\end{aligned} \tag{7.7.6b}
$$

with n-independent positive constants:

$$
C_1^\circ := \sup_{n \in \mathbb{N}} \left\{ \begin{array}{c} (1 + (C_V^{L^4})^2 |\mu_n^\circ|_{L^\infty(0,T;H)} + |\lambda_n^\circ|_{L^\infty(Q)} + |\omega_n^\circ|_{[L^\infty(Q)]^N}) \\ \cdot \left(\left|[p_n^\circ, z_n^\circ]\right|_{\mathscr{Y}} + \left|M_\eta(\eta_n^\circ - \eta_{\mathrm{ad}})\right|_{\mathscr{H}} \right) \end{array} \right\} (< \infty),
$$

and

$$
C_2^\circ := \sup_{n \in \mathbb{N}} \left\{ \begin{array}{c} (1 + \nu^2 + C_{V_0}^H |\alpha_0|_{L^\infty(Q)} + |\omega_n^\circ|_{[L^\infty(Q)]^N}) \\ \cdot \left(\left|[p_n^\circ, z_n^\circ]\right|_{\mathscr{Y}} + \left|M_\theta(\theta_n^\circ - \theta_{\mathrm{ad}})\right|_{\mathscr{H}} \right) \end{array} \right\} (< \infty),
$$

where $C_V^{L^4} > 0$ and $C_{V_0}^H > 0$ are the constants of embeddings $V \subset L^4(\Omega)$ and $V_0 \subset H$, respectively.

Due to (7.7.1d)–(7.7.1g), (7.7.6), ($\star$5), Lemma 7.1, and the compactness theory of Aubin's type (cf. [83, Corollary 4]), we can find subsequences of $\{[p_n^\circ, z_n^\circ]\}_{n=1}^\infty \subset \mathscr{Y}$, $\{\omega_n^\circ \cdot \nabla z_n^\circ\}_{n=1}^\infty \subset \mathscr{H}$, and $\{-\mathrm{div}(A_n^\circ \nabla z_n^\circ)\}_{n=1}^\infty \subset \mathscr{W}_0^*$ (not relabeled), together with the respective limits $[p^\circ, z^\circ] \in \mathscr{Y}$, $\xi^\circ \in \mathscr{H}$, and $\zeta^\circ \in \mathscr{W}_0^*$, such that:

$$
\begin{cases} [p_n^\circ, z_n^\circ] \to [p^\circ, z^\circ] \text{ weakly in } \mathscr{Y}, \\ p_n^\circ \to p^\circ \text{ in } \mathscr{H}, \text{ weakly in } W^{1,2}(0,T;V^*), \\ \quad \text{and in the pointwise sense a.e. in } Q, \end{cases} \tag{7.7.7a}
$$

$$
\mu_n^\circ p_n^\circ \to \mu^\circ p^\circ \text{ weakly in } \mathscr{V}^*, \tag{7.7.7b}
$$

$$
\lambda_n^\circ p_n^\circ \to \lambda^\circ p^\circ \text{ in } \mathscr{H}, \tag{7.7.7c}
$$

$$
p_n^\circ \omega_n^\circ \to p^\circ \alpha'(\eta^\circ) \sigma^\circ \text{ weakly in } \mathscr{H}, \tag{7.7.7d}
$$

$$
\begin{cases} \nabla f_{\varepsilon_n}(\nabla \theta_n^\circ) \cdot \nabla z_n^\circ \to \xi^\circ \text{ weakly in } \mathscr{H}, \\ \omega_n^\circ \cdot \nabla z_n^\circ = \alpha'(\eta_n^\circ) \nabla f_{\varepsilon_n}(\nabla \theta_n^\circ) \cdot \nabla z_n^\circ \to \alpha'(\eta^\circ) \xi^\circ \\ \qquad\qquad \text{weakly in } \mathscr{H}, \end{cases} \tag{7.7.7e}
$$

and

$$
-\mathrm{div}(A_n^\circ \nabla z_n^\circ) \to \zeta^\circ \text{ weakly in } \mathscr{W}_0^*, \text{ as } n \to \infty. \tag{7.7.7f}
$$

Now, the properties (7.3.18)–(7.3.21) will be verified through the limiting observations for (7.7.4), as $n \to \infty$, with use of (7.7.1) and (7.7.7).

Finally, we verify the properties as in (7.3.22), under the situation (r.s.1). To this end, we first invoke (7.1.4b) and (7.7.1c), and confirm that:

$$\begin{cases} \bigl|\varphi\bigl(\nabla f_\varepsilon(\nabla\theta_n^\circ) - \nabla f_\varepsilon(\nabla\theta^\circ)\bigr)\bigr| \to 0, \text{ in the pointwise sense,} \\ \qquad \text{a.e. in } Q, \text{ as } n \to \infty, \\ \bigl|\varphi\bigl(\nabla f_\varepsilon(\nabla\theta_n^\circ) - \nabla f_\varepsilon(\nabla\theta^\circ)\bigr)\bigr| \leq 2|\varphi|, \text{ a.e. in } Q,\ n = 1, 2, 3, \dots, \\ \qquad \text{for any } \varphi \in \mathscr{H}. \end{cases} \tag{7.7.8}$$

With (7.7.8), (r.s.1), ($\star$4), and (A4) in mind, using (7.1.4b) and (7.7.1b), and applying the dominated convergence theorem [56, Theorem 10 on page 36] yield that:

$$\begin{aligned} & \bigl|\varphi\bigl(\alpha'(\eta_n^\circ)\nabla f_{\varepsilon_n}(\nabla\theta_n^\circ) - \alpha'(\eta^\circ)\nabla f_\varepsilon(\nabla\theta^\circ)\bigr)\bigr|_{[\mathscr{H}]^N} \\ & \quad \leq \bigl|\varphi\bigl(\alpha'(\eta_n^\circ) - \alpha'(\eta^\circ)\bigr)\bigr|_{\mathscr{H}} \\ & \qquad + |\alpha'|_{L^\infty(\mathbb{R})}\bigl|\varphi\bigl(\nabla f_\varepsilon(\nabla\theta_n^\circ) - \nabla f_\varepsilon(\nabla\theta^\circ)\bigr)\bigr|_{[\mathscr{H}]^N} \\ & \qquad + |\alpha'|_{L^\infty(\mathbb{R})}\bigl|\varphi\bigl(\nabla f_{\varepsilon_n}(\nabla\theta_n^\circ) - \nabla f_\varepsilon(\nabla\theta_n^\circ)\bigr)\bigr|_{[\mathscr{H}]^N} \\ & \quad \leq \bigl|\varphi\bigl(\alpha'(\eta_n^\circ) - \alpha'(\eta^\circ)\bigr)\bigr|_{\mathscr{H}} \\ & \qquad + |\alpha'|_{L^\infty(\mathbb{R})}\bigl|\varphi\bigl(\nabla f_\varepsilon(\nabla\theta_n^\circ) - \nabla f_\varepsilon(\nabla\theta^\circ)\bigr)\bigr|_{[\mathscr{H}]^N} \\ & \qquad + \frac{2|\alpha'|_{L^\infty(\mathbb{R})}}{\varepsilon}|\varepsilon_n - \varepsilon||\varphi|_{\mathscr{H}} \\ & \quad \to 0, \text{ as } n \to \infty, \text{ for any } \varphi \in \mathscr{H}. \end{aligned} \tag{7.7.9}$$

Owing to (7.7.7a) and (7.7.9), one can further observe that:

$$\begin{aligned} & \bigl(\alpha'(\eta_n^\circ)\nabla f_{\varepsilon_n}(\nabla\theta_n^\circ) \cdot \nabla z_n^\circ, \varphi\bigr)_{\mathscr{H}} = \bigl(\nabla z_n^\circ, \varphi\alpha'(\eta_n^\circ)\nabla f_{\varepsilon_n}(\nabla\theta_n^\circ)\bigr)_{[\mathscr{H}]^N} \\ & \quad \to \bigl(\alpha'(\eta^\circ)\nabla f_\varepsilon(\nabla\theta^\circ) \cdot \nabla z^\circ, \varphi\bigr)_{\mathscr{H}} = \bigl(\nabla z^\circ, \varphi\alpha'(\eta^\circ)\nabla f_\varepsilon(\nabla\theta^\circ)\bigr)_{[\mathscr{H}]^N} \\ & \qquad \text{as } n \to \infty, \text{ for any } \varphi \in \mathscr{H}. \end{aligned} \tag{7.7.10}$$

Meanwhile, from (7.1.4c), (7.7.1b), (7.7.1c), ($\star$4), and (A4), it is inferred that:

$$\begin{aligned} & \bigl|\bigl(\alpha(\eta_n^\circ)\nabla^2 f_{\varepsilon_n}(\nabla\theta_n^\circ) - \alpha(\eta^\circ)\nabla^2 f_\varepsilon(\nabla\theta^\circ)\bigr)\nabla\psi\bigr|_{[\mathscr{H}]^N} \\ & \quad \leq |\alpha(\eta_n^\circ) - \alpha(\eta^\circ)|_{\mathscr{H}}\bigl|\nabla^2 f_{\varepsilon_n}(\nabla\theta_n^\circ)\bigr|_{L^\infty(Q;\mathbb{R}^{N\times N})}|\nabla\psi|_{C(\overline{Q};\mathbb{R}^N)} \\ & \qquad + |\alpha(\eta^\circ)|_{\mathscr{H}}\bigl|\nabla^2 f_{\varepsilon_n}(\nabla\theta_n^\circ) - \nabla^2 f_\varepsilon(\nabla\theta^\circ)\bigr|_{[\mathscr{H}]^{N\times N}}|\nabla\psi|_{C(\overline{Q};\mathbb{R}^N)} \\ & \quad \leq \frac{N+1}{\varepsilon}|\alpha'|_{L^\infty(\mathbb{R})}|\nabla\psi|_{C(\overline{Q};\mathbb{R}^N)}|\eta_n^\circ - \eta|_{\mathscr{H}} \\ & \qquad + \frac{3(N+1)^2}{\varepsilon^2}|\alpha(\eta^\circ)|_{\mathscr{H}}|\nabla\psi|_{C(\overline{Q};\mathbb{R}^N)}\left(|\nabla(\theta_n^\circ - \theta)|_{[\mathscr{H}]^N} + \frac{1}{n}\right) \\ & \quad \to 0, \text{ as } n \to \infty, \text{ for any } \psi \in C_{\mathrm{c}}^\infty(Q), \end{aligned}$$

and therefore,

$$\begin{aligned} & \bigl\langle -\mathrm{div}\bigl(\alpha(\eta_n^\circ)\nabla^2 f_{\varepsilon_n}(\nabla\theta_n^\circ)\nabla z_n^\circ\bigr), \psi\bigr\rangle = \bigl(\nabla z_n^\circ, \alpha(\eta_n^\circ)\nabla^2 f_{\varepsilon_n}(\nabla\theta_n^\circ)\nabla\psi\bigr)_{[\mathscr{H}]^N} \\ & \quad \to \bigl\langle -\mathrm{div}\bigl(\alpha(\eta^\circ)\nabla^2 f_\varepsilon(\nabla\theta^\circ)\nabla z^\circ\bigr), \psi\bigr\rangle = \bigl(\nabla z^\circ, \alpha(\eta^\circ)\nabla^2 f_\varepsilon(\nabla\theta^\circ)\nabla\psi\bigr)_{[\mathscr{H}]^N}, \end{aligned} \tag{7.7.11}$$

as $n \to \infty$, for any $\psi \in C_c^\infty(Q)$.

The fine properties as in (7.3.22) will be a consequence of (7.7.1f), (7.7.1g), (7.7.2), (7.7.7d)–(7.7.7f), (7.7.10), and (7.7.11).

Thus, we complete the proof of Main Theorem 7.4. □

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