Construction conditions of quantum error-correcting codes for various types of insertion/deletion errors

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Abstract

Quantum error-correction is a theory for protecting quantum information and is one of the essential factors in quantum information theory. Since its first construction by Shor in 1995, many quantum codes that can correct arbitrary unitary errors have been constructed. On the other hand, quantum insertion/deletion errors have only started to attract attention in 2020, and further research is expected.

This thesis discusses quantum error-correcting codes for various types of quantum insertion/deletion errors. First, the Nakayama-Hagiwara conditions are discussed, which are known as the conditions for constructing single deletion error-correcting codes, and examples of the new codes are given. Next, the construction conditions of quantum deletion codes with permutation-invariance are presented and their decoding method is described. This is the first construction of quantum codes that can correct multiple deletion errors. Furthermore, systematic construction of single insertion error-correcting codes is presented, including a decoding method. The conditions used in this construction are described only in terms of combinatorics, which means that the problem of quantum insertion errors is attributed to the problem of classical combinatorics. Finally, the equivalence between the correctability of deletions and insertions of separable states in quantum codes is proved using the Knill-Laflamme conditions, known as quantum error-correction conditions.

Keywords

Information theory, Quantum computing, Quantum error-correcting codes, Insertion/deletion errors, Combinatrics, Adjacency matrices, Kraus operators

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Chapter 1

Introduction

1.1 Background

1.1.1 Quantum computing

Quantum information technology, including quantum computers, is a new ICT field that fully utilizes the quantum mechanical properties of matter, and is expected to achieve things that conventional computers cannot, such as large-scale molecular design [6] and information retrieval systems that completely protect privacy [26]. In the field of quantum information, there is fierce competition at the national level in Europe, the U.S., China, and other countries around the world, and full-scale research and development are also underway in Japan.

Quantum computing began in 1980 when physicist Paul Benioff proposed a quantum mechanical model of Turing machines [7]. Later, Richard Feynman [24], Yuri Manin [42], and others suggested that quantum computers might be able to simulate what classical computers could not. In 1994, Peter Shor developed a quantum algorithm for factoring integers that may be able to break RSA-encrypted transmissions [71]. Since then, a great deal of research has been carried out.

Quantum computers have made tremendous progress in recent years, and in 2019, a Google research team announced that they had created a device with 53-qubit, which surpassed a supercomputer in "specific tasks" [4]. However, it has also been pointed out that the "specific task" is a very favorable problem setting for quantum computers and the view that we are still far from achieving practical computation [60]. A small-scale, noisy quantum computer like the one used in this demonstration is called a NISQ (Noisy Intermediate Scale Quantum) computer. In the future, it will be necessary to overcome a number of challenges in order to realize the full-scale practical use of quantum computers. To this end, a wide range of approaches are currently underway simultaneously, including basic research on hardware development and applied research for business applications.

1.1.2 Protection of quantum information

The most essential technology for realizing a quantum computer is error tolerance technology. There is an enormous difference in the amount of noise that can occur in quantum information processing and classical information processing. In classical information processing, errors occurred less frequently, and in some cases, processing could be done without much concern for errors. However, in quantum information processing, the frequency of errors is very high, and it is known that only small-scale calculations can be performed without error countermeasures. In order to put quantum information processing to practical use on a large scale, it is necessary to perform error tolerance processing of quantum information and to protect quantum information from noise.

One method of error handling in quantum information processing is to use quantum error-correcting codes. Quantum error-correcting codes encode qubits redundantly so that even if some errors occur, they can be corrected by decoding. This is the only method known to be capable of scalable error tolerance processing so far and is predicted to be an essential technology for the practical application of large-scale quantum information processing in the future. However, due to the inevitable increase in the number of qubits required for encoding and the number of processes in decoding, it is believed that it will still take time to realize error tolerance processing using quantum error-correction.

In addition, we have to take into account the fact that the quantum circuits must be well designed so that quantum error-correction still succeeds, assuming that the error is amplified during the decoding of the quantum error. Such a method is called fault-tolerant quantum computation (FTQC) [3, 22, 29]. According to this theory, as long as the probability of error is kept below a certain threshold, any length of calculation is possible. This is called threshold theorem, and it is one of the greatest achievements of quantum information theory [50]. Thus, the use of quantum error-correction in quantum computers must be discussed from various points of view.

1.1.3 Quantum error-correcting codes

To understand the meaning of quantum error-correcting codes, we introduce the communication channel model. The communication channel model is also an important concept in classical coding theory and was given by Shannon in 1948 as a concise and rational model for ensuring the transmission of information [63,64]. A visual representation of Shannon's communication channel model is shown in Figure 1.1. The information source selects the message to be transmitted, turns it into a signal by the encoder, and sends it to the receiver through the communication channel. During this transmission process, the information may be interfered with in various ways and may not be transmitted accurately. This model defines it as noise. The receiver decodes the signal received by the decoder back into a message and understands the message sent by the information source. When the message of the sender and the decoded message of the receiver match, the communication has been established. Weaver used Shannon's communication channel model for machines as a model to explain communication between humans, and discussed it. In 1949, Shannon and Weaver co-authored a book titled "The Mathematical Theory of Communication" [65].

The theory of error-correcting codes aims to efficiently and accurately perform two operations for situations where errors are expected to occur: adding information so that it can be corrected even if an error occurs (encoding) and inferring the original information from the incorrect information (decoding). In quantum error-correcting codes, the information to be transmitted

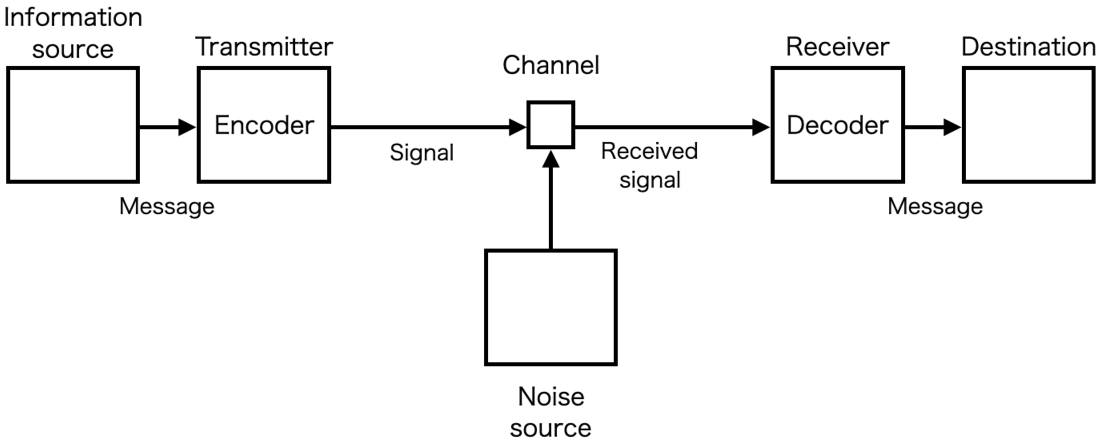

Figure 1.1: Shannon's communication channel model

is called a quantum state, which is represented as a density matrix. In addition, restrictions are placed on the encoding and decoding processes based on quantum mechanics. Therefore, the construction of quantum error-correcting codes is more difficult than that of classical error-correcting codes, and this is an area where further research for practical application is expected in the future.

In classical coding theory, errors in the original information have been corrected by adding redundancy to it, as in so-called repetition codes. On the other hand, because of the no-cloning theorem in quantum mechanics [78], it has been thought that such redundancy cannot be added in quantum error-correction. However, since 1995, when Shor explicitly denied the conjecture by providing an example of a quantum error-correcting code [72], many quantum codes have been devised, including Calderbank-Shor-Steane (CSS) codes [17, 75], the stabilizer codes [15,16,27,28], and surface codes [25]. Here, CSS code is a special case of the stabilizer code. Furthermore, the translation between quantum codes and linear spaces over finite field [15, 16] allows us to apply research results on classical codes to quantum codes. These results were later extended to non-binary quantum codes [5, 11, 46].

One attempt in quantum coding theory that is not found in classical codes is the use of quantum teleportation. Quantum teleportation is the transfer of a quantum state to a remote location using classical means of information transmission and the effects of quantum entanglement [8]. Although it is called teleportation, it does not mean that a particle moves instantaneously to another location in space, but rather that the observation of one state of two particles in a quantum entanglement relationship instantly reveals the definite state of the other. For a long time, experiments were considered difficult, but in 1997, D. Bouwmeester and his group were the first to successfully perform a quantum teleportation experiment [12]. Even recently, many successful experiments have been reported, and a variety of applications are expected [39,61,76]. Quantum teleportation can be interpreted as the transmission of quantum information through the sharing of quantum entanglement [20]. Along this line, the entanglement-assisted quantum error-correcting code was proposed [13, 77], which enables transmission of more quantum information at the cost of shared quantum entanglement.

Quantum error-correcting codes were also used in cryptography as a tool to prove the security of quantum key distribution (QKD) [73]. They were also used as an example of quantum secret sharing [30, 43]. Thus, as research on quantum error-correcting codes progresses, new applications, including cryptography, are expected in the future. In other words, it has been suggested that quantum error-correcting codes have sufficient potential for applications other than quantum computers.

There are good survey papers on the topic of quantum error-correcting codes, for example, see Reference [45].

1.1.4 Quantum insertion/deletion codes

In classical coding theory, since the first example was devised by Levenshtein in 1966 [41], many insertion/deletion error-correcting codes have been studied [14,23,32,33,36,74]. Indeed, classical insertion/deletion codes have received invigorated attention because of interesting applications such as DNA storage [14], and racetrack memories [18]. What this thesis deal with is a quantum version of this classical insertion/deletion code.

In quantum communication, the quantum state is transmitted through a quantum channel with possible errors, and it is naturally expected that some of the underlying qubits will be inadvertently lost during this process. This can be caused by temporary interruption, misalignment, or destruction of the transmitted signal. If we knew which qubit was lost, we could correct it as a quantum erasure error [9, 31], but in realistic scenarios, we often do not know which qubit was lost, which is the concept of a quantum deletion error. In quantum coding theory, erasure errors can be modeled using a partial trace where the traced qubits are known, but for deletion errors, we do not know what the traced qubits are. We can also interpret deletion errors as erasure errors implemented by an adversary who hides information about which qubits were erased. Hence, correcting deletion errors is harder than correcting erasure errors. Quantum deletion error-correction is a problem of determining the quantum state in the entire quantum system from a quantum state in a partial system. Therefore, it is related to various topics such as quantum secret sharing [43], purification of quantum state [37], and quantum cloud computing [10]. Similar to deletion errors, an insertion error occurs when a quantum state is inserted at unknown locations within a quantum code. As for quantum insertion codes, although they are mathematically very interesting as it relates to deletion, a detailed discussion of their applicability in realistic scenarios has not yet been given. However, it can also be said that it has many possibilities.

The quantum insertion/deletion error is a concept that was first introduced by Leahy et al. in 2019 [40]. They provided a way to turn quantum insertion/deletion errors into errors that can be handled by conventional methods under certain assumptions. Since then, quantum insertion/deletion error-correction has attracted the attention of researchers.

The first example of quantum deletion codes in a general scenario was given by Nakayama in 2020 [48]. Nakayama's 8-qubit code is capable of correcting a single deletion error, and this fact is shown by specifically defining the encoding and decoding. The second example of quantum deletion codes was given by Hagiwara in 2020 [35]. Hagiwara's 4-qubit code can correct a single deletion error, and it has been confirmed that this is a quantum deletion code with an optimal code length. In other words, the shortest code length of a quantum code that can correct deletion errors is 4. This is an interesting result compared to the shortest code length of a quantum code that can correct unitary errors, which is 5 [50]. Hagiwara's 4-qubit code can also be regarded as Ouyang's $(2, 2, 1)$ gnu code [51].

Later, in 2020, a systematic construction method of single quantum deletion codes including the two examples above was given by Nakayama and Hagiwara [49]. One idea for constructing quantum deletion codes is to use classical deletion codes. In fact, it was pointed out that quantum codes can be constructed from any classical code by using the framework of Movassagh and Ouyang [47]. However, the quantum deletion codes by Nakayama and Hagiwara, which were discovered for the first time, were derived based on a novel approach. According to Nakayama and Hagiwara, the three conditions (C1), (C2), and (C3) (the NH conditions) are introduced for two sets of bit sequences; those three conditions are described in combinatorial terms only. In their study, they proved that single quantum deletion error-correcting codes can be constructed by using two sets that satisfy the NH conditions. However, the NH conditions were complex and no new codes were presented in their paper. The NH conditions were studied by Shibayama, and many examples satisfying the conditions were given [66, 67]. This will be discussed in detail in Chapter 3 of this thesis. To distinguish them from the conditions of the insertion version, which will be explained later, the NH conditions for deletion will be denoted as $(C1)^-$, $(C2)^-$, and $(C3)^-$ with a minus character "$-$" as a superscript in this thesis.

Quantum error-correcting codes that can correct two or more deletion errors were first proposed in 2021 by Ouyang [53] and Shibayama [68], who studied them independently, respectively. Both of their methods focus on permutation-invariant codes, which are invariant under any permutation of the underlying particles. Permutation-invariant quantum codes have been studied as codes that can correct errors represented by unitary matrices [47, 51, 52, 55, 56, 59, 62]. Recently, permutation-invariant quantum codes have been explored for applications such as for quantum storage [54] and robust quantum metrology [58]. These are expected to be applied to physically realistic scenarios [79]. Since it is clear that erasure error and deletion error are equivalent in permutation-invariant codes, we can see that the permutation-invariant quantum codes that can correct quantum erasure errors, which have already been studied, are also quantum deletion codes. Ouyang [53] gave detailed discussions of permutation-invariant quantum deletion codes, which are defined by encodings formed by superpositions over states called Dicke states, described in Section 2.2.3 below, using quantum circuits. On the other hand, Shibayama and Hagiwara [68] proposed three conditions (D1), (D2), and (D3) that can correct multiple deletion errors for permutation-invariant codes defined by encodings that do not specify coefficients but are expressed in a more general form. Their method is a practical one that directly defines the decoder, and the details will be explained in Chapter 4 of this thesis. Later, further permutation-invariant deletion codes were given by Matsumoto and Hagiwara [44].

Several examples of quantum deletion codes have been given in the last two years, but quantum insertion codes seem to be more difficult than deletion. The first example is a 4-qubit code given by Hagiwara in 2021 [34], which is also known as a deletion code [35]. The Hagiwara code is a single insertion error-correcting code, and its decoding method for insertion errors is much more technical than that for deletion errors. Then, very recently, a systematic construction method for single quantum insertion codes was proposed by Shibayama and Hagiwara [69]. This construction can be regarded as an insertion version of the method using the Nakayama-Hagiwara conditions described earlier, and the conditions used in the insertion codes are named $(C1)^+$, $(C2)^+$, and $(C3)^+$. These conditions are described only in terms of classical combinatorics, not in terms of quantum information. It means that the problem of quantum insertion errors is attributed to the problem of classical combinatorics. This construction method will be discussed in detail in Chapter 5 of this thesis.

In classical insertion/deletion codes, the most mathematically interesting fact is that a code that can correct t deletion errors can correct t_1 insertion errors and t_2 deletion errors if $t = t_1 + t_2$. This implies the equivalence of correctability of insertion and deletion errors in classical codes. This fact was shown by Levenshtein in 1966 when the insertion/deletion code was first introduced [41]. It can be explained using the concept of Levenshtein distance [36]. It is conjectured that this equivalence, which holds for classical codes, also holds for quantum codes.

Conjecture 1. *For any positive integer t, and a quantum code Q, the followings are equivalent.*

1. *Q is a t-deletion error-correcting code.*
2. *Q is a t-insertion error-correcting code.*

To date, in quantum codes, the equivalence has not been shown, which is an open problem. However, a partial solution to this conjecture was given by Shibayama and Ouyang in 2021 [70]. Their method is to represent the Kraus operators for quantum deletion errors and quantum insertion errors and discuss them using the KL conditions, which are known as necessary and sufficient conditions for quantum error-correction. Here they discuss t-insertion, which is the repeated insertion of a single qubit state t times. In other words, they have not completely solved the problem in that they have not taken into account the insertion of quantum entangled states. However, it does provide a significant contribution in that it provides the first step towards this important conjecture. The details will be discussed in Chapter 6 of this thesis. Since then, research using the Kraus operator for quantum deletion errors and quantum insertion errors has gradually progressed [1, 2].

As mentioned above, quantum insertion/deletion codes are a very new field, and further research is expected in the future, including basic research. In the question and answer session of "An Introduction to Quantum Computation" by Chitambar, one of the tutorials at the 2020 IEEE Information Theory Workshop (ITW2020) held in April 2021, there was an exchange in which he predicted that deletion errors would be the focus of the next few years in the field of quantum codes [19]. Quantum insertion/deletion codes are an area that is gradually gaining attention.

1.2 Organization of the thesis

This thesis is organized as follows. In Chapter 2, the knowledge assumed in this research is explained. Specifically, it starts with the fundamentals of quantum information theory based on the postulates of quantum mechanics, and finally gives the definition of quantum insertion/deletion error-correcting codes. Chapter 3 gives the discussion related to the single quantum deletion error-correcting codes proposed by Nakayama and Hagiwara [49]. After giving explicit sets satisfying the Nakayama-Hagiwara (NH) conditions and proposing new codes, it is analyzed the NH conditions using the adjacency matrices of graphs. In Chapter 4, focusing on quantum codes with permutation-invariance, error-correction conditions for quantum codes that can correct multiple deletion errors are proposed and discussed, including their coding and decoding. In addition, it is given several examples of quantum codes that satisfy the error-correcting conditions and proposed the first quantum code that can correct two or more deletion errors. Chapter 5 proposes a systematic construction of single quantum insertion error-correcting codes. This method can be regarded as an insertion version of the method by Nakayama and Hagiwara. It can be said that this construction makes a significant contribution to the field of quantum insertion codes, for which only one example had been presented. In Chapter 6, quantum deletion errors and quantum insertion errors are discussed using the Kraus operators, which allows us to apply the Knill-Laflamme conditions, a necessary and sufficient condition for quantum error-correction. In particular, using this technique, we show that in quantum codes, the ability to correct deletions and insertions of separable states is equivalent. This can be said to give a partial solution for the quantum version of the famous result in classical insertion/deletion codes. At last, the conclusion is stated in Chapter 7.

Chapter 2

Fundamentals for quantum insertion/deletion error-correcting codes

The bit is the fundamental concept of classical computation and classical information. Quantum computation and quantum information are built on the similar concept of a quantum bit or a qubit for short. There are two possible states of a qubit, $|0\rangle$ and $|1\rangle$, which correspond to the states 0 and 1 for a classical bit. The notation '$|\ \rangle$' is the standard notation for states in quantum mechanics, and is called the Dirac notation [21]. The Dirac notation is used in this thesis. The purpose of this chapter is, to begin with, an explanation of the fundamental concepts of quantum information, including single qudit state, quantum system, projective measurement, etc., and finally to define quantum insertion/deletion codes.

2.1 Fundamental concepts of quantum information

Let N be a positive integer and $[N] \coloneqq \{1, 2, \dots, N\}$. The set of the non-negative integers is denoted by $\mathbb{N} \coloneqq \{n \in \mathbb{Z} \mid n \geq 0\}$. Given an N-tuple $\boldsymbol{x} = x_1 x_2 \dots x_N \in \mathbb{N}^N$, we use the notation $\mathrm{wt}(\boldsymbol{x})$ to denote the Hamming weight of $\boldsymbol{x}$, i.e.,

$$\mathrm{wt}(\boldsymbol{x}) \coloneqq |\{i \in [N] \mid x_i \neq 0\}|. \tag{2.1}$$

Here, we denote $|X|$ as the size of a finite set X. For an N_1-tuple $\boldsymbol{x} = x_1 x_2 \dots x_{N_1} \in \mathbb{N}^{N_1}$ and an N_2-tuple $\boldsymbol{y} = y_1 y_2 \dots y_{N_2} \in \mathbb{N}^{N_2}$ with positive integers N_1, N_2, we simply denote by $\boldsymbol{xy} \coloneqq x_1 x_2 \dots x_{N_1} y_1 y_2 \dots y_{N_2} \in \mathbb{N}^{N_1+N_2}$ the $(N_1 + N_2)$-tuple formed by concatenating $\boldsymbol{x}$ and $\boldsymbol{y}$.

We denote by $\mathbb{C}^l$ the l-dimensional vector space over a complex field $\mathbb{C}$. In this thesis, the inner product $(\cdot, \cdot)$ on $\mathbb{C}^l$ is defined as

$$(\boldsymbol{x}, \boldsymbol{y}) \coloneqq \overline{x_1} y_1 + \overline{x_2} y_2 + \cdots + \overline{x_l} y_l \in \mathbb{C} \tag{2.2}$$

for $\boldsymbol{x} = (x_1, x_2, \dots, x_l)^\top \in \mathbb{C}^l$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_l)^\top \in \mathbb{C}^l$. Here, $^\top$ is the transpose operation, and $\overline{c}$ is the conjugate of the complex number $c \in \mathbb{C}$.

2.1.1 Quantum message

A quantum message is assumed to be a complex vector in $\mathbb{C}^l$. A standard notation for a quantum message is $|\psi\rangle$, which is called ψ-ket. Assume that the length of the quantum message $|\psi\rangle$ is 1, i.e., $\||\psi\rangle\| = 1$. Here, the length of $|\psi\rangle$ is defined as the positive square root of the inner product with itself, i.e.,

$$\||\psi\rangle\|^2 = (|\psi\rangle, |\psi\rangle). \tag{2.3}$$

For a quantum message $|\psi\rangle \in \mathbb{C}^l$, we define $\langle\psi|$ as the conjugate transpose of $|\psi\rangle$, i.e., $\langle\psi| \coloneqq |\psi\rangle^\dagger$, and $\langle\psi|$ is called ψ-bra. Here, † is the conjugate transpose operation. Note that the inner product of quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ is denoted by $\langle\psi_1|\psi_2\rangle$, i.e.,

$$\langle\psi_1|\psi_2\rangle \coloneqq (|\psi_1\rangle, |\psi_2\rangle) \tag{2.4}$$

$$= \langle\psi_1||\psi_2\rangle. \tag{2.5}$$

The product in Equation (2.5) is the product as matrices, which is easily derived by calculating it according to the definition of the inner product in Equation (2.2).

An l-dimensional vector $|\psi\rangle \in \mathbb{C}^l$ with $\||\psi\rangle\| = 1$ is called a qudit. In particular, let $|0\rangle, |1\rangle, \dots, |l-1\rangle$ be the standard orthonormal basis of $\mathbb{C}^l$, and call them zero-ket, one-ket, $\dots$, $(l-1)$-ket, respectively. That is,

$$|0\rangle \coloneqq (1,0,0,\dots,0)^\top, |1\rangle \coloneqq (0,1,0,\dots,0)^\top, \dots, |l-1\rangle \coloneqq (0,0,0,\dots,1)^\top. \tag{2.6}$$

In general, any qudit can be written in the form $\alpha_0|0\rangle + \alpha_1|1\rangle + \cdots + \alpha_{l-1}|l-1\rangle \in \mathbb{C}^l$, where $|\alpha_0|^2 + |\alpha_1|^2 + \cdots + |\alpha_{l-1}|^2 = 1$. Here, $|c|$ is the absolute value of the complex number $c \in \mathbb{C}$. Qudits $|\psi_1\rangle$, $|\psi_2\rangle$ are considered to be the same if $|\psi_1\rangle = c|\psi_2\rangle$ for some constant $c \in \mathbb{C}$. Note that $|c| = 1$ holds since qudits have the same length.

In this thesis, we mainly deal with the case of $l = 2$, and in this case, we use the term qubit instead of qudit. In particular, for qubits, zero-ket and one-ket are written as $|0\rangle \coloneqq (1,0)^\top \in \mathbb{C}^2$ and $|1\rangle \coloneqq (0,1)^\top \in \mathbb{C}^2$, respectively.

2.1.2 Density matrix and single qudit state

For a square matrix M over the complex field $\mathbb{C}$, the sum of the diagonal elements of M is denoted by $\mathrm{Tr}(M)$ and is called the trace of M.

A square matrix M satisfying the following three conditions is called a density matrix:

- It is positive semi-definite, i.e., any eigenvalue is non-negative.
- It is Hermitian, i.e., $M = M^\dagger$.
- Its trace is equal to 1, i.e., $\mathrm{Tr}(M) = 1$.

The set of all density matrices of order l is denoted by $S(\mathbb{C}^l)$. An element of $S(\mathbb{C}^l)$ is called a single qudit state.

In general, for any positive integer t, any qudits $|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_t\rangle \in \mathbb{C}^l$, and any non-negative real numbers $p_1, p_2, \dots, p_t \in \mathbb{R}$ with $p_1 + p_2 + \cdots + p_t = 1$,

$$\sum_{i=1}^{t} p_i |\psi_i\rangle\langle\psi_i| \tag{2.7}$$

is a density matrix. In particular, any qudit $|\psi\rangle \in \mathbb{C}^l$ corresponds to the density matrix $|\psi\rangle\langle\psi| \in S(\mathbb{C}^l)$. In fact, for a qudit $|\psi\rangle \in \mathbb{C}^l$ and a constant $c \in \mathbb{C}$ with $|c| = 1$, the density matrix corresponding to $|\psi\rangle \in \mathbb{C}^l$ and $c|\psi\rangle \in \mathbb{C}^l$ are the same. A single qudit state is called pure if it can be represented in the form $|\psi\rangle\langle\psi| \in S(\mathbb{C}^l)$ using some qudit $|\psi\rangle \in \mathbb{C}^l$. We also use a complex vector $|\psi\rangle \in \mathbb{C}^l$ for representing a pure single qudit state $|\psi\rangle\langle\psi| \in S(\mathbb{C}^l)$. Conversely, a single qudit state that is not pure is called mixed. Note that the quantum message described in Section 2.1.1 is defined as a pure single qudit state.

2.1.3 Quantum system

In a physical implementation of a quantum message, a single qudit corresponds to the state of a single particle. Conversely, any single particle can be represented as a vector of length one in a complex vector space $\mathcal{H}$. This vector space $\mathcal{H}$ is called the quantum system of the particle. If the dimension of $\mathcal{H}$ is l, the quantum system is said to be of level l.

Let $\mathcal{H}_i$ be the quantum system of the single particle p_i for $i \in [N]$. The quantum system of N particles $p_1, p_2, \ldots, p_N$ is described by

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N, \tag{2.8}$$

where $\otimes$ is the tensor product operation. For simplicity, a quantum system $\mathbb{C}^l \otimes \mathbb{C}^l \otimes \cdots \otimes \mathbb{C}^l$ for N particles is denoted by $(\mathbb{C}^l)^{\otimes N}$. Set $|\boldsymbol{x}\rangle := |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_N\rangle \in (\mathbb{C}^l)^{\otimes N}$ for an N-tuple $\boldsymbol{x} = x_1x_2\ldots x_N \in \{0, 1, \ldots, l-1\}^N$. Since the standard orthonormal basis of $(\mathbb{C}^l)^{\otimes N}$ is $\{|\boldsymbol{x}\rangle \in (\mathbb{C}^l)^{\otimes N} \mid \boldsymbol{x} \in \{0, 1, \ldots, l-1\}^N\}$, we can regard $(\mathbb{C}^l)^{\otimes N}$ and $\mathbb{C}^{l^N}$ as isomorphic vector spaces. A l^N-dimensional vector $|\psi\rangle \in (\mathbb{C}^l)^{\otimes N}$ with $\||\psi\rangle\| = 1$ is called an N-qudit.

If the single qudit of the single particle p_i is represented by a vector $|\psi_i\rangle \in \mathcal{H}_i$ for $i = 1, 2, \ldots, N$, the quantum state of the N particles $p_1, p_2, \ldots, p_N$ is

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_N\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N. \tag{2.9}$$

In this way, when an N-qudit can be represented in the form of a tensor product of N single qudits, the N-qudit is called separable. Conversely, an N-qudit that is not separable is called entangled. For example,

$$|\Psi\rangle := \frac{|00\rangle + |11\rangle}{\sqrt{2}} \in (\mathbb{C}^2)^{\otimes 2} \tag{2.10}$$

cannot be expressed in the form of $|\psi_1\rangle \otimes |\psi_2\rangle \in (\mathbb{C}^2)^{\otimes 2}$, thus $|\Psi\rangle$ is an entangled 2-qubit.

The set of quantum states in a quantum system $(\mathbb{C}^l)^{\otimes N}$ is denoted by $S((\mathbb{C}^l)^{\otimes N})$, which is defined as the set of all density matrices of order l^N, that is, $S(\mathbb{C}^{l^N})$. An element of $S((\mathbb{C}^l)^{\otimes N})$ is called an N-qudit state, which represents a quantum state of N particles $p_1, p_2, \ldots, p_N$. For the N-qudit state, pure and mixed are defined in the same way as for the single qudit state. That is, an N-qudit state is called pure if it can be represented in the form $|\psi\rangle\langle\psi| \in S((\mathbb{C}^l)^{\otimes N})$ using some qudit $|\psi\rangle \in (\mathbb{C}^l)^{\otimes N}$, and an N-qudit state that is not pure is called mixed.

Furthermore, for quantum states represented as density matrices, a separable state, and an entangled state are defined as in the case of qudit represented as vectors. In other words, the definition is as follows. If the single quantum state of the single particle p_i is represented by a

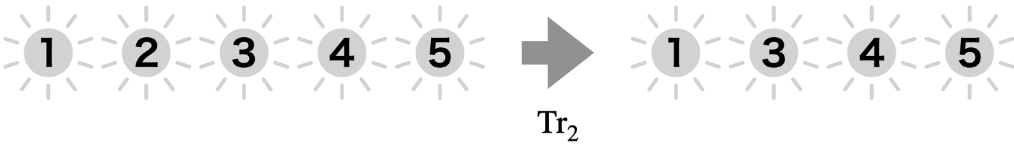

Figure 2.1: Action of the partial trace Tr_2 on a quantum state of five particles

density matrix $\rho_i \in S(\mathcal{H}_i)$ for $i = 1, 2, \ldots, N$, the quantum state of the N particles $p_1, p_2, \ldots, p_N$ is

$$\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N \in S(\mathcal{H}_1) \otimes S(\mathcal{H}_2) \otimes \cdots \otimes S(\mathcal{H}_N). \tag{2.11}$$

In this way, when an N-qudit state can be represented in the form of a tensor product of N single qudit states, the N-qudit state is called separable. Conversely, an N-qudit state that is not separable is called entangled.

Now we explain the quantum subsystem. For N particles with a quantum state $M \in S((\mathbb{C}^l)^{\otimes N})$, we want to describe the state of their sub-particles. The quantum state M is re-written as

$$M = \sum_{\boldsymbol{x},\boldsymbol{y} \in \{0,1,\ldots,l-1\}^N} m_{\boldsymbol{x},\boldsymbol{y}} |x_1\rangle\langle y_1| \otimes |x_2\rangle\langle y_2| \otimes \cdots \otimes |x_N\rangle\langle y_N| \tag{2.12}$$

with $m_{\boldsymbol{x},\boldsymbol{y}} \in \mathbb{C}$. For an integer $i \in [N]$, define the map $\mathrm{Tr}_i : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes (N-1)})$ as

$$\begin{aligned} \mathrm{Tr}_i(M) \coloneqq & \sum_{\boldsymbol{x},\boldsymbol{y} \in \{0,1,\ldots,l-1\}^N} m_{\boldsymbol{x},\boldsymbol{y}} \cdot \mathrm{Tr}(|x_i\rangle\langle y_i|) \\ & |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_{i-1}\rangle\langle y_{i-1}| \otimes |x_{i+1}\rangle\langle y_{i+1}| \otimes \cdots \otimes |x_N\rangle\langle y_N|. \end{aligned} \tag{2.13}$$

The map Tr_i is called a partial trace. The state of $N-1$ particles $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N$ is described as $\mathrm{Tr}_i(M) \in S((\mathbb{C}^l)^{\otimes (N-1)})$. In other words, the partial trace Tr_i describes the state of the subsystem of $N-1$ particles $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N$. For example, when a partial trace is applied to a quantum state consisting of five particles, it means that one particle is deleted and four particles are left, as shown in Figure 2.1.

2.1.4 Projective measurement

In quantum communication, the sender encodes a quantum message into particles. The receiver of the transmitted particles will read the whole or part of the message from the particles. Reading the message from the particles is called measurement. A Hermitian matrix P satisfying $P^2 = P$ is called a projection matrix. In this study, a measurement called projective measurement, which is represented by the projection matrices, is used in the decoding process.

Let Ω be the set of all outcomes that can be obtained by the projective measurement. A projective measurement is defined by a set $\mathbb{P} \coloneqq \{P_k \mid k \in \Omega\}$ of complex matrices of order l^N. Here, each element of $\mathbb{P}$ is a projection matrix and is assumed to satisfy the equation

$$\sum_{k \in \Omega} P_k = \mathbb{I}, \tag{2.14}$$

where $\mathbb{I}$ is the identity matrix of order l^N. Note that N here is the number of particles that the receiver has, not necessarily the code length. The index $k \in \Omega$ refers to the measurement outcomes that may occur in the experiment. If the quantum state is $\rho \in S((\mathbb{C}^l)^{\otimes N})$ immediately before the measurement then the probability that outcome k occurs is given by $\mathrm{Tr}(P_k\rho)$, and the quantum state after the measurement ρ' is

$$\rho' := \frac{P_k \rho P_k}{\mathrm{Tr}(P_k \rho)}. \tag{2.15}$$

Note here that the receiver can only know the outcome of the measurement, but not the received quantum state itself.

2.1.5 The postulates of quantum mechanics

To begin with, quantum information is a theory that uses the fundamental postulates of quantum mechanics as a starting point for constructing the basic operating principles of quantum computers. At the end of this section, we will confirm that the concepts of quantum information set up above satisfy the postulates of quantum mechanics.

The following four are the most basic postulates of quantum mechanics; for more details about them, see e.g., Nielsen-Chuang [50].

Postulate 1. Associated to any isolated physical system is a complex vector space with an inner product (that is, a Hilbert space) known as the state space of the system. The system is completely described by its density operator, which is a positive operator ρ with trace one, acting on the state space of the system. If a quantum system is in the state ρ_i with probability p_i, then the density operator for the system is $\sum_i p_i \rho_i$.

Postulate 2. The evolution of a closed quantum system is described by a unitary transformation. That is, the state ρ of the system at time t_1 is related to the state ρ' of the system at time t_2 by a unitary operator U which depends only on the time t_1 and t_2,

$$\rho' = U \rho U^\dagger \tag{2.16}$$

Postulate 3. Quantum measurements are described by a collection $\{M_k\}$ of measurement operators. These are operators acting on the state space of the system being measured. The index k refers to the measurement outcomes that may occur in the experiment then the probability that result k occurs is given by

$$p(k) = \mathrm{Tr}(M_k^\dagger M_k \rho), \tag{2.17}$$

and the state of the system after the measurement is

$$\frac{M_k \rho M_k^\dagger}{\mathrm{Tr}(M_k^\dagger M_k \rho)}. \tag{2.18}$$

The measurement operators satisfy the completeness equation,

$$\sum_k M_k^\dagger M_k = I, \tag{2.19}$$

where I is the identity operator.

Postulate 4. The state-space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through N, and system number i is prepared in the state ρ_i, then the joint state of the total system is

$$\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N. \tag{2.20}$$

In particular, the contents described in Sections 2.1.1, 2.1.2, and Section 2.1.3 satisfy Postulate 1 and Postulate 4, respectively. The projective measurements defined in Section 2.1.4 satisfy Postulate 3. Note that $P^\dagger P = P$, since the projection matrix P is a Hermitian matrix satisfying $P^2 = P$.

In quantum information theory, the operation of acting a unitary matrix on a quantum state can be performed as Equation (2.16). Here, a square matrix U is called a unitary matrix if $UU^\dagger$ is the identity matrix. For any unitary matrix U of order l^N and any quantum state $\rho \in S((\mathbb{C}^l)^{\otimes N})$, $\rho' := U\rho U^\dagger$ is a quantum state. Also, for any two quantum states $\rho_1, \rho_2 \in S((\mathbb{C}^l)^{\otimes N})$, there exists a unitary matrix U of order l^N such that $\rho' = U\rho U^\dagger$. These can be easily checked by calculation as matrices.

2.2 Quantum error-correcting codes for unitary errors

This section explains errors in quantum communication and defines error-correcting codes, which are known as techniques to protect quantum information from such errors. In particular, we define gnu codes, which appear several times in this thesis.

2.2.1 Unitary errors

This thesis focuses on deletion and insertion errors and does not deal in depth with the widely known unitary errors, including bit-flip and phase errors. The errors described by unitary matrices are briefly explained here, especially using a qubit system as an example.

The Pauli operators are introduced as basic quantum operations that act on a single qubit. The operators, which are represented by the following four unitary matrices, are called the Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.21}$$

A unitary error in quantum information is described as the action of a unitary matrix on a quantum state. For example, when an error corresponding to X occurs, we get

$$X|0\rangle = |1\rangle, \quad X|1\rangle = |0\rangle, \tag{2.22}$$

which represents a bit-flip error. Also, when an error corresponding to Z occurs, we get

$$Z|0\rangle = |0\rangle, \quad Z|1\rangle = -|1\rangle, \tag{2.23}$$

which represents a phase error. Any unitary matrix of order 2 is represented by a linear combination of I, X, Y, and Z. In fact, the Pauli matrices in Equation (2.21) form a basis of the set of 2-by-2 matrices over the complex field $\mathbb{C}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+c}{2} I + \frac{b+d}{2} X + \frac{b-d}{2} iY + \frac{a-c}{2} Z. \tag{2.24}$$

A single unitary error for an N-qudit $|\psi\rangle \in (\mathbb{C}^l)^{\otimes N}$ is represented as

$$(\mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes U \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I})|\psi\rangle \tag{2.25}$$

for some unitary matrix U of order l. Here, $\mathbb{I}$ is the identity matrix of order l. For example, for a single bit-flip error $I \otimes X \otimes I$, we get

$$(I \otimes X \otimes I)|000\rangle = |010\rangle, \quad (I \otimes X \otimes I)|111\rangle = |101\rangle, \tag{2.26}$$

which confirms that a bit-flip error has indeed occurred for the second qubit. Note that when a unitary error occurs for a quantum state represented by a density matrix, the unitary matrix acts as in Equation (2.16).

An error obtained by combining single unitary errors t times is called a t-qudit unitary error.

2.2.2 Quantum error-correcting codes

Let us define a quantum error-correcting code. A quantum message is usually defined as a single qudit state associated with a single particle, but here we define it more generally as a K-qudit state associated with K particles.

Let $\mathcal{E}$ be the set of all errors that we want to correct here. That is, each element of $\mathcal{E}$ is a map that maps an N-qudit state $\rho \in S((\mathbb{C}^l)^{\otimes N})$ to an N'-qudit state $\rho' \in S((\mathbb{C}^l)^{\otimes N'})$ with some N'. Note here that the number of particles may change before and after the error, so that $N \neq N'$ may be possible.

Definition 2.1 (Quantum error-correcting code)**.** *We call an image of* Enc *an* $(\!(N, K)\!)$ *quantum error-correcting code for the error* $\mathcal{E}$ *if the following conditions hold:*

1. *There exists a map* $\mathrm{Enc} : (\mathbb{C}^l)^{\otimes K} \to (\mathbb{C}^l)^{\otimes N}$ *defined as* $\mathrm{enc} \circ \mathrm{pad}^{N,K}$*, that is, the composition of two maps* $\mathrm{pad}^{N,K}$ *and* enc*. Here, the map* $\mathrm{pad}^{N,K} : (\mathbb{C}^l)^{\otimes K} \to (\mathbb{C}^l)^{\otimes N}$ *is defined by*

$$\mathrm{pad}^{N,K}(|\psi\rangle) := |\psi\rangle \otimes \underbrace{|0\rangle \otimes \cdots \otimes |0\rangle}_{N-K \text{ times}} \tag{2.27}$$

for a pure K*-qudit state* $|\psi\rangle \in (\mathbb{C}^l)^{\otimes K}$*, and the map* $\mathrm{enc} : (\mathbb{C}^l)^{\otimes N} \to (\mathbb{C}^l)^{\otimes N}$ *is a unitary transformation acting on* $(\mathbb{C}^l)^{\otimes N}$*.*

2. *There exists a map* Dec *defined by the operations allowed by quantum mechanics, and*

$$\mathrm{Dec} \circ E \circ \mathrm{Enc}(|\psi\rangle) = |\psi\rangle \tag{2.28}$$

holds for any error $E \in \mathcal{E}$ *and any pure* K*-qudit state* $|\psi\rangle \in (\mathbb{C}^l)^{\otimes K}$*.*

The maps Enc *and* Dec *are called the encoder and decoder for the quantum error-correcting code, respectively.*

If we take as $\mathcal{E}$ the set consisting of all t-qudit unitary errors acting on N-qudit, then an $((N, K))$ quantum error-correcting code for the error $\mathcal{E}$ is called an $((N, K))$ t-qudit error-correcting code or simply a t-qudit code. It is clear that a t-qudit code is an s-qudit code for any non-negative integer $0 \leq s \leq t$. In other words, the t-qudit code can be interpreted as a code that can correct any unitary error that occurs in t or fewer arbitrary particles.

There are three types of quantum mechanical operations used in the decoder in this thesis: projective measurements, actions of a unitary matrix, and partial traces.

In particular, to construct an encoder Enc for an $((N, 1))$ quantum error-correcting code, it is enough to choose vectors $|0_L\rangle, |1_L\rangle, \dots, |l-1_L\rangle \in (\mathbb{C}^l)^{\otimes N}$ of length 1 orthogonal to each other, and the encoder is defined as a linear map Enc such that $\mathrm{Enc}(|\psi\rangle) \coloneqq |\Psi\rangle$,

$$|\psi\rangle \coloneqq \alpha_0|0\rangle + \alpha_1|1\rangle + \cdots + \alpha_{l-1}|l-1\rangle \in \mathbb{C}^l, \tag{2.29}$$

$$|\Psi\rangle \coloneqq \alpha_0|0_L\rangle + \alpha_1|1_L\rangle + \cdots + \alpha_{l-1}|l-1_L\rangle \in (\mathbb{C}^l)^{\otimes N}. \tag{2.30}$$

Here, these vectors $|0_L\rangle, |1_L\rangle, \dots, |l-1_L\rangle$ are called a logical 0, a logical $1, \dots,$ and a logical $l-1$, respectively. These l vectors are collectively called logical codewords. In fact, since

$$\mathrm{pad}^{N,1}(|\psi\rangle) = (\alpha_0|0\rangle + \alpha_1|1\rangle + \cdots + \alpha_{l-1}|l-1\rangle) \otimes \underbrace{|0\rangle \otimes \cdots \otimes |0\rangle}_{N-1 \text{ times}} \tag{2.31}$$

$$= \alpha_0|00\dots0\rangle + \alpha_1|10\dots0\rangle + \cdots + \alpha_{l-1}|(l-1)0\dots0\rangle \tag{2.32}$$

holds, if we define enc as the unitary transformation that maps $|i0\dots0\rangle \in (\mathbb{C}^l)^{\otimes N}$ to $|i_L\rangle \in (\mathbb{C}^l)^{\otimes N}$ for each integer $0 \leq i \leq l-1$, we can easily check that $\mathrm{Enc}(|\psi\rangle) = \mathrm{enc} \circ \mathrm{pad}^{N,1}(|\psi\rangle)$ for any quantum message $|\psi\rangle \in \mathbb{C}^l$.

In other words, in order to construct an $((N, 1))$ quantum error-correcting code, the problems are how to define logical codewords $|0_L\rangle, |1_L\rangle, \dots, |l-1_L\rangle$ and how to define a decoder Dec that satisfies Equation (2.28).

2.2.3 Ouyang Δ-shifted gnu codes

An N-qudit state is called permutation-invariant if its state is invariant under any permutation of N particles. A quantum code is called a permutation-invariant code if its state after encoding is permutation-invariant. The term permutation-invariant is also called "PI" for short.

Here, the definition of gnu codes, which appears often in this thesis, is explained. Ouyang introduced gnu codes in 2014 as a family of permutation-invariant quantum codes [51]. The precise definition of permutation-invariant quantum codes will be given in Chapter 4. Ouyang succeeded in generalizing gnu codes and constructing further quantum codes in 2017 [52]. Among them, shifted gnu codes for qubit systems, which have similar properties to gnu codes, have been proposed in 2021 [53]. This code has four parameters Δ, g, n, and u, where the shift Δ is a non-negative integer, and the code gap g and the occupancy n are positive integers, and the rational number $u = \frac{N}{gn} \geq 1$ is a scaling factor that determines the length of the quantum code.

The corresponding logical codewords are

$$|0_L\rangle = \sum_{\substack{0\leq j\leq n\\ j \text{ even}}} \sqrt{\frac{\binom{n}{j}}{2^{n-1}}}|D^{gnu+\Delta}_{gj+\Delta}\rangle, \qquad |1_L\rangle = \sum_{\substack{0\leq j\leq n\\ j \text{ odd}}} \sqrt{\frac{\binom{n}{j}}{2^{n-1}}}|D^{gnu+\Delta}_{gj+\Delta}\rangle, \tag{2.33}$$

where $|D^N_w\rangle$ is called Dicke states and is defined as

$$|D^N_w\rangle \coloneqq \frac{1}{\sqrt{\binom{N}{w}}} \sum_{\substack{x_1x_2\ldots x_N\in\{0,1\}^N\\ x_1+x_2+\cdots+x_N=w}} |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_N\rangle. \tag{2.34}$$

Here, w is the weight of the Dicke state and counts the Hamming weights of its constituent computation basis states' labels. The Dicke states for qubit states are labeled by only their weights, of which there are only $N+1$ possibilities. The quantum code defined by the logical codewords in Equation (2.33) is called a Δ-shifted gnu code and is denoted by $\Delta-(g,n,u)$ with these parameters. In particular, in the case of $\Delta = 0$, a 0-shifted gnu code is simply called a gnu code.

Fact 2.2 (Ouyang [52])**.** *Fix t as a non-negative integer. Let Δ be a non-negative integer, g, n be integers, and u be a rational number, and suppose that*

$$g \geq 2t+1, \qquad n \geq 2t+1, \qquad u \geq 1. \tag{2.35}$$

Then, the $\Delta-(g,n,u)$ code is an $((N,1))$ t-qudit error-correcting code with $N = gnu+\Delta$.

Example 2.3. *The $0-(3,3,1)$ gnu code is a $((9,1))$ 1-qudit error-correcting code, and its logical codewords are*

$$|0_L\rangle = \frac{|D^9_0\rangle + \sqrt{3}|D^9_6\rangle}{2}, \qquad |1_L\rangle = \frac{\sqrt{3}|D^9_3\rangle + |D^9_9\rangle}{2}. \tag{2.36}$$

This quantum code is called the Ruskai code [62].

2.3 Quantum error-correcting codes for insertion/deletion errors

This section explains the concept of quantum insertion and deletion errors and then gives the definition of quantum insertion codes and quantum deletion codes, respectively.

2.3.1 Quantum insertion/deletion errors

Recall that in classical coding theory, for an integer $1 \leq t < N$, a t-deletion error is defined as a map from a sequence of length N to its subsequence of length $N-t$. In addition, insertion errors are defined as the inverse of deletion errors. That is, an insertion error maps a sequence $\boldsymbol{x}$ of length N to a sequence $\boldsymbol{x}'$ of length $N+t$, where $\boldsymbol{x}$ is a subsequence of $\boldsymbol{x}'$.

In quantum coding theory, deletion errors and insertion errors are defined for quantum states represented by density matrices in the same way as for classical codes.

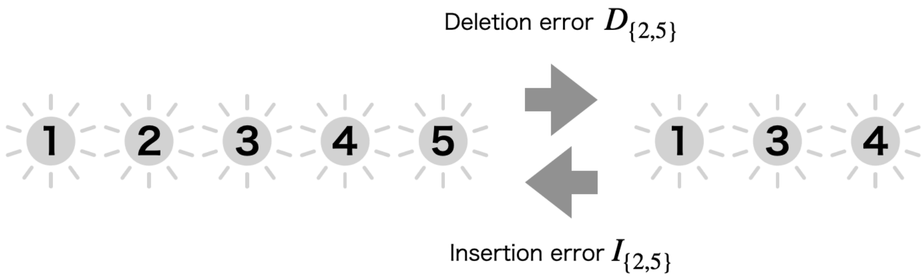

Figure 2.2: Relationship between deletion and insertion errors acting on particles

Definition 2.4 (Quantum deletion error D_P)**.** *Let $1 \leq t < N$ be a positive integer and let $P = \{p_1, p_2, \ldots, p_t\} \subset [N]$ be a set satisfying $1 \leq p_1 < p_2 < \cdots < p_t \leq N$. Let us define a map $D_P : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes(N-t)})$ as*

$$D_P(\rho) \coloneqq \underbrace{\mathrm{Tr}_{p_1} \circ \mathrm{Tr}_{p_2} \circ \cdots \circ \mathrm{Tr}_{p_t}}_{t\ times}(\rho), \tag{2.37}$$

where $\rho \in S((\mathbb{C}^l)^{\otimes N})$ is a quantum state. Here the symbol $\circ$ indicates the composition of maps. We call the map D_P a t-deletion error with the deletion position P. The cardinality $|P|$ is called the number of deletions for D_P.

In Definition 2.4, especially in the case of $P = \{p\}$ (i.e., $|P| = 1$), we denote D_p instead of $D_{\{p\}}$ and call the map D_p a single deletion error. In other words, a single deletion error D_p is a partial trace Tr_p. The identity map for quantum states is also called the 0-deletion error.

As explained in Section 2.1.3, For a quantum state $\rho \in S((\mathbb{C}^l)^{\otimes N})$ corresponding to N particles $p_1, p_2, \ldots, p_N$, the state $D_P(\rho) \in S((\mathbb{C}^l)^{\otimes(N-t)})$ represents the quantum state corresponding to $N - t$ particles, excluding the particles labeled with P.

Definition 2.5 (Quantum insertion error I_P)**.** *Let $t \geq 1$ be a positive integer and let $P = \{p_1, p_2, \ldots, p_t\} \subset [N + t]$ be a set satisfying $1 \leq p_1 < p_2 < \cdots < p_t \leq N + t$. For each quantum state $\rho \in S((\mathbb{C}^l)^{\otimes N})$, a map $I_P : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes(N+t)})$ such that*

$$D_P \circ I_P(\rho) = \rho \tag{2.38}$$

is called a t-insertion error for quantum state ρ with the insertion position P. The cardinality $|P|$ is called the number of insertions for I_P.

According to Definition 2.5, an insertion error is defined as the inverse correspondence of a deletion error. In Equation (2.38), if ρ is a quantum state consisting of three particles and $P = \{2, 5\}$, then $D_P(I_P(\rho))$ is a 3-qubit state, and therefore $I_P(\rho)$ is a 5-qubit state. In other words, $I_P(\rho)$ can be regarded as the addition of two particles to the original three. Figure 2.2 shows the transformation of the particles in this case.

In Definition 2.5, in the case of $P = \{p\}$ (i.e., $|P| = 1$), as with the deletion error, we denote I_p instead of $I_{\{p\}}$ and call the map I_p a single insertion error. Although the characteristics of the inserted particles should be written in the subscript of the map, to avoid complications, only the insertion positions are written here for a general definition. The identity map for quantum states is also called the 0-insertion error.

For any quantum state $\rho \in S((\mathbb{C}^l)^{\otimes N})$, there exists a non-negative integer k and a pure state $\rho' \in S((\mathbb{C}^l)^{\otimes (N+k)})$ such that $D_{[k]}(\rho') = \rho$. This means that an arbitrary quantum state can always be regarded as a pure state by adding an external system. This procedure is called the purification of a quantum state, which is a type of quantum insertion error.

If we limit a quantum state before an insertion error occurs to the pure state, the insertion error I_P in Definition 2.5 is expressed in a clean form as in Fact 2.6. This fact was shown by Reference [69] in 2021.

Fact 2.6. *Let $t \geq 1$ be a positive integer and let $P = \{p_1, p_2, \ldots, p_t\} \subset [N+t]$ be a set satisfying $1 \leq p_1 < p_2 < \cdots < p_t \leq N+t$. A t-insertion error for a pure state $|\psi\rangle \in (\mathbb{C}^l)^{\otimes N}$ with the insertion position P can be expressed as*

$$I_P(|\psi\rangle\langle\psi|) = \tau(\sigma \otimes |\psi\rangle\langle\psi|) \tag{2.39}$$

with some quantum state $\sigma \in S((\mathbb{C}^l)^{\otimes t})$ and a permutation $\tau \in S_{N+t}$ acting on the quantum system of the $N+t$ particles $p_1, p_2, \ldots, p_{N+t}$. Here, S_{N+t} is the symmetry group of degree $N+t$, and the permutation τ maps the ith particle to the p_ith if $i \in [t]$, and preserves the order of the particles if $i \notin [t]$.

Fact 2.6 means that the original state ρ and the state σ to be inserted are not quantum entangled, assuming that the state ρ after encoding is a pure state $\rho = |\psi\rangle\langle\psi|$. Without the assumption that the quantum state ρ is pure, quantum insertion errors cannot be expressed in a simple form. For example, as mentioned earlier, the purification of a quantum state is a type of quantum insertion error, but when ρ is a mixed state, then $\sigma \otimes \rho$ is also a mixed state, and thus the purification cannot be expressed in the form of Equation (2.39). However, the assumption that the quantum state after encoding is pure is not strong in practice. This is because most of the codes known to date encode to a pure state. This thesis will also be discussed under this assumption from now on.

An error consisting of a combination of quantum deletion errors and quantum insertion errors is called a quantum insertion/deletion error. Since this thesis does not deal with quantum codes that can simultaneously correct general insertion and deletion errors, the definitions of quantum insertion/deletion errors are kept to a brief description. However, since insertion errors of a special case and deletion errors occur simultaneously will be discussed in Section 6, we will redefine quantum insertion/deletion errors in that section.

2.3.2 Quantum insertion/deletion error-correcting codes

This thesis deals with quantum deletion codes and quantum insertion codes, which are defined as follows.

Definition 2.7 (Quantum deletion error-correcting code)**.** *Let us define*

$$\mathcal{D}_t \coloneqq \{D_P : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes (N-t)}) \mid P \subset [N], |P| = t\}. \tag{2.40}$$

Then, an $((N, K))$ quantum error-correcting code for $\mathcal{D}_t$ is called an $((N, K))$ t-deletion error-correcting code or simply a t-deletion code.

It is clear that a t-deletion code is an s-deletion code for any non-negative integer $0 \leq s \leq t$. In practice, we can choose $N-t$ particles from $N-s$ particles, and then apply the decoder for the t-deletion code. Namely, the t-deletion code can be interpreted as a code that can correct for the loss of t particles or less.

Definition 2.8 (Quantum insertion error-correcting code)**.** *Let us define*

$$\mathcal{I}_t := \{I_P : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes (N+t)}) \mid P \subset [N+t], |P| = t\}. \tag{2.41}$$

Then, an $((N,K))$ *quantum error-correcting code for* $\mathcal{I}_t$ *is called an* $((N,K))$ t*-insertion error-correcting code or simply a* t*-insetion code.*

For any non-negative integer $0 \leq s \leq t$, it is also clear that a t-insertion code is an s-insertion code, as in the case of a deletion code. In other words, the t-insertion code may be interpreted as a code that can correct any insertion error of t or less.

Chapter 3

Quantum error-correcting codes for single deletion errors

The main purpose of this chapter is to discuss the Nakayama-Hagiwara conditions, which are known as the construction conditions for single quantum deletion codes [49]. In particular, in Sections 3.2 and 3.3, we describe the contents presented at the 2020 International Symposium on Information Theory and Its Applications (ISITA2020) and published in the journal "Quantum Information Processing", respectively.

3.1 Code construction using the Nakayama-Hagiwara conditions

According to Nakayama and Hagiwara [49], the problem of single quantum deletion error can be attributed to the problem of classical combinatorics. This section briefly describes their results, which are prior work. In addition, two examples are introduced and their error-correction conditions are explained.

3.1.1 Nakayama-Hagiwara conditions

This chapter deals only with encoders that are expressed as in Definition 3.1. That is, logical 0 and logical 1 should be the uniform superpositions over sets A and B respectively.

Definition 3.1 (Encoder $\mathrm{Enc}_{A,B}$ and Code $Q_{A,B}$)**.** *Let $A, B \subset \{0,1\}^N$ be non-empty sets with $A \cap B = \emptyset$. Define an encoder as a linear map $\mathrm{Enc}_{A,B} : \mathbb{C}^2 \to (\mathbb{C}^2)^{\otimes N}$. For a quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$, $\mathrm{Enc}_{A,B}$ maps the state $|\psi\rangle$ to the state $|\Psi\rangle := \alpha|0_L\rangle + \beta|1_L\rangle \in (\mathbb{C}^2)^{\otimes N}$, where*

$$|0_L\rangle := \frac{1}{\sqrt{|A|}} \sum_{\boldsymbol{a} \in A} |\boldsymbol{a}\rangle, \qquad |1_L\rangle := \frac{1}{\sqrt{|B|}} \sum_{\boldsymbol{b} \in B} |\boldsymbol{b}\rangle. \tag{3.1}$$

Set $Q_{A,B}$ as the image of $\mathrm{Enc}_{A,B}$, i.e.,

$$Q_{A,B} := \{\mathrm{Enc}_{A,B}(|\psi\rangle) \mid |\psi\rangle \in \mathbb{C}^2, |\psi\rangle\langle\psi| \in S(\mathbb{C}^2)\}. \tag{3.2}$$

It can be checked by straightforward calculations that $\||0_L\rangle\| = \||1_L\rangle\| = 1$ and $\langle 0_L|1_L\rangle = 0$ hold for logical codewords defined by Equation (3.1).

The following Definitions 3.2 and 3.3 are given by Nakayama and Hagiwara [49].

Definition 3.2 (Deletion set $\Delta^-_{i,b}$)**.** *Let $i \in [N]$ be an integer and let $b \in \{0,1\}$ be a bit. For a non-empty set $A \subset \{0,1\}^N$, define a set $\Delta^-_{i,b}(A) \subset \{0,1\}^{N-1}$ as*

$$\Delta^-_{i,b}(A) \coloneqq \{a_1 \dots a_{i-1}a_{i+1} \dots a_N \in \{0,1\}^{N-1} \mid a_1 \dots a_{i-1}ba_{i+1} \dots a_N \in A\}. \tag{3.3}$$

In other words, $\Delta^-_{i,b}(A)$ is the set of bit sequences that deleting the ith component "b" from each sequence $\boldsymbol{a} \in \{a_1a_2 \dots a_N \in A \mid a_i = b\}$ gives. We call the set $\Delta^-_{i,b}(A)$ an (i,b) deletion set of A.

Definition 3.3 (Nakayama-Hagiwara conditions)**.** *For two non-empty sets $A, B \subset \{0,1\}^N$, define three conditions (C1)$^-$, (C2)$^-$, and (C3)$^-$ as follows:*

(C1)$^-$ (Ratio condition) For any non-empty set $I \subset [N]$ and any bit $b \in \{0,1\}$,

$$|A||B^-_{I,b}| = |B||A^-_{I,b}|, \tag{3.4}$$

where

$$A^-_{I,b} \coloneqq \bigcap_{i \in I} \Delta^-_{i,b}(A) \cap \bigcap_{i \in I^c} \Delta^-_{i,b}(A)^c, \quad B^-_{I,b} \coloneqq \bigcap_{i \in I} \Delta^-_{i,b}(B) \cap \bigcap_{i \in I^c} \Delta^-_{i,b}(B)^c \tag{3.5}$$

and X^c denotes the complement of a set X, in particular,

$$\Delta^-_{i,b}(A)^c = \{0,1\}^{N-1} \setminus \Delta^-_{i,b}(A), \quad I^c = [N] \setminus I. \tag{3.6}$$

(C2)$^-$ (Outer distance condition) For any integers $i_1, i_2 \in [N]$ and any bits $b_1, b_2 \in \{0,1\}$,

$$|\Delta^-_{i_1,b_1}(A) \cap \Delta^-_{i_2,b_2}(B)| = 0. \tag{3.7}$$

(C3)$^-$ (Inner distance condition) For any integers $i_1, i_2 \in [N]$,

$$|\Delta^-_{i_1,0}(A) \cap \Delta^-_{i_2,1}(A)| = 0, \quad |\Delta^-_{i_1,0}(B) \cap \Delta^-_{i_2,1}(B)| = 0. \tag{3.8}$$

The conditions (C1)$^-$, (C2)$^-$, and (C3)$^-$ are collectively called the Nakayama-Hagiwara conditions or simply the NH conditions.

The condition (C2)$^-$ was also considered independently by Ouyang and Rengaswamy under the name "1-disjointness condition" [57]. The sets A and B in Definition 3.3 satisfy the NH conditions even if they are exchanged, thus we consider the two pairs (A, B) and (B, A) to be identical, and in particular, assume that $|A| \leq |B|$ in this thesis. Since the NH conditions are very complicated, we will show two examples later in Examples 3.5 and 3.6. There, we will deepen our understanding by observing the examples and making sure that the three conditions are satisfied.

Fact 3.4 (Nakayama and Hagiwara [49])**.** *Let $A, B \subset \{0,1\}^N$ be non-empty sets satisfying the Nakayama-Hagiwara conditions. Then, the code $Q_{A,B}$ is an $(\!(N,1)\!)$ single quantum deletion error-correcting code.*

Fact 3.4 means that the NH conditions are sufficient conditions for constructing a single quantum deletion code. However, they are not shown to be necessary conditions. Therefore, the code construction in this study also does not characterize all possible single deletion codes, but rather a special family. For more information about the definition of the decoding method, please refer to Reference [49]. What is important here is that problems that correct the deletion errors for quantum states are reduced to problems that find the sets satisfying the three conditions just mentioned, namely the NH conditions.

Table 3.1: Deletion sets of A, B that give the Nakayama code

$\Delta^-_{i,b}$	$A = \{00001001, 01101111\}$		$B = \{00001111, 01101001\}$	
	$b = 0$	$b = 1$	$b = 0$	$b = 1$
$i = 1$	$\{0001001, 1101111\}$	$\emptyset$	$\{0001111, 1101001\}$	$\emptyset$
$i = 2$	$\{0001001\}$	$\{0101111\}$	$\{0001111\}$	$\{0101001\}$
$i = 3$	$\{0001001\}$	$\{0101111\}$	$\{0001111\}$	$\{0101001\}$
$i = 4$	$\{0001001, 0111111\}$	$\emptyset$	$\{0001111, 0111001\}$	$\emptyset$
$i = 5$	$\emptyset$	$\{0000001, 0110111\}$	$\emptyset$	$\{0000111, 0110001\}$
$i = 6$	$\{0000101\}$	$\{0110111\}$	$\{0110101\}$	$\{0000111\}$
$i = 7$	$\{0000101\}$	$\{0110111\}$	$\{0110101\}$	$\{0000111\}$
$i = 8$	$\emptyset$	$\{0000100, 0110111\}$	$\emptyset$	$\{0000111, 0110100\}$

Table 3.2: Deletion sets of A, B that give the Hagiwara code

$\Delta^-_{i,b}$	$A = \{0000, 1111\}$		$B = \{0011, 0101, 0110, 1001, 1010, 1100\}$	
	$b = 0$	$b = 1$	$b = 0$	$b = 1$
$i = 1$	$\{000\}$	$\{111\}$	$\{011, 101, 110\}$	$\{001, 010, 100\}$
$i = 2$	$\{000\}$	$\{111\}$	$\{011, 101, 110\}$	$\{001, 010, 100\}$
$i = 3$	$\{000\}$	$\{111\}$	$\{011, 101, 110\}$	$\{001, 010, 100\}$
$i = 4$	$\{000\}$	$\{111\}$	$\{011, 101, 110\}$	$\{001, 010, 100\}$

3.1.2 Two known examples

Since Definition 3.3 is very complicated, Nakayama and Hagiwara [49] provide only two examples of sets that satisfy the NH conditions. These two examples are described below.

Example 3.5. *If $N = 8$ and the sets $A, B \subset \{0, 1\}^8$ are defined as*

$$A = \{00001001, 01101111\}, \qquad B = \{00001111, 01101001\}, \tag{3.9}$$

then A and B satisfy the NH conditions.

Table 3.1 expresses the (i, b) deletion sets of A and B defined by Equation (3.9) for each integer $i \in [8]$ and each bit $b \in \{0, 1\}$; by observing it, we can check that the two sets A and B satisfy the three conditions. The 8-qubit code obtained from Example 3.5 is the first example of quantum deletion codes, given in 2020. [48]. We call this code the Nakayama code.

Example 3.6. *If $N = 4$ and the sets $A, B \subset \{0, 1\}^4$ are defined as*

$$A = \{0000, 1111\}, \qquad B = \{0011, 0101, 0110, 1001, 1010, 1100\}, \tag{3.10}$$

then A and B satisfy the NH conditions.

Similar to the previous example, by observing Table 3.2, we can check that the two sets A and B defined by Equation (3.10) satisfy the three conditions. According to the table, it can be seen that (i,b) deletion sets $\Delta^-_{i,b}(A)$ and $\Delta^-_{i,b}(B)$ are constant regardless of each integer $i \in [4]$. The 4-qubit code obtained from Example 3.6 is a single quantum deletion error-correcting code with the optimal length [35]. We call this code the Hagiwara code. From Equations (3.1) and (3.10), the logical codewords are

$$|0_L\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) = \frac{1}{\sqrt{2}}|D^4_0\rangle + \frac{1}{\sqrt{2}}|D^4_4\rangle, \tag{3.11}$$

$$|1_L\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle) = |D^4_2\rangle. \tag{3.12}$$

By comparing these with Equation (2.33), we can check that the Hagiwara code is the $0-(2,2,1)$ gnu code.

At the time when the construction by Nakayama-Hagiwara was proposed [49], only the two examples above were shown as examples of sets satisfying these conditions, and since these two examples gave already known codes, no new codes had been proposed. However, many examples satisfying the NH conditions were given by Shibayama at about the same time [66]. An attempt was also made to consider the NH conditions from the viewpoint of matrices, which led to further research [67]. From the following sections, we will describe these results in detail.

Note that observing Tables 3.1 and 3.2 above is also helpful in understanding the proof of this chapter.

3.2 Construction of sets that satisfy the NH conditions

This section gives examples of constructing sets of bit sequences that satisfy the Nakayama-Hagiwara conditions. The contents of Section 3.2 were presented at the 2020 International Symposium on Information Theory and Its Applications (ISITA2020) [66].

3.2.1 Construction 1 (Sets with the optimal cardinalities)

The following Lemma 3.7 gives a necessary condition for $(C1)^-$ and $(C2)^-$. It provides lower bounds on the cardinalities of the two sets A and B.

Lemma 3.7. *Suppose that non-empty sets $A, B \subset \{0,1\}^N$ satisfy the conditions $(C1)^-$ and $(C2)^-$. Then, $|A| \geq 2$ and $|B| \geq 2$.*

Proof. Suppose $|A| = 1$, then $(|\Delta^-_{i,0}(A)|, |\Delta^-_{i,1}(A)|) = (0,1)$ or $(1,0)$ holds for any integer $i \in [N]$. For a bit $b \in \{0,1\}$ such that $|\Delta^-_{i,b}(A)| = 0$, we have $|B^-_{\{i\},b}| = 0$ by the condition $(C1)^-$. If $|\Delta^-_{i,b}(B)| \neq 0$, we can take a sequence $\boldsymbol{b'} \in \Delta^-_{i,b}(B)$. Then, there is a subset $I \subset [N]$ where $i \in I$ and $\boldsymbol{b'} \in B^-_{I,b}$. Hence, we have $|A||B^-_{I,b}| \neq 0$ and $|B||A^-_{I,b}| = 0$; this contradicts the condition $(C1)^-$. Thus, $|\Delta^-_{i,b}(A)| = 0$ implies $|\Delta^-_{i,b}(B)| = 0$ for any integer $i \in [N]$ and any bit $b \in \{0,1\}$. Therefore, for any sequence $\boldsymbol{b} \in B$, the ith component of $\boldsymbol{b}$ is equal to the ith component of unique sequence $\boldsymbol{a} \in A$ for any integer $i \in [N]$. Thus, $B = A$ holds; this contradicts the condition $(C2)^-$. Therefore, we obtain $|A| \geq 2$. Similarly, $|B| \geq 2$ holds. $\square$

The following Theorem 3.8 gives a series of sets A and B with $|A| = |B| = 2$, which is the smallest cardinality by Lemma 3.7. The previously known Example 3.5 is also derived from

Equation (3.14), in which

$$\boldsymbol{x}_1 = 000, \quad \boldsymbol{x}_2 = 011, \quad \boldsymbol{y}_1 = 001, \quad \boldsymbol{y}_2 = 111. \tag{3.13}$$

In other words, Theorem 3.8 is a generalization of the Nakayama code [48].

Theorem 3.8. *Suppose that* $|\mathrm{wt}(\boldsymbol{x}_1) - \mathrm{wt}(\boldsymbol{x}_2)| \geq 2$ *and* $|\mathrm{wt}(\boldsymbol{y}_1) - \mathrm{wt}(\boldsymbol{y}_2)| \geq 2$ *for bit sequences* $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \{0,1\}^{m_1}$ *and bit sequences* $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \{0,1\}^{m_2}$ *with integers* $m_1 \geq 2$ *and* $m_2 \geq 2$*. Then, the sets*

$$A \coloneqq \{\boldsymbol{x}_1 01 \boldsymbol{y}_1, \boldsymbol{x}_2 01 \boldsymbol{y}_2\}, \qquad B \coloneqq \{\boldsymbol{x}_1 01 \boldsymbol{y}_2, \boldsymbol{x}_2 01 \boldsymbol{y}_1\} \tag{3.14}$$

and the sets

$$A \coloneqq \{\boldsymbol{x}_1 10 \boldsymbol{y}_1, \boldsymbol{x}_2 10 \boldsymbol{y}_2\}, \qquad B \coloneqq \{\boldsymbol{x}_1 10 \boldsymbol{y}_2, \boldsymbol{x}_2 10 \boldsymbol{y}_1\} \tag{3.15}$$

satisfy the NH conditions.

Proof. By symmetry, it is enough to show only the case of Equation (3.14). Set $N_1, N_2 \subset [m_1 + m_2 + 2]$ as $N_1 \coloneqq [m_1 + 1]$, $N_2 \coloneqq [m_1 + m_2 + 2] \setminus [m_1 + 1]$ respectively.

By $|A| = |B| = 2$, in order to show (C1)$^-$, it is enough to prove $|A^-_{I,b}| = |B^-_{I,b}|$ for any non-empty set $I \subset [m_1 + m_2 + 2]$ and any bit $b \in \{0,1\}$. Here, fix a bit $b \in \{0,1\}$. By the assumptions $|\mathrm{wt}(\boldsymbol{x}_1) - \mathrm{wt}(\boldsymbol{x}_2)| \geq 2$ and $|\mathrm{wt}(\boldsymbol{y}_1) - \mathrm{wt}(\boldsymbol{y}_2)| \geq 2$, we have

$$\left| \bigcap_{i \in I} \Delta^-_{i,b}(A) \right| = \left| \bigcap_{i \in I} \Delta^-_{i,b}(B) \right| = \begin{cases} \left| \bigcap_{i \in I} \Delta^-_{i,b}(\{\boldsymbol{x}_1 0, \boldsymbol{x}_2 0\}) \right| & \text{if } I \subset N_1, \\ \left| \bigcap_{i \in I} \Delta^-_{i-m-1,b}(\{1\boldsymbol{y}_1, 1\boldsymbol{y}_2\}) \right| & \text{if } I \subset N_2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

Note that in the case $i_1 \in N_1$ and $i_2 \in N_2$, for every element in $\Delta^-_{i_1,b}(A)$, its (m_1+1)th component is equal to 1, on the other hand, for every element in $\Delta^-_{i_2,b}(A)$, its (m_1+1)th component is equal to 0.

Thus, if $I \cap N_1 \neq \emptyset$ and $I \cap N_2 \neq \emptyset$, $|A^-_{I,b}| = |B^-_{I,b}| = 0$ holds. It is enough to consider the case $I \subset N_1$ and the case $I \subset N_2$. By symmetry, we can assume that $I \subset N_1$. Then,

$$\begin{aligned} |A^-_{I,b}| &= \left| \bigcap_{i \in I} \Delta^-_{i,b}(A) \cap \bigcap_{i \in I^c} \Delta^-_{i,b}(A)^c \right| &(3.17) \\ &= \left| \bigcap_{i \in I} \Delta^-_{i,b}(A) \cap \bigcap_{i \in N_1 \setminus I} \Delta^-_{i,b}(A)^c \cap \bigcap_{i \in N_2} \Delta^-_{i,b}(A)^c \right| &(3.18) \\ &= \left| \bigcap_{i \in I} \Delta^-_{i,b}(A) \cap \bigcap_{i \in N_1 \setminus I} \Delta^-_{i,b}(A)^c \right| &(3.19) \\ &= \left| \bigcap_{i \in I} \Delta^-_{i,b}(\{\boldsymbol{x}_1 0, \boldsymbol{x}_2 0\}) \cap \bigcap_{i \in N_1 \setminus I} \Delta^-_{i,b}(\{\boldsymbol{x}_1 0, \boldsymbol{x}_2 0\})^c \right| &(3.20) \end{aligned}$$

holds. What is important here is that $|A^-_{I,b}|$ depends only on $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and N_1. The same calculation can be done for $|B^-_{I,b}|$. Thus, we obtain $|A^-_{I,b}| = |B^-_{I,b}|$ for any non-empty set $I \subset [m_1 + m_2 + 2]$ and any bit $b \in \{0,1\}$. Therefore, (C1)$^-$ holds.

We consider the case $i_1, i_2 \in N_1$. Suppose that $|\Delta^-_{i_1,b_1}(A) \cap \Delta^-_{i_2,b_2}(B)| \neq 0$ for bits $b_1, b_2 \in \{0,1\}$. Then, $\Delta^-_{i_1,b_1}(\{\boldsymbol{x}_1 0\}) = \Delta^-_{i_2,b_2}(\{\boldsymbol{x}_2 0\}) \neq \emptyset$ or $\Delta^-_{i_1,b_1}(\{\boldsymbol{x}_2 0\}) = \Delta^-_{i_2,b_2}(\{\boldsymbol{x}_1 0\}) \neq \emptyset$ holds. This contradicts the assumption $|\mathrm{wt}(\boldsymbol{x}_1 0) - \mathrm{wt}(\boldsymbol{x}_2 0)| \geq 2$. Therefore, we obtain $|\Delta^-_{i_1,b_1}(A) \cap \Delta^-_{i_2,b_2}(B)| = 0$. The case $i_1, i_2 \in N_2$ is similar. In the case $i_1 \in N_1$ and $i_2 \in N_2$, we have $|\Delta^-_{i_1,b_1}(A) \cap \Delta^-_{i_2,b_2}(B)| = 0$ by comparing (m_1+1)th components. The case $i_1 \in N_2$ and $i_2 \in N_1$ is similar. Therefore, (C2)$^-$ holds.

In the case $i_1, i_2 \in N_1$, we have $|\Delta^-_{i_1,0}(A) \cap \Delta^-_{i_2,1}(A)| = |\Delta^-_{i_1,0}(\{\boldsymbol{x}_1 0\}) \cap \Delta^-_{i_2,1}(\{\boldsymbol{x}_1 0\})| + |\Delta^-_{i_1,0}(\{\boldsymbol{x}_2 0\}) \cap \Delta^-_{i_2,1}(\{\boldsymbol{x}_2 0\})| = 0$ by the assumptions. The case $i_1, i_2 \in N_2$ is similar. In the case $i_1 \in N_1$ and $i_2 \in N_2$, we have $|\Delta^-_{i_1,0}(A) \cap \Delta^-_{i_2,1}(A)| = 0$ by comparing (m_1+1)th components. The case $i_1 \in N_2$ and $i_2 \in N_1$ is similar. Furthermore, for any integers $i_1, i_2 \in [m_1 + m_2 + 2]$, $|\Delta^-_{i_1,0}(B) \cap \Delta^-_{i_2,1}(B)| = 0$ is given in the same way. Therefore, (C3)$^-$ holds. □

3.2.2 Construction 2 (Permutation-invariant sets)

The following Theorem 3.9 provides a family of sets A and B that give permutation-invariant quantum codes for deletion errors. They have the property that $\Delta^-_{i,b}(A)$ and $\Delta^-_{i,b}(B)$ are constant, respectively, regardless of each integer $i \in [N]$. See Table 3.2 for an example. The previously known Example 3.6 is also derived from Equation (3.21), in which $N_A = \{0, 4\}$ and $N_B = \{2\}$. In other words, Theorem 3.9 is a generalization of the Hagiwara code [35].

Theorem 3.9. *Suppose that two sets $N_A, N_B \subset \{0, 1, \ldots, N\}$ with $N_A \cap N_B = \emptyset$ satisfy the following three conditions:*

1. $w \in N_A \implies N - w \in N_A$.
2. $w \in N_B \implies N - w \in N_B$.
3. $|w_1 - w_2| > 1$, *for any integers $w_1, w_2 \in N_A \cup N_B$ with $w_1 \neq w_2$.*

Then, the sets

$$A \coloneqq \bigcup_{w \in N_A} W^N(w), \qquad B \coloneqq \bigcup_{w \in N_B} W^N(w) \tag{3.21}$$

satisfy the NH conditions. Here, for an integer $w \in \{0, 1, \ldots, N\}$,

$$W^N(w) \coloneqq \{\boldsymbol{a} \in \{0,1\}^N \mid \mathrm{wt}(\boldsymbol{a}) = w\}. \tag{3.22}$$

Proof. For any integer $i \in [N]$, it is clear that

$$\Delta^-_{i,0}(W^N(w)) = \begin{cases} W^{N-1}(w) & \text{if } w \in \{0, 1, \ldots, N-1\}, \\ \emptyset & \text{if } w = N, \end{cases} \tag{3.23}$$

$$\Delta^-_{i,1}(W^N(w)) = \begin{cases} \emptyset & \text{if } w = 0, \\ W^{N-1}(w-1) & \text{if } w \in \{1, 2, \ldots, N\}. \end{cases} \tag{3.24}$$

Thus, for any integers $i_1, i_2 \in [N]$ and any bit $b \in \{0,1\}$, $\Delta^-_{i_1,b}(A) = \Delta^-_{i_2,b}(A)$ holds.

In the case $I \neq [N]$, we have

$$A^-_{I,b} = \bigcap_{i \in I} \Delta^-_{i,b}(A) \cap \bigcap_{i \in I^c} \Delta^-_{i,b}(A)^c = \Delta^-_{1,b}(A) \cap \Delta^-_{1,b}(A)^c = \emptyset \tag{3.25}$$

for any non-empty subset $I \subset [N]$ and any bit $b \in \{0,1\}$. Similarly, $B^-_{I,b} = \emptyset$ holds. Thus, (C1)$^-$ holds. In the case $I = [N]$, we have $A^-_{I,b} = \Delta^-_{i,b}(A)$ and $B^-_{I,b} = \Delta^-_{i,b}(B)$ for any integer $i \in [N]$ and any bit $b \in \{0,1\}$. In order to show (C1)$^-$, it is enough to prove the equation

$$|A||\Delta^-_{i,b}(B)| = |B||\Delta^-_{i,b}(A)| \tag{3.26}$$

for any integer $i \in [N]$ and any bit $b \in \{0,1\}$. By the assumption $w \in N_A \Longrightarrow N - w \in N_A$, we obtain

$$|\Delta^-_{i,0}(A)| = \sum_{w \in N_A \setminus \{N\}} |W^{N-1}(w)| = \sum_{w \in N_A \setminus \{N\}} \binom{N-1}{w}, \tag{3.27}$$

$$|\Delta^-_{i,1}(A)| = \sum_{w \in N_A \setminus \{0\}} |W^{N-1}(w-1)| = \sum_{w \in N_A \setminus \{0\}} \binom{N-1}{w-1} = \sum_{w' \in N_A \setminus \{N\}} \binom{N-1}{w'}, \tag{3.28}$$

where $w' \coloneqq N - w$. Hence, $|\Delta^-_{i,0}(A)| = |\Delta^-_{i,1}(A)|$ holds. On the other hand, we have

$$\begin{aligned}
|\Delta^-_{i,0}(A)| + |\Delta^-_{i,1}(A)| &= \sum_{w \in N_A \setminus \{N\}} \binom{N-1}{w} + \sum_{w \in N_A \setminus \{0\}} \binom{N-1}{w-1} && (3.29)\\
&= \begin{cases} \displaystyle\sum_{w \in N_A \setminus \{0,N\}} \left\{ \binom{N-1}{w} + \binom{N-1}{w-1} \right\} + 2 & \text{if } 0 \in N_A \\ \displaystyle\sum_{w \in N_A \setminus \{0,N\}} \left\{ \binom{N-1}{w} + \binom{N-1}{w-1} \right\} & \text{otherwise} \end{cases} && (3.30)\\
&= \begin{cases} \displaystyle\sum_{w \in N_A \setminus \{0,N\}} \binom{N}{w} + 2 & \text{if } 0 \in N_A \\ \displaystyle\sum_{w \in N_A \setminus \{0,N\}} \binom{N}{w} & \text{otherwise} \end{cases} && (3.31)\\
&= \sum_{w \in N_A} \binom{N}{w} && (3.32)\\
&= |A|. && (3.33)
\end{aligned}$$

Thus $|\Delta^-_{i,b}(A)| = |A|/2$ holds for any integer $i \in [N]$ and any bit $b \in \{0,1\}$. Similarly, $|\Delta^-_{i,b}(B)| = |B|/2$ holds. Therefore, Equation (3.26) holds for any integer $i \in [N]$ and any bit $b \in \{0,1\}$.

By the assumption, $N'_A \cap N'_B = \emptyset$ holds, where

$$N'_A \coloneqq N_A \cup \{w - 1 \mid w \in N_A, w \geq 1\}, \tag{3.34}$$

$$N'_B \coloneqq N_B \cup \{w - 1 \mid w \in N_B, w \geq 1\}. \tag{3.35}$$

Then, we have

$$\bigcup_{\substack{i \in [N] \\ b \in \{0,1\}}} \Delta^-_{i,b}(A) \cap \bigcup_{\substack{i \in [N] \\ b \in \{0,1\}}} \Delta^-_{i,b}(B) = \bigcup_{w \in N'_A} W^{N-1}(w) \cap \bigcup_{w \in N'_B} W^{N-1}(w) = \emptyset. \tag{3.36}$$

Therefore, (C2)$^-$ holds.

Suppose that $|\Delta^-_{i_1,0}(A) \cap \Delta^-_{i_2,1}(A)| \neq 0$ holds for integers $i_1, i_2 \in [N]$, then we can take an element $\boldsymbol{a}' \in \Delta^-_{i_1,0}(A) \cap \Delta^-_{i_2,1}(A)$. Thus, the set A contains two sequences whose Hamming weights are $\mathrm{wt}(\boldsymbol{a}')$ and $\mathrm{wt}(\boldsymbol{a}')+1$; this contradicts the assumption. Hence, we obtain $|\Delta^-_{i_1,0}(A)\cap \Delta^-_{i_2,1}(A)| = 0$ for any integers $i_1, i_2 \in [N]$. Similarly, $|\Delta^-_{i_1,0}(B)\cap\Delta^-_{i_2,1}(B)| = 0$ holds. Therefore, $(\mathrm{C3})^-$ holds. □

3.2.3 Recursive construction

The following Theorem 3.10 means that if we find two sets $A, B \subset \{0,1\}^N$ which satisfy the NH conditions, then we can construct new sets $A', B' \in \{0,1\}^{N+1}$ that also satisfy the NH conditions. Constructing a new code from a certain code by Theorem 3.10 is called extending the code.

For a set $A \subset \{0,1\}^N$ and a bit $x \in \{0,1\}$, we denote $Ax \coloneqq \{\boldsymbol{a}x \in \{0,1\}^{N+1} \mid \boldsymbol{a} \in A\}$ and $xA \coloneqq \{x\boldsymbol{a} \in \{0,1\}^{N+1} \mid \boldsymbol{a} \in A\}$.

Theorem 3.10. *Suppose that non-empty sets $A, B \subset \{0,1\}^N$ satisfy the NH conditions. Then, for any bit $x \in \{0,1\}$, the sets $Ax, Bx \subset \{0,1\}^{N+1}$ and the sets $xA, xB \subset \{0,1\}^{N+1}$ also satisfy the NH conditions, respectively.*

Proof. By symmetry, it is enough to show only that the sets $Ax, Bx \subset \{0,1\}^{N+1}$ satisfy the three conditions. It is clear that

$$\Delta^-_{N+1,b}(Ax) = \begin{cases} A & \text{if } b = x, \\ \emptyset & \text{otherwise,} \end{cases} \tag{3.37}$$

$$\Delta^-_{N+1,b}(Bx) = \begin{cases} B & \text{if } b = x, \\ \emptyset & \text{otherwise,} \end{cases} \tag{3.38}$$

and $|Ax| = |A|$, $|Bx| = |B|$ for any bits $b, x \in \{0,1\}$. In order to show $(\mathrm{C1})^-$, it is enough to prove that

$$|A||\emptyset \cap (Bx)^-_{I,b}| = |B||\emptyset \cap (Ax)^-_{I,b}|, \tag{3.39}$$

$$|A||B \cap (Bx)^-_{I,x}| = |B||A \cap (Ax)^-_{I,x}| \tag{3.40}$$

for any set $I \in [N]$ and any bits $b, x \in \{0,1\}$ with $b \neq x$, and

$$|A||\emptyset^c \cap (Bx)^-_{I,b}| = |B||\emptyset^c \cap (Ax)^-_{I,b}|, \tag{3.41}$$

$$|A||B^c \cap (Bx)^-_{I,x}| = |B||A^c \cap (Ax)^-_{I,x}| \tag{3.42}$$

for any non-empty set $I \in [N]$ and any bits $b, x \in \{0,1\}$ with $b \neq x$. Equation (3.39) clearly holds. We obtain $|\emptyset^c \cap (Ax)^-_{I,b}| = |(Ax)^-_{I,b}| = |\{\boldsymbol{a}x \mid \boldsymbol{a} \in A^-_{I,b}\}| = |A^-_{I,b}|$, similarly, we have $|\emptyset^c \cap (Bx)^-_{I,b}| = |B^-_{I,b}|$. Hence, Equation (3.41) holds by the hypothesis of induction. Thus, we now show Equations (3.40) and (3.42).

In the case $N \in I$, we obtain $|A \cap (Ax)^-_{I,x}| = |(Ax)^-_{I,x}| = |A^-_{I,x}|$ and $|A^c \cap (Ax)^-_{I,x}| = 0$ by $\Delta^-_{n,x}(Ax) \subset A$ for any bit $x \in \{0,1\}$ and any non-empty set $I \subset [N]$. Similarly, we have $|B \cap (Bx)_{I,x}| = |B_{I,x}|$ and $|B^c \cap (Bx)_{I,x}| = 0$. Therefore, Equations (3.40) and (3.42) hold by the hypothesis of induction.

We consider the case $N \notin I$. In the case $I \neq \emptyset$, we can see the Nth component of every element in $A \cap \Delta^-_{n,x}(Ax)^c$ is not equal to x and the Nth component of every element in $\bigcap_{i\in I} \Delta^-_{i,x}(Ax)$

is equal to x. Hence, $|A \cap (Ax)^-_{I,x}| = 0$ holds. Similarly, we have $|B \cap (Bx)^-_{I,x}| = 0$. On the other hand, in the case $I = \emptyset$, we have

$$
\begin{aligned}
|A \cap (Ax)^-_{\emptyset,x}| &= |\{a_1a_2\dots a_N \in A \mid a_N \neq x\}| && (3.43)\\
&= |\Delta^-_{N,b}(A)| && (3.44)\\
&= \sum_{J\subset[N-1]} \left| \Delta^-_{N,b}(A) \cap \bigcap_{j\in J} \Delta^-_{j,b}(A) \cap \bigcap_{j\in[N-1]\setminus J} \Delta^-_{j,b}(A)^c \right| && (3.45)\\
&= \sum_{J\subset[N-1]} \left|A_{J\cup\{N\},b}\right|, && (3.46)
\end{aligned}
$$

where $b \neq x$. Note that in Equation (3.45), for subsets $J \subset [N-1]$, the sets $\bigcap_{j\in J} \Delta^-_{j,b}(A) \cap \bigcap_{j\in[N-1]\setminus J} \Delta^-_{j,b}(A)^c$ gives the partition of $\{0,1\}^N$. Similarly, $|B\cap(Bx)^-_{\emptyset,x}| = \sum_{J\subset[N-1]} |B^-_{J\cup\{N\},b}|$ holds. By the hypothesis of induction, we get

$$
|A||B \cap (Bx)^-_{\emptyset,x}| = \sum_{J\subset[N-1]} |A||B^-_{J\cup\{N\},b}| = \sum_{J\subset[N-1]} |B||A^-_{J\cup\{N\},b}| = |B||A \cap (Ax)^-_{\emptyset,x}|. \quad (3.47)
$$

Hence, Equation (3.40) holds. Since $|\Delta^-_{i,x}(Ax) \cap \Delta^-_{N,x}(Ax)^c \cap A^c| = |\Delta^-_{i,x}(Ax) \cap \Delta^-_{N,x}(Ax)^c|$ holds for any integer $i \in [N-1]$, we have

$$
\begin{aligned}
|A^c \cap (Ax)_{I,x}| &= \left| \bigcap_{i\in I} \Delta^-_{i,x}(Ax) \cap \bigcap_{i\in[N-1]\setminus I} \Delta^-_{i,x}(Ax)^c \cap \Delta^-_{N,x}(Ax)^c \cap A^c \right| && (3.48)\\
&= \left| \bigcap_{i\in I} \Delta^-_{i,x}(Ax) \cap \bigcap_{i\in[N-1]\setminus I} \Delta^-_{i,x}(Ax)^c \cap \Delta^-_{N,x}(Ax)^c \right| && (3.49)\\
&= |(Ax)^-_{I,x}| && (3.50)\\
&= |A^-_{I,x}| && (3.51)
\end{aligned}
$$

for any non-empty set $I \subset [N]$. Similarly, $|B^c \cap (Bx)_{I,x}| = |B_{I,x}|$ holds. Therefore, Equation (3.42) holds by the hypothesis of induction.

From the above, it is shown that (C1)$^-$ is satisfied.

By the assumption, it is clear that $|\Delta^-_{i_1,b_1}(Ax) \cap \Delta^-_{i_2,b_2}(Bx)| = 0$ for any $i_1, i_2 \in [N]$ and any $b_1, b_2 \in \{0,1\}$. By Equations (3.37) and (3.38), in order to show (C2)$^-$, it is enough to prove that the following equations hold for any $i \in [N]$:

$$
|(\Delta^-_{i,0}(Ax) \cup \Delta^-_{i,1}(Ax)) \cap B| = 0, \quad (3.52)
$$

$$
|(\Delta^-_{i,0}(Bx) \cup \Delta^-_{i,1}(Bx)) \cap A| = 0. \quad (3.53)
$$

If we take a sequence $\boldsymbol{b} \in (\Delta^-_{i,0}(Ax) \cup \Delta^-_{i,1}(Ax)) \cap B$, there is a sequence $\boldsymbol{a} \in A$ whose ith component is $y \in \{0,1\}$, such that $\boldsymbol{a}x$ maps to $\boldsymbol{b} \in B$ if we delete "y" at i. Then, there exists a sequence $\boldsymbol{b}' \in \Delta^-_{i,y}(A) \cap \Delta^-_{N,x}(B)$ such that $\boldsymbol{b} \in B$ maps to $\boldsymbol{b}'$ if we delete "x" at N. Hence, $\Delta^-_{i,y}(A) \cap \Delta_{N,x}(B) \neq \emptyset$; this contradicts the assumption. Therefore, Equation (3.52) holds. Equation (3.53) is also proved similarly. Therefore, (C2)$^-$ holds.

By Equations (3.37) and (3.38), in order to show satisfying (C3)$^-$, it is enough to prove that $A \cap \Delta^-_{i,b}(Ax) = \emptyset$ and $B \cap \Delta^-_{i,b}(Bx) = \emptyset$ for any $i \in [N]$ and any $b, x \in \{0,1\}$ with $b \neq x$. By the assumption and by comparison of the Nth component, we have

$$
A \cap \Delta^-_{i,b}(Ax) = (\Delta^-_{N,x}(Ax) \cap \Delta^-_{i,b}(Ax)) \cup \{(A \setminus \Delta^-_{N,x}(Ax)) \cap \Delta^-_{i,b}(Ax)\} = \emptyset, \quad (3.54)
$$

and similarly, $B \cap \Delta^-_{i,b}(Bx) = \emptyset$. Therefore, (C3)$^-$ holds. $\square$

3.3 Discussion of the NH conditions using adjacency matrices

This section gives a discussion of the Nakayama-Hagiwara conditions using adjacency matrices. The contents of Section 3.3 were originally published in the journal "Quantum Information Processing" in 2021 [67].

3.3.1 Characterization of distance conditions via adjacency matrices

Here, we describe the inner and outer distance conditions in terms of matrices in order to further consider the sets that satisfy the NH conditions. First, we describe the (i, b) deletion sets using matrices.

Definition 3.11 (Deletion Matrix). *Let $i \in [N]$ be an integer. Define two 2^{N-1}-by-2^N matrices $D_{i,\langle 0|}, D_{i,\langle 1|}$ as*

$$D_{i,\langle 0|} := \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{i-1 \ times} \otimes \langle 0| \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{N-i \ times}, \tag{3.55}$$

$$D_{i,\langle 1|} := \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{i-1 \ times} \otimes \langle 1| \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{N-i \ times} \tag{3.56}$$

respectively, where $\mathbb{I}_2$ is the identity matrix of order 2. For a matrix $D_{i,\langle b|}$ with a bit $b \in \{0, 1\}$, let the pth row correspond to the unique sequence $\boldsymbol{x} = x_1 x_2 \ldots x_{N-1} \in \{0, 1\}^{N-1}$ such that

$$\sum_{k=1}^{N-1} x_{N-k} 2^{k-1} = p - 1, \tag{3.57}$$

and let the qth column correspond to the unique sequence $\boldsymbol{y} = y_1 y_2 \ldots y_N \in \{0, 1\}^N$ such that

$$\sum_{k=1}^{N} y_{N-k+1} 2^{k-1} = q - 1. \tag{3.58}$$

In this case, we denote the $(\boldsymbol{x}, \boldsymbol{y})$ entry (i.e., the (p, q) entry) of the matrix $D_{i,\langle b|}$ by $D_{i,\langle b|}(\boldsymbol{x}, \boldsymbol{y})$. We call a matrix $D_{i,\langle b|}$ an (i, b) deletion matrix.

The deletion matrix $D_{i,\langle b|}$ defined above can be regarded as a special case of matrices $D^n_{P,\langle \Psi|}$, described later at the beginning of Section 6.2.1.

As you can see in Example 3.14, each row and column is naturally labeled using the binary system. We can check the elements of the (i, b) deletion set $\Delta^-_{i,b}(A)$ for a set $A \subset \{0, 1\}^N$ by observing the (i, b) deletion matrix $D_{i,\langle b|}$, since it is easy to see that for sequences $\boldsymbol{x}' \in \{0, 1\}^{N-1}$ and $\boldsymbol{x} \in \{0, 1\}^N$,

$$D_{i,\langle b|}(\boldsymbol{x}', \boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x}' \in \Delta^-_{i,b}(\{\boldsymbol{x}\}), \\ 0 & \text{otherwise.} \end{cases} \tag{3.59}$$

Lemma 3.12. *Sets $A, B \subset \{0,1\}^N$ satisfy the condition (C2)$^-$, if and only if the $(\boldsymbol{a}, \boldsymbol{b})$ entry of the matrix A_N^{C2} is equal to 0 for any sequences $\boldsymbol{a} \in A$ and $\boldsymbol{b} \in B$, where*

$$A_N^{C2} \coloneqq \sum_{i=1}^{N} \left(D_{i,\langle 0|} + D_{i,\langle 1|}\right)^\top \sum_{j=1}^{N} \left(D_{j,\langle 0|} + D_{j,\langle 1|}\right). \tag{3.60}$$

We call the square matrix A_N^{C2} of order 2^N a (C2)-adjacency matrix.

Proof. Lemma 3.12 clearly holds. Note that for integers $i, j \in [N]$ and sequences $\boldsymbol{a}, \boldsymbol{b} \in \{0,1\}^N$, the $(\boldsymbol{a}, \boldsymbol{b})$ entry of the matrix $(D_{i,\langle 0|} + D_{i,\langle 1|})^\top (D_{j,\langle 0|} + D_{j,\langle 1|})$ is not equal to 0, if and only if the subsequence obtained by deleting the ith and jth elements of sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively are the same. □

Lemma 3.13. *A set $A \subset \{0,1\}^N$ satisfies the condition (C3)$^-$, if and only if the $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ entry of the matrix A_N^{C3} is equal to 0 for any sequences $\boldsymbol{a}_1, \boldsymbol{a}_2 \in A$, where*

$$A_N^{C3} \coloneqq \sum_{i=1}^{N} D_{i,\langle 0|}{}^\top \sum_{j=1}^{N} D_{j,\langle 1|} + \sum_{i=1}^{N} D_{i,\langle 1|}{}^\top \sum_{j=1}^{N} D_{j,\langle 0|}. \tag{3.61}$$

We call the square matrix A_N^{C3} of order 2^N a (C3)-adjacency matrix.

Proof. Lemma 3.13 also clearly holds. Note that for integers $i, j \in [N]$ and sequences $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \{0,1\}^N$, the $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ entry of the matrix $D_{i,\langle 0|}{}^\top D_{j,\langle 1|}$ is not equal to 0, if and only if the subsequence obtained by deleting the ith element "0" and jth element "1" of sequences $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ respectively are the same. □

Example 3.14. *Let $N = 3$. Simple calculations show that A_3^{C2} and A_3^{C3} are as shown in Figure 3.1. The sequence written on the right side of each row is the labeling for that row. Note that the columns are also labeled in the same way.*

It is easy to check that the $(\boldsymbol{x}, \boldsymbol{y})$ entry of the matrix A_3^{C2} is equal to the number of paths with length 2 from $\boldsymbol{x} \in \{0,1\}^3$ to $\boldsymbol{y} \in \{0,1\}^3$ of the graph in Figure 3.2. Furthermore, we can see that the $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ entry of the matrix A_3^{C3} is equal to the number of paths from $\boldsymbol{x}_1 \in \{0,1\}^3$ to $\boldsymbol{x}_2 \in \{0,1\}^3$ that can be formed using only one solid line and one dashed line. From this observation, it is easy to understand that Lemmas 3.12 and 3.13 hold.

Let us explain in detail the graph that Figure 3.2 represents. The graph has as vertex set $\{0,1\}^2 \cup \{0,1\}^3$, with two vertices $\boldsymbol{x}' \in \{0,1\}^2$ and $\boldsymbol{x} \in \{0,1\}^3$ adjacent if and only if $\boldsymbol{x}'$ is obtained by deleting the ith element of the sequence $\boldsymbol{x}$ for an integer $i \in [3]$. For each vertex $\boldsymbol{x} \in \{0,1\}^3$, there are three positions that can be deleted, thus there are a total of $3 \times 8 = 24$ edges. Also note that the edges formed by deleting "0" are shown as solid lines, and the edges formed by deleting "1" are shown as dashed lines.

3.3.2 All examples of sets with length 5 or less

Using the adjacency matrices defined above, we want to examine two sets of small lengths that satisfy the NH condition. First, we show that there are no examples of sets with a length less than 4.

$$A_3^{C2} = \left(\begin{array}{cccc|cccc} 9 & 3 & 3 & 0 & 3 & 0 & 0 & 0 \\ 3 & 5 & 3 & 4 & 1 & 2 & 0 & 0 \\ 3 & 3 & 3 & 2 & 3 & 2 & 2 & 0 \\ 0 & 4 & 2 & 5 & 0 & 3 & 1 & 3 \\ \hline 3 & 1 & 3 & 0 & 5 & 2 & 4 & 0 \\ 0 & 2 & 2 & 3 & 2 & 3 & 3 & 3 \\ 0 & 0 & 2 & 1 & 4 & 3 & 5 & 3 \\ 0 & 0 & 0 & 3 & 0 & 3 & 3 & 9 \end{array}\right) \begin{array}{l} \mathit{000} \\ \mathit{001} \\ \mathit{010} \\ \mathit{011} \\ \mathit{100} \\ \mathit{101} \\ \mathit{110} \\ \mathit{111} \end{array}$$

$$A_3^{C3} = \left(\begin{array}{cccc|cccc} 0 & 3 & 3 & 0 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 4 & 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 2 & 0 & 2 & 2 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 & 0 & 3 \\ \hline 3 & 0 & 0 & 0 & 0 & 2 & 4 & 0 \\ 0 & 2 & 2 & 0 & 2 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 4 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 & 3 & 3 & 0 \end{array}\right) \begin{array}{l} \mathit{000} \\ \mathit{001} \\ \mathit{010} \\ \mathit{011} \\ \mathit{100} \\ \mathit{101} \\ \mathit{110} \\ \mathit{111} \end{array}$$

Figure 3.1: The (C2)-adjacency matrix A_3^{C2} and the (C3)-adjacency matrix A_3^{C3}

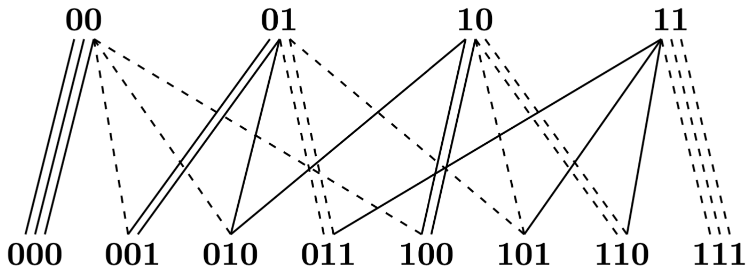

Figure 3.2: The graph representing how a single deletion occurs from $\{0,1\}^3$

Lemma 3.15. *For an integer $N \in \{2,3\}$, there is no pair of non-empty sets $A, B \subset \{0,1\}^N$ that satisfy the NH conditions.*

Proof. Since it is clear that Lemma 3.15 holds for $N = 2$, we consider the case $N = 3$. By Lemmas 3.7 and 3.12, observation of the matrix A_3^{C2} shows that there are only the following two possible pairs of sets (A, B) that satisfy the conditions $(\mathrm{C1})^-$ and $(\mathrm{C2})^-$:

$$(A, B) = (\{000, 001\}, \{110, 111\}), (\{000, 100\}, \{011, 111\}). \tag{3.62}$$

However, we can see by observing the matrix A_3^{C3} that none of these two pairs satisfy the condition $(\mathrm{C3})^-$. $\square$

Here, we have shown the non-existence of codes of length 3 or less in Nakayama and Hagiwara's construction method, but more generally, it is already known that there is no single quantum deletion error-correcting code of length 3 or less [35].

The necessary and sufficient conditions for (C2)$^-$ and (C3)$^-$ could be given in terms of adjacency matrices, however (C1)$^-$ is the most complicated of the three conditions and it is difficult to give this necessary and sufficient condition. Here, we give a simple necessary condition.

Lemma 3.16. *Suppose that non-empty sets $A, B \subset \{0,1\}^N$ satisfy the condition (C1)$^-$. Then,*

$$|A| \sum_{\boldsymbol{b} \in B} \boldsymbol{b} = |B| \sum_{\boldsymbol{a} \in A} \boldsymbol{a}. \tag{3.63}$$

Note that we use the same rules as in the N-dimensional complex vector space $\mathbb{C}^N$ to calculate the sum of the bit sequences.

For example, for the sets $A, B \subset \{0,1\}^8$ in Equation (3.9), we can check that Lemma 3.16 is indeed satisfied, since

$$|A| \sum_{\boldsymbol{b} \in B} \boldsymbol{b} = 2 \cdot (00001111 + 01101001) = 2 \cdot 01102112 = 02204224 \in \mathbb{C}^8, \tag{3.64}$$

$$|B| \sum_{\boldsymbol{a} \in A} \boldsymbol{a} = 2 \cdot (00001001 + 01101111) = 2 \cdot 01102112 = 02204224 \in \mathbb{C}^8. \tag{3.65}$$

Proof. It is enough to show that $|A||\Delta^-_{i,1}(B)| = |B||\Delta^-_{i,1}(A)|$ for any integer $i \in [N]$. We have

$$|\Delta^-_{i,b}(A)| = \sum_{J \subset [N] \setminus \{i\}} \left| \Delta^-_{i,b}(A) \cap \bigcap_{j \in J} \Delta^-_{j,b}(A) \cap \bigcap_{j \in [N] \setminus (J \cup \{i\})} \Delta^-_{j,b}(A)^c \right| \tag{3.66}$$

$$= \sum_{J \subset [N] \setminus \{i\}} \left| A^-_{J \cup \{i\}, b} \right| \tag{3.67}$$

for any integer $i \in [N]$ and any bit $b \in \{0,1\}$. Note that in Equation (3.66), there are 2^{N-1} ways to take a subset $J \subset [N] \setminus \{i\}$, and the sets $\bigcap_{j \in J} \Delta^-_{j,b}(A) \cap \bigcap_{j \in [N] \setminus (J \cup \{i\})} \Delta^-_{j,b}(A)^c$ give a partition of $\{0,1\}^N$. Similarly, $|\Delta^-_{i,b}(B)| = \sum_{J \subset [N] \setminus \{i\}} \left| B^-_{J \cup \{i\}, b} \right|$ holds. Therefore, we obtain

$$|B||\Delta^-_{i,b}(A)| = \sum_{J \subset [N] \setminus \{i\}} |B||A^-_{J \cup \{i\}, b}| = \sum_{J \subset [N] \setminus \{i\}} |A||B^-_{J \cup \{i\}, b}| = |A||\Delta^-_{i,b}(B)| \tag{3.68}$$

by the condition (C1)$^-$. □

Using Lemma 3.16, we can determine the set $A, B \subset \{0,1\}^N$ with $N = 4$ by examining the adjacency matrices as in the case of $N = 3$. Lemma 3.17 means that in Nakayama and Hagiwara's construction, there is no code with the optimal length other than the Hagiwara code.

Lemma 3.17. *Let $N = 4$. Suppose that sets $A, B \subset \{0,1\}^4$ satisfy the NH conditions. Then,*

$$A = \{0000, 1111\}, \qquad B = \{0011, 0101, 0110, 1001, 1010, 1100\}. \tag{3.69}$$

Proof. Simple calculations shows that A_4^{C2} and A_4^{C3} are as shown in Figure 3.3.

For an integer $i \in [N]$ and a bit $b \in \{0,1\}$, set $\Gamma_{i,b} := \{x_1x_2x_3x_4 \mid x_i = b\}$. By Lemma 3.7, we have $|A| \geq 2$ and $|B| \geq 2$. In the case $|A \cap \Gamma_{1,0}| \geq 2$ and $|B \cap \Gamma_{1,0}| \geq 2$, by observation of the matrix A_4^{C2}, we have $(A \cap \Gamma_{1,0}, B \cap \Gamma_{1,0}) = (\{0000, 0001\}, \{0110, 0111\}), (\{0000, 0100\}, \{0011, 0111\})$.

$$
A_4^{C2} = \left(\begin{array}{cccc|cccc|cccc|cccc}
16 & 4 & 4 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 10 & 4 & 6 & 1 & 3 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 6 & 2 & 5 & 3 & 4 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 2 & 8 & 0 & 4 & 2 & 6 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\
\hline
4 & 1 & 5 & 0 & 6 & 2 & 4 & 0 & 4 & 1 & 3 & 0 & 2 & 0 & 0 & 0 \\
0 & 3 & 3 & 4 & 2 & 4 & 3 & 3 & 0 & 3 & 2 & 3 & 0 & 2 & 0 & 0 \\
0 & 0 & 4 & 2 & 4 & 3 & 6 & 3 & 0 & 0 & 3 & 1 & 2 & 1 & 3 & 0 \\
0 & 0 & 0 & 6 & 0 & 3 & 3 & 10 & 0 & 0 & 0 & 4 & 0 & 1 & 1 & 4 \\
\hline
4 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 10 & 3 & 3 & 0 & 6 & 0 & 0 & 0 \\
0 & 3 & 1 & 2 & 1 & 3 & 0 & 0 & 3 & 6 & 3 & 4 & 2 & 4 & 0 & 0 \\
0 & 0 & 2 & 0 & 3 & 2 & 3 & 0 & 3 & 3 & 4 & 2 & 4 & 3 & 3 & 0 \\
0 & 0 & 0 & 2 & 0 & 3 & 1 & 4 & 0 & 4 & 2 & 6 & 0 & 5 & 1 & 4 \\
\hline
0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 6 & 2 & 4 & 0 & 8 & 2 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 4 & 3 & 5 & 2 & 6 & 4 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 3 & 1 & 6 & 4 & 10 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 4 & 4 & 16
\end{array}\right)
\begin{array}{l}
\mathit{0000} \\ \mathit{0001} \\ \mathit{0010} \\ \mathit{0011} \\ \mathit{0100} \\ \mathit{0101} \\ \mathit{0110} \\ \mathit{0111} \\ \mathit{1000} \\ \mathit{1001} \\ \mathit{1010} \\ \mathit{1011} \\ \mathit{1100} \\ \mathit{1101} \\ \mathit{1110} \\ \mathit{1111}
\end{array}
$$

$$
A_4^{C3} = \left(\begin{array}{cccc|cccc|cccc|cccc}
0 & 4 & 4 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 6 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 2 & 0 & 3 & 4 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 2 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
\hline
4 & 0 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 1 & 3 & 0 & 2 & 0 & 0 & 0 \\
0 & 3 & 3 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 \\
0 & 0 & 4 & 0 & 4 & 0 & 0 & 3 & 0 & 0 & 0 & 1 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & 6 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
\hline
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 6 & 0 & 0 & 0 \\
0 & 3 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 4 & 0 & 4 & 0 & 0 \\
0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 2 & 0 & 3 & 3 & 0 \\
0 & 0 & 0 & 2 & 0 & 3 & 1 & 0 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 4 \\
\hline
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 2 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 4 & 3 & 0 & 2 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 6 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 4 & 4 & 0
\end{array}\right)
\begin{array}{l}
\mathit{0000} \\ \mathit{0001} \\ \mathit{0010} \\ \mathit{0011} \\ \mathit{0100} \\ \mathit{0101} \\ \mathit{0110} \\ \mathit{0111} \\ \mathit{1000} \\ \mathit{1001} \\ \mathit{1010} \\ \mathit{1011} \\ \mathit{1100} \\ \mathit{1101} \\ \mathit{1110} \\ \mathit{1111}
\end{array}
$$

Figure 3.3: The (C2)-adjacency matrix A_4^{C2} and the (C3)-adjacency matrix A_4^{C3}

However, both of them do not satisfy the condition (C3)$^-$. In the case $|A\cap\Gamma_{1,0}| = 0$, by Equation (3.68), we obtain

$$|A||B\cap\Gamma_{1,0}| = |A||\Delta^-_{1,0}(B)| = |B||\Delta^-_{1,0}(A)| = |B||A\cap\Gamma_{1,0}|. \tag{3.70}$$

Hence, $|B\cap\Gamma_{1,0}| = 0$ holds. From Lemma 3.7, $|A\cap\Gamma_{1,1}| \geq 2$ and $|B\cap\Gamma_{1,1}| \geq 2$ holds, and therefore a contradiction follows from the same argument as above.

Now we just need to consider the case $|A\cap\Gamma_{1,0}| = 1$ and $|B\cap\Gamma_{1,0}| \geq 1$.

If $|B\cap\Gamma_{1,0}| = 1$, then $|A| = |B| = 2$ by Equation (3.70). By Lemma 3.12, there are 9 ways to take $A\cap\Gamma_{1,0}$ and $B\cap\Gamma_{1,0}$. Examining them, the sets A, B satisfying (C2)$^-$, (C3)$^-$, and Lemma 3.16 can be narrowed down to the 3 possibilities $(A, B) = (\{0000, 1111\}, \{0011, 1100\})$, $(\{0000, 1111\}, \{0101, 1010\})$, $(\{0000, 1111\}, \{0110, 1001\})$. However, all of them do not satisfy the condition (C1)$^-$.

If $|B\cap\Gamma_{1,0}| = 2$, then $|A| = 2$ and $|B| = 4$ by Equation (3.70). Then, using the same method as above, we can see that there are 6 ways to take $A\cap\Gamma_{1,0}$ and $B\cap\Gamma_{1,0}$, from which we can find the following 3 possibilities $(A, B) = (\{0000, 1111\}, \{0011, 0101, 1100, 1010\})$, $(\{0000, 1111\}, \{0011, 0110, 1100, 1001\})$, $(\{0000, 1111\}, \{0101, 0110, 1010, 1001\})$. However, all of them do not satisfy the condition (C1)$^-$.

If $|B\cap\Gamma_{1,0}| = 3$, then $|A| = 2$ and $|B| = 6$ by Equation (3.70). Then, using the same method as above, we find that there are 2 ways to take $A\cap\Gamma_{1,0}$ and $B\cap\Gamma_{1,0}$, from which we get $(A, B) = (\{0000, 1111\}, \{0011, 0101, 0110, 1001, 1010, 1100\})$. This satisfies the conditions (C1)$^-$, (C2)$^-$, and (C3)$^-$ from Theorem 3.9.

If $|B\cap\Gamma_{1,0}| \geq 4$, we cannot take $B\cap\Gamma_{1,0}$ to satisfy (C3)$^-$. □

Lemma 3.17 can be proved by counting directly as above, but it is a tough calculation. By using a computer to calculate it, we can find out more quickly. In fact, using a computer, by examining the adjacency matrices A_4^{C2} and A_4^{C3}, we can check that there are 187 different pairs of sets (A, B) that satisfy the conditions (C2)$^-$ and (C3)$^-$. Note that the interchange of two sets A and B are not distinguished and that the sets given by sequences in reverse order are considered identical. If we check whether (C1)$^-$ is satisfied for all of them, we find that only the Hagiwara code is applicable.

Similarly, the case $N = 5$ can be examined using a computer.

Lemma 3.18. *Let $N = 5$. Suppose that sets $A, B \subset \{0,1\}^5$ satisfy the NH conditions. Then the pair of sets (A, B) is one of the following:*

$$A = \{00000, 01111\}, \quad B = \{00011, 00101, 00110, 01001, 01010, 01100\}, \tag{3.71}$$
$$A = \{10000, 11111\}, \quad B = \{10011, 10101, 10110, 11001, 11010, 11100\}, \tag{3.72}$$
$$A = \{00000, 11110\}, \quad B = \{00110, 01010, 01100, 10010, 10100, 11000\}, \tag{3.73}$$
$$A = \{00001, 11111\}, \quad B = \{00111, 01011, 01101, 10011, 10101, 11001\}. \tag{3.74}$$

Proof. The proof is by computer. Examining the adjacency matrix A_4^{C3}, we can check that there are 4143 ways to take a set $A \subset \{0,1\}^5$ with $|A| \geq 2$ satisfying (C2)$^-$. Note here that a set given in reverse order is considered to be the same as the original set. For each case, examining the adjacency matrices A_4^{C2} and A_4^{C3}, we find that there are 162499 pairs of sets A and B such that (C2)$^-$ and (C3)$^-$ are satisfied, without distinguishing between the interchange of A and B. By examining all of these, we can check that only the four examples above are applicable. □

The four examples obtained in Lemma 3.18 are all extensions of the Hagiwara code by Theorem 3.10. The codes obtained from these four pairs of sets are called the extended Hagiwara codes. Lemmas 3.15, 3.17 and 3.18 can be summarized as Theorem 3.19 below. This theorem gives us all the examples of length 5 or less.

Theorem 3.19. *A single quantum deletion code with length* 5 *or less constructed by Fact 3.4 is the Hagiwara code or one of the four extended Hagiwara codes.*

It should be noted that in Theorem 3.19, we restrict ourselves to encoders expressed in the form of Definition 3.1, and furthermore, we are discussing the code construction of Nakayama and Hagiwara. For example, the $0-(2,2,5/4)$ gnu code [53] is one of the single deletion codes with length 5, but of course, it is not included in Lemma 3.18. The correctability for deletion errors of gnu codes defined in Section 2.2.3 is described in detail later in Chapter 4.

3.4 Examples

Several examples of sets discussed in Section 3.2 that have a length of 8 or less are represented in Table 3.3. Lemma 3.15 shows that there are no sets with length 3 or less. Using the method in Section 3.2, we can always construct sets that satisfy the NH conditions with length 4 or more. From Lemma 3.17, we see that the only example of sets with length 4 is the Hagiwara code, and from Lemma 3.18, we see that the only examples of sets with length 5 are the four extended Hagiwara codes. Many more examples of sets with lengths 6 or more can be constructed other than those listed in Table 3.3.

Note that the $\Delta-(g,n,u)$ code in the Remarks of Table 3.3 represents the parameters of the Δ-shifted gnu code defined in Section 2.2.3.

Example 3.20. *Assuming that*

$$A = W^6(1) \cup W^6(5), \qquad B = W^6(3), \tag{3.75}$$

the logical codewords of the code $Q_{A,B}$ are

$$|0_L\rangle = \frac{1}{\sqrt{12}} \sum_{\substack{\boldsymbol{x}\in\{0,1\}^6 \\ \mathrm{wt}(\boldsymbol{x})\in\{1,5\}}} |\boldsymbol{x}\rangle = \frac{1}{\sqrt{2}}|D^6_1\rangle + \frac{1}{\sqrt{2}}|D^6_5\rangle, \tag{3.76}$$

$$|1_L\rangle = \frac{1}{\sqrt{20}} \sum_{\substack{\boldsymbol{x}\in\{0,1\}^6 \\ \mathrm{wt}(\boldsymbol{x})\in\{3\}}} |\boldsymbol{x}\rangle = |D^6_3\rangle \tag{3.77}$$

from Equation (3.1). By comparing these with Equation (2.33), we can check that the code $Q_{A,B}$ is the $1-(2,2,5/4)$ *gnu code.*

The single quantum deletion codes constructed in this chapter include some of the Δ-shifted gnu codes such that $n=2$, but none with n greater than 2. From Fact 2.2, Δ-shifted gnu codes

Table 3.3: Several examples of sets A, B with short lengths

N	Theorem	A	B		Remarks
4	3.9	$W^4(0) \cup W^4(4)$	$W^4(2)$		Hagiwara code
					$0-(2,2,1)$ code
					Equation (3.10)
					Equation (3.69)
5	3.10	$0\{W^4(0) \cup W^4(4)\}$	$0\{W^4(2)\}$		Equation (3.71)
		$1\{W^4(0) \cup W^4(4)\}$	$1\{W^4(2)\}$		Equation (3.72)
		$\{W^4(0) \cup W^4(4)\}0$	$\{W^4(2)\}0$		Equation (3.73)
		$\{W^4(0) \cup W^4(4)\}1$	$\{W^4(2)\}1$		Equation (3.74)
6	3.8	$\{000100, 110111\}$	$\{000111, 110100\}$		Equation (3.78)
		$\{001000, 111011\}$	$\{001011, 111000\}$		Equation (3.79)
	3.9	$W^6(0) \cup W^6(6)$	$W^6(3)$		$0-(3,2,1)$ code
		$W^6(1) \cup W^6(5)$	$W^6(3)$		$1-(2,2,5/4)$ code
					Equation (3.75)
		$W^6(0) \cup W^6(6)$	$W^6(2) \cup W^6(4)$		
	3.10	$00\{W^4(0) \cup W^4(4)\}$	$00\{W^4(2)\}$		
		$1\{W^4(0) \cup W^4(4)\}0$	$1\{W^4(2)\}0$		
		$\{W^4(0) \cup W^4(4)\}01$	$\{W^4(2)\}01$	etc.	
7	3.8	$\{0001000, 1101111\}$	$\{0001111, 1101000\}$		
		$\{0000100, 1100111\}$	$\{0000111, 1100100\}$		
		$\{0010100, 1110111\}$	$\{0010111, 1110100\}$	etc.	
	3.9	$W^7(0) \cup W^7(7)$	$W^7(2) \cup W^7(5)$		
	3.10	$\{0001001, 1101111\}$	$\{0001111, 1101001\}$		
		$\{W^6(0) \cup W^6(6)\}1$	$\{W^6(2) \cup W^6(4)\}1$		
		$1\{W^4(0) \cup W^4(4)\}01$	$1\{W^4(2)\}01$	etc.	
8	3.8	$\{00001001, 01101111\}$	$\{00001111, 01101001\}$		Nakayama code
					Equation (3.9)
					Equation (3.80)
		$\{01001001, 11101111\}$	$\{01001111, 11101001\}$		
		$\{00010100, 10110111\}$	$\{00010111, 10110100\}$	etc.	
	3.9	$W^8(1) \cup W^8(7)$	$W^8(4)$		$1-(3,2,7/6)$ code
		$W^8(0) \cup W^8(8)$	$W^8(2) \cup W^8(4) \cup W^8(6)$		
		$W^8(0) \cup W^8(4) \cup W^8(8)$	$W^8(2) \cup W^8(6)$	etc.	
	3.10	$\{00010010, 11011110\}$	$\{00011110, 11010010\}$		
		$1\{W^7(0) \cup W^7(7)\}$	$1\{W^7(2) \cup W^7(5)\}$		
		$01\{W^4(0) \cup W^4(4)\}01$	$01\{W^4(2)\}01$	etc.	

with $n > 2$ and $g > 2$ can correct any single unitary error, but no single deletion code that can correct single unitary errors has been found in the construction of this chapter.

The sets given in Theorem 3.8 that have the shortest length are

$$A = \{000100, 110111\}, \qquad B = \{000111, 110100\} \tag{3.78}$$

and

$$A = \{001000, 111011\}, \qquad B = \{001011, 111000\}. \tag{3.79}$$

From Theorem 3.19, these two are important examples because they have the minimum length among sets that are neither permutation-invariant nor extended permutation-invariant.

By Theorem 3.10, for sets $A, B \subset \{0,1\}^6$ in Equation (3.78),

$$0A1 = \{00001001, 01101111\}, \qquad 0B1 = \{00001111, 01101001\} \tag{3.80}$$

also satisfy the NH conditions. These are exactly the two sets presented in Equation (3.9), which provide the Nakayama code. Nakayama's 8-qubit code can be regarded as an example of Theorem 3.8, but it can also be constructed in terms of extension in this way.

Chapter 4

Permutation-invariant quantum codes for multiple deletion errors

The purpose of this chapter is to construct quantum error-correcting codes for multiple deletion errors by focusing on permutation-invariance. The contents of Chapter 4 were presented at the 2021 International Symposium on Information Theory (ISIT2021) [68].

In this chapter, an integer $1 \leq t < N$ is fixed and denotes the number of deletions, and a set $P \subset [N]$ denotes the deletion position satisfying $|P| = t$.

4.1 Code construction using the conditions (D1), (D2), and (D3)

This section proposes the conditions, named (D1), (D2), and (D3), for constructing multiple deletion error-correcting codes, and gives a code construction using these conditions and proves their error-correctability.

4.1.1 Deletion error-correcting conditions for permutation-invariant codes

The following Definition 4.1 gives a sufficient condition for constructing an $((N, 1))$ t-deletion error-correcting code. The method of constructing a quantum code using these conditions is described in Definition 4.2. Note that for binomial coefficients, if $w < 0$ or $N < w$, we define $\binom{N}{w} := 0$.

Definition 4.1 (Conditions (D1), (D2), and (D3))**.** *For non-empty sets $\mathcal{A}, \mathcal{B} \subset \{0, 1, \ldots, N\}$ and a map $f : \mathcal{A} \cup \mathcal{B} \to \mathbb{C}$, define three conditions (D1), (D2), and (D3) as follows:*

(D1) The following equation holds:

$$\sum_{w \in \mathcal{A}} |f(w)|^2 \binom{N}{w} = \sum_{w \in \mathcal{B}} |f(w)|^2 \binom{N}{w} = 1. \tag{4.1}$$

(D2) For any integer $0 \leq k \leq t$,

$$\sum_{w \in \mathcal{A}} |f(w)|^2 \binom{N-t}{w-k} = \sum_{w \in \mathcal{B}} |f(w)|^2 \binom{N-t}{w-k} \neq 0. \tag{4.2}$$

(D3) For any integers $w_1, w_2 \in A \cup B$,

$$w_1 \neq w_2 \implies |w_1 - w_2| > t. \tag{4.3}$$

Definition 4.2 (Encoder $\mathrm{Enc}^f_{\mathcal{A},\mathcal{B}}$ and code $Q^f_{\mathcal{A},\mathcal{B}}$)**.** *Let $\mathcal{A}, \mathcal{B} \subset \{0, 1, \dots, N\}$ be non-empty sets with $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $f : \mathcal{A} \cup \mathcal{B} \to \mathbb{C}$ be a map satisfying the conditions (D1), (D2), and (D3). Define an encoder as a linear map $\mathrm{Enc}^f_{\mathcal{A},\mathcal{B}} : \mathbb{C}^2 \to (\mathbb{C}^2)^{\otimes N}$. For a quantum state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \in \mathbb{C}^2$, $\mathrm{Enc}^f_{\mathcal{A},\mathcal{B}}$ maps the state $|\psi\rangle$ to the state $|\Psi\rangle \coloneqq \alpha|0_L\rangle + \beta|1_L\rangle \in (\mathbb{C}^2)^{\otimes N}$, where*

$$|0_L\rangle \coloneqq \sum_{\substack{\boldsymbol{a}\in\{0,1\}^N \\ \mathrm{wt}(\boldsymbol{a})\in\mathcal{A}}} f(\mathrm{wt}(\boldsymbol{a}))|\boldsymbol{a}\rangle, \qquad |1_L\rangle \coloneqq \sum_{\substack{\boldsymbol{b}\in\{0,1\}^N \\ \mathrm{wt}(\boldsymbol{b})\in\mathcal{B}}} f(\mathrm{wt}(\boldsymbol{b}))|\boldsymbol{b}\rangle. \tag{4.4}$$

Set $Q^f_{\mathcal{A},\mathcal{B}}$ as the image of $\mathrm{Enc}^f_{\mathcal{A},\mathcal{B}}$, i.e.,

$$Q^f_{\mathcal{A},\mathcal{B}} \coloneqq \{\mathrm{Enc}^f_{\mathcal{A},\mathcal{B}}(|\psi\rangle) \mid |\psi\rangle \in \mathbb{C}^2, |\psi\rangle\langle\psi| \in S(\mathbb{C}^2)\}. \tag{4.5}$$

For logical 0 and logical 1 defined by Equation (4.4), it is clear from the condition (D3) that $\langle 0_L|1_L\rangle = 0$ holds. Furthermore, from the condition (D1), we have

$$\||0_L\rangle\|^2 = \sum_{\substack{\boldsymbol{a}\in\{0,1\}^N \\ \mathrm{wt}(\boldsymbol{a})\in\mathcal{A}}} |f(\mathrm{wt}(\boldsymbol{a}))|^2\langle\boldsymbol{a}|\boldsymbol{a}\rangle = \sum_{w\in\mathcal{A}} |f(w)|^2\binom{N}{w} = 1, \tag{4.6}$$

$$\||1_L\rangle\|^2 = \sum_{\substack{\boldsymbol{b}\in\{0,1\}^N \\ \mathrm{wt}(\boldsymbol{b})\in\mathcal{B}}} |f(\mathrm{wt}(\boldsymbol{b}))|^2\langle\boldsymbol{b}|\boldsymbol{b}\rangle = \sum_{w\in\mathcal{B}} |f(w)|^2\binom{N}{w} = 1. \tag{4.7}$$

The condition (D1) can be regarded as the condition that the length of both logical 0 and logical 1 is equal to 1.

The code $Q^f_{\mathcal{A},\mathcal{B}}$ defined by Equation (4.5) is a permutation-invariant code regardless of the choice of $\mathcal{A}, \mathcal{B}$ and f. The fact that the code $Q^f_{\mathcal{A},\mathcal{B}}$ is an $((N, 1))$ t-deletion error-correcting code, i.e., Theorem 4.9, which will be explained later, is the main result of this chapter.

4.1.2 State after deletion errors

The following Lemma 4.3 describes the state after a t-deletion error for a permutation-invariant state.

Lemma 4.3. *Let $|\Psi\rangle \in (\mathbb{C}^2)^{\otimes N}$ be a pure permutation-invariant state with*

$$|\Psi\rangle \coloneqq \sum_{\boldsymbol{x}\in\{0,1\}^N} c(\mathrm{wt}(\boldsymbol{x}))|\boldsymbol{x}\rangle \tag{4.8}$$

for a map $c : \{0, 1, \dots, N\} \to \mathbb{C}$. For an integer $0 \leq k \leq t$, let

$$|\Psi_k\rangle \coloneqq \sum_{\boldsymbol{x}\in\{0,1\}^{N-t}} c(\mathrm{wt}(\boldsymbol{x}) + k)|\boldsymbol{x}\rangle. \tag{4.9}$$

Then, for any deletion position $P \subset [N]$ satisfying $|P| = t$,

$$D_P(|\Psi\rangle\langle\Psi|) = \sum_{k=0}^{t} \binom{t}{k} |\Psi_k\rangle\langle\Psi_k|. \tag{4.10}$$

Proof. By Equations (4.8) and (4.9), it is clear that

$$|\Psi\rangle = \sum_{\boldsymbol{y}\in\{0,1\}^t} \left(|\boldsymbol{y}\rangle \otimes |\Psi_{\mathrm{wt}(\boldsymbol{y})}\rangle\right) \tag{4.11}$$

for any integer $1 \leq t < N$. By the permutation-invariance of $|\Psi\rangle$ and the definition of the partial trace,

$$\begin{aligned} D_P(|\Psi\rangle\langle\Psi|) &= \underbrace{\mathrm{Tr}_1 \circ \cdots \circ \mathrm{Tr}_1}_{t \text{ times}}(|\Psi\rangle\langle\Psi|) && (4.12)\\ &= \sum_{\boldsymbol{y}\in\{0,1\}^t} |\Psi_{\mathrm{wt}(\boldsymbol{y})}\rangle\langle\Psi_{\mathrm{wt}(\boldsymbol{y})}| && (4.13)\\ &= \sum_{k=0}^{t} \binom{t}{k} |\Psi_k\rangle\langle\Psi_k| && (4.14) \end{aligned}$$

holds for any deletion position $P \subset [N]$. $\square$

Note that the state $|\Psi\rangle := \alpha|0_L\rangle + \beta|1_L\rangle$ defined in Equation (4.4) can be obtained in Equation (4.8) by setting

$$c(w) = \begin{cases} \alpha f(w) & \text{if } w \in \mathcal{A},\\ \beta f(w) & \text{if } w \in \mathcal{B},\\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

The state $|\Psi_k\rangle \in (\mathbb{C}^2)^{\otimes(N-t)}$ in Equation (4.9) for the state $|\Psi\rangle \in (\mathbb{C}^2)^{\otimes N}$ encoded by Definition 4.2 can be expressed in a convenient form by the following Lemma 4.4. While the condition (D1) was described as a condition for satisfying Equations (4.6) and (4.7), the conditions (D2) and (D3) can be considered as an adaptation of the Knill-Laflamme (KL) conditions [38] to permutation-invariant codes for quantum deletion errors. Here, the KL conditions will be explained in detail later in Chapter 6.

Lemma 4.4. *Let $\mathcal{A}, \mathcal{B} \subset \{0, 1, \ldots, N\}$ be non-empty sets with $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $f : \mathcal{A} \cup \mathcal{B} \to \mathbb{C}$ be a map satisfying the conditions (D2) and (D3). Then for any integer $0 \leq k \leq t$, there exist a real number $l_k \neq 0$ and unit vectors $|u_1^k\rangle, |u_2^k\rangle \in (\mathbb{C}^2)^{\otimes(N-t)}$ that satisfy the following:*

1. *For a vector $|\Psi_k\rangle$ defined by Equations (4.9) and (4.15),*

$$|\Psi_k\rangle = l_k(\alpha|u_1^k\rangle + \beta|u_2^k\rangle). \tag{4.16}$$

2. *For integers $k_1, k_2 \in \{0, 1, \ldots, t\}$ and $b_1, b_2 \in \{1, 2\}$,*

$$\langle u_{b_1}^{k_1}|u_{b_2}^{k_2}\rangle = \begin{cases} 1 & \textit{if } (k_1, b_1) = (k_2, b_2),\\ 0 & \textit{otherwise}. \end{cases} \tag{4.17}$$

Proof. For an integer $0 \leq k \leq t$, suppose that

$$|U_1^k\rangle := \sum_{w\in\mathcal{A}} \left(\sum_{\substack{\boldsymbol{a}\in\{0,1\}^{N-t}\\ \mathrm{wt}(\boldsymbol{a})=w-k}} f(w)|\boldsymbol{a}\rangle \right), \quad |U_2^k\rangle := \sum_{w\in\mathcal{B}} \left(\sum_{\substack{\boldsymbol{b}\in\{0,1\}^{N-t}\\ \mathrm{wt}(\boldsymbol{b})=w-k}} f(w)|\boldsymbol{b}\rangle \right). \tag{4.18}$$

By the condition (D2), we have $\langle U_1^k | U_1^k \rangle = \langle U_2^k | U_2^k \rangle \neq 0$. Set real number $l_k \neq 0$ and pure states $|u_1^k\rangle, |u_2^k\rangle \in (\mathbb{C}^2)^{\otimes(N-t)}$ as

$$l_k := \sqrt{\langle U_1^k | U_1^k \rangle}, \qquad |u_1^k\rangle := \frac{|U_1^k\rangle}{l_k}, \qquad |u_2^k\rangle := \frac{|U_2^k\rangle}{l_k}. \tag{4.19}$$

Hence, we have

$$|\Psi_k\rangle = \alpha |U_1^k\rangle + \beta |U_2^k\rangle = l_k(\alpha |u_1^k\rangle + \beta |u_2^k\rangle) \tag{4.20}$$

by Equations (4.9) and (4.15), In the case $(k_1, b_1) \neq (k_2, b_2)$, we obtain $\langle u_{b_1}^{k_1} | u_{b_2}^{k_2} \rangle = 0$ by the condition (D3). □

4.1.3 Decoding algorithm

The projective measurement used for decoding the code $Q^f_{\mathcal{A},\mathcal{B}}$ is expressed in the following Definition 4.5.

Definition 4.5 (Projective measurement $\mathbb{P}_{\mathcal{A},\mathcal{B}}$)**.** *Let $\mathcal{A}, \mathcal{B} \subset \{0, 1, \ldots, N\}$ be sets. For an integer $0 \leq k \leq t$, suppose that*

$$W_k := \{\boldsymbol{x} \in \{0,1\}^{N-t} \mid \mathrm{wt}(\boldsymbol{x}) + k \in \mathcal{A} \cup \mathcal{B}\}. \tag{4.21}$$

Then, we define a set $\mathbb{P}_{\mathcal{A},\mathcal{B}} := \{P_0, P_1, \ldots, P_t, P_\emptyset\}$ of projection matrices, where

$$P_k := \begin{cases} \displaystyle\sum_{\boldsymbol{x} \in W_k} |\boldsymbol{x}\rangle\langle \boldsymbol{x}| & \text{if } k \in \{0, 1, \ldots, t\}, \\ \mathbb{I} - \displaystyle\sum_{k=0}^{t} P_k & \text{if } k = \emptyset. \end{cases} \tag{4.22}$$

Here, $\mathbb{I}$ is the identity matrix of order 2^{N-t}.

If non-empty sets $\mathcal{A}, \mathcal{B} \subset \{0, 1, \ldots, N\}$ satisfy the condition (D3), the set $\mathbb{P}_{\mathcal{A},\mathcal{B}}$ is clearly a projective measurement. The following Lemma 4.6 shows the results of the projective measurement $\mathbb{P}_{\mathcal{A},\mathcal{B}}$ under the state after a deletion error in the code $Q^f_{\mathcal{A},\mathcal{B}}$.

Lemma 4.6. *Let $\mathcal{A}, \mathcal{B} \subset \{0, 1, \ldots, N\}$ be non-empty sets with $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $f : \mathcal{A} \cup \mathcal{B} \to \mathbb{C}$ be a map satisfying the conditions (D1), (D2), and (D3). Let $|\Psi\rangle$ and $|\Psi_k\rangle$ for an integer $0 \leq k \leq t$ be defined by Equations (4.8), (4.9) and (4.15). If we perform the projective measurement $\mathbb{P}_{\mathcal{A},\mathcal{B}}$ under the quantum state $D_P(|\Psi\rangle\langle\Psi|) \in S((\mathbb{C}^2)^{\otimes(N-t)})$ for any deletion position $P \subset [N]$, the probability $p(k)$ of getting outcome $k \in \{0, 1, \ldots, t, \emptyset\}$ is*

$$p(k) = \begin{cases} \dbinom{t}{k} {l_k}^2 & \text{if } k \in \{0, 1, \ldots, t\}, \\ 0 & \text{if } k = \emptyset. \end{cases} \tag{4.23}$$

When the outcome $k \in \{0, 1, \ldots, t\}$ is obtained, the quantum state $\rho(k) \in S((\mathbb{C}^2)^{\otimes(N-t)})$ after the measurement is

$$\rho(k) = \frac{1}{{l_k}^2} |\Psi_k\rangle\langle\Psi_k|. \tag{4.24}$$

Proof. In the case $k \in \{0, 1, \ldots, t\}$, we have

$$p(k) = \mathrm{Tr}(P_k D_P(|\Psi\rangle\langle\Psi|)) \tag{4.25}$$
$$= \mathrm{Tr}\left(\sum_{\boldsymbol{x}\in W_k} |\boldsymbol{x}\rangle\langle\boldsymbol{x}| \sum_{k'=0}^{t} \binom{t}{k'} |\Psi_{k'}\rangle\langle\Psi_{k'}|\right) \tag{4.26}$$
$$= \mathrm{Tr}\left(\binom{t}{k}|\Psi_k\rangle\langle\Psi_k|\right) \tag{4.27}$$
$$= \mathrm{Tr}\left(\binom{t}{k}{l_k}^2(\alpha|u_1^k\rangle + \beta|u_2^k\rangle)(\overline{\alpha}\langle u_1^k| + \overline{\beta}\langle u_2^k|)\right) \tag{4.28}$$
$$= \binom{t}{k}{l_k}^2\left(|\alpha|^2\langle u_1^k|u_1^k\rangle + |\beta|^2\langle u_2^k|u_2^k\rangle\right) \tag{4.29}$$
$$= \binom{t}{k}{l_k}^2 \tag{4.30}$$

by Lemmas 4.3 and 4.4. In the case $k = \emptyset$, it is clear that $p(\emptyset) = \mathrm{Tr}(P_\emptyset D_P(|\Psi\rangle\langle\Psi|)) = 0$.

If we obtain an outcome $k \in \{0, 1, \ldots, t\}$, then by Lemma 4.3, the quantum state immediately after the measurement is

$$\frac{P_k D_P(|\Psi\rangle\langle\Psi|)P_k}{\mathrm{Tr}(P_k D_P(|\Psi\rangle\langle\Psi|))} = \frac{P_k\left(\sum_{k'=0}^{t}\binom{t}{k'}|\Psi_{k'}\rangle\langle\Psi_{k'}|\right)P_k}{\binom{t}{k}{l_k}^2} \tag{4.31}$$
$$= \frac{\binom{t}{k}|\Psi_k\rangle\langle\Psi_k|}{\binom{t}{k}{l_k}^2} \tag{4.32}$$
$$= \frac{1}{{l_k}^2}|\Psi_k\rangle\langle\Psi_k| \tag{4.33}$$

for any deletion position $P \subset [N]$. □

Note that in the proof above, the properties of projective measurements described in Section 2.1.4 are used.

Definition 4.7 (Error-correcting operator U_k)**.** *Suppose that the assumptions of Lemma 4.4 are satisfied. Then, for integers $k \in \{0, 1, \ldots, t\}$ and $m \in \{1, 2\}$, we can choose a unitary matrix U_k of order 2^{N-t} whose mth row is $\langle u_m^k|$. We call the matrix U_k an error-correcting operator. For that unitary matrix U_k, the mth row for $3 \le m \le 2^{N-t}$ is also denoted as $\langle u_m^k|$.*

Definition 4.7 represents the unitary matrix used for decoding. Here, we have defined the unitary matrix U_k by specifying the structure, in other words, the action of this unitary matrix U_k of order 2^{N-t} is defined as the unitary transformation such that

$$|u_1^k\rangle \mapsto |0\ldots 00\rangle, \qquad |u_2^k\rangle \mapsto |0\ldots 01\rangle. \tag{4.34}$$

The decoder for the code $Q^f_{\mathcal{A},\mathcal{B}}$ is defined by combining the projective measurement $\mathbb{P}_{\mathcal{A},\mathcal{B}}$, the error-correcting operator U_k, and the partial trace Tr_1 as follows.

Definition 4.8 (Decoder $\mathrm{Dec}^f_{\mathcal{A},\mathcal{B}}$)**.** *Let $\mathcal{A}, \mathcal{B} \subset \{0, 1, \ldots, N\}$ be non-empty sets with $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $f : \mathcal{A} \cup \mathcal{B} \to \mathbb{C}$ be a map satisfying the conditions (D1), (D2), and (D3). Define a decoder $\mathrm{Dec}^f_{\mathcal{A},\mathcal{B}}$ as a map from $\rho \in S((\mathbb{C}^2)^{\otimes(N-t)})$ to $\sigma \in S(\mathbb{C}^2)$ constructed by the following steps:*

(Step 1) Perform the projective measurement $\mathbb{P}_{\mathcal{A},\mathcal{B}}$ under the quantum state ρ. Assume that the outcome is k and that the state after the measurement is $\rho(k)$.

(Step 2) Let $\tilde{\rho} := U_k \rho(k) U_k^\dagger$. Here U_k is the error-correcting operator corresponding to the obtained outcome k.

(Step 3) At last, return $\sigma := \underbrace{\mathrm{Tr}_1 \circ \cdots \circ \mathrm{Tr}_1}_{N-t-1 \text{ times}}(\tilde{\rho})$.

The proof that the code $Q^f_{\mathcal{A},\mathcal{B}}$ is indeed error-correctable by the decoder $\mathrm{Dec}^f_{\mathcal{A},\mathcal{B}}$, i.e., that it satisfies Equation (2.28), is given in Section 4.1.4.

4.1.4 Proof of error-correctability

The following Theorem 4.9 is the main theorem of this chapter and the first given as a systematic method of constructing multiple deletion error-correcting codes.

Theorem 4.9. *Let $\mathcal{A}, \mathcal{B} \subset \{0, 1, \ldots, N\}$ be non-empty sets with $\mathcal{A} \cap \mathcal{B} = \emptyset$ and $f : \mathcal{A} \cup \mathcal{B} \to \mathbb{C}$ be a map satisfying the conditions (D1), (D2), and (D3). Then, the code $Q^f_{\mathcal{A},\mathcal{B}}$ is an $((N, 1))$ t-deletion error-correcting code with the encoder $\mathrm{Enc}^f_{\mathcal{A},\mathcal{B}}$ and the decoder $\mathrm{Dec}^f_{\mathcal{A},\mathcal{B}}$.*

In other words, for any pure quantum message $|\psi\rangle \in \mathbb{C}^2$ and any deletion position $P \subset [N]$,

$$\mathrm{Dec}^f_{\mathcal{A},\mathcal{B}} \circ D_P \circ \mathrm{Enc}^f_{\mathcal{A},\mathcal{B}}(|\psi\rangle) = |\psi\rangle \tag{4.35}$$

holds.

Proof. Set $|\Psi\rangle := \mathrm{Enc}^f_{\mathcal{A},\mathcal{B}}(|\psi\rangle) \in S((\mathbb{C}^2)^{\otimes N})$ for a pure quantum state $|\psi\rangle \in \mathbb{C}^2$. For an integer $k \in \{0, 1, \ldots, t\}$ and integers $i, j \in [2^{N-t}]$, we denote the (i, j) element of the matrix $U_k \left(\frac{1}{{l_k}^2} |\Psi_k\rangle\langle\Psi_k| \right) U_k^\dagger$ by $u_k(i, j)$. By Lemma 4.4, we have

$$\begin{aligned}
u_k(i,j) &= \langle u_i^k| \left(\frac{1}{{l_k}^2} |\Psi_k\rangle\langle\Psi_k| \right) |u_j^k\rangle &(4.36)\\
&= \langle u_i^k| (\alpha |u_1^k\rangle + \beta |u_2^k\rangle)(\overline{\alpha}\langle u_1^k| + \overline{\beta}\langle u_2^k|) |u_j^k\rangle &(4.37)\\
&= (\alpha \langle u_i^k | u_1^k\rangle + \beta \langle u_i^k | u_2^k\rangle)(\overline{\alpha}\langle u_1^k | u_j^k\rangle + \overline{\beta}\langle u_2^k | u_j^k\rangle) &(4.38)\\
&= \begin{cases} |\alpha|^2 & \text{if } (i,j) = (1,1), \\ \alpha\overline{\beta} & \text{if } (i,j) = (1,2), \\ \overline{\alpha}\beta & \text{if } (i,j) = (2,1), \\ |\beta|^2 & \text{if } (i,j) = (2,2), \\ 0 & \text{otherwise.} \end{cases} &(4.39)
\end{aligned}$$

By Lemmas 4.3, 4.6, Definition 4.8, and Equation (4.39),

$$\begin{aligned}
\mathrm{Dec}^f_{\mathcal{A},\mathcal{B}} \circ D_P \circ \mathrm{Enc}^f_{\mathcal{A},\mathcal{B}}(|\psi\rangle\langle\psi|) &= \mathrm{Dec}^f_{\mathcal{A},\mathcal{B}} \circ D_P(|\Psi\rangle\langle\Psi|) &(4.40)\\
&= \mathrm{Dec}^f_{\mathcal{A},\mathcal{B}} \left(\sum_{k=0}^{t} \binom{t}{k} |\Psi_k\rangle\langle\Psi_k| \right) &(4.41)\\
&= \mathrm{Tr}_1 \circ \cdots \circ \mathrm{Tr}_1 \left(U_k \left(\frac{1}{{l_k}^2} |\Psi_k\rangle\langle\Psi_k| \right) U_k^\dagger \right) &(4.42)
\end{aligned}$$

$$= \mathrm{Tr}_1 \circ \cdots \circ \mathrm{Tr}_1(|0\rangle\langle 0| \otimes \cdots \otimes |0\rangle\langle 0| \otimes |\psi\rangle\langle\psi|) \tag{4.43}$$

$$= |\psi\rangle\langle\psi| \tag{4.44}$$

holds for any pure quantum message $|\psi\rangle \in \mathbb{C}^2$ and any deletion position $P \in [N]$. Equation (4.44) is exactly the original quantum message. □

4.2 Examples

By Theorem 4.9, we can construct a permutation-invariant quantum code for deletion errors by finding two non-empty sets $A, B \subset \{0, 1, \ldots, N\}$ with $A \cap B = \emptyset$ and a map $f : A \cup B \to \mathbb{C}$ that satisfy the three conditions (D1), (D2), and (D3). we give two families of our codes in this section.

4.2.1 Example 1 (Multiple deletion error-correcting codes)

First, we introduce a key combinatorial equation in giving the first example. Lemma 4.10 below is Lemma 1 in Reference [51].

Lemma 4.10. *Let n be a positive integer. Then for all integers $0 \le t \le n-1$,*

$$\sum_{j=0}^{n} \binom{n}{j} j^t (-1)^j = 0. \tag{4.45}$$

Lemma 4.10 can be easily shown by induction using the binomial identity

$$\sum_{j=0}^{n} \binom{n}{j} \binom{j}{t} (-1)^j = 0, \tag{4.46}$$

which holds for any integer $0 \le t < n$.

The following Theorem 4.11 gives quantum codes for multiple deletion errors. This is an interesting example that can be proved by good use of the combinatorial equation above. Here, we fix an integer $1 \le t < N$.

Theorem 4.11. *Let g, n be integers and u be a rational number with*

$$g \ge t+1, \qquad n \ge t+1, \qquad u := \frac{N}{gn} \ge 1. \tag{4.47}$$

Suppose that sets $\mathcal{A}, \mathcal{B} \subset \{0, 1, \ldots, N\}$ and a map $f : \mathcal{A} \cup \mathcal{B} \to \mathbb{C}$ are set as

$$\mathcal{A} := \{gj \mid 0 \le j \le n,\ j : \text{even}\}, \tag{4.48}$$

$$\mathcal{B} := \{gj \mid 0 \le j \le n,\ j : \text{odd}\}, \tag{4.49}$$

$$f(gj) := \sqrt{\frac{\binom{n}{j}}{2^{n-1}\binom{gnu}{gj}}}. \tag{4.50}$$

Then, $Q^f_{\mathcal{A},\mathcal{B}}$ is an $((N, 1))$ t-deletion error-correcting code.

Proof. It is clear that $\mathcal{A} \neq \emptyset$, $\mathcal{B} \neq \emptyset$, and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Hence, it is enough to prove that the three conditions (D1), (D2), and (D3) hold by Theorem 4.9.

Simple calculations show that

$$\sum_{w \in \mathcal{A}} |f(w)|^2 \binom{N}{w} = \frac{1}{2^{n-1}} \sum_{\substack{0 \leq j \leq n \\ j \text{ even}}} \binom{n}{j} = 1. \tag{4.51}$$

Similarly, $\sum_{w \in \mathcal{B}} |f(w)|^2 \binom{N}{w} = 1$. Therefore, (D1) holds.

For an integer $0 \leq k \leq t$, we obtain

$$\sum_{w \in \mathcal{A}} |f(w)|^2 \binom{N-t}{w-k} - \sum_{w \in \mathcal{B}} |f(w)|^2 \binom{N-t}{w-k} = \sum_{j=0}^{n} \frac{\binom{n}{j}}{2^{n-1} \binom{gnu}{gj}} \binom{gnu-t}{gj-k} (-1)^j = 0 \tag{4.52}$$

by the assumption $n \geq t+1$ and Lemma 4.10. Note that the ratio of binomial coefficients $\binom{gnu-t}{gj-k} / \binom{gnu}{gj}$ is a polynomial in j of order t. On the other hand, it is obvious that

$$\sum_{w \in \mathcal{A}} |f(w)|^2 \binom{N-t}{w-k} \neq 0. \tag{4.53}$$

Therefore, (D2) holds.

It is clear that (D3) holds by the assumption $g \geq t+1$. □

The quantum code constructed by Theorem 4.11 is the already known $0-(g,n,u)$ code. However, the tolerance to deletion errors is mentioned here for the first time. Theorem 4.11 claims that any $0-(g,n,u)$ code is a t-deletion error-correcting code if $g \geq t+1$, $n \geq t+1$, and $u \geq 1$. The smallest example is precisely Hagiwara's 4-qubit single deletion code that is the $(2,2,1)$ code [35].

Theorem 4.11 is also the first published method for constructing multiple deletion error-correcting codes. Furthermore, from Fact 2.2, when $g \geq 2t+1$, $n \geq 2t+1$, and $u \geq 1$, we can see that the $0-(g,n,u)$ code can correct t-qubit errors and $2t$-deletion errors if they do not occur simultaneously. Thus, Theorem 4.11 is novel also in that we propose quantum codes that are tolerant to two types of errors, including deletion errors. Previous quantum deletion codes could only correct one of them, even if they were not simultaneous. The smallest example is precisely Ruskai's 9-qubit code [62], which is the $0-(3,3,1)$ code introduced in Example 2.3. It was already known that this code can correct 1-qubit errors, but it was shown here for the first time that it can also correct 2-deletion errors.

The relationship with Ouyang's work on permutation-invariant deletion codes [53], which was published at the same time as Theorem 4.11 in this thesis, is explained here. He pointed out that having t-deletion errors is equivalent to having t-erasure errors on any permutation-invariant code. Furthermore, from the fact that any t-qudit error-correcting code is a $2t$-erasure error-correcting code [31], the following fact can be derived from Fact 2.2.

Fact 4.12 (Ouyang [53])**.** *Fix t as a non-negative integer. Let Δ be a non-negative integer, g, n be positive integers, and u be a rational number, and suppose that*

$$g \geq t+1, \quad n \geq t+1, \quad u \geq 1. \tag{4.54}$$

Then, the $\Delta-(g,n,u)$ code is an $((N,1))$ t-deletion error-correcting code with $N = gnu + \Delta$.

Fact 4.12 is seemingly a better result than Theorem 4.11 of this thesis in that it takes the shift Δ into account. However, of course, the Δ-shifted gnu codes that take the shift Δ into account are also included in our framework and can be checked by calculations. Our argument is also practical in that we proved error-correctability by giving a systematic decoder as in References [35,48]. Ouyang pointed out that deletion errors can be corrected not only for level 2 qubit systems but also for level 3 and higher qudit systems. In addition, Ouyang gave a detailed discussion of encoding and decoding for the special case, the Δ-shifted gnu code. Thus, although some of the results in References [68] and [53] are the same, each has its uniqueness, and both were accepted and presented at the same conference, the 2021 International Symposium on Information Theory (ISIT2021).

4.2.2 Example 2 (Single deletion error-correcting codes)

Here, we introduce 1-deletion error-correcting codes. The codes constructed by Theorem 4.13 below are already known as examples [66] of the code construction given by Nakayama and Hagiwara [49] and are the same codes described in Theorem 3.9. However, they are also one of the codes constructed in this chapter. Note that Theorem 4.13 has the same content as Theorem 3.9, but the proof is slightly simpler.

Theorem 4.13. *Suppose that two non-empty sets $\mathcal{A}, \mathcal{B} \subset \{0, 1, \ldots, N\}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset$ satisfy following:*

1. $w \in \mathcal{A} \implies N - w \in \mathcal{A}$,
2. $w \in \mathcal{B} \implies N - w \in \mathcal{B}$,
3. $|w_1 - w_2| > 1$, *for any integers* $w_1, w_2 \in \mathcal{A} \cup \mathcal{B}$ *with* $w_1 \neq w_2$.

and that the map $f : \mathcal{A} \cup \mathcal{B} \to \mathbb{C}$ is set as

$$f(w) := \begin{cases} \left(\sum_{w' \in \mathcal{A}} \binom{N}{w'}\right)^{-\frac{1}{2}} & \text{if } w \in \mathcal{A}, \\ \left(\sum_{w' \in \mathcal{B}} \binom{N}{w'}\right)^{-\frac{1}{2}} & \text{if } w \in \mathcal{B}. \end{cases} \tag{4.55}$$

Then, $Q^f_{\mathcal{A},\mathcal{B}}$ is an $((N, 1))$ 1-deletion error-correcting code.

Proof. It is clear that (D1) and (D3) hold by the assumptions. Hence we show that (D2) holds. By the assumption, we have

$$\sum_{w \in \mathcal{A}} \binom{N-1}{w-0} = \sum_{w \in \mathcal{A}} \binom{N-1}{w-1}, \tag{4.56}$$

$$\sum_{w \in \mathcal{A}} \binom{N-1}{w-0} + \sum_{w \in \mathcal{A}} \binom{N-1}{w-1} = \sum_{w \in \mathcal{A}} \binom{N}{w}. \tag{4.57}$$

Similarly, the same equations for $\mathcal{B}$ are obtained. Hence,

$$\sum_{w \in \mathcal{A}} |f(w)|^2 \binom{N-t}{w-k} - \sum_{w \in \mathcal{B}} |f(w)|^2 \binom{N-t}{w-k} = \frac{\sum_{w \in \mathcal{A}} \binom{N-1}{w-k}}{\sum_{w' \in \mathcal{A}} \binom{N}{w'}} - \frac{\sum_{w \in \mathcal{B}} \binom{N-1}{w-k}}{\sum_{w' \in \mathcal{B}} \binom{N}{w'}} \tag{4.58}$$

$$= \frac{1}{2} - \frac{1}{2} \tag{4.59}$$

$$= 0 \tag{4.60}$$

holds for any integer $0 \leq k \leq 1$. Therefore, (D2) holds. □

4.3 Further considerations

This section gives two considerations for the permutation-invariant quantum codes that we have discussed. First, we describe the construction of the codes such that the number of particles in the quantum message is increased. This is a generalization of the permutation-invariant deletion codes constructed in Section 4.1. Second, we give an argument when an adversary performs a measurement on the deleted qubit and the state is changed.

4.3.1 Generalization of code construction

Here, we discuss constructions of $((N, K))$ t-deletion error-correcting codes for any positive integer K. Let l be a positive integer and $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{l-1} \subset \{0, 1, \dots, N\}$ be mutually disjoint non-empty sets and $f : \bigcup_{i=0}^{l-1} A_i \to \mathbb{C}$ be a map which satisfy the following three conditions:

(D1)* For any integer $i \in \{0, 1, \dots, l-1\}$,

$$\sum_{w \in \mathcal{A}_i} |f(w)|^2 \binom{N}{w} = 1. \tag{4.61}$$

(D2)* For any integer $0 \leq k \leq t$ and any integers $i, j \in \{0, 1, \dots, l-1\}$,

$$\sum_{w \in \mathcal{A}_i} |f(w)|^2 \binom{N-t}{w-k} = \sum_{w \in \mathcal{A}_j} |f(w)|^2 \binom{N-t}{w-k} \neq 0. \tag{4.62}$$

(D3)* For any integers $w_1, w_2 \in \bigcup_{i=0}^{l-1} \mathcal{A}_i$,

$$w_1 \neq w_2 \implies |w_1 - w_2| > t. \tag{4.63}$$

Each of the three conditions above is an extension of conditions (D1), (D2), and (D3) proposed in Definition 4.1.

Let us define an encoder as a linear map $\mathrm{Enc}^f_{\{\mathcal{A}_i\}} : \mathbb{C}^l \to (\mathbb{C}^2)^{\otimes N}$. As shown in Equation (2.6), let $|0\rangle, |1\rangle, \dots, |l-1\rangle$ be the standard orthonormal basis of $\mathbb{C}^l$, and set the quantum state $|\psi\rangle$ to $|\psi\rangle = \sum_{i=0}^{l-1} \alpha_i |i\rangle \in \mathbb{C}^l$. The encoder $\mathrm{Enc}^f_{\{\mathcal{A}_i\}}$ maps the state $|\psi\rangle$ to the state $|\Psi\rangle \coloneqq \alpha_0 |0_L\rangle + \alpha_1 |1_L\rangle + \cdots + \alpha_{l-1} |l-1_L\rangle$, where the logical cordword is

$$|i_L\rangle \coloneqq \sum_{\substack{\boldsymbol{a} \in \{0,1\}^N \\ \mathrm{wt}(\boldsymbol{a}) \in \mathcal{A}_i}} f(\mathrm{wt}(\boldsymbol{a})) |\boldsymbol{a}\rangle \tag{4.64}$$

for each integer $0 \leq i \leq l-1$. Note that this encoder is an extension of Definition 4.2. We claim that the image of $\mathrm{Enc}^f_{\{\mathcal{A}_i\}}$ is a t-deletion error-correcting code for an integer $1 \leq t < N$. A detailed explanation is not given here, but it can be proved using the same method as in Section 4.1. In particular, for the case $l = 2^K$, we obtain an $((N, K))$ t-deletion error-correcting code.

By extending Theorem 4.13, we can construct $((N, K))$ 1-deletion codes with any positive integer K, if we take the code length N to be sufficiently large.

Corollary 4.14. *Suppose that mutually disjoint non-empty sets $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{l-1} \subset \{0, 1, \ldots, N\}$ satisfy following:*

1. *$w \in \mathcal{A}_i \implies N - w \in \mathcal{A}_i$, for any integer $0 \leq i \leq l-1$,*
2. *$|w_1 - w_2| > 1$, for any integers $w_1, w_2 \in \bigcup_{i=0}^{l-1} \mathcal{A}_i$ with $w_1 \neq w_2$.*

and that the map $f : \bigcup_{i=0}^{l-1} A_i \to \mathbb{C}$ is set as

$$f(w) := \left(\sum_{w' \in \mathcal{A}_i} \binom{N}{w'} \right)^{-\frac{1}{2}} \qquad \text{if } w \in \mathcal{A}_i. \tag{4.65}$$

Then, the conditions (D1), (D2)*, and (D3)* are satisfied.*

Corollary 4.14 is immediately derived from Theorem 4.13. Here, one example is given.

Example 4.15. *Assuming that $N = 12$ and*

$$\mathcal{A}_0 = \{0, 12\}, \quad \mathcal{A}_1 = \{2, 10\}, \quad \mathcal{A}_2 = \{4, 8\}, \quad \mathcal{A}_3 = \{6\}, \tag{4.66}$$

$$f(0) = f(12) = \frac{1}{\sqrt{2}}, \quad f(2) = f(10) = \frac{1}{\sqrt{132}}, \quad f(4) = f(8) = \frac{1}{\sqrt{990}}, \quad f(6) = \frac{1}{\sqrt{924}} \tag{4.67}$$

in Corollary 4.14, from Equation (4.64), the logical codewords are as follows:

$$|0_L\rangle := \frac{1}{\sqrt{2}} \sum_{\substack{\boldsymbol{a} \in \{0,1\}^{12} \\ \mathrm{wt}(\boldsymbol{a}) \in \{0,12\}}} |\boldsymbol{a}\rangle, \qquad |1_L\rangle := \frac{1}{\sqrt{132}} \sum_{\substack{\boldsymbol{a} \in \{0,1\}^{12} \\ \mathrm{wt}(\boldsymbol{a}) \in \{2,10\}}} |\boldsymbol{a}\rangle, \tag{4.68}$$

$$|2_L\rangle := \frac{1}{\sqrt{990}} \sum_{\substack{\boldsymbol{a} \in \{0,1\}^{12} \\ \mathrm{wt}(\boldsymbol{a}) \in \{4,8\}}} |\boldsymbol{a}\rangle, \qquad |3_L\rangle := \frac{1}{\sqrt{924}} \sum_{\substack{\boldsymbol{a} \in \{0,1\}^{12} \\ \mathrm{wt}(\boldsymbol{a}) \in \{6\}}} |\boldsymbol{a}\rangle. \tag{4.69}$$

Then, by direct calculations, it can be checked that (D1), (D2)*, and (D3)* are satisfied. That is, we can construct a $((12, 2))$ 1-deletion error-correcting code by defining the encoder as a linear map that maps the state*

$$\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle \in (\mathbb{C}^2)^{\otimes 2} \tag{4.70}$$

to the state

$$\alpha_0|0_L\rangle + \alpha_1|1_L\rangle + \alpha_2|2_L\rangle + \alpha_3|3_L\rangle \in (\mathbb{C}^2)^{\otimes 12}. \tag{4.71}$$

Here, we have extended the number of particles that form a quantum message in a level 2 qubit system. This does not mean that we have increased the level of the quantum system we are considering. Such an extension in the deletion error-correcting codes was first given in Reference [35]. The general conditions to realize this extension are (D1)*, (D2)*, and (D3)*. However, to date, no $((N, K))$ t-deletion code with $K \geq 2$ and $t \geq 2$ has been reported. Hence, further research is expected in the future.

4.3.2 Error-correctability for artificial deletions

Here, we consider the case where an adversary performs a measurement on the deleted qubit and the quantum state changes. Our code can also decode exactly in such a case using the decoder defined in Definition 4.8.

We explain using the Hagiwara code as an example. The code with $(g, n, u) = (2, 2, 1)$ in Theorem 4.11 is the Hagiwara code, whose logical codewords are

$$|0_L\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle), \tag{4.72}$$

$$|1_L\rangle = \frac{1}{\sqrt{6}}(|0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle + |1100\rangle) \tag{4.73}$$

from Equation (4.4). For any two mutually perpendicular unit vectors $\alpha_1|0\rangle + \beta_1|1\rangle, \alpha_2|0\rangle + \beta_2|1\rangle \in \mathbb{C}^2$, the state after encoding is

$$\begin{aligned} |\Psi\rangle &= \alpha|0_L\rangle + \beta|1_L\rangle && (4.74)\\ &= |0\rangle \otimes |\Psi_0\rangle + |1\rangle \otimes |\Psi_1\rangle && (4.75)\\ &= (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\overline{\alpha_1}|\Psi_0\rangle + \overline{\beta_1}|\Psi_1\rangle) + (\alpha_2|0\rangle + \beta_2|1\rangle) \otimes (\overline{\alpha_2}|\Psi_0\rangle + \overline{\beta_2}|\Psi_1\rangle). && (4.76) \end{aligned}$$

Note that

$$|\Psi_0\rangle = \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\beta}{\sqrt{6}}(|011\rangle + |101\rangle + |110\rangle), \tag{4.77}$$

$$|\Psi_1\rangle = \frac{\alpha}{\sqrt{2}}|111\rangle + \frac{\beta}{\sqrt{6}}(|001\rangle + |010\rangle + |100\rangle) \tag{4.78}$$

from Equation (4.9). According to Equation (4.76), when a measurement is performed on the deleted 1-qubit, the state after the measurement of the remaining 3-qubit state is expressed in the form of a linear combination of $|\Psi_0\rangle$ and $|\Psi_1\rangle$. Therefore, by performing the projective measurement in Definition 4.5, the state after the measurement is always in the form of Equation (4.24), and thus the decoding can be performed correctly.

Here we have used the Hagiwara code, but the general case can also be expressed as a linear combination of $|\Psi_k\rangle$ for $0 \le k \le t$, thus decoding is possible in the same way.

Chapter 5

Quantum error-correcting codes for single insertion errors

The purpose of this chapter is to construct quantum error-correcting codes for single insertion errors. The contents of Chapter 5 were presented at the 44th Symposium on Information Theory and its Applications (SITA2021) in Japanese.

5.1 Code construction using the conditions $(C1)^+$, $(C2)^+$, and $(C3)^+$

This section introduces two sets A, B of bit sequences to construct quantum insertion codes, and proposes the sufficient conditions $(C1)^+$, $(C2)^+$, and $(C3)^+$ for single insertion error-correction for the sets A, B. We also give a code construction using these conditions and prove their error-correctability.

5.1.1 Single insertion error-correcting conditions

We assume that the state after encoding is pure. That is, from Fact 2.6, a single insertion error can be defined as follows.

Definition 5.1 (Single insertion error $\mathrm{In}_{p,\sigma}$)**.** *Let us denote the N-qudit state $M \in S((\mathbb{C}^l)^{\otimes N})$ as*

$$M = \sum_{\boldsymbol{x},\boldsymbol{y} \in \{0,1,\ldots,l-1\}^N} m_{\boldsymbol{x},\boldsymbol{y}} |x_1\rangle\langle y_1| \otimes |x_2\rangle\langle y_2| \otimes \cdots \otimes |x_N\rangle\langle y_N| \tag{5.1}$$

with $m_{\boldsymbol{x},\boldsymbol{y}} \in \mathbb{C}$. For an integer $i \in [N+1]$ and a quantum state $\sigma \in S(\mathbb{C}^l)$, define the map $\mathrm{In}_{i,\sigma} : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes (N+1)})$ as

$$\mathrm{In}_{i,\sigma}(M) \coloneqq \sum_{\boldsymbol{x},\boldsymbol{y} \in \{0,1,\ldots,l-1\}^N} m_{\boldsymbol{x},\boldsymbol{y}} \\ |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_{i-1}\rangle\langle y_{i-1}| \otimes \sigma \otimes |x_i\rangle\langle y_i| \otimes \cdots \otimes |x_N\rangle\langle y_N|. \tag{5.2}$$

The map $\mathrm{In}_{i,\sigma}$ is called a single insertion error.

In the above, we defined single insertion errors in a qudit system, however, this chapter will consider a level 2 qubit system in particular.

The encoder and the code used in this chapter are the ones defined in Chapter 3, i.e. $\mathrm{Enc}_{A,B}$ and $Q_{A,B}$ in Definition 3.1. The problem is to give conditions on sets A, B such that the quantum code $Q_{A,B}$ defined above is tolerant of insertion errors, and to define the corresponding decoder. The following Definition 5.2 is an insertion version of the deletion set $\Delta^-_{i,b}$ introduced in Definition 3.2.

Definition 5.2 (Insertion set $\Delta^+_{i,b}$)**.** *Let $i \in [N+1]$ be an integer and let $b \in \{0,1\}$ be a bit. For a non-empty set $A \subset \{0,1\}^N$, define a set $\Delta^+_{i,b}(A) \subset \{0,1\}^{N+1}$ as*

$$\Delta^+_{i,b}(A) := \{a_1 \dots a_{i-1} b a_i \dots a_N \in \{0,1\}^{N+1} \mid a_1 \dots a_{i-1} a_i \dots a_N \in A\}. \tag{5.3}$$

In other words, $\Delta^+_{i,b}(A)$ is the set of bit sequences obtained by inserting "b" into the ith component of each sequence $\boldsymbol{a} \in A$. We call the set $\Delta^+_{i,b}(A)$ an (i,b) insertion set of A.

Definition 5.3 (Conditions (C1)$^+$, (C2)$^+$, and (C3)$^+$)**.** *For two non-empty sets $A, B \subset \{0,1\}^N$, define three conditions (C1)$^+$, (C2)$^+$, and (C3)$^+$ as follows:*

(C1)$^+$ (Ratio condition) For any non-empty set $I \subset [N+1]$ and any bit $b \in \{0,1\}$,

$$|A||B^+_{I,b}| = |B||A^+_{I,b}|, \tag{5.4}$$

where

$$A^+_{I,b} := \bigcap_{i \in I} \Delta^+_{i,b}(A) \cap \bigcap_{i \in I^c} \Delta^+_{i,b}(A)^c, \quad B^+_{I,b} := \bigcap_{i \in I} \Delta^+_{i,b}(B) \cap \bigcap_{i \in I^c} \Delta^+_{i,b}(B)^c \tag{5.5}$$

and X^c denotes the complement of a set X, in particular,

$$\Delta^+_{i,b}(A)^c = \{0,1\}^{N+1} \setminus \Delta^+_{i,b}(A), \quad I^c = [N+1] \setminus I. \tag{5.6}$$

(C2)$^+$ (Outer distance condition) For any integers $i_1, i_2 \in [N+1]$ and any bits $b_1, b_2 \in \{0,1\}$,

$$|\Delta^+_{i_1,b_1}(A) \cap \Delta^+_{i_2,b_2}(B)| = 0. \tag{5.7}$$

(C3)$^+$ (Inner distance condition) For any integers $i_1, i_2 \in [N+1]$,

$$|\Delta^+_{i_1,0}(A) \cap \Delta^+_{i_2,1}(A)| = 0, \quad |\Delta^+_{i_1,0}(B) \cap \Delta^+_{i_2,1}(B)| = 0. \tag{5.8}$$

For the symbols defined in Chapter 3, by replacing the symbol "$-$" with the symbol "$+$", it means that we are considering an insertion version of the concept dealt with in the deletion. As in the case of the NH conditions, the two pairs (A, B) and (B, A) are considered to be identical, and thus we assume in particular that $|A| \le |B|$.

The conditions (C1)$^+$, (C2)$^+$, and (C3)$^+$ represented by Definition 5.3 are very complicated, and examples of sets satisfying them are not easy to find. Examples satisfying these conditions are given in section 5.3.

5.1.2 Decoding algorithm

The projective measurement used for decoding the code $Q_{A,B}$ for single insertion errors is expressed in Definition 5.4.

Definition 5.4 (Projective measurement $\mathbb{P}^+_{A,B}$)**.** *For non-empty sets $A, B \subset \{0,1\}^N$, suppose that $\tilde{\mathbb{P}}^+_{A,B} \coloneqq \{P_{I,b} \mid I \subset [N+1], I \neq \emptyset, b \in \{0,1\}\}$, where*

$$P_{I,b} \coloneqq \sum_{\boldsymbol{a} \in A^+_{I,b}} |\boldsymbol{a}\rangle\langle\boldsymbol{a}| + \sum_{\boldsymbol{b} \in B^+_{I,b}} |\boldsymbol{b}\rangle\langle\boldsymbol{b}|. \tag{5.9}$$

Then, we define a set $\mathbb{P}^+_{A,B} \coloneqq \tilde{\mathbb{P}}^+_{A,B} \cup \{P_\emptyset\}$ of projection matrices, where

$$P_\emptyset \coloneqq \mathbb{I} - \sum_{P \in \tilde{\mathbb{P}}^+_{A,B}} P. \tag{5.10}$$

Here, $\mathbb{I}$ is the identity matrix of order 2^{N+1}.

If non-empty sets $A, B \subset \{0,1\}^N$ satisfy the conditions $(C2)^+$ and $(C3)^+$, the set $\mathbb{P}^+_{A,B}$ is clearly a projective measurement.

Definition 5.5 (Error-correcting operator $U_{I,b}$)**.** *For a non-empty set $I \subset [N+1]$, a bit $b \in \{0,1\}$, and integer $m \in \{1,2\}$, we can choose a unitary matrix $U_{I,b}$ of order 2^{N+1} whose mth row is $\langle u^m_{I,b}|$, where*

$$|u^1_{I,b}\rangle \coloneqq \frac{1}{\sqrt{|A^+_{I,b}|}} \sum_{\boldsymbol{a} \in A^+_{I,b}} |\boldsymbol{a}\rangle, \qquad |u^2_{I,b}\rangle \coloneqq \frac{1}{\sqrt{|B^+_{I,b}|}} \sum_{\boldsymbol{b} \in B^+_{I,b}} |\boldsymbol{b}\rangle. \tag{5.11}$$

We call the matrix $U_{I,b}$ an error-correcting operator. For that unitary matrix $U_{I,b}$, the mth row for $3 \leq m \leq 2^{N+1}$ is also denoted as $\langle u^k_{I,b}|$.

From the condition $(C2)^+$, it is clear that $\langle u^i_{I,b}|u^j_{I,b}\rangle = \delta_{i,j}$ holds for any integers $i, j \in \{1,2\}$. Here, $\delta_{i,j}$ is the Kronecker delta. Thus, there exists a unitary matrix $U_{I,b}$ for any non-empty set $I \subset [N+1]$ and any bit $b \in \{0,1\}$. Definition 5.5 represents the unitary matrix used for decoding. Here we have defined the unitary matrix $U_{I,b}$ by specifying the structure, in other words, the action of this unitary matrix $U_{I,b}$ of order 2^{N+1} is defined as the unitary transformation such that

$$|u^1_{I,b}\rangle \mapsto |0\ldots00\rangle, \qquad |u^2_{I,b}\rangle \mapsto |0\ldots01\rangle. \tag{5.12}$$

The decoder for the code $Q_{A,B}$ is defined by combining the projective measurement $\mathbb{P}^+_{A,B}$, the error-correcting operator $U_{I,b}$, and the partial trace Tr_1 as follows.

Definition 5.6 (Decoder $\mathrm{Dec}_{A,B}$)**.** *Let $A, B \subset \{0,1\}^N$ be non-empty sets satisfying the conditions $(C1)^+$, $(C2)^+$, and $(C3)^+$. Define a decoder $\mathrm{Dec}_{A,B}$ as a map from $\rho \in S((\mathbb{C}^2)^{\otimes(N+1)})$ to $\sigma \in S(\mathbb{C}^2)$ constructed by the following steps:*

(Step 1) Perform the projective measurement $\mathbb{P}^+_{A,B}$ under the quantum state ρ. Assume that the outcome is (I,b) and that the state after the measurement is $\rho(I,b)$.

(Step 2) Let $\tilde{\rho} \coloneqq U_{I,b}\rho(I,b)U_{I,b}^{\dagger}$*. Here* $U_{I,b}$ *is the error-correcting operator corresponding to the obtained outcome* (I,b)*.*

(Step 3) At last, return $\sigma \coloneqq \underbrace{\mathrm{Tr}_1 \circ \cdots \circ \mathrm{Tr}_1}_{N\ times}(\tilde{\rho})$*.*

Proof that the decoder $\mathrm{Dec}_{A,B}$ is indeed error-correctable, i.e., satisfies Equation (2.28), is given in Section 5.1.3.

5.1.3 Proof of error-correctability

The following Theorem 5.7 is the main theorem of this chapter and describes the first systematic construction of quantum insertion error-correcting codes with an explicit decoder.

Theorem 5.7. *Let* $A, B \subset \{0,1\}^N$ *be non-empty sets satisfying the conditions* $(\mathrm{C1})^+$*,* $(\mathrm{C2})^+$*, and* $(\mathrm{C3})^+$*. Then, the code* $Q_{A,B}$ *is an* $(\!(N,1)\!)$ *single quantum insertion error-correcting code with the encoder* $\mathrm{Enc}_{A,B}$ *and the decoder* $\mathrm{Dec}_{A,B}$*.*

In other words, for any pure quantum message $|\psi\rangle \in \mathbb{C}^2$*, any single qubit state* $\sigma \in S(\mathbb{C}^2)$*, and any insertion position* $p \in [N+1]$*,*

$$\mathrm{Dec}_{A,B} \circ \mathrm{In}_{p,\sigma} \circ \mathrm{Enc}_{A,B}(|\psi\rangle) = |\psi\rangle \tag{5.13}$$

holds.

Proof. Set $|\Psi\rangle \coloneqq \mathrm{Enc}_{A,B}(|\psi\rangle) \in S((\mathbb{C}^2)^{\otimes N})$ for a pure quantum state $|\psi\rangle \in \mathbb{C}^2$. If we denote

$$\sigma \coloneqq \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = p_{00}|0\rangle\langle 0| + p_{01}|0\rangle\langle 1| + p_{10}|1\rangle\langle 0| + p_{11}|1\rangle\langle 1| \in S(\mathbb{C}^2), \tag{5.14}$$

then we have $p_{00} + p_{11} = 1$ from $\mathrm{Tr}(\sigma) = 1$. Then, from Definitions 5.1 and 3.1, the state after the insertion error $\mathrm{In}_{p,\sigma}$ is

$$\begin{aligned}\mathrm{In}_{p,\sigma}(|\Psi\rangle\langle\Psi|) = \sum_{\substack{b_0\in\{0,1\}\\ b_1\in\{0,1\}}} p_{b_0b_1} \Bigg(&\frac{|\alpha|^2}{|A|} \sum_{\substack{\boldsymbol{a}\in\Delta^+_{p,b_0}(A)\\ \tilde{\boldsymbol{a}}\in\Delta^+_{p,b_1}(A)}} |\boldsymbol{a}\rangle\langle\tilde{\boldsymbol{a}}| + \frac{\alpha\bar{\beta}}{\sqrt{|A||B|}} \sum_{\substack{\boldsymbol{a}\in\Delta^+_{p,b_0}(A)\\ \tilde{\boldsymbol{b}}\in\Delta^+_{p,b_1}(B)}} |\boldsymbol{a}\rangle\langle\tilde{\boldsymbol{b}}| \\ &+ \frac{\bar{\alpha}\beta}{\sqrt{|A||B|}} \sum_{\substack{\boldsymbol{b}\in\Delta^+_{p,b_0}(B)\\ \tilde{\boldsymbol{a}}\in\Delta^+_{p,b_1}(A)}} |\boldsymbol{b}\rangle\langle\tilde{\boldsymbol{a}}| + \frac{|\beta|^2}{|B|} \sum_{\substack{\boldsymbol{b}\in\Delta^+_{p,b_0}(B)\\ \tilde{\boldsymbol{b}}\in\Delta^+_{p,b_1}(B)}} |\boldsymbol{b}\rangle\langle\tilde{\boldsymbol{b}}| \Bigg).\end{aligned} \tag{5.15}$$

Observe Equation (5.15) and calculate the probability of getting each outcome when the projective measurement $\mathbb{P}^+_{A,B}$ is performed on the quantum state $\mathrm{In}_{p,\sigma}(|\Psi\rangle\langle\Psi|)$.

From the definition of $A^+_{I,b}$, i.e., Equation (5.5), we get

$$\bigcup_{\substack{\emptyset\neq I\subset[N+1]\\ b\in\{0,1\}}} A^+_{I,b} = \bigcup_{\substack{i\in[N+1]\\ b\in\{0,1\}}} \Delta^+_{i,b}(A). \tag{5.16}$$

Therefore, the probability of getting the outcome $\emptyset$ is $\mathrm{Tr}\left(P_{\emptyset}\mathrm{In}_{p,\sigma}(|\Psi\rangle\langle\Psi|)\right) = 0$. We also calculate the probability of getting the outcome (I,b) for any non-empty set $I \subset [N+1]$ and any bit

$b \in \{0,1\}$. In the case of $p \notin I$, we have $\mathrm{Tr}\,(P_{I,b}\mathrm{In}_{p,\sigma}(|\Psi\rangle\langle\Psi|)) = 0$ since $\boldsymbol{a} \notin A^{+}_{I,b}$ for any $\boldsymbol{a} \in \Delta^{+}_{i,b}(A)$. On the other hand, in the case of $p \in I$, we have

$$\mathrm{Tr}\,(P_{I,b}\mathrm{In}_{p,\sigma}(|\Psi\rangle\langle\Psi|)) = p_{bb}\left(\frac{|\alpha|^2}{|A|}|A^{+}_{I,b}| + \frac{|\beta|^2}{|B|}|B^{+}_{I,b}|\right) = p_{bb}\frac{|A^{+}_{I,b}|}{|A|} \tag{5.17}$$

since $A^{+}_{I,b} \subset \Delta^{+}_{p,b}(A)$ and the condition $(\mathrm{C1})^{+}$ are satisfied.

In the case of $\mathrm{Tr}\,(P_{I,b}\mathrm{In}_{p,\sigma}(|\Psi\rangle\langle\Psi|)) \neq 0$, the state $\rho(I,b)$ after performing the projective measurement $\mathbb{P}^{+}_{A,B}$ and obtaining the outcome (I,b) can be expressed as

$$\begin{aligned}
\frac{P_{I,b}\mathrm{In}_{p,\sigma}(|\Psi\rangle\langle\Psi|)P_{I,b}}{\mathrm{Tr}(P_{I,b}\mathrm{In}_{p,\sigma}(|\Psi\rangle\langle\Psi|))} &= \frac{|\alpha|^2}{|A^{+}_{I,b}|}\sum_{\substack{\boldsymbol{a}\in A^{+}_{I,b}\\ \tilde{\boldsymbol{a}}\in A^{+}_{I,b}}}|\boldsymbol{a}\rangle\langle\tilde{\boldsymbol{a}}| + \frac{\alpha\bar{\beta}}{\sqrt{|A^{+}_{I,b}||B^{+}_{I,b}|}}\sum_{\substack{\boldsymbol{a}\in A^{+}_{I,b}\\ \tilde{\boldsymbol{b}}\in B^{+}_{I,b}}}|\boldsymbol{a}\rangle\langle\tilde{\boldsymbol{b}}| \\
&\quad + \frac{\bar{\alpha}\beta}{\sqrt{|A^{+}_{I,b}||B^{+}_{I,b}|}}\sum_{\substack{\boldsymbol{b}\in B^{+}_{I,b}\\ \tilde{\boldsymbol{a}}\in A^{+}_{I,b}}}|\boldsymbol{b}\rangle\langle\tilde{\boldsymbol{a}}| + \frac{|\beta|^2}{|B^{+}_{I,b}|}\sum_{\substack{\boldsymbol{b}\in B^{+}_{I,b}\\ \tilde{\boldsymbol{b}}\in B^{+}_{I,b}}}|\boldsymbol{b}\rangle\langle\tilde{\boldsymbol{b}}| && (5.18)\\
&= |\Phi_{I,b}\rangle\langle\Phi_{I,b}| && (5.19)
\end{aligned}$$

from Equations (5.15) and (5.17), where

$$|\Phi_{I,b}\rangle \coloneqq \alpha|u^{1}_{I,b}\rangle + \beta|u^{2}_{I,b}\rangle \in (\mathbb{C}^2)^{\otimes(N+1)}. \tag{5.20}$$

For integers $i,j \in [2^{N+1}]$, we denote the (i,j) element of the matrix $U_{I,b}|\Phi_{I,b}\rangle\langle\Phi_{I,b}|U^{\dagger}_{I,b}$ by $u_{I,b}(i,j)$. We have

$$\begin{aligned}
u_{I,b}(i,j) &= \langle u^{i}_{I,b}|\Phi_{I,b}\rangle\langle\Phi_{I,b}|u^{j}_{I,b}\rangle && (5.21)\\
&= \langle u^{i}_{I,b}|(\alpha|u^{1}_{I,b}\rangle + \beta|u^{2}_{I,b}\rangle)(\bar{\alpha}\langle u^{1}_{I,b}| + \bar{\beta}\langle u^{2}_{I,b}|)|u^{j}_{I,b}\rangle && (5.22)\\
&= (\alpha\langle u^{i}_{I,b}|u^{1}_{I,b}\rangle + \beta\langle u^{i}_{I,b}|u^{2}_{I,b}\rangle)(\bar{\alpha}\langle u^{1}_{I,b}|u^{j}_{I,b}\rangle + \bar{\beta}\langle u^{2}_{I,b}|u^{j}_{I,b}\rangle) && (5.23)\\
&= \begin{cases} |\alpha|^2 & \text{if } (i,j) = (1,1)\\ \alpha\overline{\beta} & \text{if } (i,j) = (1,2)\\ \overline{\alpha}\beta & \text{if } (i,j) = (2,1)\\ |\beta|^2 & \text{if } (i,j) = (2,2)\\ 0 & \text{otherwise}\end{cases} && (5.24)
\end{aligned}$$

by Definition 5.5. Therefore,

$$\begin{aligned}
\mathrm{Dec}_{A,B}\circ\mathrm{In}_{p,\sigma}\circ\mathrm{Enc}_{A,B}(|\psi\rangle\langle\psi|) &= \mathrm{Dec}_{A,B}\circ\mathrm{In}_{p,\sigma}(|\Psi\rangle\langle\Psi|) && (5.25)\\
&= \mathrm{Tr}_1\circ\cdots\circ\mathrm{Tr}_1(U_{I,b}|\Phi_{I,b}\rangle\langle\Phi_{I,b}|U^{\dagger}_{I,b}) && (5.26)\\
&= \mathrm{Tr}_1\circ\cdots\circ\mathrm{Tr}_1(|0\rangle\langle 0|\otimes\cdots\otimes|0\rangle\langle 0|\otimes|\psi\rangle\langle\psi|) && (5.27)\\
&= |\psi\rangle\langle\psi| && (5.28)
\end{aligned}$$

holds for any pure quantum message $|\psi\rangle \in \mathbb{C}^2$, any single qubit state $\sigma \in S(\mathbb{C}^2)$, and any insertion position $p \in [N+1]$ from Definition 5.6. Equation (5.28) is exactly the original quantum message. □

5.2 Relationship to the Nakayama-Hagiwara conditions

This section discusses the relationship between the Nakayama-Hagiwara conditions $(\mathrm{C1})^-$, $(\mathrm{C2})^-$, and $(\mathrm{C3})^-$, which are known as error-correction conditions for single quantum deletion errors introduced in Definition 3.3, and the three conditions $(\mathrm{C1})^+$, $(\mathrm{C2})^+$, and $(\mathrm{C3})^+$ in Definition 5.3.

Lemma 5.8. *The following properties hold:*

1. *The condition* $(\mathrm{C2})^+$ *is equivalent to the condition* $(\mathrm{C2})^-$.
2. *The condition* $(\mathrm{C3})^+$ *is equivalent to the condition* $(\mathrm{C3})^-$.

Proof. The equivalence of the conditions $(\mathrm{C2})^+$ and $(\mathrm{C2})^-$ is clear from the equivalence of classical insertion codes and classical deletion codes, hence here we only show the equivalence of the conditions $(\mathrm{C3})^+$ and $(\mathrm{C3})^-$.

In the following, we can assume that $i_1 \leq i_2$. Take a bit sequence $\boldsymbol{x} \in \Delta^+_{i_1,0}(A) \cap \Delta^+_{i_2,1}(A)$ and denote

$$\boldsymbol{x} = x_1 \dots x_{i_1-1} 0 x_{i_1+1} \dots x_{i_2-1} 1 x_{i_2+1} \dots x_{N+1} \in \{0,1\}^{N+1}. \tag{5.29}$$

Then, $\boldsymbol{x'} \in \Delta^-_{i_2-1,1}(A) \cap \Delta^-_{i_1,0}(A)$ holds for the sequence

$$\boldsymbol{x'} \coloneqq x_1 \dots x_{i_1-1} x_{i_1+1} \dots x_{i_2-1} x_{i_2+1} \dots x_{N+1} \in \{0,1\}^{N-1}. \tag{5.30}$$

Therefore, if $(\mathrm{C3})^-$, then $(\mathrm{C3})^+$ holds.

On the other hand, take a bit sequence $\boldsymbol{y} = y_1 y_2 \dots y_{N-1} \in \Delta^-_{i_1,0}(A) \cap \Delta^-_{i_2,1}(A)$. Then, $\boldsymbol{y'} \in \Delta^+_{i_2+1,1}(A) \cap \Delta^+_{i_1,0}(A)$ holds for the sequence

$$\boldsymbol{y'} \coloneqq y_1 \dots y_{i_1-1} 0 y_{i_1} \dots y_{i_2-1} 1 y_{i_2} \dots y_{N-1}. \tag{5.31}$$

Therefore, if $(\mathrm{C3})^+$, then $(\mathrm{C3})^-$ holds. □

In connection with Lemma 5.8, it can be checked that the conditions $(\mathrm{C1})^+$ and $(\mathrm{C1})^-$ are not equivalent. According to Example 3.6, it is checked that the sets

$$A = \{0000, 1111\}, \qquad B = \{0011, 0101, 0110, 1001, 1010, 1100\} \tag{5.32}$$

satisfy the condition $(\mathrm{C1})^-$, but for these sets A, B, we know that the condition $(\mathrm{C1})^+$ is not satisfied because $A^+_{[5],0} = \{00000\}$ and $B^+_{[5],0} = \emptyset$ hold from Table 5.1. In other words, the proposition " $(\mathrm{C1})^- \implies (\mathrm{C1})^+$ " does not hold.

Although the 4-qubit code constructed by A, B defined in Equation (5.32) cannot be decoded at least by our method, it has been reported that the code is tolerant to single quantum insertion errors by technical decoding [34]. Namely, the non-equivalence between the conditions $(\mathrm{C1})^+$ and $(\mathrm{C1})^-$ does not conclude the non-equivalence between correctability of insertion errors and deletion errors in quantum codes.

Lemmas 3.12 and 3.13, discussed in Section 3.3, can also be used in the case where the conditions $(\mathrm{C2})^+$ and $(\mathrm{C3})^+$ are assumed respectively, by Lemma 5.8. However, it is a rather more natural idea to define the adjacency matrix of the insertion version and proceed with a similar discussion as in Section 3.3. No study using that approach has been reported yet, and it is future work.

Table 5.1: Insertion sets of A, B that give the Hagiwara code

$\Delta^+_{i,b}$	$A = \{0000, 1111\}$		$B = \{0011, 0101, 0110, 1001, 1010, 1100\}$	
	$b = 0$	$b = 1$	$b = 0$	$b = 1$
$i = 1$	$\{00000, 01111\}$	$\{10000, 11111\}$	$\{00011, 00101, 00110, 01001, 01010, 01100\}$	$\{10011, 10101, 10110, 11001, 11010, 11100\}$
$i = 2$	$\{00000, 10111\}$	$\{01000, 11111\}$	$\{00011, 00101, 00110, 10001, 10010, 10100\}$	$\{01011, 01101, 01110, 11001, 11010, 11100\}$
$i = 3$	$\{00000, 11011\}$	$\{00100, 11111\}$	$\{00011, 01001, 01010, 10001, 10010, 11000\}$	$\{00111, 01101, 01110, 10101, 10110, 11100\}$
$i = 4$	$\{00000, 11101\}$	$\{00010, 11111\}$	$\{00101, 01001, 01100, 10001, 10100, 11000\}$	$\{00111, 01011, 01110, 10011, 10110, 11010\}$
$i = 5$	$\{00000, 11110\}$	$\{00001, 11111\}$	$\{00110, 01010, 01100, 10010, 10100, 11000\}$	$\{00111, 01011, 01101, 10011, 10101, 11001\}$

5.3 Examples

This section gives examples of sets $A, B \subset \{0,1\}^N$ of bit sequences that satisfy the conditions $(\mathrm{C1})^+$, $(\mathrm{C2})^+$, and $(\mathrm{C3})^+$.

It was stated in Theorem 3.8 that the set A, B of bit sequences constructed by the following Theorem 5.9 satisfies the Nakayama-Hagiwara conditions. In other words, the quantum code $Q_{A,B}$ constructed by these sets A, B is a single deletion error-correcting code. Theorem 5.9 shows that this code is also tolerant of single quantum insertion errors. Assuming that the receiver can count the number of particles received, he can decide whether to use a deletion decoder or an insertion decoder. Therefore, the quantum code given in Theorem 5.9 is an error-correcting code that can correct a single insertion error or a single deletion error, whichever occurs once in total.

Theorem 5.9. *Suppose that* $|\mathrm{wt}(\boldsymbol{x}_1) - \mathrm{wt}(\boldsymbol{x}_2)| \geq 2$ *and* $|\mathrm{wt}(\boldsymbol{y}_1) - \mathrm{wt}(\boldsymbol{y}_2)| \geq 2$ *for bit sequences* $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \{0,1\}^{m_1}$ *and bit sequences* $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \{0,1\}^{m_2}$ *with integers* $m_1 \geq 2$ *and* $m_2 \geq 2$. *Then, the set*

$$A \coloneqq \{\boldsymbol{x}_1 01 \boldsymbol{y}_1, \boldsymbol{x}_2 01 \boldsymbol{y}_2\}, \qquad B \coloneqq \{\boldsymbol{x}_1 01 \boldsymbol{y}_2, \boldsymbol{x}_2 01 \boldsymbol{y}_1\} \tag{5.33}$$

and the sets

$$A \coloneqq \{\boldsymbol{x}_1 10 \boldsymbol{y}_1, \boldsymbol{x}_2 10 \boldsymbol{y}_2\}, \qquad B \coloneqq \{\boldsymbol{x}_1 10 \boldsymbol{y}_2, \boldsymbol{x}_2 10 \boldsymbol{y}_1\} \tag{5.34}$$

satisfy the conditions $(\mathrm{C1})^+$, $(\mathrm{C2})^+$, *and* $(\mathrm{C3})^+$.

Proof. Since $|A| = |B| = 2$ by the assumption, it is enough to show $|A^+_{I,b}| = |B^+_{I,b}|$ for any non-empty set $I \subset [m_1 + m_2 + 3]$ and any bit $b \in \{0,1\}$. In the following, we fix a bit $b \in \{0,1\}$ and let

$$N_0 \coloneqq [m_1 + 2 - b], \qquad N_1 \coloneqq [m_1 + m_2 + 3] \setminus [m_1 + 2 - b]. \tag{5.35}$$

Table 5.2: Insertion sets of $A = \{000100, 110111\}$ and $B = \{000111, 110100\}$

$\Delta^+_{i,b}$	$A = \{000100, 110111\}$		$B = \{000111, 110100\}$	
	$b = 0$	$b = 1$	$b = 0$	$b = 1$
$i = 1$	$\{0000100, 0110111\}$	$\{1000100, 1110111\}$	$\{0000111, 0110100\}$	$\{1000111, 1110100\}$
$i = 2$	$\{0000100, 1010111\}$	$\{0100100, 1110111\}$	$\{0000111, 1010100\}$	$\{0100111, 1110100\}$
$i = 3$	$\{0000100, 1100111\}$	$\{0010100, 1110111\}$	$\{0000111, 1100100\}$	$\{0010111, 1110100\}$
$i = 4$	$\{0000100, 1100111\}$	$\{0001100, 1101111\}$	$\{0000111, 1100100\}$	$\{0001111, 1101100\}$
$i = 5$	$\{0001000, 1101011\}$	$\{0001100, 1101111\}$	$\{0001011, 1101000\}$	$\{0001111, 1101100\}$
$i = 6$	$\{0001000, 1101101\}$	$\{0001010, 1101111\}$	$\{0001101, 1101000\}$	$\{0001111, 1101010\}$
$i = 7$	$\{0001000, 1101110\}$	$\{0001001, 1101111\}$	$\{0001110, 1101000\}$	$\{0001111, 1101001\}$

In the case of $i \in N_{b'}$, the $m_1 + 2$ component of any bit sequence in $\Delta^+_{i,b}(A)$ is $b' \in \{0, 1\}$. Hence, if $I \cap N_0 \neq \emptyset$ and $I \cap N_1 \neq \emptyset$, then $|\bigcap_{i\in I} \Delta^+_{i,b}(A)| = 0$ holds. Therefore, we obtain $|A^+_{I,b}| = |B^+_{I,b}| = 0$.

On the other hand, for a bit $b' \in \{0, 1\}$, if $I \subset N_{b'}$, we can express

$$|A^+_{I,b}| = \left| \bigcap_{i\in I} \Delta^+_{i,b}(A) \cap \bigcap_{i\in N_{b'}\setminus I} \Delta^+_{i,b}(A)^c \right| \tag{5.36}$$

$$= \begin{cases} \left| \bigcap_{i\in I} \Delta^+_{i,b}(S_0) \cap \bigcap_{i\in N_0\setminus I} \Delta^+_{i,b}(S_0)^c \right| & \text{if } b' = 0, \\ \left| \bigcap_{i\in I'} \Delta^+_{i,b}(S_1) \cap \bigcap_{i\in N_1'\setminus I'} \Delta^+_{i,b}(S_1)^c \right| & \text{if } b' = 1, \end{cases} \tag{5.37}$$

where $S_0 := \{\boldsymbol{x}_1 0, \boldsymbol{x}_2 0\}$, $S_1 := \{1\boldsymbol{y}_1, 1\boldsymbol{y}_2\}$, $I' := \{i - m_1 - 1 \mid i \in I\}$, and $N_1' := \{i - m_1 - 1 \mid i \in N_1\}$. The same result was obtained for $|B^+_{I,b}|$, and it was shown that $|A^+_{I,b}| = |B^+_{I,b}|$ for any non-empty set $I \in [m_1 + m_2 + 3]$. From the above, the condition $(C1)^+$ is satisfied.

Furthermore, since the sets A, B satisfy the conditions $(C2)^-$ and $(C3)^-$ from Theorem 3.8, the conditions $(C2)^+$ and $(C3)^+$ are also satisfied from Lemma 5.8. □

It is a well-known fact that in classical codes any deletion code is also an insertion code, but in general decoding for insertion errors is more difficult than decoding for deletion errors. However, for the quantum codes given in Theorem 5.9, deletions and insertions can be decoded with almost the same effort.

Note also that Theorem 5.9 gives the first example of a quantum insertion code that is not permutation-invariant. The sets $A, B \subset \{0, 1\}^N$ given in Theorem 5.9 that have the shortest length are

$$A = \{000100, 110111\}, \quad B = \{000111, 110100\} \tag{5.38}$$

and

$$A = \{001000, 111011\}, \quad B = \{001011, 111000\}, \tag{5.39}$$

where $N = 6$. In particular, the fact that the sets A, B in Equation (5.38) satisfy the conditions $(C1)^+$, $(C2)^+$, and $(C3)^+$ can be understood by observing Table 5.2. The first quantum deletion code reported by Nakayama [48] was also given as an example of Theorem 59, and it was shown that this code can also correct single quantum insertion errors.

From Fact 2.6 stated earlier and Theorem 6.5 to be discussed later, it follows that the correctability of a single insertion error and a single deletion error are equivalent in quantum codes. In other words, all the quantum codes constructed in Chapter 3 are also single insertion codes. The main contribution of this chapter is the construction of a quantum code with a simple decoder for single insertion errors.

Chapter 6

The equivalence between correctability of deletions and insertions of separable states in quantum codes

The quantum states we deal with in this chapter are not limited to level 2 qubits. That is, let l be any integer greater than or equal to 2, and consider the elements in $S(\mathbb{C}^l)$ to be single qudit states. As a set of qudit labeling, fix the set $\mathcal{L} \coloneqq \{0, 1, \ldots, l-1\}$ for the integer $l \geq 2$.

Chapter 6 is summarized as the contents presented at the 2021 Information Theory Workshop (ITW2021) [70].

6.1 Quantum error-correction using the Kraus operators

This section describes the quantum insertion and deletion errors in the form of quantum channels represented by linear operators to use the Knill-Laflamme conditions known as quantum error-correction conditions.

6.1.1 Knill-Laflamme conditions

Here, we review the Knill-Laflamme quantum error-correction criterion [38]. This section represents the quantum process as a quantum channel that maps a density matrix to a density matrix. A linear map $\Phi : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes N'})$ with positive integers N, N' is a quantum channel, if and only if it is completely positive and trace-preserving. For any quantum channel $\Phi : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes N'})$, there exist linear operators A_i such that for every N-qudit state $\rho \in S((\mathbb{C}^l)^{\otimes N})$,

$$\Phi(\rho) = \sum_i A_i \rho A_i^\dagger \tag{6.1}$$

holds and $\sum_i A_i^\dagger A_i$ is the identity operator on $S((\mathbb{C}^l)^{\otimes N})$. The linear operators A_i are known as the Kraus operators of Φ and their representation is not unique. Given a quantum channel $\mathcal{N}$, a subspace $\mathcal{C}$ of $(\mathbb{C}^l)^{\otimes N}$ is a quantum code that corrects all errors introduced by $\mathcal{N}$, if and only if there exists a quantum channel $\mathcal{R}$ such that

$$\mathcal{R}(\mathcal{N}(\rho)) = \rho \tag{6.2}$$

for every density matrix ρ supported on $\mathcal{C}$. Fact 6.1 below, which gives the necessary and sufficient conditions for quantum error-correction, was originally proved in 1997 by Knill and Laflamme [38].

Fact 6.1 (Knill and Laflamme [38]). *Let $\mathcal{C}$ be a d-dimensional subspace of the complex Hilbert space $(\mathbb{C}^l)^{\otimes N}$ with orthogonal basis vectors $|0_L\rangle, |1_L\rangle, \ldots, |d-1_L\rangle$. Let $\mathcal{N}$ be a quantum channel with the Kraus operators A_i. Suppose that for all i, j there exist $g_{i,j} \in \mathbb{C}$ such that the following conditions hold:*

1. *(Orthogonality condition) For any $a, b \in \{0, 1, \ldots, d-1\}$ with $a \neq b$,*

$$\langle a_L | A_i^\dagger A_j | b_L \rangle = 0. \tag{6.3}$$

2. *(Non-deformation condition) For any $a \in \{0, 1, \ldots, d-1\}$,*

$$\langle a_L | A_i^\dagger A_j | a_L \rangle = g_{i,j} \tag{6.4}$$

Then, for every density matrix ρ supported on $\mathcal{C}$, there exists a quantum channel $\mathcal{R}$ such that $\mathcal{R}(\mathcal{N}(\rho)) = \rho$. The two conditions above are collectively called the Knill-Laflamme conditions or simply the KL conditions.

6.1.2 Quantum insertion/deletion channels

This section also defines the single insertion error and single deletion error for the N-qudit state as in Equations (5.2) and (2.13), respectively. Note in particular that we assume that the quantum state before insertion is pure. In this section, the number of particles before insertion or deletion errors is written as a superscript for the maps $\mathrm{In}_{p_1,\sigma}$ and Tr_{p_2}. In other words, we denote

$$\mathrm{In}^N_{p_1,\sigma}(M) := \mathrm{In}_{p_1,\sigma}(M), \quad \mathrm{Tr}^N_{p_2}(M) := \mathrm{Tr}_{p_2}(M), \tag{6.5}$$

for integers $p_1 \in [N+1]$, $p_2 \in [N]$, a single qudit state $\sigma \in S(\mathbb{C}^l)$, and an N-qudit state $M \in S((\mathbb{C}^l)^{\otimes N})$.

The t-insertion error and the t-deletion error are already defined in Definitions 2.4 and 2.5, respectively, but here they are described as quantum channels in order to use the KL conditions as follows.

Definition 6.2 (t-insertion channel Ins^N_t). *Let t be a positive integer and let*

$$\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_t \in S((\mathbb{C}^l)^{\otimes t}), \tag{6.6}$$

where $\sigma_i \in S(\mathbb{C}^l)$ is a single qudit state for every $i \in [t]$. For a non-empty set $P = \{p_1, p_2, \ldots, p_t\} \subset [N+t]$ with $1 \leq p_1 < p_2 < \cdots < p_t \leq N+t$, define a map $\mathrm{Ins}^N_{P,\sigma} : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes (N+t)})$ as

$$\mathrm{Ins}^N_{P,\sigma}(\rho) := \underbrace{\mathrm{In}^{N+t-1}_{p_t,\sigma_t} \circ \cdots \circ \mathrm{In}^{N+1}_{p_2,\sigma_2} \circ \mathrm{In}^N_{p_1,\sigma_1}}_{t \text{ times}}(\rho), \tag{6.7}$$

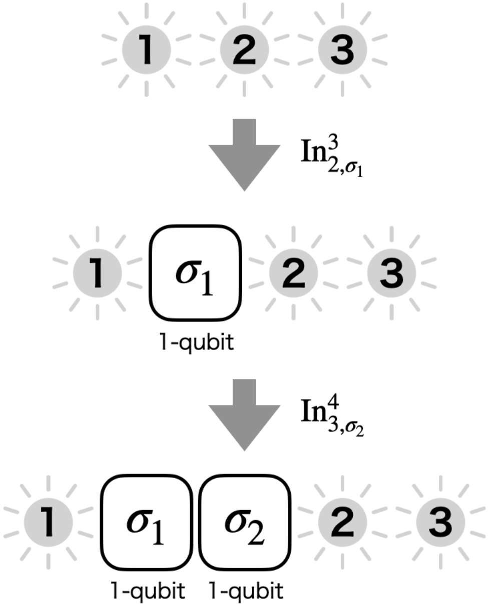

Figure 6.1: Insertion of a separable 2-qubit

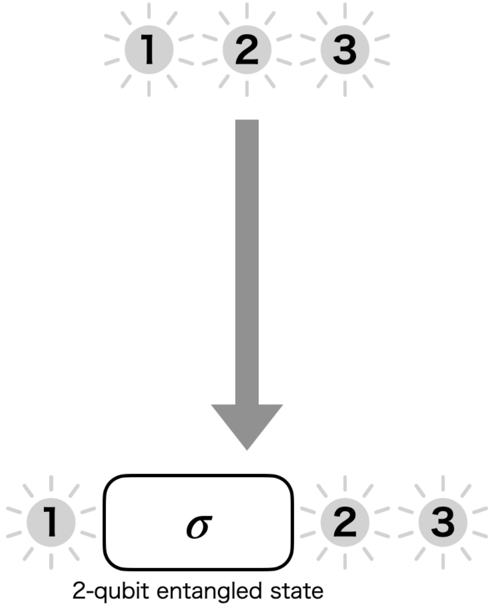

Figure 6.2: Insertion of an entangled 2-qubit

where $\rho \in S((\mathbb{C}^l)^{\otimes N})$ is an N-qudit state. We call the map $\mathrm{Ins}^N_{P,\sigma}$ a (t, P, σ)-insertion error. We define a t-insertion channel Ins^N_t as a convex combination of all (t, P, σ)-insertion errors, where a positive integer t is fixed and $|P| = t$, i.e.,

$$\mathrm{Ins}^N_t(\rho) := \int_{\sigma \in S((\mathbb{C}^l)^{\otimes t})} \mu(\sigma) \sum_{|P|=t} p_\sigma(P) \mathrm{Ins}^N_{P,\sigma}(\rho) d\sigma, \tag{6.8}$$

where $\mu(\sigma)$ and $p_\sigma(P)$ are probability distributions. Note that μ is a measure.

From Fact 2.6, we should consider insertions of arbitrary t-qudit states $\sigma \in S((\mathbb{C}^l)^{\otimes t})$, but here we only consider special insertions restricted to states $\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_t \in S((\mathbb{C}^l)^{\otimes t})$ that are not quantum entangled. This means that operations such as the insertion of two 1-qubit states $\sigma_1, \sigma_2 \in S(\mathbb{C}^l)$ independent of other particles twice, as in Figure 6.1, are allowed, but the insertion of one entangled 2-qubit $\sigma \in S((\mathbb{C}^l)^{\otimes 2})$, as in Figure 6.2, is not considered in this study. It is a future task to give a more general definition of the insertion channel for further discussion. Here, we discuss using this definition, which is limited to the insertion of separable states.

Definition 6.3 (t-deletion channel Del^N_t)**.** *Let $t < N$ be a positive integer. For a non-empty set $P = \{p_1, p_2, \ldots, p_t\} \subset [N]$ with $1 \leq p_1 < p_2 < \cdots < p_t < N$, define a map $\mathrm{Era}^N_P : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes (N-t)})$ as*

$$\mathrm{Era}^N_P(\rho) := \underbrace{\mathrm{Tr}^{N-t+1}_{p_1} \circ \cdots \circ \mathrm{Tr}^{N-1}_{p_{t-1}} \circ \mathrm{Tr}^N_{p_t}}_{t \text{ times}}(\rho), \tag{6.9}$$

where $\rho \in S((\mathbb{C}^l)^{\otimes N})$ is an N-qudit state. We call the map Era^N_P a (t, P)-erasure error. We define a t-deletion channel Del^N_t as a convex combination of all (t, P)-erasure errors, where a *positive integer t is fixed and $|P| = t$, i.e.,*

$$\mathrm{Del}_t^N(\rho) := \sum_{|P|=t} p(P)\mathrm{Era}_P^N(\rho), \tag{6.10}$$

where $p(P)$ is a probability distribution.

The map Era_P^N in Equation (6.9) is exactly the same as the map D_P in Equation (2.37), although the notation is different. Here it is rewritten as Equation (6.10) to contrast with Definition 6.2. That is, we explicitly state that the t-deletion channel does not depend on the deletion position P, but only on the number of deletions t. Extending Definitions 6.2 and 6.3, the 0-insertion and 0-deletion channels are both defined as identity maps. An insertion/deletion channel is defined by combining the insertion and deletion channels defined above.

Definition 6.4 (((t_1, t_2)-insertion/deletion channel $\mathrm{InsDel}_{t_1,t_2}^N$). *Let t_1, t_2 be non-negative integers with $t_2 < N$. We define a (t_1, t_2)-insertion/deletion channel* $\mathrm{InsDel}_{t_1,t_2}^N : S((\mathbb{C}^l)^{\otimes N}) \to S((\mathbb{C}^l)^{\otimes(N+t_1-t_2)})$ *as*

$$\mathrm{InsDel}_{t_1,t_2}^N(\rho) := \mathrm{Ins}_{t_1}^{N-t_2} \circ \mathrm{Del}_{t_2}^N(\rho), \tag{6.11}$$

where $\rho \in S((\mathbb{C}^l)^{\otimes N})$ is an N-qudit state. An insertion/deletion channel is also simply called an insdel channel.

Note that Definition 6.4 represents a special insertion/deletion channel that only considers the insertion of separable states.

6.2 Relationship between deletions and insertions of separable states

A (t_1, t_2)-insdel error-correcting code is a quantum code that can perfectly correct errors introduced by any (t_1, t_2)-insdel channel for non-negative integers t_1, t_2 with $t_2 < N$. In other words, the (t_1, t_2)-insdel error-correcting code is defined as any error $E \in \mathcal{E}$ in Equation (2.28) changed to the map $\mathrm{InsDel}_{t_1,t_2}^N$. We denote by $\mathcal{C} \subset (\mathbb{C}^l)^{\otimes N}$ the d-dimensional (t_1, t_2)-insdel error-correcting code spanned by the orthonormal logical codewords $|0_L\rangle, |1_L\rangle, \ldots, |d-1_L\rangle$. Our main theorem in this chapter concerns (t_1, t_2)-insdel error-correcting codes. In particular, we describe the equivalence between insertion and deletion error-correction capability, a well-known result in classical codes [41], from the perspective of quantum codes.

Theorem 6.5. *Let t_1, t_2, s_1, s_2 be non-negative integers satisfying $t_1 + t_2 = s_1 + s_2$. Then, any (t_1, t_2)-insdel code is an (s_1, s_2)-insdel code.*

Obviously, any (t_1, t_2)-insdel code is a (u_1, u_2)-insdel code if $u_1 \le t_1$ and $u_2 \le t_2$, therefore, Theorem 6.5 means that any (t_1, t_2)-insdel code can correct errors that occur as a combination of no more than $t_1 + t_2$ single deletions or single insertions in total. The remaining part of this chapter is devoted to proving Theorem 6.5.

6.2.1 Lemmas of tensor product calculation

Here, we introduce the rules of tensor product calculations necessary for the proof of Theorem 6.5.

Let $n \geq 0$, $t \geq 1$ be integers and let $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_t\rangle \in (\mathbb{C}^l)^{\otimes t}$, where $|\psi_i\rangle \in \mathbb{C}^l$ for every $i \in [t]$. For a set $P = \{p_1, p_2, \ldots, p_t\} \subset [n+t]$ with $1 \leq p_1 < p_2 < \cdots < p_t \leq n+t$, we define q^{n+t}-by-q^n matrix $I^n_{P,|\Psi\rangle}$ and q^n-by-q^{n+t} matrix $D^n_{P,\langle\Psi|}$ as

$$I^n_{P,|\Psi\rangle} \coloneqq A_1 \otimes A_2 \otimes \cdots \otimes A_{n+t}, \qquad D^n_{P,\langle\Psi|} \coloneqq B_1 \otimes B_2 \otimes \cdots \otimes B_{n+t}, \tag{6.12}$$

respectively, where

$$A_j \coloneqq \begin{cases} |\psi_i\rangle & \text{if } j = p_i \in P, \\ \mathbb{I}_l & \text{if } j \notin P, \end{cases} \qquad B_j \coloneqq \begin{cases} \langle\psi_i| & \text{if } j = p_i \in P, \\ \mathbb{I}_l & \text{if } j \notin P, \end{cases} \tag{6.13}$$

for $j \in [n+t]$. Here, $\mathbb{I}_l$ denotes the identity matrix of order l. Note that the superscript n of these matrices represents the number of $\mathbb{I}_l$'s included as a factor of the tensor product, which is important for the matrix operations used in later discussions. Namely, we have been treating N as the code length, but n here has a slightly different meaning. When $t = 1$, we simply denote $I^n_{\{p_1\},|\Psi\rangle}$ and $D^n_{\{p_1\},\langle\Psi|}$ as $I^n_{p_1,|\psi_1\rangle}$ and $D^n_{p_1,\langle\psi_1|}$, respectively. It is clear that

$$I^n_{P,|\Psi\rangle} = {D^n_{P,\langle\Psi|}}^\dagger, \qquad D^n_{P,\langle\Psi|} = {I^n_{P,|\Psi\rangle}}^\dagger \tag{6.14}$$

from the definitions of the matrices $I^n_{P,|\Psi\rangle}$ and $D^n_{P,\langle\Psi|}$.

The following Lemmas 6.6 and 6.7 are basic calculation rules and can be easily shown by direct calculations as matrices.

Lemma 6.6. *Let $n \geq 0$, $t \geq 1$ be integers. Suppose that $P = \{p_1, p_2, \ldots, p_t\} \subset [n+t]$ with $1 \leq p_1 < p_2 < \cdots < p_t \leq n+t$ and $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_t\rangle \in (\mathbb{C}^l)^{\otimes t}$. Then,*

$$I^n_{P,|\Psi\rangle} = I^{n+t-1}_{p_t,|\psi_t\rangle} I^{n+t-2}_{p_{t-1},|\psi_{t-1}\rangle} \cdots I^{n+1}_{p_2,|\psi_2\rangle} I^n_{p_1,|\psi_1\rangle}, \tag{6.15}$$

$$D^n_{P,\langle\Psi|} = D^n_{p_1,\langle\psi_1|} D^{n+1}_{p_2,\langle\psi_2|} \cdots D^{n+t-2}_{p_{t-1},\langle\psi_{t-1}|} D^{n+t-1}_{p_t,\langle\psi_t|}. \tag{6.16}$$

Proof. Lemma 6.6 is proved by induction on t. It clearly holds for $t = 1$. Let k be a positive integer, and it is enough to prove that

$$I^n_{P_{k+1},|\Psi_{k+1}\rangle} = I^{n+k}_{p_{k+1},|\psi_{k+1}\rangle} I^{n+k-1}_{p_k,|\psi_k\rangle} I^{n+k-2}_{p_{k-1},|\psi_{k-1}\rangle} \cdots I^{n+1}_{p_2,|\psi_2\rangle} I^n_{p_1,|\psi_1\rangle}, \tag{6.17}$$

where $P_{k+1} = \{p_1, p_2, \ldots, p_k, p_{k+1}\} \in [n+k+1]$ and $|\Psi_{k+1}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_k\rangle \otimes |\psi_{k+1}\rangle \in (\mathbb{C}^l)^{\otimes(k+1)}$. By the hypothesis of induction, we have

$$I^n_{P_k,|\Psi_k\rangle} = I^{n+k-1}_{p_k,|\psi_k\rangle} I^{n+k-2}_{p_{k-1},|\psi_{k-1}\rangle} \cdots I^{n+1}_{p_2,|\psi_2\rangle} I^n_{p_1,|\psi_1\rangle}, \tag{6.18}$$

where $P_k = P_{k+1} \setminus \{p_{k+1}\}$ and $|\Psi_k\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_k\rangle \in (\mathbb{C}^l)^{\otimes k}$. On the other hand, by simple calculation, we get

$$I^{n+k}_{p_{k+1},|\psi_{k+1}\rangle} I^n_{P_k,|\Psi_k\rangle} \tag{6.19}$$

$$= (\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_{k+1}-1} \otimes |\psi_{k+1}\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_{k+1}+k+1})(I_{P_k}^{p_{k+1}-k-1} \otimes \mathbb{I}_1 \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_{k+1}+k+1}) \tag{6.20}$$

$$= I_{P_k}^{p_{k+1}-k-1} \otimes |\psi_{k+1}\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_{k+1}+k+1} \tag{6.21}$$

$$= I^n_{P_{k+1},|\Psi_{k+1}\rangle}. \tag{6.22}$$

Hence, Equation (6.17) holds from Equations (6.18) and (6.22). Therefore, Equation (6.15) is proved by induction, and Equation (6.16) is easily shown from Equation (6.14). □

Lemma 6.7. *Let n be a non-negative integer and $p_1, p_2 \in [n+1]$ be integers with $1 \leq p_1 \leq p_2 \leq n+1$, and let $|\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^l$. Then,*

$$I^{n+1}_{p_1,|\psi_1\rangle} I^n_{p_2,|\psi_2\rangle} = I^{n+1}_{p_2+1,|\psi_2\rangle} I^n_{p_1,|\psi_1\rangle}, \tag{6.23}$$

$$D^n_{p_2,\langle\psi_2|} D^{n+1}_{p_1,\langle\psi_1|} = D^n_{p_1,\langle\psi_1|} D^{n+1}_{p_2+1,\langle\psi_2|}. \tag{6.24}$$

Proof. Simple calculations give

$$\begin{aligned} I^{n+1}_{p_1,|\psi_1\rangle} I^n_{p_2,|\psi_2\rangle} &= (\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes |\psi_1\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1} \otimes \mathbb{I}_l \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1}) \\ &\quad (\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes \mathbb{I}_1 \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1} \otimes |\psi_2\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1}) && (6.25) \\ &= \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes |\psi_1\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1} \otimes |\psi_2\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1} && (6.26) \end{aligned}$$

and

$$\begin{aligned} I^{n+1}_{p_2+1,|\psi_2\rangle} I^n_{p_1,|\psi_1\rangle} &= (\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes \mathbb{I}_l \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1} \otimes |\psi_2\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1}) \\ &\quad (\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes |\psi_1\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1} \otimes \mathbb{I}_1 \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1}) && (6.27) \\ &= \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes |\psi_1\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1} \otimes |\psi_2\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1}. && (6.28) \end{aligned}$$

Hence, we obtain $I^{n+1}_{p_1,|\psi_1\rangle} I^n_{p_2,|\psi_2\rangle} = I^{n+1}_{p_2+1,|\psi_2\rangle} I^n_{p_1,|\psi_1\rangle}$ from Equations (6.26) and (6.28). Equation (6.24) is easily shown from Equation (6.14). □

The following Lemmas 6.8 and 6.9 give commutation rules for the insertion Kraus operators I and the deletion Kraus operators D, and the proof is by simple algebraic calculations.

Lemma 6.8. *Let n be a non-negative integer and $p_1, p_2 \in [n+1]$ be integers and let $|\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^l$. Then,*

$$D^n_{p_2,\langle\psi_2|} I^n_{p_1,|\psi_1\rangle} = \begin{cases} I^{n-1}_{p_1,|\psi_1\rangle} D^{n-1}_{p_2-1,\langle\psi_2|} & \text{if } p_1 < p_2, \\ \langle\psi_2|\psi_1\rangle \mathbb{I}_{l^n} & \text{if } p_1 = p_2, \\ I^{n-1}_{p_1-1,|\psi_1\rangle} D^{n-1}_{p_2,\langle\psi_2|} & \text{if } p_1 > p_2. \end{cases} \tag{6.29}$$

Proof. When $p_1 < p_2$, simple calculations give

$$D^n_{p_2,\langle\psi_2|} I^n_{p_1,|\psi_1\rangle} = (\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes \mathbb{I}_l \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1-1} \otimes \langle\psi_2| \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1})$$

$$(\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes |\psi_1\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1-1} \otimes \mathbb{I}_l \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1}) \tag{6.30}$$

$$= \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes |\psi_1\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1-1} \otimes \langle\psi_2| \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1} \tag{6.31}$$

and

$$I^{n-1}_{p_1,|\psi_1\rangle} D^{n-1}_{p_2-1,\langle\psi_2|} = (\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes |\psi_1\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1-1} \otimes \mathbb{I}_1 \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1})$$

$$(\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes \mathbb{I}_1 \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1-1} \otimes \langle\psi_2| \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1}) \tag{6.32}$$

$$= \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_1-1} \otimes |\psi_1\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p_2-p_1-1} \otimes \langle\psi_2| \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p_2+1}. \tag{6.33}$$

Hence, we obtain $D^n_{p_2,\langle\psi_2|} I^n_{p_1,|\psi_1\rangle} = I^{n-1}_{p_1,|\psi_1\rangle} D^{n-1}_{p_2-1,\langle\psi_2|}$ from Equations (6.31) and (6.33).

The case $p_1 > p_2$ is shown similarly, and the case $p_1 = p_2$ trivially holds. □

Lemma 6.9. *Let n be a non-negative integer and $p_1, p_2 \in [n+1]$ be integers and let $|\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^l$. Then,*

$$I^n_{p_1,|\psi_1\rangle} D^n_{p_2,\langle\psi_2|} = \begin{cases} D^{n+1}_{p_2+1,\langle\psi_2|} I^{n+1}_{p_1,|\psi_1\rangle} & \text{if } p_1 \le p_2, \\ D^{n+1}_{p_2,\langle\psi_2|} I^{n+1}_{p_1+1,|\psi_1\rangle} & \text{if } p_1 \ge p_2. \end{cases} \tag{6.34}$$

Proof. Lemma 6.9 can be derived immediately from Lemma 6.8. □

Note that in the case of $p_1 = p_2$, the equation $D^{n+1}_{p_2+1,\langle\psi_2|} I^{n+1}_{p_1,|\psi_1\rangle} = D^{n+1}_{p_2,\langle\psi_2|} I^{n+1}_{p_1+1,|\psi_1\rangle}$ holds, because $|\psi_1\rangle \otimes \langle\psi_2| = \langle\psi_2| \otimes |\psi_1\rangle = |\psi_1\rangle\langle\psi_2|$ for any $|\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^l$.

6.2.2 Kraus operators for insertion/deletion errors

Here, we elucidate the properties of the Kraus operators of insdel channels. Recall that $\mathcal{L} := \{0, 1, \ldots, l-1\}$ for the integer $l \ge 2$.

Lemma 6.10. *For any N-qudit state $\rho \in S((\mathbb{C}^l)^{\otimes N})$, the state after inserting a separable t-qudit state $\sigma \in S((\mathbb{C}^l)^{\otimes t})$ in the positions labeled by $P \subset [N+t]$ can be expressed as*

$$\mathrm{Ins}^N_{P,\sigma}(\rho) = \sum_{\boldsymbol{a} \in \mathcal{L}^t} p(\boldsymbol{a}) I^N_{P,U|\boldsymbol{a}\rangle} \rho I^N_{P,U|\boldsymbol{a}\rangle}{}^{\dagger} \tag{6.35}$$

with some probability distribution $p(\boldsymbol{a})$ for $\boldsymbol{a} \in \mathcal{L}^t$ and unitary matrix U.

Proof. For any $n \ge 1$, $p \in [n+1]$, $|\psi\rangle \in \mathbb{C}^l$, and $\boldsymbol{x} \in \mathcal{L}^n$,

$$I^n_{p,|\psi\rangle}|\boldsymbol{x}\rangle = (\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p-1} \otimes |\psi\rangle \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p+1})(|x_1 \ldots x_{p-1}\rangle \otimes \mathbb{I}_1 \otimes |x_p \ldots x_n\rangle) \tag{6.36}$$

$$= |x_1\rangle \otimes \cdots \otimes |x_{p-1}\rangle \otimes |\psi\rangle \otimes |x_p\rangle \otimes \cdots \otimes |x_n\rangle \tag{6.37}$$

holds. Let $\tau = \sum_{i\in\mathcal{L}} c_i|\psi_i\rangle\langle\psi_i|$ be the spectral decomposition of $\tau \in S(\mathbb{C}^l)$, where c_i are probabilities and $\langle\psi_i|\psi_j\rangle = \delta_{i,j}$ for $i,j \in \mathcal{L}$. Here, $\delta_{i,j}$ is the Kronecker delta function. Note that there exists a unitary matrix U such that $|\psi_i\rangle = U|i\rangle$ for every $i \in \mathcal{L}$. For a quantum state $\rho = \sum_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{L}^n} m_{\boldsymbol{x},\boldsymbol{y}}|\boldsymbol{x}\rangle\langle\boldsymbol{y}|$,

$$\mathrm{In}^n_{p,\tau}(\rho) = \sum_{i\in\mathcal{L}} c_i \left(\sum_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{L}^n} m_{\boldsymbol{x},\boldsymbol{y}}|\boldsymbol{x}_i\rangle\langle\boldsymbol{y}_i| \right) \tag{6.38}$$

$$= \sum_{i\in\mathcal{L}} c_i I^n_{p,|\psi_i\rangle} \left(\sum_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{L}^n} m_{\boldsymbol{x},\boldsymbol{y}}|\boldsymbol{x}\rangle\langle\boldsymbol{y}| \right) {I^n_{p,|\psi_i\rangle}}^\dagger \tag{6.39}$$

$$= \sum_{i\in\mathcal{L}} c_i I^n_{p,U|i\rangle} \rho {I^n_{p,U|i\rangle}}^\dagger \tag{6.40}$$

holds, where

$$|\boldsymbol{x}_i\rangle = |x_1\rangle \otimes \cdots \otimes |x_{p-1}\rangle \otimes |\psi_i\rangle \otimes |x_p\rangle \otimes \cdots \otimes |x_n\rangle, \tag{6.41}$$

$$|\boldsymbol{y}_i\rangle = |y_1\rangle \otimes \cdots \otimes |y_{p-1}\rangle \otimes |\psi_i\rangle \otimes |y_p\rangle \otimes \cdots \otimes |y_n\rangle. \tag{6.42}$$

Assume that $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_t$ and $\sigma_k = \sum_{i\in\mathcal{L}} c^k_i|\psi^k_i\rangle\langle\psi^k_i|$ and $|\psi^k_i\rangle = U_k|i\rangle$ for $k \in [t]$. By Definition 6.2 and Lemma 6.6, we obtain

$$\mathrm{Ins}^N_{P,\sigma}(\rho) = \mathrm{In}^{N+t-1}_{p_t,\sigma_t} \circ \cdots \circ \mathrm{In}^{N+1}_{p_2,\sigma_2} \circ \mathrm{In}^N_{p_1,\sigma_1}(\rho) \tag{6.43}$$

$$= \sum_{i_t\in\mathcal{L}} \cdots \sum_{i_2\in\mathcal{L}} \sum_{i_1\in\mathcal{L}} c^t_{i_t} \ldots c^2_{i_2} c^1_{i_1}$$
$$I^{N+t-1}_{p_t,U_t|i_t\rangle} \ldots I^{N+1}_{p_2,U_2|i_2\rangle} I^N_{p_1,U_1|i_1\rangle} \rho {I^N_{p_1,U_1|i_1\rangle}}^\dagger {I^{N+1}_{p_2,U_2|i_2\rangle}}^\dagger \ldots {I^{N+t-1}_{p_t,U_t|i_t\rangle}}^\dagger \tag{6.44}$$

$$= \sum_{\boldsymbol{a}\in\mathcal{L}^t} p(\boldsymbol{a}) I^N_{P,U|\boldsymbol{a}\rangle} \rho {I^N_{P,U|\boldsymbol{a}\rangle}}^\dagger, \tag{6.45}$$

where $p(\boldsymbol{a}) = c^1_{a_1} c^2_{a_2} \ldots c^t_{a_t}$ and $U = U_1 \otimes U_2 \otimes \cdots \otimes U_t$. □

From Definition 6.2 and Lemma 6.10, we get the Kraus form for insertion channels, which is represented as

$$\mathrm{Ins}^N_t(\rho) = \int_U \sum_{P,\boldsymbol{a}} \mu_1(U,P,\boldsymbol{a}) I^N_{P,U|\boldsymbol{a}\rangle} \rho {I^N_{P,U|\boldsymbol{a}\rangle}}^\dagger dU, \tag{6.46}$$

where μ_1 is a probability distribution.

Lemma 6.11. *For any N-qudit state $\rho \in S((\mathbb{C}^l)^{\otimes N})$, the state after deleting the qudits labeled by $P \subset [N]$ is*

$$\mathrm{Era}^N_P(\rho) = \sum_{\boldsymbol{a}\in\mathcal{L}^t} D^{N-t}_{P,\langle\boldsymbol{a}|} \rho {D^{N-t}_{P,\langle\boldsymbol{a}|}}^\dagger. \tag{6.47}$$

Proof. For any $n \geq 2$, $p \in [n]$, $a \in \mathcal{L}$, and $\boldsymbol{x} \in \mathcal{L}^n$, we have

$$D^{n-1}_{p,\langle a|}|\boldsymbol{x}\rangle = (\underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{p-1} \otimes \langle a| \otimes \underbrace{\mathbb{I}_l \otimes \cdots \otimes \mathbb{I}_l}_{n-p})(|x_1 \ldots x_{p-1}\rangle \otimes |x_p\rangle \otimes |x_{p+1} \ldots x_n\rangle) \tag{6.48}$$

$$= |x_1\rangle \otimes \cdots \otimes |x_{p-1}\rangle \otimes \langle a|x_p\rangle \otimes |x_{p+1}\rangle \otimes \cdots \otimes |x_n\rangle \tag{6.49}$$
$$= \langle a|x_p\rangle |x_1\rangle \otimes \cdots \otimes |x_{p-1}\rangle \otimes |x_{p+1}\rangle \cdots \otimes |x_n\rangle. \tag{6.50}$$

For a quantum state $\rho = \sum_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{L}^n} m_{\boldsymbol{x},\boldsymbol{y}} |\boldsymbol{x}\rangle\langle\boldsymbol{y}|$,

$$\mathrm{Tr}_p^n(\rho) = \sum_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{L}^n} m_{\boldsymbol{x},\boldsymbol{y}} \mathrm{Tr}(|x_p\rangle\langle y_p|)|\boldsymbol{x}'\rangle\langle\boldsymbol{y}'| \tag{6.51}$$
$$= \sum_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{L}^n} m_{\boldsymbol{x},\boldsymbol{y}} \left(\sum_{a\in\mathcal{L}} \langle a|x_p\rangle\langle y_p|a\rangle \right) |\boldsymbol{x}'\rangle\langle\boldsymbol{y}'| \tag{6.52}$$
$$= \sum_{a\in\mathcal{L}} \sum_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{L}^n} m_{\boldsymbol{x},\boldsymbol{y}} \langle a|x_p\rangle |\boldsymbol{x}'\rangle\langle\boldsymbol{y}'|\langle y_p|a\rangle \tag{6.53}$$
$$= \sum_{a\in\mathcal{L}} D^{n-1}_{p,\langle a|} \left(\sum_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{L}^n} m_{\boldsymbol{x},\boldsymbol{y}} |\boldsymbol{x}\rangle\langle\boldsymbol{y}| \right) D^{n-1\,\dagger}_{p,\langle a|} \tag{6.54}$$
$$= \sum_{a\in\mathcal{L}} D^{n-1}_{p,\langle a|} \rho D^{n-1\,\dagger}_{p,\langle a|} \tag{6.55}$$

holds, where

$$|\boldsymbol{x}'\rangle = |x_1\rangle \otimes \cdots \otimes |x_{p-1}\rangle \otimes |x_{p+1}\rangle \otimes \cdots \otimes |x_n\rangle, \tag{6.56}$$
$$|\boldsymbol{y}'\rangle = |y_1\rangle \otimes \cdots \otimes |y_{p-1}\rangle \otimes |y_{p+1}\rangle \otimes \cdots \otimes |y_n\rangle. \tag{6.57}$$

Therefore, we have

$$\mathrm{Era}_P^N(\rho) = \mathrm{Tr}_{p_1}^{N-t+1} \circ \cdots \circ \mathrm{Tr}_{p_{t-1}}^{N-1} \circ \mathrm{Tr}_{p_t}^{N}(\rho) \tag{6.58}$$
$$= \sum_{a_1\in\mathcal{L}} \cdots \sum_{a_{t-1}\in\mathcal{L}} \sum_{a_t\in\mathcal{L}}$$
$$D^{N-t}_{p_1,\langle a_1|} \cdots D^{N-2}_{p_{t-1},\langle a_{t-1}|} D^{N-1}_{p_t,\langle a_t|} \rho D^{N-1}_{p_t,\langle a_t|}{}^{\dagger} D^{N-2}_{p_{t-1},\langle a_{t-1}|}{}^{\dagger} \cdots D^{N-t}_{p_1,\langle a_1|}{}^{\dagger} \tag{6.59}$$
$$= \sum_{\boldsymbol{a}\in\mathcal{L}^t} D^{N-t}_{P,\langle \boldsymbol{a}|} \rho D^{N-t\,\dagger}_{P,\langle \boldsymbol{a}|} \tag{6.60}$$

by Definition 6.3 and Lemma 6.6. □

From Definition 6.3 and Lemma 6.11, we get the Kraus form for deletion channels, which is represented as

$$\mathrm{Del}_t^N(\rho) = \sum_{P,\boldsymbol{a}} p(P) D^{N-t}_{P,\langle \boldsymbol{a}|} \rho D^{N-t\,\dagger}_{P,\langle \boldsymbol{a}|} \tag{6.61}$$
$$= \int_U \sum_{P,\boldsymbol{a}} \mu_2(U, P, \boldsymbol{a}) D^{N-t}_{P,\langle \boldsymbol{a}|U^\dagger} \rho D^{N-t}_{P,\langle \boldsymbol{a}|U^\dagger}{}^{\dagger} dU. \tag{6.62}$$

Note that by writing in integral form with a probability distribution μ_2 as in Equation (6.62), the deletion channel can be regarded as having an infinite number of Kraus operators, just like the insertion channel expressed in Equation (6.46).

Lemma 6.12 below describes the intuitive result that deleting an inserted qudit leaves the original state unchanged. This can be proved by calculating according to the definition, but it can also be easily shown by calculating using the Kraus operators. This lemma indicates that the operation of deleting after insertion is also included in the insdel error described in Definition 6.4.

Lemma 6.12. *Let $P = \{p\} \subset [N+1]$ and $\sigma \in S(\mathbb{C}^l)$. Then, for any N-qudit state $\rho \in S((\mathbb{C}^l)^{\otimes N})$,*

$$\mathrm{Era}_P^{N+1} \circ \mathrm{Ins}_{P,\sigma}^N(\rho) = \rho. \tag{6.63}$$

Proof. Let $\sigma = \sum_{i\in\mathcal{L}} c_i |\psi_i\rangle\langle\psi_i|$ be the spectral decomposition of $\sigma \in S(\mathbb{C}^l)$, where c_i are probabilities and $\langle\psi_i|\psi_j\rangle = \delta_{i,j}$ for $i, j \in \mathcal{L}$. Simple calculations show that

$$\begin{aligned}
\mathrm{Era}_P^{N+1} \circ \mathrm{Ins}_{P,\sigma}^N(\rho) &= \sum_{a\in\mathcal{L}} D_{p,\langle a|}^N \left(\sum_{i\in\mathcal{L}} c_i I_{p,|\psi_i\rangle}^N \rho {I_{p,|\psi_i\rangle}^N}^\dagger \right) {D_{p,\langle a|}^N}^\dagger && (6.64)\\
&= \sum_{a\in\mathcal{L}} \sum_{i\in\mathcal{L}} c_i D_{p,\langle a|}^N I_{p,|\psi_i\rangle}^N \rho {I_{p,|\psi_i\rangle}^N}^\dagger {D_{p,\langle a|}^N}^\dagger && (6.65)\\
&= \sum_{a\in\mathcal{L}} \sum_{i\in\mathcal{L}} c_i \langle a|\psi_i\rangle \mathbb{I}_{l^N} \rho {\mathbb{I}_{l^N}}^\dagger \langle\psi_i|a\rangle && (6.66)\\
&= \rho \sum_{a\in\mathcal{L}} \sum_{i\in\mathcal{L}} c_i \langle a|\psi_i\rangle\langle\psi_i|a\rangle && (6.67)\\
&= \rho \sum_{i\in\mathcal{L}} c_i \mathrm{Tr}(|\psi_i\rangle\langle\psi_i|) && (6.68)\\
&= \rho \sum_{i\in\mathcal{L}} c_i && (6.69)\\
&= \rho && (6.70)
\end{aligned}$$

from Equations (6.40) and (6.55) and Lemma 6.8. Therefore, Lemma 6.12 holds. □

For any N-qudit state $\rho \in S((\mathbb{C}^l)^{\otimes N})$, the state after insdel error described in Definition 6.4 can be calculated as

$$\mathrm{InsDel}_{t_1,t_2}^N(\rho) = \iint_{U,V} \sum_{P,Q,\boldsymbol{a},\boldsymbol{b}} \mu_{\boldsymbol{u}} I_{P,U|\boldsymbol{a}\rangle}^{N-t_2} D_{Q,\langle\boldsymbol{b}|V^\dagger}^{N-t_2} \rho {D_{Q,\langle\boldsymbol{b}|V^\dagger}^{N-t_2}}^\dagger {I_{P,U|\boldsymbol{a}\rangle}^{N-t_2}}^\dagger dU dV, \tag{6.71}$$

where $\mu_{\boldsymbol{u}}$ is a non-negative value that depends on $\boldsymbol{u} = (U, V, P, Q, \boldsymbol{a}, \boldsymbol{b})$ by Equations (6.46) and (6.62). We can easily calculate the matrix $I_{P,U|\boldsymbol{a}\rangle}^{N-t_2} D_{Q,\langle\boldsymbol{b}|V^\dagger}^{N-t_2}$ such as in the example below.

Example 6.13. *Let $N = 4$, $t_1 = 3$, $t_2 = 2$, $P = \{2, 3, 5\}$, $Q = \{1, 3\}$, and let*

$$\begin{aligned}
U|\boldsymbol{a}\rangle &= |\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \in (\mathbb{C}^2)^{\otimes 3}, && (6.72)\\
V|\boldsymbol{b}\rangle &= |\Phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \in (\mathbb{C}^2)^{\otimes 2}. && (6.73)
\end{aligned}$$

Then, $I_{P,U|\boldsymbol{a}\rangle}^2 D_{Q,\langle\boldsymbol{b}|V^\dagger}^2$ is one of the Kraus operators corresponding to the action of inserting the second, third, and fifth components after deleting the first and third components as follows:

$$|x_1x_2x_3x_4\rangle \to |x_2x_4\rangle \to |x_2\psi_1\psi_2x_4\psi_3\rangle. \tag{6.74}$$

The matrices $I_{P,U|\boldsymbol{a}\rangle}^2$ and $D_{Q,\langle\boldsymbol{b}|V^\dagger}^2$ can be expressed as

$$\begin{aligned}
I_{P,|\Psi\rangle}^2 &= \qquad\quad \mathbb{I}_2 \otimes |\psi_1\rangle \otimes |\psi_2\rangle \otimes \mathbb{I}_2 \otimes |\psi_3\rangle, && (6.75)\\
D_{Q,\langle\Phi|}^2 &= \langle\phi_1| \otimes \mathbb{I}_2 \otimes \langle\phi_2| \qquad\quad \otimes \mathbb{I}_2. && (6.76)
\end{aligned}$$

Note that in Equations (6.75) and (6.76) the numbers of identity matrices $\mathbb{I}_2$ included as factors of the tensor product in the matrices $I^2_{P,|\Psi\rangle}$ and $D^2_{Q,\langle\Phi|}$ are both 2, and they are aligned in the same column. Therefore, we obtain

$$I^2_{P,|\Psi\rangle} D^2_{Q,\langle\Phi|} = \langle\phi_1| \otimes \mathbb{I}_2 \otimes |\psi_1\rangle \otimes |\psi_2\rangle \otimes \langle\phi_2| \otimes \mathbb{I}_2 \otimes |\psi_3\rangle. \tag{6.77}$$

Note that $\boldsymbol{x}\boldsymbol{y}^\dagger = \boldsymbol{x} \otimes \boldsymbol{y}^\dagger = \boldsymbol{y}^\dagger \otimes \boldsymbol{x}$ holds for any vectors $\boldsymbol{x}, \boldsymbol{y}$.

6.2.3 Proof of the equivalence

Here, we will give the proof of Theorem 6.5, the main theorem of this chapter. From Equation (6.71) and Lemma 6.6, the Kraus operator $A_{\boldsymbol{u}}$ for the (t_1, t_2)-insdel channels can be expressed as a product of $t_1 + t_2$ block matrices

$$\begin{aligned} A_{\boldsymbol{u}} &= \sqrt{\mu_{\boldsymbol{u}}} I^{N-t_2}_{P,U|\boldsymbol{a}\rangle} D^{N-t_2}_{Q,\langle\boldsymbol{b}|V^\dagger} \\ &= \sqrt{\mu_{\boldsymbol{u}}} I^{N-t_2+t_1-1}_{p_{t_1},|\psi_{t_1}\rangle} \dots I^{N-t_2+1}_{p_2,|\psi_2\rangle} I^{N-t_2}_{p_1,|\psi_1\rangle} D^{N-t_2}_{q_1,\langle\phi_1|} D^{N-t_2+1}_{q_2,\langle\phi_2|} \dots D^{N-1}_{q_{t_2},\langle\phi_{t_2}|}, \end{aligned} \tag{6.78}$$

where

$$U|\boldsymbol{a}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_{t_1}\rangle \in (\mathbb{C}^l)^{\otimes t_1}, \tag{6.79}$$

$$V|\boldsymbol{b}\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_{t_2}\rangle \in (\mathbb{C}^l)^{\otimes t_2}. \tag{6.80}$$

Therefore, the KL conditions for the (t_1, t_2)-insdel channel can be written as

$$\langle i_L| {D^{N-t_2}_{Q,\langle\boldsymbol{b}|V^\dagger}}^\dagger {I^{N-t_2}_{P,U|\boldsymbol{a}\rangle}}^\dagger I^{N-t_2}_{P',U'|\boldsymbol{a}'\rangle} D^{N-t_2}_{Q',\langle\boldsymbol{b}'|V'^\dagger} |j_L\rangle = \delta_{i,j} g_{\boldsymbol{u},\boldsymbol{v}}. \tag{6.81}$$

for all $\boldsymbol{u} = (U, V, P, Q, \boldsymbol{a}, \boldsymbol{b}), \boldsymbol{v} = (U', V', P', Q', \boldsymbol{a}', \boldsymbol{b}')$ and all $i, j \in \{0, 1, \dots, d-1\}$.

The following two lemmas will help us establish the equivalence of insertion and deletions errors under the KL conditions.

Lemma 6.14. *For non-negative integers t_1, t_2 with $t_1 \geq 1$, any (t_1, t_2)-insdel quantum code is a $(t_1 - 1, t_2 + 1)$-insdel quantum code.*

Proof. From Equation (6.78), we denote any two Kraus operators $B_{\boldsymbol{u}}, B_{\boldsymbol{v}}$ for the $(t_1 - 1, t_2 + 1)$-insdel channel as

$$\begin{aligned} B_{\boldsymbol{u}} &= I^{N-(t_2+1)}_{P,U|\boldsymbol{a}\rangle} D^{N-(t_2+1)}_{Q,\langle\boldsymbol{b}|V^\dagger} \\ &= \underbrace{I^{N-t_2+t_1-3}_{p_{t_1-1},|\psi_{t_1-1}\rangle} \dots I^{N-t_2}_{p_2,|\psi_2\rangle} I^{N-t_2-1}_{p_1,|\psi_1\rangle}}_{(t_1-1) \text{ matrices}} \underbrace{D^{N-t_2-1}_{q_1,\langle\phi_1|} D^{N-t_2}_{q_2,\langle\phi_2|} \dots D^{N-1}_{q_{t_2+1},\langle\phi_{t_2+1}|}}_{(t_2+1) \text{ matrices}}, \end{aligned} \tag{6.82}$$

$$\begin{aligned} B_{\boldsymbol{v}} &= I^{N-(t_2+1)}_{P',U'|\boldsymbol{a}'\rangle} D^{N-(t_2+1)}_{Q',\langle\boldsymbol{b}'|V'^\dagger} \\ &= \underbrace{I^{N-t_2+t_1-3}_{p'_{t_1-1},|\psi'_{t_1-1}\rangle} \dots I^{N-t_2}_{p'_2,|\psi'_2\rangle} I^{N-t_2-1}_{p'_1,|\psi'_1\rangle}}_{(t_1-1) \text{ matrices}} \underbrace{D^{N-t_2-1}_{q'_1,\langle\phi'_1|} D^{N-t_2}_{q'_2,\langle\phi'_2|} \dots D^{N-1}_{q'_{t_2+1},\langle\phi'_{t_2+1}|}}_{(t_2+1) \text{ matrices}}, \end{aligned} \tag{6.83}$$

where

$$U|\boldsymbol{a}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_{t_1-1}\rangle \in (\mathbb{C}^l)^{\otimes(t_1-1)}, \tag{6.84}$$

$$
\begin{aligned}
B_{\boldsymbol{u}}^{\dagger} B_{\boldsymbol{v}} &= D^{\dagger} \ldots D^{\dagger} D^{\dagger}\ I^{\dagger}\ I^{\dagger} \ldots I^{\dagger}\ I^{\dagger}\ I^{\dagger}\ I\ I\ I \ldots I\ I\ D\ D \ldots D \\
&= I \ldots I\ I\ D\ D \ldots D\ D\ D\ I\ I\ I \ldots I\ I\ D\ D \ldots D \\
&= I \ldots I\ D\ I\ D \ldots D\ D\ D\ I\ I\ I \ldots I\ D\ I\ D \ldots D \\
&= I \ldots I\ D\ D\ I \ldots D\ D\ D\ I\ I\ I \ldots D\ I\ I\ D \ldots D \\
&= \ldots \\
&= I \ldots I\ D\ D\ D \ldots I\ D\ D\ I\ I\ D \ldots I\ I\ I\ D \ldots D \\
&= I \ldots I\ D\ D\ D \ldots D\ I\ D\ I\ D\ I \ldots I\ I\ I\ D \ldots D \\
&= I \ldots I\ D\ D\ D \ldots D\ D\ I\ D\ I\ I \ldots I\ I\ I\ D \ldots D \\
&= I \ldots I\ D\ D\ D \ldots D\ D\ D\ I\ I\ I \ldots I\ I\ I\ D \ldots D \\
&= D^{\dagger} \ldots D^{\dagger}\ I^{\dagger}\ I^{\dagger}\ I^{\dagger} \ldots I^{\dagger}\ I^{\dagger}\ I^{\dagger}\ I\ I\ I \ldots I\ I\ I\ D \ldots D.
\end{aligned}
$$

Figure 6.3: Calculation of $B_{\boldsymbol{u}}^{\dagger} B_{\boldsymbol{v}}$

$$V|\boldsymbol{b}\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_{t_2+1}\rangle \in (\mathbb{C}^l)^{\otimes(t_2+1)}, \tag{6.85}$$

$$U'|\boldsymbol{a}'\rangle = |\psi_1'\rangle \otimes |\psi_2'\rangle \otimes \cdots \otimes |\psi_{t_1-1}'\rangle \in (\mathbb{C}^l)^{\otimes(t_1-1)}, \tag{6.86}$$

$$V'|\boldsymbol{b}'\rangle = |\phi_1'\rangle \otimes |\phi_2'\rangle \otimes \cdots \otimes |\phi_{t_2+1}'\rangle \in (\mathbb{C}^l)^{\otimes(t_2+1)}. \tag{6.87}$$

Note that when considering the KL condition, we can ignore the constant multiple of the Kraus operator. By noting Equation (6.14) and using Lemma 6.9 repeatedly, we can calculate $B_{\boldsymbol{u}}^{\dagger} B_{\boldsymbol{v}}$ using Figure 6.3. Here, superscripts and subscripts are omitted to avoid confusion. However, if we consider the superscripts, we can use Lemma 6.9 in each matrix operation.

Thus, $B_{\boldsymbol{u}}^{\dagger} B_{\boldsymbol{v}}$ can be expressed as

$$B_{\boldsymbol{u}}^{\dagger} B_{\boldsymbol{v}} = \underbrace{D^{\dagger} \ldots D^{\dagger}}_{t_2+1} \underbrace{I^{\dagger} \ldots I^{\dagger}}_{t_1-1} \underbrace{I \ldots I}_{t_1-1} \underbrace{D \ldots D}_{t_2+1} \tag{6.88}$$

$$= \underbrace{D^{\dagger} \ldots D^{\dagger}}_{t_2} \underbrace{I^{\dagger} \ldots I^{\dagger}}_{t_1} \underbrace{I \ldots I}_{t_1} \underbrace{D \ldots D}_{t_2}. \tag{6.89}$$

Furthermore, repeatedly applying Lemma 6.7 gives $B_{\boldsymbol{u}}^{\dagger} B_{\boldsymbol{v}} = A_{\boldsymbol{u}'}^{\dagger} A_{\boldsymbol{v}'}$ for some Kraus operators $A_{\boldsymbol{u}'}, A_{\boldsymbol{v}'}$ of the (t_1, t_2)-insdel channel. From Equation (6.81), we get

$$\langle i_L | B_{\boldsymbol{u}}^{\dagger} B_{\boldsymbol{v}} | j_L \rangle = \delta_{i,j} g_{\boldsymbol{u}',\boldsymbol{v}'} \tag{6.90}$$

for all $i, j \in \{0, 1, \ldots, d-1\}$ and all $\boldsymbol{u}', \boldsymbol{v}'$. Since the pair $(\boldsymbol{u}', \boldsymbol{v}')$ is uniquely determined by $(\boldsymbol{u}, \boldsymbol{v})$, the KL conditions for the $(t_1 - 1, t_2 + 1)$-insdel code hold for every $\boldsymbol{u}, \boldsymbol{v}$. Fact 6.1 implies that $\mathcal{C}$ is a $(t_1 - 1, t_2 + 1)$-insdel code. □

Lemma 6.15. *For non-negative integers t_1, t_2 with $t_2 \geq 1$, any (t_1, t_2)-insdel quantum code is a $(t_1 + 1, t_2 - 1)$-insdel quantum code.*

Proof. As in the proof of Lemma 6.14, denote any two Kraus operators $C_{\boldsymbol{u}}, C_{\boldsymbol{v}}$ for the $(t_1 + 1, t_2 - 1)$-insdel channel as

$$C_{\boldsymbol{u}} = I_{P,U|\boldsymbol{a}\rangle}^{N-(t_2-1)} D_{Q,\langle \boldsymbol{b}|V^{\dagger}}^{N-(t_2-1)} \tag{6.91}$$

$$
\begin{aligned}
C_{\boldsymbol{u}}^{\dagger}C_{\boldsymbol{v}} &= D^{\dagger}\ \dots D^{\dagger}\ I^{\dagger}\ I^{\dagger}\ I^{\dagger}\ \dots\ I^{\dagger}\ I^{\dagger}\ \colorbox{lightgray}{$I^{\dagger}$}\ \colorbox{lightgray}{I}\ I\ I\ \dots\ I\ I\ I\ D\ \dots D\\
&= I\ \dots I\ D\ D\ D\ \dots\ D\ D\ \colorbox{lightgray}{D}\ \colorbox{lightgray}{I}\ I\ I\ \dots\ I\ I\ I\ D\ \dots D\\
&= I\ \dots I\ D\ D\ D\ \dots\ D\ D\ \colorbox{lightgray}{I}\ \colorbox{lightgray}{D}\ I\ I\ \dots\ I\ I\ I\ D\ \dots D\\
&= I\ \dots I\ D\ D\ D\ \dots\ D\ D\ \colorbox{lightgray}{I}\ I\ \colorbox{lightgray}{D}\ I\ \dots\ I\ I\ I\ D\ \dots D\\
&= I\ \dots I\ D\ D\ D\ \dots\ D\ D\ \colorbox{lightgray}{I}\ I\ I\ \colorbox{lightgray}{D}\ \dots\ I\ I\ I\ D\ \dots D\\
&= \dots\\
&= I\ \dots I\ D\ D\ D\ \dots\ D\ D\ \colorbox{lightgray}{I}\ I\ I\ I\ \dots\ \colorbox{lightgray}{D}\ I\ I\ D\ \dots D\\
&= I\ \dots I\ D\ D\ D\ \dots\ D\ D\ \colorbox{lightgray}{I}\ I\ I\ I\ \dots\ I\ \colorbox{lightgray}{D}\ I\ D\ \dots D\\
&= I\ \dots I\ D\ D\ D\ \dots\ D\ D\ \colorbox{lightgray}{I}\ I\ I\ I\ \dots\ I\ I\ \colorbox{lightgray}{D}\ D\ \dots D\\
&= I\ \dots I\ D\ D\ D\ \dots\ D\ \colorbox{lightgray}{I}\ D\ I\ I\ I\ \dots\ I\ I\ \colorbox{lightgray}{D}\ D\ \dots D\\
&= I\ \dots I\ D\ D\ D\ \dots\ \colorbox{lightgray}{I}\ D\ D\ I\ I\ I\ \dots\ I\ I\ \colorbox{lightgray}{D}\ D\ \dots D\\
&= \dots\\
&= I\ \dots I\ D\ D\ \colorbox{lightgray}{I}\ \dots\ D\ D\ D\ I\ I\ I\ \dots\ I\ I\ \colorbox{lightgray}{D}\ D\ \dots D\\
&= I\ \dots I\ D\ \colorbox{lightgray}{I}\ D\ \dots\ D\ D\ D\ I\ I\ I\ \dots\ I\ I\ \colorbox{lightgray}{D}\ D\ \dots D\\
&= I\ \dots I\ \colorbox{lightgray}{I}\ D\ D\ \dots\ D\ D\ D\ I\ I\ I\ \dots\ I\ I\ \colorbox{lightgray}{D}\ D\ \dots D\\
&= D^{\dagger}\ \dots D^{\dagger}\colorbox{lightgray}{$D^{\dagger}$}\ I^{\dagger}\ I^{\dagger}\ \dots\ I^{\dagger}\ I^{\dagger}\ I^{\dagger}\ I\ I\ I\ \dots\ I\ I\ \colorbox{lightgray}{D}\ D\ \dots D.
\end{aligned}
$$

Figure 6.4: Calculation of $C_{\boldsymbol{u}}^{\dagger}C_{\boldsymbol{v}}$

$$= \underbrace{I^{N-t_2+t_1+1}_{p_{t_1+1},|\psi_{t_1+1}\rangle} \dots I^{N-t_2}_{p_2,|\psi_2\rangle} I^{N-t_2+1}_{p_1,|\psi_1\rangle}}_{(t_1+1)\text{ matrices}} \underbrace{D^{N-t_2+1}_{q_1,\langle\phi_1|} D^{N-t_2}_{q_2,\langle\phi_2|} \dots D^{N-1}_{q_{t_2-1},\langle\phi_{t_2-1}|}}_{(t_2-1)\text{ matrices}}, \tag{6.92}$$

$$C_{\boldsymbol{v}} = I^{N-(t_2-1)}_{P',U'|\boldsymbol{a}'\rangle} D^{N-(t_2-1)}_{Q',\langle\boldsymbol{b}'|V'^{\dagger}} \tag{6.93}$$

$$= \underbrace{I^{N-t_2+t_1+1}_{p'_{t_1+1},|\psi'_{t_1+1}\rangle} \dots I^{N-t_2}_{p'_2,|\psi'_2\rangle} I^{N-t_2+1}_{p'_1,|\psi'_1\rangle}}_{(t_1+1)\text{ matrices}} \underbrace{D^{N-t_2+1}_{q'_1,\langle\phi'_1|} D^{N-t_2}_{q'_2,\langle\phi'_2|} \dots D^{N-1}_{q'_{t_2-1},\langle\phi'_{t_2-1}|}}_{(t_2-1)\text{ matrices}}. \tag{6.94}$$

This time using Lemma 6.8 repeatedly, we can calculate $C_{\boldsymbol{u}}^{\dagger}C_{\boldsymbol{v}}$ as Figure 6.4.

Note that, by Lemma 6.8, $DI = \langle\psi_2|\psi_1\rangle \mathbb{I}_{l^n}$ may occur in the middle of the calculation. Thus, using $c_{\boldsymbol{u},\boldsymbol{v}} \in \mathbb{C}$ depending on $(\boldsymbol{u},\boldsymbol{v})$, $C_{\boldsymbol{u}}^{\dagger}C_{\boldsymbol{v}}$ can be expressed as

$$C_{\boldsymbol{u}}^{\dagger}C_{\boldsymbol{v}} = \underbrace{D^{\dagger}\dots D^{\dagger}}_{t_2-1}\underbrace{I^{\dagger}\dots I^{\dagger}}_{t_1+1}\underbrace{I\dots I}_{t_1+1}\underbrace{D\dots D}_{t_2-1} \tag{6.95}$$

$$= \begin{cases} c_{\boldsymbol{u},\boldsymbol{v}}\underbrace{D^{\dagger}\dots D^{\dagger}}_{t_2-1}\underbrace{I^{\dagger}\dots I^{\dagger}}_{t_1}\underbrace{I\dots I}_{t_1}\underbrace{D\dots D}_{t_2-1}, \\ c_{\boldsymbol{u},\boldsymbol{v}}\underbrace{D^{\dagger}\dots D^{\dagger}}_{t_2-1}\underbrace{I^{\dagger}\dots I^{\dagger}}_{t_1-1}\underbrace{I\dots I}_{t_1}\underbrace{D\dots D}_{t_2}, \\ \underbrace{D^{\dagger}\dots D^{\dagger}}_{t_2}\underbrace{I^{\dagger}\dots I^{\dagger}}_{t_1}\underbrace{I\dots I}_{t_1}\underbrace{D\dots D}_{t_2}. \end{cases} \tag{6.96}$$

By repeatedly applying Lemma 6.7, we obtain $C_{\boldsymbol{u}}^{\dagger}C_{\boldsymbol{v}} = c_{\boldsymbol{u},\boldsymbol{v}}A_{\boldsymbol{u}'}^{\dagger}A_{\boldsymbol{v}'}$ for some Kraus operators $A_{\boldsymbol{u}'}, A_{\boldsymbol{v}'}$ for the (t_1,t_2)-insdel channel. Note that for any non-negative integers $s_1 \le t_1$ and $s_2 \le t_2$, the (t_1,t_2)-insdel code is an (s_1,s_2)-insdel code. From Equation (6.81), we get

$$\langle i_L|C_{\boldsymbol{u}}^{\dagger}C_{\boldsymbol{v}}|j_L\rangle = \delta_{i,j}c_{\boldsymbol{u},\boldsymbol{v}}g_{\boldsymbol{u}',\boldsymbol{v}'} \tag{6.97}$$

for all $i, j \in \{0, 1, \ldots, d-1\}$ and all $\boldsymbol{u}', \boldsymbol{v}'$. Since the pair $(\boldsymbol{u}', \boldsymbol{v}')$ is uniquely determined by $(\boldsymbol{u}, \boldsymbol{v})$, the KL conditions for the (t_1+1, t_2-1)-insdel code hold for every $\boldsymbol{u}, \boldsymbol{v}$. From Fact 6.1, it is shown that $\mathcal{C}$ is a (t_1+1, t_2-1)-insdel code. □

By Lemmas 6.14 and 6.15, we have completed the proof of Theorem 6.5. From the proof above, it can be seen that Equation (6.81) is satisfied for permutation-invariant codes for example [51, 52]. The analysis of decoding methods for insertion errors in permutation-invariant codes is a future task.

Chapter 7

Conclusion

In this thesis, quantum error-correcting codes for insertion/deletion errors were discussed. First, the fundamentals of quantum information theory based on quantum mechanical properties were explained, and the definition of quantum insertion/deletion codes was given.

In Chapter 3, the Nakayama-Hagiwara conditions, known as the error-correcting conditions for single quantum deletion errors, were discussed and examples of new codes were given. Furthermore, by analyzing the NH conditions in terms of the adjacency matrices of graphs, the NH codes whose length is less than or equal to 5 were determined.

Chapter 4 focused on quantum codes with permutation-invariance and gave the construction conditions for quantum codes that are tolerant to multiple deletion errors. Furthermore, we gave examples that satisfy these conditions, and for the first time, we constructed quantum codes that can correct two or more deletion errors. This construction includes multiple deletion error-correcting codes that can also correct unitary errors, and it is the first example from that perspective as well. The reason why we focused on permutation-invariance in this work is that the state after deletions does not depend on the deletion position, so there is a high possibility that it can be easily decoded. It is a future task to construct non-PI multiple-deletion codes by considering the state after 2-deletions of codes that do not have permutation-invariance. A clue is the decoding algorithm for Hagiwara's 4-qubit insertion codes. This code is a rare example of successful decoding in a technical way, even though the state after insertion error depends on the insertion position, so it has potential for application to other codes.

Chapter 5 gave a systematic construction method of quantum codes for single insertion errors and succeeded in discovering many quantum insertion codes, of which only one example had been found before. We also described the relationship between the three conditions introduced here and the NH conditions. The insertion codes presented here have been found in fewer numbers than the codes constructed based on the NH conditions, and it is expected that the present conditions will be studied in more detail in the future.

Chapter 6 showed the equivalence between the correctability of deletions and special insertions in quantum codes. This was proved by defining the Kraus operators for deletion and insertion errors and using the Knill-Laflamme conditions, which is also novel in itself. However, the fact that the general insertion error is not represented by the Kraus operators in this study is still controversial and is a future task. It should also be mentioned that all cases of single insertion errors are considered when the quantum state before insertion is assumed to be pure.

In other words, due to the facts shown in Chapter 6, we can say that the already known single deletion error-correcting codes can also correct single insertion errors. In addition, if we go into the proof of the Knill-Laflamme conditions, we may be able to understand the method of decoding, however, it is generally difficult to describe the decoder simply. For example, for the decoding of deletion errors given in Chapters 3 and 4, there is naturally a decoder for insertions, but how to represent it is not known, and we would like to study it in the future. In this respect, the decoder for insertion errors given in Chapter 5 can be described with the same simplicity as that for deletion errors, which is also an advantage in terms of practicality.

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