Appell-Lauricella Hypergeometric Functions over Finite Fields and Algebraic Varieties

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(千葉大学審査学位論文)

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AKIO NAKAGAWA

Overview

In 1812, Gauss defined hypergeometric functions ${}_2F_1$ over $\mathbb{C}$ by the power series

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(1)_m (c)_m} z^m. $$

Here, $a, b, c \in \mathbb{C}$ ($c \notin \mathbb{Z}_{\leq 0}$) are complex parameters and $(a)_m = \Gamma(a+m)/\Gamma(a)$ is the Pochhammer symbol. Generalized hypergeometric functions ${}_pF_q$ are generalizations of ${}_2F_1$ with more parameters. Each of these functions satisfies a linear differential equation, which is called the hypergeometric differential equation. Furthermore, the one has integral representations.

In 1880, Appell defined hypergeometric functions F_1, F_2, F_3 and F_4 of two variables, which are generalizations of the functions ${}_2F_1$. About ten years later, Lauricella defined hypergeometric functions $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ of n-variables, which are ${}_2F_1$ when $n = 1$ and are respectively F_2, F_3, F_4 and F_1 when $n = 2$. For example, Lauricella's functions $F_D^{(n)}$ are defined by the power series

$$ F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; z_1, \dots, z_n\right) := \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{(a)_{m_1 + \cdots + m_n} \prod_{i=1}^{n} (b_i)_{m_i}}{(c)_{m_1 + \cdots + m_n} \prod_{i=1}^{n} (1)_{m_i}} \prod_{i=1}^{n} z_i^{m_i}. $$

Similarly to the Gauss hypergeometric functions, each of Lauricella's functions satisfies a system of linear differential equations. Lauricella's functions also have integral representations such as

$$ \begin{aligned} & B(a, c-a) F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; z_1, \dots, z_n\right) \\ & = \int_0^1 \Big(\prod_{i=1}^{n} (1 - z_i u)^{-b_i} \Big) u^{a-1} (1-u)^{c-a-1} \, du, \end{aligned} $$

where $B(s, t)$ is the beta function.

In the 1980s, finite field analogues of the hypergeometric functions began to be studied due to Koblitz, Greene and Katz, independently. Recently, McCarthy, Fuselier-Long-Ramakrishna-Swisher-Tu and Otsubo also defined independently the one-variable hypergeometric functions over finite fields. Otsubo also defined Lauricella's functions over finite fields. The hypergeometric functions over a finite field κ are functions from κ to $\overline{\mathbb{Q}}$, and the parameters of them are characters of the multiplicative group $\kappa^\times$ of κ.

In this paper, we use the Otsubo's hypergeometric functions over finite fields. For characters α, ν of $\kappa^\times$, he introduced a finite field analogue of the Pochhammer symbol as

$$(\alpha)_\nu := \frac{g(\alpha\nu)}{g(\alpha)},$$

and its modification $(\alpha)^\circ_\nu$, where $g(\alpha)$ is the Gauss sum. The hypergeometric functions over κ are defined by using them. For example, Lauricella's functions $F_D^{(n)}$ over κ are defined by

$$F_D^{(n)}\left({\alpha;\beta_1,\dots,\beta_n \atop \gamma};\lambda_1,\dots,\lambda_n\right) = \frac{1}{(1-q)^n}\sum_{\nu_i\in\widehat{\kappa^\times}} \frac{(\alpha)_{\nu_1\cdots\nu_n}\prod_{i=1}^n(\beta_i)_{\nu_i}}{(\gamma)^\circ_{\nu_1\cdots\nu_n}\prod_{i=1}^n(\varepsilon)^\circ_{\nu_i}}\prod_{i=1}^n \nu_i(\lambda_i),$$

where $\widehat{\kappa^\times} = \mathrm{Hom}(\kappa^\times, \overline{\mathbb{Q}})$ (put $\eta(0)=0$ for all $\eta\in\widehat{\kappa^\times}$) and ε is the trivial character. By Otsubo, finite field analogues of the integral representations of the one-variable hypergeometric functions were given such as

$$j(\alpha,\overline{\alpha}\gamma)\,{}_2F_1\left({\alpha,\beta \atop \gamma};\lambda\right) = -\sum_{u\in\kappa^\times}\overline{\beta}(1-\lambda u)\alpha(u)\overline{\alpha}\gamma(1-u) \quad (\lambda\neq 0).$$

Here, $\overline{\alpha} := \alpha^{-1}$ and $j(\alpha,\beta)$ is the Jacobi sum.

By the integral representations, when the parameters are in $\mathbb{Q}$, a hypergeometric function over $\mathbb{C}$ is regarded as a complex period of a certain algebraic variety. On the other hand, when one-variable, the hypergeometric function over κ is regarded as the trace of Frobenius acting on the l-adic étale cohomology of the algebraic variety over κ. For example, the functions (ϕ is the quadratic character)

$${}_2F_1\left({\frac12,\frac12 \atop 1};\lambda\right) \quad (\text{over } \mathbb{C}) \quad \text{and} \quad {}_2F_1\left({\phi,\phi \atop \varepsilon};\lambda\right) \quad (\text{over } \kappa)$$

are respectively connected to the Legendre elliptic curve

$$y^2 = x(1-x)(1-\lambda x)$$

over $\mathbb{C}$ and over κ. The Frobenius trace is related with the number of κ-rational points by the Grothendieck-Lefschetz trace formula.

Our main goals in this paper are to prove finite field analogues of the integral representations of Appell-Lauricella functions, and to express the numbers of rational points on certain algebraic varieties over κ in terms of Appell-Lauricella functions over κ.

Part 1 is a survey on Appell-Lauricella functions over $\mathbb{C}$. In Section 1, we recall definitions and some properties of these functions. In Section 2, we introduce the varieties over $\mathbb{C}$ connected to these functions. More precisely, a Lauricella's $F_D^{(n)}$ is connected to each of a curve and an n-dimensional hypersurface. For example, the curve is defined by an equation of the form

$$y^d = x^a(1-x)^c\prod_{i=1}^n(1-\lambda_i x)^{b_i} \quad (a,b_i,c,d\in\mathbb{Z}_{\geq 1}).$$

A Lauricella's $F_A^{(n)}$ is connected to each of two-type n-dimensional hypersurfaces. A Lauricella's $F_B^{(n)}$ is connected to an n-dimensional hypersurface. A Lauricella's $F_C^{(n)}$ is connected to an n-dimensional hypersurface, and when $n=2$, is also connected to another surface.

Part 2 is the main part of this thesis. In Section 2, we recall definitions and some properties of generalized hypergeometric functions and Appell-Lauricella functions over κ. In Section 3, we prove finite field analogues of the integral representations of Appell-Lauricella functions. For example, we obtain

$$\begin{aligned} & -j(\alpha,\overline{\alpha}\gamma)F_D^{(n)}\left(\begin{matrix}\alpha;\beta_1,\dots,\beta_n\\ \gamma\end{matrix};\lambda_1,\dots,\lambda_n\right)\\ &= \sum_{u\in\kappa^\times}\Big(\prod_{i=1}^n\overline{\beta_i}(1-\lambda_i u)\Big)\alpha(u)\overline{\alpha}\gamma(1-u) \quad (\lambda_i\neq 0).\end{aligned}$$

Furthermore, by using this, we prove a finite field analogue of Karlsson's formula (see Subsection 1.4 in Part 1 for Karlsson's formula).

In Section 4, we consider the curve and the hypersurfaces mentioned above over κ. A subgroup of $\kappa^\times$ acts on each of the varieties. Thus, each the number of κ_r-rational points (κ_r is a degree r extension) on the varieties is decomposes into χ-components $N_r(\chi)$ for characters χ of the subgroup. Each the Artin L-function associated to χ of the varieties is a generating function of $N_r(\chi)$ ($r\geq 1$). By the results in Section 3, we express the number $N_1(\chi)$ of each the variety in terms of the corresponding Appell-Lauricella function over κ. We also express the Artin L-function of each the variety in terms of the corresponding Appell-Lauricella function over κ_r ($r\geq 1$).

Furthermore, we closely look at a smooth projective model of the curve. By the Grothendieck-Lefschetz trace formula, the Artin L-function associated to χ of the curve is the characteristic polynomial of the Frobenius acting on the χ-eigenspace of the first l-adic étale cohomology. We compute the dimension of the space, and prove that the Artin L-function is a polynomial of degree $n+1$. As a consequence,we obtain relations in field extensions of Lauricella's $F_D^{(n)}$ over κ.

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CONTENTS

Part 1. Appell-Lauricella hypergeometric functions over $\mathbb{C}$

1. APPELL-LAURICELLA HYPERGEOMETRIC FUNCTIONS OVER $\mathbb{C}$

1.1. **Definitions and integral representations.** For complex parameters $a_1, \ldots, a_p$, $b_1, \ldots, b_q$ such that $b_1, \ldots, b_q \notin \mathbb{Z}_{\leq 0}$ and complex variable z, the one variable hypergeometric function (the Gauss hypergeometric function when $p = 2$ and $q = 1$) is defined by the power series

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right) := \sum_{m=0}^{\infty} \frac{(a_1)_m (a_2)_m \cdots (a_p)_m}{(1)_m (b_1)_m \cdots (b_q)_m} z^m, $$

where $(a)_m = a(a+1)\cdots(a+m-1) = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol, in particular $(1)_m = m!$. It converges when $|z| < 1$, and if $\mathrm{Re}(\sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i) > 0$ then it converges when $|z| = 1$. Note that if $a_i \in \mathbb{Z}_{\leq 0}$ for some i, then $(a_i)_m = 0$

for $m \geq -a$, and hence the function is a polynomial. When $0 < \mathrm{Re}(a) < \mathrm{Re}(c)$, the Gauss hypergeometric function has the following integral representation (cf. [27, (1.6.6)]):

$$B(a, c-a)\,{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \int_0^1 (1-zu)^{-b} u^{a-1} (1-u)^{c-a-1}\, du, \tag{1.1}$$

where $B(s,t)$ is the beta function.

Let a, a_i, b, b_j, c, c_k be complex parameters such that $c, c_k \notin \mathbb{Z}_{\leq 0}$ and $z_1, \dots, z_n$ be complex variables. For brevity, we write $\boldsymbol{z}$ for $(z_1, \dots, z_n)$. Lauricella's functions $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$ are defined by the following power series:

$$F_A^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c_1, \dots, c_n \end{matrix}; \boldsymbol{z}\right) := \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} \prod_i (b_i)_{m_i}}{\prod_i (c_i)_{m_i} (1)_{m_i}} \prod_i z_i^{m_i},$$

which converges when $\sum_i |z_i| < 1$,

$$F_B^{(n)}\left(\begin{matrix} a_1, \dots, a_n; b_1, \dots, \beta_n \\ c \end{matrix}; \boldsymbol{z}\right) := \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{\prod_i (a_i)_{m_i} (b_i)_{m_i}}{(c)_{m_1+\cdots+m_n} \prod_i (1)_{m_i}} \prod_i z_i^{m_i},$$

which converges when $|z_i| < 1$ for all $i = 1, \dots, n$,

$$F_C^{(n)}\left(\begin{matrix} a; b \\ c_1, \dots, c_n \end{matrix}; \boldsymbol{z}\right) := \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b)_{m_1+\cdots+m_n}}{\prod_i (c_i)_{m_i} (1)_{m_i}} \prod_i z_i^{m_i},$$

which converges when $\sum_i \sqrt{|z_i|} < 1$,

$$F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; \boldsymbol{z}\right) := \sum_{m_1, \dots, m_n = 0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} \prod_i (b_i)_{m_i}}{(c)_{m_1+\cdots+m_n} \prod_i (1)_{m_i}} \prod_i z_i^{m_i},$$

which converges when $|z_i| < 1$ for all i. When $n = 2$, these are called Appell's functions and these are written

$$\begin{aligned}
F_1(a; b_1, b_2; c; x, y) &= F_D^{(2)}\left(\begin{matrix} a; b_1, b_2 \\ c \end{matrix}; x, y\right), \\
F_2(a; b_1, b_2; c_1, c_2; x, y) &= F_A^{(2)}\left(\begin{matrix} a; b_1, b_2 \\ c_1, c_2 \end{matrix}; x, y\right), \\
F_3(a_1, a_2; b_1, b_2; c; x, y) &= F_B^{(2)}\left(\begin{matrix} a_1, a_2; b_1, b_2 \\ c \end{matrix}; x, y\right), \\
F_4(a; b; c_1, c_2; x, y) &= F_C^{(2)}\left(\begin{matrix} a; b \\ c_1, c_2 \end{matrix}; x, y\right).
\end{aligned}$$

Lauricella's functions $F_D^{(n)}$ have following integral representations, which are the integral representation (1.1) when $n = 1$.

Proposition 1.1 (cf. [18, Theorem 3.4.1]).

(i) *If* $0 < \mathrm{Re}(a) < \mathrm{Re}(c)$, *then*

$$\begin{aligned}
&B(a, c-a) F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; \boldsymbol{z}\right) \\
&= \int_0^1 \Big(\prod_{i=1}^{n} (1 - z_i u)^{-b_i}\Big) u^{a-1} (1-u)^{c-a-1}\, du.
\end{aligned}$$

(ii) *If* $0 < \mathrm{Re}(b_i)$ *for all* i *and* $\mathrm{Re}(\sum_i b_i) < \mathrm{Re}(c)$, *then*

$$\frac{\left(\prod_{i=1}^n \Gamma(b_i)\right)\Gamma(c-\sum_{i=1}^n b_i)}{\Gamma(c)} F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; \boldsymbol{z}\right)$$
$$= \int_\Delta \Big(1-\sum_{i=1}^n u_i z_i\Big)^{-a}\Big(\prod_{i=1}^n u_i^{b_i-1}\Big)\Big(1-\sum_{i=1}^n u_i\Big)^{c-\sum_i b_i-1} du_1 \cdots du_n,$$

where $\Delta := \{(u_1, \dots, u_n) \in \mathbb{R}^n \mid u_i \geq 0,\ \sum_i u_i \leq 1\}$.

Proof. Note that (cf. [27, (2.2.2.2)])

$$ {}_1F_0\left(\begin{matrix} a \\ \ \end{matrix}; z\right) = (1-z)^{-a}. \tag{1.2}$$

(i) By (1.2),

$$\int_0^1 \Big(\prod_{i=1}^n (1-z_i u)^{-b_i}\Big) u^{a-1}(1-u)^{c-a-1}\, du$$
$$= \sum_{m_1,\dots,m_n=0}^{\infty} \int_0^1 \Big(\prod_{i=1}^n \frac{(b_i)_{m_i}}{(1)_{m_i}}(z_i u)^{m_i}\Big) u^{a-1}(1-u)^{c-a-1}\, du$$
$$= \sum_{m_1,\dots,m_n} \Big(\prod_i \frac{(b_i)_{m_i}}{(1)_{m_i}} z_i^{m_i}\Big) \int_0^1 u^{a+\sum_i m_i-1}(1-u)^{c-a-1}\, du$$
$$= \sum_{m_1,\dots,m_n} \Big(\prod_i \frac{(b_i)_{m_i}}{(1)_{m_i}} z_i^{m_i}\Big) \frac{\Gamma(a+\sum_i m_i)\Gamma(c-a-1)}{\Gamma(c+\sum_i m_i)}$$
$$= B(a, c-a) F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; \boldsymbol{z}\right).$$

(ii) Using (1.2),

$$\Big(1-\sum_{i=1}^n z_i u_i\Big)^{-a} = \Big(1-\sum_{i=2}^n z_i u_i\Big)^{-a} {}_1F_0\left(\begin{matrix} a \\ \ \end{matrix}; \frac{z_1 u_1}{1-\sum_{i=2}^n z_i u_i}\right)$$
$$= \sum_{m_1} \frac{(a)_{m_1}}{(1)_{m_1}} (z_1 u_1)^{m_1} \Big(1-\sum_{i=2}^n z_i u_i\Big)^{-a-m_1}.$$

Repeating this and noting that

$$(a)_{m+n} = (a)_m (a+m)_n, \tag{1.3}$$

we have

$$\Big(1-\sum_{i=1}^n z_i u_i\Big)^{-a} = \sum_{m_1,\dots,m_n} \frac{(a)_{m_1+\cdots+m_n}}{\prod_i (1)_{m_i}} \prod_i (z_i u_i)^{m_i}.$$

Therefore,

$$\int_\Delta \Big(1-\sum_i z_i u_i\Big)^{-a}\Big(\prod_i u_i^{b_i-1}\Big)\Big(1-\sum_i u_i\Big)^{c-\sum_i b_i-1} du_1 \cdots du_n,$$
$$= \sum_{m_1,\dots,m_n} \frac{(a)_{m_1+\cdots+m_n}}{\prod_i (1)_{m_i}} \Big(\prod_i z_i^{m_i}\Big) \int_\Delta \Big(\prod_i u_i^{b_i+m_i-1}\Big)\Big(1-\sum_i u_i\Big)^{c-\sum_i b_i-1} du_1 \cdots du_n$$

$$= \sum_{m_1,\dots,m_n} \frac{(a)_{m_1+\cdots+m_n}}{\prod_i (1)_{m_i}} \Big(\prod_i z_i^{m_i}\Big) \frac{\Big(\prod_i \Gamma(b_i+m_i)\Big)\Gamma(c-\sum_i b_i)}{\Gamma(c+\sum_i m_i)}$$

$$= \frac{\Big(\prod_i \Gamma(b_i)\Big)\Gamma(c-\sum_i b_i)}{\Gamma(c)} F_D^{(n)}\left(\begin{matrix} a; b_1,\dots,b_n \\ c \end{matrix}; \boldsymbol{z}\right).$$

At the second equality, we used the relation (cf. [1, p. 610]):

$$\int_\Delta \Big(\prod_{i=1}^n u_i^{z_i-1}\Big)\Big(1-\sum_i u_i\Big)^{z_{n+1}-1} du_1\cdots du_n = \frac{\Big(\prod_{i=1}^n \Gamma(z_i)\Big)\Gamma(z_{n+1})}{\Gamma\Big(\sum_{i=1}^{n+1} z_i\Big)}.$$

□

Similarly, the Lauricella functions $F_A^{(n)}, F_B^{(n)}$ and $F_C^{(n)}$ have integral representations as follows.

Proposition 1.2 (cf. [18, Theorem 3.4.1] and [16, Theorems 3 and 4]).

(i) *If* $0 < \mathrm{Re}(b_i) < \mathrm{Re}(c_i)$ *for all* i*, then*

$$\Big(\prod_{i=1}^n B(b_i, c_i-b_i)\Big) F_A^{(n)}\left(\begin{matrix} a; b_1,\dots,b_n \\ c_1,\dots,c_n \end{matrix}; \boldsymbol{z}\right)$$
$$= \int_0^1 \cdots \int_0^1 \Big(1-\sum_{i=1}^n z_i u_i\Big)^{-a} \prod_{i=1}^n u_i^{b_i-1}(1-u_i)^{c_i-b_i-1}\, du_1\cdots du_n.$$

(ii) *If* $c_1,\dots,c_n, \sum_i c_{i=1}^n - a \notin \mathbb{Z}$*, then*

$$\frac{\Big(\prod_{i=1}^n \Gamma(1-c_i)\Big)\Gamma(\sum_{i=1}^n c_i - a - n + 1)}{\Gamma(1-a)} F_A^{(n)}\left(\begin{matrix} a; b_1,\dots,b_n \\ c_1,\dots,c_n \end{matrix}; \boldsymbol{z}\right)$$
$$= \int_{\Delta'} \Big(\prod_{i=1}^n \Big(1-\frac{z_i}{u_i}\Big)^{-b_i}\Big)\Big(\prod_{i=1}^n u_i^{-c_i}\Big)\Big(1-\sum_{i=1}^n u_i\Big)^{\sum_{i=1}^n c_i - a - n} du_1\cdots du_n,$$

where Δ' *is a regularization of* Δ*.*

(iii) *If* $0 < \mathrm{Re}(b_i)$ *for all* i*, then*

$$\frac{\Big(\prod_{i=1}^n \Gamma(b_i)\Big)\Gamma(c-\sum_{i=1}^n b_i)}{\Gamma(c)} F_B^{(n)}\left(\begin{matrix} a_1,\dots,a_n; b_1,\dots,b_n \\ c \end{matrix}; \boldsymbol{z}\right)$$
$$= \int_\Delta \Big(\prod_{i=1}^n (1-u_i z_i)^{-a_i}\Big)\Big(\prod_{i=1}^n u_i^{b_i-1}\Big)\Big(1-\sum_{i=1}^n u_i\Big)^{c-\sum_i b_i - 1} du_1\cdots du_n.$$

(iv) *If* $c_1,\dots,c_n, \sum_{i=1}^n c_{i=1}^n - a \notin \mathbb{Z}$*, then*

$$\frac{\Big(\prod_{i=1}^n \Gamma(1-c_i)\Big)\Gamma(\sum_{i=1}^n c_i - a - n + 1)}{\Gamma(1-a)} F_C^{(n)}\left(\begin{matrix} a; b \\ c_1,\dots,c_n \end{matrix}; \boldsymbol{z}\right)$$
$$= \int_{\Delta'} \Big(1-\sum_{i=1}^n \frac{z_i}{u_i}\Big)^{-b}\Big(\prod_{i=1}^n u_i^{-c_i}\Big)\Big(1-\sum_{i=1}^n u_i\Big)^{\sum_{i=1}^n c_i - a - n} du_1\cdots du_n.$$

For Appell's functions F_4, the following propositions are known due to Burchnall-Chaundy.

Proposition 1.3 ([10, (54)]). *We have*

$$F_4(a;b;c_1,c_2;x(1-y),y(1-x))$$
$$=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(1+a+b-c_1-c_2)_r}{(1)_r(c_1)_r(c_2)_r}x^ry^r\,{}_2F_1\left(\begin{matrix}a+r,b+r\\c_1+r\end{matrix};x\right){}_2F_1\left(\begin{matrix}a+r,b+r\\c_2+r\end{matrix};y\right).$$

Remark 1.4. The case when $1+a+b-c_1-c_2=0$ of the proposition above is given by Bailey [3, (2.1)] as

$$\begin{aligned}F_4(a;b;c,1+a+b-c;x(1-y),y(1-x))\\={}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right){}_2F_1\left(\begin{matrix}a,b\\1+a+b-c\end{matrix};y\right).\end{aligned}\tag{1.4}$$

Burchnall-Chaundy proves the proposition using some operators, and Bailey [6] gives an alternative proof by the coefficient comparison.

Proof. We recall Bailey's proof. Put

$$\Phi(x,y)=(1-x)^{-a}(1-y)^{-b}F_4\Big(a;b;c_1,c_2;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\Big).$$

Let $C_{m,n}$ be the coefficient of x^my^n in the expansion in series of $\Phi(x,y)$. Using (1.3), one shows that

$$\begin{aligned}C_{m,n}&=\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-1)^{r+s}(a)_{m+s}(b)_{n+r}}{(1)_r(1)_s(c_1)_r(c_2)_s(1)_{m-r}(1)_{n-s}}\\&=\frac{(a)_m(b)_n}{(1)_m(1)_n}\sum_r\frac{(-1)^r(b+n)_r}{(1)_r(c_1)_r(1+m)_{-r}}\sum_s\frac{(-1)^s(a+m)_s}{(1)_s(c_2)_s(1+n)_{-s}}.\end{aligned}$$

Noting that (cf. [27, (I.7)])

$$(a)_{-n}=\frac{(-1)^n}{(1-a)_n},\tag{1.5}$$

the last member above is equal to

$$\frac{(a)_m(b)_n}{(1)_m(1)_n}{}_2F_1\left(\begin{matrix}b+n,-m\\c_1\end{matrix};1\right){}_2F_1\left(\begin{matrix}a+m,-n\\c_2\end{matrix};1\right).$$

Therefore, using the Vandermonde theorem (cf. [27, (1.7.7)])

$${}_2F_1\left(\begin{matrix}a,-m\\c\end{matrix};1\right)=\frac{(c-a)_m}{(c)_m}\quad(m\in\mathbb{Z}_{>0}),\tag{1.6}$$

we have

$$C_{m,n}=\frac{(a)_m(b)_n(c_1-b-n)_m(c_2-a-m)_n}{(1)_m(1)_n(c_1)_m(c_2)_n}.\tag{1.7}$$

By the Saalschutz theorem (cf. [27, (2.3.1.3)])

$${}_3F_2\left(\begin{matrix}a,b,-n\\c,1+a+b-c-n\end{matrix};1\right)=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}\quad(n\in\mathbb{Z}_{>0}),$$

(1.5) and (1.3), we have

$$\begin{aligned}&(c_1-b-n)_m(c_2-a-m)_n\\&=(c_1-b)_m(c_2-a)_n\cdot{}_3F_2\left(\begin{matrix}1+a+b-c_1-c_2,-m,-n\\1+b-c_1-m,1+a-c_2-n\end{matrix};1\right)\end{aligned}$$

$$= \sum_{r=0}^{\min(m,n)} \frac{(1+a+b-c_1-c_2)_r(c_1-b)_{m-r}(c_2-a)_{n-r}}{(1)_r(1+m)_{-r}(1+n)_{-r}}.$$

Thus, substituting this into the (1.7), we have

$$C_{m,n} = \sum_{r=1}^{\min(m,n)} \frac{(1+a+b-c_1-c_2)_r(a)_m(c_1-b)_{m-r}(b)_n(c_2-a)_{n-r}}{(1)_r(1)_{m-r}(c_1)_m(1)_{n-r}(c_2)_n}.$$

Thus, change of the order of sums and replacing m and n with $r+s$ and $r+t$ respectively, we have

$$\begin{aligned}\Phi(x,y) &= \sum_{m,n} C_{m,n}x^m y^n \\ &= \sum_{r=0}^{\infty}\sum_{s,t=0}^{\infty} \frac{(1+a+b-c_1-c_2)_r(a)_{r+t}(c_1-b)_s(b)_{r+s}(c_2-a)_t}{(1)_r(1)_s(c_1)_{r+s}(1)_t(c_2)_{r+t}} x^{r+s}y^{r+t} \\ &= \sum_r \frac{(a)_r(b)_r(1+a+b-c_1-c_2)_r}{(1)_r(c_1)_r(c_2)_r} x^r y^r \\ &\qquad\qquad \times {}_2F_1\left(\begin{matrix} a+r, c_1-b \\ c_1+r \end{matrix}; x\right) {}_2F_1\left(\begin{matrix} b+r, c_2-a \\ c_2+r \end{matrix}; y\right).\end{aligned}$$

Applying the Pfaff formula (cf. [27, (1.7.1.3)])

$$(1.8) \qquad {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = (1-z)^{-a}{}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}; \frac{-z}{1-z}\right),$$

to the last two ${}_2F_1$ functions above, we obtain

$$\begin{aligned}\Phi(x,y) = (1-x)^{-a}(1-y)^{-b} &\sum_r \frac{(a)_r(b)_r(1+a+b-c_1-c_2)_r}{(1)_r(c_1)_r(c_2)_r}\left(\frac{x}{1-x}\right)^r\left(\frac{y}{1-y}\right)^r \\ &\times {}_2F_1\left(\begin{matrix} a+r, b+r \\ c_1+r \end{matrix}; \frac{-x}{1-x}\right) {}_2F_1\left(\begin{matrix} a+r, b+r \\ c_2+r \end{matrix}; \frac{-y}{1-y}\right).\end{aligned}$$

Therefore, the proof completes by putting $x = -x/(1-x)$ and $y = -y/(1-y)$. $\square$

PROPOSITION 1.5 (cf. [10, (68)]). *If* $0 < \mathrm{Re}(a) < \mathrm{Re}(c_1)$ *and* $0 < \mathrm{Re}(b) < \mathrm{Re}(c_2)$, *then*

$$\begin{aligned}&B(a, c_1-a)B(b, c_2-b)F_4(a; b; c_1, c_2; x(1-y), y(1-x)) \\ &= \int_0^1\int_0^1 u^{a-1}v^{b-1}(1-u)^{c_1-a-1}(1-v)^{c_2-b-1} \\ &\qquad \times (1-ux)^{a-c_1-c_2+1}(1-vy)^{b-c_1-c_2+1}(1-ux-vy)^{c_1+c_2-a-b-1}dudv.\end{aligned}$$

PROOF. By Proposition 1.3,

$$\begin{aligned}&F_4(a; b; c_1, c_2; x(1-y), y(1-x)) \\ &= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(1+a+b-c_1-c_2)_r}{(1)_r(c_1)_r(c_2)_r} x^r y^r {}_2F_1\left(\begin{matrix} a+r, b+r \\ c_1+r \end{matrix}; x\right) {}_2F_1\left(\begin{matrix} a+r, b+r \\ c_2+r \end{matrix}; y\right).\end{aligned}$$

Applying the integral representation (1.1) to the functions ${}_2F_1$ above, we have

$$F_4(a; b; c_1, c_2; x(1-y), y(1-x))$$

$$
\begin{aligned}
&= B(a, c_1 - a)^{-1} B(b, c_2 - b)^{-1} \sum_{r=0}^{\infty} \frac{(1 + a + b - c_1 - c_2)_r}{(1)_r} \Big(\frac{uvxy}{(1-ux)(1-vy)}\Big)^r \\
&\times \int_0^1 \int_0^1 u^{a-1}(1-u)^{c_1-a-1} v^{b-1} (1-v)^{c_2-b-1} (1-ux)^{-b} (1-vy)^{-a} \, dudv \\
&= B(a, c_1 - a)^{-1} B(b, c_2 - b)^{-1} {}_1F_0 \left(\begin{matrix} 1 + a + b - c_1 - c_2 \\ \end{matrix} ; \frac{uvxy}{(1-ux)(1-vy)} \right) \\
&\times \int_0^1 \int_0^1 u^{a-1}(1-u)^{c_1-a-1} v^{b-1} (1-v)^{c_2-b-1} (1-ux)^{-b} (1-vy)^{-a} \, dudv.
\end{aligned}
$$

Thus we obtain the proposition by applying (1.2) to the function ${}_1F_0$ above. □

1.2. **Differential equations.** Recall that, noting (1.3),

$$
\frac{d}{dz} {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; z \right) = \frac{ab}{c} {}_2F_1 \left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix} ; z \right).
$$

Thus, letting $\theta_z := z \frac{d}{dz}$ be the Euler operator,

$$
\begin{aligned}
(\theta_z + a) {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; z \right) &= a {}_2F_1 \left(\begin{matrix} a+1, b \\ c \end{matrix} ; z \right), \\
(\theta_z + c - 1) {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; z \right) &= (c-1) {}_2F_1 \left(\begin{matrix} a, b \\ c-1 \end{matrix} ; z \right) \quad \text{(when } c \neq 1\text{)}.
\end{aligned}
$$

Therefore, the function ${}_2F_1 \left(\begin{smallmatrix} a, b \\ c \end{smallmatrix} ; z \right)$ is a solution of the differential equation (cf. [27, (1.2.3)])

$$
\theta_z (\theta_z + c - 1) f = z (\theta_z + a)(\theta_z + b) f,
$$

which is of rank 2. This equation can be rewritten

$$
z(1-z) \frac{d^2 f}{dz^2} + \{c - (1 + a + b) z\} \frac{df}{dz} - abf = 0,
$$

and hence, we can see that 0 and 1 are regular singularities. By writing $1/z$ for z, we find that the point at infinity is also a regular singularity.

In this subsection, we recall differentials of Lauricella's functions and differential equations satisfied by Lauricella's functions. First, we consider Lauricella's functions $F_D^{(n)}$. Similarly to the Gauss hypergeometric functions, we have

$$
\frac{\partial}{\partial z_i} F_D^{(n)} \left(\begin{matrix} a; b_1, \ldots, b_n \\ c \end{matrix} ; \boldsymbol{z} \right) = \frac{ab_i}{c} F_D^{(n)} \left(\begin{matrix} a+1; b_1, \ldots, b_i + 1, \ldots, b_n \\ c+1 \end{matrix} ; \boldsymbol{z} \right). \tag{1.9}
$$

Put

$$
\theta_i = z_i \frac{\partial}{\partial z_i} \quad (i = 1, \ldots, n), \qquad \theta = \sum_{i=1}^{n} \theta_i.
$$

Noting that

$$
(a)_{m_1 + \cdots + m_n} \Big(a + \sum_{i=1}^{n} m_i \Big) = a (a+1)_{m_1 + \cdots + m_n},
$$

and

$$
\frac{c - 1 + \sum_{i=1}^{n} m_i}{(c)_{m_1 + \cdots + m_n}} = \frac{c-1}{(c-1)_{m_1 + \cdots + m_n}},
$$

we deduce the following.

Proposition 1.6.

(i) *We have*

$$(\theta+a)F_D^{(n)}\left({a;b_1,\dots.b_n \atop c};\boldsymbol{z}\right)=aF_D^{(n)}\left({a+1;b_1,\dots,b_n \atop c};\boldsymbol{z}\right).$$

(ii) *For each* i*,*

$$(\theta_i+b_i)F_D^{(n)}\left({a;b_1,\dots,b_n \atop c};\boldsymbol{z}\right)=b_iF_D^{(n)}\left({a;b_1,\dots,b_i+1,\dots,b_n \atop c};\boldsymbol{z}\right).$$

(iii) *If* $c\neq 1$ *then*

$$(\theta+c-1)F_D^{(n)}\left({a;b_1,\dots,b_n \atop c};\boldsymbol{z}\right)=(c-1)F_D^{(n)}\left({a;b_1,\dots,b_n \atop c-1};\boldsymbol{z}\right).$$

By the proposition above, we obtain the following.

Proposition 1.7 (cf. [22, subsection 1.1]). *Lauricella's function* $F_D^{(n)}\left({a;b_1,\dots,b_n \atop c};\boldsymbol{z}\right)$ *satisfies the system* E_D*:*

$$\begin{cases}\theta_i(\theta+c-1)f=z_i(\theta_i+b_i)(\theta+a)f, & i=1,\dots,n,\\ z_i(\theta_i+b_i)\theta_jf=z_j(\theta_j+b_j)\theta_if, & i,j=1,\dots,n.\end{cases}$$

Let f be a solution of E_D. By the first n equations

$$\theta_i(\theta+c-1)f=z_i(\theta_i+b_i)(\theta+a)f,\quad i=1,\dots,n,$$

any partial derivatives of f are determined by the derivatives of the form

$$\frac{\partial^r f}{\partial z_{i_1}\cdots\partial z_{i_r}},\qquad 0\le r\le n,\ i_1<i_2\cdots<i_r.$$

Furthermore, by the other equations

$$z_i(\theta_i+b_i)\theta_jf=z_j(\theta_j+b_j)\theta_if,\quad i,j=1,\dots,n,$$

these derivatives are determined by

$$f,\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}.$$

From this, we have that the rank of E_D is not higher than $n+1$. Lauricella gives the following.

Theorem 1.8 (cf. [19]). *The rank of* E_D *is* $n+1$*.*

An irreducibility condition of the system E_D is known as follows.

Theorem 1.9 ([23, Theorem 0.1]). *The system* E_D *is irreducible if and only if none of* $a, c-a, c-\sum_{i=1}^n b_i$ *and* b_i $(i=1,\dots,n)$ *are integers.*

Next, we consider Lauricella's functions $F_A^{(n)}, F_B^{(n)}$ and $F_C^{(n)}$. These functions satisfy similar relations as Proposition 1.6.

Proposition 1.10.

(i) *For each i, we have the followings:*

$$
\begin{aligned}
&(\theta+a)F_A^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\c_1,\dots,c_n\end{matrix};\boldsymbol{z}\right)=aF_A^{(n)}\left(\begin{matrix}a+1;b_1,\dots,b_n\\c_1,\dots,c_n\end{matrix};\boldsymbol{z}\right),\\
&(\theta_i+b_i)F_A^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\c_1,\dots,c_n\end{matrix};\boldsymbol{z}\right)=b_iF_A^{(n)}\left(\begin{matrix}a;b_1,\dots,b_i+1,\dots,b_n\\c_1,\dots,c_n\end{matrix};\boldsymbol{z}\right),\\
&(\theta_i+c_i-1)F_A^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\c_1,\dots,c_n\end{matrix};\boldsymbol{z}\right)\\
&\qquad=(c_i-1)F_A^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\c_1,\dots,c_i-1,\dots,c_n\end{matrix};\boldsymbol{z}\right)\quad(\text{when } c_i\neq 1).
\end{aligned}
$$

(ii) *For each i, we have the followings:*

$$
\begin{aligned}
&(\theta_i+a_i)F_B^{(n)}\left(\begin{matrix}a_1,\dots,a_n;b_1,\dots,b_n\\c\end{matrix};\boldsymbol{z}\right)\\
&\qquad=a_iF_B^{(n)}\left(\begin{matrix}a_1,\dots,a_i+1,\dots,a_n;b_1,\dots,b_n\\c\end{matrix};\boldsymbol{z}\right)\quad(\textit{similar for } b_i),\\
&(\theta+c-1)F_B^{(n)}\left(\begin{matrix}a_1,\dots,a_n;b_1,\dots,b_n\\c\end{matrix};\boldsymbol{z}\right)\\
&\qquad=(c-1)F_B^{(n)}\left(\begin{matrix}a_1,\dots,a_n;b_1,\dots,b_n\\c-1\end{matrix};\boldsymbol{z}\right)\quad(\text{when } c\neq 1).
\end{aligned}
$$

(iii) *For each i, we have the followings:*

$$
\begin{aligned}
&(\theta+a)F_C^{(n)}\left(\begin{matrix}a;b\\c_1,\dots,c_n\end{matrix};\boldsymbol{z}\right)=aF_C^{(n)}\left(\begin{matrix}a+1;b\\c_1,\dots,c_n\end{matrix};\boldsymbol{z}\right)\quad(\textit{similar for } b),\\
&(\theta_i+c_i-1)F_C^{(n)}\left(\begin{matrix}a;b\\c_1,\dots,c_n\end{matrix};\boldsymbol{z}\right)\\
&\qquad=(c_i-1)F_C^{(n)}\left(\begin{matrix}a;b\\c_1,\dots,c_i-1,\dots,c_n\end{matrix};\boldsymbol{z}\right)\quad(\text{when } c_i\neq 1).
\end{aligned}
$$

Thus, we derive the following.

Proposition 1.11 (cf. [22, section 1.1]).

(i) *Lauricella's function* $F_A^{(n)}\left(\begin{smallmatrix}a;b_1,\dots,b_n\\c_1,\dots,c_n\end{smallmatrix};\boldsymbol{z}\right)$ *satisfies the system* E_A*:*

$$\theta_i(\theta_i+c_i-1)f=z_i(\theta_i+b_i)(\theta+a)f,\quad i=1,\dots,n.$$

(ii) *Lauricella's function* $F_B^{(n)}\left(\begin{smallmatrix}a_1,\dots,a_n;b_1,\dots,b_n\\c\end{smallmatrix};\boldsymbol{z}\right)$ *satisfies the system* E_B*:*

$$\theta_i(\theta+c-1)f=z_i(\theta_i+a_i)(\theta_i+b_i)f,\quad i=1,\dots,n.$$

(iii) *Lauricella's function* $F_C^{(n)}\left(\begin{smallmatrix}a;b\\c_1,\dots,c_n\end{smallmatrix};\boldsymbol{z}\right)$ *satisfies the system* E_C*:*

$$\theta_i(\theta_i+c_i-1)f=z_i(\theta+a)(\theta+b)f,\quad i=1,\dots,n.$$

Similarly as above, one sees easily that each the rank of E_A, E_B and E_C is not higher than 2^r. Lauricella gives the following.

Theorem 1.12 (cf. [19]). *For each of the systems* E_A*,* E_B *and* E_C*, its rank is* 2^n*.*

Irreducibility conditions of these systems are also known.

THEOREM 1.13 (cf. [22, (2.8), (3.8) and (4.8)]).

(i) *The system* E_A *irreducible if and only if* $b_i, c_i - b_i, \sum_{j\in J} b_j - a \notin \mathbb{Z}$ *for all* $i = 1, \dots, n$ *and all* $J \subset \{1, \dots, n\}$.

(ii) *The system* E_B *irreducible if and only if* $a_i, b_i, c - \sum_{j=1}^{n} d_j \notin \mathbb{Z}$ *for all* i *and* $d_j \in \{a_j, b_j\}$.

(iii) *The system* E_C *irreducible if and only if* $\sum_{j\in J} c_j - a, \sum_{j\in J} c_j - b \notin \mathbb{Z}$ *for all* $J \subset \{1, \dots, n\}$.

1.3. **Transformation formulas.** By the integral representations of $F_D^{(n)}$ and $F_A^{(n)}$ in Proposition 1.1 (i) and Proposition 1.2 (i), we have the following formulas.

PROPOSITION 1.14 (cf. [18, Proposition 3.7.1]).

(i) *We have*

$$\begin{aligned}
&F_D^{(n)}\left({a; b_1, \dots, b_n \atop c}; \boldsymbol{z}\right)\\
&= \left(\prod_{i=1}^{n}(1-z_i)^{-b_i}\right) F_D^{(n)}\left({c-a; b_1, \dots .b_n \atop c}; \frac{-z_1}{1-z_1}, \dots, \frac{-z_n}{1-z_n}\right).
\end{aligned}$$

(ii) *For each* $i = 1, \dots, n$,

$$\begin{aligned}
&F_D^{(n)}\left({a; b_1, \dots, b_n \atop c}; \boldsymbol{z}\right)\\
&= (1-z_i)^{-a} F_D^{(n)}\left({a; b_1, \dots, b_{i-1}, c - \sum_{j=1}^{n} b_j, b_{i+1}, \dots, b_n \atop c}; \boldsymbol{z}'_i\right),
\end{aligned}$$

where

$$\boldsymbol{z}'_i = \left(\frac{z_1 - z_i}{1-z_i}, \dots, \frac{z_{i-1}-z_i}{1-z_i}, \frac{-z_i}{1-z_i}, \frac{z_{i+1}-z_i}{1-z_i}, \dots, \frac{z_n - z_i}{1-z_i}\right).$$

(iii) *For each* $i = 1, \dots, n$,

$$\begin{aligned}
F_D^{(n)}\left({a; b_1, \dots, b_n \atop c}; \boldsymbol{z}\right) &= (1-z_i)^{c-a}\left(\prod_{j=1}^{n}(1-z_j)^{-b_j}\right)\\
&\times F_D^{(n)}\left({c-a; b_1, \dots, b_{i-1}, c - \sum_{j=1}^{n} b_j, b_{i+1}, \dots, b_n \atop a}; \boldsymbol{z}''_i\right),
\end{aligned}$$

where

$$\boldsymbol{z}''_i = \left(\frac{z_i - z_1}{1-z_1}, \dots, \frac{z_i - z_{i-1}}{1-z_{i-1}}, z_i, \frac{z_i - z_{i+1}}{1-z_{i+1}}, \dots, \frac{z_i - z_n}{1-z_n}\right).$$

(iv) *For any subset* $I_r \subseteq \{1, \dots, n\}$ *with* r*-elements* $(1 \leq r \leq n)$,

$$\begin{aligned}
&F_A^{(n)}\left({a; b_1, \dots, b_n \atop c_1, \dots, c_n}; \boldsymbol{z}\right)\\
&= \left(1 - \sum_{i\in I_r} z_i\right)^{-a} F_A^{(n)}\left({a, d_1, \dots .d_n \atop c_1, \dots, c_n}; \frac{(-1)^{\delta_1} z_1}{1 - \sum_{i\in I_r} z_i}, \dots, \frac{(-1)^{\delta_n} z_n}{1 - \sum_{i\in I_r} z_i}\right),
\end{aligned}$$

where

$$d_i = \begin{cases} c_i - b_i & (i \in I_r) \\ b_i & (i \notin I_r) \end{cases},$$

and $\delta_i = 1$ *if* $i \in I_r$ *and* $\delta_i = 0$ *otherwise.*

Proof. Putting $u = 1 - v$ in the integral in Proposition 1.1 (i),

$$\begin{aligned} &B(a, c-a) F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; \boldsymbol{z}\right) \\ &= \int_0^1 \Big(\prod_{i=1}^n (1 - z_i u)^{-b_i}\Big) u^{a-1}(1-u)^{c-a-1}\, du \\ &= \int_0^1 \Big(\prod_{i=1}^n (1 - z_i + z_i v)^{-b_i}\Big) v^{c-a-1}(1-v)^{a-1}\, dv \\ &= \Big(\prod_{i=1}^n (1 - z_i)^{-b_i}\Big) \int_0^1 \Big(\prod_{i=1}^n \Big(1 - \frac{-z_i}{1-z_i}\Big) v\Big)^{-b_i}\Big) v^{c-a-1}(1-v)^{a-1}\, dv. \end{aligned}$$

Thus, we obtain (i). Similarly, we can also prove (ii) (resp. (iii)) by putting $u = v/(1 - z_i + v z_i)$ (resp. $(1 - v)/(1 - z_i v)$) in the integral in Proposition 1.1 (i). Putting $u_i = 1 - v_i$ for each $i \in I_r$ in the integral in Proposition 1.2 (i), we obtain (iv). □

Remark 1.15.

(i) The formulas (i), (ii) and (iv) above are generalizations of Pfaff's formula (1.8), and the formula (iii) above is a generalization of Euler's formula (cf. [27, (1.3.15)])

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = (1-z)^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right). \tag{1.10}$$

(ii) It seems that transformation formulas as above for Lauricella's functions $F_B^{(n)}$ and $F_C^{(n)}$ are not known.

Recall Kummer's quadratic formula (cf. [24, (4.1)])

$$ {}_2F_1\left(\begin{matrix} a, b \\ 2b \end{matrix}; 2z\right) = (1-z)^{-a}\, {}_2F_1\left(\begin{matrix} \frac{a}{2}, \frac{a}{2} + \frac{1}{2} \\ b + \frac{1}{2} \end{matrix}; \frac{z^2}{(1-z)^2}\right). \tag{1.11}$$

The following is a generalization of this formula.

Proposition 1.16 ([7, (4.7) and (4.8)]). *We have*

$$\begin{aligned} &F_A^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ 2b_1, \dots, 2b_n \end{matrix}; 2z_1, \dots, 2z_n\right) \\ &= \Big(1 - \sum_{i=1}^n z_i\Big)^{-a} F_C^{(n)}\left(\begin{matrix} \frac{a}{2}; \frac{a}{2} + \frac{1}{2} \\ b_1 + \frac{1}{2}, \dots, b_n + \frac{1}{2} \end{matrix}; \frac{z_1^2}{(1 - \sum_{i=1}^n z_i)^2}, \dots, \frac{z_n^2}{(1 - \sum_{i=1}^n z_i)^2}\right). \end{aligned}$$

Proof. We prove this by induction for n. When $n = 1$, the proposition is (1.11). Using (1.3),

$$F_A^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ 2b_1, \dots, 2b_n \end{matrix}; 2z_1, \dots, 2z_n\right) \tag{1.12}$$

$$= \sum_{m_1,\ldots,m_{n-1}} \frac{(a)_{m_1+\cdots+m_{n-1}} \prod_{i=1}^{n-1}(b_i)_{m_i}}{\prod_{i=1}^{n-1}(2b_i)_{m_i}(1)_{m_i}} \Big(\prod_{i=1}^{n-1}(2z_i)^{m_i}\Big)$$
$$\times {}_2F_1\left(\begin{matrix} a+\sum_{i=1}^{n-1} m_i, b_n \\ 2b_n \end{matrix}; 2z_n\right).$$

By (1.11) and that (cf. [27, I.25])

$$\left(\frac{a}{2}\right)_m\left(\frac{a}{2}+\frac{1}{2}\right)_m = 2^{-2m}(a)_{2m}, \tag{1.13}$$

we have

$${}_2F_1\left(\begin{matrix} a+\sum_{i=1}^{n-1} m_i, b_n \\ 2b_n \end{matrix}; 2z_n\right)$$
$$= (1-z_n)^{-a-\sum_{i=1}^{n-1} m_i} \sum_{m_n} \frac{(a+\sum_{i=1}^{n-1} m_i)_{2m_n}}{2^{2m_n}(1)_{m_n}(b_n+\frac{1}{2})_{m_n}} \Big(\frac{z_n}{1-z_n}\Big)^{2m_n}.$$

Thus, using (1.3) again, the right-hand side of (1.12) is equal to

$$(1-z_n)^{-a} \sum_{m_n} \frac{(a)_{2m_n}}{2^{2m_n}(1)_{m_n}(b_n+\frac{1}{2})_{m_n}} \Big(\frac{z_n}{1-z_n}\Big)^{2m_n}$$
$$\times F_A^{(n-1)}\left(\begin{matrix} a+2m_n; b_1,\ldots,b_{n-1} \\ 2b_1,\ldots,2b_{n-1} \end{matrix}; \frac{2z_1}{1-z_n},\ldots,\frac{2z_{n-1}}{1-z_n}\right). \tag{1.14}$$

By the induction hypothesis, we have

$$F_A^{(n-1)}\left(\begin{matrix} a+2m_n; b_1,\ldots,b_{n-1} \\ 2b_1,\ldots,2b_{n-1} \end{matrix}; \frac{2z_1}{1-z_n},\ldots,\frac{2z_{n-1}}{1-z_n}\right)$$
$$= \Big(1-\frac{\sum_{i=1}^{n-1} z_i}{1-z_n}\Big)^{-a-2m_n}$$
$$\times F_C^{(n-1)}\left(\begin{matrix} \frac{a}{2}+m_n; \frac{a}{2}+\frac{1}{2}+m_n \\ b_1+\frac{1}{2},\ldots,b_{n-1}+\frac{1}{2} \end{matrix}; \frac{z_1^2}{(1-\sum_{i=1}^n z_i)^2},\ldots,\frac{z_{n-1}^2}{(1-\sum_{i=1}^n z_i)^2}\right)$$
$$= (1-z_n)^{a+2m_n}\Big(1-\sum_{i=1}^n z_i\Big)^{-a-2m_n}$$
$$\times \sum_{m_1,\ldots,m_{n-1}} \frac{(\frac{a}{2}+m_n)_{m_1+\cdots+m_{n-1}}(\frac{a}{2}+\frac{1}{2}+m_n)_{m_1+\cdots+m_{n-1}}}{\prod_{i=1}^{n-1}(1)_{m_i}(b_i+\frac{1}{2})_{m_i}} \prod_{i=1}^{n-1}\Big(\frac{z_i}{1-\sum_{i=1}^n z_i}\Big)^{2m_i}.$$

Substituting this into (1.14), applying (1.13) to $(a)_{2m_n}$ and using (1.3), (1.14) is equal to

$$\Big(1-\sum_{i=1}^n z_i\Big)^{-a} \sum_{m_1,\ldots,m_n} \frac{(\frac{a}{2})_{m_1+\cdots+m_n}(\frac{a}{2}+\frac{1}{2})_{m_1+\cdots+m_n}}{\prod_{i=1}^{n}(1)_{m_i}(b_i+\frac{1}{2})_{m_i}} \prod_{i=1}^{n}\Big(\frac{z_i}{1-\sum_{i=1}^n z_i}\Big)^{2m_i},$$

and hence, we obtain the proposition. □

Matsumoto-Ohara [21] prove the following.

Theorem 1.17 ([21, Theorems 1–3]).

(i) *Let ω be a cubic root of unity. Then*

$$F_1\Big(\frac{c}{3};\frac{c+1}{6},\frac{c+1}{6};\frac{c+1}{2};1-x^3,1-y^3\Big)$$
$$=\Big(\frac{3}{1+x+y}\Big)^c F_1\Big(\frac{c}{3};\frac{c+1}{6},\frac{c+1}{6};\frac{c+5}{6};\Big(\frac{1+\omega x+\omega^2 y}{1+x+y}\Big)^3,\Big(\frac{1+\omega^2 x+\omega y}{1+x+y}\Big)^3\Big).$$

(ii) *Put*

$$x'=\frac{\Big(\sqrt{(1-x^2)(1-y^2)}+\sqrt{-1}(x-y)\Big)^2}{(1+\sqrt{xy})^4},$$
$$y'=\frac{\Big(\sqrt{(1-x^2)(1-y^2)}-\sqrt{-1}(x-y)\Big)^2}{(1+\sqrt{xy})^4}.$$

Then,

$$F_1\Big(\frac{c}{2};\frac{2c-1}{4},\frac{2c-1}{4};c;1-x^2,1-y^2\Big)$$
$$=\Big(\frac{1+\sqrt{z_1z_2}}{2}\Big)^{2c-1}F_1\Big(\frac{c}{2};\frac{2c-1}{4},\frac{2c-1}{4};\frac{c+1}{2};x',y'\Big).$$

(iii) *Put*

$$x'=\Big(\frac{1-x-y+z}{1+x+y+z}\Big)^2,\quad y'=\Big(\frac{1-x+y-z}{1+x+y+z}\Big)^2,\quad z'=\Big(\frac{1+x-y-z}{1+x+y+z}\Big)^2.$$

Then,

$$F_D^{(3)}\left(\begin{matrix}\frac{c}{4};\frac{c+2}{12},\frac{c+2}{12},\frac{c+2}{12}\\ \frac{c+2}{3}\end{matrix};1-x^3,1-y^3,1-z^3\right)$$
$$=\Big(\frac{4}{1+x+y+z}\Big)^{c/2}F_D^{(3)}\left(\begin{matrix}\frac{c}{4};\frac{c+2}{12},\frac{c+2}{12},\frac{c+2}{12}\\ \frac{c+5}{6}\end{matrix};x',y',z'\right).$$

Remark 1.18.

(i) When $x=y$, the functions F_1 reduce to ${}_2F_1$ by Proposition 1.20 below. Then the formula (i) reduces to Ramanujan's formula [26, Second Notebook, p. 258] when $c=1$. For general c, the formula reduces to Berndt-Bhargava-Garvan [8, Theorem 2.3] (see also [24, Theorem 1.1 (1.2)] for a simple proof):

$$\Big(\frac{1+2z}{3}\Big)^c{}_2F_1\left(\begin{matrix}\frac{c}{3},\frac{c+1}{3}\\ \frac{c+1}{2}\end{matrix};1-z^3\right)={}_2F_1\left(\begin{matrix}\frac{c}{3},\frac{c+1}{3}\\ \frac{c+5}{6}\end{matrix};\Big(\frac{1-z}{1+2z}\Big)^3\right).$$

Koike-Shiga [17] prove the case $c=1$ of (i) by using the differential system E_D in Theorem 1.7 as well as Matsumoto-Ohara.

(ii) When $x=y$, the formula (ii) reduces to the formula [21, Corollary 2]

$$\Big(\frac{1+z}{2}\Big)^{2c-1}{}_2F_1\left(\begin{matrix}\frac{c}{2},\frac{2c-1}{4}\\ c\end{matrix};1-z^4\right)={}_2F_1\left(\begin{matrix}\frac{c}{2},\frac{2c-1}{2}\\ \frac{c+1}{2}\end{matrix};-\Big(\frac{1-z}{1+z}\Big)^2\right).$$

(iii) When $x=y=z$, the functions $F_D^{(3)}$ reduce to ${}_2F_1$. Then the formula (iii) reduces to Ramanujan's formula [26, Second Notebook, p. 260] when

$c=1$. For general c, the formula (iii) reduces to the formula [21, Corollary 3]

$$\left(\frac{1+3z}{4}\right)^{\frac{c}{2}} {}_2F_1\left(\begin{matrix}\frac{c}{4}, \frac{c+2}{4}\\ \frac{c+2}{3}\end{matrix};1-z^2\right)={}_2F_1\left(\begin{matrix}\frac{c}{4}, \frac{c+2}{4}\\ \frac{c+5}{6}\end{matrix};\left(\frac{1-z}{1+3z}\right)^2\right).$$

1.4. **Reducible cases and values at $z=1$.** We use the following.

LEMMA 1.19 (Multinomial theorem for Pochhammer symbols). *For any $m\geq 1$,*

$$\frac{(\sum_{i=1}^n b_i)_m}{(1)_m}=\sum_{m_1+\cdots+m_n=m}\frac{\prod_{i=1}^n (b_i)_{m_i}}{\prod_{i=1}^n (1)_{m_i}}.$$

PROOF. It is trivial for $m=1$. For each $m\geq 2$, we have

$$\frac{(\sum_{i=1}^n b_i)_m}{(1)_m}=\frac{1}{m}\sum_{i=1}^n b_i\frac{(b_1+\cdots+b_n+1)_{m-1}}{(1)_{m-1}}.$$

By the induction hypothesis, the right-hand side above is

$$\begin{aligned}&\frac{1}{m}\sum_{i=1}^n b_i\sum_{m_1+\cdots+m_n=m-1}\frac{(b_1)_{m_1}\cdots(b_i+1)_{m_i}\cdots(b_n)_{m_n}}{\prod_{i=1}^n (1)_{m_i}}\\ &=\frac{1}{m}\sum_{i=1}^n\sum_{m_1+\cdots+m_n=m-1}\frac{(b_1)_{m_1}\cdots(b_i)_{m_i+1}\cdots(b_n)_{m_n}}{\prod_{i=1}^n (1)_{m_i}}\\ &=\frac{1}{m}\sum_{m_1+\cdots+m_n=m}\frac{\prod_{i=1}^n (b_i)_{m_i}}{\prod_{i=1}^n (1)_{m_i}}\sum_{i=1}^n m_i=\sum_{m_1+\cdots+m_n=m}\frac{\prod_{i=1}^n (b_i)_{m_i}}{\prod_{i=1}^n (1)_{m_i}}.\end{aligned}$$

Thus, we obtain the lemma. □

Recall the Euler-Gauss summation formula (cf. [27, (1.7.6)])

$$ {}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};1\right)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\quad (\mathrm{Re}(c-a-b)>0). \tag{1.15}$$

For cases when $z_i=z_j$ $(i\neq j)$ or $z_i=1$ for some $i=1,\dots,n-1$, the following is known.

PROPOSITION 1.20 ([19, p. 149–151]).

(i) *For any $i=0,\dots,n-1$,*

$$\begin{aligned}&F_D^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\ c\end{matrix};z_1,\dots,z_i,x,\dots,x\right)\\ &=F_D^{(i+1)}\left(\begin{matrix}a;b_1,\dots,b_i,\sum_{j=i+1}^n b_j\\ c\end{matrix};z_1,\dots,z_i,x\right).\end{aligned}$$

(ii) *For any $i=0,\dots,n-1$, if $\mathrm{Re}(c-a-\sum_{j=i+1}^n b_j)>0$, then*

$$\begin{aligned}&F_D^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\ c\end{matrix};z_1,\dots,z_i,1,\dots,1\right)\\ &=\frac{\Gamma(c)\Gamma(c-a-\sum_{j=i+1}^n b_j)}{\Gamma(c-a)\Gamma(c-\sum_{j=i+1}^n b_j)}F_D^{(i)}\left(\begin{matrix}a;b_1,\dots,b_i\\ c-\sum_{j=i+1}^n b_j\end{matrix};z_1,\dots,z_i\right).\end{aligned}$$

In particular, if $\mathrm{Re}(c-a-\sum_{j=1}^{n} b_j) > 0$ *then*

$$F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; 1, \dots, 1\right) = \frac{\Gamma(c)\Gamma(c-a-\sum_j b_j)}{\Gamma(c-a)\Gamma(c-\sum_j b_j)}.$$

Proof of Proposition 1.20. (i) By Lemma 1.19,

$$\begin{aligned} & F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; z_1, \dots, z_i, x, \dots, x\right) \\ &= \sum_{m_1,\dots,m_n} \frac{(a)_{m_1+\cdots+m_n} \prod_{j=1}^{n} (b_j)_{m_j}}{(c)_{m_1+\cdots+m_n} \prod_{j=1}^{n} (1)_{m_j}} x^{m_{i+1}+\cdots+m_n} \prod_{j=1}^{i} z_j^{m_j} \\ &= \sum_{m_1,\dots,m_i,m} \frac{(a)_{m_1+\cdots+m_i+m} \prod_{j=1}^{i} (b_j)_{m_j}}{(c)_{m_1+\cdots+m_i+m} \prod_{j=1}^{i} (1)_{m_j}} x^m \Big(\prod_{j=1}^{i} z_j^{m_j}\Big) \sum_{m_{i+1}+\cdots+m_n=m} \frac{\prod_{j=i+1}^{n} (b_j)_{m_j}}{\prod_{j=i+1}^{n} (1)_{m_j}} \\ &= \sum_{m_1,\dots,m_i,m} \frac{(a)_{m_1+\cdots+m_i+m} \prod_{j=1}^{i} (b_j)_{m_j}}{(c)_{m_1+\cdots+m_i+m} \prod_{j=1}^{i} (1)_{m_j}} x^m \Big(\prod_{j=1}^{i} z_j^{m_j}\Big) \frac{(\sum_{j=i+1}^{n} b_j)_m}{(1)_m}. \end{aligned}$$

Thus, we obtain the formula.

(ii) By (i), we have

$$\begin{aligned} & F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; z_1, \dots, z_i, 1, \dots, 1\right) \\ &= F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_i, \sum_{j=i+1}^{n} b_j \\ c \end{matrix}; z_1, \dots, z_i, 1\right) \\ &= \sum_{m_1,\dots,m_i} \frac{(a)_{m_1+\cdots+m_i} \prod_{j=1}^{i} (b_j)_{m_j}}{(c)_{m_1+\cdots+m_i} \prod_{j=1}^{i} (1)_{m_j}} \Big(\prod_{j=1}^{i} z_j^{m_j}\Big) {}_2F_1\left(\begin{matrix} a+\sum_{j=1}^{i} m_j, \sum_{j=i+1}^{n} b_j \\ c+\sum_{j=1}^{i} m_j \end{matrix}; 1\right). \end{aligned}$$

Therefore, we derive the formula by the Euler-Gauss summation formula (1.15) and (1.3) .

□

For Appell's functions F_3, we have the following.

Proposition 1.21. *If* $\mathrm{Re}(c-a_2-b_2) > 0$,

$$F_3(a_1, a_2; b_1, b_2; c; x, 1) = \frac{\Gamma(c)\Gamma(c-a_2-b_2)}{\Gamma(c-a_2)\Gamma(c-b_2)} {}_3F_2\left(\begin{matrix} a_1, b_1, c-a_2-b_2 \\ c-a_2, c-b_2 \end{matrix}; x\right).$$

Proof. By (1.15),

$$\begin{aligned} F_3(a_1, a_2; b_1, b_2; c; x, 1) &= \sum_{m} \frac{(a_1)_m (b_1)_m}{(c)_m (1)_m} {}_2F_1\left(\begin{matrix} a_2, b_2 \\ c+m \end{matrix}; 1\right) x^m \\ &= \sum_{m} \frac{(a_1)_m (b_1)_m}{(c)_m (1)_m} \cdot \frac{\Gamma(c+m)\Gamma(c-a_2-b_2+m)}{\Gamma(c-a_2+m)\Gamma(c-b_2+m)} x^m \\ &= \frac{\Gamma(c)\Gamma(c-a_2-b_2)}{\Gamma(c-a_2)\Gamma(c-b_2)} {}_3F_2\left(\begin{matrix} a_1, b_1, c-a_2-b_2 \\ c-a_2, c-b_2 \end{matrix}; x\right). \end{aligned}$$

□

Next, we recall some cases when parameters are special. By (1.2),

$$F_D^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\ a\end{matrix};\boldsymbol{z}\right)=\prod_{i=1}^{n}(1-z_i)^{-b_i}. \tag{1.16}$$

Noting that $(0)_n=0$ when $n>0$ and $(0)_0=1$, we have

$$F_D^{(n)}\left(\begin{matrix}0;b_1,\dots,b_n\\ c\end{matrix};\boldsymbol{z}\right)=1.$$

If $b_i=0$ for some $i=1,\dots,n$, then

$$F_D^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\ c\end{matrix};\boldsymbol{z}\right)=F_D^{(n-1)}\left(\begin{matrix}a;b_1,\dots,\widehat{b_i},\dots,b_n\\ c\end{matrix};z_1,\dots,\widehat{z_i},\dots,z_n\right), \tag{1.17}$$

where $\widehat{b_i}$ and $\widehat{z_i}$ mean that b_i and z_i are omitted. The other Lauricella's functions also satisfy similar relations. Thus, Proposition 1.14 gives the following.

Proposition 1.22.

(i) *For each $i=1,\dots,n$, if $z_i\neq 1$,*

$$\begin{aligned}&F_D^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\ \sum_{j=1}^n b_j\end{matrix};\boldsymbol{z}\right)\\ &=(1-z_i)^{-a}F_D^{(n-1)}\left(\begin{matrix}a;b_1,\dots,\widehat{b_i},\dots,b_n\\ \sum_{j=1}^n b_j\end{matrix};\frac{1-z_1}{1-z_i},\dots,\widehat{\frac{1-z_i}{1-z_i}},\dots,\frac{1-z_n}{1-z_i}\right).\end{aligned}$$

(ii) *For each subset I_r as in Proposition 1.14 (iv), if $b_i=c_i$ for all $i\in I_r$, then*

$$\begin{aligned}&F_A^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\ c_1,\dots,c_n\end{matrix};\boldsymbol{z}\right)\\ &=\left(1-\sum_{i\in I_r}z_i\right)^{-a}F_A^{(n-r)}\left(\begin{matrix}a;(b_j)_{j\notin I_r}\\ (c_j)_{j\notin I_r}\end{matrix};\left(\frac{z_j}{1-\sum_{i\in I_r}z_i}\right)_{j\notin I_r}\right).\end{aligned}$$

Proof. By the Proposition 1.14 (i),

$$\begin{aligned}&F_D^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\ \sum_{j=1}^n b_j\end{matrix};\boldsymbol{z}\right)\\ &=(1-z_i)^{-a}F_D^{(n)}\left(\begin{matrix}a;b_1,\dots,b_{i-1},0,b_{i+1},\dots,b_n\\ \sum_{j=1}^n b_j\end{matrix};\frac{1-z_1}{1-z_i},\dots,\frac{1-z_n}{1-z_i}\right).\end{aligned}$$

Thus, (i) follows by (1.17). (ii) follows by Proposition 1.14 (iv) and (1.17) similarly. □

Karlsson [15] gives some reduction formulas as follows.

Theorem 1.23 ([15, (4.6), (4.8) and (4.10)]).

(i) *Suppose $y\neq 1$ and put*

$$p_1=\frac{y-1+\sqrt{(y-1)(y+3)}}{2},\quad p_2=\frac{y-1-\sqrt{(y-1)(y+3)}}{2}.$$

Then, we have

$$F_D^{(5)}\left(\begin{matrix}3a; c-b, 2b-1, b, 1+a-c, 1+a-c\\ 2a+c\end{matrix}; y, \frac{2y}{3}, -\frac{y}{3}, p_1, p_2\right)$$
$$= (1-y)^{-a}\frac{\Gamma(a)\Gamma(2a+c)}{3\Gamma(3a)\Gamma(c)}\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; \frac{4y^3}{27(y-1)}\right),$$

and

$$F_D^{(4)}\left(\begin{matrix}3a; c, -1, 1+a-c, 1+a-c\\ 2a+c\end{matrix}; y, \frac{2y}{3}, p_1, p_2\right)$$
$$= (1-y)^{-a}\frac{\Gamma(a)\Gamma(2a+c)}{3\Gamma(3a)\Gamma(c)}.$$

(ii) *Let ω be a cubic root of unity. Then,*

$$F_D^{(5)}\left(\begin{matrix}3a; 1+a-c, 1+a-c, b, b, b\\ 2a+c\end{matrix}; \omega, \omega^2, \zeta, \omega\zeta, \omega^2\zeta\right)$$
$$= \frac{\Gamma(a)\Gamma(2a+c)}{3\Gamma(3a)\Gamma(c)}\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; \zeta^3\right).$$

PROOF. By the change of variable $u = (1-y)v^3/(1-yv)$ in the integral (1.1), we have

$$(1.18)\quad B(a, c-a)\,{}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right)$$
$$= 3(1-y)^a\int_0^1 v^{3a-1}(1-v)^{c-a-1}(1-yv)^{b-c}(1-\frac{2}{3}yv)(1-p_1v)^{c-a-1}$$
$$\times (1-p_2v)^{c-a-1}(1-r_1v)^{-b}(1-r_2v)^{-b}(1-r_3v)^{-b}\,dv,$$

where r_i $(i = 1, 2, 3)$ are defined by

$$1 - zu = \frac{(1-r_1v)(1-r_2v)(1-r_3v)}{1-yv}.$$

If we set

$$z = \frac{4y^3}{27(y-1)} \quad \text{(resp. 0)}$$

in (1.18), we obtain the first (resp. second) formula in (i) by the integral representation in Proposition 1.1 (i). If we set $y = 0$ and $z = \zeta^3$ in (1.18), we obtain (ii) by loc. cit.. □

By the second formula in (i) above with $y = -3$ and Proposition 1.22 (i), he gives the formula ([15, (4.9a)])

$$F_1(3a; c, 1+2a-2c; 2a+c; -3, -2) = 4^{-a}\frac{\Gamma(a)\Gamma(2a+c)}{3\Gamma(3a)\Gamma(c)}.$$

Similarly, by taking $b = 0$ or $\zeta = 0$ in (iii) above, he gives ([15, (4.11a)])

$$F_1(3a; 1+a-c, 1+a-c; 2a+c; \omega, \omega^2) = \frac{\Gamma(a)\Gamma(2a+c)}{3\Gamma(3a)\Gamma(c)}.$$

Furthermore, applying Proposition 1.14 (i) and (ii), he also gives many special values formula ([15, (4.9b)–(4.9f) and (4.11a)–(4.11f)]).

Theorem 1.24 ([15, (5.6), (5.8) and (6.1)]).

(i) *Suppose that* $x \neq 1$ *and put*

$$p_1 = \frac{-1+\sqrt{(1+3x)/(1-x)}}{2}, \quad p_2 = \frac{-1-\sqrt{(1+3x)/(1-x)}}{2}.$$

Then, we have

$$F_D^{(5)}\left(\begin{matrix}2a;1-a,2b-1,b,1+a-c,1+a-c\\ a+c\end{matrix};x,\frac{3}{2}x,-3x,p_1,p_2\right)$$
$$=(1-x)^a\frac{\Gamma(a)\Gamma(a+c)}{2\Gamma(2a)\Gamma(c)}\,{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};\frac{27x^2(1-x)}{4}\right),$$

and

$$F_D^{(4)}\left(\begin{matrix}2a;1-a,-1,1+a-c,1+a-c\\ a+c\end{matrix};x,\frac{3}{2}x,p_1,p_2\right)$$
$$=(1-x)^a\frac{\Gamma(a)\Gamma(a+c)}{2\Gamma(2a)\Gamma(c)}.$$

(ii) *We have*

$$F_D^{(3)}\left(\begin{matrix}2a;1+a-c,b,b\\ a+c\end{matrix};-1,\zeta,-\zeta^2\right)=\frac{\Gamma(a)\Gamma(a+c)}{2\Gamma(2a)\Gamma(c)}\,{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};\zeta^2\right).$$

Proof. By the change of variable $u = v^2(1-xv)/(1-x)$ in the integral (1.1), we have

$$B(a,c-a)\,{}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix};z\right) \tag{1.19}$$
$$=2(1-x)^{-a}\int_0^1 v^{2a-1}(1-v)^{c-a-1}(1-xv)^{a-1}(1-\frac{3}{2}xv)(1-p_1v)^{c-a-1}$$
$$\times(1-p_2v)^{c-a-1}(1-r_1v)^{-b}(1-r_2v)^{-b}(1-r_3v)^{-b}\,dv,$$

where r_i $(i=1,2,3)$ are as in Theorem 1.23. If we set

$$z=\frac{27x^2(1-x)}{4} \quad \text{(resp. 0)}$$

in (1.19), we obtain the first (resp. second) formula in (i). If we set $x = 0$ and $z = \zeta^2$, we obtain (ii). □

Remark 1.25. When $b = 0$ or $\zeta = 0$, (ii) of the theorem above reduces to Kummer's theorem (cf. [4, subsection 2.3 (1)])

$${}_2F_1\left(\begin{matrix}2a,b\\ 1+2a-b\end{matrix};-1\right)=\frac{\Gamma(1+2a-b)\Gamma(1+a)}{\Gamma(1+2a)\Gamma(1+a-b)}. \tag{1.20}$$

By taking $x = -\frac{1}{3}$ and $x = \frac{2}{3}$ respectively in the second formula in (i) above, Karlsson deduces the formulas ([15, (5.9a) and (5.10a)])

$$F_1\Big(2a;1-a,1+2a-2c;a+c;-\frac{1}{3},-\frac{1}{2}\Big)=\Big(\frac{4}{3}\Big)^a\frac{\Gamma(a)\Gamma(a+c)}{2\Gamma(2a)\Gamma(c)},$$

and

$$F_1\Big(2a;1-a,1+a-c;2c;\frac{2}{3},-2\Big)=3^{-a}\frac{\Gamma(a)\Gamma(c-a)\Gamma(2c)}{2\Gamma(2a)\Gamma(2c-2a)\Gamma(c)}.$$

Furthermore, by using Proposition 1.14 (i) and (ii), he also deduces many formulas ([15, (5.9b)–(5.9f), (5.10b) and (5.10c)].

THEOREM 1.26 ([15, (6.3)]). *We have*

$$F_D^{(3)}\left(\begin{matrix} a; 1-a, 1+a-c, 2b-1 \\ c \end{matrix}; x, \frac{x}{1-x}, 2x\right) = (1-x)^a {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 4x(1-x)\right).$$

PROOF. By the change of variable $u = v(1-xv)/(1-x)$ and setting $z = 4x(1-x)$ in the integral (1.1), we can prove the theorem similar to the proofs of the theorems above. □

Appell's functions F_2, F_3 and F_4 can be reduced to F_1 and one-variable hypergeometric functions in particular cases as follows.

PROPOSITION 1.27 (cf. [27, subsection 8.3.1]).

(i) *For Appell's functions* F_2,

$$F_2(a; b_1, b_2; c, a; x, y) = (1-y)^{-b_2} F_1\Big(b_1; a-b_2, b_2; c; x, \frac{x}{1-y}\Big).$$

In particular, by Proposition 1.14 (i),

$$F_2(a; b_1, b_2; a, a; x, y) = (1-x)^{-b_1}(1-y)^{-b_2} {}_2F_1\left(\begin{matrix} b_1, b_2 \\ a \end{matrix}; \frac{xy}{(1-x)(1-y)}\right).$$

(ii) *For Appell's functions* F_3,

$$F_3(a, c-a; b_1, b_2; c; x, y) = (1-y)^{-b_2} F_1\Big(a; b_1, b_2; c; x, \frac{y}{y-1}\Big).$$

In particular, by Proposition 1.14 (i),

$$F_3(a, c-a; b, c-b; c; x, y) = (1-y)^{a+b-c} {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x+y-xy\right).$$

PROOF. By (1.3) and (1.8),

$$\begin{aligned} &F_2(a; b_1, b_2; c, a; x, y) \\ &= \sum_{m=0}^{\infty} \frac{(a)_m (b_1)_m}{(c)_m (1)_m} x^m {}_2F_1\left(\begin{matrix} a+m, b_2 \\ a \end{matrix}; y\right) \\ &= (1-y)^{-b_2} \sum_m \frac{(a)_m (b_1)_m}{(c)_m (1)_m} x^m {}_2F_1\left(\begin{matrix} -m, b_2 \\ a \end{matrix}; \frac{y}{y-1}\right). \end{aligned} \tag{1.21}$$

Using that

$${}_2F_1\left(\begin{matrix} a, -m \\ b \end{matrix}; z\right) = \frac{(b-a)_m}{(b)_m} {}_2F_1\left(\begin{matrix} a, -m \\ 1+a-b-m \end{matrix}; 1-z\right) \quad (m \in \mathbb{Z}_{\geq 0}),$$

(1.21) is equal to

$$\begin{aligned} &(1-y)^{-b_2} \sum_m \frac{(b_1)_m (a-b_2)_m}{(c)_m (1)_m} x^m {}_2F_1\left(\begin{matrix} b_2, -m \\ 1+b_2-a-m \end{matrix}; \frac{1}{1-y}\right) \\ &= (1-y)^{-b_2} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(b_1)_m (a-b_2)_m (b_2)_n (-m)_n}{(c)_m (1)_m (1+b_2-a-m)_n (1)_n} x^m (1-y)^{-n}. \end{aligned}$$

Noting (1.5) and (1.3) and putting $s = m - n$, the right-hand side above is

$$
\begin{aligned}
&(1-y)^{-b_2}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(b_1)_m(a-b_2)_{m-n}(b_2)_n}{(c)_m(1)_{m-n}(1)_n}x^m(1-y)^{-n}\\
&=(1-y)^{-b_2}\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}\frac{(b_1)_{s+n}(a-b_2)_s(b_2)_n}{(c)_{s+n}(1)_s(1)_n}x^s\left(\frac{x}{1-y}\right)^n.
\end{aligned}
$$

Thus, we obtain (i). Using the Pfaff formula (1.8), (ii) can be proved similar as (i). ☐

Let $\binom{m}{n}$ be the binomial coefficient. Hereafter, we use that

$$
\binom{m}{n}=\frac{(1)_m}{(1)_n(1)_{m-n}}=\frac{(-1)^n(-m)_n}{(1)_n}. \tag{1.22}
$$

Bailey [7] proves the following formulas for Appell's F_2 and F_3 by using formulas for ${}_3F_2(1)$.

Proposition 1.28 ([7, (4.4), (4.5) and (4.3)]).

(i) *We have*

$$
F_2(a;b,c;2b,2c;x,-x)={}_4F_3\left({\frac{a}{2},\frac{a+1}{2},\frac{b+c}{2},\frac{b+c+1}{2} \atop b+\frac{1}{2},c+\frac{1}{2},b+c};x^2\right).
$$

(ii) *We have*

$$
F_2(a;b,b;c,c;x,-x)={}_4F_3\left({\frac{a}{2},\frac{a+1}{2},b,c-b \atop c,\frac{c}{2},\frac{c+1}{2}};x^2\right).
$$

(iii) *We have*

$$
F_3(a,a;b,b;c;x,-x)={}_4F_3\left({\frac{a+b}{2},\frac{a+b+1}{2},a,b \atop a+b,\frac{c}{2},\frac{c+1}{2}};x^2\right).
$$

Proof. Recall Watson's formula (cf. [4, subsection 3.3 (1)])

$$
{}_3F_2\left({a,b,c \atop \frac{a+b+1}{2},2c};1\right)=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+c)\Gamma(\frac{1+a+b}{2})\Gamma(\frac{1-a-b}{2}+c)}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1+b}{2})\Gamma(\frac{1-a}{2}+c)\Gamma(\frac{1-b}{2}+c)}.
$$

Note that the left-hand side above is 0 if a is a negative odd integer. By the formula with $a = -n$ and $b = 1 - 2b - n$, we have

$$
\begin{aligned}
&\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{(b)_{n-r}(c)_r}{(2b)_{n-r}(2c)_r}\\
&=\begin{cases}2^{-2m}\dfrac{(\frac{1}{2})_m(b+c)_{2m}}{(b+\frac{1}{2})_m(c+\frac{1}{2})_m(b+c)_m} & (n=2m),\\[2ex] 0 & (n\text{ is odd}).\end{cases}
\end{aligned}
$$

Therefore, using this and (1.13),

$$
F_2(a;b,c;2b,2c;x,-x)=\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r\frac{(a)_n(b)_{n-r}(c)_r}{(1)_{n-r}(1)_r(2b)_{n-r}(2c)_r}x^n
$$

$$\begin{aligned}
&= \sum_{m=0}^{\infty} \frac{(a)_{2m}}{(1)_{2m}} x^{2m} \sum_{r=0}^{2m} (-1)^r \binom{2m}{r} \frac{(b)_{2m-r}(c)_r}{(2b)_{2m-r}(2c)_r} \\
&= \sum_{m} \frac{(\frac{1}{2})_m (a)_{2m} (b+c)_{2m}}{2^{2m}(1)_{2m}(b+\frac{1}{2})_m (c+\frac{1}{2})_m (b+c)_m} x^{2m} \\
&= {}_4F_3\left({\frac{a}{2}, \frac{a}{2}+\frac{1}{2}, \frac{b}{2}, \frac{b}{2}+\frac{1}{2} \atop b+\frac{1}{2}, c+\frac{1}{2}, b+c} ; x^2 \right).
\end{aligned}$$

Thus, we obtain (i).

Dixon's formula (cf. [4, subsection 3.1 (1)]) is

$$\begin{aligned}
&{}_3F_2\left({a, b, c \atop 1+a-b, 1+a-c} ; 1 \right) \\
&\quad = \frac{\Gamma(1+\frac{a}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{a}{2}-b-c)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)\Gamma(1+\frac{a}{2}-c)\Gamma(1+a-b-c)}.
\end{aligned} \tag{1.23}$$

By this formula with $a=-n$ and $c=1-c-n$ and (1.3), we have

$$\begin{aligned}
&\sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{(b)_{n-r}(b)_r}{(c)_{n-r}(c)_r} \\
&\quad = \begin{cases} \left(\dfrac{1}{2}\right)_m \dfrac{(b)_m (c-b)_m}{(c)_m (\frac{c}{2})_m (\frac{c+1}{2})_m} & (n=2m), \\ 0 & (n \text{ is odd}). \end{cases}
\end{aligned} \tag{1.24}$$

Similarly, using the formula (1.23) with $a=-n$ and $c=a$, we have

$$\begin{aligned}
&\sum_{r=0}^{n} (-1)^r \binom{n}{r} (a)_r (b)_r (a)_{n-r} (b)_{n-r} \\
&\quad = \begin{cases} \dfrac{(1)_{2m} (a)_m (b)_m (a+b)_{2m}}{(1)_m (a+b)_m} & (n=2m), \\ 0 & (n \text{ is odd}). \end{cases}
\end{aligned} \tag{1.25}$$

We obtain (ii) and (iii) by (1.24) and (1.25) respectively similar to (i). □

The following is also known due to Bailey.

PROPOSITION 1.29 ([5, (4.1)-(4.4)]). *We have*

$$\begin{aligned}
&F_4\Big(a; b; c, b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\Big) \\
&\quad = (1-x)^a (1-y)^a F_1(a; c-b, 1+a-c; c; x, xy).
\end{aligned}$$

In particular,

$$F_4\Big(a; b; a, b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\Big) = (1-xy)^{-1}(1-x)^b(1-y)^a,$$

$$\begin{aligned}
&F_4\Big(a; b; b, b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\Big) \\
&\qquad\qquad = (1-x)^a (1-y)^a {}_2F_1\left({a, 1+a-b \atop b} ; xy \right),
\end{aligned}$$

$$
\begin{aligned}
&F_4\Big(a; b; 1+a-b, b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\Big)\\
&\qquad\qquad = (1-y)^a {}_2F_1\left(\begin{matrix} a, b \\ 1+a-b \end{matrix}; -\frac{x(1-y)}{1-x}\right).
\end{aligned}
$$

Proof. Put $C_{m,n}$ is as in the proof of Proposition 1.3. Then, as mentioned at (1.7), we have

$$
C_{m,n} = \frac{(a)_m (b)_n (c_1-b-n)_m (c_2-a-m)_n}{(1)_m (1)_n (c_1)_m (c_2)_n}.
$$

Thus, when $c_2 = b$, we have (rewrite c for c_1)

$$
\begin{aligned}
&(1-x)^{-a}(1-y)^{-b} F_4\Big(a; b; c, b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\Big)\\
&= \sum_m \sum_n \frac{(a)_m (c-b-n)_m (b-a-m)_n}{(1)_m (1)_n (c)_m} x^m y^n\\
&= \sum_m \frac{(a)_m (c-b)_m}{(1)_m (c)_m} x^m {}_2F_1\left(\begin{matrix} b-a-m, 1+b-c \\ 1+b-c-m \end{matrix}; y\right)\\
&= \sum_m \frac{(a)_m (c-b)_m}{(1)_m (c)_m} x^m (1-y)^{a-b} {}_2F_1\left(\begin{matrix} -m, 1+a-c \\ 1+b-c-m \end{matrix}; y\right)\\
&= (1-y)^{a-b} \sum_m \sum_{r=0}^{m} \frac{(a)_m (c-b)_m}{(1)_m (c)_m} \cdot \frac{(-m)_r (1+a-c)_r}{(1)_r (1+b-c-m)_r} x^m y^r. \qquad (1.26)
\end{aligned}
$$

Here, we used (1.3) and (1.5) at the second equality, and we used Euler's formula (1.10) at the third equality. Write $m = r+s$. Then, (1.26) is equal to

$$
\begin{aligned}
&(1-y)^{a-b} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_{r+s} (c-b)_s (1+a-c)_r}{(1)_r (1)_s (c)_{r+s}} x^{r+s} y^r\\
&= (1-y)^{a-b} F_1(a; c-b, 1+a-c; c; x, xy).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&F_4\Big(a; b; c, b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\Big)\\
&\qquad = (1-x)^a (1-y)^a F_1(a; c-b, 1+a-c; c; x, xy).
\end{aligned}
$$

For the particular cases, we obtain the first formula by (1.16). The second formula can be obtained by (1.17), and the third formula follows by Proposition 1.14 (i) and (1.17). □

Srivastava [28] gives the following ((i) was first found by Burchnall [9]).

Proposition 1.30 (cf. [28, (8) and (9)]).

(i) *We have*

$$
F_4(a; b; c_1, c_2; x, x) = {}_4F_3\left(\begin{matrix} a, b, \frac{c_1+c_2-1}{2}, \frac{c_1+c_2}{2} \\ c_1, c_2, c_1+c_2-1 \end{matrix}; 4x\right).
$$

(ii) *We have*

$$F_4(a;b;c,c;x,-x) = {}_4F_3\left(\begin{matrix}\frac{a}{2}, \frac{a+1}{2}, \frac{b}{2}, \frac{b+1}{2} \\ c, \frac{c}{2}, \frac{c+1}{2}\end{matrix}; -4x^2\right).$$

Proof. (i) By the Vandermonde theorem (1.6) and (1.22), we have

$$\sum_{r=0}^{m} \binom{m}{r} \frac{1}{(a)_{m-r}(b)_r} = \frac{(a+b+m-1)_m}{(a)_m(b)_m}.$$

Thus, putting $m = r+s$,

$$\begin{aligned} F_4(a;b;c_1,c_2;x,x) &= \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_{r+s}}{(c_1)_r(c_2)_s(1)_r(1)_s} x^{r+s} \\ &= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(1)_m} x^m \sum_{r=0}^{m} \binom{m}{r} \frac{1}{(c_1)_{m-r}(c_2)_r} \\ &= \sum_{m} \frac{(a)_m(b)_m(c_1+c_2+m-1)_m}{(c_1)_m(c_2)_m(1)_m} x^m \\ &= \sum_{m} \frac{(a)_m(b)_m(c_1+c_2-1)_{2m}}{(c_1)_m(c_2)_m(c_1+c_2-1)_m(1)_m} x^m. \end{aligned}$$

Hence, we obtain the formula (i) by applying (1.13) to $(c_1+c_2-1)_{2m}$.

(ii) By Kummer's theorem (1.20) with $a = -m$, we have

$$\begin{aligned} &\sum_{r=0}^{n} (-1)^n \binom{n}{r} \frac{1}{(c)_{n-r}(c)_r} \\ &= \begin{cases} (-1)^m \dfrac{(1)_{2m}}{(1)_m(c)_m(c)_{2m}} & (n = 2m), \\ 0 & (n \text{ is odd}), \end{cases} \end{aligned}$$

(for the case when n is odd, it is clear by the symmetry). Therefore,

$$\begin{aligned} F_4(a;b;c,c;x,-x) &= \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(1)_n} x^n \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{1}{(c)_{n-r}(c)_r} \\ &= \sum_{m=0}^{\infty} \frac{(a)_{2m}(b)_{2m}}{(1)_m(c)_m(c)_{2m}} (-x)^m. \end{aligned}$$

Thus, we complete the proof by applying (1.13) to $(a)_{2m}$, $(b)_{2m}$ and $(c)_{2m}$. □

Remark 1.31. Some finite field analogues of the results so far are known.

(i) Frechette-Swisher-Tu [12, Propositions 6.6 and 6.7] give analogues of Proposition 1.14 (i) and (iii). Furthermore, they also give an analogue [12, Theorem 1.1] of the Koike-Shiga formula (see Remark 1.18 (i)).

(ii) He [13, Corollary 2.2] proves an analogue of Proposition 1.21.

(iii) He-Li-Zhang [14, Theorems 3.3 and 3.4] give analogues of the case $n = 2$ of Propositions 1.14 (iv) and 1.22 (ii). For general n, Chetry-Kalita [11, Theorems 3.7 and 3.8] prove analogues of the case $r = 1$ of Propositions 1.14 (iv) and 1.22 (ii).

(iv) Tripathi-Saikia-Barman [31, Theorems 1.2 and 1.3] give analogues of the formulas in Proposition 1.27.

(v) Tripathi-Barman [29, Theorems 1.2 and 1.3] prove analogue of Proposition 1.29 and (1.4). An alternative proof of the analogue of (1.4) is given by Otsubo-Senoue [25, Theorem 4.1]. Furthermore, Tripathi-Barman [30, Theorems 1.1, 1.5, 1.6 and 1.7] give analogues of the formulas in Propositions 1.28 and 1.30.

(vi) Analogues of Propositions 1.1, 1.2, 1.3 and 1.5 and Theorem 1.23 will be obtained in the next part.

2. Algebraic varieties connected to Appell-Lauricella functions

Throughout this section, let $a, a_i, b, b_i, c, c_i \in \mathbb{Z}_{\geq 1}$ and $\lambda_i \in \mathbb{C}$ $(i = 1, \ldots, n)$ and write $\lambda = (\lambda_1, \ldots, \lambda_n)$.

2.1. **Algebraic varieties connected to Appell-Lauricella functions.**

We consider the case when the parameters of Appell-Lauricella functions are rational numbers. By the integral representation in Propositions 1.1 (i), each of Lauricella's functions $F_D^{(n)}$ is regarded as a period of a curve defined by an equation of the form

$$C_{D,\lambda} : y^d = \Big(\prod_{i=1}^{n}(1-\lambda_i x)^{b_i}\Big)x^a(1-x)^c. \tag{2.1}$$

Furthermore, it is also regarded as a period of an n-dimensional hypersurface defined by an equation of the form

$$y^d = \Big(1-\sum_{i=1}^{n}\lambda_i x_i\Big)^a\Big(\prod_{i=1}^{n}x_i^{b_i}\Big)\Big(1-\sum_{i=1}^{n}x_i\Big)^c,$$

by the integral representation in Proposition 1.1 (ii).

Similarly, by the integral representations in Proposition 1.2, we have the followings. Each of Lauricella's functions $F_A^{(n)}$ is connected to each of two-types n-dimensional hypersurfaces defined respectively by an equation of the forms

$$y^d = \Big(1-\sum_{i=1}^{n}\lambda_i x_i\Big)^a\prod_{i=1}^{n}x_i^{b_i}(1-x_i)^{c_i},$$

$$y^d = \Big(\prod_{i=1}^{n}(x_i-\lambda_i)^{b_i}\Big)\Big(\prod_{i=1}^{n}x_i^{c_i}\Big)\Big(1-\sum_{i=1}^{n}x_i\Big)^a.$$

Each of Lauricella's functions $F_B^{(n)}$ is connected to an n-dimensional hypersurface defined by an equation of the form

$$y^d = \Big(\prod_{i=1}^{n}(1-\lambda_i x_i)^{a_i}\Big)\Big(\prod_{i=1}^{n}x_i^{b_i}\Big)\Big(1-\sum_{i=1}^{n}x_i\Big)^c.$$

Each of Lauricella's functions $F_C^{(n)}$ is connected to an n-dimensional hypersurface defined by an equation of the form

$$y^d = \Big(\prod_{i=1}^{n}x_i^{c_i}\Big)\Big(1-\sum_{i=1}^{n}x_i\Big)^a\Big(\prod_{i=1}^{n}x_i-\sum_{i=1}^{n}\lambda_i\prod_{j\neq i}x_j\Big)^b.$$

By the integral representation in Proposition 1.5, each of Appell's functions $F_4(\lambda_1(1-\lambda_2), \lambda_2(1-\lambda_1))$ is connected to a surface defined by an equation of the form

$$\begin{aligned} y^d =& x_1^{\langle a\rangle} x_2^{\langle b\rangle} (1-x_1)^{\langle c_1-a\rangle} (1-x_2)^{\langle c_2-b\rangle} \\ & \times (1-\lambda_1 x_1)^{\langle a-c_1-c_2\rangle} (1-\lambda_2 x_2)^{\langle b-c_1-c_2\rangle} (1-\lambda_1 x_1-\lambda_2 x_2)^{\langle c_1+c_2-a-b\rangle}. \end{aligned}$$

Here, for $n \in \mathbb{Z}$, $\langle n\rangle \in \{0, \ldots, d-1\}$ denotes the representative of n mod d.

2.2. **Smooth compactification of $C_{D,\lambda}$.** A smooth compactification of $C_{D,\lambda}$ is given explicitly in Archinard [2]. Let $\overline{C}_{D,\lambda}$ be the projective curve obtained by the homogenization of the equation (2.1). We denote the coordinates of $\overline{C}_{D,\lambda}$ by $(X : Y : Z)$, where $x = X/Z$ and $y = Y/Z$. Without loss of generality, we assume that $\prod_{i=1}^n \lambda_i(1-\lambda_i)\prod_{j\neq i}(\lambda_j-\lambda_i) \neq 0$. Put $e = |a+c+\sum_i b_i - d|$ and suppose that $e \neq 0$, then the curve $\overline{C}_{D,\lambda}$ has the only one point at infinity ∞. If $a > 1$ (resp. $b_i > 1$, $c > 1$, $e > 1$) then $\overline{C}_{D,\lambda}$ is singular at $P_0 := (0 : 0 : 1)$ (resp. $P_{\lambda_i} := (\lambda_i^{-1} : 0 : 1)$, $P_1 := (1 : 0 : 1)$, ∞). Archinard [2] constructs a desingularization $\pi : X_{D,\lambda} \to \overline{C}_{D,\lambda}$ by the blowup. For each singular points, we have (see [2, Remarks 3, 5])

$$\begin{aligned} \#\pi^{-1}(P_0) &= (d,a), \\ \#\pi^{-1}(P_{\lambda_i}) &= (d,b_i), \\ \#\pi^{-1}(P_1) &= (d,c), \\ \#\pi^{-1}(\infty) &= (d,e), \end{aligned}$$

where we write $(m,n) = \gcd(m,n)$. Let $p : \overline{C}_{D,\lambda} \to \mathbb{P}^1$ be the projection given by $(X : Y : Z) \mapsto (X : Z)$, and put

$$\nu := p\circ\pi : X_{D,\lambda} \longrightarrow \mathbb{P}^1.$$

Applying the Hurwitz formula (cf. [20, subsection 7.4.2]) to ν, we have the following.

THEOREM 2.1 ([2, Theorem 4.1]). *Suppose that $(d, a, b_1, \ldots, b_n, c) = 1$. Then,*

$$\text{genus } X_{D,\lambda} = 1 + \frac{1}{2}\Big(d(n+1) - (d,a) - \sum_{i=1}^n (d,b_i) - (d,c) - (d,e)\Big).$$

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Part 2. Appell-Lauricella hypergeometric functions over finite fields and algebraic varieties

1. Introduction

Generalized hypergeometric functions ${}_{n+1}F_n(z)$ (the Gauss hypergeometric functions when $n = 1$) over $\mathbb{C}$ are defined by the power series

$$ {}_{n+1}F_n\left(\begin{matrix} a_0, a_1, \dots, a_n \\ b_1, \dots, b_n \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_n)_k}{(1)_k (b_1)_k \cdots (b_n)_k} z^k. $$

Here, a_i, b_j are complex parameters with $b_j \notin \mathbb{Z}_{\leq 0}$, and $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol. Lauricella's hypergeometric functions $F_D^{(n)}, F_A^{(n)}, F_B^{(n)}$ and $F_C^{(n)}$ with n variables (Appell's functions F_1, F_2, F_3 and F_4 respectively, when $n = 2$) are generalizations of the Gauss hypergeometric functions. For example,

$$ F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; z_1, \dots, z_n\right) := \sum_{k_i \geq 0} \frac{(a)_{k_1+\cdots+k_n} (b_1)_{k_1} \cdots (b_n)_{k_n}}{(c)_{k_1+\cdots+k_n} (1)_{k_1} \cdots (1)_{k_n}} z_1^{k_1} \cdots z_n^{k_n}, $$

where $a, b_i, c \in \mathbb{C}$ with $c \notin \mathbb{Z}_{\leq 0}$. These functions have integral representations of Euler type, such as

$$ \begin{aligned} & F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; z_1, \dots, z_n\right) \\ & = B(a, c-a)^{-1} \int_0^1 \left(\prod_{i=1}^{n} (1 - z_i u)^{-b_i}\right) u^{a-1} (1-u)^{c-a-1}\, du. \end{aligned} $$

Over finite fields, one-variable hypergeometric functions were defined independently by Koblitz [15], Katz [13], Greene [8], McCarthy [19], Fuselier-Long-Ramakrishna-Swisher-Tu [7] and Otsubo [23]. Appell's functions were defined by Li-Li-Mao [17], He [9], He-Li-Zhang [11] and Ma [18] as generalizations of Greene's functions, and were defined by Tripathi-Barman [27] and Tripathi-Saikia-Barman [29] as generalizations of McCarthy's functions. For general n, $F_D^{(n)}$ were defined by Frechette-Swisher-Tu [5] and He [10], and $F_A^{(n)}$ were defined by Chetry-Kalita [4] as generalizations of Greene's functions. Otsubo [23] gave a definition of all the Lauricella functions, which will be used in this paper (see subsection 2.1).

In this paper, we prove finite field analogues of integral representations of $F_D^{(n)}$, $F_A^{(n)}$, $F_B^{(n)}$ and $F_C^{(n)}$ (Theorems 3.1, 3.3, 3.4, 3.5 and 3.7). As a corollary, we prove a finite analogue of Karlsson's formula which relates certain $F_D^{(n)}$ with Gauss hypergeometric functions (Theorem 3.2). Furthermore, we show a finite field analogue (Theorem 3.12) of an integral representation of $F_4(x(1-y), y(1-x))$ due to Burchnall-Chaundy [3].

The reason for the strong analogy between a hypergeometric function over $\mathbb{C}$ and a hypergeometric function over a finite field is that they come from a same algebraic variety. The former is the complex period of the variety and the latter is the trace of Frobenius acting on the l-adic étale cohomology. By the Grothendieck-Lefschetz formula, the Frobenius trace is related with the number of rational points on the variety. For example, one-variable hypergeometric functions, over $\mathbb{C}$ and over finite fields, are associated with the variety of the form

$$y^d = (1-\lambda x_1\cdots x_n)^{a_0}\prod_{i=1}^n x_i^{a_i}(1-x_i)^{b_i}.$$

By computing the number of its rational points over finite fields, Koblitz [15] arrived at his definition of the hypergeometric function.

For the Appell-Lauricella functions, we find naturally corresponding algebraic varieties from the complex integral representations. For example, an algebraic curve $C_{D,\lambda}$ related to $F_D^{(n)}$ is given by

$$y^d = \Big(\prod_{i=1}^n (1-\lambda_i x)^{b_i}\Big) x^a (1-x)^c.$$

They admit an action of the group μ_d of dth roots of unity, and each of the numbers of κ-rational points decomposes into χ-components for characters χ of μ_d, where κ is a finite field. By the analogues of integral representations mentioned above, such numbers are expressed in terms of Appell-Lauricella functions over κ (Theorems 4.2, 4.4, 4.8, 4.9, 4.11 and 4.13).

According to the decomposition of the numbers, each of the zeta functions decomposes into the Artin L-functions. As corollaries of the theorems, we express the Artin L-functions in terms of the Appell-Lauricella functions over κ_r $(r \geq 1)$, where κ_r is a degree r extension of κ (Corollaries 4.5, 4.10, 4.12 and 4.14).

Furthermore, under some conditions, we will closely look at the curve $X_{D,\lambda}$ which is a smooth projective model of $C_{D,\lambda}$. For each non-trivial character χ of μ_d, using the result above, the Artin L-function $L(X_{D,\lambda},\chi;t)$ is written in terms of Lauricella functions $F_D^{(n)}(\lambda_1,\dots,\lambda_n)_{\kappa_r}$ over κ_r $(r \geq 1)$. By the Grothendieck-Lefschetz formula, the Artin L-function $L(X_{D,\lambda},\chi;t)$ is essentially the characteristic polynomial of the Frobenius acting on the χ-eigenspace $H^1(X_{D,\lambda},\overline{\mathbb{Q}_l})(\chi)$ of the first l-adic étale cohomology. By computing its dimension, we will show that the degree of $L(X_{D,\lambda},\chi;t)$ is $n+1$ (Theorem 4.7), and hence it follows that $F_D^{(n)}(\lambda_1,\dots,\lambda_n)_{\kappa_r}$ $(r \geq 1)$ are written as symmetric polynomials of the first $n+1$ functions.

2. Hypergeometric functions over finite fields

Throughout this paper, let κ be a finite field with q elements of characteristic p. Let $\widehat{\kappa^\times} = \mathrm{Hom}(\kappa^\times, \overline{\mathbb{Q}}^\times)$ denote the group of multiplicative characters of κ, and write $\varepsilon \in \widehat{\kappa^\times}$ for the trivial character. For any $\eta \in \widehat{\kappa^\times}$, we set $\eta(0)=0$ and write $\overline{\eta} = \eta^{-1}$. Put, for $\eta \in \widehat{\kappa^\times}$,

$$\delta(\eta) = \begin{cases} 1 & (\eta = \varepsilon), \\ 0 & (\eta \neq \varepsilon). \end{cases}$$

Throughout this paper, we fix a non-trivial additive character $\psi \in \mathrm{Hom}(\kappa, \overline{\mathbb{Q}}^\times)$.

2.1. Definitions.

In this subsection, we recall definitions [23] of hypergeometric functions over finite fields.

For $\eta, \eta_1, \dots, \eta_n \in \widehat{\kappa^\times}$ $(n \geq 2)$, *the Gauss sum* $g(\eta)$ and *the Jacobi sum* $j(\eta_1,\dots,\eta_n)$ are defined by

$$g(\eta) = -\sum_{x\in\kappa^\times} \psi(x)\eta(x) \ \in \mathbb{Q}(\mu_{p(q-1)}),$$

$$j(\eta_1,\dots,\eta_n)=(-1)^{n-1}\sum_{\substack{x_i\in\kappa^\times\\ x_1+\cdots+x_n=1}}\prod_{i=1}^{n}\eta_i(x_i)\ \in\mathbb{Q}(\mu_{q-1}).$$

Note that $g(\varepsilon)=1$. Put $g^\circ(\eta)=q^{\delta(\eta)}g(\eta)$. Then (cf. [23, Proposition 2.2 (iii)])

$$g(\eta)g^\circ(\overline{\eta})=\eta(-1)q. \tag{2.1}$$

For $\eta_1,\dots,\eta_n\in\widehat{\kappa^\times}$, we have (cf. [23, Proposition 2.2 (iv)])

$$j(\eta_1,\dots,\eta_n)=\begin{cases}\dfrac{1-(1-q)^n}{q} & (\eta_1=\cdots=\eta_n=\varepsilon),\\[2mm] \dfrac{g(\eta_1)\cdots g(\eta_n)}{g^\circ(\eta_1\cdots\eta_n)} & (\text{otherwise}).\end{cases} \tag{2.2}$$

As an analogue of the Pochhammer symbol $(a)_n=\Gamma(a+n)/\Gamma(a)$, put

$$(\alpha)_\nu=\frac{g(\alpha\nu)}{g(\alpha)},\qquad (\alpha)^\circ_\nu=\frac{g^\circ(\alpha\nu)}{g^\circ(\alpha)}$$

for $\alpha,\ \nu\in\widehat{\kappa^\times}$. Then, these satisfy

$$(\alpha)_{\nu\mu}=(\alpha)_\nu(\alpha\nu)_\mu,\quad (\alpha)^\circ_{\nu\mu}=(\alpha)^\circ_\nu(\alpha\nu)^\circ_\mu, \tag{2.3}$$

and

$$(\alpha)_\nu(\overline{\alpha})^\circ_{\overline{\nu}}=\nu(-1). \tag{2.4}$$

Definition 2.1. For $\alpha_0,\dots,\alpha_n,\beta_1,\dots,\beta_n\in\widehat{\kappa^\times}$, the hypergeometric function over κ is defined by

$${}_{n+1}F_n\left({\alpha_0,\alpha_1,\dots,\alpha_n \atop \beta_1,\dots,\beta_n};\lambda\right)=\frac{1}{1-q}\sum_{\nu\in\widehat{\kappa^\times}}\frac{(\alpha_0)_\nu(\alpha_1)_\nu\cdots(\alpha_n)_\nu}{(\varepsilon)^\circ_\nu(\beta_1)^\circ_\nu\cdots(\beta_n)^\circ_\nu}\nu(\lambda)\quad(\lambda\in\kappa).$$

Definition 2.2. For $\alpha,\alpha_1,\dots,\alpha_n,\beta,\beta_1,\dots,\beta_n,\gamma,\gamma_1,\dots,\gamma_n\in\widehat{\kappa^\times}$, Lauricella's functions over κ are defined as follows. For $\lambda_1,\dots,\lambda_n\in\kappa$,

$$F_A^{(n)}\left({\alpha;\beta_1,\dots,\beta_n \atop \gamma_1,\dots,\gamma_n};\lambda_1,\dots,\lambda_n\right)$$
$$=\frac{1}{(1-q)^n}\sum_{\nu_i\in\widehat{\kappa^\times}}\frac{(\alpha)_{\nu_1\cdots\nu_n}(\beta_1)_{\nu_1}\cdots(\beta_n)_{\nu_n}}{(\gamma_1)^\circ_{\nu_1}\cdots(\gamma_n)^\circ_{\nu_n}(\varepsilon)^\circ_{\nu_1}\cdots(\varepsilon)^\circ_{\nu_n}}\nu_1(\lambda_1)\cdots\nu_n(\lambda_n),$$

$$F_B^{(n)}\left({\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n \atop \gamma};\lambda_1,\dots,\lambda_n\right)$$
$$=\frac{1}{(1-q)^n}\sum_{\nu_i\in\widehat{\kappa^\times}}\frac{(\alpha_1)_{\nu_1}\cdots(\alpha_n)_{\nu_n}(\beta_1)_{\nu_1}\cdots(\beta_n)_{\nu_n}}{(\gamma)^\circ_{\nu_1\cdots\nu_n}(\varepsilon)^\circ_{\nu_1}\cdots(\varepsilon)^\circ_{\nu_n}}\nu_1(\lambda_1)\cdots\nu_n(\lambda_n),$$

$$F_C^{(n)}\left({\alpha;\beta \atop \gamma_1,\dots,\gamma_n};\lambda_1,\dots,\lambda_n\right)$$
$$=\frac{1}{(1-q)^n}\sum_{\nu_i\in\widehat{\kappa^\times}}\frac{(\alpha)_{\nu_1\cdots\nu_n}(\beta)_{\nu_1\cdots\nu_n}}{(\gamma_1)^\circ_{\nu_1}\cdots(\gamma_n)^\circ_{\nu_n}(\varepsilon)^\circ_{\nu_1}\cdots(\varepsilon)^\circ_{\nu_n}}\nu_1(\lambda_1)\cdots\nu_n(\lambda_n),$$

$$F_D^{(n)}\left({\alpha;\beta_1,\dots,\beta_n \atop \gamma};\lambda_1,\dots,\lambda_n\right)$$

$$= \frac{1}{(1-q)^n} \sum_{\nu_i \in \widehat{\kappa^\times}} \frac{(\alpha)_{\nu_1\cdots\nu_n}(\beta_1)_{\nu_1}\cdots(\beta_n)_{\nu_n}}{(\gamma)^\circ_{\nu_1\cdots\nu_n}(\varepsilon)^\circ_{\nu_1}\cdots(\varepsilon)^\circ_{\nu_n}} \nu_1(\lambda_1)\cdots\nu_n(\lambda_n).$$

Analogues of Appell's functions are defined by

$$\begin{aligned}
F_1(\alpha;\beta_1,\beta_2;\gamma;\lambda_1,\lambda_2) &= F_D^{(2)}\left(\begin{matrix}\alpha;\beta_1,\beta_2\\ \gamma\end{matrix};\lambda_1,\lambda_2\right),\\
F_2(\alpha;\beta_1,\beta_2;\gamma_1,\gamma_2;\lambda_1,\lambda_2) &= F_A^{(2)}\left(\begin{matrix}\alpha;\beta_1,\beta_2\\ \gamma_1,\gamma_2\end{matrix};\lambda_1,\lambda_2\right),\\
F_3(\alpha_1,\alpha_2;\beta_1,\beta_2;\gamma;\lambda_1,\lambda_2) &= F_B^{(2)}\left(\begin{matrix}\alpha_1,\alpha_2;\beta_1,\beta_2\\ \gamma\end{matrix};\lambda_1,\lambda_2\right),\\
F_4(\alpha;\beta;\gamma_1,\gamma_2;\lambda_1,\lambda_2) &= F_C^{(2)}\left(\begin{matrix}\alpha,\beta\\ \gamma_1,\gamma_2\end{matrix};\lambda_1,\lambda_2\right).
\end{aligned}$$

Remark 2.3. A priori, the functions ${}_{n+1}F_n$, $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ are $\mathbb{Q}(\mu_{p(q-1)})$-valued, but in fact they take values in $\mathbb{Q}(\mu_{q-1})$ and are independent of the choice of ψ (see [23, Lemma 2.5 (iii)]).

Remark 2.4. By (2.4), (2.3) and (2.1), one shows that, for $\lambda_i \in \kappa^\times$,

$$\begin{aligned}
&F_B^{(n)}\left(\begin{matrix}\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n\\ \gamma\end{matrix};\lambda_1,\dots,\lambda_n\right)\\
&= (\overline{\gamma})_{\beta_1\cdots\beta_n}\Big(\prod_{i=1}^n(\alpha_i)_{\overline{\beta_i}}\overline{\beta_i}(\lambda_i)\Big)F_A^{(n)}\left(\begin{matrix}\beta_1\cdots\beta_n\overline{\gamma};\beta_1,\dots,\beta_n\\ \overline{\alpha_1}\beta_1,\dots,\overline{\alpha_n}\beta_n\end{matrix};\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_n}\right).
\end{aligned}$$

2.2. **Properties.**

We recall some formulas on ${}_{n+1}F_n$ which will be used in the next section.

Proposition 2.5 ([23, Corollary 3.4 and Corollary 3.5]).

(i) *For each* $\alpha \in \widehat{\kappa^\times}$ *and* $\lambda \in \kappa^\times$,

$${}_1F_0\left(\begin{matrix}\alpha\\ \end{matrix};\lambda\right) = \begin{cases}\overline{\alpha}(1-\lambda) & (\alpha \neq \varepsilon \text{ or } \lambda \neq 1),\\ 1-q & (\alpha = \varepsilon \text{ and } \lambda = 1).\end{cases}$$

(ii) *Suppose that* $\beta \neq \gamma$. *Then, for* $\lambda \neq 0$,

$$\begin{aligned}
&-j(\beta,\overline{\beta}\gamma)\,{}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};\lambda\right)\\
&\quad= \sum_{u\in\kappa^\times}\beta(u)\overline{\beta}\gamma(1-u)\overline{\alpha}(1-\lambda u) + \delta(\alpha)(1-q)\overline{\gamma}(\lambda)\overline{\beta}\gamma(\lambda-1).
\end{aligned}$$

Remark 2.6. In (ii) above, the case when $\alpha = \varepsilon$ is not contained in [23, Corollary 3.5], but one shows the case easily by Lemma 2.8.

Proposition 2.7 (cf. [23, Theorem 3.2]). *If* $n \geq 1$,

$$\begin{aligned}
&{}_{n+1}F_n\left(\begin{matrix}\alpha_1,\dots,\alpha_n,\gamma\\ \beta_1,\dots,\beta_{n-1},\gamma\end{matrix};\lambda\right)\\
&\quad= q^{\delta(\gamma)}\left({}_nF_{n-1}\left(\begin{matrix}\alpha_1,\dots,\alpha_n\\ \beta_1,\dots,\beta_{n-1}\end{matrix};\lambda\right) + \frac{1}{q}\cdot\frac{\prod_{i=1}^n(\alpha_i)_{\overline{\gamma}}}{(\varepsilon)^\circ_{\overline{\gamma}}\prod_{i=1}^{n-1}(\beta_i)^\circ_{\overline{\gamma}}}\overline{\gamma}(\lambda)\right).
\end{aligned}$$

Lemma 2.8. *For $\lambda \in \kappa^\times$,*

$$
{}_2F_1\left(\begin{matrix}\alpha,\varepsilon\\ \gamma\end{matrix};\lambda\right)=\begin{cases}\dfrac{g(\alpha\overline{\gamma})g^\circ(\gamma)}{g(\alpha)}\overline{\gamma}(\lambda)\overline{\alpha}\gamma(1-\lambda)+1 & (\lambda\neq 1 \text{ or } \alpha\neq\gamma),\\ 1+q^{\delta(\alpha)}(1-q) & (\lambda=1 \text{ and } \alpha=\gamma).\end{cases}
$$

Proof. By letting $\mu=\gamma\nu$ and using (2.3), we have

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix}\alpha,\varepsilon\\ \gamma\end{matrix};\lambda\right)&=\frac{1}{1-q}\sum_{\nu}q^{1-\delta(\nu)}\frac{(\alpha)_\nu}{(\gamma)^\circ_\nu}\nu(\lambda)\\
&=\frac{q}{1-q}\cdot\frac{(\alpha)_{\overline{\gamma}}}{(\gamma)^\circ_{\overline{\gamma}}}\overline{\gamma}(\lambda)\sum_{\mu}\frac{(\alpha\overline{\gamma})_\mu}{(\varepsilon)^\circ_\mu}\mu(\lambda)+1\\
&=\frac{g(\alpha\overline{\gamma})g^\circ(\gamma)}{g(\alpha)}\overline{\gamma}(\lambda){}_1F_0\left(\begin{matrix}\alpha\overline{\gamma}\\ \end{matrix};\lambda\right)+1.
\end{aligned}
$$

Thus, we obtain the lemma by Proposition 2.5 (i). ☐

The following propositions are slight generalizations of Otsubo's results [23]. A finite analogue of the Pfaff formula is the following.

Proposition 2.9 (cf. [23, Theorem 3.14]). *Suppose that $\beta\neq\varepsilon$, $\alpha\neq\gamma$. Then, for $\lambda\neq 1$,*

$$
\begin{aligned}
&\alpha(1-\lambda){}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};\lambda\right)\\
&={}_2F_1\left(\begin{matrix}\alpha,\overline{\beta}\gamma\\ \gamma\end{matrix};\frac{\lambda}{\lambda-1}\right)+\delta(\overline{\beta}\gamma)(1-q)\frac{g^\circ(\gamma)}{g(\alpha)g(\overline{\alpha}\gamma)}\overline{\gamma}(\lambda)\alpha(\lambda-1).
\end{aligned}
$$

Proof. By Proposition 2.5 (ii) and letting $v=u(1-\lambda)/(1-\lambda u)$, we have

$$
\begin{aligned}
-j(\alpha,\overline{\alpha}\gamma){}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};\lambda\right)&=\sum_{u}\alpha(u)\overline{\alpha}\gamma(1-u)\overline{\beta}(1-\lambda u)\\
&=\overline{\alpha}(1-\lambda)\sum_{v}\alpha(v)\overline{\alpha}\gamma(1-v)\beta\overline{\gamma}\Big(1-\frac{\lambda v}{\lambda-1}\Big).
\end{aligned}
$$

Thus the proposition follows from Proposition 2.5 (ii). ☐

The following is a finite analogue of the Vandermonde theorem (cf. [26, (1.7.7)]).

Proposition 2.10 (cf. [23, Theorem 4.3 and Remark 4.4]).

(i) *If $\{\alpha,\overline{\mu}\}\neq\{\varepsilon,\gamma\}$, then*

$$
{}_2F_1\left(\begin{matrix}\alpha,\overline{\mu}\\ \gamma\end{matrix};1\right)=q^{-\delta(\overline{\alpha}\gamma)}\frac{(\overline{\alpha}\gamma)_\mu}{(\gamma)^\circ_\mu}.
$$

(ii) *If $\{\alpha,\overline{\mu}\}=\{\varepsilon,\gamma\}$ then,*

$$
{}_2F_1\left(\begin{matrix}\alpha,\overline{\mu}\\ \gamma\end{matrix};1\right)=q^{-\delta(\overline{\alpha}\gamma)}\frac{(\overline{\alpha}\gamma)_\mu}{(\gamma)^\circ_\mu}-\frac{(1-q)^2(1+q)^{\delta(\gamma)}}{q}.
$$

Proof. (i) follows by [23, Theorem 4.3], hence we only have to prove (ii). Suppose that $\{\alpha,\overline{\mu}\}=\{\varepsilon,\gamma\}$. By [23, Theorem 4.3] again, it follows that

$${}_2F_1\left({\alpha,\overline{\mu} \atop \gamma};1\right)=1+q^{\delta(\gamma)}(1-q).$$

On the other hand, if $\alpha=\varepsilon$ and $\overline{\mu}=\gamma$ then,

$$q^{-\delta(\overline{\alpha}\gamma)}\frac{(\overline{\alpha}\gamma)_\mu}{(\gamma)^\circ_\mu}=q^{-\delta(\gamma)}\frac{(\gamma)_{\overline{\gamma}}}{(\gamma)^\circ_{\overline{\gamma}}}=\frac{1}{q},$$

and if $\alpha=\gamma$ and $\overline{\mu}=\varepsilon$ then,

$$q^{-\delta(\overline{\alpha}\gamma)}\frac{(\overline{\alpha}\gamma)_\mu}{(\gamma)^\circ_\mu}=q^{-1}\frac{(\varepsilon)_\varepsilon}{(\gamma)^\circ_\varepsilon}=\frac{1}{q}.$$

Thus, we have

$$q^{-\delta(\overline{\alpha}\gamma)}\frac{(\overline{\alpha}\gamma)_\mu}{(\gamma)^\circ_\mu}-\frac{(1-q)^2(1+q)^{\delta(\gamma)}}{q}=1+q^{\delta(\gamma)}(1-q).$$

Therefore, we obtain the proposition. □

A finite analogue of the Saalschütz theorem (cf. [26, (2.3.1.3)]) is the following.

Proposition 2.11 (cf. [23, Theorem 4.11]). *Suppose that* $\alpha\neq\varepsilon$, $\beta\neq\gamma$ *and* $\alpha\beta\overline{\gamma}\neq\varepsilon$. *Then,*

$$\begin{aligned}{}_3F_2\left({\alpha,\beta,\overline{\nu} \atop \gamma,\alpha\beta\overline{\gamma\nu}};1\right)=&q^{-\delta(\overline{\alpha}\gamma)}\frac{(\overline{\alpha}\gamma)_\nu(\overline{\beta}\gamma)_\nu}{(\gamma)^\circ_\nu(\overline{\alpha\beta}\gamma)_\nu}+\frac{g^\circ(\gamma)g^\circ(\alpha\beta\overline{\gamma\nu})}{g(\alpha)g(\beta)g(\overline{\nu})}\\&-(\delta(\overline{\alpha}\gamma)\delta(\nu)+\delta(\beta)\delta(\gamma\nu))\frac{(1-q)^2}{q}.\end{aligned}$$

Proof. By [23, Theorem 4.11], we only have to prove for the case when $\{\alpha,\beta,\overline{\nu}\}=\{\varepsilon,\gamma,\alpha\beta\overline{\gamma\nu}\}$ (i.e. $\overline{\alpha}\gamma=\nu=\varepsilon$ or $\beta=\gamma\nu=\varepsilon$). If $\overline{\alpha}\gamma=\nu=\varepsilon$, then the right-hand side of the proposition is equal to $3-q$. On the other hand, by Proposition 2.7 and Lemma 2.8, we have

$${}_3F_2\left({\alpha,\beta,\overline{\nu} \atop \gamma,\alpha\beta\overline{\gamma\nu}};1\right)={}_3F_2\left({\alpha,\beta,\varepsilon \atop \alpha,\beta};1\right)=\frac{1}{q}\cdot\frac{(\beta)_{\overline{\alpha}}(\varepsilon)_{\overline{\alpha}}}{(\varepsilon)^\circ_{\overline{\alpha}}(\beta)^\circ_{\overline{\alpha}}}+2-q=3-q.$$

Here, note that $\alpha\neq\beta$ and $\beta\neq\varepsilon$ by the assumptions. Similarly, we can prove for $\beta=\gamma\nu=\varepsilon$. □

3. FINITE ANALOGUES OF INTEGRAL REPRESENTATIONS

3.1. The case of F_D. For a function $f:(\kappa^\times)^n\to\mathbb{C}$, its *Fourier transform* is a function on $(\widehat{\kappa^\times})^n$ defined by

$$\widehat{f}(\nu_1,\dots,\nu_n)=\sum_{t_i\in\kappa^\times}f(t_1,\dots,t_n)\prod_{i=1}^n\overline{\nu_i}(t_i).$$

Then,

$$f(\lambda_1,\dots,\lambda_n)=\frac{1}{(q-1)^n}\sum_{\nu_i\in\widehat{\kappa^\times}}\widehat{f}(\nu_1,\dots,\nu_n)\prod_{i=1}^n\nu_i(\lambda_i).\tag{3.1}$$

Over $\mathbb{C}$, Lauricella's functions $F_D^{(n)}$ have the following integral representations (cf. [16, Theorem 3.4.1]). If $0 < \mathrm{Re}(a) < \mathrm{Re}(c)$,

$$B(a, c-a) F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; z_1, \dots, z_n\right) = \int_0^1 \Big(\prod_{i=1}^n (1 - z_i u)^{-b_i}\Big) u^{a-1}(1-u)^{c-a-1}\, du. \tag{3.2}$$

If $0 < \mathrm{Re}(b_i)$ for all i and $\mathrm{Re}(\sum_i b_i) < \mathrm{Re}(c)$, then

$$\frac{\Big(\prod_{i=1}^n \Gamma(b_i)\Big)\Gamma(c - \sum_{i=1}^n b_i)}{\Gamma(c)} F_D^{(n)}\left(\begin{matrix} a; b_1, \dots, b_n \\ c \end{matrix}; z_1, \dots, z_n\right) = \int_\Delta \Big(1 - \sum_{i=1}^n z_i u_i\Big)^{-a} \prod_{i=1}^n u_i^{b_i - 1} \Big(1 - \sum_{i=1}^n u_i\Big)^{c - \sum_{i=1}^n b_i - 1} du_1 \cdots du_n, \tag{3.3}$$

where $\Delta := \{(u_1, \dots, u_n) \in \mathbb{R}^n \mid u_i \geq 0,\ \sum_i u_i \leq 1\}$. Their finite analogues are as follows.

Theorem 3.1.

(i) *Suppose that* $\alpha \neq \gamma$ *and* $\beta_i \neq \varepsilon$ *for all* i*. Then, for* $\lambda_1, \dots, \lambda_n \in \kappa^\times$,

$$-j(\alpha, \overline{\alpha}\gamma) F_D^{(n)}\left(\begin{matrix} \alpha; \beta_1, \dots, \beta_n \\ \gamma \end{matrix}; \lambda_1, \dots, \lambda_n\right) = \sum_{u \in \kappa^\times} \Big(\prod_{i=1}^n \overline{\beta_i}(1 - \lambda_i u)\Big) \alpha(u) \overline{\alpha}\gamma(1-u).$$

(ii) *Suppose that* $\alpha \neq \varepsilon$ *and* $\beta_1 \cdots \beta_n \neq \gamma$*. Then, for* $\lambda_1, \dots, \lambda_n \in \kappa^\times$,

$$(-1)^n \frac{\Big(\prod_{i=1}^n g(\beta_i)\Big) g(\overline{\beta_1 \cdots \beta_n}\gamma)}{g^\circ(\gamma)} F_D^{(n)}\left(\begin{matrix} \alpha; \beta_1, \dots, \beta_n \\ \gamma \end{matrix}; \lambda_1, \dots, \lambda_n\right) = \sum_{u_1, \dots, u_n \in \kappa^\times} \overline{\alpha}\Big(1 - \sum_{i=1}^n \lambda_i u_i\Big) \Big(\prod_{i=1}^n \beta_i(u_i)\Big) \overline{\beta_1 \cdots \beta_n}\gamma\Big(1 - \sum_{i=1}^n u_i\Big).$$

Proof. (i) Put

$$f(\lambda_1, \dots, \lambda_n) = \sum_{u \in \kappa^\times} \Big(\prod_{i=1}^n \overline{\beta_i}(1 - \lambda_i u)\Big) \alpha(u) \overline{\alpha}\gamma(1-u).$$

Letting $s_i = t_i u$ for all i and using (2.2) (note that $\alpha \neq \gamma$ and $\beta_i \neq \varepsilon$) and (2.4), we have

$$\begin{aligned}
\widehat{f}(\nu_1, \dots, \nu_n) &= \sum_{t_1, \dots, t_n \in \kappa^\times} \sum_{u \in \kappa^\times} \alpha(u) \overline{\alpha}\gamma(1-u) \prod_i \overline{\beta_i}(1 - t_i u) \overline{\nu_i}(t_i) \\
&= \sum_u \alpha\nu_1 \cdots \nu_n(u) \overline{\alpha}\gamma(1-u) \sum_{s_1, \dots, s_n \in \kappa^\times} \prod_i \overline{\beta_i}(1 - s_i) \overline{\nu_i}(s_i) \\
&= (-1)^{n+1} j(\overline{\alpha}\gamma, \alpha\nu_1 \cdots \nu_n) \prod_i j(\overline{\beta_i}, \overline{\nu_i}) \\
&= (-1)^{n+1} j(\alpha, \overline{\alpha}\gamma) \cdot \frac{(\alpha)_{\nu_1 \cdots \nu_n}}{(\gamma)^\circ_{\nu_1 \cdots \nu_n}} \cdot \frac{\prod_i (\beta_i)_{\nu_i}}{\prod_i (\varepsilon)^\circ_{\nu_i}}.
\end{aligned}$$

Thus, (i) follows by (3.1).

(ii) Put

$$g(\lambda_1,\dots,\lambda_n)=\sum_{u_1,\dots,u_n\in\kappa^\times}\overline{\alpha}\Big(1-\sum_i\lambda_iu_i\Big)\Big(\prod_{i}^{n}\beta_i(u_i)\Big)\overline{\beta_1\cdots\beta_n\gamma}\Big(1-\sum_i u_i\Big).$$

Letting $s_i=t_iu_i$ for all i and using (2.2) (note that $\alpha\neq\varepsilon$ and $\beta_1\cdots\beta_n\neq\gamma$) and (2.4), we obtain

$$\begin{aligned}
&\widehat{g}(\nu_1,\dots,\nu_n)\\
&=\sum_{t_1,\dots,t_n\in\kappa^\times}\ \sum_{u_1,\dots,u_n\in\kappa^\times}\overline{\alpha}\Big(1-\sum_i t_iu_i\Big)\Big(\prod_i\beta_i(u_i)\Big)\overline{\beta_1\cdots\beta_n\gamma}\Big(1-\sum_i u_i\Big)\prod_i\overline{\nu_i}(t_i)\\
&=\sum_{u_1,\dots,u_n}\Big(\prod_i\beta_i\nu_i(u_i)\Big)\overline{\beta_1\cdots\beta_n\gamma}\Big(1-\sum_i u_i\Big)\sum_{s_1,\dots,s_n}\overline{\alpha}\Big(1-\sum_i s_i\Big)\prod_i\overline{\nu_i}(s_i)\\
&=j(\overline{\beta_1\cdots\beta_n\gamma},\beta_1\nu_1,\dots,\beta_n\nu_n)\cdot j(\overline{\alpha},\overline{\nu_1},\dots,\overline{\nu_n})\\
&=\frac{\Big(\prod_i g(\beta_i)\Big)g(\overline{\beta_1\cdots\beta_n\gamma})}{g^\circ(\gamma)}\cdot\frac{\prod_i(\beta_i)_{\nu_i}}{(\gamma)^\circ_{\nu_1\cdots\nu_n}}\cdot\frac{(\alpha)_{\nu_1\cdots\nu_n}}{\prod_i(\varepsilon)^\circ_{\nu_i}}.
\end{aligned}$$

Thus, (ii) follows by (3.1). □

Let $d\in\mathbb{Z}_{\geq1}$. Over $\mathbb{C}$, the Gauss hypergeometric functions have the integral representation (cf. [26, (1.6.6)])

$$B(a,c-a)\,{}_2F_1\left({a,b\atop c};z\right)=\int_0^1t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt. \tag{3.4}$$

If we put $\zeta=\exp(2\pi\sqrt{-1}/d)$, by the change of variable $t=\tau^d$ in (3.4) and using (3.2), we obtain

$$\begin{aligned}
&F_D^{(2d-1)}\left({da;\overbrace{a-c,\dots,a-c}^{d-1\ times},\overbrace{b,\dots,b}^{d\ times}\atop (d-1)a+c};\zeta,\dots,\zeta^{d-1},z,\zeta z,\dots,\zeta^{d-1}z\right)\\
&\quad=\frac{\Gamma(a)\Gamma((d-1)a+c)}{d\Gamma(da)\Gamma(c)}\,{}_2F_1\left({a,b\atop c};z^d\right).
\end{aligned}$$

This is a generalization of Karlsson's formula proved for $d=2,3$ [12, (4.10) and (6.1)]. As an application of Theorem 3.1, we obtain a finite analogue of this formula.

Theorem 3.2. *Suppose that $d\mid q-1$, $\alpha\neq\gamma$ and $\beta\neq\varepsilon$. Let $\varphi_d\in\widehat{\kappa^\times}$ be a character of exact order d and $\xi\in\kappa^\times$ be a primitive dth root of unity. Then, for any $\lambda\in\kappa$,*

$$\begin{aligned}
&F_D^{(2d-1)}\left({\alpha^d;\overbrace{\alpha\overline{\gamma},\dots,\alpha\overline{\gamma}}^{d-1\ times},\overbrace{\beta,\dots,\beta}^{d\ times}\atop \alpha^{d-1}\gamma};\xi,\dots,\xi^{d-1},\lambda,\xi\lambda,\dots,\xi^{d-1}\lambda\right)\\
&\quad=\sum_{i=0}^{d-1}\frac{g(\varphi_d^i\alpha)g^\circ(\alpha^{d-1}\gamma)}{g(\alpha^d)g^\circ(\varphi_d^i\gamma)}\,{}_2F_1\left({\varphi_d^i\alpha,\beta\atop \varphi_d^i\gamma};\lambda^d\right).
\end{aligned}$$

Proof. For $\lambda = 0$, it is clear. Suppose that $\lambda \neq 0$. By Theorem 3.1 (i), we have

$$\begin{aligned}
&-j(\alpha^d,\overline{\alpha}\gamma)F_D^{(2d-1)}\left(\begin{matrix}\alpha^d; \overbrace{\alpha\overline{\gamma},\dots,\alpha\overline{\gamma}}^{d-1}, \overbrace{\beta,\dots,\beta}^{d}\\ \alpha^{d-1}\gamma\end{matrix}; \xi,\dots,\xi^{d-1},\lambda,\xi\lambda,\dots,\xi^{d-1}\lambda\right)\\
&= \sum_{t\in\kappa^\times}\alpha(t^d)\overline{\alpha}\gamma(1-t^d)\overline{\beta}(1-\lambda^d t^d)\\
&= d\sum_{\substack{t\in\kappa^\times\\ \varphi_d(t)=1}}\alpha(t)\overline{\alpha}\gamma(1-t)\overline{\beta}(1-\lambda^d t)\\
&= \sum_{i=0}^{d-1}\sum_{t\in\kappa^\times}\varphi_d^i\alpha(t)\overline{\alpha}\gamma(1-t)\overline{\beta}(1-\lambda^d t).
\end{aligned}$$

Here, note that

$$\sum_{i=0}^{d-1}\varphi_d^i(t) = \begin{cases} d & (\varphi_d(t)=1),\\ 0 & (\text{otherwise}).\end{cases}$$

Thus, the theorem follows from Proposition 2.5 (ii). □

3.2. **The cases of F_A and F_B.** In the complex case, Lauricella's functions $F_A^{(n)}$ have the integral representation (cf. [16, Theorem 3.4.1])

$$\begin{aligned}
&\Big(\prod_{i=1}^n B(b_i,c_i-b_i)\Big)F_A^{(n)}\left(\begin{matrix}a;b_1,\dots,b_n\\ c_1,\dots,c_n\end{matrix};z_1,\dots,z_n\right)\\
&=\int_0^1\cdots\int_0^1\Big(1-\sum_{i=1}^n z_iu_i\Big)^{-a}\prod_{i=1}^n u_i^{b_i-1}(1-u_i)^{c_i-b_i-1}\,du_1\cdots du_n,
\end{aligned}$$

if $0<\mathrm{Re}(b_j)<\mathrm{Re}(c_j)$ for all j.

Theorem 3.3. *Suppose that $\alpha\neq\varepsilon$ and $\beta_i\neq\gamma_i$ for all i. Then, for $\lambda_i\in\kappa^\times$,*

$$\begin{aligned}
&\Big(\prod_{i=1}^n -j(\beta_i,\overline{\beta_i}\gamma_i)\Big)F_A^{(n)}\left(\begin{matrix}\alpha;\beta_1,\dots,\beta_n\\ \gamma_1,\dots,\gamma_n\end{matrix};\lambda_1,\dots,\lambda_n\right)\\
&=\sum_{u_1,\dots,u_n\in\kappa^\times}\overline{\alpha}\Big(1-\sum_{i=1}^n\lambda_iu_i\Big)\prod_{i=1}^n\beta_i(u_i)\overline{\beta_i}\gamma_i(1-u_i).
\end{aligned}$$

Proof. Write $f(\lambda_1,\dots,\lambda_n)$ for the right-hand side of the theorem. Then, putting $s_i=t_iu_i$ and using (2.2) and (2.4), we have

$$\begin{aligned}
\widehat{f}(\nu_1,\dots,\nu_n) &= \sum_{t_1,\dots,t_n\in\kappa^\times}\sum_{u_1,\dots,u_n\in\kappa^\times}\overline{\alpha}\Big(1-\sum_i t_iu_i\Big)\prod_i\beta_i(u_i)\overline{\beta_i}\gamma_i(1-u_i)\overline{\nu_i}(t_i)\\
&=\Big(\prod_i\sum_{u_i\in\kappa^\times}\beta_i\nu_i(u_i)\overline{\beta_i}\gamma_i(1-u_i)\Big)\sum_{s_1,\dots,s_n\in\kappa^\times}\overline{\alpha}\Big(1-\sum_i s_i\Big)\prod_i\overline{\nu_i}(s_i)\\
&=\Big(\prod_i j(\overline{\beta_i}\gamma_i,\beta_i\nu_i)\Big)j(\overline{\alpha},\overline{\nu_1},\dots,\overline{\nu_n})\\
&=\Big(\prod_i j(\beta_i,\overline{\beta_i}\gamma_i)\frac{(\beta_i)_{\nu_i}}{(\gamma_i)^\circ_{\nu_i}}\Big)\frac{(\alpha)_{\nu_1\cdots\nu_n}}{(\varepsilon)^\circ_{\nu_1}\cdots(\varepsilon)^\circ_{\nu_n}}.
\end{aligned}$$

Thus, we obtain the theorem by (3.1). □

Lauricella's $F_A^{(n)}$ have another integral representation ([14, Theorem 3], see also [16, Theorem 3.4.1]) as

$$\frac{\Big(\prod_{i=1}^n \Gamma(1-c_i)\Big)\Gamma(\sum_{i=1}^n c_i - a - n + 1)}{\Gamma(1-a)} F_A^{(n)}\left({a; b_1, \dots, b_n \atop c_1, \dots, c_n}; z_1, \dots, z_n\right) \tag{3.5}$$
$$= \int_{\Delta'} \Big(\prod_{i=1}^n \Big(1 - \frac{z_i}{u_i}\Big)^{-b_i}\Big)\Big(\prod_{i=1}^n u_i^{-c_i}\Big)\Big(1 - \sum_{i=1}^n u_i\Big)^{\sum_{i=1}^n c_i - a - n} du_1 \cdots du_n,$$

where Δ' is a twisted cycle constructed in [14], if $c_1, \dots, c_n, \sum_i c_i - a \notin \mathbb{Z}$.

Theorem 3.4. *Suppose that* $\overline{\alpha}\gamma_1 \cdots \gamma_n, \beta_i \neq \varepsilon$ *for all* i. *Then, for* $\lambda_i \in \kappa^\times$,

$$(-1)^n \frac{\Big(\prod_{i=1}^n g(\overline{\gamma_i})\Big) g(\overline{\alpha}\gamma_1 \cdots \gamma_n)}{g^\circ(\overline{\alpha})} F_A^{(n)}\left({\alpha; \beta_1, \dots, \beta_n \atop \gamma_1, \dots, \gamma_n}; \lambda_1, \dots, \lambda_n\right)$$
$$= \sum_{u_1, \dots, u_n \in \kappa^\times} \Big(\prod_{i=1}^n \overline{\beta_i}\Big(1 - \frac{\lambda_i}{u_i}\Big)\Big)\Big(\prod_{i=1}^n \overline{\gamma_i}(u_i)\Big)\overline{\alpha}\gamma_1 \cdots \gamma_n\Big(1 - \sum_{i=1}^n u_i\Big).$$

Proof. Write $f(\lambda_1, \dots, \lambda_n)$ for the right-hand side of the theorem. Then, putting $s_i = t_i/u_i$ and similarly as the proof of Theorem 3.3, we have

$$\begin{aligned}
&\widehat{f}(\nu_1, \dots, \nu_n) \\
&= \sum_{t_1, \dots, t_n} \sum_{u_1, \dots, u_n} \Big(\prod_{i=1}^n \overline{\beta_i}\Big(1 - \frac{t_i}{u_i}\Big)\overline{\nu_i}(t_i)\Big)\Big(\prod_{i=1}^n \overline{\gamma_i}(u_i)\Big)\overline{\alpha}\gamma_1 \cdots \gamma_n\Big(1 - \sum_{i=1}^n u_i\Big) \\
&= \sum_{u_1, \dots, u_n} \Big(\prod_{i=1}^n \overline{\gamma_i \nu_i}(u_i)\Big)\overline{\alpha}\gamma_1 \cdots \gamma_n\Big(1 - \sum_{i=1}^n u_i\Big) \sum_{s_1, \dots, s_n} \prod_{i=1}^n \overline{\beta_i}(1 - s_i)\overline{\nu_i}(s_i) \\
&= j(\overline{\gamma_1 \nu_1}, \dots, \overline{\gamma_n \nu_n}, \overline{\alpha}\gamma_1 \cdots \gamma_n) \prod_{i=1}^n j(\overline{\beta_i}, \overline{\nu_i}) \\
&= \frac{\Big(\prod_{i=1}^n g(\overline{\gamma_i})\Big) g(\overline{\alpha}\gamma_1 \cdots \gamma_n)}{g^\circ(\overline{\alpha})} \cdot \frac{(\alpha)_{\nu_1 \cdots \nu_n} \prod_{i=1}^n (\beta_i)_{\nu_i}}{\prod_{i=1}^n (\gamma_i)^\circ_{\nu_i} (\varepsilon)^\circ_{\nu_i}}.
\end{aligned}$$

Thus, we obtain the theorem by (3.1). □

Lauricella's functions $F_B^{(n)}$ have the integral representation (cf. [16, Theorem 3.4.1])

$$\frac{\Big(\prod_{i=1}^n \Gamma(b_i)\Big)\Gamma(c - \sum_{i=1}^n b_i)}{\Gamma(c)} F_B^{(n)}\left({a_1, \dots, a_n; b_1, \dots, b_n \atop c}; z_1, \dots, z_n\right)$$
$$= \int_\Delta \Big(\prod_{i=1}^n (1 - z_i u_i)^{-a_i}\Big)\Big(\prod_{i=1}^n u_i^{b_i - 1}\Big)\Big(1 - \sum_{i=1}^n u_i\Big)^{c - \sum_{i=1}^n b_i - 1} du_1 \cdots du_n,$$

where Δ is as in (3.3), if $0 < \mathrm{Re}(b_i)$ for all i, and $\mathrm{Re}(\sum_i b_i) < \mathrm{Re}(c)$.

Theorem 3.5. *Suppose that $\alpha_i \neq \varepsilon$ for all i and $\beta_1\cdots\beta_n \neq \gamma$. Then, for $\lambda_i \in \kappa^\times$,*

$$(-1)^n \frac{\left(\prod_{i=1}^n g(\beta_i)\right) g(\overline{\beta_1\cdots\beta_n}\gamma)}{g^\circ(\gamma)} F_B^{(n)}\left({\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_n \atop \gamma};\lambda_1,\dots,\lambda_n\right)$$
$$= \sum_{u_1,\dots,u_n\in\kappa^\times}\left(\prod_{i=1}^n \overline{\alpha_i}(1-\lambda_i u_i)\right)\left(\prod_{i=1}^n \beta_i(u_i)\right)\overline{\beta_1\cdots\beta_n}\gamma\Big(1-\sum_{i=1}^n u_i\Big).$$

Proof. Write $f(\lambda_1,\dots,\lambda_n)$ for the right-hand side of the theorem. Letting $s_i = t_i u_i$ for all i and using (2.2) (note that $\alpha_i \neq \varepsilon$ and $\beta_1\cdots\beta_n\neq\gamma$) and (2.4), we have

$$\widehat{f}(\nu_1,\dots,\nu_n)$$
$$= \sum_{t_1,\dots,t_n\in\kappa^\times}\sum_{u_1,\dots,u_n\in\kappa^\times}\Big(\prod_i \beta_i(u_i)\Big)\overline{\beta_1\cdots\beta_n}\gamma\Big(1-\sum_i u_i\Big)\prod_i \overline{\alpha_i}(1-t_iu_i)\overline{\nu_i}(t_i)$$
$$= \sum_{u_1,\dots,u_n\in\kappa^\times}\Big(\prod_i \beta_i\nu_i(u_i)\Big)\overline{\beta_1\cdots\beta_n}\gamma\Big(1-\sum_i u_i\Big)\prod_i\Big(\sum_{s_i\in\kappa^\times}\overline{\alpha_i}(1-s_i)\overline{\nu_i}(s_i)\Big)$$
$$= j(\overline{\beta_1\cdots\beta_n}\gamma,\beta_1\nu_1,\dots,\beta_n\nu_n)\prod_i j(\overline{\alpha_i},\overline{\nu_i})$$
$$= \frac{\left(\prod_{i=1}^n g(\beta_i)\right)g(\overline{\beta_1\cdots\beta_n}\gamma)}{g^\circ(\gamma)}\cdot\frac{\prod_i(\beta_i)_{\nu_i}}{(\gamma)^\circ_{\nu_1\cdots\nu_n}}\cdot\prod_i\frac{(\alpha_i)_{\nu_i}}{(\varepsilon)^\circ_{\nu_i}}.$$

Thus, we obtain the theorem by (3.1). □

Remark 3.6. Theorem 3.5 is equivalent to Theorem 3.3 via Remark 2.4.

3.3. **The case of F_C.** In the complex case, Lauricella's functions $F_C^{(n)}$ have the integral representation ([14, Theorem 4], see also [20, Remark 4.4])

$$\frac{\left(\prod_{i=1}^n \Gamma(1-c_i)\right)\Gamma(c_1+\cdots+c_n+1-n-a)}{\Gamma(1-a)} F_C^{(n)}\left({a;b \atop c_1,\dots,c_n};z_1,\dots,z_n\right)$$
$$= \int_{\Delta'}\Big(1-\sum_{i=1}^n\frac{z_i}{t_i}\Big)^{-b}\Big(\prod_{i=1}^n t_i^{-c_i}\Big)\Big(1-\sum_{i=1}^n t_i\Big)^{\sum_{i=1}^n c_i-a-n} dt_1\cdots dt_n,$$

where Δ' is as in (3.5), if $c_1,\dots,c_n,\sum_i c_i - a \notin \mathbb{Z}$.

Theorem 3.7. *Suppose that $\overline{\alpha}\gamma_1\cdots\gamma_n, \beta \neq \varepsilon$. Then, for $\lambda_i\in\kappa^\times$,*

$$(-1)^n\frac{\left(\prod_{i=1}^n g(\overline{\gamma_i})\right)g(\overline{\alpha}\gamma_1\cdots\gamma_n)}{g^\circ(\overline{\alpha})}F_C^{(n)}\left({\alpha;\beta \atop \gamma_1,\dots,\gamma_n};\lambda_1,\dots,\lambda_n\right)$$
$$= \sum_{u_1,\dots,u_n\in\kappa^\times}\overline{\beta}\Big(1-\sum_{i=1}^n\frac{\lambda_i}{u_i}\Big)\Big(\prod_{i=1}^n\overline{\gamma_i}(u_i)\Big)\overline{\alpha}\gamma_1\cdots\gamma_n\Big(1-\sum_{i=1}^n u_i\Big).$$

Proof. Write $f(\lambda_1,\dots,\lambda_n)$ for the right-hand side of the theorem. Letting $s_i = t_i/u_i$ and using (2.2) (note that $\overline{\alpha}\gamma_1\cdots\gamma_n,\beta\neq\varepsilon$) and (2.4), we have

$$\widehat{f}(\nu_1,\dots,\nu_n)$$
$$= \sum_{t_1,\dots,t_n}\sum_{u_1,\dots,u_n}\overline{\alpha}\gamma_1\cdots\gamma_n\Big(1-\sum_i u_i\Big)\overline{\beta}\Big(1-\sum_i\frac{t_i}{u_i}\Big)\prod_i\overline{\gamma_i}(u_i)\overline{\nu_i}(t_i)$$

$$
\begin{aligned}
&= \sum_{u_1,\dots,u_n} \Big(\prod_i \overline{\gamma_i\nu_i}(u_i)\Big)\overline{\alpha}\gamma_1\cdots\gamma_n\Big(1-\sum_i u_i\Big) \sum_{s_1,\dots,s_n} \Big(\prod_i \overline{\nu_i}(s_i)\Big)\overline{\beta}\Big(1-\sum_i s_i\Big) \\
&= j(\overline{\gamma_1\nu_1},\dots,\overline{\gamma_n\nu_n},\overline{\alpha}\gamma_1\cdots\gamma_n)j(\overline{\nu_1},\dots,\overline{\nu_n},\overline{\beta}) \\
&= \frac{\Big(\prod_i g(\overline{\gamma_i})\Big)g(\overline{\alpha}\gamma_1\cdots\gamma_n)}{g^\circ(\overline{\alpha})} \cdot \frac{(\alpha)_{\nu_1\cdots\nu_n}}{\prod_i(\gamma_i)^\circ_{\nu_i}} \cdot \frac{(\beta)_{\nu_1\cdots\nu_n}}{\prod_i(\varepsilon)^\circ_{\nu_i}}.
\end{aligned}
$$

Thus, we obtain the theorem by (3.1). □

In the complex case, Burchnall-Chaundy [3] proved the expansion formula

$$
\begin{aligned}
&F_4(a;b;c_1,c_2;x(1-y),y(1-x)) \\
&= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r(1+a+b-c_1-c_2)_r}{(1)_r(c_1)_r(c_2)_r} x^r y^r \\
&\qquad \times {}_2F_1\left(\begin{matrix} a+r, b+r \\ c_1+r \end{matrix}; x\right) {}_2F_1\left(\begin{matrix} a+r, b+r \\ c_2+r \end{matrix}; y\right),
\end{aligned} \tag{3.6}
$$

(an alternative proof was given by Bailey [2]). From this they deduced, by using (3.4) and ${}_1F_0\left({}^{a};z\right) = (1-z)^{-a}$, the integral representation

$$
\begin{aligned}
&B(a,c_1-a)B(b,c_2-b)F_4(a;b;c_1,c_2;x(1-y),y(1-x)) \\
&= \int_0^1\int_0^1 u^{a-1}v^{b-1}(1-u)^{c_1-a-1}(1-v)^{c_2-b-1} \\
&\qquad \times (1-xu)^{a-c_1-c_2+1}(1-yv)^{b-c_1-c_2+1}(1-xu-yv)^{c_1+c_2-a-b-1}dudv,
\end{aligned} \tag{3.7}
$$

provided that $0 < \mathrm{Re}(a) < \mathrm{Re}(c_1)$, $0 < \mathrm{Re}(b) < \mathrm{Re}(c_2)$ and $|x|, |y|$ are small enough to make the double integral convergent. We prove finite analogues of these formulas.

The following lemmas will be used in the proof of Proposition 3.10, from which we will deduce finite analogues of (3.6) and (3.7) (Theorem 3.11 and Theorem 3.12, respectively). Since the function F_4^* in [27] coincides with ours by (2.1), we have the following.

LEMMA 3.8 ([27, Theorem 1.1]). *For any $x, y \neq 1$,*

$$
\begin{aligned}
&\overline{\alpha}(1-x)\overline{\beta}(1-y)F_4\left(\alpha;\beta;\gamma_1,\gamma_2;\frac{-x}{(1-x)(1-y)},\frac{-y}{(1-x)(1-y)}\right) \\
&= \frac{1}{(1-q)^2}\sum_{\mu,\nu}\frac{(\alpha)_\mu(\beta)_\nu}{(\varepsilon)^\circ_\mu(\varepsilon)^\circ_\nu} {}_2F_1\left(\begin{matrix} \beta\nu, \overline{\mu} \\ \gamma_1 \end{matrix}; 1\right) {}_2F_1\left(\begin{matrix} \alpha\mu, \overline{\nu} \\ \gamma_2 \end{matrix}; 1\right)\mu(x)\nu(y).
\end{aligned}
$$

LEMMA 3.9. *Suppose that $\alpha, \beta \notin \{\varepsilon, \gamma_1, \gamma_2\}$ and $\alpha\beta\overline{\gamma_1\gamma_2} \neq \varepsilon$. For any $\mu, \nu \in \widehat{\kappa^\times}$,*

$$
\begin{aligned}
&{}_2F_1\left(\begin{matrix} \beta\nu, \overline{\mu} \\ \gamma_1 \end{matrix}; 1\right) {}_2F_1\left(\begin{matrix} \alpha\mu, \overline{\nu} \\ \gamma_2 \end{matrix}; 1\right) \\
&= \frac{(\overline{\beta}\gamma_1)_\mu(\overline{\alpha}\gamma_2)_\nu}{(\gamma_1)^\circ_\mu(\gamma_2)^\circ_\nu} {}_3F_2\left(\begin{matrix} \alpha\beta\overline{\gamma_1\gamma_2}, \overline{\mu}, \overline{\nu} \\ \beta\overline{\gamma_1}\mu, \alpha\overline{\gamma_2}\nu \end{matrix}; 1\right) - j(\alpha\overline{\gamma_2},\beta\overline{\gamma_1})\frac{(\varepsilon)^\circ_\mu(\varepsilon)^\circ_\nu}{(\gamma_1)^\circ_\mu(\gamma_2)^\circ_\nu} \\
&\quad - \frac{(1-q)^2}{q}\left(\delta(\gamma_1\mu)\delta(\beta\nu)C_1 + \delta(\alpha\mu)\delta(\gamma_2\nu)C_2\right),
\end{aligned}
$$

where

$$C_1 := q^{\delta(\gamma_1)} \frac{g(\overline{\alpha\beta}\gamma_1\gamma_2)g^\circ(\gamma_2)}{g^\circ(\overline{\alpha}\gamma_1\gamma_2)g(\overline{\beta}\gamma_2)}, \qquad C_2 := q^{\delta(\gamma_2)} \frac{g(\overline{\alpha\beta}\gamma_1\gamma_2)g^\circ(\gamma_1)}{g^\circ(\overline{\beta}\gamma_1\gamma_2)g(\overline{\alpha}\gamma_1)}.$$

PROOF. Put

$$L(\mu,\nu) = {}_2F_1\left(\begin{matrix}\beta\nu,\overline{\mu}\\ \gamma_1\end{matrix};1\right){}_2F_1\left(\begin{matrix}\alpha\mu,\overline{\nu}\\ \gamma_2\end{matrix};1\right),$$

and

$$M(\mu,\nu) = q^{-\delta(\beta\overline{\gamma_1}\nu)-\delta(\alpha\overline{\gamma_2}\mu)} \frac{(\overline{\beta}\gamma_1\overline{\nu})_\mu(\overline{\alpha}\gamma_2\overline{\mu})_\nu}{(\gamma_1)^\circ_\mu(\gamma_2)^\circ_\nu}.$$

First, if $\{\beta\nu,\overline{\mu}\} \neq \{\varepsilon,\gamma_1\}$ and $\{\alpha\mu,\overline{\nu}\} \neq \{\varepsilon,\gamma_2\}$, then by Proposition 2.10 (i), we have

$$L(\mu,\nu) = M(\mu,\nu).$$

Using Proposition 2.11 (note that $\{\alpha\beta\overline{\gamma_1\gamma_2},\overline{\mu},\overline{\nu}\} \neq \{\varepsilon,\beta\overline{\gamma_1\mu},\alpha\overline{\gamma_2\nu}\}$), we have

$$\begin{aligned} M(\mu,\nu) &= \frac{(\overline{\beta}\gamma_1)_\mu(\overline{\alpha}\gamma_2)_\nu}{(\gamma_1)^\circ_\mu(\gamma_2)^\circ_\nu}\left({}_3F_2\left(\begin{matrix}\alpha\beta\overline{\gamma_1\gamma_2},\overline{\mu},\overline{\nu}\\ \beta\overline{\gamma_1\mu},\alpha\overline{\gamma_2\nu}\end{matrix};1\right) - \frac{g^\circ(\beta\overline{\gamma_1\mu})g^\circ(\alpha\overline{\gamma_2\nu})}{g(\alpha\beta\overline{\gamma_1\gamma_2})g(\overline{\mu})g(\overline{\nu})}\right)\\ &= N(\mu,\nu), \end{aligned}$$

where

$$N(\mu,\nu) := \frac{(\overline{\beta}\gamma_1)_\mu(\overline{\alpha}\gamma_2)_\nu}{(\gamma_1)^\circ_\mu(\gamma_2)^\circ_\nu}{}_3F_2\left(\begin{matrix}\alpha\beta\overline{\gamma_1\gamma_2},\overline{\mu},\overline{\nu}\\ \beta\overline{\gamma_1\mu},\alpha\overline{\gamma_2\nu}\end{matrix};1\right) - j(\alpha\overline{\gamma_2},\beta\overline{\gamma_1})\frac{(\varepsilon)^\circ_\mu(\varepsilon)^\circ_\nu}{(\gamma_1)^\circ_\mu(\gamma_2)^\circ_\nu}.$$

Therefore, we obtain the formula of the lemma.

Next, if $\{\beta\nu,\overline{\mu}\} = \{\varepsilon,\gamma_1\}$ (then $\{\alpha\mu,\overline{\nu}\} \neq \{\varepsilon,\gamma_2\}$), then by Proposition 2.10 (ii),

$$L(\mu,\nu) = M(\mu,\nu) - \frac{(1-q)^2(1+q)^{\delta(\gamma_1)}}{q}q^{-\delta(\overline{\alpha}\gamma_2\overline{\mu})}\frac{(\overline{\alpha}\gamma_2\overline{\mu})_\nu}{(\gamma_2)^\circ_\nu}. \tag{3.8}$$

By Proposition 2.11, if $\overline{\mu} = \varepsilon$ and $\beta\nu = \gamma_1$, then

$$M(\mu,\nu) = N(\mu,\nu) + \frac{(1-q)^2}{q}\frac{(\overline{\alpha}\gamma_2)_{\overline{\beta}\gamma_1}}{(\gamma_2)^\circ_{\overline{\beta}\gamma_1}},$$

and if $\overline{\mu} = \gamma_1 \neq \varepsilon$ and $\beta\nu = \varepsilon$ (then $\{\alpha\beta\overline{\gamma_1\gamma_2},\overline{\mu},\overline{\nu}\} \neq \{\varepsilon,\beta\overline{\gamma_1\mu},\alpha\overline{\gamma_2\nu}\}$), then

$$M(\mu,\nu) = N(\mu,\nu).$$

Consequently, by (3.8), we have

$$L(\mu,\nu) = N(\mu,\nu) - \delta(\gamma_1\mu)\delta(\beta\nu)\frac{(1-q)^2}{q}C_1.$$

Similarly, if $\{\alpha\mu,\overline{\nu}\} = \{\varepsilon,\gamma_2\}$, then we have

$$L(\mu,\nu) = N(\mu,\nu) - \delta(\alpha\mu)\delta(\gamma_2\nu)\frac{(1-q)^2}{q}C_2.$$

Thus, we obtain the lemma. □

For brevity, put

$$J := j(\alpha,\overline{\alpha}\gamma_1)j(\beta,\overline{\beta}\gamma_2).$$

Proposition 3.10. *Suppose that* $\alpha, \beta \notin \{\varepsilon, \gamma_1, \gamma_2\}$ *and* $\alpha\beta\overline{\gamma_1\gamma_2} \neq \varepsilon$. *Then, for any* $x, y \in \kappa^\times \setminus \{1\}$,

$$\begin{aligned}
&J \cdot F_4\left(\alpha; \beta; \gamma_1, \gamma_2; x(1-y), y(1-x)\right)\\
&\quad = \overline{\alpha}(1-x)\overline{\beta}(1-y)\frac{J}{1-q}\sum_{\eta \in \widehat{\kappa^\times}} \frac{(\alpha)_\eta(\beta)_\eta(\alpha\beta\overline{\gamma_1\gamma_2})_\eta}{(\varepsilon)^\circ_\eta(\gamma_1)^\circ_\eta(\gamma_2)^\circ_\eta}\eta\Big(\frac{xy}{(x-1)(y-1)}\Big)\\
&\qquad\qquad \times {}_2F_1\left(\begin{matrix}\alpha\eta, \overline{\beta}\gamma_1\\ \gamma_1\eta\end{matrix}; \frac{x}{x-1}\right) {}_2F_1\left(\begin{matrix}\beta\eta, \overline{\alpha}\gamma_2\\ \gamma_2\eta\end{matrix}; \frac{y}{y-1}\right)\\
&\qquad - S_0(x,y) - S_1(x,y) - S_2(x,y),
\end{aligned}$$

where

$$\begin{aligned}
S_0(x,y) &:= \alpha\beta(-1)j(\alpha\overline{\gamma_2}, \beta\overline{\gamma_1})\overline{\gamma_1}(x)\overline{\gamma_2}(y),\\
S_1(x,y) &:= j(\overline{\alpha\beta}\gamma_1\gamma_2, \beta)\overline{\gamma_1}(x)\overline{\alpha}\gamma_1(x-1)\overline{\beta}(y),\\
S_2(x,y) &:= j(\overline{\alpha\beta}\gamma_1\gamma_2, \alpha)\overline{\gamma_2}(y)\overline{\beta}\gamma_2(y-1)\overline{\alpha}(x).
\end{aligned}$$

Proof. By Lemma 3.8 (replace x, y with $x/(x-1), y/(y-1)$ respectively),

$$\begin{aligned}
&J \cdot \alpha(1-x)\beta(1-y)F_4\left(\alpha; \beta; \gamma_1, \gamma_2; x(1-y), y(1-x)\right)\\
&= \frac{J}{(1-q)^2}\sum_{\mu,\nu}\frac{(\alpha)_\mu(\beta)_\nu}{(\varepsilon)^\circ_\mu(\varepsilon)^\circ_\nu}{}_2F_1\left(\begin{matrix}\beta\nu, \overline{\mu}\\ \gamma_1\end{matrix}; 1\right){}_2F_1\left(\begin{matrix}\alpha\mu, \overline{\nu}\\ \gamma_2\end{matrix}; 1\right)\mu\Big(\frac{x}{x-1}\Big)\nu\Big(\frac{y}{y-1}\Big).
\end{aligned}$$

Thus, by Lemma 3.9, we obtain that

$$\begin{aligned}
&J \cdot \alpha(1-x)\beta(1-y)F_4\left(\alpha; \beta; \gamma_1, \gamma_2; x(1-y), y(1-x)\right)\\
&\quad = \Phi(x,y) - \alpha(1-x)\beta(1-y)\big(S_0(x,y) + S_1(x,y) + S_2(x,y)\big),
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
&\Phi(x,y)\\
&:= \frac{J}{(1-q)^3}\sum_{\mu,\nu,\eta}\frac{(\alpha)_\mu(\beta)_\nu(\overline{\beta}\gamma_1)_\mu(\overline{\alpha}\gamma_2)_\nu(\alpha\beta\overline{\gamma_1\gamma_2})_\eta(\overline{\mu})_\eta(\overline{\nu})_\eta}{(\varepsilon)^\circ_\mu(\varepsilon)^\circ_\nu(\gamma_1)^\circ_\mu(\gamma_2)^\circ_\nu(\varepsilon)^\circ_\eta(\beta\overline{\gamma_1\mu})^\circ_\eta(\alpha\overline{\gamma_2\nu})^\circ_\eta}\mu\Big(\frac{x}{x-1}\Big)\nu\Big(\frac{y}{y-1}\Big).
\end{aligned}$$

In proving (3.9), note that, by replacing μ, ν with $\overline{\gamma_1}\mu$, $\overline{\gamma_2}\nu$ respectively and using (2.3) and Proposition 2.5 (i),

$$\begin{aligned}
&\frac{J}{(1-q)^2}j(\alpha\overline{\gamma_2}, \beta\overline{\gamma_1})\sum_{\mu,\nu}\frac{(\alpha)_\mu(\beta)_\nu}{(\gamma_1)^\circ_\mu(\gamma_2)^\circ_\nu}\mu\Big(\frac{x}{x-1}\Big)\nu\Big(\frac{y}{y-1}\Big)\\
&= \alpha\beta(-1)j(\alpha\overline{\gamma_2}, \beta\overline{\gamma_1})\overline{\gamma_1}\Big(\frac{x}{x-1}\Big)\overline{\gamma_2}\Big(\frac{y}{y-1}\Big){}_1F_0\left(\begin{matrix}\alpha\overline{\gamma_1}\\ \end{matrix}; \frac{x}{x-1}\right){}_1F_0\left(\begin{matrix}\beta\overline{\gamma_2}\\ \end{matrix}; \frac{y}{y-1}\right)\\
&= \alpha(1-x)\beta(1-y)S_0(x,y).
\end{aligned}$$

By (2.1), for any $\varphi, \chi \in \widehat{\kappa^\times}$,

$$\frac{(\chi)_\varphi}{(\overline{\chi\varphi})^\circ_\varphi} = \varphi(-1). \tag{3.10}$$

Replacing μ, ν with $\mu\eta$, $\nu\eta$ respectively, and using (2.3) and (3.10), we have

$$\frac{(1-q)^3}{J}\Phi(x,y)$$

$$= \sum_{\eta,\mu,\nu} \frac{(\alpha)_\eta(\beta)_\eta(\alpha\beta\overline{\gamma_1\gamma_2})_\eta(\alpha\eta)_\mu(\overline{\beta}\gamma_1)_\mu(\beta\eta)_\nu(\overline{\alpha}\gamma_2)_\nu}{(\varepsilon)^\circ_\eta(\gamma_1)^\circ_\eta(\gamma_2)^\circ_\eta(\varepsilon)^\circ_\mu(\gamma_1\eta)^\circ_\mu(\varepsilon)^\circ_\nu(\gamma_2\eta)^\circ_\nu} \mu\eta\Big(\frac{x}{x-1}\Big)\nu\eta\Big(\frac{y}{y-1}\Big),$$

and hence we have

$$\Phi(x,y) = \frac{J}{1-q}\sum_{\eta} \frac{(\alpha)_\eta(\beta)_\eta(\alpha\beta\overline{\gamma_1\gamma_2})_\eta}{(\varepsilon)^\circ_\eta(\gamma_1)^\circ_\eta(\gamma_2)^\circ_\eta}\eta\Big(\frac{xy}{(x-1)(y-1)}\Big)$$
$$\times {}_2F_1\left(\begin{matrix}\alpha\eta, \overline{\beta}\gamma_1\\ \gamma_1\eta\end{matrix}; \frac{x}{x-1}\right){}_2F_1\left(\begin{matrix}\beta\eta, \overline{\alpha}\gamma_2\\ \gamma_2\eta\end{matrix}; \frac{y}{y-1}\right).$$

Thus, the proposition follows from (3.9). ☐

By Proposition 3.10, we obtain a finite analogue of the Burchnall-Chaundy expansion (3.6), under the assumption $\alpha\beta\overline{\gamma_1\gamma_2} \neq \varepsilon$, as follows.

THEOREM 3.11. *Suppose that $\alpha, \beta \notin \{\varepsilon, \gamma_1, \gamma_2\}$ and $\alpha\beta\overline{\gamma_1\gamma_2} \neq \varepsilon$. Then, for any $x, y \in \kappa\backslash\{1\}$, we have*

$$J \cdot F_4(\alpha;\beta;\gamma_1,\gamma_2;x(1-y),y(1-x))$$
$$= \frac{J}{1-q}\sum_{\eta\in\widehat{\kappa^\times}} \frac{(\alpha)_\eta(\beta)_\eta(\alpha\beta\overline{\gamma_1\gamma_2})_\eta}{(\varepsilon)^\circ_\eta(\gamma_1)^\circ_\eta(\gamma_2)^\circ_\eta}\eta(xy){}_2F_1\left(\begin{matrix}\alpha\eta, \beta\eta\\ \gamma_1\eta\end{matrix}; x\right){}_2F_1\left(\begin{matrix}\alpha\eta, \beta\eta\\ \gamma_2\eta\end{matrix}; y\right)$$
$$- S_0(x,y) + R_1(x,y) + q^{\delta(\alpha\overline{\beta})}R_2(x,y).$$

Here, J and S_0 are as in Proposition 3.10 and

$$R_1(x,y) := j(\overline{\alpha\beta}\gamma_1\gamma_2, \beta)j(\alpha\overline{\gamma_2}, \overline{\beta}\gamma_2)\overline{\alpha}\gamma_1(x-1)\overline{\alpha}\gamma_2(1-y)\overline{\gamma_1}(x)\overline{\gamma_2}(y),$$
$$R_2(x,y) := j(\overline{\alpha\beta}\gamma_1\gamma_2, \alpha)j(\overline{\alpha}\gamma_1, \beta\overline{\gamma_1})\overline{\beta}\gamma_1(1-x)\overline{\beta}\gamma_2(y-1)\overline{\gamma_1}(x)\overline{\gamma_2}(y).$$

PROOF. Using Proposition 2.9 with $\lambda = x/(x-1)$ and $\lambda = y/(y-1)$, we have

$${}_2F_1\left(\begin{matrix}\alpha\eta, \overline{\beta}\gamma_1\\ \gamma_1\eta\end{matrix}; \frac{x}{x-1}\right){}_2F_1\left(\begin{matrix}\beta\eta, \overline{\alpha}\gamma_2\\ \gamma_2\eta\end{matrix}; \frac{y}{y-1}\right)$$
$$= \left(\alpha\eta(1-x){}_2F_1\left(\begin{matrix}\alpha\eta, \beta\eta\\ \gamma_1\eta\end{matrix}; x\right) + \delta(\beta\eta)\alpha\beta(-1)(1-q)\frac{g^\circ(\overline{\beta}\gamma_1)}{g(\alpha\overline{\beta})g(\overline{\alpha}\gamma_1)}\beta\overline{\gamma_1}\Big(\frac{x}{x-1}\Big)\right)$$
$$\times \left(\beta\eta(1-y){}_2F_1\left(\begin{matrix}\alpha\eta, \beta\eta\\ \gamma_2\eta\end{matrix}; y\right) + \delta(\alpha\eta)\alpha\beta(-1)(1-q)\frac{g^\circ(\overline{\alpha}\gamma_2)}{g(\overline{\alpha}\beta)g(\overline{\beta}\gamma_2)}\alpha\overline{\gamma_2}\Big(\frac{y}{y-1}\Big)\right)$$
$$= \alpha\eta(1-x)\beta\eta(1-y){}_2F_1\left(\begin{matrix}\alpha\eta, \beta\eta\\ \gamma_1\eta\end{matrix}; x\right){}_2F_1\left(\begin{matrix}\alpha\eta, \beta\eta\\ \gamma_2\eta\end{matrix}; y\right)$$
$$+ \delta(\alpha\eta)\alpha\beta(-1)(1-q)\frac{g^\circ(\overline{\alpha}\gamma_2)}{g(\overline{\alpha}\beta)g(\overline{\beta}\gamma_2)}\alpha\overline{\gamma_2}\Big(\frac{y}{y-1}\Big){}_2F_1\left(\begin{matrix}\overline{\alpha}\beta, \varepsilon\\ \overline{\alpha}\gamma_1\end{matrix}; x\right)$$
$$+ \delta(\beta\eta)\alpha\beta(-1)(1-q)\frac{g^\circ(\overline{\beta}\gamma_1)}{g(\alpha\overline{\beta})(\overline{\alpha}\gamma_1)}\beta\overline{\gamma_1}\Big(\frac{x}{x-1}\Big){}_2F_1\left(\begin{matrix}\alpha\overline{\beta}, \varepsilon\\ \overline{\beta}\gamma_2\end{matrix}; y\right)$$
$$+ \delta(\alpha\overline{\beta})\delta(\beta\eta)(1-q)^2\alpha\overline{\gamma_1}\Big(\frac{x}{x-1}\Big)\alpha\overline{\gamma_2}\Big(\frac{y}{y-1}\Big).$$

Thus, by Proposition 3.10, (2.1) and (2.2), we have

$$J \cdot F_4(\alpha;\beta;\gamma_1,\gamma_2;x(1-y),y(1-x)) + S_0(x,y) + S_1(x,y) + S_2(x,y)$$

$$\begin{aligned}&= \frac{J}{1-q}\sum_{\eta\in\widehat{\kappa^\times}} \frac{(\alpha)_\eta(\beta)_\eta(\alpha\beta\overline{\gamma_1\gamma_2})_\eta}{(\varepsilon)^\circ_\eta(\gamma_1)^\circ_\eta(\gamma_2)^\circ_\eta}\eta(xy)\,{}_2F_1\left({\alpha\eta,\beta\eta \atop \gamma_1\eta};x\right){}_2F_1\left({\alpha\eta,\beta\eta\atop\gamma_2\eta};y\right)\\ &\quad + S_1(x,y)\,{}_2F_1\left({\alpha\overline{\beta},\varepsilon\atop \overline{\beta}\gamma_2};y\right)+S_2(x,y)\,{}_2F_1\left({\overline{\alpha}\beta,\varepsilon\atop\overline{\alpha}\gamma_1};x\right)+\delta(\alpha\overline{\beta})(1-q)R_1(x,y).\end{aligned}$$

By Lemma 2.8, (2.1) and (2.2), we have

$$S_1(x,y)\,{}_2F_1\left({\alpha\overline{\beta},\varepsilon\atop \overline{\beta}\gamma_2};y\right)=q^{\delta(\alpha\overline{\beta})}R_1(x,y)+S_1(x,y),$$

and

$$S_2(x,y)\,{}_2F_1\left({\overline{\alpha}\beta,\varepsilon\atop\overline{\alpha}\gamma_1};x\right)=q^{\delta(\alpha\overline{\beta})}R_2(x,y)+S_2(x,y).$$

Therefore, we obtain the theorem. □

A finite analogue of (3.7) is the following.

THEOREM 3.12. *Suppose that* $\alpha,\beta\notin\{\varepsilon,\gamma_1,\gamma_2\}$. *Then, for any* $x,y\in\kappa^\times\backslash\{1\}$,

$$\begin{aligned}&J\cdot F_4(\alpha;\beta;\gamma_1,\gamma_2;x(1-y),y(1-x))\\ &\quad=\sum_{u,v\in\kappa^\times}\alpha(u)\beta(v)\overline{\alpha}\gamma_1(1-u)\overline{\beta}\gamma_2(1-v)\\ &\qquad\qquad\times\alpha\overline{\gamma_1\gamma_2}(1-xu)\beta\overline{\gamma_1\gamma_2}(1-yv)\overline{\alpha\beta}\gamma_1\gamma_2(1-xu-yv)\\ &\qquad\quad-S_0(x,y)-S_1(x,y)-S_2(x,y).\end{aligned}$$

Here, J *and* $S_i(x,y)$ $(i=0,1,2)$ *are as in Proposition 3.10.*

PROOF. First, suppose that $\alpha\beta\overline{\gamma_1\gamma_2}=\varepsilon$. Then, we have a result of Tripathi-Barman [28, Theorem 3.1] (see also [24, Theorem 4.1])

$$\begin{aligned}&J\cdot F_4(\alpha;\beta;\gamma_1,\gamma_2;x(1-y),y(1-x))\\ &\quad=J\cdot{}_2F_1\left({\alpha,\beta\atop\gamma_1};x\right){}_2F_1\left({\alpha,\beta\atop\gamma_2};y\right)-\delta(1-x-y)qS_0(x,y),\end{aligned}\tag{3.11}$$

where $\delta(u)=0$ for $u\in\kappa^\times$ and $\delta(0)=1$. On the other hand, by using Proposition 2.5 (ii) and letting $t=ux$, the first term of the right-hand side of the theorem is

$$\begin{aligned}&\sum_{u,v}\alpha(u)\beta(v)\overline{\alpha}\gamma_1(1-u)\overline{\beta}\gamma_2(1-v)\alpha\overline{\gamma_1\gamma_2}(1-xu)\beta\overline{\gamma_1\gamma_2}(1-yv)\varepsilon(1-xu-yv)\\ &=\sum_u\alpha(u)\overline{\alpha}\gamma_1(1-u)\overline{\beta}(1-xu)\sum_v\beta(v)\overline{\beta}\gamma_2(1-v)\overline{\alpha}(1-yv)\\ &\qquad-\sum_{\substack{u,v\\ ux+vy=1}}\alpha(u)\beta(v)\overline{\alpha}\gamma_1(1-u)\overline{\beta}\gamma_2(1-v)\overline{\beta}(1-xu)\overline{\alpha}(1-yv)\\ &=J\cdot{}_2F_1\left({\alpha,\beta\atop\gamma_1};x\right){}_2F_1\left({\alpha,\beta\atop\gamma_2};y\right)-\overline{\gamma_1}(x)\overline{\gamma_2}(y)\sum_{\substack{t\in\kappa\\ t\neq0,1,1-y}}\overline{\alpha}\gamma_1\Big(\frac{x-t}{y-1+t}\Big).\end{aligned}$$

If $x+y\neq 1$, then $(x-t)/(y-1+t)$ runs through $\kappa\setminus\{-1, x/(y-1), (x-1)/y\}$, and hence we have

$$\overline{\gamma_1}(x)\overline{\gamma_2}(y)\sum_{\substack{t\in\kappa\\ t\neq 0,1,1-y}}\overline{\alpha}\gamma_1\Big(\frac{x-t}{y-1+t}\Big) = -S_0(x,y)-S_1(x,y)-S_2(x,y).$$

On the other hand, if $x+y=1$, then $S_0(x,y)=S_1(x,y)=S_2(x,y)$ and

$$\overline{\gamma_1}(x)\overline{\gamma_2}(y)\sum_{\substack{t\in\kappa\\ t\neq 0,1,1-y}}\overline{\alpha}\gamma_1\Big(\frac{x-t}{y-1+t}\Big) = (q-3)S_0(x,y).$$

Therefore, the right-hand side of the theorem is equal to the right-hand side of (3.11), and hence we obtain the theorem.

Secondly, suppose that $\alpha\beta\overline{\gamma_1\gamma_2}\neq\varepsilon$, and put

$$\begin{aligned}\Phi(x,y)=\frac{J}{1-q}\sum_{\eta}\frac{(\alpha)_\eta(\beta)_\eta(\alpha\beta\overline{\gamma_1\gamma_2})_\eta}{(\gamma_1)^\circ_\eta(\gamma_2)^\circ_\eta(\varepsilon)^\circ_\eta}\eta\Big(\frac{xy}{(x-1)(y-1)}\Big)&\\ \times\,{}_2F_1\left({\alpha\eta,\overline{\beta}\gamma_1 \atop \gamma_1\eta};\frac{x}{x-1}\right){}_2F_1\left({\beta\eta,\overline{\alpha}\gamma_2 \atop \gamma_2\eta};\frac{y}{y-1}\right)&.\end{aligned}$$

Then, by Proposition 3.10, the left-hand side of the theorem is equal to

$$\overline{\alpha}(1-x)\overline{\beta}(1-y)\Phi(x,y)-S_0(x,y)-S_1(x,y)-S_2(x,y).$$

By Proposition 2.5 (ii) and letting $u=s/(sx-x+1)$ and $v=t/(ty-y+1)$,

$$\begin{aligned}&J\frac{(\alpha)_\eta(\beta)_\eta}{(\gamma_1)^\circ_\eta(\gamma_2)^\circ_\eta}\cdot{}_2F_1\left({\alpha\eta,\overline{\beta}\gamma_1 \atop \gamma_1\eta};\frac{x}{x-1}\right){}_2F_1\left({\beta\eta,\overline{\alpha}\gamma_2 \atop \gamma_2\eta};\frac{y}{y-1}\right)\\ =&\sum_{s,t}\alpha(s)\overline{\alpha}\gamma_1(1-s)\beta\overline{\gamma_1}\Big(1-\frac{xs}{x-1}\Big)\beta(t)\overline{\beta}\gamma_2(1-t)\alpha\overline{\gamma_2}\Big(1-\frac{yt}{y-1}\Big)\eta(st)\\ =&\alpha(1-x)\beta(1-y)\sum_{u,v}\alpha(u)\beta(v)\overline{\alpha}\gamma_1(1-u)\overline{\beta}\gamma_2(1-v)\overline{\beta}(1-xu)\overline{\alpha}(1-yv)\\ &\times\eta\Big(\frac{(x-1)(y-1)uv}{(1-xu)(1-yv)}\Big).\end{aligned}$$

Thus, we obtain, by Proposition 2.5 (i),

$$\begin{aligned}&\overline{\alpha}(1-x)\overline{\beta}(1-y)\Phi(x,y)\\ &=\sum_{u,v}\alpha(u)\beta(v)\overline{\alpha}\gamma_1(1-u)\overline{\beta}\gamma_2(1-v)\overline{\beta}(1-xu)\overline{\alpha}(1-yv)\\ &\qquad\times{}_1F_0\left({\alpha\beta\overline{\gamma_1\gamma_2} \atop };\frac{xyuv}{(1-xu)(1-yv)}\right)\\ &=\sum_{u,v}\alpha(u)\beta(v)\overline{\alpha}\gamma_1(1-u)\overline{\beta}\gamma_2(1-v)\\ &\qquad\times\alpha\overline{\gamma_1\gamma_2}(1-xu)\beta\overline{\gamma_1\gamma_2}(1-yv)\overline{\alpha\beta}\gamma_1\gamma_2(1-xu-yv).\end{aligned}$$

Therefore, we obtain the theorem. □

4. The number of rational points on some algebraic varieties.

4.1. Rational points and Artin L-functions. In this subsection, we recall the definitions of zeta functions and Artin L-functions of a variety and their properties. For more details, see [25] and [30].

Fix an algebraic closure $\overline{\kappa}$ of κ and let $\kappa_r \subset \overline{\kappa}$ be the degree r extension of κ. Let V be a variety over κ and put $N_r(V) = \#V(\kappa_r)$. Then, *the zeta function of* V is defined by

$$Z(V,t) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r(V)}{r} t^r\right) \in \mathbb{Q}[[t]].$$

Let G be a finite abelian group and suppose that G acts on V over κ. Let F be the q-Frobenius acting on $V(\overline{\kappa})$. For $\chi \in \widehat{G} := \mathrm{Hom}(G, \overline{\mathbb{Q}}^{\times})$ and $r \in \mathbb{Z}_{\geq 1}$, put

$$N_r(V;\chi) := \frac{1}{\#G} \sum_{g \in G} \chi(g) \#\{x \in V(\overline{\kappa}) \mid F^r(x) = g(x)\} \in \overline{\mathbb{Q}}.$$

The Artin L-function of V associated to χ is defined by

$$L(V,\chi;t) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r(V;\chi)}{r} t^r\right) \in \overline{\mathbb{Q}}[[t]].$$

Since $N_r(V) = \sum_{\chi \in \widehat{G}} N_r(V;\chi)$, we have $Z(V,t) = \prod_{\chi \in \widehat{G}} L(V,\chi;t)$.

Remark 4.1. Let D_λ be the diagonal hypersurface in $\mathbb{P}^{n-1}$ defined by the equation

$$X_1^d + \cdots + X_n^d = d\lambda X_1^{h_1} \cdots X_n^{h_n},$$

where $\lambda \in \kappa^{\times}$, $h_i \in \mathbb{Z}_{\geq 1}$ and $\sum_i h_i = d$. A subquotient G of $(\mu_d)^n$ acts on D_λ. The author [22] expresses $N_r(D_\lambda;\chi)$ ($\chi \in \widehat{G}$) in terms of one-variable hypergeometric functions ${}_dF_{d-1}(\lambda^d)$ over κ_r.

4.2. Algebraic varieties related to F_D. In this subsection, let $d, a, b_1, \dots, b_n, c$ be positive integers and let $\lambda_1, \dots, \lambda_n \in \kappa^{\times}$. Write $\lambda = (\lambda_1, \dots, \lambda_n)$. We consider the affine curve $C_{D,\lambda}$ over κ defined by the equation

$$y^d = \Big(\prod_{i=1}^{n} (1-\lambda_i x)^{b_i}\Big) x^a (1-x)^c. \tag{4.1}$$

Without loss of generality, we assume that $\lambda_1, \dots, \lambda_n$ are not 1 and distinct. Suppose that $d \mid q-1$ and let $\mu_d \subset \kappa^{\times}$ be the subgroup consisting of all the dth roots of unity. Then, μ_d acts on $C_{D,\lambda}$ by $(x,y) \mapsto (x, \xi y)$ ($\xi \in \mu_d$). Fix a generator φ of $\widehat{\kappa^{\times}}$, and put $\varphi_d = \varphi^{(q-1)/d} \in \widehat{\kappa^{\times}}$ and $\chi = \varphi|_{\mu_d} \in \widehat{\mu_d}$. Note that $\widehat{\mu_d} = \{\chi^m \mid m \in \mathbb{Z}/d\mathbb{Z}\}$.

Theorem 4.2. *Suppose that $gcd(d,c) = gcd(d,b_i) = 1$ for all i. Then,*

$$N_1(C_{D,\lambda};\chi^m) = \begin{cases} q & (m=0), \\ -j\big(\varphi_d^{ma}, \varphi_d^{mc}\big) F_D^{(n)}\left(\begin{matrix} \varphi_d^{ma}; \overline{\varphi_d}^{mb_1}, \dots, \overline{\varphi_d}^{mb_n} \\ \varphi_d^{m(a+c)} \end{matrix}; \lambda_1, \dots, \lambda_n\right) & (m \neq 0). \end{cases}$$

Proof. Put $f(x) = \left(\prod_{i=1}^{n}(1-\lambda_i x)^{b_i}\right)x^a(1-x)^c$. Then, for each $m \in \mathbb{Z}/d\mathbb{Z}$,

$$
\begin{aligned}
&N_1(C_{D,\lambda};\chi^m)\\
&\quad= \frac{1}{d}\sum_{\xi\in\mu_d}\chi^m(\xi)\#\{(x,y)\in C_{D,\lambda}(\overline{\kappa}) \mid F(x,y)=(x,\xi y)\}\\
&\quad= \frac{1}{d}\sum_{\xi\in\mu_d}\chi^m(\xi)\#\{(x,y)\in C_{D,\lambda}(\overline{\kappa}) \mid x\in\kappa \text{ and } \ y^{q-1}=\xi \text{ or } y=0\}\\
&\quad= \sum_{\xi\in\mu_d}\chi^m(\xi)\#\{x\in\kappa \mid f(x)^{(q-1)/d}=\xi\} + \left(\frac{1}{d}\sum_{\xi\in\mu_d}\chi^m(\xi)\right)\#\{x\in\kappa \mid f(x)=0\}.
\end{aligned}
$$

Therefore, if $m=0$, then

$$
\begin{aligned}
&N_1(C_{D,\lambda};\chi^m)\\
&\quad= \sum_{\xi\in\mu_d}\#\{x\in\kappa \mid f(x)^{(q-1)/d}=\xi\} + \#\{x\in\kappa \mid f(x)=0\} = \#\kappa = q.
\end{aligned}
$$

If $m\neq 0$, then since $\sum_{\xi\in\mu_d}\chi^m(\xi)=0$, we have

$$
\begin{aligned}
N_1(C_{D,\lambda};\chi^m) &= \sum_{\xi\in\mu_d}\chi^m(\xi)\#\{x\in\kappa \mid f(x)^{(q-1)/d}=\xi\}\\
&= \sum_{x\in\kappa}\varphi_d^m(f(x))\\
&= \sum_{x\in\kappa}\left(\prod_{i=1}^{n}\varphi_d^{mb_i}(1-\lambda_i x)\right)\varphi_d^{ma}(x)\varphi_d^{mc}(1-x).
\end{aligned}
$$

Thus, noting that $\varphi_d^{mb_i}, \varphi_d^{mc} \neq \varepsilon$ for all i by the assumption, the theorem follows from Theorem 3.1 (i). □

Remark 4.3. Frechette-Swisher-Tu [5, Theorem 5.3] expresses $N_1(C_{D,\lambda})$ in terms of a sum of their Appell-Lauricella functions $\mathbb{F}_D^{(n)}$ over κ. Since our $F_D^{(n)}$ coincides with their $\mathbb{F}_D^{(n)}$ under the assumption of Theorem 4.2 by Theorem 3.1 (i), Theorem 4.2 is a refinement of their result.

Next, we consider the affine hypersurface $S_{D,\lambda}$ of dimension n over κ defined by the equation

$$
y^d = \left(1-\sum_{i=1}^{n}\lambda_i x_i\right)^a\left(\prod_{i=1}^{n}x_i^{b_i}\right)\left(1-\sum_{i=1}^{n}x_i\right)^c.
$$

The group μ_d acts on $S_{D,\lambda}$ similarly as $C_{D,\lambda}$.

Theorem 4.4. *Suppose that $gcd(d,a)=gcd(d,c)=1$. Then,*

$$
\begin{aligned}
&N_1(S_{D,\lambda};\chi^m)\\
&\quad= \begin{cases} q^n & (m=0),\\ (-1)^n J_D\cdot F_D^{(n)}\left(\begin{matrix}\overline{\varphi_d}^{ma};\varphi_d^{mb_1},\dots,\varphi_d^{mb_n}\\ \varphi_d^{m(b_1+\cdots+b_n+c)}\end{matrix};\lambda_1,\dots,\lambda_n\right) & (m\neq 0),\end{cases}
\end{aligned}
$$

where $J_D := j(\varphi_d^{mb_1},\dots,\varphi_d^{mb_n},\varphi_d^{mc})$.

Proof. Similarly as the proof of Theorem 4.2, we have

$$N_1(S_{D,\lambda};\chi^0)=\#\kappa^n=q^n,$$

and if $m\neq 0$, then

$$N_1(S_{D,\lambda};\chi^m)=\sum_{x_1,\dots,x_n\in\kappa^\times}\varphi_d^{ma}\Big(1-\sum_i\lambda_i x_i\Big)\Big(\prod_i\varphi_d^{mb_i}(x_i)\Big)\varphi_d^{mc}\Big(1-\sum_i x_i\Big).$$

Thus, the theorem follows from Theorem 3.1 (ii). $\square$

Fix an integer $r\geq 1$. Write $\varphi_{d,r}=\varphi_d\circ N_{\kappa_r/\kappa}\in\widehat{\kappa_r}$ where $N_{\kappa_r/\kappa}$ is the norm map.

Corollary 4.5. *Put hypergeometric functions over κ_r as*

$$f_r(\lambda)=F_D^{(n)}\left(\begin{matrix}\varphi_{d,r}^{ma};\overline{\varphi_{d,r}}^{mb_1},\dots,\overline{\varphi_{d,r}}^{mb_n}\\ \varphi_{d,r}^{m(c+a)}\end{matrix};\lambda_1,\dots,\lambda_n\right),$$

$$g_r(\lambda)=F_D^{(n)}\left(\begin{matrix}\overline{\varphi_{d,r}}^{ma};\varphi_{d,r}^{mb_1},\dots,\varphi_{d,r}^{mb_n}\\ \varphi_{d,r}^{m(b_1+\cdots+b_n+c)}\end{matrix};\lambda_1,\dots,\lambda_n\right).$$

(i) *Suppose that $gcd(d,c)=gcd(d,b_i)=1$ for all i. Then,*

$$L(C_{D,\lambda},\chi^m;t)=\begin{cases}\dfrac{1}{1-qt} & (m=0),\\ \exp\Big(\sum_{r=1}^\infty j(\varphi_d^{ma},\varphi_d^{mc})^r f_r(\lambda)\dfrac{t^r}{r}\Big)^{-1} & (m\neq 0).\end{cases}$$

(ii) *Suppose that $gcd(d,a)=gcd(d,c)=1$. Then,*

$$L(S_{D,\lambda},\chi^m;t)=\begin{cases}\dfrac{1}{1-q^n t} & (m=0),\\ \exp\Big(\sum_{r=1}^\infty J_D^r\cdot g_r(\lambda)\dfrac{t^r}{r}\Big)^{(-1)^n} & (m\neq 0),\end{cases}$$

where J_D is as in Theorem 4.4.

Proof. For each $r\geq 1$, let φ' be a generator of $\widehat{\kappa_r^\times}$ such that $\varphi'|_{\kappa^\times}=\varphi$. By applying Theorems 4.2 and 4.4 for κ_r and φ', we obtain the formulas for $N_r(C_{D,\lambda};\chi^m)$ and $N_r(S_{D,\lambda};\chi^m)$. Note that φ_d is replaced with $\varphi'^{(q^r-1)/d}=\varphi_{d,r}$. For $\eta\in\widehat{\kappa^\times}$ and $\eta_r:=\eta\circ N_{\kappa_r/\kappa}$, we have the Davenport-Hasse theorem (cf. [30])

$$g(\eta_r)=g(\eta)^r.$$

By this, we have

$$j(\varphi_{d,r}^{ma},\varphi_{d,r}^{mc})=j(\varphi_d^{ma},\varphi_d^{mc})^r,\quad j(\varphi_{d,r}^{mb_1},\dots,\varphi_{d,r}^{mb_n},\varphi_{d,r}^{mc})=J_D^r.$$

Thus, the corollary follows formally. $\square$

4.3. **Smooth compactification of $C_{D,\lambda}$.** Let $\overline{C}_{D,\lambda}$ be the projective curve defined by the homogenization of (4.1) with $x = X/Z$, $y = Y/Z$:

$$\begin{cases} Y^d = Z^e X^a (Z-X)^c \prod_i (Z-\lambda_i X)^{b_i} & (\text{if } d \geq a + \sum_i b_i + c), \\ Z^e Y^d = X^a (Z-X)^c \prod_i (Z-\lambda_i X)^{b_i} & (\text{if } d < a + \sum_i b_i + c), \end{cases}$$

where

$$e := |a + \sum_i b_i + c - d|.$$

Recall that $\prod_i \lambda_i (1-\lambda_i) \prod_{j\neq i} (\lambda_j - \lambda_i) \neq 0$. The group μ_d acts on $\overline{C}_{D,\lambda}$ by $\xi \cdot (X : Y : Z) = (X : \xi Y : Z)$ $(\xi \in \mu_d)$. Suppose that $e > 0$. Then, $\overline{C}_{D,\lambda}$ has the only one point at infinity, denoted by ∞. Since μ_d and F acts on ∞ trivially, we have

$$N_r(\overline{C}_{D,\lambda}; \chi^m) - N_r(C_{D,\lambda}; \chi^m) = \begin{cases} 1 & (m = 0), \\ 0 & (m \neq 0). \end{cases} \tag{4.2}$$

If $a > 1$ (resp. $b_i > 1$, $c > 1$, $e > 1$) then $\overline{C}_{D,\lambda}$ is singular at $(0 : 0 : 1)$ (resp. at $(\lambda_i^{-1} : 0 : 1)$, $(1 : 0 : 1)$, ∞). Archinard [1] constructs a desingularization $\pi : X_{D,\lambda} \to \overline{C}_{D,\lambda}$ under the assumption that $\gcd(d, a, b_1, \dots, b_n, c) = 1$. She works over $\mathbb{C}$, but the same argument is valid in characteristic p if $p \nmid d$, which holds by our assumption that $d \mid q-1$. Moreover, $X_{D,\lambda}$ has a smooth model over $\mathbb{Z}_p$. Now we suppose

$$\gcd(d, a) = \gcd(d, b_i) = \gcd(d, c) = \gcd(d, e) = 1. \tag{4.3}$$

Then, we have $\#\pi^{-1}(P) = 1$ for all $P \in \{(0 : 0 : 1), (\lambda_i^{-1} : 0 : 1), (1 : 0 : 1), \infty\}$ (see [1, subsection 3.1]), and we obtain, for all m,

$$N_r(X_{D,\lambda}; \chi^m) = N_r(\overline{C}_{D,\lambda}; \chi^m). \tag{4.4}$$

By (4.2), (4.4) and Theorem 4.2, we obtain the following corollary similarly as Corollary 4.5.

Corollary 4.6. *Under the assumption* (4.3), *we have*

$$N_r(X_{D,\lambda}; \chi^m)$$
$$= \begin{cases} 1 + q^r & (m = 0), \\ -j\left(\varphi_d^{ma}, \varphi_d^{mc}\right)^r F_D^{(n)}\left({\varphi_{d,r}^{ma}; \overline{\varphi_{d,r}}^{mb_1}, \dots, \overline{\varphi_{d,r}}^{mb_n} \atop \varphi_{d,r}^{m(a+c)}} ; \lambda_1, \dots, \lambda_n \right) & (m \neq 0). \end{cases}$$

Therefore, the Artin L-function $L(X_{D,\lambda}, \chi^m; t)$ is expressed in terms of the hypergeometric functions over κ_r $(r \geq 1)$ and the Jacobi sum. In fact, we show that the first $n+1$ functions are sufficient.

Let $l \neq p$ be a prime number and $H^i(\overline{X}_{D,\lambda}, \overline{\mathbb{Q}_l})(\chi^m)$ be the χ^m-eigencomponent of the l-adic étale cohomology of $\overline{X}_{D,\lambda} = X_{D,\lambda} \otimes_\kappa \overline{\kappa}$, where we fixed an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_l}$. By the Grothendieck-Lefschetz trace formula (cf. [6, Theorem 2.9])

$$N_r(X_{D,\lambda}; \chi^m) = \sum_{i=0}^{2} (-1)^i \mathrm{Tr}\Big((F^*)^r \mid H^i(\overline{X}_{D,\lambda}, \overline{\mathbb{Q}_l})(\chi^m) \Big),$$

we have

$$L(X_{D,\lambda},\chi^m;t)=\prod_{i=0}^{2}\det\Big(1-F^*t\mid H^i(\overline{X}_{D,\lambda},\overline{\mathbb{Q}_l})(\chi^m)\Big)^{(-1)^{i+1}}.$$

By the following theorem, it follows that the $F_D^{(n)}$ functions in Corollary 4.6 for $r=1,2,\dots$ are written as symmetric polynomials of the first $n+1$ functions.

Theorem 4.7. *Under the assumption* (4.3)*, if* $m\neq 0$*, then* $L(X_{D,\lambda},\chi^m;t)$ *is a polynomial of degree* $n+1$*.*

Proof. Since $H^i(\overline{X}_{D,\lambda},\overline{\mathbb{Q}_l})=H^i(\overline{X}_{D,\lambda},\overline{\mathbb{Q}_l})(\chi^0)$ for $i=0,2$, it suffices to show

$$d_m:=\dim_{\overline{\mathbb{Q}_l}}H^1(\overline{X}_{D,\lambda},\overline{\mathbb{Q}_l})(\chi^m)=n+1.$$

Since the quotient $X_{D,\lambda}/\mu_d$ is a rational curve, $H^1(\overline{X}_{D,\lambda},\overline{\mathbb{Q}_l})(\chi^0)=0$ and

$$\sum_{m=1}^{d-1}d_m=2\cdot\mathrm{genus}(X_{D,\lambda})=(d-1)(n+1),$$

by [1, Theorem 4.1] (note that d and n are not both even by the assumption (4.3)). Hence, it suffices to show that $d_m\geq n+1$.

By a standard argument using the smooth base change theorem (cf. [6, Theorem 7.3]) and the Artin comparison theorem (cf. [6, Proposition 11.6]), we are reduced to characteristic 0. Regard κ as a residue field of a number field in such a way that the character of $\mu_d(\mathbb{C})\cong\mu_d(\kappa)$ induced by χ is the inclusion. Put

$$S=\Big\{(t_1,\dots,t_n)\in\mathbb{C}^n\mid\prod_{i=1}^{n}t_i(1-t_i)\prod_{j\neq i}(t_j-t_i)\neq 0\Big\},$$

and let $f:X_D\to S$ be the relative projective curve over $\mathbb{C}$ defined by the equation (4.1). Since f is smooth, the relative algebraic de Rham cohomology $\mathcal{H}^1_{\mathrm{dR}}(X_D/S)=R^1f_*\Omega^\bullet_{X_D/S}$ is a locally free $\mathcal{O}_S$-module and $\mathrm{rank}_{\mathcal{O}_S}\mathcal{H}^1_{\mathrm{dR}}(X_D/S)(\chi^m)=d_m$.

For $m=1,\dots,d-1$, put a differential 1-form on the fibre $X_{D,\lambda}$ as

$$\omega_m=\frac{y^m}{x(1-x)}dx.$$

We show that it is of the second kind. It may have a pole only at ∞. A local parametrization of $X_{D,\lambda}$ at ∞ is given by (cf. [1, (7) and (8)])

$$(x,y)=\Big(s^{-d},s^{-(a+c+\sum b_j)}(s^d-1)^{\frac{c}{d}}\prod_i(s^d-\lambda_i)^{\frac{b_i}{d}}\Big),$$

where $s\in\mathbb{C}$ takes values in a neighbourhood of 0 on which $(s^d-1)\prod(s^d-\lambda_i)\neq 0$. Then, we have

$$\omega_m=-d\cdot s^{-m(a+c+\sum b_j)+d-1}(s^d-1)^{\frac{mc}{d}-1}\prod_i(s^d-\lambda_i)^{\frac{mb_i}{d}}\,ds.$$

Since $(s^d-1)^{mc/d}\prod(s^d-\lambda_i)^{mb_i/d}$ is a power series in s^d and $\gcd(d,a+c+\sum b_j)=\gcd(d,e)=1$ by the assumption (4.3), ω_m has the trivial residue, thus is of the second kind. Hence, it defines a section of $\mathcal{H}^1_{\mathrm{dR}}(X_D/S)(\chi^m)$.

Define a path $\delta:[0,1]\to X_{D,\lambda}(\mathbb{C})$ by $\delta(t)=(t,\sqrt[d]{t^a(1-t)^c\prod_i(1-\lambda_it)^{b_i}})$, where the branch of the dth root is taken by setting $|\arg(t^a(1-t)^c\prod_i(1-\lambda_it)^{b_i})|<\pi$

when λ_i are close to 0, and continued analytically. Choose a primitive root $\xi \in \mu_d$ and put $\gamma = \delta - \xi_* \delta$. Then, we have the period by (3.2),

$$\int_\gamma \omega_m = (1-\xi^m) B\Big(\frac{ma}{d}, \frac{mc}{d}\Big) F_D^{(n)} \left(\begin{matrix} \frac{ma}{d}; -\frac{mb_1}{d}, \dots, -\frac{mb_n}{d} \\ \frac{m(a+c)}{d} \end{matrix} ; \lambda_1, \dots, \lambda_n \right).$$

This $F_D^{(n)}$ function satisfies a system of differential equations of rank $n+1$, which is irreducible by a result of Mimachi-Sasaki [21, Theorem 3.1] and our assumption (4.3). This shows that $\mathcal{H}^1_{\mathrm{dR}}(X_D/S)(\chi^m)$ contains an $\mathcal{O}_S$-submodule of rank $n+1$. Hence $d_m \geq n+1$ and the theorem is proved. □

4.4. **Algebraic varieties related to F_A and F_B.** We consider n-dimensional affine hypersurfaces $S^1_{A,\lambda}$, $S^2_{A,\lambda}$ and $S_{B,\lambda}$ over κ defined by the equations

$$\begin{aligned} S^1_{A,\lambda} &: y^d = \Big(1 - \sum_{i=1}^n \lambda_i x_i\Big)^a \prod_{i=1}^n x_i^{b_i} (1-x_i)^{c_i}, \\ S^2_{A,\lambda} &: y^d = \Big(\prod_{i=1}^n (x_i - \lambda_i)^{b_i}\Big)\Big(\prod_{i=1}^n x_i^{c_i}\Big)\Big(1 - \sum_{i=1}^n x_i\Big)^a, \\ S_{B,\lambda} &: y^d = \Big(\prod_{i=1}^n (1-\lambda_i x_i)^{a_i}\Big)\Big(\prod_{i=1}^n x_i^{b_i}\Big)\Big(1 - \sum_{i=1}^n x_i\Big)^c, \end{aligned}$$

where $d, a, a_1, \dots, a_n, b_1, \dots, b_n, c, c_1, \dots, c_n \in \mathbb{Z}_{\geq 1}$, and $\lambda_1, \dots, \lambda_n \in \kappa^\times$. Suppose that $d \mid q-1$. In the same way as the previous subsection, the group μ_d acts on these hypersurfaces. Similarly as in the proof of Theorem 4.4, we can show the followings by using Theorems 3.3, 3.4 and 3.5.

Theorem 4.8.

(i) *Suppose that $gcd(d,a) = gcd(d,c_i) = 1$ for all i. Then,*

$$N_1(S^1_{A,\lambda}; \chi^m) = \begin{cases} q^n & (m=0), \\ \Big(\prod_{i=1}^n -j(\varphi_d^{mb_i}, \varphi_d^{mc_i})\Big) F_A^{(n)} \left(\begin{matrix} \overline{\varphi_d}^{ma}; \varphi_d^{mb_1}, \dots, \varphi_d^{mb_n} \\ \varphi_d^{m(b_1+c_1)}, \dots, \varphi_d^{m(b_n+c_n)} \end{matrix} ; \lambda_1, \dots, \lambda_n \right) & (m \neq 0). \end{cases}$$

(ii) *Suppose that $gcd(d,a) = gcd(d,b_i) = 1$ for all i. Then,*

$$N_1(S^2_{A,\lambda}; \chi^m) = \begin{cases} q^n & (m=0), \\ (-1)^n J_A \cdot F_A^{(n)} \left(\begin{matrix} \overline{\varphi_d}^{m(a+\sum_{i=1}^n (b_i+c_i))}; \overline{\varphi_d}^{mb_1}, \dots, \overline{\varphi_d}^{mb_n} \\ \overline{\varphi_d}^{m(b_1+c_1)}, \dots, \overline{\varphi_d}^{m(b_n+c_n)} \end{matrix} ; \lambda_1, \dots, \lambda_n \right) & (m \neq 0), \end{cases}$$

where $J_A := j\big(\varphi_d^{ma}, \varphi_d^{m(b_1+c_1)}, \dots, \varphi_d^{m(b_n+c_n)}\big)$.

Theorem 4.9. *Suppose that $gcd(d,a_i) = gcd(d,c) = 1$ for all i. Then,*

$N_1(S_{B,\lambda}; \chi^m)$

$$= \begin{cases} q^n & (m=0), \\ (-1)^n J_B \cdot F_B^{(n)}\left({\overline{\varphi_d}^{ma_1}, \dots, \overline{\varphi_d}^{ma_n}; \varphi_d^{mb_1}, \dots, \varphi_d^{mb_n} \atop \varphi_d^{m(b_1+\cdots+b_n+c)}} ; \lambda_1, \dots, \lambda_n \right) & (m \neq 0), \end{cases}$$

where $J_B := j\big(\varphi_d^{mb_1}, \dots, \varphi_d^{mb_n}, \varphi_d^{mc}\big)$.

Similarly as Corollary 4.5, we have the following.

Corollary 4.10. *Put*

$$f_r(\lambda) = F_A^{(n)}\left({\overline{\varphi_{d,r}}^{ma}; \varphi_{d,r}^{mb_1}, \dots, \varphi_{d,r}^{mb_n} \atop \varphi_{d,r}^{m(b_1+c_1)}, \dots, \varphi_{d,r}^{m(b_n+c_n)}} ; \lambda_1, \dots, \lambda_n \right),$$

$$g_r(\lambda) = F_A^{(n)}\left({\overline{\varphi_{d,r}}^{m(a+\sum_{i=1}^n (b_i+c_i))}; \overline{\varphi_{d,r}}^{mb_1}, \dots, \overline{\varphi_{d,r}}^{mb_n} \atop \overline{\varphi_{d,r}}^{m(b_1+c_1)}, \dots, \overline{\varphi_{d,r}}^{m(b_n+c_n)}} ; \lambda_1, \dots, \lambda_n \right),$$

$$h_r(\lambda) = F_B^{(n)}\left({\overline{\varphi_{d,r}}^{ma_1}, \dots, \overline{\varphi_{d,r}}^{ma_n}; \varphi_{d,r}^{mb_1}, \dots, \varphi_{d,r}^{mb_n} \atop \varphi_{d,r}^{m(b_1+\cdots+b_n+c)}} ; \lambda_1, \dots, \lambda_n \right).$$

(i) *Suppose that* $gcd(d,a) = gcd(d,c_i) = 1$ *for all* i. *Then,*

$$L(S^1_{A,\lambda}, \chi^m; t) = \begin{cases} \dfrac{1}{1-q^n t} & (m=0), \\ \exp\Big(\sum_{r=1}^\infty \big(\prod_{i=1}^n -j(\varphi_d^{mb_i}, \varphi_d^{mc_i})^r \big) \cdot f_r(\lambda) \dfrac{t^r}{r} \Big) & (m \neq 0). \end{cases}$$

(ii) *Suppose that* $gcd(d,a) = gcd(d,b_i) = 1$ *for all* i. *Then,*

$$L(S^2_{A,\lambda}, \chi^m; t) = \begin{cases} \dfrac{1}{1-q^n t} & (m=0), \\ \exp\Big(\sum_{r=1}^\infty (-1)^n J_A^r \cdot g_r(\lambda) \dfrac{t^r}{r} \Big) & (m \neq 0), \end{cases}$$

where J_A *is as in Theorem 4.8 (ii).*

(iii) *Suppose that* $gcd(d,a_i) = gcd(d,c) = 1$ *for all* i. *Then,*

$$L(S_{B,\lambda}, \chi^m; t) = \begin{cases} \dfrac{1}{1-q^n t} & (m=0), \\ \exp\Big(\sum_{r=1}^\infty (-1)^n J_B^r \cdot h_r(\lambda) \dfrac{t^r}{r} \Big) & (m \neq 0), \end{cases}$$

where J_B *is as in Theorem 4.9.*

4.5. **Algebraic varieties related to** F_C**.** Let $d, a, b, c_1, \dots, c_n \in \mathbb{Z}_{\geq 1}$ be integers and let $\lambda_1, \dots, \lambda_n \in \kappa^\times$. Write $S_{C,\lambda}$ for the n-dimensional affine hypersurface over κ defined by the equation

$$y^d = \Big(\prod_{i=1}^n x_i^{c_i} \Big) \Big(1 - \sum_{i=1}^n x_i \Big)^a \Big(\prod_{i=1}^n x_i - \sum_{i=1}^n \lambda_i \prod_{j \neq i} x_j \Big)^b.$$

Similarly as in the previous subsections, suppose that $d \mid q-1$ and hence, the group μ_d acts on $S_{C,\lambda}$, and we obtain the following theorem and corollary.

Theorem 4.11. *Suppose that* $gcd(d,a) = gcd(d,b) = 1$. *Then,*

$$N_1(S_{C,\lambda};\chi^m) = \begin{cases} q^n & (m=0),\\ (-1)^n J_C \cdot F_C^{(n)}\left(\begin{matrix}\overline{\varphi_d}^{m(a+nb+\sum_{i=1}^n c_i)}; \overline{\varphi_d}^{mb} \\ \overline{\varphi_d}^{m(b+c_1)},\dots,\overline{\varphi_d}^{m(b+c_n)}\end{matrix}; \lambda_1,\dots,\lambda_n\right) & (m\neq 0),\end{cases}$$

where $J_C = j(\varphi_d^{ma}, \varphi_d^{m(b+c_1)}, \dots, \varphi_d^{m(b+c_n)})$.

Corollary 4.12. *Put*

$$f_r(\lambda) = F_C^{(n)}\left(\begin{matrix}\overline{\varphi_{d,r}}^{m(a+nb+\sum_{i=1}^n c_i)}; \overline{\varphi_{d,r}}^{mb} \\ \overline{\varphi_{d,r}}^{m(b+c_1)},\dots,\overline{\varphi_{d,r}}^{m(b+c_n)}\end{matrix}; \lambda_1,\dots,\lambda_n\right),$$

where $\lambda_1,\dots,\lambda_n \in \kappa^\times$. *Suppose that* $gcd(d,a) = gcd(d,b) = 1$. *Then,*

$$L(S_{C,\lambda},\chi^m;t) = \begin{cases} \dfrac{1}{1-q^n t} & (m=0),\\ \exp\left(\sum_{r=1}^\infty (-1)^n J_C^r \cdot f_r(\lambda)\dfrac{t^r}{r}\right) & (m\neq 0),\end{cases}$$

where J_C *is as in Theorem 4.11.*

Suppose that $\lambda_1, \lambda_2 \neq 1$. Let $S_{4,\lambda}$ be the affine surface over κ defined by the equation

$$\begin{aligned} y^d =& x_1^{\langle a\rangle} x_2^{\langle b\rangle}(1-x_1)^{\langle c_1-a\rangle}(1-x_2)^{\langle c_2-b\rangle} \\ &\times (1-\lambda_1x_1)^{\langle a-c_1-c_2\rangle}(1-\lambda_2x_2)^{\langle b-c_1-c_2\rangle}(1-\lambda_1x_1-\lambda_2x_2)^{\langle c_1+c_2-a-b\rangle}.\end{aligned}$$

Here, for $n \in \mathbb{Z}$, $\langle n\rangle \in \{0,\dots,d-1\}$ denotes the representative of n mod d.

Theorem 4.13. *Suppose that* $gcd(d,a) = gcd(d,b) = gcd(d,c_i-a) = gcd(d,c_i-b) = 1$ *for* $i = 1,2$. *Then,*

$$\begin{aligned} &N_1(S_{4,\lambda};\chi^m) \\ &= \begin{cases} q^2 & (m=0),\\ J\cdot F_4\big(\varphi_d^{ma};\varphi_d^{mb};\varphi_d^{mc_1},\varphi_d^{mc_2};\lambda_1(1-\lambda_2),\lambda_2(1-\lambda_1)\big) & \\ +\sum_{i=0}^2 S_i(\lambda_1,\lambda_2) & (m\neq 0).\end{cases}\end{aligned}$$

Here, J *and* S_i *are as in Theorem 3.12 with* $\alpha = \varphi_d^{ma}$, $\beta = \varphi_d^{mb}$, $\gamma_i = \varphi_d^{mc_i}$.

Proof. Similarly as in the proof of Theorem 4.4, we have

$$N_1(S_{4,\lambda};\chi^0) = q^2.$$

For $m \neq 0$,

$$\begin{aligned} N_1(S_{4,\lambda};\chi^m) = &\sum_{u,v} \varphi_d^{ma}(u)\varphi_d^{mb}(v)\varphi_d^{m(c_1-a)}(1-u)\varphi_d^{m(c_2-b)}(1-v) \\ &\times \varphi_d^{m(a-c_1-c_2)}(1-\lambda_1u)\varphi_d^{m(b-c_1-c_2)}(1-\lambda_2v)\varphi_d^{m(c_1+c_2-a-b)}(1-\lambda_1u-\lambda_2v).\end{aligned}$$

Here, note that $\varphi_d^{\langle n\rangle} = \varphi_d^n$. Thus, the theorem follows by Theorem 3.12. □

Corollary 4.14. *Let the assumptions and notations be as in Theorem 4.13. Put*

$$f_r(\lambda_1,\lambda_2)=F_4\big(\varphi_{d,r}^{ma};\varphi_{d,r}^{mb};\varphi_{d,r}^{mc_1},\varphi_{d,r}^{mc_2};\lambda_1(1-\lambda_2),\lambda_2(1-\lambda_1)\big).$$

Then,

$$L(S_{4,\lambda},\chi^m;t)=\begin{cases}\dfrac{1}{1-q^2t} & (m=0),\\ \exp\Big(\sum_{r=1}^{\infty}J^r\cdot f_r(\lambda_1,\lambda_2)\dfrac{t^r}{r}\Big)\prod_{i=0}^{2}(1-S_i(\lambda_1,\lambda_2)t) & (m\neq 0).\end{cases}$$

Proof. Note that, for $\eta\in\widehat{\kappa^\times}$ and $\eta_r=\eta\circ N_{\kappa_r/\kappa}$, if $\lambda\in\kappa$ then $\eta_r(\lambda)=\eta^r(\lambda)$. Similarly as Corollary 4.5, we obtain

$$N_r(S_{4,\lambda},\chi^m;t)=\begin{cases}q^{2r} & (m=0),\\ J^rf_r(\lambda_1,\lambda_2)+\sum_{i=0}^{2}S_i(\lambda_1,\lambda_2)^r & (m\neq 0),\end{cases}$$

by Theorem 4.13, and hence the corollary follows formally. □

As far as the author knows, explicit constructions of smooth projective models of $S_{D,\lambda},S^1_{A,\lambda},S^2_{A,\lambda},S_{B,\lambda},S_{C,\lambda}$ and $S_{4,\lambda}$ are not known. If we know the Betti numbers of such models, we could apply a similar argument as in Subsection 4.3.

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